Algorithms and Data Structures
Coursework 3 Report

TSP Algorithms

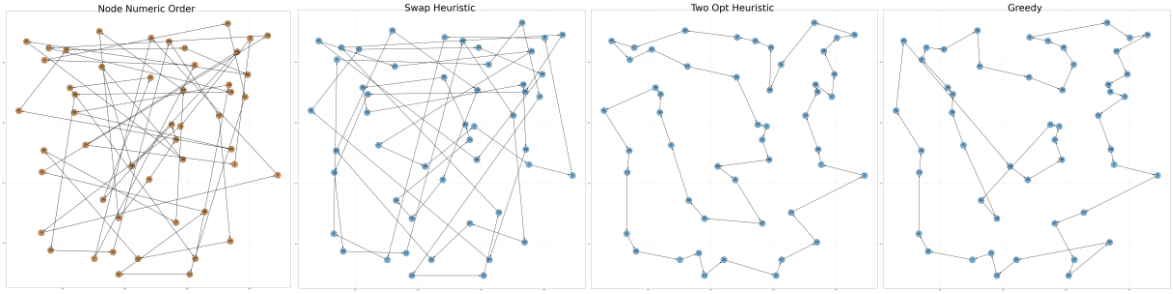
The travelling salesman problem (TSP) is a well-known problem, one which is NP-complete in its decision form. Therefore, an algorithm to reliably find the perfect solution would have to run in exponential time at best.

As such to find an answer to the solution in reasonable time, approximation algorithms must be used instead. 3 such methods of producing approximate solutions using heuristics have already been explored in the 2-opt, swap and greedy algorithms.

The swap and greedy algorithms are self-explanatory in their workings, using comparisons to adjacent vertices to shorten the length of the tour. The swap attempting to shorten the path by swapping adjacent vertices and the greedy selecting the next closest adjacent vertex

The 2-opt heuristic takes advantage of the Euclidean quality ensuring that a tour with paths crossing is invariably at least as long as the same tour with the paths uncrossed. By locally swapping the order of visiting any combination of two vertices in a tour, graphically, a cross in the paths will be caused or reversed at each attempted swap. Due to this, 2-opt performs very well in "real world" examples of the TSP.

The impacts of the algorithm on the tour are visualized below for the cities50 dataset.



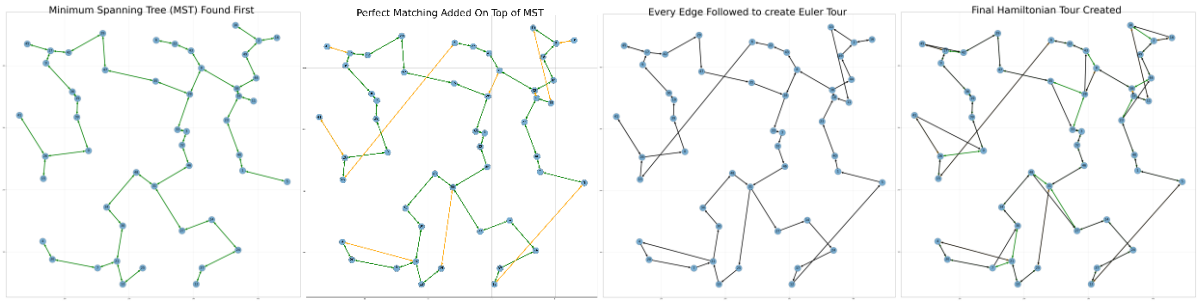
Christofides

The above algorithms all tend to come closer to a better solution but they provide no strong guarantees on the maximum relative tour length. After doing some research I came across the Christofides algorithm which, for an optimal tour, OPT, guarantees a tour length of no more than 1.5*cost(OPT). It does so while running in polynomial time, thus meeting the required conditions.

Algorithm

The steps of the algorithm are as follows.

1. Create a maximum spanning tree (MST), T of the graph (shown in green)
2. Find the vertices with odd degree in T, let us call it O
3. Find the minimum weight perfect matching, M, from the vertices O. A perfect matching is always possible due to the handshaking lemma. (shown in orange)
4. Combine M and T to form an Eulerian Circuit (shown in black)
5. Remove repeated vertices from the combined tour to find the desired Hamilton tour. (shown in black)



Implementation

1. Prim's algorithm was used to find the MST. The MST and M were represented by their edges in a 2D array. The number of edges, n, between two vertices u and v were represented by setting $edges_{uv}$ and $edges_{vu}$ to n as the graphs are bidirectional.
2. The sum of a row $edge_u$ represented the degree of u and a u with an odd sum was added to a list of odd vertices. The distance matrix was sliced using the odd vertices as indexes. This new array served as input for the algorithms to find M.
3. A greedy algorithm to find M was used in the basic graph.py file for simplicity's sake. However, by setting the default argument optimal to False when calling the Christofides function, an optimal version from the test.py file is used. The optimal version in the test.py file is an implementation of the Hungarian algorithm written by Timon Knigge. Every edge found in M was added to the total in its respective cell in the edge array.
4. To get an Eulerian tour the edge array served as a map. Starting from any row. the column with a non-zero value was chosen as the next destination in the tour. That row was then chosen as the origin for the next iteration. The last row in this case was chosen as the starting row but as it is a tour it did not make a significant difference. To cover the loops that could be left out in tour creation, the edge array was passed over and rows with positive sums were reselected as starting points for the creation of a new tour. This new tour in each iteration is added to the original tour at the position of the index at which the new loop starts.
5. The Hamiltonian tour is simply a nubbed version of the Eulerian tour. This is simply achieved by converting the perm list to a dictionary and back to a list.

Proof of Polynomial Time Complexity.

Each step is separate from the last in terms of time complexity for vertices, V.

1. Prim's algorithm has time complexity $O(V^2)$.
2. As each vertex is visited once this has time complexity O(V).
3. The greedy algorithm to find M goes through the list of vertices once for every vertex in the odd vertex list and as such has time complexity $O(V^2)$. The Hungarian algorithm has slightly higher time complexity $O(V^3)$
4. The cost of finding the Eulerian tour is the cost of traversing every edge is $O(V^2)$
5. The cost of removing repetitions in the Eulerian tour to get the Hamiltonian tour is $O(V^2)$.

Proof of 1.5 Performance Bound

As the optimal tour with one edge forms a minimum spanning tree. As there can be no negative edges in the graph, cost(MST) ≤ cost(OPT). The cost of the Christofides tour can be no greater than the cost of the matchings and the MST.

Therefore, cost(Christofides) ≤ cost(MST) + cost(M) ≤ cost(OPT) + cost(M)

Any tour can be broken into two perfect matchings with the possibility of a single spare vertex for an odd number of vertices. As the matching, M is minimized, cost(M) is at least as low as the cost of the lesser of the two matchings in OPT. This is because M is essentially taking shortcuts of the tour as it uses edges between the odd vertices and the triangle equality means that any edge in M cannot have an alternative path shorter than it. As such cost(M) ≤ 0.5*cost(OPT).

Evaluating:

cost(Christofides) ≤ cost(MST) + cost(M) ≤ cost(OPT) + cost(M) ≤ cost(OPT) + 0.5*cost(OPT)

cost(Christofides) ≤ 1.5 cost(OPT) QED

Experiments

To evaluate the performance differences in the algorithms, the mean tour lengths given by the algorithms at varying numbers of nodes and densities were evaluated. The heuristics compared were the ones mentioned in this article as well as Christofides with optimal matching followed by Two-Opt. Optimal solutions were derived using an exponential algorithm. As such, for higher node quantities, calculating optimal values proved infeasible and those values are blacked out in the results table. Repeats were also done where feasible given available resources i.e., time and computational power. 5 repeats were done up to 16 nodes, 2 thereafter up to 18 and a single run thereafter. The tests were done with randomly produced Euclidean graphs, and the details of their generation and the test parameters are apparent in the test file where the supporting algorithms used are also credited. The tour lengths are shown in the table below, coloured by their relative values for each node quantity.

I hypothesized that the best performing algorithms would have their advantage heightened with a larger number of nodes, but density would have little impact on the relative performance levels. This is because while in reality higher density implies less bidirectionality in routes, in the simulated conditions all edges are completely bidirectional.

Effect of Increasing the Number of Nodes on Tour Lengths									
Num of Nodes	Density (nodes/unit²)	No Heuristic	Swap Heuristic	Two-Opt Heuristic	Greedy	Christofides Greedy	Christofides Optimal	Christofides and Two-Opt	Optimal
3	0.0002	224	224	224	224	224	224	224	224
4	0.0002	287	272	272	282	287	278	272	272
5	0.0002	420	341	341	356	352	351	341	341
6	0.0002	515	381	381	389	394	386	378	378
7	0.0002	694	540	451	479	479	473	450	449
8	0.0002	667	591	473	500	490	484	472	472
9	0.0002	1129	872	645	723	683	675	642	634
10	0.0002	1348	907	659	735	713	685	651	651
11	0.0002	1164	935	712	751	841	790	711	708
12	0.0002	1449	1126	774	842	880	841	775	772
13	0.0002	1754	1338	837	912	953	910	834	817
14	0.0002	2072	1718	917	993	973	962	910	904
15	0.0002	2186	1775	937	1030	1046	967	923	908
16	0.0002	2545	1938	1092	1216	1160	1103	1053	1048
17	0.0002	2677	2235	1009	1009	1015	1015	1009	
18	0.0002	2987	2502	1304	1425	1420	1420	1234	
19	0.0002	3565	2343	1080	1227	1143	1143	1072	
20	0.0002	3538	2673	1367	1515	1473	1481	1353	
25	0.0002	5163	3485	1536	1619	1854	1854	1582	
50	0.0002	11681	9516	3241	3498	3237	3503	2948	
75	0.0002	25084	19244	4152	4905	4115	4108	3877	
100	0.0002	37306	29262	5672	7075	6077	6140	5588	
125	0.0002	51029	37243	7547	7942	7558	7609	7004	
150	0.0002	65357	48800	8891	10148	9681	9448	8559	
175	0.0002	88227	67131	10743	11003	11992	11680	9939	
200	0.0002	103402	80565	12057	12717	12441	12117	11207	
225	0.0002	114969	90598	13897	15929	14723	15582	13043	
250	0.0002	145076	116217	14224	16474	16135	15611	13787	
275	0.0002	168096	131909	16871	17717	18271	16929	15523	
300	0.0002	194781	144541	17730	18367	19403	18886	16868	
325	0.0002	214035	168736	19338	21780	22146	21286	18526	
350	0.0002	257713	188849	20872	22983	23207	20569	19053	
375	0.0002	281593	207621	22317	25709	24783	23536	21602	
400	0.0002	303747	236951	23532	25093	25093	25712	22747	
425	0.0002	335507	254570	25051	29417	26932	25899	23520	
450	0.0002	352053	279574	27510	30987	29687	28532	25760	
475	0.0002	376670	292460	27690	30212	29983	30932	26135	
500	0.0002	408771	321177	29654	32221	32451	31847	28068	

Effect of Increasing the Number of Nodes on Tour Lengths									
Num of Nodes	Density	No Heuristic	Swap Heuristic	Two-Opt Heuristic	Greedy	Christofides Greedy	Christofides Optimal	Christofides and Two-Opt	Optimal
12	0.083333	79	62	40	43	44	42	40	39
12	0.000689	735	600	404	455	453	427	392	389
12	0.000189	1654	1205	774	886	873	821	775	774
12	0.000087	2438	1856	1135	1249	1262	1183	1135	1127
12	0.00005	3446	2486	1551	1616	1641	1621	1528	1520
12	0.000032	4025	3345	1996	2160	2176	2133	2002	1985
12	0.000022	4691	3680	2279	2349	2452	2425	2236	2201
12	0.000017	5120	4193	2543	3013	2869	2813	2567	2536
12	0.000013	5992	4724	2994	3308	3302	3201	2922	2917
12	0.00001	6776	5459	3233	3560	3703	3498	3241	3182
12	0.000008	7455	5713	3798	4539	4388	4182	3754	3700
250	0.004	34252	26483	3322	3840	3634	3376	3100	
250	0.000033	345804	272742	34006	39651	37054	36200	32946	
250	0.000009	699940	565483	68754	74776	80015	74558	63984	
250	0.000004	1020714	801792	100655	106880	113531	109271	98089	
250	0.000002	1320790	1068541	134941	156025	148468	146540	128838	
250	0.000002	1619095	1234398	165948	188342	185038	190577	160703	
250	0.000001	1995279	1598574	213269	245498	229769	232211	204964	
250	0.000001	2343939	1809610	236775	266093	268802	255544	225667	
250	0.000001	2727502	2156014	275688	307544	296497	291720	260704	
250	0.0000005	3009142	2297245	290616	310113	335307	318840	278923	
250	0.0000001	3326150	2628261	327768	382523	388641	355757	325502	

Visualisation Analysis

From the visualization, it can be seen that the swap heuristic followed by the simple greedy algorithms are the worst performing for almost all cases. Furthermore, at higher node quantities the relative performance of the swap heuristic worsens.

The Christofides algorithm with optimal matching performs slightly better than the aforementioned, especially as the number of nodes increases. The two opt algorithm when preceded by the swap heuristic performs very well in all scenarios but is consistently outperformed when preceded by the Christofides algorithm with optimal matching. This discrepancy is exacerbated as the number of nodes increases. Additionally, at both high and low quantities of nodes, no conclusive pattern could be seen in the relative performance levels of the algorithms when density was varied.

Additionally, in the scenarios tested, the optimal Christofides algorithm never exceeded its promised bound of 1.5 times the optimal tour length where the tour length was available. Furthermore, the optimal Christofides algorithm never returned a length more than 13% higher than the best length found by any other algorithm. However, this pales in comparison to the two-opt heuristic which strays above 7% above the best tour length when preceded by the swap-heuristic. Even then, the tour length is only 17 units longer as this occurs at a low node count at just 6 nodes.

When the optimal Christofides and two-opt are combined, the tour length never exceeds 2.7%. Although this was predicted to be the best performing, the observed bound on performance is impressive.

Conclusion

This analysis demonstrates that while the Christofides algorithm provides a well-performing backbone through its minimum spanning tree, the Christofides algorithm suffers from performance loss in the matching phase. The two-opt heuristic performs well in the area that the Christofides algorithm has its shortcomings. The effects of the additional Christofides algorithm only offers minimal tour length benefits when used in tandem with the two-opt heuristic compared to the two-opt heuristic on its own. As such while it can be noted that a benefit is found with the Christofides algorithm, the two-opt heuristic appears to be by far the best performing algorithm in head-to-head comparisons. These conclusions are mostly in line with the hypothesis. However, the excellence of the two-opt heuristic was not predicted and should be considered in further experiments.

It should also be noted that none of the algorithms besides the swap heuristic on its own routinely went close to exceeding the best route length by more than 50% despite only the Christofides algorithm promising this

Possible Extensions

The advantages two-opt provides in Euclidean graphs is useful as the scenario is a common enough one. However, the performance of the heuristic in other conditions should be investigated as the benefits of “uncrossing” the path can be mitigated. These non-Euclidean based experiments can be analysed not just in the case of networks but of focused scenarios, such as intra-city road networks vs inter-city ones as one would expect a truer reflection of the effects of density in such scenarios. The algorithms could also be tested to limits where it is more challenging to come closer to an optimal solution, providing a more testing analysis.