Calibration of the Hull and White model

Murex – MACS*

First Version: Thursday 16th December, 2010
This Version: Tuesday 14th August, 2018

Abstract

This document details MACS' implementation of the Hull-White model calibration

Contents

1 Presentation of the model

1.1 Introduction

The Hull and White model (HW) is a short rate model. It is used to price financial products based on underlyings depending on the short rate being modelled. It is a 1-factor model, it can be calibrated to match the volatility of target swaption but not its whole smile.

1.2 Definition

The HW model is mean-reverting just like Vasicek model, but instead of reverting to a constant mean, it reverts to a deterministic time-dependent mean. The short rate can be expressed as the sum of an Ornstein-Uhlenbeck process and a deterministic function.

$$r_t = X_t + \varphi(t) + f^M(0,t), \tag{1.1}$$

where

$$\begin{aligned} dX_t &= -aX_t dt + \sigma(t) dW_t, \\ X_0 &= 0 \end{aligned} \tag{1.2}$$

$$\varphi(t) = \int_0^t \sigma^2(s) e^{-a(t-s)} \frac{1-e^{-a(t-s)}}{a} ds, \tag{1.3}$$

$$f^M(0,t) = -\frac{\partial}{\partial T} \ln P^M(0,T) \tag{1.4}$$

where the mean reversion rate a is a constant, the time-dependent parameter $\sigma(t)$ is taken piecewise constant on $]\mathcal{T}_{i-1}, \mathcal{T}_i]$, and W_t is a Brownian motion.

1.3 Choice of φ

The function φ enables to reprice correctly market zero-coupon bonds ! To see that, we will need the fact[1] that

$$\int_t^T X_u du \bigg| \mathcal{F}_t \sim \mathcal{N}\left(\frac{1-e^{-a(T-t)}}{a} X_t, \int_t^T \left(\frac{1-e^{-a(T-s)}}{a}\right)^2 \sigma_s^2 ds\right), \tag{1.5}$$

and in particular for $t = 0$

$$\int_0^T X_u du \sim \mathcal{N}\left(0, \int_0^T \left(\frac{1-e^{-a(T-s)}}{a}\right)^2 \sigma_s^2 ds\right). \tag{1.6}$$

When computing the price of a zero-coupon maturing at T, denoted $P(0,T)$, we get:

$$\begin{aligned} P(0,T) &= \mathbb{E}\left[e^{-\int_0^T r_u du}\right] \\ &= \mathbb{E}\left[e^{-\int_0^T X_u du}\right] e^{-\int_0^T \varphi(u) du} \underbrace{e^{-\int_0^T f^M(0,u) du}}_{P^M(0,T)}, \end{aligned}$$

[1] cf. Appendix A.2

so in order to match the market zero-coupon prices, we must have that

$$\begin{aligned}\int_0^T \varphi(u)du &= \ln\left(\mathbb{E}\left[e^{-\int_0^T X_u du}\right]\right)\\ &= \tfrac{1}{2}\int_0^T \frac{\left(1-e^{-a(T-s)}\right)^2}{a^2}\sigma_s^2 ds,\end{aligned}$$

where the last step follows from (1.6). Taking the derivative with respect to T will give the definition (1.3) we took for φ, so the model indeed reprices zero-coupons correctly.

1.4 Zero-coupon price

More generally, we can now price zero-coupons in the future, conditionnally on the future value of X_t:

$$\begin{aligned}P(t,T) &= \mathbb{E}\left[e^{-\int_t^T r_u du}\middle|\mathcal{F}_t\right]\\ &= \mathbb{E}\left[e^{-\int_t^T X_u du}\middle|\mathcal{F}_t\right] e^{-\int_t^T \varphi(u)du}\underbrace{e^{-\int_t^T f^M(0,u)du}}_{\frac{P^M(0,T)}{P^M(0,t)}}.\end{aligned}$$

Let's introduce the following useful notations:

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}, \tag{1.7}$$

$$V(t,T,U) = \int_t^T \frac{\left(1-e^{-a(U-s)}\right)^2}{a^2}\sigma_s^2 ds. \tag{1.8}$$

So now (1.5) can be rewritten as

$$\begin{aligned}\int_t^T X_u du\middle|\mathcal{F}_t &\sim \mathcal{N}\left(B(t,T)X_t, V(t,T,T)\right),\\ \mathbb{E}\left[e^{-\int_t^T X_u du}\middle|\mathcal{F}_t\right] &= e^{-B(t,T)X_t+\frac{1}{2}V(t,T,T)}.\end{aligned}$$

We have also seen that

$$\int_0^T \varphi(u)du = \tfrac{1}{2}V(0,T,T),$$

so we have that

$$e^{-\int_t^T \varphi(u)du} = e^{\frac{1}{2}(V(0,t,t)-V(0,T,T))}.$$

All in all, since $V(0,T,T) = V(0,t,T) + V(t,T,T)$, we end up with:

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)}e^{-B(t,T)X_t+\frac{1}{2}(V(0,t,t)-V(0,t,T))} \tag{1.9}$$

$$P(t,T) = A(t,T)e^{-B(t,T)X_t}, \tag{1.10}$$

where A is defined as

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}e^{\frac{1}{2}(V(0,t,t)-V(0,t,T))}. \tag{1.11}$$

2 Swaption valuation

2.1 Swaption valuation in a mono-curve framework

Swaptions prices are computed using the Jamishidian trick derived in [2].

We consider a receiver swaption:

$$Sw(t) = \left(\sum_i \alpha_i K P(t, T_i) - \mathrm{FloatingLeg}(t)\right)^+ .$$

In a monocurve framework, $\mathrm{FloatingLeg}(t) = P(t, T_0) - P(t, T_N)$. Hence

$$\begin{aligned} Sw(t) &= \left(\sum_i \alpha_i K P(t, T_i) - (P(t, T_0) - P(t, T_N))\right)^+ \\ &= \left(\sum_i \alpha_i K A_i e^{-B_i X_t} - \left(A_0 e^{-B_0 X_t} - A_N e^{-B_N X_t}\right)\right)^+ \\ &= \left(\sum_{i=0}^{N} C_i e^{-B_i X_t}\right)^+ \end{aligned}$$

Where

$$\begin{aligned} C_0 &= -A_0 \\ C_i &= \alpha_i K A_i, 0 < i < N \\ C_N &= (1 + \alpha_N K) A_N \end{aligned}$$

The expected value of the swaption can be expressed as an integral:

$$\mathrm{Swaption} = \int_{-\infty}^{\infty} \left(\sum_{i=0}^{N} C_i e^{-B_i \sqrt{Var[X_t]} x}\right)^+ \phi(x) dx \tag{2.1}$$

Since C_i are coefficients which are all *negative* up to a certain i (in this example: $i = 0$), then all *positive*, and B_i are positive, increasing functions of i, it can be shown that there exists x^* satisfying:

$$\sum_{i=0}^{N} C_i e^{-B_i \sqrt{Var[X_t]} x^*} = 0$$

and such that $Sw(t)$ is *positive* for $x < x^*$ and *negative* for $x > x^*$.
The receiver swaption value can thus be estimated analytically:

$$\mathrm{Swaption} = \int_{-\infty}^{x^*} \left(\sum_{i=0}^{N} C_i e^{-B_i \sqrt{Var[X_t]} x}\right) \phi(x) dx$$

$$\text{Swaption} = \sum_{i=0}^{N} C_i e^{\frac{1}{2}B_i^2 Var[X_t]} \Phi\left(x^* + B_i\sqrt{Var[X_t]}\right) \tag{2.2}$$

Shall we consider a payer swaption:

$$Sw(t) = \left(\text{FloatingLeg}(t) - \sum_i \alpha_i K P(t, T_i)\right)^+$$

We would end up estimating:

$$Sw(t) = \left(\sum_{i=0}^{N} C_i e^{-B_i X_t}\right)^+$$

Where

$$\begin{aligned} C_0 &= A_0 \\ C_i &= -\alpha_i K A_i, 0 < i < N \\ C_N &= -(1 + \alpha_N K) A_N \end{aligned}$$

Since C_i are coefficients which are all *positive* up to a certain i (in this example: $i = 0$), then all *negative*, and B_i are positive, increasing functions of i, it can be shown that there exists x^* satisfying:

$$\sum_{i=0}^{N} C_i e^{-B_i\sqrt{Var[X_t]}x^*} = 0$$

and such that $Sw(t)$ is *negative* for $x < x^*$ and *positive* for $x > x^*$.
The analytic value of the payer swaption is thus:

$$\text{Swaption} = \int_{x^*}^{\infty} \left(\sum_{i=0}^{N} C_i e^{-B_i\sqrt{Var[X_t]}x}\right) \phi(x) dx$$

$$\text{Swaption} = \sum_{i=0}^{N} C_i e^{\frac{1}{2}B_i^2 Var[X_t]} \Phi\left(-(x^* + B_i\sqrt{Var[X_t]})\right) \tag{2.3}$$

2.2 Swaption valuation in a multi-curve framework

In a multi-curve framework, the libor rates are estimated on an evaluation rate curve different from the discount rate curve used to discount the expected fixed and floating flows.
Notations for the libor rates estimated at date t and fixed at date T (omitting the libor tenor):

- $\tilde{L}(t, T)$ on the evaluation curve

- $L(t,T)$ on the discount curve

Let K be the number of floating flows paid during each fixed period. The floating leg value becomes

$$\begin{aligned}
\mathrm{FloatingLeg}(t) &= \sum_i^N \sum_k^K \alpha_{i,k}\tilde{L}(t,T_{i,k})P(t,T_{i,k}) \\
&= \sum_i^N \sum_k^K \alpha_{i,k}(\tilde{L}(t,T_{i,k}) - L(t,T_{i,k})P(t,T_{i,k})) + P(t,T_0) - P(t,T_N) \\
&= \sum_i^N \alpha_i \left(\sum_k^K \frac{\alpha_{i,k}(\tilde{L}(t,T_{i,k}) - L(t,T_{i,k})P(t,T_{i,k}))}{\alpha_i P(t,T_i)} \right) P(t,T_i) + P(t,T_0) - P(t,T_N) \\
&\approx \sum_i^N \alpha_i \left(\sum_k^K \frac{\alpha_{i,k}(\tilde{L}(0,T_{i,k}) - L(0,T_{i,k})P(0,T_{i,k}))}{\alpha_i P(0,T_i)} \right) P(t,T_i) + P(t,T_0) - P(t,T_N)
\end{aligned}$$

Where the last part has been obtained by freezing the libor and discount estimations at their time 0 expected values. The floating rates adjustments are noted $f_i = \sum_k^K \frac{\alpha_{i,k}(\tilde{L}(0,T_{i,k}) - L(0,T_{i,k})P(0,T_{i,k}))}{\alpha_i P(0,T_i)}$.
The swaption payoff may thus be written, in a multi-curve framework:

$$\begin{aligned}
Sw(t) &= \left(\mathrm{FloatingLeg}(t) - \sum_i \alpha_i K P(t,T_i) \right)^+ \\
&= \left(\sum_i^N \alpha_i f_i P(t,T_i) + P(t,T_0) - P(t,T_N) - \sum_i \alpha_i K P(t,T_i) \right)^+ \\
&= \left(P(t,T_0) - P(t,T_N) - \sum_i \alpha_i (K - f_i) P(t,T_i) \right)^+ \\
&= \left(\sum_{i=0}^N \tilde{C}_i e^{-B_i X_t} \right)^+
\end{aligned}$$

Where

$$\begin{aligned}
\tilde{C}_0 &= A_0 \\
\tilde{C}_i &= -\alpha_i (K - f_i) A_i, 0 < i < N \\
\tilde{C}_N &= -(1 + \alpha_N (K - f_N) A_N)
\end{aligned}$$

In a multi-curve framework, the $\tilde{C}_i$ are no longer necessarily all of the same sign up to a certain i, then the opposite sign. If they change sign several times, analytical formulae (2.2) or (2.3) cannot be used and the integral (2.1) needs to be estimated numerically (using Gauss-Legendre integration for instance).

2.3 Implementation details

During the mean reversion calibration (cf. section 3) or the Hull-White piecewise volatility calibration (cf. section 4), market volatilities of swaptions are used[2]. Their model values are computed using (2.1), (2.2) or (2.3) depending on the signs of the swaptions coefficients C_i used.

2.3.1 Analytic formula

Shall the C_i be of the same sign up to a certain i, then the opposite sign, (2.2) or (2.3) will be used.

Algorithm to compute x^*:

- Solver: bisection solver.
- Target: $Sw(x) = 0.0$
- Target precision: $1.0e^{-6}$
- First guess: 0.0
- Max number of iterations: 50
- Solver bounds: [-6;+6] extended to [-9;+9] if no solution is found (otherwise the solution which makes $Sw(x)$ the closest to 0 is kept).

2.3.2 Numerical integration

If the C_i change several times of sign, (2.1) is evaluated numerically using a Gauss-Legendre integration with 100 points between -6 and 6 standard deviations. A control variable is used to remove some noise due to the numerical integration: the difference between the analytic price and numeric price of a degenerate swaption close to the initial one is added to the numeric price of the original swaption.

$$\begin{aligned} Swaption = & \ Swaption^{NumericIntegration} \\ & + DegenerateSwaption^{Analytic} \\ & - DegenerateSwaption^{NumericIntegration} \end{aligned}$$

Let i_{min} be the first index at which the C_i coefficients of the original swaption change sign and i_{max} the last index at which the C_i change sign. The degenerate swaption is defined by the new coefficients $\tilde{C}_i$ where:

$$\begin{aligned} \tilde{C}_i &= C_i, i < i_{min}, i \geqslant i_{max} \\ \tilde{C}_i &= 0, i_{min} \leqslant i < i_{max} \end{aligned}$$

Assuming $\tilde{C}_0$ and $\tilde{C}_N$ are of opposite signs, the degenerate swaption can be computed analytically using (2.2) or (2.3).

[2]The calibrations are always performed on out of the money swaptions.

3 Best fit calibration

3.1 Objective

The objective of the best fit is to find the mean reversion parameter a that will best reprice a given calibration basket. Since the model is not entirely defined by a, we need to choose $\sigma(t)$ as well. During the best fit phase, σ is chosen to be constant and is determined as the minimum of an error function, for each possible value of a. The error function reflects how well the model reprices a given calibration basket:

$$\sum_{i \in \text{Best fit basket}} \left(\sigma_i^{\text{Bachelier}} - \sigma_i^{\text{Calibrated}}\right)^2,$$

where:

- $\sigma_i^{\text{Calibrated}}$ is the implied *normal* volatility corresponding to the *model* price of the instrument,
- $\sigma_i^{\text{Bachelier}}$ is the implied *normal* volatility corresponding to the *market* price of the instrument (if a normal vol is quoted, it is this market vol, if a log-normal vol is quoted, it is implied from the market price).

So, chronologically:

1. For each possible value of the mean reversion a, a minimization is done to find the constant value of σ.
2. For a given a and a given σ, each instrument in the calibration basket is evaluated via the model.
3. For each instrument premium, the associated implied *normal* volatility is computed.
4. The error function accumulates the errors made on all instruments' implied volatilities.

3.2 Instruments used

Swaptions are used, the user is able to calibrate the mean reversion on all instruments from the calibration basket or only on instruments from a given calibration level.

3.3 Implementation details

In order not to have several nested minimizations, a is not directly determined by a minimization. Instead, the search interval $I = [-.3, .3]$ is discretized in 61 points, and the error function is evaluated on each of these points. For each of the values of $a_i \in I$, the error function is minimized in σ. Then we look for the point in the discretization which gives the lowest error, and perform a quadratic expansion on it and and its two neighbours.

More formally, let $\mathrm{err}(a, \sigma)$ be the error function defined above, and let

$$a_{i^*} = \arg\min_{a_i \in I} \left(\min_{\sigma} \mathrm{err}(a_i, \sigma) \right),$$

then the result of the best fit calibration is given by the minimum a^* of the quadratic function $f(a)$ that interpolates a_{i^*-1}, a_{i^*}, and a_{i^*+1} at their corresponding errors ε_{i^*-1}, ε_{i^*}, and ε_{i^*+1}. It is given by

$$a^* = a_{i^*} - f'(a_{i^*})/f''(a_{i^*}),$$

which, when the discretization of I is uniform with a Δa step, simplifies into

$$a^* = a_{i^*} - \Delta a \frac{(\varepsilon_{i^*+1} - \varepsilon_{i^*-1})}{2\,(\varepsilon_{i^*+1} - 2\varepsilon_{i^*} + \varepsilon_{i^*-1})}.$$

If no bootstrap is to be performed, then σ (constant) needs to be computed once again with the final value of $a = a^*$:

$$\sigma^* = \arg\min_{\sigma} \left(\mathrm{err}(a^*, \sigma) \right).$$

Algorithm to compute the constant σ for a given mean reversion:

- Minimizer: brent.
- Target: $\sum_{i=1}^{N} (\sigma^{\mathrm{implied}}_{\mathrm{normal}}(\mathrm{Swaption}^{HW}_i(\sigma)) - \sigma^{\mathrm{market}}_{i,\mathrm{normal}})^2$
- Target precision: $1.0e^{-7}$
- First guess: 0.011
- Max number of iterations: 100
- Solver bounds: $[1.0e^{-7}, 0.1]$
- Implied normal vol solver: brent solver with a precision of $1.0e{-6}$

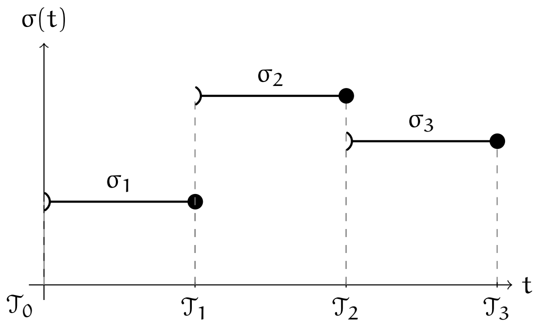
4 Bootstrap calibration

4.1 Objective

Once the constant mean-reversion parameter a has been found or chosen, the time-dependent volatility σ remains to be calibrated.

The idea of the boostrap calibration is to use instruments with different maturities $\mathcal{T}_i$, and to determine the value of σ on each interval successively. This way, even though σ takes as many values as there are instruments in the bootstrap calibration basket, each minimization is still one-dimensional. This enables a far more efficient calibration than a minimization on all values at once. However, the main drawback is that in some cases, earlier instruments will be better matched than later ones, for no good reason.

So let $\mathcal{T}_i$ be the maturity of the i-th instrument, and $\mathcal{T}_0 = 0$. The volatility function $\sigma(t)$ is chosen to be piecewise constant: let σ_i be its value on the interval $]\mathcal{T}_{i-1}, \mathcal{T}_i]$. The value σ_i will be directly determined so that the model exactly reprices instrument i (when possible).



4.2 Instruments used

As in the best-fit calibration, vanilla options on a sum of bonds maturing at the same expiry date may be used. However, for stability reasons, the following instruments will not be used:

- if their market value is below 0.1bp;
- if their market vega (approximately computed) is below 0.001bp.

4.3 First guess

The idea of the first guess is to use an approximation of the model price of the current instrument (i) as a function of σ_i, and to find the value of σ_i for which the approximated model price will be equal to the market price. For this we use an approximation of swaption prices by Schrager and Pelsser [3].

4.3.1 Swaption price approximation

Let S_t be a swap rate defined by

$$S_t = \frac{\sum_{i=1}^{N} \alpha_i f_i P(t,T_i) + P(t,T_0) - P(t,T_N)}{\sum_{i=1}^{N} \alpha_i P(t,T_i)} = \frac{\sum_{i=1}^{N} \alpha_i f_i P(t,T_i) + P(t,T_0) - P(t,T_N)}{\hat{P}_t},$$

then assuming the absence of arbitrage, the swap rate is a martingale under the swap measure, so it has zero drift, and the application of Itō's formula yields

$$
\begin{aligned}
dS_t &= (\dots)dt + \frac{\partial S_t}{\partial X_t} dX_t \\
&= \left(\frac{-\sum_{i=1}^{N} \alpha_i f_i B(t,T_i)P(t,T_i) - B(t,T_0)P(t,T_0) + B(t,T_N)P(t,T_N)}{\hat{P}_t} \right) * \sigma(t) dW_t^P \\
&+ \left(S_t \sum_{i=1}^{N} \alpha_i B(t,T_i) \frac{P(t,T_i)}{\hat{P}_t} \right) * \sigma(t) dW_t^P \\
&= (-B(t,T_0)\frac{P(t,T_0))}{\hat{P}_t} + B(t,T_N)\frac{P(t,T_N)}{\hat{P}_t} + \sum_{i=1}^{N} \alpha_i (S_t - f_i) B(t,T_i) \frac{P(t,T_i)}{\hat{P}_t}) * \sigma(t) dW_t^P \\
&\approx \left(-B(t,T_0)\frac{P(0,T_0)}{\hat{P}_0} + B(t,T_N)\frac{P(0,T_N)}{\hat{P}_0} + \sum_{i=1}^{N} \alpha_i (S_0 - f_i) B(t,T_i) \frac{P(0,T_i)}{\hat{P}_0} \right) * \sigma(t) dW_t^P \\
&= \underbrace{\left(\frac{e^{-aT_0}}{a} \frac{P(0,T_0)}{\hat{P}_0} - \frac{e^{-aT_N}}{a} \frac{P(0,T_N)}{\hat{P}_0} - \sum_{i=1}^{N} \alpha_i (f_i - S_0) \frac{e^{-aT_i}}{a} \frac{P(0,T_i)}{\hat{P}_0} \right)}_{C(a)} e^{at} \sigma(t) dW_t^P,
\end{aligned}
$$

where the $\frac{P(t,T_i)}{\hat{P}_t}$ terms are approximated by $\frac{P(0,T_i)}{\hat{P}_0}$, their conditional expectation under the swap measure.

Under this approximation, S_T is a *normal* random variable with mean S_0 and variance $C(a)^2 V(T)$, where

$$V(T) = \int_0^T \sigma^2(s) e^{2as} ds.$$

Let us now consider our i-th instrument, which we suppose to be a swaption maturing at $T = \mathcal{T}_i$. We know that its payoff can be easily expressed as an expectation under the swap measure:

$$\hat{P}_0 \mathbb{E}^P \left[(S_{\mathcal{T}_i} - K)^+ \right].$$

Since $S_{\mathcal{T}_i}$ is approximately normal under this measure, we can use the Bachelier formula[3] to approximate its model premium (Premium_{HW}). Since we already know its market premium (Premium_M), the idea is to find the value of σ_i for which the approximated model premium will match the market premium. Equivalently, we can find the value of σ_i for which the approximated model normal variance will match the market implied normal variance.

4.3.2 Volatilitiy first guess

We seek to find the value of σ_i such that

$$\text{Variance}^i_M = \text{Variance}^i_{HW}. \tag{4.1}$$

[3] cf. appendix A.3

with, for the maturity $\mathcal{T}_i$ of the i^{th} calibration instrument:

$$\text{Variance}_M^i = (\sigma_M^i)^2 * \mathcal{T}_i. \tag{4.2}$$

and

$$\text{Variance}_{HW}^i \approx C(a)^2 * V(\mathcal{T}_i). \tag{4.3}$$

Let us compute $V(\mathcal{T}_i)$:

$$\begin{aligned} V(\mathcal{T}_i) &= \sum_{k=1}^{i} \int_{\mathcal{T}_{k-1}}^{\mathcal{T}_k} \sigma_k^2 e^{2as} ds \\ &= \sum_{k=1}^{i} \sigma_k^2 \frac{e^{2a\mathcal{T}_k} - e^{2a\mathcal{T}_{k-1}}}{2a} \\ &= V(\mathcal{T}_{i-1}) + \sigma_i^2 \frac{e^{2a\mathcal{T}_i} - e^{2a\mathcal{T}_{i-1}}}{2a}. \end{aligned}$$

At the i-th step of the bootstrap, all values before σ_i have been determined, so in other words, $V(\mathcal{T}_{i-1})$ is fixed. Supposing that the value $V(\mathcal{T}_i)$ such that equation (4.1) is respected is bigger than $V(\mathcal{T}_{i-1})$, we can then choose σ_i accordingly:

$$\begin{aligned} \sigma_i^2 &= \frac{2a}{e^{2a\mathcal{T}_i} - e^{2a\mathcal{T}_{i-1}}} \left(V(\mathcal{T}_i) - V(\mathcal{T}_{i-1})\right) \\ &= \frac{2a}{e^{2a\mathcal{T}_i} - e^{2a\mathcal{T}_{i-1}}} \left(\frac{(\sigma_M^i)^2 \mathcal{T}_i}{C(a)^2} - \sum_{k=1}^{i-1} \sigma_k^2 \frac{e^{2a\mathcal{T}_k} - e^{2a\mathcal{T}_{k-1}}}{2a} \right). \end{aligned}$$

Remark. *In case the right hand side is negative, there is a good chance that there isn't enough variance left to match the instrument. Previous first guess is used.*

4.4 Implementation details

Algorithm to compute σ_i for which the instrument i will be repriced by the model:

- Solver: Newton-Raphson method, coupled with a bisection in case it fails.
- Target: $Swaption_{HW} - Swaption_{market} = 0.0$
- Target precision: $1.0e^{-9} * \max(1, 10 * Vega_{market})$
- Max number of iterations: 80
- Derivative: computed by finite difference with an upside shift of 0.001.
- First guess: $\sigma_{Schrager}$ for the first maturity, σ_{i-1} for the following ones.
- Lower bound: 0.0000001 for the first instrument, $0.1 * \max_{k<i} \sigma_k$ for the following ones.
- Upper bound: $10 * \sigma_{Schrager}$ for the first instrument, $10 * \sigma_{i-1}$ for the following ones.

A Ornstein-Uhlenbeck processes

In this section, X is an Ornstein-Uhlenbeck process defined as in (1.2).

A.1 Law of X

Using Itō's formula

$$\begin{aligned} d(e^{at}X_t) &= e^{at}\left(dX_t + aX_t\,dt\right) \\ &= e^{at}\sigma_t\,dW_t. \end{aligned}$$

Integrating between t and T gives

$$e^{aT}X_T = e^{at}X_t + \int_t^T e^{as}\sigma_s\,dW_s$$

$$X_T = e^{-a(T-t)}X_t + \int_t^T e^{-a(T-s)}\sigma_s\,dW_s, \tag{A.1}$$

in particular, when $t = 0$,

$$X_T = \int_0^T e^{-a(T-s)}\sigma_s\,dW_s, \tag{A.2}$$

which proves that X is a martingale on any finite time interval, at least for a piecewise constant σ. In particular, it is adapted to the filtration $\mathcal{F}$ of W.

Returning to the general case $t \leqslant T$, we then deduce that X_T conditional on $\mathcal{F}_t$ is Gaussian, with the following mean and variance

$$X_T|\mathcal{F}_t \sim \mathcal{N}\left(e^{-a(T-t)}X_t, \int_t^T e^{-2a(T-s)}\sigma_s^2\,ds\right) \tag{A.3}$$

A.2 Law of integrated X

Let $t \leqslant T$, then integrating (A.1) between t and T,

$$\begin{aligned} \int_t^T X_u\,du &= \int_t^T e^{-a(u-t)}X_t\,du + \int_t^T \int_t^u e^{-a(u-s)}\sigma_s\,dW_s\,du \\ &= \frac{1-e^{-a(T-t)}}{a}X_t + \int_t^T \left(\int_s^T e^{-a(u-s)}\,du\right)\sigma_s\,dW_s \\ &= \frac{1-e^{-a(T-t)}}{a}X_t + \int_t^T \frac{1-e^{-a(T-s)}}{a}\sigma_s\,dW_s, \end{aligned}$$

where the Fubini-like interversion is legitimate since the integrand is deterministic and bounded, but the result can be proven in a much more general context using integration by parts. See for example [1].

From this we deduce that $\int_t^T X_u\,du$ conditional on $\mathcal{F}_t$ is Gaussian, with the following mean and variance

$$\int_t^T X_u\,du\bigg|\mathcal{F}_t \sim \mathcal{N}\left(\frac{1-e^{-a(T-t)}}{a}X_t, \int_t^T \left(\frac{1-e^{-a(T-s)}}{a}\right)^2\sigma_s^2\,ds\right). \tag{A.4}$$

A.3 Bachelier formula

Let S_T be a normal random variable with mean S_0 and variance Σ^2, and suppose we wish to compute the expectation of a call of strike K.

$$\begin{aligned}
\mathbb{E}\left[(S_T - K)^+\right] &= \int_K^\infty (s-K)\,\frac{e^{\frac{-(s-S_0)^2}{2\Sigma^2}}}{\sqrt{2\pi}\Sigma}\,ds \\
&= \int_{\frac{K-S_0}{\Sigma}}^\infty (S_0 + \Sigma z - K)\,\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,dz \\
&= \int_{\frac{K-S_0}{\Sigma}}^\infty \Sigma z\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,dz + \int_{\frac{K-S_0}{\Sigma}}^\infty (S_0 - K)\,\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,dz \\
&= \frac{\Sigma}{\sqrt{2\pi}}\left[-e^{\frac{-z^2}{2}}\right]_{\frac{K-S_0}{\Sigma}}^\infty + (S_0 - K)\left[1 - \Phi\left(\frac{K-S_0}{\Sigma}\right)\right],
\end{aligned}$$

and so finally, we get the Bachelier formula:

$$\mathbb{E}\left[(S_T - K)^+\right] = \frac{\Sigma}{\sqrt{2\pi}}\exp\left(\frac{-(S_0-K)^2}{2\Sigma^2}\right) + (S_0 - K)\,\Phi\left(\frac{S_0-K}{\Sigma}\right). \tag{A.5}$$

In the common case of an ATM option, the strike K is equal to S_0. Under this condition, the formula simplifies to

$$\mathbb{E}\left[(S_T - S_0)^+\right] = \frac{\Sigma}{\sqrt{2\pi}}. \tag{A.6}$$

References

[1] D. Brigo and F. Mercurio, *Interest rate models: theory and practice: with smile, inflation, and credit*, Springer Verlag, 2006.

[2] Henrard, *Efficient swaptions price in hull-white one factor model*, (2009).

[3] D. Schrager and A. Pelsser, *Pricing swaptions and coupon bond options in affine term structure models*, Mathematical Finance, 16 (2006), pp. 673–694.