Mathematical Review for EN.540.301: Kinetic Processes

Noah J. Wichrowski

Johns Hopkins University

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Vectors

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- In the context of engineering, a **vector** is an ordered collection of numbers, either real ($\mathbb{R}$) or complex ($\mathbb{C}$).
- In most practical applications, real numbers will suffice.
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Example

$$\boldsymbol{v} = \begin{pmatrix} \pi \\ 0 \\ -2 \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{v} = \begin{pmatrix} \pi & 0 & -2 \end{pmatrix}$$

Vectors

- For each positive integer n, we write $\mathbb{R}^n$ to denote the set of all real-valued vectors with n entries:

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| \; x_1, \ldots, x_n \in \mathbb{R} \right\}.$$

- Subscripts are used to denote the entries in a vector.

Computing with Vectors

- There are two main operations involving vectors:
  - vector addition: for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, the sum $\boldsymbol{v} + \boldsymbol{w} \in \mathbb{R}^n$.
  - scalar multiplication: for any $\boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, the product $c\boldsymbol{v} \in \mathbb{R}^n$.

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  - scalar multiplication: for any $\boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, the product $c\boldsymbol{v} \in \mathbb{R}^n$.
- These are performed element-by-element:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} \qquad \text{and} \qquad c \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} cv_1 \\ \vdots \\ cv_n \end{pmatrix}.$$

Linear Combinations

Definition

If we have a collection of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$, then for any choice of scalars $c_1, \ldots, c_k \in \mathbb{R}$, the quantity

$$\sum_{i=1}^{k} c_i \boldsymbol{v}_i = c_1 \boldsymbol{v}_1 + \cdots + c_k \boldsymbol{v}_k$$

is called a **linear combination** of $\{v_1, \ldots, v_k\}$.

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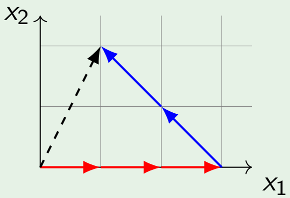
Example

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Span

Definition

The **span** of a set of vectors $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^n$ is the set of all linear combinations of vectors in S:

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} c_i \boldsymbol{v}_i \;\middle|\; c_1, \ldots, c_k \in \mathbb{R} \right\}.$$

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$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \,\middle|\, x_1 \in \mathbb{R} \right\} = \text{“the } x_1\text{-axis (in } \mathbb{R}^2\text{)”}$$

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- Every vector $\boldsymbol{v}_i \in S$ is also in $\operatorname{span}(S)$: $\boldsymbol{v}_i = 1 \cdot \boldsymbol{v}_i$.
- In all cases, the zero vector $\mathbf{0} = \sum_i 0 \cdot \boldsymbol{v}_i \in \operatorname{span}(S)$.

Linear Independence

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Example

Two vectors that are linearly independent:

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies c_2 = 0 \implies c_1 = 0$$

Linear Independence

- If $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is linearly independent, then there is only one way to write $\boldsymbol{w} \in \text{span}(S)$ as a linear combination of $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$.
- Otherwise, we say S is **linearly dependent**.
  - Then we can write one vector as a linear combination of the others.
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Example

Three vectors that are linearly dependent:

$$3\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Vector Spaces

Definition

A **vector space** is a set $V \subseteq \mathbb{R}^n$ that is closed under linear combination: for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $c_1, c_2 \in \mathbb{R}$, we have $c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 \in V$.

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Example

- For any positive integer n, $\mathbb{R}^n$ is a vector space.
- For any set of vectors $S \subset \mathbb{R}^n$, $\text{span}(S)$ is a vector space.
- The set of only the zero vector, $\{\mathbf{0}\}$ is a vector space.

Basis

Definition

A set of vectors $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^n$ is a **basis** for vector space V if:

- $\text{span}(S) = V$, and
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Example

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 + x_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Standard Basis Vectors

Definition

The **standard basis** for $\mathbb{R}^n$ is denoted $\mathcal{E}_n = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ and consists of the n vectors that form the columns of an $n \times n$ identity matrix. That is,

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \cdots, \qquad \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

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- Exercise: Confirm that $\mathcal{E}_n$ satisfies the defining properties of a basis.
- For any $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{e}_k \in \mathbb{R}^n$,

$$A\boldsymbol{e}_k = \text{the } k^{\text{th}} \text{ column of } A.$$

Dimension

Theorem

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Example

- For any positive integer n, $\dim(\mathbb{R}^n) = n$.
- If $V = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_2 = x_3\}$, $\dim(V) = 2$. (Can you find a basis for V?)

Matrices

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- A **matrix** is a rectangular array of numbers, indexed by two values.
- We can add and scale matrices in analogous fashion to vectors.
- Some (but not all) matrices can be multiplied together.

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Example

$$B = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -3 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 3}$$

is a two-by-three matrix with $b_{1,3} = -2$, $b_{2,1} = 0$, *etc.*

Matrix Multiplication

Definition

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ be matrices. We define the product $C = AB \in \mathbb{R}^{m \times p}$ with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \, .$$

$$\begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}$$

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Matrices as Functions

- We often associate a matrix $A \in \mathbb{R}^{m \times n}$ to the function $f(\boldsymbol{v}) = A\boldsymbol{v}$.
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Definition

The **range** of a matrix $A \in \mathbb{R}^{m \times n}$ (also called **image** or **column space**) is the set of all possible outputs of matrix-vector multiplication:

$$\mathcal{R}(A) = \{A\boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m .$$

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Definition

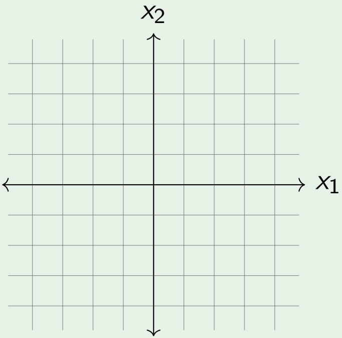
The **null space** of a matrix $A \in \mathbb{R}^{m \times n}$ (also called **kernel**) is the set of all vectors for which matrix-vector multiplication yields the zero vector:

$$\mathcal{N}(A) = \{\boldsymbol{v} \in \mathbb{R}^n \mid A\boldsymbol{v} = \mathbf{0}\} \subseteq \mathbb{R}^n .$$

Range and Nullspace

Example

Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then for $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $A\boldsymbol{x} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_1 \end{pmatrix}$.



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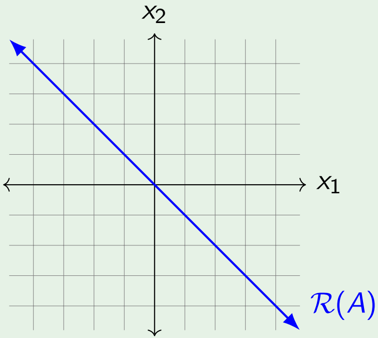
- For all $\boldsymbol{x}$, $(A\boldsymbol{x})_1 = -(A\boldsymbol{x})_2$.

$$\mathcal{R}(A) = \left\{ \begin{pmatrix} z \\ -z \end{pmatrix} \middle| \, z \in \mathbb{R} \right\}$$



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- $A\boldsymbol{x} = \boldsymbol{0} \iff x_1 = x_2$.

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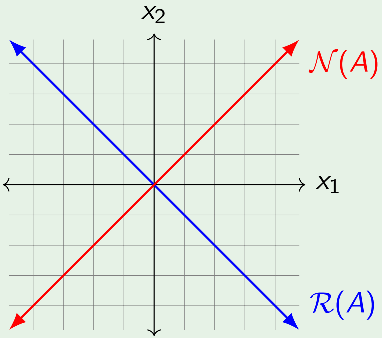

Range and Nullspace

Theorem

Let $A \in \mathbb{R}^{m \times n}$ *be a matrix. Then* $\mathcal{R}(A)$ *and* $\mathcal{N}(A)$ *are vector spaces.*

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Proof.

(i) Suppose $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{R}(A)$. Then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \boldsymbol{u}$ and $A\boldsymbol{y} = \boldsymbol{v}$. For any scalars $c_1, c_2 \in \mathbb{R}$, we have

$$c_1\boldsymbol{u} + c_2\boldsymbol{v} = c_1A\boldsymbol{x} + c_2A\boldsymbol{y} = A(c_1\boldsymbol{x} + c_2\boldsymbol{y}) \in \mathcal{R}(A)\,.$$

□

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$$c_1\boldsymbol{u} + c_2\boldsymbol{v} = c_1A\boldsymbol{x} + c_2A\boldsymbol{y} = A(c_1\boldsymbol{x} + c_2\boldsymbol{y}) \in \mathcal{R}(A)\,.$$

(ii) Suppose $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{N}(A)$. Then $A\boldsymbol{u} = \boldsymbol{0}$ and $A\boldsymbol{v} = \boldsymbol{0}$. For any scalars $c_1, c_2 \in \mathbb{R}$, we have

$$A(c_1\boldsymbol{u} + c_2\boldsymbol{v}) = c_1A\boldsymbol{u} + c_2A\boldsymbol{v} = c_1\boldsymbol{0} + c_2\boldsymbol{0} = \boldsymbol{0}\,,$$

so $c_1\boldsymbol{u} + c_2\boldsymbol{v} \in \mathcal{N}(A)$. $\square$

Linear Systems

- Consider a set of m linear equations with n variables:

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad \ddots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_1 \end{matrix}$$

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- We can write the system as a single matrix equation: $A\boldsymbol{x} = \boldsymbol{b}$.
- Given A and $\boldsymbol{b}$, how many solution vectors $\boldsymbol{x}$ exist?
  - zero?
  - one?
  - infinitely many?

Gaussian Elimination

- Suppose we want to solve the following system:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_1 + x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned} \tag{$\star$}$$

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- Begin by building the **augmented matrix** of coefficients:

$$
\left(\begin{array}{rrr|r}
1 & -2 & 3 & 7 \\
-1 & 1 & -2 & -5 \\
2 & -1 & -1 & 4
\end{array}\right)
$$

Row Reduction

- There are three **row operations** we can use to convert the augmented matrix into a form that reveals the solution:
  - Multiply a row by a constant $c \neq 0$,
  - Add a multiple of one row to another row,
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  - Multiply a row by a constant $c \neq 0$,
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  - Swap two rows.
- Performing any combination of these operations on the rows leaves the set of solutions unchanged.

Row Reduction

Example

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right) \xrightarrow{\texttt{R2 += R1}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 2 & -1 & -1 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 2 & -1 & -1 & 4 \end{array}\right) \xrightarrow{\texttt{R3 += (-2)R1}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

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Row Reduction

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$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{array}\right) \xrightarrow{\texttt{R3 *= -1/4}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Row Reduction

Example

The total effect of our row operations:

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 7 \\ -x_1 & + & x_2 & - & 2x_3 & = & -5 \\ 2x_1 & - & x_2 & - & x_3 & = & 4 \end{array}\right\} \longrightarrow \left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 7 \\ & & x_2 & - & x_3 & = & -2 \\ & & & & x_3 & = & 1 \end{array}\right\}$$

"Back substitution" yields the answer:

$$\begin{aligned} x_3 = 1 &\implies x_2 = -2 + x_3 = -2 + 1 = -1 \\ &\implies x_1 = 7 + 2x_2 - 3x_3 = 2 \end{aligned} \qquad \boldsymbol{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Systems with No Solutions

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ -x_1 + x_2 - 2x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 1 \end{aligned}$$

- Performing row reduction, we obtain

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -1 & 1 & -2 & 1 \\ 2 & -1 & 3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right).$$

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- The bottom row implies $0 = 5$, so the system is inconsistent.
- There is no $\boldsymbol{x} \in \mathbb{R}^3$ that satisfies all three equations.

Systems with Multiple Solutions

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 2 \\
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- The bottom row has become the trivial equation $0 = 0$.
- We must analyze the remaining equations.

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- We can find a solution for any choice of $x_3 \in \mathbb{R}$.
- The set of all solutions is

$$\left\{ z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}.$$

Homogeneous Linear Systems

Definition

A linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$ is called **homogeneous** if $\boldsymbol{b} = \mathbf{0}$. Otherwise, the system is called **inhomogeneous**.

- A homogeneous system always has the trivial solution $\boldsymbol{x} = \mathbf{0}$.

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- Recall: if $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ are linearly dependent, there are multiple ways to write $\boldsymbol{w} \in \text{span}(S)$ as a linear combination:

$$\boldsymbol{w} = \sum_{i=1}^{k} c_i \boldsymbol{v}_i = \sum_{i=1}^{k} c_i \boldsymbol{v}_i + \mathbf{0} = \sum_{i=1}^{k} c_i \boldsymbol{v}_i + \sum_{i=1}^{k} d_i \boldsymbol{v}_i = \sum_{i=1}^{k} (c_i + d_i) \boldsymbol{v}_i \,.$$

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- A nontrivial null space indicates multiple solutions to a system:
  - Suppose $\boldsymbol{y}$ is one solution: $A\boldsymbol{y} = \boldsymbol{b}$.
  - Then the set of all solutions is $\{\boldsymbol{y} + \boldsymbol{x} \,|\, \boldsymbol{x} \in \mathcal{N}(A)\}$, since

$$A(\boldsymbol{y} + \boldsymbol{x}) = A\boldsymbol{y} + A\boldsymbol{x} = \boldsymbol{b} + A\boldsymbol{x} = \boldsymbol{b} \iff A\boldsymbol{x} = \mathbf{0}\,.$$

Matrix Inverses

- Consider the system $A\boldsymbol{x} = \boldsymbol{b}$ with $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^n$.
- If the rows of A are linearly independent, then there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that

$$AB = BA = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

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Definition

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. If $B \in \mathbb{R}^{n \times n}$ satisfies $AB = BA = I$, we call B the **inverse** of A and write $B = A^{-1}$.

Matrix Inverses: Comments

- We can solve linear systems by multiplying with the inverse matrix:

$$\boldsymbol{x} = I\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}\,.$$

- There are infinitely many square matrices that do not have an inverse, so "matrix division" is a misnomer.
- If a matrix has an inverse, it is unique: suppose B and C are both inverses of A; then $B = IB = CAB = CI = C$.

Fundamental Theorem of Linear Algebra

Definition

The **rank** of a matrix $A \in \mathbb{R}^{m \times n}$ is the dimension of its column space:

$$\mathrm{rk}(A) = \dim(\mathcal{R}(A))\,.$$

Theorem

The rank of any matrix A equals both

- *the maximum number of linearly independent columns of A, and*
- *the maximum number of linearly independent rows of A.*

Example

$$\mathrm{rk}\left[\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\right] = 1 \qquad \mathrm{rk}\left[\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{pmatrix}\right] = 2 \qquad \mathrm{rk}\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] = 0$$

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Theorem (FToLA)

Let $A \in \mathbb{R}^{m\times n}$ be any matrix.

$$\mathrm{rk}(A) + \dim(\mathcal{N}(A)) = n\,.$$

A Detailed Example

- Consider a linear system of the form $A\boldsymbol{x} = \boldsymbol{b}$.
- For now, let's suppose the system is homogeneous, *i.e.*, $\boldsymbol{b} = \mathbf{0}$.

$$A = \begin{pmatrix} 0 & -1 & 2 & 2 & -2 \\ -2 & -5 & -2 & 0 & 2 \\ -1 & -1 & -4 & 1 & 0 \\ 1 & 3 & 0 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{4\times 5}$$

Row Operations

$$\begin{pmatrix} 0 & -1 & 2 & 2 & -2 \\ -2 & -5 & -2 & 0 & 2 \\ -1 & -1 & -4 & 1 & 0 \\ 1 & 3 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{\texttt{R1} \leftrightarrow \texttt{R4}} \xrightarrow{\texttt{R2 += (2)R1}} \xrightarrow{\texttt{R3 += R1}} \xrightarrow{\texttt{R3 += (-2)R2}}$$

$$\xrightarrow{\texttt{R4 += R2}} \xrightarrow{\texttt{R4 += (-2)R3}} \xrightarrow{\texttt{R1 += (-3)R2}} \begin{pmatrix} 1 & 0 & 6 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced Row Echelon Form

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 6 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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A matrix $B \in \mathbb{R}^{m \times n}$ is in **reduced row echelon form** (RREF) if:

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- Each leading entry is a 1 and is the only non-zero entry in the corresponding column.

RREF and the Fundamental Theorem of Linear Algebra

$$\mathrm{rref}\left[\begin{pmatrix} 0 & -1 & 2 & 2 & -2 \\ -2 & -5 & -2 & 0 & 2 \\ -1 & -1 & -4 & 1 & 0 \\ 1 & 3 & 0 & 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 6 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Recall: for any $A \in \mathbb{R}^{m \times n}$, $\mathrm{rk}(A) + \dim(\mathcal{N}(A)) = n$.
- Each column corresponds either to a pivot or to a free variable.

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RREF and Range of a Matrix

$$\operatorname{rref}\left[\begin{pmatrix} 0 & -1 & 2 & 2 & -2 \\ -2 & -5 & -2 & 0 & 2 \\ -1 & -1 & -4 & 1 & 0 \\ 1 & 3 & 0 & 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 6 & 0 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{R}(A) &= \operatorname{span}\{\text{columns from } A \text{ that contain pivots in } \operatorname{rref}(A)\} \\ &= \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

RREF and Null Space of a Matrix

$$\mathcal{N}(A) = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \mathbf{0}\} = \{\text{solutions to homogeneous system}\}\,.$$

- For our example system, $A\boldsymbol{x} = \mathbf{0}$ entails

$$\begin{aligned} x_1 \quad + 6x_3 \quad - x_5 &= 0 \\ x_2 - 2x_3 \quad &= 0 \\ x_4 - x_5 &= 0 \end{aligned} \implies \begin{aligned} -6x_3 + x_5 &= x_1 \\ 2x_3 \quad &= x_2 \\ x_5 &= x_4 \end{aligned}$$

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- We can find a basis for $\mathcal{N}(A)$ by setting one free variable at a time to the value 1 and solving for the "pivot" variables.

$$\mathcal{N}(A) = \text{span}\left\{ \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Eigenvalues

Definition

Let $A \in \mathbb{R}^{n \times n}$. A scalar $\lambda \in \mathbb{R}$ is called an **eigenvalue** of A if there exists a *non-zero* vector $\boldsymbol{v} \in \mathbb{R}^n$ satisfying $A\boldsymbol{v} = \lambda\boldsymbol{v}$. In such case, $\boldsymbol{v}$ is called an **eigenvector** of A associated with λ.

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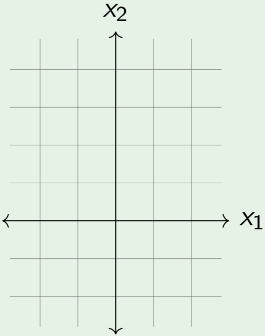
Example

Consider the matrix

$$A = \begin{pmatrix} 1 & 0.5 \\ 1 & 1.5 \end{pmatrix} .$$



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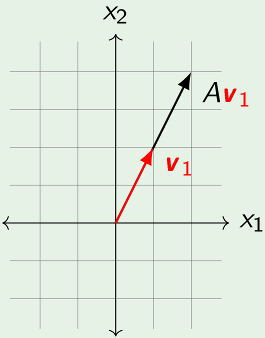
Consider the matrix

$$A = \begin{pmatrix} 1 & 0.5 \\ 1 & 1.5 \end{pmatrix}.$$

The eigenvalues of A are 2 and 0.5, since

$$\begin{pmatrix} 1 & 0.5 \\ 1 & 1.5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$



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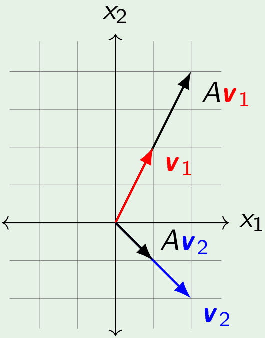

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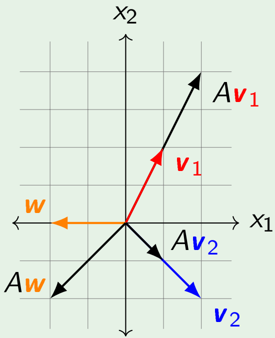

Eigenvalues

- What do eigenvalues and eigenvectors tell us about a matrix?
  - Eigenvectors act as a sort of "natural coordinate system."
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- What do eigenvalues and eigenvectors tell us about a matrix?
  - Eigenvectors act as a sort of "natural coordinate system."
  - Eigenvalues indicate stability or instability in dynamical systems.
  - Eigenvalues can measure importance in data analysis.
- How can we determine the eigenvalues and eigenvectors of a matrix?
  - If $A\boldsymbol{v} = \lambda\boldsymbol{v}$, then $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$.
  - Since $\boldsymbol{v} \neq \mathbf{0}$, we know $A - \lambda I$ is *not* invertible.
  - We usually express this condition as: $\det(\lambda I - A) = 0$.
  - Knowing λ, we can solve $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$.

Example

Consider the matrix from our previous example:

$$A = \begin{pmatrix} 1 & 0.5 \\ 1 & 1.5 \end{pmatrix} \implies \lambda I - A = \begin{pmatrix} \lambda - 1 & -0.5 \\ -1 & \lambda - 1.5 \end{pmatrix}.$$

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 1)(\lambda - 1.5) - (-1)(-0.5) \\ &= \lambda^2 - 2.5\lambda + 1 \\ &= (\lambda - 2)(\lambda - 0.5) = 0 \end{aligned}$$

Therefore, the eigenvalues of A are $\{0.5, 2\}$.

Example

- Find an eigenvector associated with $\lambda_1 = 2$:

$$(\lambda_1 I - A)\boldsymbol{v} = \begin{pmatrix} 1 & -0.5 \\ -1 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 0.5y \\ -x + 0.5y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both equations imply $x = 0.5y$, so $\boldsymbol{v}_1 = (1,\ 2)^\top$ is an eigenvector.

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  Both equations imply $x = 0.5y$, so $\boldsymbol{v}_1 = (1,\ 2)^\top$ is an eigenvector.

- Find an eigenvector associated with $\lambda_2 = 0.5$:

$$(\lambda_2 I - A)\boldsymbol{v} = \begin{pmatrix} -0.5 & -0.5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.5x - 0.5y \\ -x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  Both equations imply $x = -y$, so $\boldsymbol{v}_2 = (1,\ -1)^\top$ is an eigenvector.

Differential Equations

Types of Differential Equations

Definition

The **order** of a differential equation

$$F\left[x,\, y(x),\, \frac{dy}{dx},\, \cdots,\, \frac{d^n y}{dx^n}\right] = 0$$

is the order of the highest derivative that appears in the expression for F.

Example

$$\frac{dC}{dt} = -kC$$

first-order

$$\frac{d^2 u}{dx^2} - x\,\frac{du}{dx} + u = 2$$

second-order

Types of Differential Equations

Definition

A first-order differential equation is called **separable** if it can be written in the following equivalent forms:

$$\begin{aligned} f(x) + g(y)\,\frac{dy}{dx} &= 0\,, \\ g(y)\,dy &= -f(x)\,dx\,. \end{aligned}$$

Example

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2}{1+y^2} \\ (1+y^2)\frac{dy}{dx} &= x^2 \\ (1+y^2)\,dy &= x^2\,dx \end{aligned}$$

Types of Differential Equations

Definition

An ordinary differential equation is called **linear** if it can be written as a linear combination of the function $y(x)$ and its derivatives:

$$\sum_{k=0}^{n} a_k(x) y^{(k)}(x) = f(x)\,.$$

Note that the coefficients $a_k(x)$ and “source term” $f(x)$ may depend on the independent variable x.

Example

$$(x^2+1)\,\frac{d^2y}{dx^2} + y = \sin(x)$$

linear

$$\left(\frac{dy}{dx}\right)^2 + y\,\frac{d^2y}{dx^2} = 0$$

non-linear

Types of Differential Equations

Definition

A linear differential equation is called **homogeneous** if it can be written as a linear combination of $y(x)$ and its derivatives with no source term:

$$\mathcal{L}y := \sum_{k=0}^{n} a_k(x) y^{(k)}(x) = 0\,.$$

Otherwise, the equation is called **inhomogeneous**.

Types of Differential Equations

Definition

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Otherwise, the equation is called **inhomogeneous**.

Parallels to Matrix Equations

- A homogeneous ODE always has the trivial solution $y(x) \equiv 0$.
- If y_p is one solution to the linear equation $\mathcal{L}y = f(x)$, then the set of all such solutions is $\{y_p + y_h \,|\, \mathcal{L}y_h = 0\}$.

Integrating Factor

- Suppose we want to solve the following equation:

$$\frac{dy}{dx} + g(x)\, y(x) = f(x)\,.$$

- If $v(x) = \int g(x)\, dx$, then we have

$$\frac{d}{dx}\left[e^{v(x)}\, y(x)\right] = e^{v(x)}\, \frac{dy}{dx} + e^{v(x)}\, \frac{dv}{dx}\, y(x)$$

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&= e^{v(x)}\, \left[\frac{dy}{dx} + g(x)\, y(x)\right] \\
&= e^{v(x)}\, f(x)\,.
\end{aligned}$$

- We can solve for $y(x)$ explicitly:

$$y(x) = e^{-v(x)} \left(\int e^{v(x)}\, f(x)\, dx + C\right)\,.$$

Solving Differential Equations

Example

$$\frac{dy}{dx} + 2\,y(x) = e^{-3x} \qquad y(0) = 0$$

Our integrating factor is $e^{\int 2\,dx} = e^{2x}$. Hence,

$$\begin{aligned}
e^{2x}\left(\frac{dy}{dx} + 2\,y(x)\right) &= e^{2x}e^{-3x} \\
\frac{d}{dx}\left[e^{2x}\,y(x)\right] &= e^{-x} \\
e^{2x}\,y(x) &= \int e^{-x}\,dx \\
e^{2x}\,y(x) &= -e^{-x} + C \qquad\qquad y(0) = 0 \implies C = 1 \\
y(x) &= e^{-2x}(1 - e^{-x}) \\
y(x) &= e^{-2x} - e^{-3x}
\end{aligned}$$

Appendix

Many Ways to Say a Matrix is Invertible

Theorem

Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:

- *A is invertible: there exists A^{-1} such that $AA^{-1} = A^{-1}A = I$.*
- *A has full rank: $\mathrm{rk}(A) = n$.*
- *A has a trivial null space: $\mathcal{N}(A) = \{\mathbf{0}\}$.*
- *The rows of A are linearly independent.*
- *The columns of A are linearly independent.*
- *The columns of A span $\mathbb{R}^n$.*
- *The columns of A form a basis for $\mathbb{R}^n$.*
- *The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\boldsymbol{x} = \mathbf{0}$.*
- *For each $\boldsymbol{b} \in \mathbb{R}^n$, there is exactly one solution to $A\boldsymbol{x} = \boldsymbol{b}$.*
- *None of the eigenvalues of A is 0.*
- $\det(A) \neq 0$.

Additional Resources

Textbook

- Jim Hefferon has written an open-source Linear Algebra textbook that is available for free online.
- Ancillary materials can be found on the book's webpage.

MATLAB Tutorial

- The MathWorks company offers an interactive online course on the MATLAB functions used to solve systems and compute eigenvalues.
- They also have courses for various other topics.

Symbolic Package for Differential Equations

MATLAB Can Solve ODEs Symbolically

```
>> syms y(t); % Define y as a symbolic function of t.
>> equation = diff(y, t) + 2*y == exp(-3*t);
>> dsolve(eqn) % Finds a general solution.
ans(t) =
        C1*exp(-2*t) - exp(-3*t)
>> IC = y(0) == 0; % Specify initial condition.
>> dsolve(eqn, IC) % Find particular solution.
ans(t) =
        exp(-3*t)*(exp(t) - 1)
>> eqn2 = diff(y, t, 2) == cos(2*t) - y;
>> dydt = diff(y, t);
>> ICs = [y(0) == 1, dydt(0) == 0];
>> soln = dsolve(eqn2, ICs);
>> simplify(soln)
    1 - (8*sin(x/2)^4)/3
```

Textbook Problem 2.9a

Problem Statement

Consider the element balance matrix equation for a system of s chemical species, comprised of e elements: $\nu\mathcal{A} = \mathbf{0}$ or

$$\begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_s \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1e} \\ a_{21} & a_{22} & \cdots & a_{2e} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{se} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Use the fundamental theorem of linear algebra to show that the number of linearly independent reactions that satisfy this equation is

$$i = s - \mathrm{rk}(\mathcal{A}).$$

Textbook Problem 2.9a

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Solution

- We can assume the stoichiometric matrix $\nu \in \mathbb{R}^{i\times s}$ contains a "full set" of i linearly independent reactions.

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- We can assume the stoichiometric matrix $\nu \in \mathbb{R}^{i \times s}$ contains a "full set" of i linearly independent reactions.
- Transpose the balance equation: $\mathcal{A}^\top \nu^\top = \mathbf{0}$.
- Apply FToLA to $\mathcal{A}^\top$: $\text{rk}(\mathcal{A}^\top) + \dim(\mathcal{N}(\mathcal{A}^\top)) = s$.

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- Apply FToLA to $\mathcal{A}^\top$: $\mathrm{rk}(\mathcal{A}^\top) + \dim(\mathcal{N}(\mathcal{A}^\top)) = s$.
- Each of the i columns of $\nu^\top$ is in $\mathcal{N}(\mathcal{A}^\top)$, so $i = \dim(\mathcal{N}(\mathcal{A}^\top))$.
- Since $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{A}^\top)$, we obtain $s = i + \mathrm{rk}(\mathcal{A})$.