

Chapter 28: Lossy Transmission Lines and Dispersion

Chapter Learning Objectives: After completing this chapter the student will be able to:

- Calculate the propagation constant, attenuation constant, and phase constant of a lossy transmission line.
- Explain the phenomenon of dispersion and the difference between group velocity and phase velocity.
- Design a solution to make a lossy line distortionless.

You can watch the video associated with this chapter at the following link:



Historical Perspective: John William Strutt (1842-1919) was a British physicist who did research in acoustic engineering, and electromagnetic waves. He received the Nobel Prize in Physics in 1904 for the discovery of Argon, he discovered why the sky is blue (Rayleigh Scattering), and he also formalized the concepts of dispersion and group velocity.

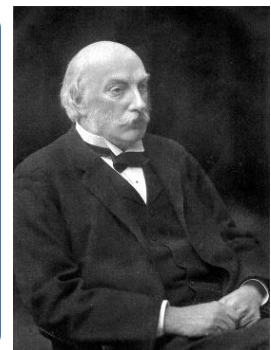


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28.1 Lossy Transmission Lines

Hopefully, you remember that all of our discussions regarding transmission lines are based on the Transmission Line Equations, which are in turn based on a lossless distributed model of the inductance and capacitance of a transmission line. This lossless model does not include any resistance or any possibility of leakage current flowing between the conductors. This model, which is shown in Figure 23.1, is very good, but it is not perfect. When transmission lines become very long or are of relatively poor quality, we must account for the losses on the line. A model that includes these losses is shown in Figure 28.1.

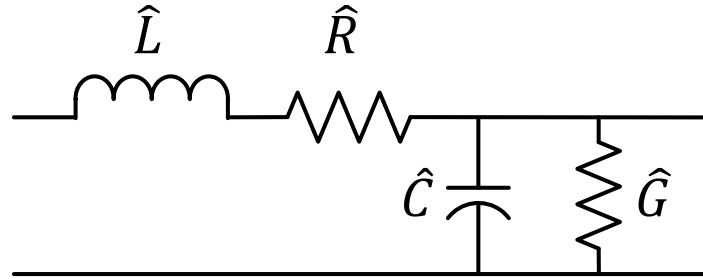


Figure 28.1. Model of a Lossy Transmission Line

In this model, \hat{L} and \hat{C} represent the parasitic inductance and capacitance, \hat{R} represents the resistance of the line preventing current from flowing along its length, and \hat{G} (the symbol for conductance, the inverse of resistance) represents the possibility that current could flow through the imperfect insulation between the conductors. All quantities are per unit length, and this model will be repeated several times to represent the distributed nature of these values.

Following the same procedure we used for lossless lines, we can write KCL and KVL equations and take the limit as $\Delta z \rightarrow 0$ to obtain the following two lossy transmission line equations:

$$\frac{\partial I(z, t)}{\partial z} = -\hat{C} \frac{\partial V(z, t)}{\partial t} - \hat{G} V(z, t) \quad (\text{Equation 28.1})$$

$$\frac{\partial V(z, t)}{\partial z} = -\hat{L} \frac{\partial I(z, t)}{\partial t} - \hat{R} I(z, t) \quad (\text{Equation 28.2})$$

The derivation of these equations follows very closely the derivation of Equations 23.9 and 23.13 with an extra term in each. Notice that if \hat{R} and \hat{G} go to zero (indicating a lossless line), then Equations 28.1 and 28.2 reduce to Equations 23.9 and 23.13, the lossless transmission line equations.

If we assume that the signal will be time-harmonic, then the partial derivatives with respect to t can be written in phasor form, and these two equations reduce to:

$$\frac{\partial I(z)}{\partial z} = -[\hat{G} + j\omega\hat{C}]V(z) \quad (\text{Equation 28.3})$$

$$\frac{\partial V(z)}{\partial z} = -[\hat{R} + j\omega\hat{L}]I(z) \quad (\text{Equation 28.4})$$

If we define the total admittance and total impedance of the line to be:

$$\hat{Y} = \hat{G} + j\omega\hat{C} \quad (\text{Equation 28.5})$$

$$\hat{Z} = \hat{R} + j\omega\hat{L} \quad (\text{Equation 28.6})$$

Then Equations 28.3 and 28.4 can be simplified as follows:

$$\frac{\partial I(z)}{\partial z} = -\hat{Y}V(z) \quad (\text{Equation 28.7})$$

$$\frac{\partial V(z)}{\partial z} = -\hat{Z}I(z) \quad (\text{Equation 28.8})$$

If we substitute Equation 28.8 into Equation 28.7, we find:

$$\frac{\partial^2 I(z)}{\partial z^2} = \hat{Z}\hat{Y}I(z) \quad (\text{Equation 28.9})$$

Similarly, if we substitute Equation 28.7 into Equation 28.8, we obtain:

$$\frac{\partial^2 V(z)}{\partial z^2} = \hat{Z}\hat{Y}V(z) \quad (\text{Equation 28.10})$$

Each of these equations takes the form of a complex Helmholtz equation, with the following solutions:

$$V(z) = V_1e^{-\gamma z} + V_2e^{+\gamma z} \quad (\text{Equation 28.11})$$

$$I(z) = I_1e^{-\gamma z} + I_2e^{+\gamma z} \quad (\text{Equation 28.12})$$

Where the propagation constant γ is defined to be:

$$\gamma = \sqrt{\hat{Z}\hat{Y}} \quad (\text{Equation 28.13})$$

The propagation constant can also be broken into a real and an imaginary component, represented by α and β respectively:

$$\gamma = \alpha + j\beta \quad (\text{Equation 28.14})$$

Substituting Equations 28.5 and 28.6 into Equation 28.13 yields:

$$\gamma = \sqrt{(\hat{R} + j\omega\hat{L})(\hat{G} + j\omega\hat{C})} \quad (\text{Equation 28.15})$$

Substituting Equation 28.14 into Equations 28.11 and 28.12 allows us to see that the form of the solution is an exponentially decaying sinusoid for both voltage and current:

$$V(z, t) = V_1 e^{-\alpha z} \cos(\omega t - \beta z) + V_2 e^{+\alpha z} \cos(\omega t + \beta z) \quad (\text{Equation 28.16})$$

$$I(z, t) = I_1 e^{-\alpha z} \cos(\omega t - \beta z) + I_2 e^{+\alpha z} \cos(\omega t + \beta z) \quad (\text{Equation 28.17})$$

As we have seen before, this most general form of the solution also allows for a superposition of waves moving toward positive z (to the right) and those moving toward negative z (to the left).

? **Example 28.1:** Consider a lossy transmission line that can be modeled as follows: $\hat{L}=10\text{nH/m}$, $\hat{C}=200\text{pF/m}$, $\hat{R}=100\text{m}\Omega/\text{m}$, and $\hat{G}=10\text{mS/m}$. Calculate and describe the voltage on this line if the input signal is moving to the right with a frequency of 10MHz and an initial amplitude of 5V.

Figure 28.2 shows the solution to this exercise. The solid blue line is the complete solution at $t=0$, and the dashed black line represents the exponential envelope $5e^{-0.0408z}$.

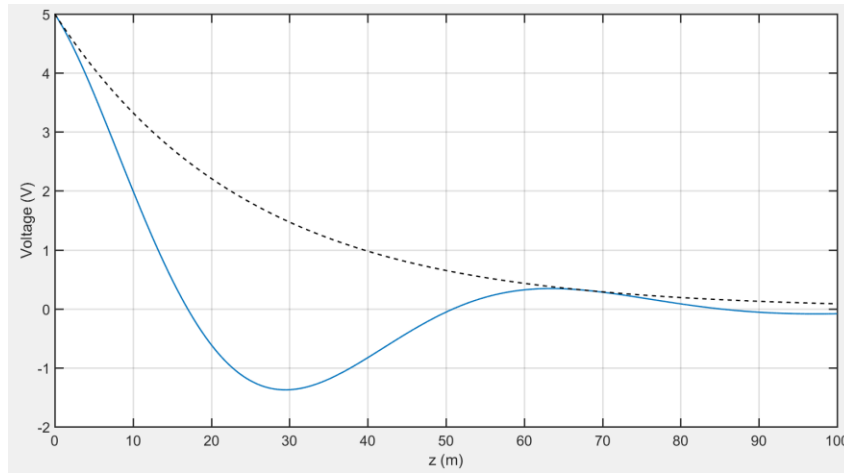


Figure 28.2. Voltage on a lossy transmission line

28.2 Dispersion

The general form for the phase velocity of a signal on a lossy transmission line can be written as follows:

$$v_p = \frac{\omega}{\beta} \quad (\text{Equation 28.18})$$

Furthermore, we saw in the previous section that β (which acts just like the wave number k for a lossless line) can be written as follows:

$$\beta = \text{Im} \left[\sqrt{(\hat{R} + j\omega\hat{L})(\hat{G} + j\omega\hat{C})} \right] \quad (\text{Equation 28.19})$$

These two equations, taken together, illustrate that the phase velocity will be a function of frequency. This is not good. It means that some frequencies will travel at a higher velocity than others, which means that as the signal is traveling along the line, it is being spread out and

dispersed. In fact, this phenomenon is known as **dispersion**, and it is a major problem for transmission lines, especially long ones. When the telegraph was first invented, dispersion caused the signals to be unrecognizable if they were transmitted for more than a mile or so, meaning that telegraph operators had to be set up in remote stations to receive the signal and retransmit it to the next station. This made telegrams very slow, error-prone, and expensive. By the end of this chapter, we'll see how a genius electrical engineer fixed this problem.

Dispersion also has an interesting side-effect—when multiple sinusoids of different frequencies interact with each other, their constructive and destructive interference makes it appear that the waves cluster together into groups, and these groups can have a different velocity than the phase velocity defined in Equation 28.18. The following figure (which is also linked to a YouTube animation) illustrates this concept of group velocity.

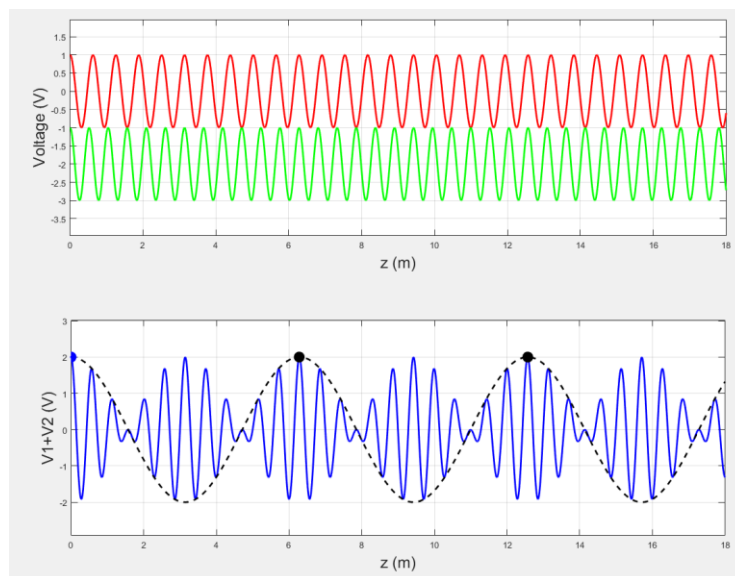


Figure 28.3. Dispersion and Group Velocity

You can calculate the group velocity as the partial derivative of ω with respect to β :

$$v_g = \frac{\partial \omega}{\partial \beta} \quad (\text{Equation 28.20})$$

In practice, this equation is not nearly as important as Equation 28.18, because group velocity is a symptom of a system with dispersion, and a system with dispersion is poorly designed. We will take advantage of two different ways to eliminate dispersion and group velocity from our systems: reducing the losses until they can be neglected and using distortionless lines.

28.3 Nearly Lossless Lines

The lossy terms (\hat{R} and \hat{G}) will often be quite small, especially if the transmission line is of good quality and appropriately sized for the length of transmission.. In this case, we can develop approximate expressions to represent the voltage and current on the line when the losses are very small.

We will begin by factoring Equation 28.15:

$$\gamma = \alpha + j\beta = \sqrt{(j\omega\hat{L})(j\omega\hat{C})} \sqrt{\left(1 + \frac{\hat{R}}{j\omega\hat{L}}\right) \left(1 + \frac{\hat{G}}{j\omega\hat{C}}\right)} \quad (\text{Equation 28.21})$$

When the factors under the square root are cross-multiplied, we will neglect the term that contains both \hat{R} and \hat{G} , since the product of two very small numbers is very, very small. This produces:

$$\gamma \approx \sqrt{(j\omega\hat{L})(j\omega\hat{C})} \sqrt{1 - j\left(\frac{\hat{R}}{\omega\hat{L}} + \frac{\hat{G}}{\omega\hat{C}}\right)} \quad (\text{Equation 28.22})$$

Mathematicians have shown that, for small values of x, the following is approximately true:

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \quad (\text{Equation 28.23})$$

Applying Equation 28.23 to Equation 28.22 gives:

$$\gamma \approx j\omega\sqrt{\hat{L}\hat{C}} \left[1 - j\left(\frac{\hat{R}}{2\omega\hat{L}} + \frac{\hat{G}}{2\omega\hat{C}}\right)\right] \quad (\text{Equation 28.24})$$

Focusing on the attenuation constant α , which is the real part of the propagation constant γ , we find:

$$\alpha \approx \omega\sqrt{\hat{L}\hat{C}} \left(\frac{\hat{R}}{2\omega\hat{L}} + \frac{\hat{G}}{2\omega\hat{C}}\right) \quad (\text{Equation 28.25})$$

This can be rewritten as:

$$\alpha \approx \frac{1}{2}\sqrt{\hat{L}\hat{C}} \left(\frac{\hat{R}}{\hat{L}} + \frac{\hat{G}}{\hat{C}}\right) \quad (\text{Equation 28.26})$$

Notice that this quantity is independent of frequency, which means that all frequencies will be attenuated by the same amount. This will prevent distortion of the signal.

We can also write an expression for β from Equation 28.24 in the nearly lossless case:

$$\beta \approx \omega \sqrt{\hat{L}\hat{C}} \quad (\text{Equation 28.27})$$

Substituting this equation into Equation 28.18, we find:

$$v_p \approx \frac{\omega}{\omega \sqrt{\hat{L}\hat{C}}} \quad (\text{Equation 28.28})$$

This can be simplified to:

$$v_p \approx \frac{1}{\sqrt{\hat{L}\hat{C}}} \quad (\text{Equation 28.29})$$

So, for the nearly lossless case, we find that not only is the attenuation constant independent of frequency (preventing certain frequencies from being over-attenuated and distorting the signal, but the phase constant β is linearly dependent on ω , meaning that the phase velocity is independent of frequency, preventing dispersion.

? Example 28.2: Calculate α and β for the situation in Example 28.1, this time using the approximate equations. How do these result compare to the result for α and β obtained from the full equations in Example 28.1?

Of course, to have a nearly lossless line, you will have to pay extra for larger conductors and better insulators. Throwing money at a problem is one way to solve it, but engineers usually prefer to throw math at the problem. It's cheaper, and it leaves more money to pay the engineers.

28.3 Distortionless Lines

There is a very special and very desirable condition under which a line will be “distortionless,” meaning that both the attenuation constant (α) and the characteristic impedance (Z_c) will be independent of frequency (ω), and the phase constant (β) is linearly proportional to frequency. The condition for this to be true is:

$$\boxed{\frac{\hat{R}}{\hat{L}} = \frac{\hat{G}}{\hat{C}}} \quad (\text{Equation 28.30})$$

Let's prove that this condition leads to both an attenuation constant and a characteristic impedance that are independent of frequency. We will begin by solving Equation 28.30 for \hat{G} and substituting it into Equation 28.15:

$$\gamma = \sqrt{\left(\hat{R} + j\omega\hat{L}\right)\left(\frac{\hat{R}\hat{C}}{\hat{L}} + j\omega\hat{C}\right)} \quad (\text{Equation 28.31})$$

Rearranging this expression gives:

$$\gamma = \left(\hat{R} + j\omega\hat{L}\right)\sqrt{\frac{\hat{C}}{\hat{L}}} \quad (\text{Equation 28.32})$$

The attenuation constant is simply the real part of the propagation constant:

$$\alpha = \text{Re}(\gamma) = \hat{R}\sqrt{\frac{\hat{C}}{\hat{L}}} \quad (\text{Equation 28.33})$$

Notice that this attenuation constant is independent of frequency, which means that all frequency components will be attenuated at the same rate, causing no distortion. The phase constant is the imaginary part of the propagation constant:

$$\beta = \text{Im}(\gamma) = \omega\hat{L}\sqrt{\frac{\hat{C}}{\hat{L}}} = \omega\sqrt{\hat{L}\hat{C}} \quad (\text{Equation 28.34})$$

Substituting this equation into Equation 28.18, we find:

$$v_p = \frac{\omega}{\omega \sqrt{\hat{L}\hat{C}}} \quad (\text{Equation 28.35})$$

This can be reduced to:

$$v_p = \frac{1}{\sqrt{\hat{L}\hat{C}}} \quad (\text{Equation 28.36})$$

Thus, we have determined that the phase velocity of a distortionless line is identical to that of a lossless line, and most importantly that it is independent of frequency, meaning there will be no dispersion.

Of course, you might ask yourself, “How likely is it that we will be lucky enough to have a transmission line that satisfies Equation 28.30, giving us distortionless transmission?” The answer, unfortunately, is that it is extremely rare to find such a transmission line. Most typically, we find that $R/L > G/C$, meaning that the line is not distortionless. Furthermore, it is difficult to decrease R without using more expensive materials or larger cables. But notice that there is another way to correct this inequality. Since it is difficult to decrease R , we can instead choose to increase L . Doing so will make the left side of this equation smaller, and if we add just the right amount of inductance, we can satisfy Equation 28.30, making the line distortionless.

So, after working out all the math of dispersion, group velocity, and distortionless lines, we reach the conclusion that the solution is to add a small amount of extra inductance to the line at regular intervals. This is the conclusion reached by Oliver Heaviside (the genius electrical engineer who was already spotlighted in the Historical Perspectives in Chapters 1 and 14). When he installed these extra inductors on the telegraph lines in the mid-1800s, he found that they could transmit without distortion for hundreds of miles, eliminating the need for all those repeater stations. We still use this method today in long-distance telephone lines.

? **Example 28.3:** Consider a lossy transmission line that can be modeled as follows: $\hat{L}=10\text{nH/m}$, $\hat{C}=50\text{nF/m}$, $\hat{R}=100\text{m}\Omega/\text{m}$, and $\hat{G}=10\text{mS/m}$. We wish to make this line distortionless by adding an inductor every 100 meters along the line. What value must this inductor have?

28.4 Summary

- A lossy transmission line includes a term \hat{R} to represent the resistance of the signal flowing down the line and a conductance \hat{G} to represent the possibility of a leakage current between the conductors through the insulator. A lossy transmission line has a propagation constant γ that is calculated as follows:

$$\gamma = \sqrt{(\hat{R} + j\omega\hat{L})(\hat{G} + j\omega\hat{C})}$$

- γ can be broken down into a real part (a, the attenuation constant) and an imaginary part (b, the phase constant):

$$\gamma = \alpha + j\beta$$

- The solution to the lossy transmission line equations is an exponentially decaying sinusoid, with the decay envelope controlled by α and the oscillation controlled by β . β fills a role very similar to k in lossless lines.

$$V(z, t) = V_1 e^{-\alpha z} \cos(\omega t - \beta z) + V_2 e^{+\alpha z} \cos(\omega t + \beta z)$$

$$I(z, t) = I_1 e^{-\alpha z} \cos(\omega t - \beta z) + I_2 e^{+\alpha z} \cos(\omega t + \beta z)$$

- The phase velocity (which we have just called velocity until now) can be calculated as follows:

$$v_p = \frac{\omega}{\beta}$$

- For a typically lossy transmission line, this leads to a phase velocity that depends on the signal's frequency, which means that different frequency components of a signal will propagate at different rates, leading to dispersion. A symptom of dispersion is a clustering of the signal into groups, which propagate at a group velocity v_g . The group velocity can be different from the phase velocity when there is dispersion
- For nearly lossless lines, the attenuation constant and phase velocity are both approximately constant. This means that we can eliminate distortion (frequency-dependent attenuation) and dispersion (frequency-dependent phase velocity) if we improve the quality of the cable.
- Another solution is to ensure that the line is distortionless, which requires:

$$\frac{\hat{R}}{\hat{L}} = \frac{\hat{G}}{\hat{C}}$$

- If this condition is satisfied (typically by increasing \hat{L}), the attenuation constant and phase velocity are both perfectly constant, meaning that both distortion and dispersion are eliminated entirely.