Under confounding, KLM conditions are not necessary for consistency

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Overview

The following example illustrates the fact that Assumptions 1–4 in Knox et al. [2020] are not necessary assumptions for the "naive estimator" (i.e., the stratified difference in means) to yield a consistent estimate of the $\textsc{cde}_{\text{Ob}}$, even when there is unobserved confounding between the outcome, Y, and the arrest decision, M. This example is virtually identical to Case 2 in the proof of Theorem 9 in our paper [Gaebler et al., 2020].

More specifically, we give an example in which: (1) the joint distribution of random variables is captured by the following causal DAG:

$$\begin{tikzcd} & & U \arrow[dl] \arrow[d] \\ Z \arrow[r] \arrow[rr, bend right] & M \arrow[r] & Y \end{tikzcd}$$

(2) the conjunction of Assumptions 1–4 is violated; and (3) the stratified difference in means is a consistent estimator of the $\textsc{cde}_{\text{Ob}}$.

Construction of the Example

We set $X = 1$ (i.e., X is constant), and define Z, $M(b)$, $M(w)$, $Y(z,m)$, and U to all be binary variables. The joint distribution of the variables in our example factors as follows:

$$\begin{aligned} &\Pr(Z = z, U = u, M(b) = m_b, M(w) = m_w, Y(z,m) = y_{zm}) \\ &\quad = \Pr(Y(z,m) = y_{zm} \mid U = u) \cdot \Pr(Z = z) \cdot \Pr(U = u) \\ &\qquad \cdot \Pr(M(b) = m_b \mid Z = z, U = u) \\ &\qquad \cdot \Pr(M(w) = m_w \mid Z = z, U = u). \end{aligned} \tag{1}$$

Here we suppress the argument u in the expressions $Y(z,m)$ and $M(z)$ for consistency with our earlier notation.

Now, we set $\Pr(Z = z) = \frac{1}{2}$ for $z \in \{w, b\}$ and $\Pr(U = u) = \frac{1}{2}$ for $u \in \{0, 1\}$. Then, the conditional distributions of the remaining variables are defined as follows:

$$\Pr(M(z) = 1 \mid Z, U) = \begin{cases} \frac{1}{4} \cdot (1 + U) \cdot (1 + \mathbb{1}_{Z=b}) & z = b, \\ \frac{1}{8} \cdot (1 + U) \cdot (1 + \mathbb{1}_{Z=b}) & z = w. \end{cases} \tag{2}$$

Likewise,

$$\Pr(Y(z, m) = 1 \mid U) = \begin{cases} \frac{1}{2}(1 + U) & z = b,\ m = 1 \\ \frac{1}{4}(1 + U) & z = w,\ m = 1 \\ 0 & m = 0. \end{cases} \tag{3}$$

Together, Eqs. (1), (2), and (3), along with the aforementioned distributions of Z and U, fully define the joint probability distribution. Lastly, we set $M = M(Z, U)$ and $Y = Y(Z, M, U)$.

Analysis

"Necessary assumptions" are violated

This example does not satisfy treatment ignorability, as $M(z) \not\perp Z$, contrary to Assumption 4(a) in Knox et al. [2020]. (Because X is constant, we need not condition on it when evaluating the treatment ignorability criterion.) Therefore, in particular, this example does not satisfy the conjunction of Assumptions 1–4.

To see this, we compare $\Pr(M(w) = 1 \mid Z = w)$ and $\Pr(M(w) = 1 \mid Z = b)$.

$$\begin{aligned} \Pr(M(w) = 1 \mid Z = w) &= \sum_{u \in \{0,1\}} \Pr(M(w) = 1 \mid Z = w, U = u) \cdot \Pr(U = u) \\ &= \left(\frac{1}{8} \cdot (1 + 1) \cdot (1 + 0) \cdot \frac{1}{2}\right) + \left(\frac{1}{8} \cdot (1 + 0) \cdot (1 + 0) \cdot \frac{1}{2}\right) \\ &= \frac{1}{8} + \frac{1}{16} \\ &= \frac{3}{16} \\ \Pr(M(w) = 1 \mid Z = b) &= \sum_{u \in \{0,1\}} \Pr(M(w) = 1 \mid Z = b, U = u) \cdot \Pr(U = u) \\ &= \left(\frac{1}{8} \cdot (1 + 1) \cdot (1 + 1) \cdot \frac{1}{2}\right) + \left(\frac{1}{8} \cdot (1 + 0) \cdot (1 + 1) \cdot \frac{1}{2}\right) \\ &= \frac{1}{4} + \frac{1}{8} \\ &= \frac{3}{8} \end{aligned}$$

Subset ignorability holds

However, despite the unobserved confounding between M and Y, subset ignorability still holds in this example. To see this, note that by Eq. (1):

$$\begin{aligned} &\Pr(Y(z, m) = 1, Z = z', M = 1) \\ &\quad = \Pr(Y(z, m) = 1) \cdot \Pr(M(z') = 1 \mid Z = z') \cdot \Pr(Z = z') \\ &\quad = \sum_{u \in \{0,1\}} \big(\Pr(Y(z, m) = 1 \mid U = u) \cdot \Pr(U = u) \\ &\qquad\qquad \cdot \Pr(M(z') = 1 \mid Z = z', U = u) \cdot \Pr(Z = z')\big) \end{aligned} \tag{4}$$

Now, assume $m = 1$. Then, if $z = b$ and $z' = b$, Eq. (4) equals

$$\begin{aligned}\left(\frac{1}{2}(1+1)\cdot\frac{1}{2}\cdot\frac{1}{4}(1+1)(1+1)\cdot\frac{1}{2}\right)+\left(\frac{1}{2}(1+0)\cdot\frac{1}{2}\cdot\frac{1}{4}(1+0)(1+1)\cdot\frac{1}{2}\right)&=\frac{1}{4}+\frac{1}{16}\\&=\frac{5}{16}\end{aligned}$$

where here we use the definitions in Eqs. (2) and (3). If $z = b$ and $z' = w$, then Eq. (4) equals

$$\begin{aligned}\left(\frac{1}{2}(1+1)\cdot\frac{1}{2}\cdot\frac{1}{4}(1+1)(1+0)\cdot\frac{1}{2}\right)+\left(\frac{1}{2}(1+0)\cdot\frac{1}{2}\cdot\frac{1}{4}(1+0)(1+0)\cdot\frac{1}{2}\right)&=\frac{1}{8}+\frac{1}{32}\\&=\frac{5}{32}\end{aligned}$$

Virtually identical calculations show that if $z = w$ and $z' = b$, Eq. (4) equals $\frac{5}{32}$; and if $z = w$ and $z' = w$, then Eq. (4) equals $\frac{5}{64}$ as well.

Next, we see that

$$\begin{aligned}\Pr(Z=b, M=1) &= \Pr(Z=b, M(b)=1)\\&=\sum_{u\in\{0,1\}}\Pr(Z=b, M(b)=1\mid U=u)\cdot\Pr(U=u)\\&=\sum_{u\in\{0,1\}}\Pr(M(b)=1\mid Z=b, U=u)\cdot\Pr(Z=b)\cdot\Pr(U=u)\\&=1\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\\&=\frac{3}{8}\end{aligned}$$

while

$$\begin{aligned}\Pr(Z=w, M=1) &= \Pr(Z=w, M(w)=1)\\&=\sum_{u\in\{0,1\}}\Pr(Z=w, M(w)=1\mid U=u)\cdot\Pr(U=u)\\&=\sum_{u\in\{0,1\}}\Pr(M(w)=1\mid Z=w, U=u)\cdot\Pr(Z=w)\cdot\Pr(U=u)\\&=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\\&=\frac{3}{16}.\end{aligned}$$

Therefore

$$\begin{aligned}\Pr(Y(b,1)=1\mid Z=b, M=1)&=\frac{\Pr(Y(b,1)=1, Z=b, M=1)}{\Pr(Z=b, M=1)}\\&=\frac{5\ /\ 16}{3\ /\ 8}\\&=\frac{5}{6},\\\Pr(Y(b,1)=1\mid Z=w, M=1)&=\frac{\Pr(Y(b,1)=1, Z=w, M=1)}{\Pr(Z=w, M=1)}\\&=\frac{5\ /\ 32}{3\ /\ 16}\\&=\frac{5}{6}.\end{aligned}$$

Also,

$$\begin{aligned}
\Pr(Y(w,1) = 1 \mid Z = b, M = 1) &= \frac{\Pr(Y(w,1) = 1, Z = b, M = 1)}{\Pr(Z = b, M = 1)} \\
&= \frac{5 \ / \ 32}{3 \ / \ 8} \\
&= \frac{5}{12}, \\
\Pr(Y(w,1) = 1 \mid Z = w, M = 1) &= \frac{\Pr(Y(w,1) = 1, Z = w, M = 1)}{\Pr(Z = w, M = 1)} \\
&= \frac{5 \ / \ 64}{3 \ / \ 16} \\
&= \frac{5}{12}.
\end{aligned}$$

This is equivalent to the statement that $Y(z,1)$ is independent of Z given $M = 1$, i.e., $Y(z,1) \perp\!\!\!\perp Z \mid M = 1$. Therefore subset ignorability is satisfied, and so Δ_n is a consistent estimator of the CDE_{Ob}.

References

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