Logic: Exercises

Andrzej Szałas

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Contents

Some remarks

This ebook contains exercises for lectures on logic. There are two types of exercises:

- warm-up exercises: for practicing useful techniques;
- exam-like exercises: similar exercises are likely to appear as (parts of) exam exercises.

Exam-like exercises are marked by ¡E!.

Solutions for exercises are provided but it is a good idea to first try to solve them yourself. Note that typically there are many different equally good solutions.

Recall that to simplify notation we often omit brackets, assuming that the order of precedence from high to low is: $\neg, \wedge, \vee, \leftrightarrow, \rightarrow$. For example, $A \vee \neg B \wedge C \rightarrow \neg D \leftrightarrow E \vee F$ abbreviates $\big[A \vee \big((\neg B) \wedge C\big)\big] \rightarrow \big[(\neg D) \leftrightarrow (E \vee F)\big]$.

Paper and font sizes are selected to make the manual suitable for ebook readers.

Lecture 1

Introduction to logics

1.1 Exercise

Let $high\_temp, high_pressure, signal\_on$ be propositional variables. Let $\mathcal{M}_1$ be a model, such that:

- $\mathcal{M}_1 \models high\_temp$, $\mathcal{M}_1 \models signal$;
- $\mathcal{M}_1 \not\models high\_pressure$, $\mathcal{M}_1 \not\models danger$, $\mathcal{M}_1 \not\models reduce\_speed$,

and let $\mathcal{M}_2$ be a model such that:

- $\mathcal{M}_2 \models high\_temp$, $\mathcal{M}_2 \models signal$, $\mathcal{M}_2 \models reduce\_speed$;
- $\mathcal{M}_2 \not\models high\_pressure$, $\mathcal{M}_2 \not\models danger$.

Are the following statements true?

1a. $\mathcal{M}_1 \models (high\_temp \wedge high\_pressure) \rightarrow danger$;

1b. $\mathcal{M}_1 \models (high\_temp \wedge signal) \rightarrow reduce\_speed$;

2a. $\mathcal{M}_2 \models (high\_temp \wedge high\_pressure) \rightarrow danger$;

2b. $\mathcal{M}_2 \models (high\_temp \wedge signal) \rightarrow reduce\_speed.$

1.1 Solution

1

In $\mathcal{M}_1$:

- *high_temp* and *signal* take the value T;
- *high_pressure*, *danger* and *reduce_speed* take the value F.

1a

In $\mathcal{M}_1$:

$$\begin{array}{ccccccccc}
(\mathit{high_temp} & \wedge & \mathit{high_pressure}) & \rightarrow & \mathit{danger} & \leftrightarrow \\
(\quad \mathrm{T} & \wedge & \mathrm{F} \quad) & \rightarrow & \mathrm{F} & \leftrightarrow \\
 & \mathrm{F} & & \rightarrow & \mathrm{F} & \leftrightarrow \\
 & \mathrm{T}. & & & &
\end{array}$$

1b

In $\mathcal{M}_1$:

$$\begin{array}{cccccc}
(\mathit{high_temp} & \wedge & \mathit{signal}) & \rightarrow & \mathit{reduce_speed} & \leftrightarrow \\
(\quad \mathrm{T} & \wedge & \mathrm{T} \quad) & \rightarrow & \mathrm{F} & \leftrightarrow \\
 & \mathrm{T} & & \rightarrow & \mathrm{F} & \leftrightarrow \\
 & \mathrm{F}. & & & &
\end{array}$$

2

In $\mathcal{M}_2$:

- $high\_temp$, $signal$ and $reduce\_speed$ take the value T;
- $high\_pressure$ and $danger$ take the value F.

2a

In $\mathcal{M}_2$:

$$\begin{array}{ccccccc}
(high_temp \wedge high_pressure) & \to & danger & \leftrightarrow \\
(\quad T \quad \wedge \quad F \quad) & \to & F & \leftrightarrow \\
F & \to & F & \leftrightarrow \\
T. & & &
\end{array}$$

2b

In $\mathcal{M}_2$:

$$\begin{array}{cccc}
(high_temp \wedge signal) & \to & reduce_speed & \leftrightarrow \\
(\quad T \quad \wedge \quad T \quad) & \to & T & \leftrightarrow \\
T & \to & T & \leftrightarrow \\
T. & & &
\end{array}$$

1.2 Exercise

Using the truth table method prove that:

1. $$(p \to \neg p) \to \neg p. \tag{1.1}$$

2. $$\big[(p \vee q) \wedge (\neg p \vee q)\big] \to q; \tag{1.2}$$

3. $$\big[(p \to q) \wedge (r \to q)\big] \leftrightarrow \big[(p \vee r) \to q\big]; \tag{1.3}$$

4. $$\big[p \to (q \wedge r)\big] \leftrightarrow \big[(p \to q) \wedge (p \to r)\big]. \tag{1.4}$$

1

There is only one variable in (1.1). A suitable truth table is:

p	$\neg p$	$p \to \neg p$	$(p \to \neg p) \to \neg p$
T	F	F	T
F	T	T	T

2

There are two variables in (1.2). A suitable truth table is:

p	q	$p \vee q$	$\neg p$	$\neg p \vee q$	$(p \vee q) \wedge (\neg p \vee q)$	(1.2)
T	T	T	F	T	T	T
T	F	T	F	F	F	T
F	T	T	T	T	T	T
F	F	F	T	T	F	T

3

There are three variables in (1.3). A suitable truth table is:

p	q	r	$p \to q$	$r \to q$	$(p \to q) \wedge (r \to q)$	$p \vee r$	$(p \vee r) \to q$	(1.3)
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	F	F	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T	T
F	F	T	T	F	F	T	F	T
F	F	F	T	T	T	F	T	T

4

There are three variables in (1.4). A suitable truth table is:

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$	(1.4)
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	F	F	F	T	F	T
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

1.3 Exercise

Translate the following sentence into propositional logic:[1]

> "She asked me to stay and she told me to sit anywhere.
> So I looked around and I noticed there wasn't a chair."

Remark: when we translate natural language into logic, we usually have to simplify and rephrase sentences to make them closer to a logical form. This is often done by understanding what sentences actually say and then formulating them appropriately. In propositional logic only fragments of natural language can be expressed, so we frequently only approximate the meaning of more complicated sentences.

[1]Taken from "Norwegian wood" lyrics.

1.3 Solution

Simplifying a bit, we have the following atomic sentences:[2]

- "she asked me to stay";
- "she told me to sit anywhere';
- "I looked around";
- "there was a chair".

The last two sentences are implied by the second one being a reason for them.

Denoting the above sentences respectively by *asked*, *told*, *looked*, *chair*, we can translate the initial sentence into:

$$\begin{array}{l} \mathit{asked} \land \mathit{told} \\ \qquad \land \\ \mathit{told} \to (\mathit{looked} \land \neg \mathit{chair}). \end{array}$$

---

[2]That is, sentences not involving connectives.

1.4 Exercise ¡E!

1. Translate the following sentences into a set of propositional formulas:

   "Move small, medium or large objects."
   "Move small objects to the first container,
   medium objects to the second container
   and large objects to the third container."

   "Small objects are green."
   "Medium objects are red."
   "Large objects are blue."
   "Move objects to the second or to the third
   container."
   "Do not move blue objects."

2. Assuming that each object can be moved to at most one container, determine what objects can be moved.

1.4 Solution

1

We have atomic sentences like "move small objects", "move medium objects", "move large objects", ...

We denote these sentences by *small*, *medium*, *large*, *first*, *second*, *third*, *green*, *red*, *blue*.

It is important to see that sentences as

"large objects are blue"

mean that whenever we move a large object then it is blue, so should be translated into implication like:

$$large \to blue.$$

The translation is:

$$small \vee medium \vee large \tag{1.5}$$

$$small \to first \tag{1.6}$$

$$medium \to second \tag{1.7}$$

$$large \to third \tag{1.8}$$

$$small \to green \tag{1.9}$$

$$medium \to red \tag{1.10}$$

$$large \to blue \tag{1.11}$$

$$second \vee third \tag{1.12}$$

$$\neg blue. \tag{1.13}$$

2

The assumption that each object can be moved to at most one container is translated into:

$$first \to \neg second \tag{1.14}$$
$$first \to \neg third \tag{1.15}$$
$$second \to \neg first \tag{1.16}$$
$$second \to \neg third \tag{1.17}$$
$$third \to \neg first \tag{1.18}$$
$$third \to \neg second. \tag{1.19}$$

By (1.13) the object cannot be blue so, by (1.11), cannot be large. Can it be small? By (1.5), small objects are moved to the first container which violates (1.12) together with (1.14) and (1.15). Therefore, only medium (and, therefore, red) objects can be moved (if any).

1.5 Exercise ¡E!

Let L, T and John be persons such that that L always lies and T always tells the truth.

L claims that on Saturdays:

> "John works."
> "John does not read books and he does not watch TV and he cooks."

T claims that on Saturdays:

> "When John does not work, he does not watch TV either."
> "John reads books or watches TV or cooks."

1. Translate the above claims into a set of propositional formulas.
2. Determine what is John's activity on Saturdays.

1.5 Solution

1

First we translate claims of L. We have atomic sentences:

- "John works";
- "John reads books";
- "John watches TV";
- "John cooks".

Denoting these sentences by *works*, *books*, *tv* and *cooks*, respectively, we translate L's claims to:

$$works \text{ and } (\neg books \wedge \neg tv \wedge cooks). \tag{1.20}$$

Since L lies, in reality, rather than (1.20) we have negations:

$$\neg works \text{ and } \neg(\neg books \wedge \neg tv \wedge cooks). \tag{1.21}$$

Using a DeMorgan Law and removing double negation we translate L's claims into:

$$\neg works; \tag{1.22}$$

$$books \vee tv \vee \neg cooks. \tag{1.23}$$

Claims of T can be translated into:

$$\neg works \to \neg tv; \tag{1.24}$$

$$books \vee tv \vee cooks. \tag{1.25}$$

2

Observe that, by (1.22), we know that John does not work. Therefore, by (1.24), he does not watch tv either. Now (1.23) and (1.25) reduce to:

$$books \vee \neg cooks \tag{1.26}$$

$$books \vee cooks. \tag{1.27}$$

Suppose $\neg books$. Then the above reduces to:

$$\neg cooks \tag{1.28}$$

$$cooks, \tag{1.29}$$

which is a contradiction. Therefore, $\neg books$ cannot be true and the only option that remains is that John actually reads books.

1.6 Exercise

Translate the following sentence into propositional logic:[3]

> "If I burden myself with a companion in my various little inquiries it is not done out of sentiment or caprice, but it is that Watson has some remarkable characteristics of his own."

[3] After Sherlock Holmes.

1.6 Solution

We first have to find out what, in fact, Holmes said. The meaning is more or less the following:

> "If I burden myself with a companion in my various little inquiries *then* it is not done out of sentiment or caprice, but *because* Watson has some remarkable characteristics of his own."

The sentence after "but" can be approximately rephrased:

> "*The reason for burdening myself* is that Watson has some remarkable characteristics of his own."

We have the following atomic sentences:

- "I burden myself with a companion in my various little inquiries",
- "it is done out of sentiment",
- "it is done out of caprice",
- "Watson has some remarkable characteristics of his own".

We denote these sentences by $burden$, $sentiment$, $caprice$ and $watson$, respectively.

Using the rephrased sentence, one can translate Holmes' sentence to:

$$\begin{aligned} &\mathit{burden} \rightarrow \neg(\mathit{sentiment} \vee \mathit{caprice}) \qquad (1.30) \\ &\qquad \wedge \\ &\mathit{watson} \rightarrow \mathit{burden}. \qquad (1.31) \end{aligned}$$

Using a DeMorgan law and tautology (1.4) (see page 7), we can simplify (1.30) to:

$$\begin{aligned} &\mathit{burden} \rightarrow \neg \mathit{sentiment} \\ &\qquad \wedge \\ &\mathit{burden} \rightarrow \neg \mathit{caprice}. \end{aligned}$$

1.7 Exercise

Verify whether the following sentence is valid:

> "If I'll go to a cinema then I'll take a bus, or
> if I'll take a bus then I'll go to a cinema".

1.7 Solution

The above sentence looks a bit strange but it actually is a tautology. To see it, lets translate it into propositional logic. We have two atomic sentences:

- "I'll go to a cinema";
- "I'll take a bus".

Denoting these sentences respectively by *cinema* and *bus*, the translation is:

$$(cinema \rightarrow bus) \vee (bus \rightarrow cinema). \tag{1.32}$$

To convince ourselves that (1.32) is a tautology we can use truth tables or the following reasoning:

$$(cinema \rightarrow bus) \vee (bus \rightarrow cinema) \leftrightarrow \tag{1.33}$$

$$\neg cinema \vee bus \vee \neg bus \vee cinema. \tag{1.34}$$

Sentence (1.34) reduces to T since $\neg cinema \vee cinema$ as well as $bus \vee \neg bus$ are T. So disjunction (1.34) is T, too.

1.8 Exercise

Verify the following reasoning:

> It is not true that:
>   John bought a house, or
>   if he bought a house then he had money.
> Therefore John didn't have money.

1.8 Solution

To verify the reasoning we first translate it into logic. We have two atomic sentences:

- "John bought a house";
- "John had money".

We denote these sentences by h and l, respectively. The translation is:

$$\big[\neg\big(b \vee (b \to l)\big)\big] \to \neg l. \tag{1.35}$$

We apply truth table method, showing that (1.35) is a tautology:

b	l	$b \to l$	$b \vee (b \to l)$	$\neg\big(b \vee (b \to l)\big)$	$\neg l$	(1.35)
T	T	T	T	F	F	T
T	F	F	T	F	T	T
F	T	T	T	F	F	T
F	F	T	T	F	T	T

Lecture 2

Tableaux and SAT

2.1 Exercise

Using tableaux transform the following formula into the disjunctive normal form:

$$(p \vee q) \rightarrow (p \wedge q). \tag{2.1}$$

A disjunctive normal form is:

$$(a_1 \wedge a_2 \wedge \ldots \wedge a_k) \vee \ldots \vee (c_1 \wedge c_2 \wedge \ldots \wedge c_m),$$

where each $a_1, a_2, a_k, \ldots, c_1, c_2, \ldots, c_m$ are literals (possibly negated propositional variables).

2.1 Solution

We construct a tableau for (2.1):

$$
\begin{array}{ccc}
 & (p \vee q) \to (p \wedge q) & \\
 & \swarrow \qquad\quad \searrow \ (\to) & \\
\neg(p \vee q) & & p \wedge q \\
(\neg\vee) \ \downarrow & & \downarrow \ (\wedge) \\
\neg p, \neg q & & p, q
\end{array}
$$

Recall that a tableau represents disjunction of conjunctions of literals in leaves. Therefore, the above tableau represents:

$$(\neg p \wedge \neg q) \vee (p \wedge q). \tag{2.2}$$

Of course, formula (2.2) is equivalent to (2.1) and is in disjunctive normal form.

2.2 Exercise ¡E!

Using tableaux check whether the following formula is satisfiable:

$$(p \vee q) \wedge (\neg p \vee \neg q). \tag{2.3}$$

If it is, provide a satisfying valuation.

2.2 Solution

We construct a tableau for (2.3):

$$
\begin{array}{ccccccc}
 & & & (p \vee q) \wedge (\neg p \vee \neg q) & & & \\
 & & & \downarrow\ (\wedge) & & & \\
 & & & p \vee q, \neg p \vee \neg q & & & \\
 & & \swarrow & & \searrow\ (\vee) & & \\
 & p, \neg p \vee \neg q & & & & q, \neg p \vee \neg q & \\
 \swarrow & & \searrow\ (\vee) & & \swarrow & & \searrow\ (\vee) \\
 \underbrace{p, \neg p}_{\text{closed}} & & p, \neg q & & q, \neg p & & \underbrace{q, \neg q}_{\text{closed}}
\end{array}
$$

The formula is satisfiable since the tableau has open leaves. Each open leaf provides a satisfying valuation:

- $p, \neg q$: $p = \mathrm{T}$ and $q = \mathrm{F}$;
- $q, \neg p$: $p = \mathrm{F}$ and $q = \mathrm{T}$.

Note that checking satisfiability requires only one satisfying valuation, so one can interrupt constructing the tableau after obtaining the first open leaf.

2.3 Exercise ¡E!

Using tableaux check whether the following formulas are tautologies. If not, extract a counterexample from the constructed tableau.

1. $$(p \vee q) \to p. \tag{2.4}$$

2. $$(p \vee q) \to (p \wedge r). \tag{2.5}$$

3. $$(p \to q) \vee (q \to p). \tag{2.6}$$

4. $$(p \vee q) \to \big[(p \vee q \vee \neg r) \wedge (r \vee p \vee q)\big] \tag{2.7}$$

2.3 Solution

1

We construct a tableau for negated (2.4):

$$
\begin{array}{ccc}
 & \neg[(p \vee q) \to p] & \\
 & \downarrow (\neg\!\to) & \\
 & p \vee q, \neg p & \\
 \swarrow & & \searrow (\vee) \\
 \underbrace{p, \neg p}_{\text{closed}} & & q, \neg p
\end{array}
$$

The above tableau is not closed. The leaf containing $q, \neg p$ provides a counterexample for (2.4): $q = \mathrm{T}$ and $p = \mathrm{F}$. Indeed, in such a case:

$$
\begin{aligned}
&(p \vee q) \to p &\leftrightarrow \\
&(\mathrm{F} \vee \mathrm{T}) \to \mathrm{F} &\leftrightarrow \\
&\underbrace{\mathrm{T} \to \mathrm{F}}_{\mathrm{F}}.
\end{aligned}
$$

2

We construct a tableau for negated (2.5):

$$
\begin{array}{c}
\neg[(p \vee q) \to (p \wedge r)] \\
\downarrow (\neg\!\to) \\
p \vee q, \neg(p \wedge r) \\
\swarrow \qquad\qquad \searrow (\neg\wedge) \\
\begin{array}{cc}
p \vee q, \neg p & p \vee q, \neg r \\
\swarrow \quad \searrow (\vee) & \swarrow \quad \searrow (\vee) \\
\underbrace{p, \neg p}_{\text{closed}} \quad q, \neg p & p, \neg r \quad q, \neg r
\end{array}
\end{array}
$$

The above tableau is not closed. All open leaves provide counterexamples for (2.5).

- The leaf containing $q, \neg p$ provides a counterexample $q = \mathrm{T}$ and $p = \mathrm{F}$. Note that the value r is not determined: when $q = \mathrm{T}$ and $p = \mathrm{F}$, formula (2.5) is false no matter whether r is T or F.

- The leaf containing $p, \neg r$ provides a counterexample $p = \mathrm{T}$ and $r = \mathrm{F}$. In this case the value of (2.5) is F no matter what the value of q is.

- The leaf containing $q, \neg r$ provides a counterexample $q = \mathrm{T}$ and $r = \mathrm{F}$. Again, in this case the value of (2.5) is F no matter what the value of p is.

3

We construct a tableau for the negation of (2.6):

$$\begin{array}{c}
\neg\big[(p \to q) \lor (q \to p)\big] \\
\downarrow (\neg\lor) \\
\neg(p \to q), \neg(q \to p) \\
\downarrow (\neg\to) \\
p, \neg q, \neg(q \to p) \\
\downarrow (\neg\to) \\
\underbrace{p, \neg q, q, \neg p}_{\text{closed}}
\end{array}$$

The tableau is closed, so formula (2.6) is a tautology.

4

We construct a tableau for the negation (2.7):

$$\begin{array}{c}
\neg\big[(p \lor q) \to \big[(p \lor q \lor \neg r) \land (r \lor p \lor q)\big]\big] \\
\downarrow (\neg\to) \\
p \lor q, \neg\big[(p \lor q \lor \neg r) \land (r \lor p \lor q)\big]\big] \\
\swarrow \qquad \searrow (\neg\land) \\
\begin{array}{cc}
p \lor q, \neg(p \lor q \lor \neg r) & p \lor q, \neg(r \lor p \lor q) \\
\downarrow (\neg\lor) & \downarrow (\neg\lor) \\
p \lor q, \neg p, \neg q, \neg\neg r & p \lor q, \neg r, \neg p, \neg q \\
\swarrow \qquad \searrow (\lor) & \swarrow \qquad \searrow (\lor) \\
\underbrace{p, \neg p, \neg q, \neg\neg r}_{\text{closed}} \quad \underbrace{q, \neg p, \neg q, \neg\neg r}_{\text{closed}} & \underbrace{p, \neg r, \neg p, \neg q}_{\text{closed}} \quad \underbrace{q, \neg r, \neg p, \neg q}_{\text{closed}}
\end{array}
\end{array}$$

The tableau is closed, so (2.7) is a tautology.

2.4 Exercise

Consider the following relationships among concepts:

> "Children and adults are persons."
> "Children and adults are humans."

According to the above specification:

1. Are persons humans?
2. Are humans persons?

2.4 Solution

We have the following concepts:[1] *person* (p), *child* (c), *adult* (a) and *human* (h).

Lets first translate relationships among concepts into propositional formulas:

$$c \to p \tag{2.8}$$

$$a \to p \tag{2.9}$$

$$c \to h \tag{2.10}$$

$$a \to h. \tag{2.11}$$

[1] In brackets we give their abbreviations used later on.

1

We have to verify whether $p \to h$ follows from (2.8)–(2.11), so construct a tableau:

$$
\begin{array}{c}
\neg\big[[(c \to p) \wedge (a \to p) \wedge (c \to h) \wedge (a \to h)] \to (p \to h)\big] \\
\downarrow (\neg\to) \\
(c \to p) \wedge (a \to p) \wedge (c \to h) \wedge (a \to h), \neg(p \to h) \\
\downarrow (\wedge) \\
c \to p, (a \to p) \wedge (c \to h) \wedge (a \to h), \neg(p \to h) \\
\downarrow (\wedge) \\
c \to p, a \to p, (c \to h) \wedge (a \to h), \neg(p \to h) \\
\downarrow (\wedge) \\
c \to p, a \to p, c \to h, (a \to h), \neg(p \to h) \\
\downarrow (\neg\to) \\
c \to p, a \to p, c \to h, a \to h, p, \neg h \\
\swarrow \quad (\to) \quad \searrow
\end{array}
$$

$$
\begin{array}{ll}
\neg c, a \to p, c \to h, a \to h, p, \neg h & \ldots \\
\downarrow \quad (\to) \quad \searrow & \\
\neg c, a \to p, c \to h, a \to h, p, \neg h & \ldots \\
\downarrow \quad (\to) \quad \searrow & \\
\neg c, \neg a, c \to h, a \to h, p, \neg h & \ldots \\
\downarrow \quad (\to) \quad \searrow & \\
\neg c, \neg a, \neg c, a \to h, p, \neg h & \ldots \\
\downarrow \quad (\to) \quad \searrow & \\
\neg c, \neg a, \neg c, \neg a, p, \neg h & \ldots
\end{array}
$$

We have an open leaf, so $p \to h$ does not follow from formulas (2.8)–(2.11). The open leaf provides a counterexample: $c = \mathrm{F}$, $a = \mathrm{F}$, $p = \mathrm{T}$ and $h = \mathrm{F}$.

2

We have to verify whether $h \to p$ follows from (2.8)–(2.11), so have to construct a tableau for:

$$\neg\big[[(c \to p)\wedge(a \to p)\wedge(c \to h)\wedge(a \to h)] \to (p \to h)\big].$$

A tableau can be similar to that constructed previously with $p \to h$ replaced by $h \to p$. Repeating the construction we will obtain an open leaf:

$$\neg c, \neg a, \neg c, \neg a, h, \neg p,$$

providing a counterexample: $c = \mathrm{F}$, $a = \mathrm{F}$, $h = \mathrm{T}$ and $p = \mathrm{F}$.

2.5 Exercise

Consider the following relationships among concepts:

> "A good scientific paper is a paper containing outstanding results or published in a very good scientific journal, and only such papers are good scientific papers."

> "Papers containing outstanding results and papers published in very good scientific journals are publications."

According to the above specification:

1. Are good scientific papers also publications?
2. Are publications good scientific papers?

2.5 Solution

We have concepts *good scientific paper* (g), *paper containing outstanding results* (o), *paper published in a very good scientific journal* (j) and *publication* (p).

Translating our specification into propositional formulas results in:

$$g \leftrightarrow (o \vee j) \tag{2.12}$$

$$o \rightarrow p \tag{2.13}$$

$$j \rightarrow p. \tag{2.14}$$

Note that equivalence in (2.12) comes from "and only such paper" meaning that such papers are limited to those explicitly specified. On the other hand, this is not stated about publications, so publications can include other types of papers/books, too. Therefore we have inclusions expressed by implications.

1

The first question is expressed by $g \to p$. We then have to verify whether (2.12)–(2.14) imply $g \to p$. We construct a tableau showing that this indeed is the case:

$$
\begin{array}{c}
\neg\Big[\big[\big(g \leftrightarrow (o \lor j)\big) \land (o \to p) \land (j \to p)\big] \to (g \to p)\Big] \\
\downarrow (\neg\!\to) \\
\big(g \leftrightarrow (o \lor j)\big) \land (o \to p) \land (j \to p) \land \neg(g \to p) \\
\downarrow (\land) \\
g \leftrightarrow (o \lor j), o \to p \land (j \to p) \land \neg(g \to p) \\
\downarrow (\land) \\
g \leftrightarrow (o \lor j), o \to p, (j \to p) \land \neg(g \to p) \\
\downarrow (\land) \\
g \leftrightarrow (o \lor j), o \to p, j \to p, \neg(g \to p) \\
\downarrow (\land) \\
g \leftrightarrow (o \lor j), o \to p, j \to p, g, \neg p \\
\swarrow \qquad\qquad \searrow (\to)
\end{array}
$$

$$
\begin{array}{cc}
g \leftrightarrow (o \lor j), \neg o, j \to p, g, \neg p & \underbrace{g \leftrightarrow (o \lor j), p, j \to p, g, \neg p}_{\text{closed}} \\
\downarrow \qquad (\to) \qquad \searrow & \\
g \leftrightarrow (o \lor j), \neg o, \neg j, g, \neg p & \underbrace{g \leftrightarrow (o \lor j), \neg o, p, g, \neg p}_{\text{closed}} \\
\downarrow \ (\leftrightarrow) & \\
\neg g \lor o \lor j, g \lor \neg(o \lor j), \neg o, \neg j, g, \neg p & \\
\downarrow \qquad (\lor) \qquad \searrow & \\
\underbrace{\neg g, g \lor \neg(o \lor j), \neg o, \neg j, g, \neg p}_{\text{closed}} & o \lor j, g \lor \neg(o \lor j), \neg o, \neg j, g, \neg p \\
\swarrow \qquad (\lor) \qquad \downarrow & \\
\underbrace{o, g \lor \neg(o \lor j), \neg o, \neg j, g, \neg p}_{\text{closed}} & \underbrace{j, g \lor \neg(o \lor j), \neg o, \neg j, g, \neg p}_{\text{closed}}
\end{array}
$$

2

The second question is expressed by $p \to g$. We have to verify whether (2.12)–(2.14) imply $p \to g$. We construct a tableau showing that this is not the case:

$$
\begin{array}{c}
\neg\Big[\big[\big(g \leftrightarrow (o \vee j)\big) \wedge (o \to p) \wedge (j \to p)\big] \to (p \to g)\Big] \\
\downarrow (\neg\to) \\
\big(g \leftrightarrow (o \vee j)\big) \wedge (o \to p) \wedge (j \to p) \wedge \neg(p \to g) \\
\downarrow (\wedge) \\
g \leftrightarrow (o \vee j), o \to p \wedge (j \to p) \wedge \neg(p \to g) \\
\downarrow (\wedge) \\
g \leftrightarrow (o \vee j), o \to p, (j \to p) \wedge \neg(p \to g) \\
\downarrow (\wedge) \\
g \leftrightarrow (o \vee j), o \to p, j \to p, \neg(p \to g) \\
\downarrow (\wedge) \\
g \leftrightarrow (o \vee j), o \to p, j \to p, p, \neg g \\
\swarrow \qquad\qquad\qquad \searrow \; (\to) \\
g \leftrightarrow (o \vee j), \neg o, j \to p, p, \neg g \qquad \ldots \\
\downarrow \; (\to) \quad \searrow \\
g \leftrightarrow (o \vee j), \neg o, \neg j, p, \neg g \qquad \ldots \\
\downarrow \; (\leftrightarrow) \\
\neg g \vee o \vee j, g \vee \neg(o \vee j), \neg o, \neg j, p, \neg g \\
\swarrow \quad (\vee) \quad \searrow \\
\neg g, g \vee \neg(o \vee j), \neg o, \neg j, p, \neg g \qquad \ldots \\
\swarrow \quad (\vee) \quad \searrow \\
\ldots \qquad \neg g, \neg(o \vee j), \neg o, \neg j, p, \neg g \\
\downarrow (\neg\vee) \\
\neg g, \neg o, \neg j, \neg o, \neg j, p, \neg g
\end{array}
$$

The above open leaf gives a valuation satisfying (2.12)–(2.14) and falsifying $p \to g$: $g = \mathrm{F}$, $o = \mathrm{F}$, $j = \mathrm{F}$, $p = \mathrm{T}$.

Lecture 3

Resolution

3.1 Exercise

Transform into an equivalent set of clauses:

$$\neg\big[(\neg p \leftrightarrow r) \rightarrow [(q \wedge r) \vee p]\big]. \tag{3.1}$$

3.1 Solution

We first transform (3.1) into the conjunctive normal form:

$$\begin{array}{ll}
\neg\big[(\neg p \leftrightarrow r) \to [(q \wedge r) \vee p]\big] & \leftrightarrow \\
(\neg p \leftrightarrow r) \wedge \neg[(q \wedge r) \vee p] & \leftrightarrow \\
(\neg\neg p \vee r) \wedge (\neg p \vee \neg r) \wedge \neg[(q \wedge r) \vee p] & \leftrightarrow \\
(p \vee r) \wedge (\neg p \vee \neg r) \wedge \neg[(q \wedge r) \vee p] & \leftrightarrow \\
(p \vee r) \wedge (\neg p \vee \neg r) \wedge \neg(q \wedge r) \wedge \neg p & \leftrightarrow \\
(p \vee r) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r) \wedge \neg p. &
\end{array}$$

So we have the following clauses:

$$p \vee r$$

$$\neg p \vee \neg r$$

$$\neg q \vee \neg r$$

$$\neg p.$$

3.2 Exercise ¡E!

Using resolution prove the following propositional formula:

$$(p \lor q) \to \big[(p \lor q \lor \neg r) \land (r \lor p \lor q)\big] \tag{3.2}$$

3.2 Solution

First, we negate (3.2) and transform it into a conjunctive normal form:

$$
\begin{aligned}
&\neg\Big[(p \vee q) \rightarrow \big[(p \vee q \vee \neg r) \wedge (r \vee p \vee q)\big]\Big] \leftrightarrow \\
&(p \vee q) \wedge \neg\big[(p \vee q \vee \neg r) \wedge (r \vee p \vee q)\big] \leftrightarrow \\
&(p \vee q) \wedge [\neg(p \vee q \vee \neg r) \vee \neg(r \vee p \vee q)] \leftrightarrow \\
&(p \vee q) \wedge [(\neg p \wedge \neg q \wedge \neg\neg r) \vee (\neg r \wedge \neg p \wedge \neg q)] \leftrightarrow \\
&(p \vee q) \wedge [(\neg p \wedge \neg q \wedge r) \vee (\neg r \wedge \neg p \wedge \neg q)] \leftrightarrow \\
&(p \vee q) \wedge [((\neg p \wedge \neg q \wedge r) \vee \neg r) \wedge \\
&\qquad\qquad ((\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q))] \leftrightarrow \\
&(p \vee q) \wedge [((\neg p \wedge \neg q \wedge r) \vee \neg r) \wedge \\
&\qquad\qquad ((\neg p \wedge \neg q \wedge r) \vee \neg p) \wedge \\
&\qquad\qquad ((\neg p \wedge \neg q \wedge r) \vee \neg q)] \leftrightarrow \\
&(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r) \wedge (r \vee \neg r) \wedge \\
&\qquad\qquad (\neg p \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (r \vee \neg p) \wedge \\
&\qquad\qquad (\neg p \vee \neg q) \wedge (\neg q \vee \neg q) \wedge (r \vee \neg q).
\end{aligned}
$$

So we have the following clauses:

1. $p \vee q$
2. $\neg p \vee \neg r$
3. $\neg q \vee \neg r$
4. $r \vee \neg r$
5. $\neg p \vee \neg p$
6. $\neg q \vee \neg p$
7. $r \vee \neg p$
8. $\neg p \vee \neg q$
9. $\neg q \vee \neg q$
10. $r \vee \neg q$.

A proof by resolution:

11. $\neg p$ – (fctr): 5
12. $\neg q$ – (fctr): 9
13. q – (res): 1, 11
14. $\emptyset$ – (res): 12, 13

3.3 Exercise

Using resolution check whether the following formula is a tautology:

$$(p \vee q) \to \big(q \to (p \vee (p \wedge q))\big). \tag{3.3}$$

3.3 Solution

We first transform negated (3.3) into conjunctive normal form:

$$
\begin{array}{ll}
\neg\big[(p \vee q) \to \big(q \to (p \vee (p \wedge q))\big)\big] & \leftrightarrow \\
(p \vee q) \wedge \neg\big(q \to (p \vee (p \wedge q))\big) & \leftrightarrow \\
(p \vee q) \wedge q \wedge \neg(p \vee (p \wedge q)) & \leftrightarrow \\
(p \vee q) \wedge q \wedge \neg p \wedge \neg(p \wedge q) & \leftrightarrow \\
(p \vee q) \wedge q \wedge \neg p \wedge (\neg p \vee \neg q). &
\end{array}
$$

In the following reasoning we will apply an optimization technique: whenever we have a clause containing proposition and its negation - we remove this clause.

This follows from the fact that such clauses are equivalent to T since the subexpression $r \vee \neg r$ in a clause of the form $(\ldots \vee r \vee \ldots \vee \ldots \vee \neg r \vee \ldots)$ is T and $(\ldots \vee \mathrm{T} \vee \ldots)$ is equivalent to T. Therefore any clause containing a proposition and its negation is equivalent to T, so can be removed (T in conjunction with other clauses is equivalent to other clauses).

Clauses:

1. $p \vee q$
2. q
3. $\neg p$
4. $\neg p \vee \neg q$.

Resolution can be applied to:

- $1, 3$: resulting in q which is already present (as clause 2.);
- $1, 4$: resulting in $q \vee \neg q$ (removed as containing a proposition and its negation);
- $2, 4$: resulting in $\neg p$ which is already present (as clause 3.).

Therefore no matter how we apply resolution, the empty clause $\emptyset$ cannot be obtained. So, formula (3.3) is not a tautology.

3.4 Exercise

Using resolution check whether the following formula is satisfiable:

$$(p \lor q) \land (\neg p \lor q) \land (p \lor \neg q) \land (\neg p \lor \neg q). \tag{3.4}$$

3.4 Solution

Formula (3.4) is in conjunctive normal form and consists of the following clauses:

1 $p \vee q$
2 $\neg p \vee q$
3 $p \vee \neg q$
4 $\neg p \vee \neg q$.

Lets check conclusions that can be obtained using resolution:

5.	$q \vee q$	– (res): 1, 2
6.	q	– (fctr): 5
7.	p	– (res): 3, 6
8.	$\neg p$	– (res): 4, 6
9.	$\emptyset$	– (res): 7, 8.

The above derivation shows that the set of clauses implies the empty clause $\emptyset$, equivalent to F. So (3.4) is not satisfialble.

3.5 Exercise ¡E!

Using resolution prove the conclusion of Exercise 1.4 (page 13). Recall formulas resulting from translation considered there:

$$small \vee medium \vee large \tag{3.5}$$
$$small \to first \tag{3.6}$$
$$medium \to second \tag{3.7}$$
$$large \to third \tag{3.8}$$
$$small \to green \tag{3.9}$$
$$medium \to red \tag{3.10}$$
$$large \to blue \tag{3.11}$$
$$second \vee third \tag{3.12}$$
$$\neg blue. \tag{3.13}$$
$$first \to \neg second \tag{3.14}$$
$$first \to \neg third \tag{3.15}$$
$$second \to \neg first \tag{3.16}$$
$$second \to \neg third \tag{3.17}$$
$$third \to \neg first \tag{3.18}$$
$$third \to \neg second. \tag{3.19}$$

3.5 Solution

The translation given by (1.5)–(1.19) (on page 14) implied *medium*. The same translation is recalled as formulas (3.5)–(3.19) (on page 54), so to apply resolution we first transform (3.5)–(3.19) into the following set of clauses:

1. $small \vee medium \vee large$
2. $\neg small \vee first$
3. $\neg medium \vee second$
4. $\neg large \vee third$
5. $\neg small \vee green$
6. $\neg medium \vee red$
7. $\neg large \vee blue$
8. $second \vee third$
9. $\neg blue$
10. $\neg first \vee \neg second$
11. $\neg first \vee \neg third$
12. $\neg second \vee \neg first$
13. $\neg second \vee \neg third$
14. $\neg third \vee \neg first$
15. $\neg third \vee \neg second.$

Next, we add negated conclusion:

16. $\neg medium$

Proof by resolution:

17.	$\neg large$	– (res): 7, 9
18.	$small \vee large$	– (res): 1, 16
19.	$small$	– (res): 17, 18
20.	$first$	– (res): 2, 19
21.	$\neg second$	– (res): 10, 20
22.	$\neg third$	– (res): 11, 20
23.	$third$	– (res): 8, 21
24.	$\emptyset$	– (res): 22, 23.

3.6 Exercise ¡E!

Using resolution prove the conclusion of Exercise 1.5 (page 16). Recall formulas resulting from translation considered there:

$$\neg works; \tag{3.20}$$

$$books \vee tv \vee \neg cooks. \tag{3.21}$$

$$\neg works \rightarrow \neg tv; \tag{3.22}$$

$$books \vee tv \vee cooks. \tag{3.23}$$

3.6 Solution

We add negated conclusion $\neg books$ to formulas (3.20)–(3.23) we obtain the following set of clauses:

1. $\neg works$
2. $books \vee tv \vee \neg cooks$
3. $works \vee \neg tv$
4. $books \vee tv \vee cooks$
5. $\neg books.$

Proof by resolution:

6. $\neg tv$	– (res): 1, 3
7. $books \vee \neg cooks$	– (res): 2, 6
8. $books \vee cooks$	– (res): 4, 6
9. $books \vee books$	– (res): 7, 8
10. $books$	– (fctr): 9
11. $\emptyset$	– (res): 5, 10

3.7 Exercise ¡E!

Using resolution verify (1.35) (from Exercise 1.8, page 24). Recall that (1.35) was:

$$\left[\neg\big(b \vee (b \to l)\big)\right] \to \neg l. \tag{3.24}$$

3.7 Solution

We have to negate (3.24) and transform into the clausal form:

$$
\begin{array}{ll}
\neg\Big[\big[\neg\big(b \vee (b \to l)\big)\big] \to \neg l\Big] & \leftrightarrow \\
\neg\big(b \vee (b \to l)\big) \wedge \neg\neg l & \leftrightarrow \\
\neg b \wedge \neg(b \to l) \wedge \neg\neg l & \leftrightarrow \\
\neg b \wedge b \wedge \neg l \wedge \neg\neg l & \leftrightarrow \\
\neg b \wedge b \wedge \neg l \wedge l. &
\end{array}
$$

We have clauses:

1. $\neg b$
2. b
3. $\neg l$
4. l.

Proof by resolution:

4. $\emptyset$ – (res): 1, 2.

3.8 Exercise

Using resolution solve Exercise 2.5 (page 39). That is, we have to verify whether formulas:

$$g \leftrightarrow (o \vee j) \tag{3.25}$$

$$o \rightarrow p \tag{3.26}$$

$$j \rightarrow p \tag{3.27}$$

imply:

1. $g \rightarrow p$;
2. $p \rightarrow g$.

3.8 Solution

First, we transform formulas (3.25)–(3.27) into clauses:

1.	$\neg g \vee o \vee j$	– obtained from $g \leftrightarrow (o \vee j)$
2.	$\neg o \vee g$	– obtained from $g \leftrightarrow (o \vee j)$
3.	$\neg j \vee g$	– obtained from $g \leftrightarrow (o \vee j)$
4.	$\neg o \vee p$	– obtained from $o \to p$
5.	$\neg j \vee p$	– obtained from $j \to p$.

1

We first verify whether the obtained set of clauses implies $g \to p$, so negate conclusion:

$$\neg(g \to p) \leftrightarrow (g \wedge \neg p).$$

Therefore we add clauses:

6.	g
7.	$\neg p$

and use resolution:

8.	$o \vee j$	– (res): 1, 6
9.	$\neg o$	– (res): 4, 7
10.	$\neg j$	– (res): 5, 7
11.	j	– (res): 8, 9
12.	$\emptyset$	– (res): 10, 11.

2

Below we shall use another optimization. Namely, we shall remove subsumed clauses. A clause C subsumes a clause D if every literal in C occurs also in D. In such a cease we have that $C \to D$ and so clause D can be removed.

No we verify whether the obtained set of clauses implies $p \to g$, so negate conclusion:

$$\neg(p \to g) \leftrightarrow (p \land \neg g).$$

Therefore we add clauses:

$6'$. p
$7'$. $\neg g$

and use resolution:

$8'$. $\neg o$ – (res): 2, 7'
$9'$. $\neg j$ – (res): 3, 7'.

Using clauses 1–5 and $6'$–$11'$ one can obtain clauses:

- which are already among $1–5, 6'–11'$;
- which can be removed as containing both g and $\neg g$;
- which can be removed as subsumed by clause $7'$.

Therefore, no matter how we apply resolution and factorization, we cannot obtain the empty clause $\emptyset$, so $p \to g$ is not implied by the considered set of clauses.

3.9 Exercise ¡E!

Consider the following relationships among concepts:

> “persons who consume neither meat nor fish food are vegetarians;”
> “vegans consume no animal food nor milk products,”
> “meat is animal food,”
> “fish food is animal food.”

Do the above relationships among concepts imply that vegans are vegetarians?

3.9 Solution

In the sentences we have concepts:[1] *meat* (m), *fish food* (f), *animal food* (a), *milk products* (mp), *vegetarian* (vt), *vegan* (v).

We have the following concept inclusions (represented by implications):

$$(\neg m \wedge \neg f) \rightarrow vt \tag{3.28}$$
$$v \rightarrow (\neg a \wedge \neg mp) \tag{3.29}$$
$$m \rightarrow a \tag{3.30}$$
$$f \rightarrow a. \tag{3.31}$$

We want to verify whether (3.28)–(3.31) imply $v \rightarrow vt$.

We transform concept inclusions into clauses:

$$\begin{aligned}
(\neg m \wedge \neg f) \rightarrow vt \quad & \leftrightarrow \neg(\neg m \wedge \neg f) \vee vt \\
& \leftrightarrow \neg\neg m \vee \neg\neg f \vee vt \\
& \leftrightarrow m \vee f \vee vt \\
v \rightarrow (\neg a \wedge \neg mp) \quad & \leftrightarrow \neg v \vee (\neg a \wedge \neg mp) \\
& \leftrightarrow (\neg v \vee \neg a) \wedge (\neg v \vee \neg mp) \\
m \rightarrow a \quad & \leftrightarrow \neg m \vee a \\
f \rightarrow a \quad & \leftrightarrow \neg f \vee a.
\end{aligned}$$

[1] In brackets we give their abbreviations used later on.

Next, we add negated conclusion (which is $v \wedge \neg vt$):

1. $m \vee f \vee vt$
2. $\neg v \vee \neg a$
3. $\neg v \vee \neg mp$
4. $\neg m \vee a$
5. $\neg f \vee a$
6. v
7. $\neg vt$.

Proof by resolution:

8.	$m \vee f$	– (res): 1, 7
9.	$\neg a$	– (res): 2, 6
10.	$\neg m$	– (res): 4, 9
11.	$\neg f$	– (res): 5, 9
12.	f	– (res): 8, 10
13.	$\emptyset$	– (res): 11, 12.

We then conclude that (3.28)–(3.31) imply $v \to vt$.

Lecture 4

First-order logic

4.1 Exercise

Translate the following sentences into first-order logic:

1. "All employees have income."
2. "Some employees are on holidays."
3. "No employees are unemployed."
4. "Some employees are not satisfied by their salary policy."

4.1 Solution

Note that the sentences are in Aristotelian forms. Possible translations are given below.

1 $\forall x[employee(x) \rightarrow has\_income(x)]$.

2 $\exists x[employee(x) \wedge on\_holidays(x)]$.

3 $\forall x[employee(x) \rightarrow \neg unemployed(x)]$.

4 $\exists x[employee(x) \wedge \neg satisfied(x, salary\_policy)]$,

where '$salary\_policy$' is a constant denoting a particular salary policy.

4.2 Exercise

Translate the following sentences into first-order logic:

1. "I always travel somewhere not far from the city where I live."
2. "I sometimes travel somewhere not far from the city where I live."
3. "I sometimes visit all my favourite places not far from the city where I live."

4.2 Solution

Concepts occurring in sentences are time and/or location dependent. So, we have the following concepts (vocabulary):

- $travel(t, p)$ – "at time t I travel to place p";
- $far\_from(p_1, p_2)$ – "places p_1, p_2 are far from each other";
- $favorite(p)$ – "p is one of my favourite places".

1 $\forall t \exists p \big[travel(t, p) \wedge \neg far\_from(p, my\_city)\big],$

where 'my_city' is a constant denoting "my city".

2 $\exists t \exists p \big[travel(t, p) \wedge \neg far\_from(p, my\_city)\big].$

3 $$\exists t \forall p \big[(favorite(p) \wedge \neg far\_from(p, my\_city)) \rightarrow travel(t, p)\big].$$

4.3 Exercise

Translate the following formulas into natural language:

$$\forall x[(strongEngine(x) \wedge car(x) \wedge wheels(x,4)) \\ \rightarrow fast(x)]; \tag{4.1}$$

$$\forall x \forall y[(parent(x,y) \wedge ancestor(y)) \\ \rightarrow ancestor(x)]; \tag{4.2}$$

$$\forall x \forall y[(car(x) \wedge onRoad(x,y) \wedge highway(y) \wedge \\ normalConditions(y)) \rightarrow fastSpeedAllowed(x)]. \tag{4.3}$$

4.3 Solution

(4.1) Four-wheels cars with strong engines are fast.

(4.2) A parent of an ancestor is an ancestor, too.

(4.3) Under normal conditions, for cars on a highway fast speed is allowed.

4.4 Exercise

Negate formulas obtained in Exercise 4.2 (page 70), simplify them by moving negation inside and translate them into natural language.

That is, we have to negate the following formulas considered in Exercise 4.2:

1. $\forall t \exists p \big[travel(t,p) \wedge \neg far\_from(p, my\_city) \big]$;
2. $\exists t \exists p \big[travel(t,p) \wedge \neg far\_from(p, my\_city) \big]$;
3. $\exists t \forall p \big[(favorite(p) \wedge \neg far\_from(p, my\_city)) \rightarrow travel(t,p) \big]$.

4.4 Solution

1

$$
\begin{aligned}
&\neg\forall t\exists p\big[travel(t,p) \wedge \neg far\_from(p, my\_city)\big] &\leftrightarrow\\
&\exists t\neg\exists p\big[travel(t,p) \wedge \neg far\_from(p, my\_city)\big] &\leftrightarrow\\
&\exists t\forall p\neg\big[travel(t,p) \wedge \neg far\_from(p, my\_city)\big] &\leftrightarrow\\
&\exists t\forall p\big[\neg travel(t,p) \vee \neg\neg far\_from(p, my\_city)\big] &\leftrightarrow\\
&\exists t\forall p\big[\neg travel(t,p) \vee far\_from(p, my\_city)\big].
\end{aligned}
$$

The resulting sentence can be translated into:

> “Sometimes I don’t travel anywhere or I travel far from the city I live in.”

Note that the resulting sentence can be presented in a more readable form:

$$\exists t\forall p\big[travel(t,p) \rightarrow far\_from(p, my\_city)\big], \quad (4.4)$$

which can be translated into an equivalent sentence but perhaps more elegant than before:

> “Sometimes I travel far from the city I live in, if anywhere.”

2

$$
\begin{aligned}
&\neg\exists t \exists p\big[travel(t,p) \wedge \neg far\_from(p, my\_city)\big] && \leftrightarrow\\
&\forall t \neg\exists p\big[travel(t,p) \wedge \neg far\_from(p, my\_city)\big] && \leftrightarrow\\
&\forall t \forall p \neg\big[travel(t,p) \wedge \neg far\_from(p, my\_city)\big] && \leftrightarrow\\
&\forall t \forall p\big[\neg travel(t,p) \vee \neg\neg far\_from(p, my\_city)\big] && \leftrightarrow\\
&\forall t \forall p\big[\neg travel(t,p) \vee far\_from(p, my\_city)\big].
\end{aligned}
$$

The resulting sentence can be translated into:

> "Always I don't travel anywhere or I travel far from the city I live in."

3

$$
\begin{aligned}
&\neg\exists t \forall p\big[(favorite(p) \wedge \neg far\_from(p, my\_city))\\
&\qquad\qquad\qquad \rightarrow travel(t,p)\big] && \leftrightarrow\\
&\forall t \neg\forall p\big[(favorite(p) \wedge \neg far\_from(p, my\_city))\\
&\qquad\qquad\qquad \rightarrow travel(t,p)\big] && \leftrightarrow\\
&\forall t \exists p \neg\big[(favorite(p) \wedge \neg far\_from(p, my\_city))\\
&\qquad\qquad\qquad \rightarrow travel(t,p)\big] && \leftrightarrow\\
&\forall t \exists p\big[favorite(p) \wedge \neg far\_from(p, my\_city)\\
&\qquad\qquad\qquad \wedge \neg travel(t,p)\big].
\end{aligned}
$$

The resulting sentence can be translated into:

> "Always there is a favorite place of mine, not far from the city I live in, where I don't travel to."

4.5 Exercise ¡E!

1. Verify informally the following reasoning:

   > "Every animal in this room is a dog or a cat.
   > Therefore there is an animal in this room being a dog or a cat."

2. Does correctness of the above reasoning change when we add the assumption:

   > "There is an animal in this room"?

4.5 Solution

1

We first translate sentences into first-order formulas:

$$\forall x[(animal(x) \wedge in\_room(x)) \rightarrow (dog(x) \vee cat(x))]; \quad (4.5)$$

$$\exists x[animal(x) \wedge in\_room(x) \wedge (dog(x) \vee cat(x))]. \quad (4.6)$$

The reasoning is formalized as:

$$(4.5) \rightarrow (4.6). \quad (4.7)$$

In such exercises a good idea is to start with constructing a counterexample.

Consider a situation when there are no animals in the considered room. In this case formula (4.5) is true, since:

$$\forall x[\underbrace{(animal(x) \wedge in\_room(x))}_{\text{false}} \rightarrow (dog(x) \vee cat(x))],$$

so no matter what value of implication conclusion is, the whole implication is true.

On the other hand, formula (4.6) is false, since we have:

$$\exists x[\underbrace{animal(x) \wedge in\_room(x)}_{\text{false}} \wedge (dog(x) \vee cat(x))],$$

so the whole conjunction under $\exists x$ is false.

The reasoning expressed by (4.7) reduces in this case to T $\rightarrow$ F which is F. Therefore the reasoning, in general, is incorrect.

2

The additional assumption can be translated into:

$$\exists x(animal(x) \wedge in\_room(x)). \tag{4.8}$$

The reasoning is now formalized as:

$$((4.5) \wedge (4.8)) \rightarrow (4.6). \tag{4.9}$$

We will show that the above reasoning is correct.

Assume that (4.5) $\wedge$ (4.8) is true. In particular, (4.8) is true. Denote by jd the existing x. So we have that:

$$animal(jd) \wedge in\_room(jd). \tag{4.10}$$

By (4.5), we have:

$$(animal(jd) \wedge in\_room(jd)) \rightarrow (dog(jd) \vee cat(jd)). \tag{4.11}$$

Applying modus ponens to (4.10) and (4.11) we derive:

$$(dog(jd) \vee cat(jd)), \tag{4.12}$$

which, in conjunction with (4.10) gives us:

$$animal(jd) \wedge in\_room(jd) \wedge (dog(jd) \vee cat(jd)), \tag{4.13}$$

so (4.6) holds (the existing x is jd).

4.6 Exercise ¡E!

Consider games, in which players move stones between places. A move from place p to place q is possible if there is an arrow between p and q. For example, in the game represented in Figure 4.1, there are moves from a to b, from b to c, from c to a and from c to c and there are no moves from b to a, from c to b, from a to a and from b to b.

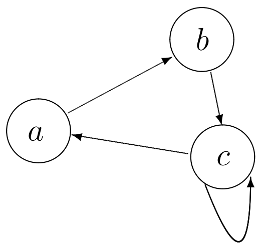


Figure 4.1: Sample game.

Different games are obtained by drawing different connections between places. A particular game can, for example, satisfy the following conditions:

(a) "From every place there is a move."
(b) "Every two places x, y have the property that whenever there is a move from x to y then there is also a move from y to x."
(c) "For every places x, y, z, if there are moves from x to y and from x to z then there is also a move from y to z."

1. Express in predicate logic properties (a), (b), (c).

2. Check informally whether the conjunction of (a) and (c) implies that "From every place there is a move to itself."

3. Check informally whether the conjunction of (a), (b) and (c) implies that "From every place there is a move to itself."

4.6 Solution

1

(a) $\forall x \exists y [move(x, y)]$;

(b) $\forall x \forall y [move(x, y) \to move(y, x)]$;

(c) $\forall x \forall y \forall z [(move(x, y) \wedge move(x, z)) \to move(y, z)]$.

2

We start with constructing a counterexample. Consider a place, say p. By (a), there is q such that $move(p, q)$ is true. By (c) we conclude $move(q, q)$ (when $x = p$ and $y = z = q$). We cannot obtain any other conclusions, so try to construct a counterexample:

- the set of places is $\{p, q\}$,
- we have $move(p, q), move(q, q)$.

In such a case, formulas (a) and (c) are true but it is not the case that from every place there is a move to itself (there is no move from p to p).

3

Attempts to construct a counterexample fail, so lets try to prove that the conclusion is correct.

Take arbitrary place p. By (a) there is a place q such that there is a move from p to q (i.e., $move(p, q)$) is true. Therefore, by (b), there is a move from q to p. The formula under quantifiers in (c) holds for every x, y, z so, in particular, for $x = q$, $y = z = p$. That is, we have:

$$(move(q, p) \wedge move(q, p)) \rightarrow move(p, p). \tag{4.14}$$

The antecedent of implication (4.14) is true, so we conclude that $move(p, p)$ must be true, too. Since p was arbitrarily chosen, we have that $\forall p(move(p, p))$, meaning that from every place there is a move to itself.

4.7 Exercise ¡E!

Consider Exercise 4.6 with (a) substituted by:

(a’) “To every place there is a move”.

1. Express in predicate logic property (a’).
2. Check informally whether the conjunction of (a’) and (c) implies that “From every place there is a move to itself.”

Recall that, in Exercise 4.6, (c) is the formula:

$$\forall x \forall y \forall z[(move(x,y) \wedge move(x,z)) \rightarrow move(y,z)].$$

4.7 Solution

1

(a') $\forall x \exists y[move(y, x)]$.

2

Again, attempts to construct a counterexample fail, so we try to prove that the conclusion is correct.

Take arbitrary place p. By (a') there is a place q such that there is a move from q to p (i.e., $move(q, p)$) is true. The formula under quantifiers in (c) holds for every x, y, z so, as before, we have (4.14) and we conclude that $move(p, p)$ must be true. Since p was arbitrarily chosen, we have that $\forall p(move(p, p))$, meaning that from every place there is a move to itself.

4.8 Exercise ¡E!

1. Translate into first-order formulas:

   > "Every considered object is red, green or blue."
   > "If x is to the left of y then x is not to the right of y."
   > "Red objects are to the left of green objects."
   > "Object a is to the right of object b."
   > "Object b is green and object a is not green."

2. Verify informally whether object a is blue.

4.8 Solution

1

A natural translation is:

$$\forall x[red(x) \vee green(x) \vee blue(x)]; \tag{4.15}$$
$$\forall x \forall y[left(x,y) \rightarrow \neg right(x,y)]; \tag{4.16}$$
$$\forall x \forall y[(red(x) \wedge green(y)) \rightarrow left(x,y)]; \tag{4.17}$$
$$right(a,b); \tag{4.18}$$
$$green(b) \wedge \neg green(a). \tag{4.19}$$

2

Assumung (4.15)–(4.19) we can conclude that $blue(a)$ is true. By (4.19) we have that a is not green so, by (4.15), it must be red or blue. By (4.18), a is to the right of b where, by (4.19), b is green. Contraposing (4.16), we have that a is not to the left of b. Therefore, by (4.17), a cannot be red. So it has to be blue.

4.9 Exercise ¡E!

Consider the following properties of binary relations:

$$\forall x[R(x,x) \to \forall y R(x,y)]; \tag{4.20}$$
$$\forall x \forall y[R(x,y) \to R(y,x)]; \tag{4.21}$$
$$\forall x \exists y[\neg R(y,x)]; \tag{4.22}$$
$$\forall x[\neg R(x,x)]. \tag{4.23}$$

1. Check informally whether the conjunction of (4.20), (4.21) and (4.22) implies (4.23).

2. Check informally whether the conjunction of (4.20), (4.22) and (4.23) implies (4.21).

4.9 Solution

1

Attempts to construct a counterexample fail, so we try to prove that 1 holds.

We proceed using the reduction to absurdity method. Suppose that (4.23) does not follow from the conjunction of (4.20)–(4.22). This means that (4.20)–(4.22) hold while there is a domain element a such that $R(a, a)$ holds. From $R(a, a)$ and (4.20) we deduce that $\forall y R(a, y)$ holds. Using(4.21) we deduce $\forall y R(y, a)$. This contradicts (4.22) because, for $x = a$, (4.22) implies $\exists y[\neg R(y, a)]$ being equivalent to $\neg\forall y R(y, a)$.

The assumption that (4.23) is not true leads to contradiction. Therefore (4.23) follows form the conjunction of (4.20)–(4.22).

2

We construct a counterexample. That is, we assume that (4.20), (4.22) and (4.23) are true and (4.21) is false.

Observe that (4.21) is false when there are elements a, b in the domain such that $R(a, b)$ is true and $R(b, a)$ is false. Now, to satisfy (4.22), we have to add a domain element, say c, such that $\neg R(c, b)$ is true. This gives rise to a counterexample: we assume that the domain is $\{a, b, c\}$, and that $R(x, y)$ is true only for $x = a, y = b$. In particular, $\forall x[\neg R(x, x)]$ holds, so (4.23) is true. This, in turn, makes (4.20) true since the antecedent of implication is always false.

Therefore, in the constructed example formulas (4.20), (4.22) and (4.23) are true and (4.21) is false. Hence, (4.21) does not follow from the conjunction of (4.20), (4.22) and (4.23).

Lecture 5

Tableaux: FOL

5.1 Exercise

Consider the following statement concerning a barber (actually the only barber in town):[1]

> "The barber is a man in town who shaves all those, and only those, men in town who do not shave themselves."

Using tableaux check whether the above statement is satisfiable.

[1] Derived from famous Russell's paradox.

5.1 Solution

The translation of the considered statement is:

$$\forall x[shaves(barber, x) \leftrightarrow \neg shaves(x, x)]. \tag{5.1}$$

We construct a tableau for (5.1):

$$\begin{array}{c}
\forall x[shaves(barber, x) \leftrightarrow \neg shaves(x, x)] \\
\downarrow (\forall x \text{ with } x = barber) \\
shaves(barber, barber) \leftrightarrow \neg shaves(barber, barber) \\
\downarrow (\leftrightarrow) \\
\neg shaves(barber, barber) \vee \neg shaves(barber, barber), \\
shaves(barber, barber) \vee \neg\neg shaves(barber, barber) \\
\downarrow (\neg\neg) \\
\neg shaves(barber, barber) \vee \neg shaves(barber, barber), \\
shaves(barber, barber) \vee shaves(barber, barber) \\
\downarrow (\text{fctr}) \\
\neg shaves(barber, barber), \\
shaves(barber, barber) \vee shaves(barber, barber) \\
\downarrow (\text{fctr}) \\
\underbrace{\neg shaves(barber, barber), shaves(barber, barber)}_{\text{closed}}
\end{array}$$

The tableau is closed, so formula (5.1) is unsatisfiable. Thus the initial sentence is unsatisfiable, too.

5.2 Exercise

Using tableaux check whether the following formula is satisfiable:

$$\forall x R(x, a) \land \forall z \neg R(z, z), \tag{5.2}$$

where a is a constant.

5.2 Solution

We construct a tableau for (5.2):

$$\begin{array}{c}
\forall x R(x,a) \land \forall z[\neg R(z,z)] \\
\downarrow (\land) \\
\forall x R(x,a), \forall z[\neg R(z,z)] \\
\downarrow (\forall x \text{ with } x = a) \\
R(a,a), \forall z[\neg R(z,z)] \\
\downarrow (\forall z \text{ with } z = a) \\
R(a,a), \neg R(a,a)
\end{array}$$

The tableau is closed, so formula (5.2) is not satisfiable.

Note: in general, checking first-order satisfiability cannot be done using tableaux (as well as any other computable method). For example consider the formula:

$$\forall x \exists y R(x,y). \tag{5.3}$$

Constructing a tableau we obtain:

$$\begin{array}{c}
\forall x \exists y R(x,y) \\
\downarrow (\forall x \text{ with } x = a) \\
\exists y R(a,y) \\
\downarrow (\exists y \text{ with } y = b) \\
R(a,b)
\end{array}$$

However, the obtained leaf contains only $R(a,b)$ so does not provide a model for (5.3) since there is no y such that $R(b,y)$ holds.

Of course, we can continue the construction using rules quantifiers, e.g., next nodes can contain:

$$\begin{array}{c}
\downarrow (\forall x \text{ with } x = b) \\
R(a, b), \exists y R(b, y) \\
\downarrow (\exists y \text{ with } y = c) \\
R(a, b), R(b, c)
\end{array}$$

Note that eliminating $\exists y$ we have to use a fresh constant (we have used c). Now there is no y such that $R(c, y)$ holds. The construction loops forever, not providing a model for (5.3). On the other hand, (5.3) is satisfiable, e.g., assuming that the domain consists of $\{a, b\}$ and such that $R(a, b)$ and $R(b, a)$ hold.

5.3 Exercise ¡E!

Using tableaux prove the following first-order formula:

$$\begin{aligned}\big[\forall x \exists y [R(x,y)] \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)]\big] \\ \rightarrow \forall x \forall y R(x,y).\end{aligned} \tag{5.4}$$

5.3 Solution

We construct a tableau for the negation of (5.4):

$$
\begin{array}{c}
\neg\Big[\big[\forall x \exists y [R(x,y)] \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)]\big] \rightarrow \forall x \forall y R(x,y)\Big] \\
\downarrow (\neg \rightarrow) \\
\forall x \exists y [R(x,y)] \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)], \neg \forall x \forall y R(x,y) \\
\downarrow (\neg \forall x \text{ with } x = a) \\
\forall x \exists y [R(x,y)] \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)], \neg \forall y R(a,y) \\
\downarrow (\neg \forall y \text{ with } y = b) \\
\forall x \exists y [R(x,y)] \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)], \neg R(a,b) \\
\downarrow (\forall x \text{ with } x = a) \\
\exists y [R(a,y)] \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)], \neg R(a,b) \\
\downarrow (\exists y \text{ with } y = c) \\
R(a,c) \wedge \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)], \neg R(a,b) \\
\downarrow (\wedge) \\
R(a,c), \forall x \forall y [R(x,y) \rightarrow \forall z R(x,z)], \neg R(a,b) \\
\downarrow (\forall x \text{ with } x = a) \\
R(a,c), \forall y [R(a,y) \rightarrow \forall z R(a,z)], \neg R(a,b) \\
\downarrow (\forall y \text{ with } y = c) \\
R(a,c), R(a,c) \rightarrow \forall z R(a,z), \neg R(a,b) \\
\begin{array}{cc}
\swarrow & \searrow (\rightarrow) \\
\underbrace{R(a,c), \neg R(a,c), \neg R(a,b)}_{\text{closed}} & R(a,c), \forall z R(a,z), \neg R(a,b) \\
 & \downarrow (\forall z \text{ with } z = b) \\
 & \underbrace{R(a,c), R(a,b), \neg R(a,b)}_{\text{closed}}
\end{array}
\end{array}
$$

The tableau is closed, so formula (5.4) is a tautology.

5.4 Exercise ¡E!

Using tableaux check whether formula:

$$\forall x \exists y \exists z [P(x, y, z) \vee \neg \exists z \exists u (\neg P(x, z, u))] \tag{5.5}$$

implies:

$$\exists x \exists y P(a, x, y), \tag{5.6}$$

where a is a constant.

5.4 Solution

We construct a tableau for $\neg[(5.5) \rightarrow (5.6)]$:

$$
\begin{array}{c}
\neg\big[\forall x \exists y \exists z[P(x,y,z) \vee \neg\exists z \exists u(\neg P(x,z,u))] \rightarrow \exists x \exists y P(a,x,y)\big] \\
\downarrow (\neg \rightarrow) \\
\forall x \exists y \exists z[P(x,y,z) \vee \neg\exists z \exists u(\neg P(x,z,u))], \neg\exists x \exists y P(a,x,y) \\
\downarrow (\forall x \text{ with } x = a) \\
\exists y \exists z[P(a,y,z) \vee \neg\exists z \exists u(\neg P(a,z,u))], \neg\exists x \exists y P(a,x,y) \\
\downarrow (\exists y \text{ with } y = b) \\
\exists z[P(a,b,z) \vee \neg\exists z \exists u(\neg P(a,z,u))], \neg\exists x \exists y P(a,x,y) \\
\downarrow (\exists z \text{ with } z = c) \\
P(a,b,c) \vee \neg\exists z \exists u(\neg P(a,z,u)), \neg\exists x \exists y P(a,x,y) \\
\downarrow (\neg\exists x \text{ with } x = b) \\
P(a,b,c) \vee \neg\exists z \exists u(\neg P(a,z,u)), \neg\exists y P(a,b,y) \\
\downarrow (\neg\exists y \text{ with } y = c) \\
P(a,b,c) \vee \neg\exists z \exists u(\neg P(a,z,u)), \neg P(a,b,c) \\
\swarrow \qquad \searrow (\vee)
\end{array}
$$

$$
\begin{array}{cc}
\underbrace{P(a,b,c), \neg P(a,b,c)}_{\text{closed}} & \neg\exists z \exists u(\neg P(a,z,u)), \neg P(a,b,c) \\
 & \downarrow (\neg\exists z \text{ with } z = b) \\
 & \neg\exists u(\neg P(a,b,u)), \neg P(a,b,c) \\
 & \downarrow (\neg\exists u \text{ with } u = c) \\
 & \neg\neg P(a,b,c), \neg P(a,b,c) \\
 & \downarrow (\neg\neg) \\
 & \underbrace{P(a,b,c), \neg P(a,b,c)}_{\text{closed}}
\end{array}
$$

The tableau is closed, so formula (5.5) implies (5.6).

5.5 Exercise ¡E!

Given the following assumptions about persons:

$$\forall x[child(x) \leftrightarrow \exists y(parent(y, x))]; \tag{5.7}$$

$$\forall x \forall y[parent(x, y) \to (person(x) \land person(y))], \tag{5.8}$$

check whether concept *child* is a subconcept of *person*.

5.5 Solution

We verify whether:

$$[(5.7) \wedge (5.8)] \rightarrow \forall x[child(x) \rightarrow person(x)], \quad (5.9)$$

so construct a tableau for:

$$\neg\big[[(5.7) \wedge (5.8)] \rightarrow \forall x[child(x) \rightarrow person(x)]\big]. \quad (5.10)$$

For the tableau to better fit the page we shall use the following abbreviations:

- *child* is denoted by C;
- *parent* is denoted by P;
- *person* is denoted by S.

The tableau:

$$\begin{array}{c}
\neg\big[\forall x[C(x) \leftrightarrow \exists y(P(y,x))] \wedge \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))] \\
\rightarrow \forall x[C(x) \rightarrow S(x)]\big] \\
\downarrow (\neg \rightarrow) \\
\forall x[C(x) \leftrightarrow \exists y(P(y,x))] \wedge \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], \\
\neg \forall x[C(x) \rightarrow S(x)] \\
\downarrow (\wedge) \\
\forall x[C(x) \leftrightarrow \exists y(P(y,x))], \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], \\
\neg \forall x[C(x) \rightarrow S(x)] \\
\downarrow (\neg \forall x \text{ with } x = a) \\
\forall x[C(x) \leftrightarrow \exists y(P(y,x))], \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], \\
\neg(C(a) \rightarrow S(a)) \\
\downarrow (\neg \rightarrow) \\
\forall x[C(x) \leftrightarrow \exists y(P(y,x))], \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], \\
C(a), \neg S(a) \\
\downarrow (\forall x \text{ with } x = a) \\
C(a) \leftrightarrow \exists y(P(y,a)), \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], \\
C(a), \neg S(a) \\
\downarrow (\leftrightarrow) \\
\neg C(a) \vee \exists y(P(y,a)), C(a) \vee \neg \exists y(P(y,a)), \\
\forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], C(a), \neg S(a)
\end{array}$$

$$\swarrow \qquad\qquad \searrow (\vee)$$

$$\underbrace{\begin{array}{c}\neg C(a), C(a) \vee \neg \exists y(P(y,a)), \\ \forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], C(a), \neg S(a)\end{array}}_{\text{closed}}$$

$$\begin{array}{c}
\exists y(P(y,a)), C(a) \vee \neg \exists y(P(y,a)), \\
\forall x \forall y[P(x,y) \rightarrow (S(x) \wedge S(y))], C(a), \neg S(a) \\
\swarrow \qquad \searrow (\vee) \\
(*) \qquad\qquad (**)
\end{array}$$

We continue starting with branch $(*)$:

$$
\begin{array}{c}
\exists y(P(y,a)), C(a), \forall x \forall y[P(x,y) \to (S(x) \wedge S(y))], C(a), \neg S(a) \\
\downarrow (\exists y \text{ with } y = b) \\
P(b,a), C(a), \forall x \forall y[P(x,y) \to (S(x) \wedge S(y))], C(a), \neg S(a) \\
\downarrow (\forall x \text{ with } x = b) \\
P(b,a), C(a), \forall y[P(b,y) \to (S(b) \wedge S(y))], C(a), \neg S(a) \\
\downarrow (\forall y \text{ with } y = a) \\
P(b,a), C(a), P(b,a) \to (S(b) \wedge S(a)), C(a), \neg S(a)
\end{array}
$$

$$
\begin{array}{cc}
\swarrow & \searrow (\vee) \\
\underbrace{P(b,a), C(a), \neg P(b,a), C(a), \neg S(a)}_{\text{closed}} & \downarrow \\
 & P(b,a), C(a), S(b) \wedge S(a), C(a), \neg S(a) \\
 & \downarrow (\wedge) \\
 & \underbrace{P(b,a), C(a), S(b), S(a), C(a), \neg S(a)}_{\text{closed}}
\end{array}
$$

Branch $(*)$ is closed, so we continue with branch $(**)$:

$$
\begin{array}{c}
\exists y(P(y,a)), \neg \exists y(P(y,a)), \\
\forall x \forall y[P(x,y) \to (S(x) \wedge S(y))], C(a), \neg S(a) \\
\downarrow (\exists y \text{ with } y = b) \\
P(b,a), \neg \exists y(P(y,a)), \\
\forall x \forall y[P(x,y) \to (S(x) \wedge S(y))], C(a), \neg S(a) \\
\downarrow (\neg \exists y \text{ with } y = b) \\
\underbrace{P(b,a), \neg P(b,a), \forall y[P(b,y) \to (S(b) \wedge S(y))], C(a), \neg S(a)}_{\text{closed}}
\end{array}
$$

Branch $(**)$ is also closed which completes the tableau and shows that *child* is a subconcept of *person*.

5.6 Exercise

Let the following taxonomy, reflecting a particular situation, be given:

- “color” is a subconcept of “red” or “brown”;
- “color” is a subconcept of red or not “brown”.

Using first-order logic verify whether this taxonomy implies that “color” is a subconcept of “red”.

5.6 Solution

Let B and R denote "brown" and "red", respectively.

We have to verify:

$$\begin{aligned}\forall x[&(color(x) \to (R(x) \vee B(x)))\wedge \\ &(color(x) \to (R(x) \vee \neg B(x))) \\ &\to (color(x) \to R(x))].\end{aligned} \tag{5.11}$$

We construct a closed tableau for the negation of (5.11):

$$\neg\big[\forall x[(\mathit{color}(x) \rightarrow (R(x) \vee B(x)))\wedge$$
$$(\mathit{color}(x) \rightarrow (R(a) \vee \neg B(a))) \rightarrow (\mathit{color}(a) \rightarrow R(a))\big]$$
$$\downarrow (\neg\rightarrow)$$
$$\forall x[(\mathit{color}(x) \rightarrow (R(x) \vee B(x)))\wedge$$
$$(\mathit{color}(x) \rightarrow (R(x) \vee \neg B(x))], \neg(\mathit{color}(x) \rightarrow R(x))$$
$$\downarrow (\forall x \text{ with } x = a)$$
$$(\mathit{color}(a) \rightarrow (R(a) \vee B(a))\wedge$$
$$(\mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \neg(\mathit{color}(a) \rightarrow R(a))$$
$$\downarrow (\wedge)$$
$$(\mathit{color}(a) \rightarrow (R(a) \vee B(a)),$$
$$(\mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \mathit{color}(a), \neg R(a)$$
$$\downarrow (\neg\rightarrow)$$
$$(\mathit{color}(a) \rightarrow (R(a) \vee B(a)),$$
$$(\mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \mathit{color}(a), \neg R(a)$$
$$\swarrow \qquad\qquad \searrow (\rightarrow)$$
$$\underbrace{\neg\mathit{color}(a), (\mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \mathit{color}(a), \neg R(a)}_{\text{closed}}$$
$$R(a) \vee B(a), \mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \mathit{color}(a), \neg R(a)$$
$$\swarrow \qquad\qquad \searrow (\vee)$$
$$\underbrace{R(a), \mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \mathit{color}(a), \neg R(a)}_{\text{closed}}$$
$$B(a), \mathit{color}(a) \rightarrow (R(a) \vee \neg B(a)), \mathit{color}(a), \neg R(a)$$
$$\swarrow \searrow (\rightarrow)$$
$$\underbrace{B(a), \neg\mathit{color}(a), \mathit{color}(a), \neg R(a)}_{\text{closed}} \qquad B(a), R(a) \vee \neg B(a), \mathit{color}(a), \neg R(a)$$
$$\swarrow \qquad\qquad \searrow (\vee)$$
$$\underbrace{B(a), R(a), \mathit{color}(a), \neg R(a)}_{\text{closed}} \qquad \underbrace{B(a), \neg B(a), \mathit{color}(a), \neg R(a)}_{\text{closed}}$$

5.7 Exercise ¡E!

Verify your informal reasoning form Exercise 4.5, point 1 (page 77) using tableaux.

Recall that we considered implication (4.7), that is:

$$\begin{gathered}\forall x[(animal(x)\wedge in\_room(x))\rightarrow(dog(x)\vee cat(x))]\\ \rightarrow\exists x[animal(x)\wedge in\_room(x)\wedge(dog(x)\vee cat(x))].\end{gathered}\tag{5.12}$$

5.7 Solution

For the tableau to better fit the page we shall use the following abbreviations:

- *animal* is denoted by a;
- *in_room* is denoted by i;
- *dog* is denoted by d;
- *cat* is denoted by c.

In Exercise 5.2 we have indicated that, in general, showing satisfiability of a first order formula is impossible. However, as we show below, this can sometimes be done.

We construct a tableau for the negation of (5.12):

$$
\begin{array}{c}
\neg\big[\forall x[(A(x) \wedge I(x)) \rightarrow (D(x) \vee C(x))] \\
\qquad\qquad \rightarrow \exists x[A(x) \wedge I(x) \wedge (D(x) \vee C(x))]\big] \\
\downarrow (\neg \rightarrow) \\
\forall x[(A(x) \wedge I(x)) \rightarrow (D(x) \vee C(x))], \\
\neg\exists x[A(x) \wedge I(x) \wedge (D(x) \vee C(x))] \\
\downarrow (\forall x \text{ with } x = e) \\
(A(e) \wedge I(e)) \rightarrow (D(e) \vee C(e)), \\
\neg\exists x[A(x) \wedge I(x) \wedge (D(x) \vee C(x))] \\
\downarrow (\neg\exists x \text{ with } x = e) \\
(A(e) \wedge I(e)) \rightarrow (D(e) \vee C(e)), \\
\neg[A(e) \wedge I(e) \wedge (D(e) \vee C(e))]
\end{array}
$$

$$
\begin{array}{ll}
\swarrow & \searrow (\rightarrow) \\
\neg(A(e) \wedge I(e)), \neg[A(e) \wedge I(e) \wedge (D(e) \vee C(e))] & \ldots \\
\swarrow & \searrow (\neg\wedge) \\
\neg A(e), \neg[A(e) \wedge I(e) \wedge (D(e) \vee C(e))] & \ldots \\
\swarrow & \searrow (\neg\wedge) \\
\neg A(e), \neg A(e) & \ldots
\end{array}
$$

We have obtained an open leaf, where $animal(e)$ is false. Using this information we can construct a counterexample being a model where:

- the domain is $\{e\}$;
- $animal(e) = \mathrm{F}$.

In this case, no matter what truth values of other literals are, formula (5.12) is false.

5.8 Exercise ¡E!

Verify your informal reasoning form Exercise 4.8 (page 86) using tableaux. Recall that we have considered:

$$\forall x[red(x) \vee green(x) \vee blue(x)]; \tag{5.13}$$

$$\forall x \forall y[left(x,y) \rightarrow \neg right(x,y)]; \tag{5.14}$$

$$\forall x \forall y[(red(x) \wedge green(y)) \rightarrow left(x,y)]; \tag{5.15}$$

$$right(a,b); \tag{5.16}$$

$$green(b) \wedge \neg green(a), \tag{5.17}$$

with the goal to show $blue(a)$.

5.8 Solution

We have to show that:

$$[(5.13)\wedge(5.14)\wedge(5.15)\wedge(5.16)\wedge(5.17)] \to blue(a). \quad (5.18)$$

We construct a tableau for the negation of (5.18). For the tableau to better fit the page we shall use the following abbreviations:

- red is denoted by R;
- $green$ is denoted by G;
- $blue$ is denoted by B;
- $left$ is denoted by l;
- $right$ is denoted by H.

The tableau is constructed on next pages.

$$\neg\Big[\big[\forall x[R(x) \vee G(x) \vee B(x)] \wedge \forall x \forall y[L(x,y) \to \neg H(x,y)] \wedge$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)] \wedge$$
$$H(a,b) \wedge G(b) \wedge \neg G(a)\big] \to B(a)\Big]$$
$$\downarrow (\neg \to)$$
$$\forall x[R(x) \vee G(x) \vee B(x)] \wedge \forall x \forall y[L(x,y) \to \neg H(x,y)] \wedge$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)] \wedge$$
$$H(a,b) \wedge G(b) \wedge \neg G(a), \neg B(a)$$
$$\downarrow (\forall x \text{ with } x = a)$$
$$R(a) \vee G(a) \vee B(a) \wedge \forall x \forall y[L(x,y) \to \neg H(x,y)] \wedge$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)] \wedge$$
$$H(a,b) \wedge G(b) \wedge \neg G(a), \neg B(a)$$
$$\downarrow (\wedge)$$
$$R(a) \vee G(a) \vee B(a), \forall x \forall y[L(x,y) \to \neg H(x,y)] \wedge$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)] \wedge$$
$$H(a,b) \wedge G(b) \wedge \neg G(a), \neg B(a)$$
$$\downarrow (\wedge)$$
$$R(a) \vee G(a) \vee B(a), \forall x \forall y[L(x,y) \to \neg H(x,y)],$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)] \wedge$$
$$H(a,b) \wedge G(b) \wedge \neg G(a), \neg B(a)$$
$$\downarrow (\wedge)$$
$$R(a) \vee G(a) \vee B(a), \forall x \forall y[L(x,y) \to \neg H(x,y)],$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)],$$
$$H(a,b) \wedge G(b) \wedge \neg G(a), \neg B(a)$$
$$\downarrow (\wedge)$$
$$R(a) \vee G(a) \vee B(a), \forall x \forall y[L(x,y) \to \neg H(x,y)],$$
$$\forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)],$$
$$H(a,b), G(b) \wedge \neg G(a), \neg B(a)$$
$$\downarrow (\wedge)$$

$$\downarrow (\wedge)$$

$$R(a) \vee G(a) \vee B(a), \forall x \forall y[L(x,y) \rightarrow \neg H(x,y)],$$
$$\forall x \forall y[(R(x) \wedge G(y)) \rightarrow L(x,y)],$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$

$$\swarrow \qquad\qquad\qquad\qquad \searrow (\vee)$$

$$R(a), \forall x \forall y[L(x,y) \rightarrow \neg H(x,y)], \qquad (*)$$
$$\forall x \forall y[(R(x) \wedge G(y)) \rightarrow L(x,y)],$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$
$$\downarrow (\forall x \text{ with } x = a)$$
$$R(a), \forall y[L(a,y) \rightarrow \neg H(a,y)],$$
$$\forall x \forall y[(R(x) \wedge G(y)) \rightarrow L(x,y)],$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$
$$\downarrow (\forall y \text{ with } y = b)$$
$$R(a), L(a,b) \rightarrow \neg H(a,b), \forall x \forall y[(R(x) \wedge G(y)) \rightarrow L(x,y)],$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$

$$\swarrow \qquad\qquad\qquad \searrow (\rightarrow)$$

$$\underbrace{\neg H(a,b), \forall x \forall y[(R(x) \wedge G(y)) \rightarrow L(x,y)], \quad H(a,b), G(b), \neg G(a), \neg B(a)}_{\text{closed}}$$

$$\downarrow$$
$$R(a), \neg L(a,b), \forall x \forall y[(R(x) \wedge G(y)) \rightarrow L(x,y)],$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$
$$\downarrow (\forall x \text{ with } x = a)$$
$$R(a), \neg L(a,b), \forall y[(R(a) \wedge G(y)) \rightarrow L(a,y)],$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$
$$\downarrow (\forall y \text{ with } y = b)$$
$$R(a), \neg L(a,b), (R(a) \wedge G(b)) \rightarrow L(a,b),$$
$$H(a,b), G(b), \neg G(a), \neg B(a)$$

$$\swarrow \qquad \searrow (\rightarrow)$$

$$(**) \qquad (***)$$

The continuation of $(*)$:

$$\begin{array}{c} G(a) \vee B(a), \forall x \forall y[L(x,y) \to \neg H(x,y)], \\ \forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)], \\ H(a,b), G(b), \neg G(a), \neg B(a) \end{array}$$

$$\swarrow \qquad\qquad \searrow (\vee)$$

$$\underbrace{\begin{array}{c} G(a), \forall x \forall y[L(x,y) \to \neg H(x,y)], \\ \forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)], \\ H(a,b), G(b), \neg G(a), \neg B(a) \end{array}}_{\text{closed}}$$

$$\underbrace{\begin{array}{c} B(a), \forall x \forall y[L(x,y) \to \neg H(x,y)], \\ \forall x \forall y[(R(x) \wedge G(y)) \to L(x,y)], \\ H(a,b), G(b), \neg G(a), \neg B(a) \end{array}}_{\text{closed}}$$

The continuation of $(**)$:

$$R(a), \neg L(a,b), \neg(R(a) \wedge G(b)), H(a,b), G(b), \neg G(a), \neg B(a)$$

$$\swarrow \qquad\qquad \searrow (\neg\wedge)$$

$$\underbrace{R(a), \neg L(a,b), \neg R(a), H(a,b), G(b), \neg G(a), \neg B(a)}_{\text{closed}}$$

$$\underbrace{R(a), \neg L(a,b), \neg G(b), H(a,b), G(b), \neg G(a), \neg B(a)}_{\text{closed}}$$

The continuation od $(***)$:

$$\underbrace{R(a), \neg L(a,b), L(a,b), H(a,b), G(b), \neg G(a), \neg B(a)}_{\text{closed}}$$

5.9 Exercise ¡E!

Verify your informal reasoning form Exercise 4.9, point 1 (page 88) using tableaux. That is, we have to prove that:

$$\left.\begin{array}{l}\forall x[R(x,x) \rightarrow \forall y R(x,y)]\wedge\\ \forall x\forall y[R(x,y) \rightarrow R(y,x)]\wedge\\ \forall x\exists y[\neg R(y,x)]\end{array}\right\} \rightarrow \forall x[\neg R(x,x)]. \quad (5.19)$$

5.9 Solution

We construct a tableau for the negation of (5.19):

$$
\begin{gathered}
\neg\big[[\forall x[R(x,x) \to \forall y R(x,y)] \wedge \forall x \forall y[R(x,y) \to R(y,x)] \wedge \\
\forall x \exists y[\neg R(y,x)]] \to \forall x[\neg R(x,x)]\big] \\
\downarrow (\neg\to) \\
\forall x[R(x,x) \to \forall y R(x,y)] \wedge \forall x \forall y[R(x,y) \to R(y,x)] \wedge \\
\forall x \exists y[\neg R(y,x)], \neg\forall x[\neg R(x,x)] \\
\downarrow (\wedge) \\
\forall x[R(x,x) \to \forall y R(x,y)], \forall x \forall y[R(x,y) \to R(y,x)] \wedge \\
\forall x \exists y[\neg R(y,x)], \neg\forall x[\neg R(x,x)] \\
\downarrow (\wedge) \\
\forall x[R(x,x) \to \forall y R(x,y)], \forall x \forall y[R(x,y) \to R(y,x)], \\
\forall x \exists y[\neg R(y,x)], \neg\forall x[\neg R(x,x)] \\
\downarrow (\neg\forall x \text{ with } x = a) \\
\forall x[R(x,x) \to \forall y R(x,y)], \forall x \forall y[R(x,y) \to R(y,x)], \\
\forall x \exists y[\neg R(y,x)], \neg\neg R(a,a) \\
\downarrow (\neg\neg) \\
\forall x[R(x,x) \to \forall y R(x,y)], \forall x \forall y[R(x,y) \to R(y,x)], \\
\forall x \exists y[\neg R(y,x)], R(a,a) \\
\downarrow (\forall x \text{ with } x = a) \\
\forall x[R(x,x) \to \forall y R(x,y)], \forall x \forall y[R(x,y) \to R(y,x)], \\
\exists y[\neg R(y,a)], R(a,a) \\
\downarrow (\exists y \text{ with } y = b) \\
\forall x[R(x,x) \to \forall y R(x,y)], \forall x \forall y[R(x,y) \to R(y,x)], \\
\neg R(b,a), R(a,a) \\
\downarrow (\forall x \text{ with } x = a) \\
R(a,a) \to \forall y R(a,y), \forall x \forall y[R(x,y) \to R(y,x)], \\
\neg R(b,a), R(a,a) \\
\swarrow \qquad\qquad\qquad \searrow (\to) \\
\downarrow \qquad\qquad\qquad\qquad (*) \\
\underbrace{\neg R(a,a), \forall x \forall y[R(x,y) \to R(y,x)], \neg R(b,a), R(a,a)}_{\text{closed}}
\end{gathered}
$$

We continue with branch $(*)$:

$$
\begin{array}{c}
\forall y R(a,y), \forall x \forall y [R(x,y) \to R(y,x)], \neg R(b,a), R(a,a) \\
\downarrow (\forall y \text{ with } y = b) \\
R(a,b), \forall x \forall y [R(x,y) \to R(y,x)], \neg R(b,a), R(a,a) \\
\downarrow (\forall x \text{ with } x = a) \\
R(a,b), \forall y [R(a,y) \to R(y,a)], \neg R(b,a), R(a,a) \\
\downarrow (\forall y \text{ with } y = b) \\
R(a,b), R(a,b) \to R(b,a), \neg R(b,a), R(a,a) \\
\swarrow \qquad\qquad\qquad \searrow (\to)
\end{array}
$$

$$
\underbrace{R(a,b), \neg R(a,b), \neg R(b,a), R(a,a)}_{\text{closed}}
\qquad
\underbrace{R(a,b), R(b,a), \neg R(b,a), R(a,a)}_{\text{closed}}
$$

The tableau is closed which proves implication (5.19).

Lecture 6

Resolution: FOL

6.1 Exercise

Transform the following formula into an equisatisfiable set of clauses:[1]

$$\neg\forall x\exists y\big[\forall z(R(x,y,z) \to \forall u(S(x,u) \vee T(y,z,u))) \\ \wedge\exists v S(x,y,v)\big]. \tag{6.1}$$

[1] Formulas A and B are equisatisfiable when A is satisfiable iff B is satisfiable.

6.1 Solution

We first transform (6.1) into the prenex normal form:

$$
\begin{aligned}
&\neg\forall x\exists y\big[\forall z(R(x,y,z)\to\forall u(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\wedge\exists vS(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\neg\exists y\big[\forall z(R(x,y,z)\to\forall u(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\wedge\exists vS(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\neg\big\{\big[\forall z(R(x,y,z)\to\forall u(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\wedge\exists vS(x,y,v)\big]\big\} \quad\leftrightarrow \\
&\exists x\forall y\big[\neg\forall z(R(x,y,z)\to\forall u(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\neg\exists vS(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\big[\exists z\neg(R(x,y,z)\to\forall u(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\forall v\neg S(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\big[\exists z(R(x,y,z)\wedge\neg\forall u(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\forall v\neg S(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\big[\exists z(R(x,y,z)\wedge\exists u\neg(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\forall v\neg S(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\exists z\big[(R(x,y,z)\wedge\exists u\neg(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\forall v\neg S(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\exists z\forall v\big[(R(x,y,z)\wedge\exists u\neg(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\neg S(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\exists z\forall v\big[\exists u(R(x,y,z)\wedge\neg(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\neg S(x,y,v)\big] \quad\leftrightarrow \\
&\exists x\forall y\exists z\forall v\exists u\big[(R(x,y,z)\wedge\neg(S(x,u)\vee T(y,z,u))) \\
&\qquad\qquad\qquad\vee\neg S(x,y,v)\big].
\end{aligned}
$$

Observe that in the last steps we could move to the prefix quantifier $\exists u$ before moving $\forall v$. The resulting formula would still be equivalent to (6.1).

The resulting formula is equivalent (thus also equisatisfiable) to (6.1). We have to eliminate existential quantifiers $\exists x$, $\exists z$ and $\exists u$, so apply Skolemization, where a is a constant and f, g are function symbols:

$$\begin{aligned}
&\exists x \forall y \exists z \forall v \exists u \big[(R(x,y,z) \wedge \neg(S(x,u) \vee T(y,z,u))) \\
&\qquad\qquad\qquad\qquad \vee \neg S(x,y,v)\big]; \\
&\forall y \exists z \forall v \exists u \big[(R(a,y,z) \wedge \neg(S(a,u) \vee T(y,z,u))) \\
&\qquad\qquad\qquad\qquad \vee \neg S(a,y,v)\big]; \\
&\forall y \forall v \exists u \big[(R(a,y,f(y)) \wedge \neg(S(a,u) \vee T(y,f(y),u))) \\
&\qquad\qquad\qquad\qquad \vee \neg S(a,y,v)\big]; \\
&\forall y \forall v \big[(R(a,y,f(y)) \wedge \neg(S(a,g(y,v)) \vee T(y,f(y),g(y,v)))) \\
&\qquad \vee \neg S(a,y,v)\big].
\end{aligned}$$

Note that Skolemization preserves satisfiability but not equivalence, so the resulting formula is equisatisfiable (but not equivalent) to (6.1).

Universal quantifiers $\forall y \forall v$ can now be removed and we transform the resulting formula into the conjunctive normal form:

$$
\begin{array}{l}
(R(a,y,f(y)) \wedge \neg(S(a,g(y,v)) \vee T(y,f(y),g(y,v)))) \\
\qquad \vee \neg S(a,y,v)
\end{array}
\quad \leftrightarrow
$$

$$
\begin{array}{l}
(R(a,y,f(y)) \wedge \neg S(a,g(y,v)) \wedge \neg T(y,f(y),g(y,v))) \\
\qquad \vee \neg S(a,y,v)
\end{array}
\quad \leftrightarrow
$$

$$
\begin{array}{l}
(R(a,y,f(y)) \vee \neg S(a,y,v)) \wedge \\
\qquad (\neg S(a,g(y,v)) \vee \neg S(a,y,v)) \wedge \\
\qquad (\neg T(y,f(y),g(y,v)) \vee \neg S(a,y,v)).
\end{array}
$$

We then have the following set of clauses equisatisfiable to (6.1) (all transformations we applied are satisfiability preserving):

1. $R(a,y,f(y)) \vee \neg S(a,y,v)$
2. $\neg S(a,g(y,v)) \vee \neg S(a,y,v)$
3. $\neg T(y,f(y),g(y,v)) \vee \neg S(a,y,v)$.

6.2 Exercise

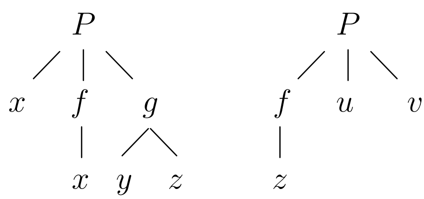
Unify the following literals:

1. $P(x, f(x), g(y, z))$ and $P(f(z), u, v))$;
2. $P(x, f(x), v)$ and $P(f(z), x, v))$.

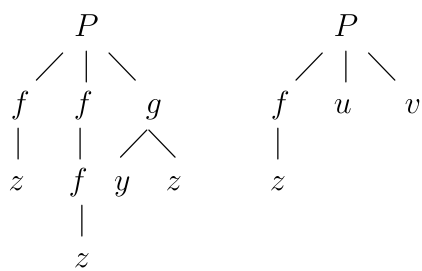
6.2 Solution

1

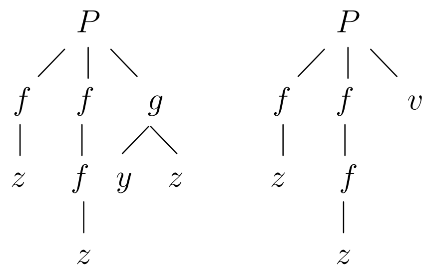
Expression trees are:



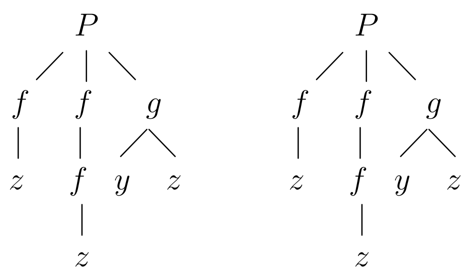
The first difference is between x and f so we set $x = f(z)$ and obtain trees:



To eliminate the next difference, we set $u = f(f(z))$ and obtain:



The last difference is eliminated by setting $v = g(y, z)$:



Summing up, both literals are unifiable by substitution $x = f(z), u = f(f(z)), v = g(y, z)$.

2

Expression trees are:

$$
\begin{tikzcd}
& P \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & & & P \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
x & f \arrow[d, dash] & v & & f \arrow[d, dash] & x & v \\
& x & & & z & &
\end{tikzcd}
$$

The first difference is between x and f so we set $x = f(z)$ and obtain trees:

$$
\begin{tikzcd}
& P \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & & & P \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
f \arrow[d, dash] & f \arrow[d, dash] & v & & f \arrow[d, dash] & f \arrow[d, dash] & v \\
z & f \arrow[d, dash] & & & z & z & \\
& z & & & & &
\end{tikzcd}
$$

The last difference is requires to set $z = f(z)$. Since z appears in $f(z)$ such substitution is not allowed (occurrence check). Therefore the considered literals are not unifiable.

6.3 Exercise

Using resolution solve Exercise 5.1 (page 92). That is, we have to chect the satisfiability of:

$$\forall x[shaves(barber, x) \leftrightarrow \neg shaves(x, x)]. \tag{6.2}$$

6.3 Solution

We transform (6.2) into a set of clauses:

$$
\begin{aligned}
&\forall x[shaves(barber, x) \leftrightarrow \neg shaves(x, x)] && \leftrightarrow \\
&shaves(barber, x) \leftrightarrow \neg shaves(x, x) && \leftrightarrow \\
&\neg shaves(barber, x) \vee \neg shaves(x, x) \wedge \\
&shaves(barber, x) \vee \neg\neg shaves(x, x) && \leftrightarrow \\
&\neg shaves(barber, x) \vee \neg shaves(x, x) \wedge \\
&shaves(barber, x) \vee shaves(x, x).
\end{aligned}
$$

A resolution proof (after renaming variables):

1. $\neg shaves(barber, x_1) \vee \neg shaves(x_1, x_1)$
2. $shaves(barber, x_2) \vee shaves(x_2, x_2)$
3. $\neg shaves(barber, barber)$
   – (fctr): 1 with $x_1 = barber$
4. $shaves(barber, barber)$
   – (fctr): 2 with $x_2 = barber$
5. $\emptyset$ – (res): 3, 4.

6.4 Exercise ¡E!

Using resolution check whether the following formula is a tautology:

$$\begin{aligned}\forall x \exists y \exists z \big[[P(x,y,z) \vee \neg \exists z \exists u (\neg P(x,z,u))] \\ \to \exists x \exists y P(f(a),x,y) \big],\end{aligned} \tag{6.3}$$

where a is a constant.

6.4 Solution

We transform negated (6.3) into the prenex normal form:

$$
\begin{aligned}
&\neg\forall x\exists y\exists z\big[(P(x,y,z) \vee \neg\exists z\exists u(\neg P(x,z,u))) \\
&\qquad\qquad \rightarrow \exists x\exists y P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\neg\exists y\exists z\big[(P(x,y,z) \vee \neg\exists z\exists u(\neg P(x,z,u))) \\
&\qquad\qquad \rightarrow \exists x\exists y P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\forall y\neg\exists z\big[(P(x,y,z) \vee \neg\exists z\exists u(\neg P(x,z,u))) \\
&\qquad\qquad \rightarrow \exists x\exists y P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\neg\big[(P(x,y,z) \vee \neg\exists z\exists u(\neg P(x,z,u))) \\
&\qquad\qquad \rightarrow \exists x\exists y P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\big[(P(x,y,z) \vee \neg\exists z\exists u(\neg P(x,z,u))) \\
&\qquad\qquad \wedge\neg\exists x\exists y P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\big[(P(x,y,z) \vee \forall z\neg\exists u(\neg P(x,z,u))) \\
&\qquad\qquad \wedge\forall x\neg\exists y P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\big[(P(x,y,z) \vee \forall z\forall u\neg(\neg P(x,z,u))) \\
&\qquad\qquad \wedge\forall x\forall y\neg P(f(a),x,y)\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\big[(P(x,y,z) \vee \forall z'\forall u P(x,z;,u)) \\
&\qquad\qquad \wedge\forall x'\forall y'\neg P(f(a),x',y')\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\big[(P(x,y,z) \vee \forall z'\forall u P(x,z',u)) \\
&\qquad\qquad \wedge\forall x'\forall y'\neg P(f(a),x',y')\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\forall z'\forall u\big[(P(x,y,z) \vee P(x,z',u)) \\
&\qquad\qquad \wedge\forall x'\forall y'\neg P(f(a),x',y')\big] \quad \leftrightarrow \\
&\exists x\forall y\forall z\forall z'\forall u\forall x'\forall y'\big[(P(x,y,z) \vee P(x,z',u)) \\
&\qquad\qquad \wedge\neg P(f(a),x',y')\big]. \quad \leftrightarrow
\end{aligned}
$$

We Skolemize (b is a fresh constant):

$$\forall y \forall z \forall z' \forall u \forall x' \forall y' \big[(P(b, y, z) \vee P(b, z', u)) \\ \wedge \neg P(f(a), x', y') \big].$$

We then have the following clauses (after renaming variables):

1. $P(b, y_1, z) \vee P(b, z_1, u)$
2. $\neg P(f(a), x_2, y_2)$.

Clause 2 contains the only negative literal. However it does not unify with any positive literal, so the empty clause cannot be obtained. Therefore (6.3) is not a tautology.

6.5 Exercise ¡E!

Prove the following formula using resolution, where b is a constant:

$$\Big[\forall y Q(b,y) \wedge \forall x \forall y [Q(x,y) \rightarrow Q(s(x),s(y))]\Big] \rightarrow \\ \exists z[Q(b,z) \wedge Q(z,s(s(b)))]. \tag{6.4}$$

6.5 Solution

We first negate formula (6.4):

$$
\begin{aligned}
&\neg\Big\{\Big[\forall y Q(b,y) \wedge \forall x \forall y[Q(x,y) \rightarrow Q(s(x), s(y))]\Big] \rightarrow \\
&\qquad\qquad \exists z[Q(b,z) \wedge Q(z, s(s(b)))]\Big\} \quad \leftrightarrow \\
&\Big[\forall y Q(b,y) \wedge \forall x \forall y[Q(x,y) \rightarrow Q(s(x), s(y))]\Big] \wedge \\
&\qquad\qquad \neg\exists z[Q(b,z) \wedge Q(z, s(s(b)))] \quad \leftrightarrow \\
&\Big[\forall y Q(b,y) \wedge \forall x \forall y[Q(x,y) \rightarrow Q(s(x), s(y))]\Big] \wedge \\
&\qquad\qquad \forall z \neg[Q(b,z) \wedge Q(z, s(s(b)))].
\end{aligned}
$$

This gives the following clauses (after renaming variables):

1. $Q(b, y_1)$;
2. $\neg Q(x_2, y_2) \vee Q(s(x_2), s(y_2))$;
3. $\neg Q(b, z_3) \vee \neg Q(z_3, s(s(b)))$.

Proof by resolution:

4. $\neg Q(z_4, s(s(b)))$ – (res): 1, 3 with $y_1 = z_3$ and renaming z_3/z_4
5. $\emptyset$ – (res): 1, 4 with $z_4 = b, y_1 = s(s(b))$.

6.6 Exercise ¡E!

Provide a resolution proof for Exercise 5.5 (page 101). That is, given the following assumptions:

$$\forall x[child(x) \leftrightarrow \exists y(parent(y,x))]; \tag{6.5}$$
$$\forall x \forall y[parent(x,y) \rightarrow (person(x) \wedge person(y))], \tag{6.6}$$

we have to check whether concept *child* is a subconcept of $person$. Thus we verify whether:

$$[(6.5) \wedge (6.6)] \rightarrow \forall x[child(x) \rightarrow person(x)]. \tag{6.7}$$

6.6 Solution

We have to prove that implication (6.7) indeed holds, so we negate it:

$$
\begin{aligned}
&\neg\Big[\forall x[child(x) \leftrightarrow \exists y(parent(y,x))] \wedge \\
&\forall x \forall y[parent(x,y) \rightarrow (person(x) \wedge person(y))] \\
&\qquad\qquad\qquad \rightarrow \forall x[child(x) \rightarrow person(x)]\Big] \leftrightarrow \\
&\forall x[child(x) \leftrightarrow \exists y(parent(y,x))] \wedge \\
&\quad \forall x \forall y[parent(x,y) \rightarrow (person(x) \wedge person(y))] \wedge \\
&\quad \neg\forall x[child(x) \rightarrow person(x)] \leftrightarrow \\
&\forall x[(\neg child(x) \vee \exists y(parent(y,x))) \wedge \\
&\quad\ (\neg\exists y(parent(y,x)) \vee child(x))] \wedge \\
&\quad \forall x \forall y[\neg parent(x,y) \vee (person(x) \wedge person(y))] \wedge \\
&\quad \exists x \neg[child(x) \rightarrow person(x)] \leftrightarrow \\
&\forall x \exists y[(\neg child(x) \vee parent(y,x)) \wedge \\
&\quad\ (\forall y(\neg parent(y,x)) \vee child(x))] \wedge \\
&\quad \forall x \forall y[(\neg parent(x,y) \vee person(x)) \wedge \\
&\qquad\qquad (\neg parent(x,y) \vee person(y))] \wedge \\
&\quad \exists x[child(x) \wedge \neg person(x)].
\end{aligned}
$$

We Skolemize and obtain the following clauses (after renaming variables):

1. $\neg child(x_1) \vee parent(f(x_1), x_1)$;
2. $\neg parent(y_2, x_2) \vee child(x_2)$;
3. $\neg parent(x_3, y_3) \vee person(x_3)$;
4. $\neg parent(x_4, y_4) \vee person(y_4)$;
5. $child(a)$;
6. $\neg person(a)$.

Proof by resolution:

7.	$parent(f(a), a)$	– (res): 1, 5 with $x_1 = a$
8.	$person(a)$	– (res): 4, 7 with $x_4 = f(a), y_4 = a$
9.	$\emptyset$	– (res): 6, 8.

6.7 Exercise ¡E!

Verify your informal reasoning form Exercise 4.5 (page 77) using resolution.

Recall that we considered implication (4.7), that is:

$$\begin{array}{l}\forall x[(animal(x) \wedge in\_room(x)) \rightarrow (dog(x) \vee cat(x))] \\ \rightarrow \exists x[animal(x) \wedge in\_room(x) \wedge (dog(x) \vee cat(x))].\end{array} \tag{6.8}$$

6.7 Solution

We first transform negated (6.8) into the conjunctive normal form:

$$
\begin{aligned}
&\neg\big[\forall x[(animal(x)\wedge in\_room(x))\rightarrow(dog(x)\vee cat(x))]\\
&\;\rightarrow\exists x[animal(x)\wedge in\_room(x)\wedge(dog(x)\vee cat(x))]\big] \leftrightarrow\\
&\forall x[(animal(x)\wedge in\_room(x))\rightarrow(dog(x)\vee cat(x))]\\
&\;\wedge\neg\exists x[animal(x)\wedge in\_room(x)\wedge(dog(x)\vee cat(x))] \leftrightarrow\\
&\forall x[\neg(animal(x)\wedge in\_room(x)) \vee dog(x)\vee cat(x)]\\
&\;\wedge\forall x\neg[animal(x)\wedge in\_room(x)\wedge(dog(x)\vee cat(x))] \leftrightarrow\\
&\forall x[\neg animal(x) \vee \neg in\_room(x) \vee dog(x) \vee cat(x)]\\
&\;\wedge\forall x[\neg animal(x) \vee \neg in\_room(x) \vee \neg(dog(x)\vee cat(x))] \leftrightarrow\\
&\forall x[\neg animal(x) \vee \neg in\_room(x) \vee dog(x) \vee cat(x)]\\
&\;\wedge\forall x[\neg animal(x)\vee\neg in\_room(x)\vee(\neg dog(x)\wedge\neg cat(x))] \leftrightarrow\\
&\forall x[\neg animal(x) \vee \neg in\_room(x) \vee dog(x) \vee cat(x)]\\
&\;\wedge\forall x[(\neg animal(x)\vee\neg in\_room(x)\vee\neg dog(x))\\
&\qquad\wedge (\neg animal(x)\vee\neg in\_room(x) \vee \neg cat(x))].
\end{aligned}
$$

We then have the following clauses (after renaming variables):

1. $\neg animal(x_1) \vee \neg in\_room(x_1) \vee dog(x_1) \vee cat(x_1)$
2. $\neg animal(x_2)\vee\neg in\_room(x_2)\vee\neg dog(x_2)$
3. $\neg animal(x_3)\vee\neg in\_room(x_3) \vee \neg cat(x_3)$.

We use resolution and factorization:

4. $\neg animal(x_4) \lor \neg in\_room(x_4) \lor$
   $\neg animal(x_4) \lor \neg in\_room(x_4) \lor cat(x_4)$
   – (res): 1, 2 with $x_1 = x_2$ and renaming x_2/x_4
5. $\neg animal(x_5) \lor \neg in\_room(x_5) \lor\ cat(x_5)$
   – (fctr): 4 and renaming x_4/x_5
6. $\ldots$

Note that no matter how we apply resolution and factorization, the empty clause cannot be obtained. In fact, clauses 1–3 contain literals $\neg animal(\ldots)$, $\neg in\_room(\ldots)$ which cannot be resolved out since there are no their positive occurrences.

Therefore, implication (6.8) indeed is not a tautology.

6.8 Exercise ¡E!

Verify your informal reasoning form point 3 of Exercise 4.6 (page 80) using resolution.

Recall that we have considered:

(a) $\forall x \exists y[move(x, y)]$;

(b) $\forall x \forall y[move(x, y) \rightarrow move(y, x)]$;

(c) $\forall x \forall y \forall z[(move(x, y) \wedge move(x, z)) \rightarrow move(y, z)]$.

We have to show that (a), (b), (c) imply $\forall x[move(x, x)]$.

6.8 Solution

We negate the claim and obtain formulas:

$$\forall x \exists y[move(x, y)]; \tag{6.9}$$

$$\forall x \forall y[move(x, y) \to move(y, x)]; \tag{6.10}$$

$$\forall x \forall y \forall z[(move(x, y) \wedge move(x, z)) \to move(y, z)]; \tag{6.11}$$

$$\neg \forall x[move(x, x)]. \tag{6.12}$$

We transform them into:

$$\forall x[move(x, f(x))] \text{ – Skolemization}; \tag{6.13}$$

$$\forall x \forall y[\neg move(x, y) \vee move(y, x)]; \tag{6.14}$$

$$\forall x \forall y \forall z[\neg(move(x, y) \wedge move(x, z)) \vee move(y, z)]; \tag{6.15}$$

$$\exists x[\neg move(x, x)]. \tag{6.16}$$

So we obtain clauses, where a is a constant obtained by Skolemizing formula (6.16) and renaming variables:

1. $move(x_1, f(x_1))$;
2. $\neg move(x_2, y_2) \vee move(y_2, x_2)$;
3. $\neg move(x_3, y_3) \vee \neg move(x_3, z_3) \vee move(y_3, z_3)$;
4. $\neg move(a, a)$.

Proof by resolution:

5. $move(f(x_5), x_5)$ – (res): 1, 2 with $y_2 = f(x_1)$
$x_2 = x_1$ and renaming x_1/x_5
6. $\neg move(f(x_6), z_6) \vee move(x_6, z_6)$
– (res): 3, 5 with $x_3 = f(x_5), y_3 = x_5$
and renaming $x_5/x_6, z_3/z_6$
7. $move(x_7, x_7)$ – (res): 5, 6 with $x_5 = x_6 = z_6$
and renaming x_6/x_7
8. $\emptyset$ – (res): 4, 7 with $x_7 = a$.

The empty clause is derived, so (a), (b), (c) indeed imply $\forall x[move(x, x)]$.

6.9 Exercise ¡E!

Verify your informal reasoning form Exercise 4.8 (page 86) using resolution.

Recall that we have considered:

$$\forall x[red(x) \vee green(x) \vee blue(x)]; \tag{6.17}$$

$$\forall x \forall y[left(x, y) \rightarrow \neg right(x, y)]; \tag{6.18}$$

$$\forall x \forall y[(red(x) \wedge green(y)) \rightarrow left(x, y)]; \tag{6.19}$$

$$right(a, b); \tag{6.20}$$

$$green(b) \wedge \neg green(a), \tag{6.21}$$

with the goal to show $blue(a)$. That is, we want to show that:

$$[(6.17) \wedge (6.18) \wedge (6.19) \wedge (6.20) \wedge (6.21)] \rightarrow blue(a). \tag{6.22}$$

6.9 Solution

We transform the negated (6.22) into the clausal form:

1. $red(x_1) \vee green(x_1) \vee blue(x_1)$
2. $\neg left(x_2, y_2) \vee \neg right(x_2, y_2)$
3. $\neg red(x_3) \vee \neg green(y_3) \vee left(x_3, y_3)$
4. $right(a, b)$
5. $green(b)$
6. $\neg green(a)$
7. $\neg blue(a)$.

Resolution proof:

8. $red(a) \vee green(a)$ – (res): 1, 7 with $x_1 = a$
9. $red(a)$ – (res): 6, 8
10. $\neg green(y_{10}) \vee left(a, y_{10})$
    – (res): 3, 9 with $x_3 = a$ and renaming y_3/y_{10}
11. $left(a, b)$ – (res): 5, 10 with $y_{10} = b$
12. $\neg right(a, b)$ – (res): 2, 11 with $x_2 = a, y_2 = b$
13. $\emptyset$ – (res): 4, 12

6.10 Exercise ¡E!

Verify your informal reasoning form Exercise 4.9, point 1 (page 88) using resolution. That is, we have to prove that:

$$\left.\begin{array}{l}\forall x[R(x,x) \to \forall y R(x,y)] \wedge \\ \forall x \forall y[R(x,y) \to R(y,x)] \wedge \\ \forall x \exists y[\neg R(y,x)]\end{array}\right\} \to \forall x[\neg R(x,x)]. \quad (6.23)$$

6.10 Solution

We transform negated (6.23) into clausal form:

1.	$\neg R(x_1, x_1) \vee R(x_1, y_1)$	(with $\forall y$ moved to the prefix)
2.	$\neg R(x_2, y_2) \vee R(y_2, x_2)$	
3.	$\neg R(f(x_3), x_3)$	(Skolemized)
4.	$R(a, a)$	(negated and Skolemized conclusion).

Resolution proof:

5. $R(a, y_5)$ – (res): 1, 4 with $x_1 = a$ and renaming y_1/y_5
6. $R(y_6, a)$ – (res): 2, 5
   with $x_2 = a, y_2 = y_5$ and renaming y_5/y_6
7. $\emptyset$ – (res): 3, 6 with $x_3 = a, y_6 = f(a)$.

Lecture 7

Deductive databases

7.1 Exercise

Translate the following rule into a first-order formula:

$$\begin{aligned} brother(X,Y) :- \quad & male(X), \\ & parent(Z,X), \\ & parent(Z,Y). \end{aligned} \tag{7.1}$$

7.1 Solution

In rule (7.1) variable Z occurs only in the rule's body, so it is existentially quantified. The body is translated into:

$$\exists z[male(x) \land parent(z,x) \land parent(z,y)]. \tag{7.2}$$

Rules are understood as *body* $\rightarrow$ *head*, so rule (7.1) is translated (almost) into:

$$\begin{split}\exists z[male(x) \land parent(z,x) \land parent(z,y)] \\ \rightarrow brother(x,y).\end{split} \tag{7.3}$$

In (7.3) variables x, y are not yet quantified. Since variables X, Y appear both in rule's (7.1) head and body, we universally quantify them and the required translation is:

$$\begin{split}\forall x \forall y \Big[\exists z[male(x) \land parent(z,x) \land parent(z,y)] \\ \rightarrow brother(x,y)\Big].\end{split} \tag{7.4}$$

7.2 Exercise

Transform the following formulas into DATALOG rules:

1. $$\forall x \forall y \big[(mother(x,y) \vee father(x,y)) \\ \rightarrow parent(x,y)\big]; \tag{7.5}$$

2. $$\forall x \forall y \big[[\neg(parent(x,y) \rightarrow \neg male(y))] \\ \rightarrow son(y)\big]; \tag{7.6}$$

3. $$\forall x \forall y \big[mother(x,y) \\ \rightarrow (parent(x,y) \wedge female(x))\big]. \tag{7.7}$$

4. $$\forall y \forall z \big[\exists x(parent(y,x) \wedge parent(x,z))] \\ \rightarrow grandparent(y,z)\big]. \tag{7.8}$$

7.2 Solution

1

First note that, according to our convention, universal quantifiers in (7.5) can be removed:

$$(mother(x, y) \vee father(x, y)) \rightarrow parent(x, y). \tag{7.9}$$

Now we use tautology $\big[(p \vee q) \rightarrow r\big] \leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$ according to which (7.9) is equivalent to:

$$\begin{array}{l} mother(x, y) \rightarrow parent(x, y) \ \wedge \\ father(x, y) \rightarrow parent(x, y). \end{array} \tag{7.10}$$

Now it is easy to see that (7.10) is equivalent to the following DATALOG rules:

$$\begin{array}{l} parent(X, Y) :- \ mother(X, Y). \\ parent(X, Y) :- \ father(X, Y). \end{array} \tag{7.11}$$

2

As before, we remove universal quantifiers from (7.6):

$$[\neg(parent(x, y) \rightarrow \neg male(y))] \rightarrow son(y). \tag{7.12}$$

Using the implication law we obtain the following formula equivalent to (7.12):

$$[parent(x, y) \wedge \neg\neg male(y)] \rightarrow son(y). \tag{7.13}$$

We remove double negation:

$$[parent(x, y) \wedge male(y)] \rightarrow son(y). \tag{7.14}$$

and obtain the following DATALOG rule:

$$son(Y) :-\ parent(x, y), male(y). \tag{7.15}$$

3

As before, we remove universal quantifiers from (7.7):

$$mother(x, y) \rightarrow (parent(x, y) \wedge female(x)). \tag{7.16}$$

Now we use tautology $\big[p \rightarrow (q \wedge r)\big] \leftrightarrow [(p \rightarrow r) \wedge (p \rightarrow r)]$ according to which (7.16) is equivalent to:

$$\begin{aligned} &mother(x, y) \rightarrow parent(x, y)\ \wedge \\ &mother(x, y) \rightarrow female(x). \end{aligned} \tag{7.17}$$

Now it is easy to see that (7.17) is equivalent to the following DATALOG rules:

$$\begin{aligned} &parent(X, Y) :-\ mother(X, Y). \\ &female(X) :-\ mother(X, Y). \end{aligned} \tag{7.18}$$

4

First we remove universal quantifiers from (7.8):

$$\exists x(parent(y, x) \land parent(x, z)) \\ \rightarrow grandparent(y, z). \qquad (7.19)$$

Existential quantifier $\exists x$ binds x in the antecedent of implication (7.19) and x does not appear in its conclusion. Therefore we can remove $\exists x$:[1]

$$(parent(y, x) \land parent(x, z)) \rightarrow grandparent(y, z). \quad (7.20)$$

Observe that (7.20) is equivalent to the following DATALOG rule:

$$grandparent(Y, Z) \text{ :- } \; parent(Y, X), \\ parent(X, Z). \qquad (7.21)$$

[1]Note that this is not Skolemization. We simply use the fact that variables appearing in rule's body and not appearing in its head are implicitly existentially quantified in the body.

7.3 Exercise ¡E!

1. Design a Datalog database for storing information about objects on a town street. Objects might be cars, bikes, etc. Each object is characterized by its size (small, medium or large) and velocity (slow, moderate, fast). In addition, the database should contain information about direct precedence between objects.[2]

2. Express in predicate calculus the constraint

   "every object has unique (exactly one) size and unique velocity".

3. Provide an integrity constraint concerning the precedence relation between objects.

4. Formulate in logic queries selecting:

   a) all bikes which are directly between two large cars;

   b) all medium cars which precede (not necessarily directly[3]) a fast large car.

---

[2]Object a "directly precedes" object b, denoted by $a \leftarrow b$, if a precedes b and there are no objects between a and b.

[3]It is assumed that object c *precedes* object e if $c \leftarrow e$ or there is $n \geq 1$ and objects $d_1, \ldots, d_n$ such that $c \leftarrow d_1 \leftarrow d_2 \leftarrow \ldots \leftarrow d_n \leftarrow e$.

7.3 Solution

1

The database contains relations:

- $object(id, type, size, velocity)$, where id is a (unique) object identifier, $type \in String$ is an object type (car, bike, etc.), $size \in Numbers$ is its size, and $velocity \in String$ is its velocity (small, moderate, fast);
- $d\_prec(id1, id2)$, where $id1, id2$ are object identifiers.

Example:

$object(2, \text{‘}car\text{’}, \text{‘}medium\text{’}, \text{‘}moderate\text{’})$
– object 2 is a medium car whose velocity is moderate;
$object(6, \text{‘}bike\text{’}, \text{‘}small\text{’}, \text{‘}slow\text{’})$
– object 2 is a small bike whose velocity is slow;
$d\_prec(2, 6)$ – object 2 directly precedes object 6.

2

Every object has unique (exactly one) size and unique velocity:

$$\forall i \forall t \forall s \forall v \forall t' \forall s' \forall v' \big[(object(i,t,s,v) \land object(i,t',s',v')) \\ \rightarrow (s = s' \land v = v')\big].$$

3

An integrity constraint concerning the precedence relation between objects:

$$\forall x \forall y [d\_prec(x,y) \rightarrow \neg d\_prec(y,x)].$$

4(a)

Select all bikes which are directly between two large cars (recall that '_' is an anonymous "don't care" variable):

$$\begin{array}{ll} answer(Id) :- & object(Id1, \text{'}car\text{'}, \text{'}large\text{'}, \_), \\ & object(Id, \text{'}bike\text{'}, \_, \_), \\ & object(Id2, \text{'}car\text{'}, \text{'}large\text{'}, \_), \\ & d\_prec(Id1, Id), \\ & d\_prec(Id, Id2). \end{array}$$

Now $answer(X)$ provides all required objects.

4(b)

Select all medium cars which precede (not necessarily directly) a fast large car.

We first define the precedence relation:

$$\begin{aligned} prec(Id1, Id2) :- &\quad d\_prec(Id1, Id2). \\ prec(Id1, Id2) :- &\quad d\_prec(Id1, Id), \\ &\quad prec(Id, Id2). \end{aligned}$$

Now we are ready to formulate the required query:

$$\begin{aligned} answer(Id) :- &\quad object(Id, \text{`}car\text{'}, \text{`}medium\text{'}, \_), \\ &\quad object(Id1, \text{`}car\text{'}, \text{`}large\text{'}, \text{`}fast\text{'}), \\ &\quad prec(Id, Id1). \end{aligned}$$

7.4 Exercise ¡E!

Consider regions on a screen. We say that a region r is *contained in* region t if all pixels of region r are located within region t and $r \neq t$. We say that r is *directly contained in* region t when

- r is contained in region t, and
- there is no region s such that r is contained in s and s is contained t.

For example, in Figure 7.1 region a is contained in region b and in region c. Region a is directly contained in b but not directly contained in c. Region b is directly contained in c. Region d is contained (also directly) in c but neither in a nor in b.

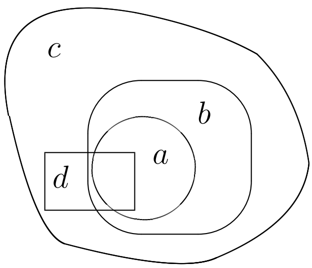


Figure 7.1: Sample screen regions.

1. Design a Datalog database for storing information about screen regions, where each region is characterized by its shape, size and color. In addition, for each pair of regions r, s, the database contains information whether r is directly contained in s.

2. Express in predicate calculus the constraint:

   > "the relation of containment is antisymmetric"

3. Provide another integrity constraint concerning the relation of region containment.

4. Formulate in logic queries selecting:

   a) all small regions directly contained in a large circle;

   b) all small objects contained[4] in a big green square.

---

[4]Directly or not.

7.4 Solution

1

The database contains relations:

- $region(id, shape, size, color)$, where id is a (unique) identifier for each region, $shape, size, color \in String$ are its shape, size and color;
- $d\_cont(id1, id2)$, where $id1, id2$ are region identifiers.

Example:

$region(18, \text{‘}oval\text{'}, small, red)$
– region 18 is a small, red oval;
$street(22, \text{‘}square\text{'}, large, green)$ – region 22 is a large, green square;
$d\_cont(18, 22)$ – region 18 is directly contained in region 22.

2

The relation of containment is antisymmetric:

$$\forall x \forall y [d\_cont(x, y) \to \neg d\_cont(y, x)].$$

3

Another integrity constraint concerning the relation of region containment:

$$\forall x[\neg d\_cont(x, x)].$$

4(a)

Select all small regions directly contained in a large circle:

$$\begin{aligned} answer(R) :- \quad & region(R, \_, \text{`}small\text{'}, \_), \\ & region(R1, \text{`}circle\text{'}, \text{`}large\text{'}, \_), \\ & d\_cont(R, R1). \end{aligned}$$

Now $answer(X)$ provides all required regions.

4(b)

Select all small objects contained in a big green square. We first define:

$$\begin{aligned} contained(R1, R2) :- \quad & d\_cont(R1, R2). \\ contained(R1, R2) :- \quad & d\_cont(R1, R3), \\ & contained(R3, R2). \end{aligned} \tag{7.22}$$

The required query can be formulated as:

$$\begin{aligned} answer(R) :- \quad & region(R, \_, \text{`}small\text{'}, \_), \\ & region(R1, \text{`}square\text{'}, \text{`}big\text{'}, \text{`}green\text{'}), \\ & contained(R, R1). \end{aligned}$$

7.5 Exercise ¡E!

Consider road segments and landmarks. Road segments are chosen in such a way that each landmark is close to exactly one road segment. Road segments are directly connected if they have a common border. In particular, each road segment is directly connected to itself.

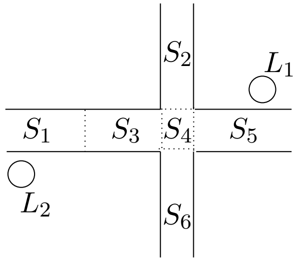


Figure 7.2: Sample road segments $S_1, \ldots, S_6$ and landmarks L_1, L_2.

For example, in Figure 7.2:

- segments S_1 and S_3, segments S_3 and S_4, segments S_4 and S_5, segments S_4 and S_6 as well as S_4 and S_2 are directly connected;
- for $1 \leq i \leq 6$ we have that S_i is directly connected to S_i;
- for example, segments S_1 and S_6, segments S_2 and S_5 as well as S_3 and S_5 are not directly connected
- landmark L_1 is close to segment S_5 and landmark L_2 is close to road segment S_1.

A landmark L is *accessible by a road trip* from landmark L' if L is close to a road segment S and L' is close to a road segment S' and there is a sequence of road segments $R_1, \ldots, R_n$ with $n \geq 1$ such that $S = R_0$, $S' = R_n$ and for all i such that $1 \leq i < n$, R_i is directly connected to R_{i+1}.

1. Design a database storing information about road segments, their direct connections and landmarks close to particular road segments. Road segments are characterized by their quality (*low, medium, high*) and average speed. The database contains also information about direct connections between road segments.
2. Express in predicate calculus the constraint:

   "each landmark is close to exactly one road segment".

3. Provide another integrity constraint concerning relationships in the database.
4. Formulate in logic queries selecting:
   a) all landmarks close to roads which are of high quality with average speed greater than $65\frac{km}{h}$;
   b) find all landmarks which are accessible by a road trip from a given landmark.

7.5 Solution

1

The database contains relations:

- $segment(id, quality, avs)$, where id is a (unique) road segment identifier, $quality \in String$ is its quality (low, medium or high), and $avs \in Numbers$ is its average speed;
- $d\_con(id1, id2)$, where $id1, id2$ are street identifiers;
- $close(idL, idS)$, where idL is a landmark identifier and idS is a street identifier.

Example:

$segment(64, \text{‘}low\text{’}, 50)$ – road segment with $id = 64$ is of low quality and average speed $50\frac{km}{h}$;

$segment(12, \text{‘}medium\text{’}, 60)$ – road segment with $id = 12$ is of medium quality and average speed $60\frac{km}{h}$;

$d\_con(12, 64)$ – road segments 12 and 64 are directly connected;

$close(11, 64)$ – landmark 11 is close to road segment 64.

2

Each landmark is close to exactly one road segment:

$$\forall x \exists y \big[close(x, y) \land \forall z(close(x, z) \to z = y)\big]. \quad (7.23)$$

3

Another integrity constraint concerning relationships in the database:

$$\begin{aligned}\forall x \forall y \forall z \forall y' \forall z' [(segment(x, y, z) \land segment(x, y', z'))\\ \to (y = y' \land z = z')]\end{aligned}$$

4(a)

Select all all landmarks close to roads which are of high quality with average speed greater than $65\frac{km}{h}$:

$$\begin{aligned} answer(L) :- \quad & segment(S, \text{`}high\text{'}, Av),\\ & Av > 65,\\ & close(L, S).\end{aligned}$$

Now $answer(L)$ provides all required landmarks.

4(b)

Select all landmarks which are accessible by a road trip from a given landmark.

We first define accessibility among road segments (*acc*) and accessibility among landmarks (*accessible*):

$$
\begin{array}{ll}
acc(S1, S2) :- & d\_con(S1, S2). \\
acc(S1, S2) :- & d\_con(S1, S3), \\
 & acc(S3, S2). \\
accessible(L1, L2) :- & close(L1, S1), \\
 & close(L2, S2), \\
 & acc(S1, S2).
\end{array}
$$

Now, given a landmark, say 'Huge Statue', we can formulate the required query:

$$answer(N) :- \quad accessible(\text{'}HugeStatue\text{'}, N).$$

Lecture 8

Sequent (Gentzen) calculus

8.1 Exercise

Using Gentzen system solve Exercise 2.2, i.e., check whether the following formula is satisfiable:

$$(p \vee q) \wedge (\neg p \vee \neg q). \tag{8.1}$$

If it is, provide a satisfying valuation.

8.1 Solution

To check satisfiability we use the "inverted" sequent:

$$
\begin{array}{c}
(\wedge l)\ \dfrac{(p \vee q) \wedge (\neg p \vee \neg q) \Rightarrow \emptyset}{p \vee q, \neg p \vee \neg q \Rightarrow \emptyset} \\
(\vee l)\ \dfrac{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}{
(\vee l)\ \dfrac{p, \neg p \vee \neg q \Rightarrow \emptyset}{(\neg l)\ \dfrac{p, \neg p \Rightarrow \emptyset}{\underbrace{p \Rightarrow p}_{\text{axiom}}} \quad (\neg l)\ \dfrac{p, \neg q \Rightarrow \emptyset}{p \Rightarrow q}}
\qquad
(\vee l)\ \dfrac{q, \neg p \vee \neg q \Rightarrow \emptyset}{(\neg l)\ \dfrac{q, \neg p \Rightarrow \emptyset}{q \Rightarrow p} \quad (\neg l)\ \dfrac{q, \neg q \Rightarrow \emptyset}{\underbrace{q \Rightarrow q}_{\text{axiom}}}}
}
\end{array}
$$

Leaves containing non-axioms provide counterexamples for the negated formula (represented by the "inverted" sequent). Each such a counterexample provides a satisfying valuation of the initial formula:

- $p \Rightarrow q$ gives $p = \mathrm{T}$ and $q = \mathrm{F}$;
- $q \Rightarrow p$ gives $q = \mathrm{T}$ and $p = \mathrm{F}$.

8.2 Exercise ¡E!

Using Gentzen system check whether formulas given in Exercise 2.3 are tautologies. If not, extract a counterexample from the constructed proof.

Recall that in Exercise 2.3 we considered the following formulas:

1. $$(p \vee q) \to p. \tag{8.2}$$

2. $$(p \vee q) \to (p \wedge r). \tag{8.3}$$

3. $$(p \to q) \vee (q \to p). \tag{8.4}$$

4. $$(p \vee q) \to \big[(p \vee q \vee \neg r) \wedge (r \vee p \vee q)\big] \tag{8.5}$$

8.2 Solution

1

We construct a proof tree for (8.2):

$$(\to r)\ \cfrac{\emptyset \Rightarrow (p \vee q) \to p}{(\vee l)\ \cfrac{p \vee q \Rightarrow p}{\underbrace{p \Rightarrow p}_{\text{axiom}} \qquad q \Rightarrow p}}$$

There is a leaf containing a non-axiom $q \Rightarrow p$. This leaf provides a counterexample $q = \mathrm{T}$, $p = \mathrm{F}$.

2

We construct a proof tree for (8.3):

$$
(\to r)\ \cfrac{\emptyset \Rightarrow (p \vee q) \to (p \wedge r)}{(\vee l)\ \cfrac{p \vee q \Rightarrow p \wedge r}{(\wedge r)\ \cfrac{p \Rightarrow p \wedge r}{\underbrace{p \Rightarrow p}_{\text{axiom}} \qquad p \Rightarrow r} \qquad (\wedge r)\ \cfrac{q \Rightarrow p \wedge r}{q \Rightarrow p \qquad q \Rightarrow r}}}
$$

There are leaves containing non-axioms and providing counterexamples for (8.3):

- $p \Rightarrow r$: a counterexample $p = \mathrm{T}, r = \mathrm{F}$ (the value of q does not matter);
- $q \Rightarrow p$: a counterexample $q = \mathrm{T}, p = \mathrm{F}$ (the value of r does not matter);
- $q \Rightarrow r$: a counterexample $q = \mathrm{T}, r = \mathrm{F}$ (the value of p does not matter).

3

We construct a proof tree for (8.4):

$$
\cfrac{\emptyset \Rightarrow (p \to q) \vee (q \to p)}{\cfrac{\emptyset \Rightarrow p \to q, q \to p}{\cfrac{p \Rightarrow q, q \to p}{\underbrace{p, q \Rightarrow q, p}_{\text{axom}}}\,(\to r)}\,(\to r)}\,(\vee r)
$$

The only leaf contains an axiom, so formula (8.4) is a tautology.

4

We construct a proof tree for (8.5):

$$
(\to r)\ \dfrac{\emptyset \Rightarrow (p \vee q) \to \big[(p \vee q \vee \neg r) \wedge (r \vee p \vee q)\big]}{(\wedge r)\ \dfrac{p \vee q \Rightarrow (p \vee q \vee \neg r) \wedge (r \vee p \vee q)}{(\vee r)\ \dfrac{p \vee q \Rightarrow p \vee q \vee \neg r}{(\vee r)\ \dfrac{p \vee q \Rightarrow p, q \vee \neg r}{(\vee l)\ \dfrac{p \vee q \Rightarrow p, q, \neg r}{\underbrace{p \Rightarrow p, q, \neg r}_{\text{axiom}} \quad \underbrace{q \Rightarrow p, q, \neg r}_{\text{axiom}}}}} \qquad (\vee r)\ \dfrac{p \vee q \Rightarrow r \vee p \vee q}{(\vee r)\ \dfrac{p \vee q \Rightarrow r, p \vee q}{(\vee l)\ \dfrac{p \vee q \Rightarrow r, p, q}{\underbrace{p \Rightarrow r, p, q}_{\text{axiom}} \quad \underbrace{q \Rightarrow r, p, q}_{\text{axiom}}}}}}}
$$

All leaves contain axioms, so formula (8.5) is a tautology.

8.3 Exercise

Derive rules for the connective "exclusive or" $\underline{\vee}$ defined by:

$$[A \underline{\vee} B] \leftrightarrow (A \vee B) \wedge (\neg B \vee \neg A). \tag{8.6}$$

8.3 Solution

Derivation of rule $(\underline{\vee}\, l)$:

$$
(def)\ \dfrac{A, B \,\underline{\vee}\, C, D \Rightarrow E}{(\wedge l)\ \dfrac{A, (B \vee C) \wedge (\neg B \vee \neg C), D \Rightarrow E}{(\vee l)\ \dfrac{A, B \vee C, \neg B \vee \neg C, D \Rightarrow E}{(\vee l)\ \dfrac{A, B, \neg B \vee \neg C, D \Rightarrow E}{(\neg l)\ \dfrac{A, B, \neg B, D \Rightarrow E}{\underbrace{A, B, D \Rightarrow B, E}_{\text{axiom}}} \quad (\neg l)\ \dfrac{A, B, \neg C, D \Rightarrow E}{A, B, D \Rightarrow C, E}} \qquad \dfrac{(*)}{\ }}}}
$$

$$
(\vee l)\ \dfrac{(*) : A, C, \neg B \vee \neg C, D \Rightarrow E}{(\neg l)\ \dfrac{A, C, \neg B, D \Rightarrow E}{A, C, D \Rightarrow B, E} \qquad (\neg l)\ \dfrac{A, C, \neg C, D \Rightarrow E}{\underbrace{A, C, D \Rightarrow C, E}_{\text{axiom}}}}
$$

The rule:

$$
(\underline{\vee}\, l)\ \dfrac{A, B \,\underline{\vee}\, C, D \Rightarrow E}{A, B, D \Rightarrow C, E \qquad A, C, D \Rightarrow B, E}
$$

Derivation of rule $(\underline{\vee}\, r)$:

$$(def)\ \dfrac{A \Rightarrow B, C \,\underline{\vee}\, D, E}{(\wedge r)\ \dfrac{A \Rightarrow B, (C \vee D) \wedge (\neg C \vee \neg D), E}{(\vee r)\ \dfrac{A \Rightarrow B, C \vee D, E}{(\vee r)\ \dfrac{A \Rightarrow B, C, D, E}{}} \qquad \dfrac{A \Rightarrow B, \neg C \vee \neg D, E}{(\neg r)\ \dfrac{A \Rightarrow B, \neg C, \neg D, E}{(\neg r)\ \dfrac{A, C \Rightarrow B, \neg D, E}{A, C, D \Rightarrow B, E}}}}}$$

The rule:

$$(\underline{\vee}\, r)\ \dfrac{A \Rightarrow B, C \,\underline{\vee}\, D, E}{A \Rightarrow B, C, D, E \qquad A, C, D \Rightarrow B, E}$$

8.4 Exercise

Using rules derived in Exercise 8.3, verify whether:

1. $(p \veebar q) \to (p \vee q)$;
2. $(p \vee q) \to (p \veebar q)$.

8.4 Solution

1

$$(\to r)\ \dfrac{\emptyset \Rightarrow (p \underline{\vee} q) \to (p \vee q)}{(\vee r)\ \dfrac{p \underline{\vee} q \Rightarrow p \vee q}{(\underline{\vee} l)\ \dfrac{p \underline{\vee} q \Rightarrow p, q}{\underbrace{p \Rightarrow q, p, q}_{\text{axiom}} \qquad \underbrace{q \Rightarrow p, p, q}_{\text{axiom}}}}}$$

All leaves are axioms, so formula $(p \underline{\vee} q) \to (p \vee q)$ is a tautology.

2

$$(\to r)\ \dfrac{\emptyset \Rightarrow (p \vee q) \to (p \underline{\vee} q)}{(\underline{\vee} r)\ \dfrac{p \vee q \Rightarrow p \underline{\vee} q}{(\vee l)\ \dfrac{p \vee q \Rightarrow p, q}{\underbrace{p \Rightarrow p, q}_{\text{axiom}} \quad \underbrace{q \Rightarrow p, q}_{\text{axiom}}} \qquad (\vee l)\ \dfrac{p \vee q, p, q \Rightarrow \emptyset}{p, p, q \Rightarrow \emptyset \quad q, p, q \Rightarrow \emptyset}}}$$

The formula is not a tautology. Both non-axiom leaves provide the same counterexample for $(p \vee q) \to (p \underline{\vee} q)$, which is $p = q = \mathrm{T}$.

8.5 Exercise ¡E!

Using Gentzen system prove the following first-order formula (see Exercise 5.4, page 97 for a tableaux proof):

$$\begin{aligned}\big[\forall x\exists y[R(x,y)] \wedge \forall x\forall y[R(x,y) \to \forall z R(x,z)]\big]& \\ \to \forall x\forall y R(x,y).&\end{aligned} \tag{8.7}$$

8.5 Solution

A proof:

$$
(\to r)\ \dfrac{\emptyset \Rightarrow \big[\forall x\exists y[R(x,y)] \land \forall x\forall y[R(x,y) \to \forall z R(x,z)]\big] \to \forall x\forall y R(x,y)}{\forall x\exists y[R(x,y)] \land \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow \forall x\forall y R(x,y)}
$$

$$
(\land l)\ \dfrac{\forall x\exists y[R(x,y)] \land \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow \forall x\forall y R(x,y)}{\forall x\exists y[R(x,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow \forall x\forall y R(x,y)}
$$

$$
(\forall x\, r)\ \dfrac{\forall x\exists y[R(x,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow \forall x\forall y R(x,y)}{\forall x\exists y[R(x,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow \forall y R(a,y)}
$$

$$
(\forall y\, r)\ \dfrac{\forall x\exists y[R(x,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow \forall y R(a,y)}{\forall x\exists y[R(x,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow R(a,b)}
$$

$$
(\forall x\, l)\ \dfrac{\forall x\exists y[R(x,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow R(a,b)}{\exists y[R(a,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow R(a,b)}
$$

$$
(\exists y\, l)\ \dfrac{\exists y[R(a,y)], \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow R(a,b)}{R(a,c), \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow R(a,b)}
$$

$$
(\forall x\, l)\ \dfrac{R(a,c), \forall x\forall y[R(x,y) \to \forall z R(x,z)] \Rightarrow R(a,b)}{R(a,c), \forall y[R(a,y) \to \forall z R(a,z)] \Rightarrow R(a,b)}
$$

$$
(\forall y\, l)\ \dfrac{R(a,c), \forall y[R(a,y) \to \forall z R(a,z)] \Rightarrow R(a,b)}{R(a,c), R(a,c) \to \forall z R(a,z) \Rightarrow R(a,b)}
$$

$$
(\to l)\ \dfrac{R(a,c), R(a,c) \to \forall z R(a,z) \Rightarrow R(a,b)}{\underbrace{R(a,c) \Rightarrow R(a,c), R(a,b)}_{\text{axiom}} \qquad (\forall z\, l)\ \dfrac{R(a,c), \forall z R(a,z) \Rightarrow R(a,b)}{\underbrace{R(a,c), R(a,b) \Rightarrow R(a,b)}_{\text{axiom}}}}
$$

8.6 Exercise ¡E!

Using Gentzen system prove that formula:

$$\forall x \exists y \exists z [P(x, y, z) \vee \neg \exists z \exists u (\neg P(x, z, u))] \tag{8.8}$$

implies:

$$\exists x \exists y P(a, x, y), \tag{8.9}$$

where a is a constant.

(The same implication is considered in Exercise 5.4, page 99).

8.6 Solution

$$
\cfrac{\emptyset \Rightarrow \forall x \exists y \exists z [P(x,y,z) \vee \neg \exists z \exists u(\neg P(x,z,u))] \to \exists x \exists y P(a,x,y)}{(\to r)\quad \cfrac{\forall x \exists y \exists z [P(x,y,z) \vee \neg \exists z \exists u(\neg P(x,z,u))] \Rightarrow \exists x \exists y P(a,x,y)}{(\forall x\, l)\quad \cfrac{\exists y \exists z [P(a,y,z) \vee \neg \exists z \exists u(\neg P(a,z,u))] \Rightarrow \exists x \exists y P(a,x,y)}{(\exists y\, l)\quad \cfrac{\exists z [P(a,b,z) \vee \neg \exists z \exists u(\neg P(a,z,u))] \Rightarrow \exists x \exists y P(a,x,y)}{(\exists z\, l)\quad \cfrac{P(a,b,c) \vee \neg \exists z \exists u(\neg P(a,z,u)) \Rightarrow \exists x \exists y P(a,x,y)}{(\exists x\, r)\quad \cfrac{P(a,b,c) \vee \neg \exists z \exists u(\neg P(a,z,u)) \Rightarrow \exists y P(a,b,y)}{(\exists y\, r)\quad P(a,b,c) \vee \neg \exists z \exists u(\neg P(a,z,u)) \Rightarrow P(a,b,c)}}}}}}
$$

$$
(\vee l)\quad \underbrace{P(a,b,c) \Rightarrow P(a,b,c)}_{\text{axiom}} \qquad (\neg l)\quad \cfrac{\neg \exists z \exists u(\neg P(a,z,u)) \Rightarrow P(a,b,c)}{(\exists z\, r)\quad \cfrac{\emptyset \Rightarrow P(a,b,c), \exists z \exists u(\neg P(a,z,u))}{(\exists u\, r)\quad \cfrac{\emptyset \Rightarrow P(a,b,c), \exists u(\neg P(a,b,u))}{(\neg r)\quad \cfrac{\emptyset \Rightarrow P(a,b,c), \neg P(a,b,c)}{\underbrace{P(a,b,c) \Rightarrow P(a,b,c)}_{\text{axiom}}}}}}
$$

Both leaves contain axioms which proves that the considered formula is a tautology.

Lecture 9

Sample exams

Exam rules

1. You can use your own copies of compendium extracted from lectures as well as an English-Swedish dictionary.

2. Exercises are formulated in English, but answers can be given in English or Swedish.

3. You are not allowed to:

    - use any writing material other than indicated in point 1, in particular you cannot use this ebook with exercises and solutions;
    - use calculators, mobile phones or any other electronic devices;

- lend/borrow/exchange anything during the exam.

4. If an exercise has not been specified completely as you see it, state which (reasonable) assumptions you have made.

5. Begin each exercise on a new sheet of paper. Write only on one side of the paper. Write clearly and make sure to give adequate explanations for all your answers.

6. There are 4 exercises, each exercise gives maximum 10 points (40 points together). Grading is provided in the following table.

number of points (n)	grade
$34 \leq n \leq 40$	5
$27 \leq n < 34$	4
$20 \leq n < 27$	3
$n < 20$	U (not passed)

Note: exercises for the first two exams are solved while for the third one are intentionally left unsolved.

9.1 Exercise (exam 1)

1. Prove the following propositional formula

$$\big[p \land (\neg q \lor r)\big] \to \big[\neg(\neg p \land q) \lor r\big] \tag{9.1}$$

   a) (*2 points*) using Gentzen system;

   b) (*2 points*) using resolution.

2. Prove the following formula of predicate logic, where a, b are constants:

$$\exists x \forall y \forall z \big[R(x,y,z) \land S(x) \land T(y)\big] \to \exists x \big[R(x,a,b) \land T(a)\big] \tag{9.2}$$

   a) (*3 points*) using tableaux;

   b) (*3 points*) using resolution.

9.1 Solution

1(a)

$$
(\to r)\,\dfrac{\emptyset \Rightarrow \big[p \wedge (\neg q \vee r)\big] \to \big[\neg(\neg p \wedge q) \vee r\big]}{(\vee r)\,\dfrac{p, \neg q \vee r \Rightarrow \neg(\neg p \wedge q) \vee r}{(\neg r)\,\dfrac{p, \neg q \vee r \Rightarrow \neg(\neg p \wedge q), r}{(\wedge l)\,\dfrac{p, \neg q \vee r, \neg p \wedge q \Rightarrow r}{(\neg l)\,\dfrac{p, \neg q \vee r, \neg p, q \Rightarrow r}{\underbrace{p, \neg q \vee r, q \Rightarrow r, p}_{\text{axiom}}}}}}}
$$

1(b)

We first negate formula (9.1):

$$\begin{aligned}
&\neg\Big[\big[p \wedge (\neg q \vee r)\big] \rightarrow \big[\neg(\neg p \wedge q) \vee r\big]\Big] \leftrightarrow \\
&\big[p \wedge (\neg q \vee r)\big] \wedge \neg\big[\neg(\neg p \wedge q) \vee r\big] \leftrightarrow \\
&p \wedge (\neg q \vee r) \wedge \neg\neg(\neg p \wedge q) \wedge \neg r \leftrightarrow \\
&p \wedge (\neg q \vee r) \wedge \neg p \wedge q \wedge \neg r.
\end{aligned} \tag{9.3}$$

Clausal form of (9.3):

1. p
2. $\neg q \vee r$
3. $\neg p$
4. q
5. $\neg r$

Proof by resolution:

6. $\neg q$ $\quad (res) : 5, 2$
7. $\emptyset$ $\quad (res) : 6, 4$

2(a)

We construct a tableau for the negated formula (9.2):

$$\begin{array}{c}
\neg\big[\exists x\forall y\forall z\big[R(x,y,z)\wedge S(x)\wedge T(y)\big]\rightarrow\exists x\big[R(x,a,b)\wedge T(a)\big]\big]\\
\downarrow(\neg\rightarrow)\\
\exists x\forall y\forall z\big[R(x,y,z)\wedge S(x)\wedge T(y)\big],\neg\exists x\big[R(x,a,b)\wedge T(a)\big]\\
\downarrow(\exists x\text{ with }x=c)\\
\forall y\forall z\big[R(c,y,z)\wedge S(c)\wedge T(y)\big],\neg\exists x\big[R(x,a,b)\wedge T(a)\big]\\
\downarrow(\neg\exists x\text{ with }x=c)\\
\forall y\forall z\big[R(c,y,z)\wedge S(c)\wedge T(y)\big],\neg\big[R(c,a,b)\wedge T(a)\big]\\
\downarrow(\forall y\text{ with }y=a)\\
\forall z\big[R(c,a,z)\wedge S(c)\wedge T(a)\big],\neg\big[R(c,a,b)\wedge T(a)\big]\\
\downarrow(\forall z\text{ with }z=b)\\
R(c,a,b)\wedge S(c)\wedge T(a),\neg\big[R(c,a,b)\wedge T(a)\big]\\
\downarrow(\wedge)\\
R(c,a,b),S(c)\wedge T(a),\neg\big[R(c,a,b)\wedge T(a)\big]\\
\downarrow(\wedge)\\
R(c,a,b),S(c),T(a),\neg\big[R(c,a,b)\wedge T(a)\big]\\
\swarrow\qquad\searrow(\neg\wedge)\\
\underbrace{R(c,a,b),S(c),T(a),\neg R(c,a,b)}_{\text{closed}}\qquad\underbrace{R(c,a,b),S(c),T(a),\neg T(a)}_{\text{closed}}
\end{array}$$

The tableau is closed, so formula (9.2) is a tautology.

2(b)

We first negate the formula:

$$\neg\Big[\exists x\forall y\forall z\big[R(x,y,z)\wedge S(x)\wedge T(y)\big]\rightarrow\exists x\big[R(x,a,b)\wedge T(a)\big]\Big]\leftrightarrow$$
$$\exists x\forall y\forall z\big[R(x,y,z)\wedge S(x)\wedge T(y)\big]\wedge\neg\exists x\big[R(x,a,b)\wedge T(a)\big]\leftrightarrow$$
$$\exists x\forall y\forall z\big[R(x,y,z)\wedge S(x)\wedge T(y)\big]\wedge\forall x\neg\big[R(x,a,b)\wedge T(a)\big]\leftrightarrow$$
$$\exists x\forall y\forall z\big[R(x,y,z)\wedge S(x)\wedge T(y)\big]\wedge\forall x\big[\neg R(x,a,b)\vee\neg T(a)\big].$$

Now Skolemiztion for $\exists x$ results in:

$$\forall y\forall z\big[R(c,y,z)\wedge S(c)\wedge T(y)\big]\wedge\forall x\big[\neg R(x,a,b)\vee\neg T(a)\big],$$

so we have clauses (after renaming variables):

1. $R(c,y_1,z_1)$
2. $S(c)$
3. $T(y_3)$
4. $\neg R(x_4,a,b)\vee\neg T(a)$

A proof by resolution:

5. $\neg R(x_5,a,b)$ – (res): 3, 4 with $y_3=a$ and renaming x_4/x_5
6. $\emptyset$ – (res): 1, 5 with $x_5=c, y_1=a, z_1=b.$

9.2 Exercise (exam 1)

1. (*4 points*) Translate the following sentences into a set of propositional formulas:

   "Boxes are small or medium."
   "Each box is red, green or blue."
   "Small boxes are blue or red."
   "Medium boxes are green or blue."
   "For shipment robots chose red boxes."
   "For activities other than shipment, robots neither chose blue nor green boxes."

2. (*2 points*) Assuming that exactly one box is to be chosen and that each box can have exactly one color, hypothesize what choice (size and color) can be made and explain your reasoning informally.

3. (*4 points*) Prove your claim formally using a proof system of your choice (tableaux or resolution). Please do not use truth table method, as this will give no points).

9.2 Solution

1

A possible translation:

$$\begin{array}{l}
small \vee medium \\
red \vee green \vee blue \\
small \rightarrow (blue \vee red) \\
medium \rightarrow (green \vee blue) \\
shipment \rightarrow red \\
\neg shipment \rightarrow (\neg blue \wedge \neg green).
\end{array} \tag{9.4}$$

2

Observe that for shipment *red* boxes are chosen. For non-shipment neither *blue* nor *green* boxes are chosen, so only *red* boxes remain. Therefore no matter whether for *shipment* or not, *red* boxes are chosen.

We have only *small* or *medium* boxes. The chosen boxes are *red*, which excludes *medium* boxes, (if a box is red then it is neither blue nor green). So one can chose only boxes being both *small* and *red*.

3

We use resolution. We first show that the set of formulas (9.4) implies *red*. We first transform (9.4) into a set of clauses:

$$\begin{array}{ll} 1. & small \vee medium \\ 2. & red \vee green \vee blue \\ 3. & \neg small \vee blue \vee red \\ 4. & \neg medium \vee green \vee blue \\ 5. & \neg shipment \vee red \\ 6. & shipment \vee \neg blue \\ 7. & shipment \vee \neg green. \end{array} \tag{9.5}$$

Now we negate formula:

$$[1. \wedge 2. \wedge 3. \wedge 4. \wedge 5. \wedge 6. \wedge 7.] \rightarrow red$$

and obtain:

$$[1. \wedge 2. \wedge 3. \wedge 4. \wedge 5. \wedge 6. \wedge 7.] \wedge \neg red.$$

Therefore, we consider clauses (1) – (7) together with

$$8.\ \neg red$$

A possible proof by resolution:

$$\begin{array}{rll} 9. & \neg shipment & \text{– (res): } 5, 8 \\ 10. & \neg blue & \text{– (res): } 6, 9 \\ 11. & \neg green & \text{– (res): } 7, 9 \\ 12. & green \vee blue & \text{– (res): } 2, 8 \\ 13. & green & \text{– (res): } 10, 12 \\ 14. & \emptyset & \text{– (res): } 11, 13. \end{array}$$

To prove that the chosen box is small, we need formulas:[1]

$$\begin{array}{l} red \to \neg green \\ red \to \neg blue \\ green \to \neg blue \\ green \to \neg red \\ blue \to \neg green \\ blue \to \neg red. \end{array}$$

We could add all of them, but it suffices to take the first two, so for the sake of simplicity we skip the remaining ones. We also add the conclusion *red* which is already proved to be the consequence of the initial set of clauses. Since we want to prove that *small* is implied by our clauses, we also add its negation to the set of clauses. The new clauses are:

$$\begin{array}{ll} 15. & red \\ 16. & \neg red \lor \neg green \\ 17. & \neg red \lor \neg blue \\ 18. & \neg small. \end{array} \tag{9.6}$$

[1]For the sake of simplicity, we consider the assumption about uniqueness of colors separately. The formulas expressing uniqueness could be added to clauses 1–8 immediately but the proof would be less readable.

A possible proof that the set of clauses (9.5) together with (9.6) leads to contradiction:

19.	$\neg green$	– (res): 15, 16
20.	$\neg blue$	– (res): 15, 17
21.	$medium$	– (res): 1, 18
22.	$green \vee blue$	– (res): 4, 21
23.	$blue$	– (res): 19, 22
24.	$\emptyset$	– (res): 20, 23

which completes the proof.

9.3 Exercise (exam 1)

Consider a set of persons and a relation R of "being a relative of". Let:

(a) the relation *relative* has the property that whenever person A is a relative of both B and C then also B is a relative of C.

Consider the relation "the same family member", denoted by F. We assume that:

(b) whenever a person A is a relative of a person B then they are in the same family;

(c) everybody is in his/her family.

Using the provided information:

(1) (*3 points*) express in predicate logic the properties (a), (b), (c);

(2) (*2 points*) provide informal arguments that R (as understood in "real life") is transitive and express in predicate logic the transitivity of R;

(3) (*5 points*) using resolution, facts from points (1), (2) and the property:

$$\forall x \forall y \Big[R(x, y) \lor \exists z [R(x, z) \land R(z, y)] \Big],$$

prove that $\forall x \forall y [F(x, y)]$.

9.3 Solution

1

(a) $\forall x \forall y \forall z[(R(x,y) \wedge R(x,z)) \rightarrow R(y,z)]$;

(b) $\forall x \forall y[R(x,y) \rightarrow F(x,y)]$;

(c) $\forall x[F(x,x)]$.

2

The transitivity of R is expressed by:

$$\forall x \forall y \forall z[(R(x,y) \wedge R(y,z)) \rightarrow R(x,z)].$$

The informal argument can be as follows. A *relative* is a person related by blood or marriage. So, if x, y are relatives, there is a chain of persons $x_1 - x_2 - \ldots - x_k$ such that:

– x is related by blood or marriage to x_1,
– x_1 related by blood or marriage to x_2,
– $\ldots$
– x_{k-1} is related by blood or marriage to x_k,
– x_k is related by blood or marriage to y.

If y and z are relatives then there is a similar chain between y and z. Therefore, there is a chain of persons relating x and z (the chain from x to y plus the chain from y to z), i.e., x and z are relatives, too.

3

We have to show that the conjunction of formulas:

$$\begin{array}{l}
\forall x \forall y \forall z[(R(x,y) \land R(x,z)) \to R(y,z)] \\
\forall x \forall y[R(x,y) \to F(x,y)] \\
\forall x[F(x,x)] \\
\forall x \forall y \forall z[(R(x,y) \land R(y,z)) \to R(x,z)] \\
\forall x \forall y\Big[R(x,y) \lor \exists z[R(x,z) \land R(z,y)]\Big]
\end{array}$$

implies $\forall x \forall y[F(x,y)]$.

To apply resolution we negate the conclusion, translate formulas into clauses and try to derive the empty clause $\emptyset$. Below clauses 5., 6. are obtained from:

$$\forall x \forall y[R(x,y) \lor \exists z[R(x,z) \land R(z,y)]]$$

by Skolemizing and transforming the formula into clauses. Clause 7. is a Skolemized negation of the conclusion, i.e., Skolemized $\exists x \exists y[\neg F(x,y)]$. As usually, variables are renamed.

The clauses:

1. $\neg R(x_1,y_1) \lor \neg R(x_1,z_1) \lor R(y_1,z_1)$
2. $\neg R(x_2,y_2) \lor F(x_2,y_2)$
3. $F(x_3,x_3)$
4. $\neg R(x_4,y_4) \lor \neg R(y_4,z_4) \lor R(x_4,z_4)$
5. $R(x_5,y_5) \lor R(x_5,f(x_5,y_5))$
6. $R(x_6,y_6) \lor R(f(x_6,y_6),y_6)$
7. $\neg F(a,b)$.

A proof by resolution:

8. $\neg R(a, b)$ – (res): 2, 7 with $x_2 = a, y_2 = b$
9. $R(a, f(a, b))$ – (res): 5, 8 with $x_5 = a, y_5 = b$
10. $R(f(a, b), b)$ – (res): 6, 8 with $x_6 = a, y_6 = b$
11. $\neg R(f(a, b), z_{11}) \vee R(a, z_{11})$
    – (res): 4, 9 with $x_4 = a, y_4 = f(a, b)$
    and renaming z_4/z_{11}
12. $R(a, b)$ – (res): 10, 11 with $z_{11} = b$
13. $\emptyset$ – (res): 8, 12,

which completes the proof.

9.4 Exercise (exam 1)

1. *(2 points)* Design a DATALOG database for storing information about streets in a town. Each street is characterized by its name, width and length. In addition, for each street s the database contains information whether s is closed for traffic as well as information about all streets intersecting s.

2. *(1 point)* Express in predicate calculus the constraint:

   > "the relation of road intersection is symmetric."

3. *(1 point)* Provide another integrity constraint concerning the relation of road intersection.

4. Formulate in logic queries selecting:

   a) *(2 points)* all streets longer than 7km and wider than 3m, intersecting the street named "Kungsgatan";

   b) *(4 points)* all streets accessible by car from a given street, assuming that driving through closed streets is impossible.

9.4 Solution

1

The database contains relations:

- $street(id, name, width, length, closed)$, where id is a (unique) identifier for a street, $name \in String$ is its name, $width, length \in Numbers$ are its width and length, $closed \in \{true, false\}$ indicates whether a given street is closed for traffic,
- $intersects(id1, id2)$, where $id1, id2$ are street identifiers.

Example:

$street(21, \text{‘}HanburySt.\text{’}, 5, 300, False)$
 ‘Hanbury St.’ is a street with $id = 21$, $width = 5$, $length = 300$, not closed for traffic;
$street(46, \text{‘}NobleSt.\text{’}, 7, 200, True)$ – ‘Noble St’ is a street with $id = 46$, $width = 7$, $length = 200$, closed for traffic;
$intersects(21, 46)$ – streets 21 and 46 intersect each other.

2

The relation of road intersection is symmetric:

$$\forall x \forall y[intersects(x, y) \to intersect(y, x)].$$

3

Another integrity constraint concerning the relation of road intersection:

$$\forall x[\neg intersects(x, x)].$$

4(a)

Select all streets longer than 700m and wider than 3m, intersecting the street named "Kungsgatan" (recall that '_' is an anonymous "don't care" variable):

$$\begin{array}{ll} answer(Id) :- & street(Id, \_, W, L, \_), W > 3, L > 700, \\ & street(Id1, `Kungsgatan', \_, \_, \_), \\ & intersects(Id, Id1). \end{array}$$

Now $answer(X)$ provides all required streets.

4(b)

Select streets accessible by car from a given street, assuming that it is impossible to drive through closed streets. We first define:

$$
\begin{array}{ll}
accessible(Id1, Id2) :- & street(Id1, \_, \_, \_, C1), \\
 & street(Id2, \_, \_, \_, C2), \\
 & C1 \neq True, C2 \neq True, \\
 & intersects(Id1, Id2). \\
accessible(Id1, Id2) :- & street(Id1, \_, \_, \_, C1), \\
 & street(Id2, \_, \_, \_, C2), \\
 & C1 \neq True, C2 \neq True, \\
 & intersects(Id1, Id3), \\
 & accessible(Id3, Id2).
\end{array}
$$

Now, given a street, say 'Stone St.', we can formulate the required query (assuming that we are interested in street names, not identifiers):

$$
\begin{array}{ll}
answer(N) :- & street(Id, \text{`}StoneSt.\text{'}, \_, \_, \_), \\
 & street(Id1, N, \_, \_, \_), \\
 & accessible(Id, Id1).
\end{array}
$$

9.5 Exercise (exam 2)

1. Prove the following propositional formula:

$$\big[(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge r\big] \rightarrow \big[\neg q \wedge r\big] \tag{9.7}$$

   a) (*2 points*) using tableaux;

   b) (*2 points*) using resolution.

2. Prove the following formula of predicate logic, where a is a constant:

$$\begin{aligned} &\forall x\big[P(x,x,x)\big] \wedge \\ &\qquad \neg\exists x \exists y \exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big] \\ &\qquad\qquad \rightarrow \exists z\big[P(a,z,a)\big] \end{aligned} \tag{9.8}$$

   a) (*3 points*) using Gentzen system;

   b) (*3 points*) using resolution.

9.5 Solution

1(a)

We construct a tableau for the negated formula (9.7):

$$
\begin{array}{c}
\neg\big[\big[(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge r\big] \to \big[\neg q \wedge r\big]\big] \\
\downarrow (\neg \to) \\
(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge r, \neg\big[\neg q \wedge r\big] \\
\downarrow (\wedge) \\
(\neg p \vee \neg q), (p \vee \neg q) \wedge r, \neg\big[\neg q \wedge r\big] \\
\downarrow (\wedge) \\
(\neg p \vee \neg q), (p \vee \neg q), r, \neg\big[\neg q \wedge r\big] \\
\swarrow \qquad \searrow (\neg\wedge)
\end{array}
$$

$$
\begin{array}{cc}
\begin{array}{c}
(\neg p \vee \neg q), (p \vee \neg q), r, \neg\neg q \\
\downarrow (\neg\neg) \\
(\neg p \vee \neg q), (p \vee \neg q), r, q \\
\downarrow \qquad\qquad \searrow (\vee) \\
\begin{array}{cc}
\begin{array}{c}
\neg p, (p \vee \neg q), r, q \\
\swarrow \quad \searrow (\vee) \\
\underbrace{\neg p, p, r, q}_{\text{closed}} \quad \underbrace{\neg p, \neg q, r, q}_{\text{closed}}
\end{array}
&
\underbrace{\neg q, (p \vee \neg q), r, q}_{\text{closed}}
\end{array}
\end{array}
&
\underbrace{(\neg p \vee \neg q), (p \vee \neg q), r, \neg r}_{\text{closed}}
\end{array}
$$

The tableau is closed, so formula (9.7) is a tautology.

1(b)

We first negate formula (9.7):

$$\begin{aligned}&\neg\{\big[(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge r\big] \rightarrow \big[\neg q \wedge r\big]\} \leftrightarrow\\&(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge r \wedge \neg\big[\neg q \wedge r\big] \leftrightarrow\\&(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge r \wedge (q \vee \neg r).\end{aligned} \tag{9.9}$$

Clausal form of (9.9):

1. $\neg p \vee \neg q$
2. $p \vee \neg q$
3. r
4. $q \vee \neg r$.

A possible proof by resolution:

5. q – (res): 3, 4
6. p – (res): 2, 5
7. $\neg p$ – (res): 1, 5
8. $\emptyset$ – (res): 6, 7.

2(a)

$$
\begin{array}{rc}
 & \emptyset \Rightarrow \forall x\big[P(x,x,x)\big] \wedge \neg\exists x\exists y\exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big] \to \exists z\big[P(a,z,a)\big] \\
\cline{2-2}
(\to r) & \forall x\big[P(x,x,x)\big] \wedge \neg\exists x\exists y\exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big] \Rightarrow \exists z\big[P(a,z,a)\big] \\
\cline{2-2}
(\wedge l) & \forall x\big[P(x,x,x)\big], \neg\exists x\exists y\exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big] \Rightarrow \exists z\big[P(a,z,a)\big] \\
\cline{2-2}
(\neg l) & \forall x\big[P(x,x,x)\big] \Rightarrow \exists z\big[P(a,z,a)\big], \exists x\exists y\exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big] \\
\cline{2-2}
(\forall x\, l) & P(a,a,a) \Rightarrow \exists z\big[P(a,z,a)\big], \exists x\exists y\exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big] \\
\cline{2-2}
(\exists z\, r) & \underbrace{P(a,a,a) \Rightarrow P(a,a,a), \exists x\exists y\exists z\big[P(x,y,z) \wedge \neg P(z,x,a)\big]}_{\text{axiom}}
\end{array}
$$

2(b)

We first negate the formula:

$$
\begin{aligned}
&\neg\{\forall x\big[P(x,x,x)\big]\wedge\neg\exists x\exists y\exists z\big[P(x,y,z)\wedge\neg P(z,x,a)\big]\rightarrow \\
&\qquad\qquad\qquad\qquad\qquad\qquad \exists z\big[P(a,z,a)\big]\}\leftrightarrow \\
&\forall x\big[P(x,x,x)\big]\wedge\neg\exists x\exists y\exists z\big[P(x,y,z)\wedge\neg P(z,x,a)\big]\wedge \\
&\qquad\qquad\qquad\qquad\qquad\qquad \neg\exists z\big[P(a,z,a)\big]\leftrightarrow \\
&\forall x\big[P(x,x,x)\big]\wedge\forall x\forall y\forall z\big[\neg P(x,y,z)\vee P(z,x,a)\big]\wedge \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall z\big[\neg P(a,z,a)\big].
\end{aligned}
$$

Clausal form (after renaming variables):

1. $P(x_1,x_1,x_1)$
2. $\neg P(x_2,y_2,z_2)\vee P(z_2,x_2,a)$
3. $\neg P(a,z_3,a)$

A proof by resolution:

4. $\emptyset$ – (res): 1, 3 with $x_1=z_3=a$.

9.6 Exercise (exam 2)

1. (*4 points*) Translate the following sentences into a set of propositional formulas:

   "chose one of three roads: short, medium or long"
   "the short road is always crowded"
   "the medium road is not comfortable, but fast"
   "the long road is comfortable"
   "the chosen road should be comfortable."

2. (*2 points*) Assuming that fast roads are not crowded and crowded roads are not comfortable, hypothesize what roads can be chosen explain your reasoning informally.

3. (*4 points*) Prove your claim formally using a proof system of your choice (tableaux, or resolution. Please do not use truth table method, as this will give no points).

9.6 Solution

1

A possible translation:

$$\begin{array}{l} short \vee medium \vee long \\ short \rightarrow crowded \\ medium \rightarrow (\neg comfortable \wedge fast) \\ long \rightarrow comfortable \\ comfortable. \end{array} \tag{9.10}$$

2

The chosen road should be *comfortable.* Therefore it cannot be *medium* since medium roads are not *comfortable.*

The chosen road cannot be *short* since *short* roads are *crowded* and *crowded* roads are not comfortable.

Therefore the only road that can be chosen if the *long* one.

3

To prove our hypothesis, we also need the translation of assumptions provided in point 2 of Exercise:

$$\begin{array}{l} fast \rightarrow \neg crowded \\ crowded \rightarrow \neg comfortable \end{array} \tag{9.11}$$

We use resolution. We first transform (9.10) and (9.11) into a set of clauses:

1. $short \vee medium \vee long$
2. $\neg short \vee crowded$
3. $\neg medium \vee \neg comfortable$
4. $\neg medium \vee fast$
5. $\neg long \vee comfortable$
6. $comfortable$
7. $\neg fast \vee \neg crowded$
8. $\neg crowded \vee \neg comfortable.$

Now we negate formula

$$[1. \wedge 2. \wedge 3. \wedge 4. \wedge 5. \wedge 6. \wedge 7. \wedge 8.] \rightarrow long$$

and obtain:

$$[1. \wedge 2. \wedge 3. \wedge 4. \wedge 5. \wedge 6. \wedge 7. \wedge 8.] \wedge \neg long.$$

Therefore, we consider clauses 1. – 8. together with

9. $\neg long.$

A possible proof by resolution:

10.	$short \vee medium$	– (res): 1, 9
11.	$crowded \vee medium$	– (res): 2, 10
12.	$\neg medium$	– (res): 3, 6
13.	$crowded$	– (res): 11, 12
14.	$\neg comfortable$	– (res): 8, 13
15.	$\emptyset$	– (res): 6, 14.

9.7 Exercise (exam 2)

Consider a relation R and properties:

(a) $\forall x \forall y \forall z[(R(x,y) \land R(x,z)) \rightarrow R(y,z)]$
(b) $\forall x \forall y \big[R(x,y) \rightarrow \exists u[R(u,x) \land R(u,y)]\big]$
(c) $\forall x \forall y[R(x,y) \rightarrow R(y,x)]$.

1. (*4 points*) Check informally whether the conjunction of (a) and (b) implies (c).
2. (*6 points*) Verify your informal reasoning using resolution.

9.7 Solution

1

To check whether (c) holds assume that for arbitrarily chosen elements s and t we have that $R(s,t)$ holds. To prove (c), we have to show that given assumptions (a) and (b), $R(t,s)$ holds, too.

1. First, using (b), we observe that $R(s,t)$ implies that there exists u such that $R(u,s)$ and $R(u,t)$ hold.

2. Second, using (a) we know that for all x, y and z, we have that $R(x,y)$ and $R(x,z)$ imply $R(y,z)$. Now, for $y = t, z = s$ and $x = u$ we have that $R(u,t)$ and $R(u,s)$ imply $R(t,s)$.

3. From the first point we have that $R(u,s)$ and $R(u,t)$ hold. From the second point we have that these imply $R(t,s)$. Therefore we conclude that $R(t,s)$.

That way we have shown that indeed the conjunction of (a) and (b) implies (c).

2

To apply resolution we negate the implication:
((a)$\wedge$ (b))$\rightarrow$ (c):

$$
\begin{aligned}
&\neg\{\forall x\forall y\forall z[(R(x,y)\wedge R(x,z))\rightarrow R(y,z)]\wedge\\
&\qquad\forall x\forall y\big[R(x,y)\rightarrow\exists u[R(u,x)\wedge R(u,y)]\big]\rightarrow\\
&\qquad\qquad\qquad\forall x\forall y[R(x,y)\rightarrow R(y,x)]\}\leftrightarrow\\
&\forall x\forall y\forall z[(R(x,y)\wedge R(x,z))\rightarrow R(y,z)]\wedge\\
&\qquad\forall x\forall y\big[R(x,y)\rightarrow\exists u[R(u,x)\wedge R(u,y)]\big]\wedge\\
&\qquad\qquad\qquad\neg\forall x\forall y[R(x,y)\rightarrow R(y,x)]\leftrightarrow\\
&\forall x\forall y\forall z[(R(x,y)\wedge R(x,z))\rightarrow R(y,z)]\wedge\\
&\qquad\forall x\forall y\big[R(x,y)\rightarrow\exists u[R(u,x)\wedge R(u,y)]\big]\wedge\\
&\qquad\qquad\qquad\exists x\exists y[R(x,y)\wedge\neg R(y,x)].
\end{aligned}
$$

We deal with the following conjuncts:

$$\forall x\forall y\forall z[(R(x,y)\wedge R(x,z))\rightarrow R(y,z)] \tag{9.12}$$

$$\forall x\forall y\big[R(x,y)\rightarrow\exists u[R(u,x)\wedge R(u,y)]\big] \tag{9.13}$$

$$\exists x\exists y[R(x,y)\wedge\neg R(y,x)]. \tag{9.14}$$

We first Skolemize (9.13) and (9.14):

$$\forall x\forall y\big[R(x,y)\rightarrow\big(R(f(x,y),x)\wedge R(f(x,y),y)\big)\big] \tag{9.15}$$

$$R(s,t)\wedge\neg R(t,s), \tag{9.16}$$

where s,t are constants.

Formulas (9.12), (9.15) and (9.16) are transformed into the following clauses (where variables are renamed):

1. $\neg R(x_1, y_1) \vee \neg R(x_1, z_1) \vee R(y_1, z_1)$
2. $\neg R(x_2, y_2) \vee R(f(x_2, y_2), x_2)$
3. $\neg R(x_3, y_3) \vee R(f(x_3, y_3), y_3)$
4. $R(s, t)$
5. $\neg R(t, s)$.

A proof by resolution:

6. $R(f(s, t), s)$ – (res): 2, 4 with $x_2 = s, y_2 = t$
7. $R(f(s, t), t)$– (res): 3, 4 with $x_3 = s, y_3 = t$
8. $\neg R(f(s, t), z_8) \vee R(t, z_8)$
   – (res): 1, 7 with $x_1 = f(s, t), y_1 = t$
   and renaming z_1/z_8
9. $R(t, s)$ – (res): 6, 8 with $z_8 = s$
10. $\emptyset$ – (res): 5, 9.

9.8 Exercise (exam 2)

1. *(2 points)* Design a DATALOG database for storing information about scientific papers, containing, among others, title, authors and information about direct references between papers (i.e., for each paper, information about papers listed in its bibliography section).

   By $p \rightsquigarrow q$ we denote that paper q is among references listed in paper p.

   We define that paper q is *indirectly referenced* by paper p (and denote this by $p \twoheadrightarrow q$) if there is $k \geq 1$ and papers $p_1, p_2, \ldots, p_k$ such that

$$p \rightsquigarrow p_1 \rightsquigarrow p_2 \rightsquigarrow \ldots \rightsquigarrow p_{k-1} \rightsquigarrow p_k \rightsquigarrow q.$$

2. *(1 point)* Provide a sample integrity constraint concerning $\rightsquigarrow$.

3. *(1 point)* Provide a sample integrity constraint concerning $\twoheadrightarrow$.

4. Formulate in logic queries selecting:

   a) *(2 points)* all pairs p_1, p_2 of papers both (co-)authored by 'J.Smith' such that $p_1 \rightsquigarrow p_2$

   b) *(4 points)* all pairs p_1, p_2 of papers both (co-)authored by 'J.Smith' such that $p_1 \twoheadrightarrow p_2$.

9.8 Solution

1

The database contains relations:

- $paper(id, title)$, where id is a (unique) identifier for a paper, and $title \in String$ is its title;
- $author(id, name)$, where id is a paper identifier and $name \in String$ is a name of its (co-)author;
- $refers\_to(id1, id2)$, where $id1, id2$ are paper identifiers.

Example:

$paper(11, \text{‘}Logic\ in\ Databases\text{’})$ – paper entitled *Logic in Databases* has $id = 11$;
$paper(22, \text{‘}Fixpoint\ Logics\text{’})$ – paper entitled *Fixpoint Logics* has $id = 22$;
$author(11, \text{‘}J.Smith\text{’})$ – *J.Smith* (co-)authored paper 11;
$author(11, \text{‘}M.Jones\text{’})$ – *M.Jones* (co-)authored paper 11;
$refers\_to(11, 22)$ – paper 11 refers to paper 22.

2

The relation of road intersection is irreflexive:

$$\forall x[\neg refers\_to(x, x)].$$

3

Relation $\twoheadrightarrow$ is transitive:

$$\forall x \forall y \forall z\big[\big(x \twoheadrightarrow y \wedge y \twoheadrightarrow z\big) \rightarrow x \twoheadrightarrow z\big].$$

4(a)

Select all pairs p_1, p_2 of papers both (co-)authored by 'J.Smith' such that $p_1 \rightsquigarrow p_2$ (recall that '_' is an anonymous "don't care" variable):

$$\begin{aligned} answer(Id1, Id2) :- \quad & paper(Id1, \_), paper(Id2, \_), \\ & author(Id1, \text{`}J.Smith\text{'}), \\ & author(Id2, \text{`}J.Smith\text{'}), \\ & refers\_to(Id1, Id2). \end{aligned}$$

Now $answer(X, Y)$ provides all required pairs of papers.

Note that the above query can be simplified to:

$$\begin{aligned} answer(Id1, Id2) :- \quad & author(Id1, \text{`}J.Smith\text{'}), \\ & author(Id2, \text{`}J.Smith\text{'}), \\ & refers\_to(Id1, Id2). \end{aligned}$$

4(b)

Select all pairs p_1, p_2 of papers both (co-)authored by 'J.Smith' such that $p_1 \twoheadrightarrow p_2$.

We need to define the relation $p_1 \twoheadrightarrow p_2$ (we give it the name $indirectly\_ref$):

$$\begin{array}{ll}
indirectly_ref(Id1, Id2) :- & refers_to(Id1, Id), \\
 & refers_to(Id, Id_2). \\
indirectly_ref(Id1, Id2) :- & refers_to(Id1, Id), \\
 & indirectly_ref(Id, Id2).
\end{array}$$

Now we can formulate the required query:

$$\begin{array}{ll}
answer(Id1, Id2) :- & paper(Id1, _), paper(Id2, _), \\
 & author(Id1, \text{`}J.Smith\text{'}), \\
 & author(Id2, \text{`}J.Smith\text{'}), \\
 & indirectly_ref(Id1, Id2).
\end{array}$$

As before, the above query can be simplified to:

$$\begin{array}{ll}
answer(Id1, Id2) :- & author(Id1, \text{`}J.Smith\text{'}), \\
 & author(Id2, \text{`}J.Smith\text{'}), \\
 & indirectly_ref(Id1, Id2).
\end{array}$$

9.9 Exercise (exam 3)

1. Prove the following propositional formula:

$$\big[(\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee R\big] \rightarrow \big[\neg Q \vee R\big]$$

   a) (*2 points*) using tableaux;

   b) (*2 points*) using Gentzen system.

2. Prove the following formula of predicate logic, where a is a constant:

$$\begin{aligned} &\forall x\big[P(x,x,x)\big] \wedge \\ &\qquad \neg\exists x \exists y \exists z\big[P(x,y,z) \wedge \neg P(f(x), y, g(z))\big] \rightarrow \\ &\qquad\qquad \exists z\big[P(f(a), z, g(a))\big] \end{aligned}$$

   a) (*3 points*) using Gentzen system;

   b) (*3 points*) using resolution.

9.10 Exercise (exam 3)

1. (*4 points*) Translate the following sentences into a set of propositional formulas:

   "objects are light, medium or heavy"
   "one can pack light objects to the first truck, medium objects to the second truck and heavy objects to the third truck"
   "light objects are green"
   "medium objects are blue"
   "heavy objects are red"
   "pack objects to the second or to the third truck"
   "do not pack red objects".

2. (*2 points*) Assuming that each object can be packed to at most one truck, hypothesize what choice as to object's weight can be made and explain your reasoning informally.

3. (*4 points*) Prove your claim formally using a proof system of your choice (tableaux or resolution. Do not use truth table method, as this will give no points).

9.11 Exercise (exam 3)

Consider a drawing which consists of lines connecting points and satisfying the following conditions:

(a) "Every point is connected to a point."
(b) "Connection is symmetric."
(c) "For every points x, y, z, if there is a connection between x and y and between x and z then there is also a connection between y and z."

1. (*1.5 points*) Express in predicate logic properties (a), (b) and (c).

2. (*3.5 points*) Check informally whether the conjunction of (a), (b) and (c) implies that "every point is connected to itself".

3. (*5 points*) Verify your informal reasoning using resolution.

9.12 Exercise (exam 3)

1. *(2 points)* Design a Datalog database for storing information about employees (including position and salary) as well as information about the *direct supervisor* relationship among employees.

   We define that employee e' is an *indirect supervisor* of employee e'' if there is $k \geq 1$ and employees $e_1, e_2, \ldots, e_k$ such that:[2]

   $$e' \rightsquigarrow e_1 \rightsquigarrow e_2 \rightsquigarrow \ldots \rightsquigarrow e_{k-1} \rightsquigarrow e_k \rightsquigarrow e''.$$

2. *(1 point)* Express in logic the constraint:

   > "the relationship of being an indirect supervisor is transitive."

3. *(1 point)* Provide another integrity constraint concerning direct supervisor relationship.

4. Formulate in logic queries selecting:

   a) *(2 points)* all pairs of employees consisting of executive officers with one being a direct supervisor of another

   b) *(4 points)* all pairs of employees X, Y such that the X is a direct or indirect supervisor of Y and X has a lower salary than Y.

[2]By $e' \rightsquigarrow e''$ we denote that e' is a direct supervisor of e''.