Let

$$A = \left(\begin{array}{c|c|c} A_{00} & \alpha_{10} e_L & 0 \\ \hline \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ \hline 0 & \alpha_{21} e_F & A_{22} \end{array}\right), L = \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & 0 \\ \hline 0 & \lambda_{21} e_F & L_{22} \end{array}\right), U = \left(\begin{array}{c|c|c} U_{00} & \upsilon_{01} e_L & 0 \\ \hline 0 & 1 & \upsilon_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right),$$

$$D = \left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \delta_1 & 0 \\ \hline 0 & 0 & D_{22} \end{array}\right), \text{ and } E = \left(\begin{array}{c|c|c} E_{00} & 0 & 0 \\ \hline 0 & \varepsilon_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right),$$

where all the partitioning is "conformal".

Exercise 6.1 *Provided ϕ_1 is chosen appropriately,*

$$\left(\begin{array}{c|c|c} A_{00} & \alpha_{10} e_L & 0 \\ \hline \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ \hline 0 & \alpha_{21} e_F & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right) \left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \phi_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right) \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)^T.$$

- *Show that $\phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$. (Hint: multiply out $A = LDL^T$ and $A = UEU^T$ with the partitioned matrices first. Then multiply out the above. Compare and match...)*

- *What is the cost of computing* **one** *twisted factorization* **given that you have already computed the** *LDL^T* **and** *UEU^T* **factorizations?** *A "Big O" estimate is sufficient. (Do not count the cost of copying data. Only count floating point computations).*

- *What is the cost of computing* **all** *twisted factorizations* **given that you have already computed the** *LDL^T* **and** *UEU^T* **factorizations?** *A "Big O" estimate is sufficient. (Do not count the cost of copying data. Only count floating point computations).*

Solving $A = LDL^T$

$$= \left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & 0 \\ \hline 0 & \lambda_{12} e_F & L_{22} \end{array}\right] \left[\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \delta_1 & 0 \\ \hline 0 & 0 & D_{22} \end{array}\right] \left[\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & 0 \\ \hline 0 & \lambda_{12} e_F & L_{22} \end{array}\right]^T$$

$$= \left[\begin{array}{c|c|c} L_{00} D_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T D_{00} & \delta_1 & 0 \\ \hline 0 & \lambda_{12} e_F \delta_1 & L_{22} D_{22} \end{array}\right] \left[\begin{array}{c|c|c} L_{00}^T & \lambda_{10} e_L & 0 \\ \hline 0 & 1 & \lambda_{12} e_F^T \\ \hline 0 & 0 & L_{22}^T \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} L_{00} D_{00} L_{00}^T & L_{00} D_{00} \lambda_{10} e_L & 0 \\ \hline \lambda_{10} e_L^T D_{00} L_{00}^T & \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \delta_1 & \delta_1 \lambda_{12} e_F^T \\ \hline 0 & \lambda_{12} e_F \delta_1 & \begin{array}{c} \lambda_{12} e_F \delta_1 \lambda_{12} e_F^T \\ + L_{22} D_{22} L_{22}^T \end{array} \end{array}\right]$$

Solving $A = UEU^T$

$$= \left[\begin{array}{c|c|c} U_{00} & v_{01} & 0 \\ \hline 0 & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right] \left[\begin{array}{c|c|c} E_{00} & 0 & 0 \\ \hline 0 & \varepsilon_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right] \left[\begin{array}{c|c|c} U_{00} & v_{01} & 0 \\ \hline 0 & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right]^T$$

$$= \left[\begin{array}{c|c|c} U_{00} E_{00} & v_{01} \varepsilon_1 & 0 \\ \hline 0 & \varepsilon_1 & v_{12} e_F^T E_{22} \\ \hline 0 & 0 & U_{22} E_{22} \end{array}\right] \left[\begin{array}{c|c|c} U_{00}^T & 0 & 0 \\ \hline v_{01} & 1 & 0 \\ \hline 0 & v_{12} e_F & U_{22}^T \end{array}\right]$$

$$
= \left[\begin{array}{c|c|c} U_{00} \mathcal{E}_{00} & v_{01}\, \varepsilon_1 & 0 \\ \hline 0 & \varepsilon_1 & v_{12}\, e_F^T \mathcal{E}_{22} \\ \hline 0 & 0 & U_{22} \mathcal{E}_{22} \end{array}\right] \left[\begin{array}{c|c|c} U_{00}^T & 0 & 0 \\ \hline v_{01} & 1 & 0 \\ \hline 0 & v_{12}\, e_F & U_{22}^T \end{array}\right]
$$

$$
= \left[\begin{array}{c|c|c} U_{00} \mathcal{E}_{00} U_{00}^T + v_{01}^2 \epsilon_1 & v_{01} \epsilon_1 & 0 \\ \hline \epsilon_1 v_{01} & \epsilon_1 + v_{12}\, e_F^T \mathcal{E}_{22} v_{12} e_F & v_{12}\, e_F^T \mathcal{E}_{22} U_{22}^T \\ \hline 0 & U_{22} \mathcal{E}_{22}\, v_{12}\, e_F & U_{22} \mathcal{E}_{22} U_{22}^T \end{array}\right]
$$

Also given

$$
\begin{bmatrix} A_{00} & \alpha_{10} e_L & 0 \\ \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ 0 & \alpha_{21} e_F & A_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ 0 & 0 & U_{22} \end{bmatrix} \begin{bmatrix} D_{00} & 0 & 0 \\ 0 & \phi_1 & 0 \\ 0 & 0 & \mathcal{E}_{22} \end{bmatrix} \begin{bmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ 0 & 0 & U_{22} \end{bmatrix}^T
$$

$$
= \begin{bmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & \phi_1 & v_{12} e_F^T \mathcal{E}_{22} \\ 0 & 0 & U_{22} \mathcal{E}_{22} \end{bmatrix} \begin{bmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & v_{12} e_F & U_{22}^T \end{bmatrix}
$$

$$
= \left[\begin{array}{c|c|c} L_{00} D_{00} L_{00}^T & L_{00} D_{00} \lambda_{10} e_L & 0 \\ \hline \lambda_{10} e_L^T D_{00} L_{00}^T & \lambda_{10}^2 e_L^T D_{00} e_L + \phi_1 + v_{12}^2 e_F^T \mathcal{E}_{22} e_F & v_{12} e_F^T \mathcal{E}_{22} U_{22} \\ \hline 0 & U_{22} \mathcal{E}_{22} v_{12} e_F & U_{22} \mathcal{E}_{22} U_{22}^T \end{array}\right]
$$

Comparing all three results, we see

$\alpha_{11} = \lambda_{10}^2 e_L^T D_{00} e_L + \delta_1$

$\Rightarrow \quad \lambda_{10}^2 e_L^T D_{00} e_L = \alpha_{11} - \delta_1$

$\alpha_{11} = \epsilon_1 + v_{12}^2 e_F^T E_{22} e_F$

$\Rightarrow \quad v_{12}^2 e_F^T E_{22} e_F = \alpha_{11} - \epsilon_1$

$\alpha_{11} = \lambda_{10}^2 e_L^T D_{00} e_L + \phi_1 + v_{12}^2 e_F^T E_{22} e_F$

$\alpha_{11} = \alpha_{11} - \delta_1 + \phi_1 + \alpha_{11} - \epsilon_1$

$\alpha_{11} = 2\alpha_{11} - (\delta_1 + \epsilon_1) + \phi_1$

Solving the above equation gives

$\phi_1 = \delta_1 + \epsilon_1 - \alpha_{11}$

Part 2:

Cost of computing one twisted factorization given that you have already computed LDL^T and UEU^T factorizations.

Solution :- To compute one twisted factorization all that is needed is to compute ϕ_1 since the factorizations are already given.

Solution for ϕ_1 is given by $\phi_1 = \delta_1 + \epsilon_1 - \alpha_{11}$

which needs $O(1)$ computation as there's no need to loop through or iterate.

Part 3:

Cost of computing all twisted factorizations given that you have already computed LDL^T and UEU^T factorizations.

Solution:-

The twist in twisted factorization depends on where ϕ_i, δ_i and ϵ_i are placed.

So all these twists are placed on the main diagonal.

Hence cost to compute all the twisted factorizations is $O(n^2)$ given $A \in \mathbb{R}^{n*n}$