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An analytical formula is derived to calculate the inductance of a finite-width superconducting strip transmission line. The formula gives an accurate inductance value when the aspect ratio (the ratio of the linewidth to the insulation thickness) exceeds about unity.

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I. INTRODUCTION

The calculation of the electrical parameters of a superconducting strip transmission line is very important for the design of circuitry and devices in Josephson junction technology.[1,2] The field solution of a wide strip line was obtained by Swihart.[3] Meyers[4] further showed that the inductance of such a line can be obtained by consideration of total free energy including the static magnetic field energy and the superconducting electron kinetic energy. These analyses, however, do not take into account the contribution of the fringe field due to the finite dimension of the strip line. As the strip line becomes narrower, the fringe field effects become more important and may dominate if the width of the strip line is comparable in size to that of the dielectric thickness. In the past, the lack of simple mathematical formulas led Baechtold[5] to use an analog method to study the approximate field distribution of such a structure. However, an experiment has to be performed and no analog exists for a thin superconducting strip line. In this paper, an analysis leading to an accurate analytical inductance formula is given. The formula gives an accurate inductance value if the aspect ratio (the ratio of the linewidth to the thickness of the insulation) of the strip line exceeds about unity. The accuracy of the formula deteriorates as the aspect ratio becomes smaller.

II. FIELD SOLUTION OF A STRIP LINE

Consider a superconducting strip line of width W and thickness t_1, located at a distance h above a second superconducting ground plane of thickness t_2 as shown in Fig. 1. The field solution of the strip-line structure must satisfy Maxwell's equations. In addition, the London equations[6] are assumed to be satisfied in the superconducting regions. We assume that the superconducting currents flow only in the z-direction so that vector potential A has a z component A_z only.

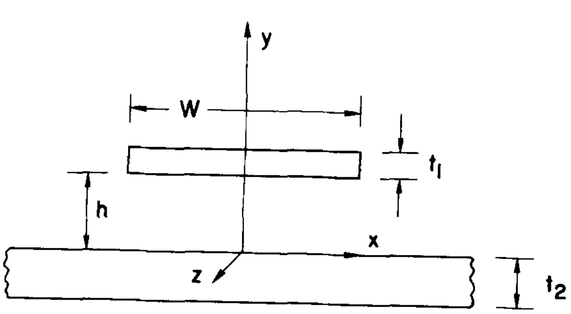

FIG. 1. Superconducting strip-line structure.

In general, the normal magnetic field B_n does not vanish along the dielectric-superconductor boundary. But for a wide superconductor strip line, it is negligibly small except near the corner of the strip line. So to a good approximation, the normal magnetic field along the dielectric-superconductor boundary may be assumed to be zero,

$$B_n = \frac{\partial A_z}{\partial t} = 0, \tag{1}$$

where t indicates tangential direction. Integrating Eq. (1) in tangential direction, we find that A_z is constant along the dielectric-superconductor boundary. We also have the relationship that the vector potential A_z satisfies the Laplace equation in the dielectric region. Thus the problem of obtaining the magnetic field distribution in the dielectric region reduces to that of solving the Laplace equation for A_z in the dielectric region with the boundary conditions

$A_z = 1$ at the strip-line surface,

$A_z = 0$ at the ground-plane surface.

The two constants can be chosen arbitrarily, but are chosen

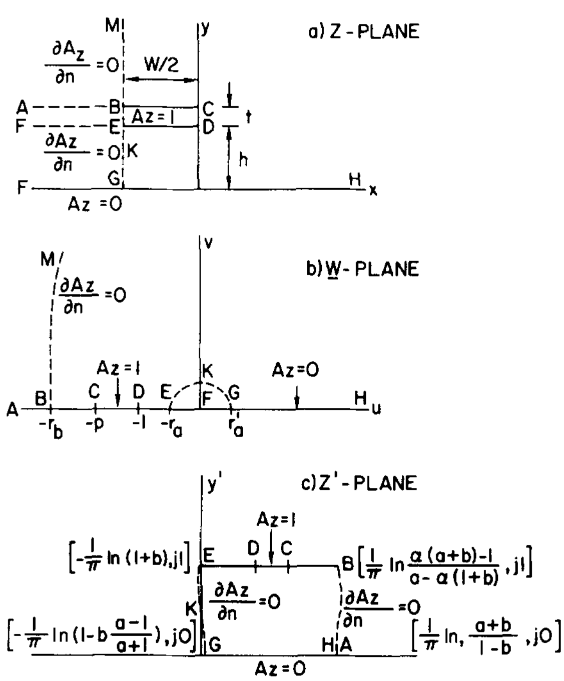

FIG. 2. Transformations for Fig. 1.

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to be 0 and 1 for convenience. The complete solution suitable for $W/h \gtrsim 1$ has been obtained analytically[7] by use of a conformal mapping technique. The required transformations used are shown in Fig. 2. First, half of the strip line in the z plane is mapped into the w plane by the transformation

$$\frac{\pi z}{h} = \frac{p+1}{p^{1/2}}\tanh^{-1}R + \frac{p-1}{p^{1/2}}\frac{R}{1-R^2} + \ln\left(\frac{Rp^{1/2}-1}{Rp^{1/2}+1}\right), \tag{2}$$

$$R = \left(\frac{w+1}{w+p}\right)^{1/2}. \tag{3}$$

Then the w plane is mapped into the z' plane by the transformation

$$z' = \frac{1}{\pi}\ln\left(\frac{r_a + aw}{ar_a + w}\right), \tag{4}$$

where

$$a = \frac{r_b}{r_a} + \left[\left(\frac{r_b}{r_a}\right)^2 - 1\right]^{1/2}, \tag{5}$$

$$\ln r_a = -1 - \frac{\pi W}{2h} - \frac{p+1}{p^{1/2}}\tanh^{-1}(p^{-1/2}) - \ln\left(\frac{p-1}{4p}\right), \tag{6}$$

$$r_b = r_{bo} \quad \text{for } W/h \geqslant 5, \tag{7}$$

$$r_b = r_{bo} - [(r_{bo}-1)(r_{bo}-p)]^{1/2} + (p+1) \times \tanh^{-1}\left(\frac{r_{bo}-p}{r_{bo}-1}\right)^{1/2} - 2p^{1/2}\tanh^{-1}\left(\frac{r_{bo}-p}{p(r_{bo}-1)}\right)^{1/2} + \frac{\pi W}{2h}p^{1/2}, \quad \text{for } 5 > \frac{W}{h} \gtrsim 1, \tag{8}$$

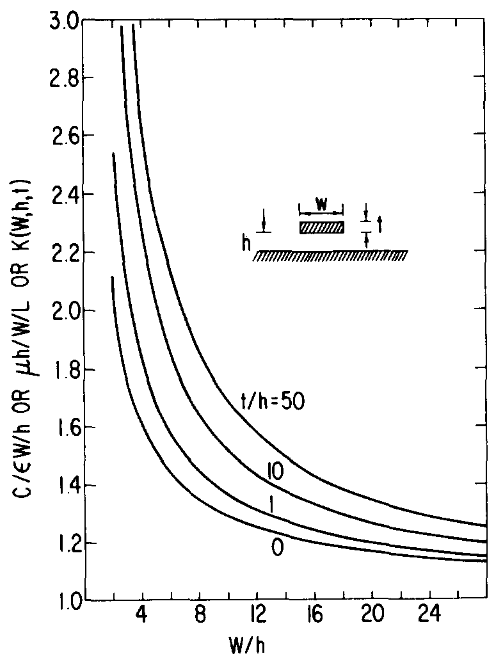

FIG. 3. The fringe factor $K(W,h,t)$ versus aspect ratio W/h.

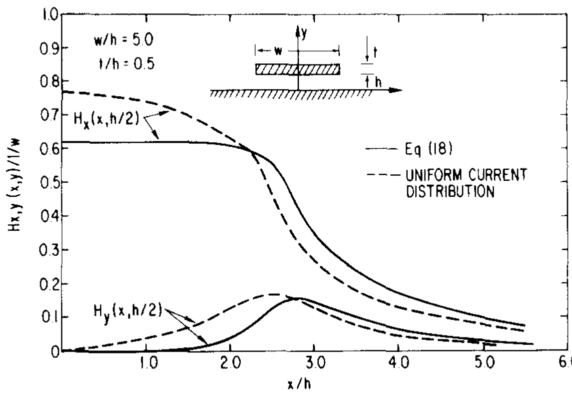

FIG. 4. Magnetic field distribution under the strip line.

$$r_{bo} = \eta + \frac{p+1}{2}\ln\Delta, \tag{9}$$

$$\Delta = \text{larger value, } \eta \text{ or } p, \tag{10}$$

$$\eta = p^{1/2}\left\{\frac{\pi W}{2h} + \frac{p+1}{2p^{1/2}}\left[1 + \ln\left(\frac{4}{p-1}\right)\right] - 2\tanh^{-1}p^{1/2}\right\}, \tag{11}$$

$$p = 2\beta^2 - 1 + [(2\beta^2-1)^2 - 1]^{1/2}, \tag{12}$$

$$\beta = 1 + \frac{t_1}{h}. \tag{13}$$

The structure in the z' plane is an almost parallel-plate structure. In Fig. 2, the b $(r'_a - r_a)/(r'_a + r_a)$, is very small and α is given approximately as r_b/r_a. As the aspect ratio W/h becomes larger, with r_a becoming smaller and r_b becoming much larger than unity, the structure would approach an exact parallel plate. The field solution of a parallel-plate structure can be obtained easily since no fringe field exists. By transforming the field solution of the parallel-plate structure back to the z plane, the field solution of the strip-line structure can be easily obtained. The total current I in the strip line for the case under consideration can be obtained by applying Ampere's law to the peripheral of the strip line in the z' plane. The result is given approximately as

$$I = \int_{\text{stripline}} \mathbf{H}_t \cdot d\mathbf{t} = \frac{1}{\mu_0}\int_{\text{stripline}} \frac{\partial A_z}{\partial n}dt = \frac{2}{\mu_0}\int_{BCDE_{z'\text{plane}}} \frac{\partial A_z}{\partial y'}dx' = \frac{2}{\mu_0}\overline{BCDE}_{z'\text{plane}} = \frac{K(W,h,t_1)}{\mu_0 h/W}, \tag{14}$$

where

$$K(W,h,t_1) = \frac{h}{W}\frac{2}{\pi}\ln\frac{2r_b}{r_a}. \tag{15}$$

We refer to $K(W,h,t_1)$ as the fringe field factor. Equation (15) with r_a and r_b evaluated using Eqs. (6)–(13) is plotted in Fig. 3.

III. MAGNETIC FIELD DISTRIBUTION

The magnetic field in the dielectric region can be derived by use of complex variable theory[8] in Fig. 2. Note the complex variable theory

$$\frac{\partial z'}{\partial z} = \frac{\partial x'}{\partial x} + j\frac{\partial y'}{\partial x} = \frac{\partial y'}{\partial y} + j\frac{\partial y'}{\partial x}$$
$$= \frac{\partial A_z}{\partial y} + j\frac{\partial A_z}{\partial x} = \frac{1}{\mu_0}(H_x - jH_y), \tag{16}$$

where the Cauchy-Riemann equation is used to obtain the second equality in Eq. (16), and

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} \tag{17}$$

is used to obtain the last equality. The function $\partial z'/\partial z$ can be obtained from Eqs. (2) and (4),

$$H_x - jH_y = \frac{p^{1/2}r_a(a^2-1)w}{\mu_0 h\,(w+1)^{1/2}(w+p)^{1/2}(w+ar_a)(aw+r_a)}. \tag{18}$$

The field above the strip line can be obtained by substituting $w = -r_b$ in Eq. (18) and noting that $r_b \gg p \gg 1$ for $W/h \gg 1$. The result after taking the first term in a Taylor series expansion of Eq. (18) is, for $x = 0$ and $y = h + t_1$,

$$H_x(0,h+t_1) \approx -\frac{2p^{1/2}}{r_b\mu_0 h}. \tag{19}$$

The field under the strip line along the vertical line passing through the center of the strip line can be obtained by substituting $w = r_a\exp(i\theta)$ in Eq. (18) with $\pi \geqslant \theta \geqslant 0$, and noting that $r_a \ll 1$ if $W/h \gg 1$. We obtain approximately

$$H_x(0,h \geqslant y \geqslant 0) \approx \frac{1}{\mu_0 h}\left(1 - \frac{2}{a}\cos\theta\right)$$
$$\approx \frac{1}{\mu_0 h}. \tag{20}$$

The analysis thus far assumes that the total current flowing in the strip line is that given by Eq. (14). For an arbitrary current I flowing in the strip line (i.e., for arbitrary constant value of A_z at the stripline surface), Eqs. (18)–(20) should be modified by multiplying the right-hand side of each equation by $\mu_0 hI/WK(W,h,t_1)$. Since $K(W,h,t_1) \gg 1$, the magnetic field is reduced as though the conductor width is increased by a factor $K(W,h,t_1)$. A typical magnetic field distribution is shown in Fig. 4. Comparison is also made with the field calculated by assuming a uniform current sheet at the bottom edge of the strip conductor. Equation (18) predicts a more uniform field under the strip line, but its magnitude is decreased from that predicted by assuming a uniform current sheet.

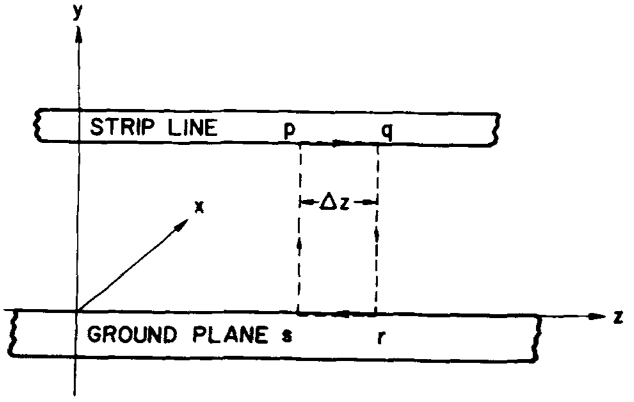

FIG. 5. Fluxoid integration path used for inductance evaluation.

IV. SUPERCONDUCTING STRIP LINE

Within the superconducting strip line, the magnetic field satisfies the London equations. Since the peripheral magnetic field [within the approximation given in Eq. (1)] has been obtained from the previous analysis of the external field, the London equation can be solved analytically by the method of separation of variables. The analysis is shown in the Appendix. The complete field and the current distributions within the strip line can be expressed as a function of its peripheral magnetic field.

V. INDUCTANCE

The inductance per unit length, L, and be obtained by evaluating the fluxoid of the strip line using the integration path shown in Fig. 5,

$$L = \frac{1}{I}\frac{1}{\Delta z}\int_{pqrsp}(\mathbf{A} + \mu_0\lambda^2\mathbf{J})\cdot d\mathbf{l}$$
$$= \frac{1}{I}\left[A_z(p) + \mu_0\lambda_1^2 J_z(p) - A_z(s) - \mu_0\lambda_2^2 J_z(s)\right]$$
$$= \frac{1}{I}\left[1 + \mu_0\lambda_1^2 J_z(p) - \mu_0\lambda_2^2 J_z(s)\right], \tag{21}$$

where λ_1 and λ_2 are the effective London penetration depths of the strip superconductor and the ground plane, respectively. The portions of the fluxoid within the superconductor are known to be constants, independent of integration path. Those portions when calculated from Eqs. (A8) and (A19) of the appendix, however, are not constants due to the approximation used in Eq. (1). When $W \gg h$, however, the error introduced is very small except near the corner of the strip line, and the portion of the fluxoid within the stripline is nearly a constant. For simplicity, the integration path for Eq. (21) is chosen just above the bottom surface of the strip line and just below the surface of the ground plane. With Eq. (A19),

$$J_z(p) = J_z\left(\frac{W}{2},0\right)$$
$$= \sum_{m=0}^{\infty}\left(\frac{(-1)^m\gamma_{2m}}{\sinh(\gamma_{2m}t_1)}(a_{2m-}\cosh\gamma_{2m}t_1 - a_{2m+})\right)$$
$$+ \sum_{m=0}^{\infty} b_m\xi_m\left[\sinh\left(\frac{\xi_m W}{2}\right)\right]^{-1}$$
$$\approx \sum_{m=0}^{\infty}(-1)^m\gamma_{2m}\left[a_{2m-}\coth\gamma_{2m}t_1 - a_{2m+}\right.$$
$$\left.\times\operatorname{csch}(\gamma_{2m}t_1)\right] + \sum_{m=0}^{\infty}\frac{2b_m}{\lambda_1}\exp\left(-\frac{W}{2\lambda_1}\right). \tag{22}$$

To a first approximation and assuming that $W \gg t_1$ and $W \gg \lambda_1$, only the first two terms are retained, and

$$J_z(p) \approx \frac{1}{\lambda_1}\left[a_{0-}\coth\left(\frac{t_1}{\lambda_1}\right) - a_{0+}\operatorname{csch}\left(\frac{t_1}{\lambda_1}\right)\right]. \tag{23}$$

From Eqs. (A12) and (A13), a_{0-} and a_{0+} are known to be

TABLE I. Comparison of the inductance values calculated by the formula and the numerical method.

W (μm)	t_1 (μm)	h (μm)	W/h	λ_2 (μm)	λ_1 (μm)	l_{num} [a] (pH/μm)	l_{for} [b] (pH/μm)
14.0	0.2	0.18	77.8	0.0	0.135	0.0282	0.0279
14.0	0.5	0.18	77.8	0.0	0.135	0.0266	0.0265
4.5	0.5	0.18	25.0	0.0	0.135	0.0730	0.0742
2.7	0.5	0.18	15.0	0.0	0.135	0.1115	0.1135
1.8	0.5	0.18	10.0	0.0	0.135	0.1522	0.1553
0.9	0.5	0.18	5.0	0.0	0.135	0.2456	0.2446
0.45	0.5	0.18	2.5	0.0	0.135	0.3683	0.3504
0.18	0.5	0.18	1.0	0.0	0.135	0.5880	0.4772
1.5	0.22	2.851	0.53	0.086	0.137	0.6207	0.5882
2.0	0.22	2.851	0.70	0.086	0.137	0.5518	0.5263
2.5	0.22	2.851	0.88	0.086	0.137	0.5010	0.4671
5.0	0.22	2.851	1.75	0.086	0.137	0.3573	0.3423
12.8	0.22	2.851	4.49	0.086	0.137	0.1928	0.1911
25.4	0.22	2.851	8.91	0.086	0.137	0.1191	0.1146
1.5	0.22	0.9995	1.50	0.086	0.137	0.4335	0.4152
2.0	0.22	0.9995	2.00	0.086	0.137	0.3687	0.3569
2.5	0.22	0.9995	2.50	0.086	0.137	0.3226	0.3142
5.0	0.22	0.9995	5.00	0.086	0.137	0.2034	0.2011
12.8	0.22	0.9995	12.81	0.086	0.137	0.0966	0.0969
25.4	0.22	0.9995	25.41	0.086	0.137	0.0540	0.0535
1.5	0.22	0.1775	8.45	0.086	0.137	0.2398	0.2492
2.0	0.22	0.1775	11.27	0.086	0.137	0.1915	0.1990
2.5	0.22	0.1775	14.08	0.086	0.137	0.1599	0.1659
5.0	0.22	0.1775	28.17	0.086	0.137	0.0885	0.0911
12.8	0.22	0.1775	72.11	0.086	0.137	0.0374	0.0381
1.5	0.217	0.1582	9.48	0.086	0.124	0.2184	0.2316
2.0	0.217	0.1582	12.64	0.086	0.124	0.1746	0.1856
2.5	0.217	0.1582	15.80	0.086	0.124	0.1458	0.1543
5.0	0.217	0.1582	31.61	0.086	0.124	0.0808	0.0841
12.8	0.217	0.1582	80.91	0.086	0.124	0.0341	0.0350

$t_2 = 0.3\,\mu$m

[a] l_{num} = inductance per unit length calculated by the numerical method of Ref.9.
[b] L_{for} = inductance per unit length calculated from Eq. (26).

the average magnetic fields on the lower and upper surfaces of the strip conductor, respectively. Further approximations can be made by assuming that the average magnetic field is equal to that which exists at the middle of the strip conductor, and by using Eqs. (19) and (20) to obtain

$$J_z(p) \approx \frac{1}{\mu_0 \lambda_1 h}\left[\coth\left(\frac{t_1}{\lambda_1}\right) + \frac{2p^{1/2}}{r_b}\operatorname{csch}\left(\frac{t_1}{\lambda_1}\right)\right]. \quad (24)$$

Since the second term is usually much smaller than the first term, the replacement of the average magnetic field on the upper surface of the strip conductor by Eq. (19) introduces only a small error.

Similarly, on the ground plane, it can be shown that

$$J_z(s) = -\frac{1}{\mu_0 h \lambda_2}\coth\left(\frac{t_2}{\lambda_2}\right). \quad (25)$$

Substituting Eqs. (24), (25), and (14) into Eq. (21), we obtain

$$L = \frac{\mu_0}{WK(W,h,t_1)}\left\{h + \lambda_1\left[\coth\left(\frac{t_1}{\lambda_1}\right) + \frac{2p^{1/2}}{r_b}\operatorname{csch}\left(\frac{t_1}{\lambda_1}\right)\right] + \lambda_2 \coth\left(\frac{t_2}{\lambda_2}\right)\right\}. \quad (26)$$

The first term represents the contribution due to the external field, the second and the third terms are the contribution from the strip conductor, and the fourth term is the contribution from the ground plane. The third term is usually small and can be neglected. The $K(W,h,t_1)$ factor is plotted in Fig. 3. When the aspect ratio $W/h \rightarrow \infty$, then $K(W,h,t_1) \rightarrow 1$, and Eq. (26) reduces to that derived by Swihart[3] and Meyers.[4] When the aspect ratio is not so large that the fringe field cannot be neglected, the inductance is reduced as though the width of the strip line is increased by the $K(W,h,t_1)$ factor.

VI. CAPACITANCE

The capacitance per unit length of a superconducting strip line is the same as that of a normal conducting strip line. It was analyzed in Ref. 7 and is found to be

$$C = \frac{\varepsilon\varepsilon_0 W}{h} K(W,h,t_1), \quad (27)$$

where ε is the dielectric constant of the insulation, and ε_0 is the permittivity of the free space.

VII. DISCUSSION

The analytical formula [Eq. (26)] was compared with the results obtained by a numerical method[9] which simultaneously solves the coupled Maxwell's equations and the London equations. Table I compares the results obtained

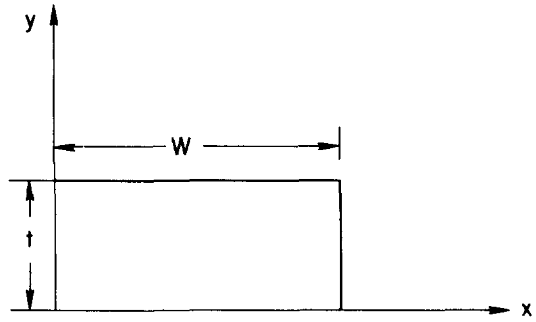

FIG. 6. Superconducting strip-line coordinate system.

between the two methods. The agreement between the two results are quite good. The accuracy of the formula, however, deteriorates when the aspect ratio becomes smaller than unity, indicating that the approximations used become less accurate.

Equation (26) was used to predict the threshold curves of superconducting Josephson junction interferometer devices,[10] which depend strongly on the loop inductance of the device. Good agreement was obtained between the calculated results and the experimental results. Preliminary direct inductance measurements[11] using Josephson junction interferometer circuits also show good agreement with inductances calculated from Eq. (26). For the example cited in Ref. 11, using nominal given dimensions $W = 3.3\,\mu$m, $h = 0.937\,\mu$m, $t_1 = 0.778\,\mu$m, $t_2 = 0.27\,\mu$m, $\lambda_1 = 0.119\,\mu$m, and $\lambda_2 = 0.086\,\mu$m, the fringe factor calculated from Eqs. (6)–(13) and (15) is 1.9, whereas the measured fringe factor is 2.00 ± 0.14.

From Fig. 3, a significant fringing field exists for most practical strip lines. Even for an aspect ratio of 20, the fringe field reduces the strip line inductance by as much as 20% from that calculated by assuming no fringe field. The reduction of strip line inductance is even more dramatic as the aspect ratio of the line becomes small. The reduction can be a factor of 3 or 4, if the aspect ratio is near unity. This is due to the fact that for small aspect ratios, the inductance increases logarithmically with the inverse of the aspect ratio, rather than linearly with the inverse of the aspect ratio, as is the case for the large aspect ratios.

VIII. CONCLUSIONS

An accurate analytical formula has been derived to calculate the inductance of a finite-width superconducting strip transmission line. The formula gives an accurate inductance value when the aspect ratio of the strip line exceeds about unity. The accuracy of the formula improves as the aspect ratio of the strip superconducting line becomes large.

ACKNOWLEDGMENTS

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APPENDIX

Consider the coordinate system shown in Fig. 6. Both H_x and H_y satisfy the London equation

$$\nabla^2 H_{x,y} = \frac{1}{\lambda^2} H_{x,y} . \tag{A1}$$

The boundary conditions are

$$H_x = h_x(x,0), \quad y = 0 \tag{A2}$$

$$H_x = h_x(x,t), \quad y = t \tag{A3}$$

$$\frac{\partial H_x}{\partial x} = 0, \quad x = 0;\ x = W \tag{A4}$$

$$H_y = h_y(0,y), \quad x = 0 \tag{A5}$$

$$H_y = h_y(W,y), \quad x = W \tag{A6}$$

$$\frac{\partial H_y}{\partial y} = 0, \quad y = 0;\ y = t . \tag{A7}$$

Applying the method of separation of variables, the solution of Eq. (A1) with boundary conditions Eqs. (A2)–(A4) is

$$H_x(x,y) = \sum_{n=0}^{\infty} \frac{\cos(n\pi x/W)}{\sinh(\gamma_n t)} \times \{a_{n+} \sinh(\gamma_n y) + a_{n-} \sinh[\gamma_n(t-y)]\}, \tag{A8}$$

where

$$\gamma_n = \left[\frac{1}{\lambda^2} + \left(\frac{n\pi}{W}\right)^2\right]^{1/2}, \tag{A9}$$

$$a_{n-} = \frac{2}{W}\int_0^W h_x(x,0) \cos\left(\frac{n\pi x}{W}\right) dx , \quad n \neq 0, \tag{A10}$$

$$a_{n+} = \frac{2}{W}\int_0^W h_x(x,t) \cos\left(\frac{n\pi x}{W}\right) dx, \quad n \neq 0, \tag{A11}$$

$$a_{0-} = \frac{1}{W}\int_0^W h_x(x,0)\, dx , \tag{A12}$$

$$a_{0+} = \frac{1}{W}\int_0^W h_x(x,t)\, dx . \tag{A13}$$

Similarly, the solution of Eq. (A1) with the boundary conditions [Eqs. (A5)–(A7)] is

$$H_y(x,y) = \sum_{m=0}^{\infty} \frac{b_m \cos(m\pi y/t)}{\sinh(\xi_m W)} \times \{\sinh(\xi_m x) - \sinh[\xi_m(W-x)]\}, \tag{A14}$$

where

$$\xi_m = \left[\frac{1}{\lambda^2} + \left(\frac{m\pi}{t}\right)^2\right]^{1/2}, \tag{A15}$$

$$b_m = \frac{2}{t}\int_0^t h_y(W,y) \cos\left(\frac{m\pi y}{t}\right) dy, \quad m \neq 0, \tag{A16}$$

$$b_0 = \frac{1}{t}\int_0^t h_y(W,y)\, dy , \tag{A17}$$

and $h_y(0,y) = -h_y(W,y)$ is used. The current distribution inside the stripline can be obtained by setting

$$J_z(x,y) = \nabla \times \mathbf{H}|_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} . \tag{A18}$$

Substituting Eqs. [(A8) and (A14)] into Eq. [(A18)], we obtain

$$J_z(x,y) = \sum_{m=0}^{\infty} \frac{b_m \xi_m \cos(m\pi y/t)}{\sinh(\xi_m W)}$$

$$\times\{\cosh \xi_m x + \cosh[\xi_m(W-x)]\} - \sum_{m=0}^{\infty} \frac{\gamma_m \cos(m\pi x/W)}{\sinh(\gamma_m t)} \times \{a_{m+} \cosh(\gamma_m y) - a_{m-} \cosh[\gamma_m(t-y)]\}\,. \tag{A19}$$

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