Exercise 7.1

Compute x_0, χ_1 and x_2 such that

$$\left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right] \left[\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right] \left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right]^T \begin{bmatrix} x_0 \\ \chi_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:- Computing

$$\left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right] \left[\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right] = \begin{bmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & 0 & v_{12} e_F^T E_{22} \\ 0 & 0 & U_{22} E_{22} \end{bmatrix}$$

Solving the next step

$$\begin{bmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & 0 & v_{12} e_F^T E_{22} \\ 0 & 0 & U_{22} E_{22} \end{bmatrix} \begin{bmatrix} L_{00}^T x_0 + \lambda_{10} e_L \chi_1 \\ \chi_1 \\ v_{12} e_F \chi_1 + U_{22}^T x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Given that $x = \left[\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right]$ is not a zero vector.

the above matrix-vector multiplication satisfies only when

$$L_{00}^T x_0 + \lambda_{10} e_L \chi_1 = 0$$

$$\chi_1 = 1$$

and

$$v_{12} e_F \chi_1 + U_{22}^T x_2 = 0$$

Solving

$$\begin{bmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & v_{12} e_F & U_{22}^T \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} L_{00}^T x_0 + \lambda_{10} e_L x_1 \\ x_1 \\ v_{12} e_F x_1 + U_{22}^T x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solving further, we get $x_1 = 1$

$$L_{00}^T x_0 + \lambda_{10} e_L = 0$$

and $$v_{12} e_F + U_{22}^T x_2 = 0$$

L_{00} is strictly lower triangular and is bidiagonal.
Hence it has 1's on its diagonal.

$$L_{00}^T x_0 = -\lambda_{10} e_L$$

$$x_0 = -\lambda_{10} e_L L_{00}^{-T}$$

Similarly, we know U_{22} is strictly upper triangular, which means diagonal has 1's on it. and it is also bidiagonal.

$$v_{12} e_F + U_{22}^T x_2 = 0$$

$$U_{22}^T x_2 = - v_{12} e_F$$

$$x_2 = - v_{12} e_F U_{22}^{-T}$$

The final value of x is

$$x = \begin{bmatrix} -x_{10} e_L L_{00}^{-T} \\ 1 \\ -v_{12} e_F U_{22}^{-T} \end{bmatrix}$$

Note:- Although equation is expressed in terms of inverse. In reality we can just solve it directly since L_{00} and U_{22} are strictly lower triangular and bidiagonal matrix. It can be solved much more easier and simply than triangular solves.

Cost of computation given L_{00} and U_{22} have special (bidiagonal) structure.

- All the computation is in the final matrix vector multiplication. which is

$$\left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right]^T \left[\begin{array}{c} x_0 \\ \hline x_1 \\ \hline x_2 \end{array}\right]$$

1) First step is the matrix-vector multiplication which takes $O(m^2)$ for a matrix of size m. just to multiply out all the elements of matrix with the vector x.

2) Next step is computing individual elements of the vector x.

2a) Solution of x_0 is expressed in terms of the following steps:-

$$L_{00}^T x_0 + \lambda_{10} e_L = 0$$

The solution of this step is a triangular solve of a bidiagonal matrix. and also has ones on diagonal and only non zero entries are on the sub diagonal. Solving the above using back subitution should take only $O(2m)$ complexity.

2b) Solution of x_2 is expressed in terms of following step:-

$$V_{12} e_F + U_{22}^T x_2 = 0$$

The solution of this equation is an upper triangular solve which has non-zero only on the super diagonal.

So this can be solved in $O(2m)$ time complexity.

The total cost of computing x is

$$O(n^2) + O(2n) + O(2n)$$