Exercise 7.1

Compute x_0, χ_1 and x_2 such that

$$\left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right] \left[\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right] \left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right]^T \begin{bmatrix} x_0 \\ \chi_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:- Computing

$$\left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right] \left[\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right] = \begin{bmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & 0 & v_{12} e_F^T E_{22} \\ 0 & 0 & U_{22} E_{22} \end{bmatrix}$$

Solving the next step

$$\begin{bmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & 0 & v_{12} e_F^T E_{22} \\ 0 & 0 & U_{22} E_{22} \end{bmatrix} \begin{bmatrix} L_{00}^T x_0 + \lambda_{10} e_L \chi_1 \\ \chi_1 \\ v_{12} e_F \chi_1 + U_{22}^T x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Given that $x = \left[\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right]$ is not a zero vector.

the above matrix-vector multiplication satisfies only when

$$L_{00}^T x_0 + \lambda_{10} e_L \chi_1 = 0$$

$$\chi_1 = 1$$

and

$$v_{12} e_F \chi_1 + U_{22}^T x_2 = 0$$

Solving

$$\begin{bmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & v_{12} e_F & U_{22}^T \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} L_{00}^T x_0 + \lambda_{10} e_L x_1 \\ x_1 \\ v_{12} e_F x_1 + U_{22}^T x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solving further, we get $x_1 = 1$

$$L_{00}^T x_0 + \lambda_{10} e_L = 0$$

and $\quad v_{12} e_F + U_{22}^T x_2 = 0$

L_{00} is strictly lower triangular and is bidiagonal.
Hence it has 1's on its diagonal.
So if we choose x_0 to be $-\lambda_{10} e_L$, we satisfy

$$L_{00}^T x_0 + \lambda_{10} e_L = 0$$

Hence $x_0 = -\lambda_{10} e_L$

Basically x_0 is standard basis vector e_L scaled by $-\lambda_{10}$

Similarly, we know U_{22} is strictly upper triangular, which means diagonal has 1's on it. and it is also bidiagonal.
So if we choose x_2 to be $-v_{12} e_F$ we satisfy $v_{12} e_F + U_{22}^T x_2 = 0$

Hence $x_2 = -v_{12} e_F$

x_2 is standard basis vector e_F scaled by $-v_{12}$

The final value of x is therefore

$$x = \begin{bmatrix} -x_{10} e_L \\ \hline 1 \\ \hline -v_{12} e_F \end{bmatrix}$$

Cost of computation given L_{00} and U_{22} have special (bidiagonal) structure.

- All the computation is in the final matrix vector multiplication. which is

$$\left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right]^T \left[\begin{array}{c} x_0 \\ \hline x_1 \\ \hline x_2 \end{array}\right]$$

1) First step is the matrix-vector multiplication which takes $O(m^2)$ for a matrix of size m. just to multiply out all the elements of matrix with the vector x.

2) Next step is computing individual elements of the vector x.

2a) Solution of x_0 is expressed in terms of the following steps:-

$$L_{00}^T x_0 + \lambda_{10} e_L = 0$$

The solution for x_0 is $-\lambda_{10}$ times standard basis vector e_L which is $O(1)$ complexity.

2b) Solution of x_2 is expressed in terms of following step:-

$$v_{12} e_F + U_{22}^T x_2 = 0$$

The solution for x_2 is $-v_{12}$ times standard basis vector e_F which is $O(1)$ complexity.

So total cost of this computation is $O(m^2) + O(1) + O(1)$ which is equal to $O(m^2)$.

The total cost of computing x is

$O(n^2)$