

## 6.825 Exercise Solutions: Week 3

### Solutions

September 27, 2004

#### Converting to CNF

Convert the following sentences to conjunctive normal form.

1.  $(A \rightarrow B) \rightarrow C$

Answer:

$$\neg(\neg A \vee B) \vee C$$

$$(A \wedge \neg B) \vee C$$

$$(A \vee C) \wedge (\neg B \vee C)$$

2.  $A \rightarrow (B \rightarrow C)$

Answer:

$$\neg A \vee \neg B \vee C$$

3.  $(A \rightarrow B) \vee (B \rightarrow A)$

Answer:

$$(\neg A \vee B) \vee (\neg B \vee A)$$

*True*

4.  $(\neg P \rightarrow (P \rightarrow Q))$

Answer:

$$\neg\neg P \vee (\neg P \vee Q)$$

$$P \vee \neg P \vee Q$$

*True*

5.  $(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow (R \rightarrow Q))$

Answer:

$$\neg(\neg P \vee \neg Q \vee R) \vee (\neg P \vee \neg R \vee Q)$$

$$(P \wedge Q \wedge \neg R) \vee (\neg P \vee \neg R \vee Q)$$

$$(P \vee \neg P \vee \neg R \vee Q) \wedge (Q \vee \neg P \vee \neg R \vee Q) \wedge (\neg R \vee \neg P \vee \neg R \vee Q)$$

$$\neg P \vee Q \vee \neg R$$

6.  $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

Answer:

$$\neg(\neg P \vee Q) \vee (\neg(\neg Q \vee R) \vee (\neg P \vee R))$$

$$(P \wedge \neg Q) \vee ((Q \wedge \neg R) \vee (\neg P \vee R))$$

$$(P \wedge \neg Q) \vee ((Q \vee \neg P \vee R) \wedge (\neg R \vee \neg P \vee R))$$

$$(P \wedge \neg Q) \vee Q \vee \neg P \vee R$$

$$(P \vee \neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg Q \vee R)$$

*True*

## First Order Logic Sentences

For each of the following English sentences, write a corresponding sentence in FOL.

1. The only good extraterrestrial is a drunk extraterrestrial.  
 $\forall x. ET(x) \wedge Good(x) \rightarrow Drunk(x)$
2. The Barber of Seville shaves all men who do not shave themselves.  
 $\forall x. \neg Shaves(x, x) \rightarrow Shaves(BarberOfSeville, x)$
3. There are at least two mountains in England.  
 $\exists x, y. Mountain(x) \wedge Mountain(y) \wedge InEngland(x) \wedge InEngland(y) \wedge x \neq y$
4. There is exactly one coin in the box.  
 $\exists x. Coin(x) \wedge InBox(x) \wedge \forall y. (Coin(y) \wedge InBox(y) \rightarrow x = y)$
5. There are exactly two coins in the box.  
 $\exists x, y. Coin(x) \wedge InBox(x) \wedge Coin(y) \wedge InBox(y) \wedge x \neq y \wedge \forall z. (Coin(z) \wedge InBox(z) \rightarrow (x = z \vee y = z))$
6. The largest coin in the box is a quarter.  
 $\exists x. Coin(x) \wedge InBox(x) \wedge Quarter(x) \wedge \forall y. (Coin(y) \wedge InBox(y) \wedge \neg Quarter(y) \rightarrow Smaller(y, x))$
7. No mountain is higher than itself.  
 $\forall x. Mountain(x) \rightarrow \neg Higher(x, x)$
8. All students get good grades if they study.  
 $\forall x. Student(x) \wedge Study(x) \rightarrow GetGoodGrade(x)$

## FOL Interpretations, Part 1

For each group of sentences, write an interpretation under which the last sentence is false and all the rest are true.

1.  $\forall x. h(x) \rightarrow g(x)$   
 $\forall x. f(x) \rightarrow g(x)$   
 $\exists x. f(x) \wedge h(x)$

An interpretation that makes the first two sentences true and the third false:

$$U = \{A, B\}$$

$$I(f) = \{A\}$$

$$I(g) = \{A, B\}$$

$$I(h) = \{B\}$$

2.  $\forall x. \exists y. f(x, y)$   
 $\exists y. \forall x. f(x, y)$

An interpretation that makes the first sentence true and the second sentence false:

$$U = \{A, B, C\}$$

$$I(f) = \{ \langle A, B \rangle, \langle B, C \rangle, \langle C, A \rangle \}$$

3.  $\forall x. (f(x) \rightarrow g(A))$   
 $(\forall x. f(x)) \rightarrow g(A)$

There is no interpretation that makes the first sentence true and the second sentence false.

Reason: For the second sentence to be false,  $\forall x. f(x)$  has to be *true*, **and**  $g(A)$  has to be *false*. With these two requirements, we can see that the first sentence cannot be true because  $f(x)$  is true for  $\forall x$ , and  $g(A)$  is false.

However, if we replace  $\forall x$  with  $\exists x$ ,

$$\exists x.(f(x) \rightarrow g(A))$$

$$(\exists x.f(x)) \rightarrow g(A)$$

Then the following interpretation makes the first sentence true and the second sentence false.

$$U = \{A, B\}$$

$$f = \{B\}$$

$$g = \{B\}$$

## FOL Interpretations, Part 2

For each group of sentences, give an interpretation in which all sentences are true.

1.  $(\forall x.p(x) \vee q(x)) \rightarrow \exists x.r(x)$

$$\forall x.r(x) \rightarrow q(x)$$

$$\exists x.p(x) \wedge \neg q(x)$$

Interpretation:

$$U = \{A, B\}$$

$$I(p) = \{A\}$$

$$I(q) = \{B\}$$

$$I(r) = \{B\}$$

2.  $\forall x.\neg f(x, x)$

$$\forall x, y, z.f(x, y) \wedge f(y, z) \rightarrow f(x, z)$$

$$\forall x.\exists y.f(x, y)$$

There is no interpretation in a finite universe that makes all of these sentences true. However, if you consider an infinite universe, (e.g., real numbers) and a *greater than* function ( $>$ ), these sentences are all true.

Interpretation:

$$U = \mathbb{R}$$

$$I(f) = >$$

3.  $\forall x.\exists y.f(x, y)$

$$\forall x.(g(x) \rightarrow \exists y.f(y, x))$$

$$\exists x.g(x)$$

$$\forall x.\neg f(x, x)$$

Interpretation:

$$U = \{A, B\}$$

$$I(f) = \{< A, B >, < B, A >\}$$

$$I(g) = \{A\}$$

## FOL Semantics

(6) Consider a world with objects **A**, **B**, and **C**. We'll look at a logical language with constant symbols  $X$ ,  $Y$ , and  $Z$ , function symbols  $f$  and  $g$ , and predicate symbols  $p$ ,  $q$ , and  $r$ . Consider the following interpretation:

- $I(X) = \mathbf{A}$ ,  $I(Y) = \mathbf{A}$ ,  $I(Z) = \mathbf{B}$
- $I(f) = \{\langle \mathbf{A}, \mathbf{B} \rangle, \langle \mathbf{B}, \mathbf{C} \rangle, \langle \mathbf{C}, \mathbf{C} \rangle\}$
- $I(p) = \{\mathbf{A}, \mathbf{B}\}$
- $I(q) = \{\mathbf{C}\}$
- $I(r) = \{\langle \mathbf{B}, \mathbf{A} \rangle, \langle \mathbf{C}, \mathbf{B} \rangle, \langle \mathbf{C}, \mathbf{C} \rangle\}$

For each of the following sentences, say whether it is true or false in the given interpretation  $I$ :

1.  $q(f(Z))$   
Answer: T
2.  $r(X, Y)$   
Answer: F
3.  $\exists w. f(w) = Y$   
Answer: F
4.  $\forall w. r(f(w), w)$   
Answer: T
5.  $\forall u, v. r(u, v) \rightarrow (\forall w. r(u, w) \rightarrow v = w)$   
Answer: F
6.  $\forall u, v. r(u, v) \rightarrow (\forall w. r(w, v) \rightarrow u = w)$   
Answer: T

## Clausal form

(6) Convert each sentence below to clausal form.

1.  $\forall y. \exists x. r(x, y) \vee s(x, y)$   
Answer:  $r(f(y), y) \vee s(f(y), y)$
2.  $\forall y. (\exists x. r(x, y)) \rightarrow p(y)$   
Answer:  $\neg r(x, y) \vee p(y)$
3.  $\forall y. \exists x. (r(x, y) \rightarrow p(x))$   
Answer:  $\neg r(f(y), y) \vee p(f(y))$

## Implication vs. Entailment

Show that  $P \models Q \leftrightarrow (True \models P \rightarrow Q)$ .

Let  $M(P)$  and  $M(Q)$  be the sets of interpretations (models) under which  $P$  and  $Q$  are true, respectively.

1. Assume  $P \models Q$ . By the definition of entailment, we have  $M(P) \subseteq M(Q)$ . Because  $M(Q)$  and  $M(\neg Q)$  are disjoint (there are no interpretations under which both  $Q$  and  $\neg Q$  are true), it follows that  $M(P) \cap M(\neg Q) = \emptyset$ . Therefore there are no interpretations under which  $P$  is true and  $Q$  is false, and so  $P \rightarrow Q$  is true under all interpretations:  $M(P \rightarrow Q) = M(True)$  and consequently  $M(True) \subseteq M(P \rightarrow Q)$ . By the definition of entailment, this means that  $True \models P \rightarrow Q$ , and so we have shown that  $P \models Q \rightarrow (True \models P \rightarrow Q)$ .
2. Assume  $True \models P \rightarrow Q$ . By the definition of entailment, this means that  $P \rightarrow Q$  is true under all models, and so there can be no model such that  $P$  is true and  $Q$  is false:  $M(P) \cap M(\neg Q) = \emptyset$ . Therefore  $M(P) \subseteq M(Q)$  and we can conclude that  $P \models Q$ . Thus we have shown that  $(True \models P \rightarrow Q) \rightarrow P \models Q$ .

(1) and (2) together prove the statement  $P \models Q \leftrightarrow (True \models P \rightarrow Q)$ .