An Elementary Introduction to Groups and Representations Solutions

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1 Chapter 1: Groups

1. To show that the center H is a subgroup of G, we first show that the identity of G is in H. This can be seen as $eg = ge = g, \forall g \in G$. Next we show that for all $h \in H$, the inverses exist in H too. By defn. $gh = hg, \forall g \ inG, h \in H$. Multiplying each side of the equation on the left and the right by h^{-1}, we get $h^{-1}ghh^{-1} = h^{-1}hgh^{-1}$, $h^{-1}g = gh^{-1}$ which shows that $h^{-1} \in H$. Finally we have to show closure, consider 2 arbitrary elements of H, namely h_1, h_2. $h_1h_2g = h_1(h_2g) = h_1gh_2 = gh_1h_2$ showing that $h_1h_2 \in H$.

2. (a) The set of odd integers is not a subgroup since it doesn't contain the identity of G viz 0.
   (b) nZ for any integer n is a subgroup of Z.
   (c) Since det(A) = 0 only when A is non-invertible, every element in H is invertible. The identity has determinant of 1 so it's part of H too. And the determinant of product of 2 matrices is the determinant of each individual matrix showing closure.
   (d) Since G is SL with det 1, the inverse exists and is composed of integers by Cramer's rule. By similar arguments to prev exercise we can show closure and identity.
   (e) It is a valid subgroup, same argument as prev exercise.
   (f) Not a subgroup since it isn't closed $8 + 4(mod 9) = 3$ which isn't part of H.

3. $gg^{-1} = g^{-1}g = e$ so g is the inverse of $(g^{-1})^{-1}$. $ghh^{-1}g^{-1} = e$, so $(gh)^{-1} = h^{-1}g^{-1}$.

4. Define the map from H to G as $f(h) = \phi^{-1}(h)$. Since phi is bijective, phi^{-1} is bijective too. We just need to show f is a homomorphism. Let $x = phi^{-1}(h_1)$ and $y = phi^{-1}(h_2)$ and $xy = z$, then $f(h_1h_2) = z = xy = f(h_1)f(h_2)$.

5. The set of positive real numbers has an inverse $1/p$, they are a closed set and contain the identity 1. The map $f : R \to R*$ can be defined as e^x. $e^{(x1+x2)} = e^{x1}e^{x2}$ shows that it is a homomorphism. Since the exponential map is bijetive, it is an isomorphism.

6. Consider two automorphisms h_1, h_2. Since an automorphism is an isomorphism, composition, inverse and identity trivially hold.

7. To show that ϕ_g is an automorphism we first show it is a homomorphism. For this $\phi(h_1)\phi(h_2) = gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} = \phi(h_1h_2)$. Next we need to show that this map is one-one and onto: $\phi(h_1) = \phi(h_2) \implies gh_1g^{-1} = gh_2g^{-1}$. Multiplying left and right of both sides by g^{-1} and g respectively we get $h_1 = h_2$. For any element h, $\phi(g^{-1}hg) = h$ showing that it is onto, completing our automorphism proof.
   To show that this map is a homomorphism we need to show $\phi_{g_1g_2} = \phi_{g_1} \circ \phi_{g_2}$. Consider any element in the LHS map $g_1g_2hg_2^{-1}g_1^{-1}$ and the RHS map is $\phi_{g_1} \circ (g_2hg_2^{-1}) = g_1g_2hg_2^{-1}g_1^{-1}$ which completes our proof.
   Finally, the kernel of the map is the maps s.t. $\phi_g(h) = h$, $ghg^{-1} = h$, multiplying both sides on the right by g, we get $gh = hg$ which is the center of the G.

8. The matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ don't commute.

9. Same question as Q1.

10. The two matrices are $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ showing that S_3 isn't commutative.

11. Let $f(n) = 2n$ and $g(n) = n/2$ when n is even and 0 otherwise. Here $g \circ f(n) = I$ while $f \circ g(n)$ isn't.

12. Consider 2 elements $a_1 = k_1n + b_1 \& a_2 = k_2n + b_2$. Now, $(a_1 + a_2) mod n = (k_1n + b_1 + k_2n + b_2) mod n = (b_1 + b_2) mod n$. Now, $(a_1 mod n + a_2 mod n) mod n = (b1 + b2) mod n$ completing the proof.

13. Consider a subgroup H, now for any element $g \in G$, $ghg^{-1} = h \in H$ (since G is commutative, we can rearrange terms) showing that any subgroup H is normal. By closure ghg^{-1} must be in G, showing that G is normal. $geg^{-1} = e \forall g \in G$, showing that e is a normal subgroup too. Consider the center H of G. Now, $ghg^{-1} = hgg^{-1} = h$ since $h$is part of the center, showing that the center is a normal subgroup too. Finally, consider the kernel K of ϕ viz, $\phi(K) = e_H$, to show that it is a normal subgroup, we need to show $\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = e_H$.
Finally, we need to show $SL(n, R)$ is a normal subgroup of $GL(n, R)$, consider the conjugate ghg^{-1} where $g \in GL$ and $h \in SL$. By property of determinants, $det(g) = x, det(g^{-1}) = 1/x \& det(h) = 1$, leading to the det of $ghg^{-1} = 1$ showing that it is in $SL(n, R)$.

2 Matrix Lie Groups

1.

2. $\langle Bx, y\rangle = (Bx)^T y = x^T B^T y = \langle x, B^T y >$. Now $\langle Ax, Ay\rangle = \langle x, A^T Ay\rangle$ If $A^T A = I$ then we get $\langle Ax, Ay\rangle = \langle x, y\rangle$. And the other direction is if we know that $\langle Ax, Ay\rangle = \langle x, y\rangle$ we can easily equate $A^T A = I$.

3. Analogous to prev exercise replacing transpose with *.

4. $\langle x, gy\rangle = \sum_i x_i g_{ii} y_i = \sum_{i=1}^n x_i g_{ii} y_i + \sum i = n+1^{n+k} x_i g_{ii} y_i = \sum_{i=1}^n x_i yi - \sum_{i=n+1}^{n+k} x_i yi = (x, y)_{n,k}$ TO FINISH

5.

6. $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ $|A| = \cos^2(\theta) + \sin^2(\theta) = 1$.
$A^T A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} * \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{pmatrix} = I$
$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} * \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\phi)\cos(\theta) - \sin(\theta)\sin(\phi) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\phi)\sin(\theta) + \cos(\phi)\cos(\theta) \end{pmatrix}$
$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$
We denote a matrix in $O(2)$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The following conditions hold since it is an orthogonal matrix: $a^2 + c^2 = 1, b^2 + d^2 = 1, ad - bc = \pm 1$ (Determinant). We can parameterize the solution to these equations by the functions $\sin(t)$ and $\cos(t)$, the exact signs of each term is dependant on considering $ad - bc = 1$ and $ad - bc = -1$. Leading to the only possible solutions as $\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ and $\begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}$.

7. Since $A = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$ satisfies $\langle A_1, A_2\rangle = \cosh(t)\sinh(t) - \sinh(t)\cosh(t) = 0.\langle A_1, A_1 = 1\rangle = \cosh(t)^2 - \sinh(t)^2 = 1, \langle A_2, A_2\rangle = \sinh(t)^2 - \cosh^2(t) = -1$ it is part of $SO(1,1)$.
Next, $\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix} = \begin{pmatrix} \cosh(s)\cosh(t) + \sinh(t)\sinh(s) & \cosh(t)\sinh(s) + \sinh(t)\cosh(s) \\ \sinh(t)\cosh(s) + \cosh(t)\sinh(s) & \sinh(t)\sinh(s) + \cosh(t)\cosh(s) \end{pmatrix} = \begin{pmatrix} \cosh(t+s) & \sinh(t+s) \\ \sinh(t+s) & \cosh(t+s) \end{pmatrix}$

8. To show that $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ is orthonormal, we can easily verify that the norms of each column is 1 by the condition given that $|\alpha|^2 + |\beta|^2$. Now represent α as $x_1 + ix_2$ and β as $y_1 + iy_2$. Now to verify that the dot product of the columns is 0, let's expand $(\alpha, \beta) * (-\bar{\beta}, \bar{\alpha}) = (x_1 + ix_2)(y_1 - iy_2) + (y_1 + iy_2)(x_1 - ix_2) = (x_1y_1 - iy_1x_2 + iy_2x_1 + y_2x_2) - (x_1y_1 - ix_1y_2 + ix_2y_1 + x_2y_2) = 0$.
For a unitary matrix, the inverse of the matrix is equal to it's transpose of complex conjugate. Let the matrix A be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now transpose of complex conjugate is $\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$ and inverse of original matrix is $\begin{pmatrix} d & -b - c & a \end{pmatrix}$. Equating both we get $d = \bar{a}, \bar{b} = -c$. Now using the fact that the matrix has det 1 we get $ad - bc = 1$. Replaving in this equation, $a\bar{a} + b\bar{b} = 1$. Similarly we can get $c\bar{c} + d\bar{d} = 1$. This shows that all matrices in $SU(2)$ are of the form $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$.

9.

10. Consider 2 elements from the Heisenberg group $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ $\&B = \begin{pmatrix} 1 & h_1 & h_2 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{pmatrix}$. Now for A to be part of the center for any matrix B we need $AB - BA = 0$
$AB - BA = \begin{pmatrix} 0 & 0 & ah_3 - ch_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Now for this matrix to be 0 for all values of $h_3 \& h_1$, a and c must be equal to 0.
So the center is all matrices of the form $\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

11.

12. By SVD we can represent any real matrix $A = W \sum V^T$ where W and V^T are both orthogonal. Now we can rewrite this as $A = WV^TV \sum V^T$, which we can factorize as $(WV^T)(V \sum V^T)$. The first is a rotation matrix since both W and V^T are orthogonal. The second is PSD since $\sum$ is a matrix containing only real positive values along the diagonal, so when we multiply with V and V^T the matrix is PSD. Hence, we have shown that there exists a decomposition of the matrix A into RH where R is a rotation matrix and H is a PSD matrix.

13. (from prev exercise) We an write any matrix in $SL(n, R)$ in the form of RH. Then we can diagonalize H into $R_1 D R_1^{-1}$ where D is a diagonal matrix. Using the same argument as in Proposition 2.8, we can connect this matrix to $RR_1 I R_1^{-1} = R$. That is we are able to connect any matrix of $SL(n, R)$ to a matrix in $SO(n)$ which is connected in turn showing that $SL(n, R)$ is connected.

14. Consider a matrix A in $GL(n, R)$, we can find a path from A to A^* where A^* is defined as $1/\sqrt[n]{det(A)}A$. A^* is clearly in $SL(n, R)$ showing that $GL(n, R)$ is connected as well since we have shown in the previous exercise that $SL(n, R)$ is connected.

15. To show that the translation group is a normal subgroup of the Euclidean group, we need to show that $ghg^{-1} \in H$ where $g \in E \& h \in T$. From the theory above we can denote any Euclidean transformation by $\{x, R\}$ where x is the translation and R is the rotation. Similarly, we can write an element of the translation group in this form as $\{x, I\}$. Now the above can be written as $\{x, R\} * \{t, I\} * \{x, R\}^{-1}$ where $\{x, R\}^{-1} = \{R^{-1}x, R^{-1}\}$. The composition is given as $\{x_1, R_1\}\{x_2, R_2\} = \{x_1 + R_1 x_2, R_1 R_2\} \implies \{x, R\}\{t, I\}\{R^{-1}x, R^{-1}\} = \{x + Rt, R\}\{R^{-1}x, R^{-1}\} = \{x + Rt + RR^{-1}x, RR^{-1}\} = \{x + Rt + RR^{-1}x, I\}$ which belongs the translation group completing our proof. TODO part 2.