

# An Elementary Introduction to Groups and Representations Solutions

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## 1 Chapter 1: Groups

1. To show that the center  $H$  is a subgroup of  $G$ , we first show that the identity of  $G$  is in  $H$ . This can be seen as  $eg = ge = g, \forall g \in G$ . Next we show that for all  $h \in H$ , the inverses exist in  $H$  too. By defn.  $gh = hg, \forall g \in G, h \in H$ . Multiplying each side of the equation on the left and the right by  $h^{-1}$ , we get  $h^{-1}ghh^{-1} = h^{-1}hgh^{-1}$ ,  $h^{-1}g = gh^{-1}$  which shows that  $h^{-1} \in H$ . Finally we have to show closure, consider 2 arbitrary elements of  $H$ , namely  $h_1, h_2$ .  $h_1h_2g = h_1(h_2g) = h_1gh_2 = gh_1h_2$  showing that  $h_1h_2 \in H$ .
2. (a) The set of odd integers is not a subgroup since it doesn't contain the identity of  $G$  viz 0.  
(b)  $nZ$  for any integer  $n$  is a subgroup of  $Z$ .  
(c) Since  $\det(A) = 0$  only when  $A$  is non-invertible, every element in  $H$  is invertible. The identity has determinant of 1 so it's part of  $H$  too. And the determinant of product of 2 matrices is the determinant of each individual matrix showing closure.  
(d) Since  $G$  is  $SL$  with  $\det 1$ , the inverse exists and is composed of integers by Cramer's rule. By similar arguments to prev exercise we can show closure and identity.  
(e) It is a valid subgroup, same argument as prev exercise.  
(f) Not a subgroup since it isn't closed  $8 + 4(\text{mod}9) = 3$  which isn't part of  $H$ .
3.  $gg^{-1} = g^{-1}g = e$  so  $g$  is the inverse of  $(g^{-1})^{-1}$ .  $ghh^{-1}g^{-1} = e$ , so  $(gh)^{-1} = h^{-1}g^{-1}$ .
4. Define the map from  $H$  to  $G$  as  $f(h) = \phi^{-1}(h)$ . Since  $\phi$  is bijective,  $\phi^{-1}$  is bijective too. We just need to show  $f$  is a homomorphism. Let  $x = \phi^{-1}(h_1)$  and  $y = \phi^{-1}(h_2)$  and  $xy = z$ , then  $f(h_1h_2) = z = xy = f(h_1)f(h_2)$ .
5. The set of positive real numbers has an inverse  $1/p$ , they are a closed set and contain the identity 1. The map  $f : R \rightarrow R^*$  can be defined as  $e^x$ .  $e^{(x_1+x_2)} = e^{x_1}e^{x_2}$  shows that it is a homomorphism. Since the exponential map is bijective, it is an isomorphism.
6. Consider two automorphisms  $h_1, h_2$ . Since an automorphism is an isomorphism, composition, inverse and identity trivially hold.
7. To show that  $\phi_g$  is an automorphism we first show it is a homomorphism. For this  $\phi(h_1)\phi(h_2) = gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} = \phi(h_1h_2)$ . Next we need to show that this map is one-one and onto:  $\phi(h_1) = \phi(h_2) \implies gh_1g^{-1} = gh_2g^{-1}$ . Multiplying left and right of both sides by  $g^{-1}$  and  $g$  respectively we get  $h_1 = h_2$ . For any element  $h$ ,  $\phi(g^{-1}hg) = h$  showing that it is onto, completing our automorphism proof.  
To show that this map is a homomorphism we need to show  $\phi_{g_1g_2} = \phi_{g_1} \circ \phi_{g_2}$ . Consider any element in the LHS map  $g_1g_2hg_2^{-1}g_1^{-1}$  and the RHS map is  $\phi_{g_1} \circ (g_2hg_2^{-1}) = g_1g_2hg_2^{-1}g_1^{-1}$  which completes our proof.  
Finally, the kernel of the map is the maps s.t.  $\phi_g(h) = h$ ,  $ghg^{-1} = h$ , multiplying both sides on the right by  $g$ , we get  $gh = hg$  which is the center of the  $G$ .
8. The matrices  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  &  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  don't commute.
9. Same question as Q1.
10. The two matrices are  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$  &  $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$  showing that  $S_3$  isn't commutative.
11. Let  $f(n) = 2n$  and  $g(n) = n/2$  when  $n$  is even and 0 otherwise. Here  $g \circ f(n) = I$  while  $f \circ g(n)$  isn't.
12. Consider 2 elements  $a_1 = k_1n + b_1$  &  $a_2 = k_2n + b_2$ . Now,  $(a_1 + a_2) \text{mod} n = (k_1n + b_1 + k_2n + b_2) \text{mod} n = (b_1 + b_2) \text{mod} n$ . Now,  $(a_1 \text{mod} n + a_2 \text{mod} n) \text{mod} n = (b_1 + b_2) \text{mod} n$  completing the proof.

13. Consider a subgroup  $H$ , now for any element  $g \in G$ ,  $ghg^{-1} = h \in H$  (since  $G$  is commutative, we can rearrange terms) showing that any subgroup  $H$  is normal. By closure  $ghg^{-1}$  must be in  $G$ , showing that  $G$  is normal.  $geg^{-1} = e \forall g \in G$ , showing that  $e$  is a normal subgroup too. Consider the center  $Z$  of  $G$ . Now,  $ghg^{-1} = hgg^{-1} = h$  since  $h$  is part of the center, showing that the center is a normal subgroup too. Finally, consider the kernel  $K$  of  $\phi$  viz,  $\phi(K) = e_H$ , to show that it is a normal subgroup, we need to show  $\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = e_H$ . Finally, we need to show  $SL(n, R)$  is a normal subgroup of  $GL(n, R)$ , consider the conjugate  $ghg^{-1}$  where  $g \in GL$  and  $h \in SL$ . By property of determinants,  $\det(g) = x, \det(g^{-1}) = 1/x$  &  $\det(h) = 1$ , leading to the  $\det$  of  $ghg^{-1} = 1$  showing that it is in  $SL(n, R)$ .

## 2 Matrix Lie Groups

1.

2.  $\langle Bx, y \rangle = (Bx)^T y = x^T B^T y = \langle x, B^T y \rangle$ . Now  $\langle Ax, Ay \rangle = \langle x, A^T Ay \rangle$  If  $A^T A = I$  then we get  $\langle Ax, Ay \rangle = \langle x, y \rangle$ . And the other direction is if we know that  $\langle Ax, Ay \rangle = \langle x, y \rangle$  we can easily equate  $A^T A = I$ .

3. Analogous to prev exercise replacing transpose with  $*$ .

4.

5.

6.  $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} |A| = \cos^2(\theta) + \sin^2(\theta) = 1$ .

$$A^T A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} * \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{pmatrix} = I$$

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} * \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\phi)\cos(\theta) - \sin(\theta)\sin(\phi) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\phi)\sin(\theta) + \cos(\phi)\cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}$$

We denote a matrix in  $O(2)$  by  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . The following conditions hold since it is an orthogonal matrix:  $a^2 + c^2 = 1, b^2 + d^2 = 1, ad - bc = \pm 1$  (Determinant). TO FINISH

7.

8.

9.

10. Consider 2 elements from the Heisenberg group  $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \& B = \begin{pmatrix} 1 & h_1 & h_2 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{pmatrix}$ . Now for  $A$  to be part of the

center for any matrix  $B$  we need  $AB - BA = 0$

$$AB - BA = \begin{pmatrix} 0 & 0 & ah_3 - ch_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ Now for this matrix to be 0 for all values of } h_3 \& h_1, a \text{ and } c \text{ must be equal to 0.}$$

11.

12.

13.

14.

15. To show that the translation group is a normal subgroup of the Euclidean group, we need to show that  $ghg^{-1} \in H$  where  $g \in E \& h \in T$ . From the theory above we can denote any Euclidean transformation by  $\{x, R\}$  where  $x$  is the translation and  $R$  is the rotation. Similarly, we can write an element of the translation group in this form as  $\{x, I\}$ . Now the above can be written as  $\{x, R\} * \{t, I\} * \{x, R\}^{-1}$  where  $\{x, R\}^{-1} = \{R^{-1}x, R^{-1}\}$ . The composition is given as  $\{x_1, R_1\}\{x_2, R_2\} = \{x_1 + R_1x_2, R_1R_2\} \implies \{x, R\}\{t, I\}\{R^{-1}x, R^{-1}\} = \{x + Rt, R\}\{R^{-1}x, R^{-1}\} = \{x + Rt + RR^{-1}x, RR^{-1}\} = \{x + Rt + RR^{-1}x, I\}$  which belongs the translation group completing our proof. TODO part 2.

16.