CMS Replication

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Contents

1 Pricing of CMS products by static replication

By observating the similarity between a CMS caplet and a payer swaption, we would like to demonstrate how to perform the static replication of a CMS caplet using a portfolio of payer swpations with discretese strikes.

First we fix the mathematical notations for the tenor Structure, bond price function and swap rate. The spot time is taken to be time 0 and the tenor structure of the reference swap rate is assumed to be $\{T_0,T_1,...,T_n\}$.

Here T_0 is the start date of the swap and $\{T_i, i = 1,...,n\}$ are the payment dates. Wet let $\delta_i = T_i - T_{i-1}$ be the year fratcion of the time interval $[T_{i-1}, T_{i-1}]$, and write δ for all δ_i if a constant year fraction is assumed.

The time-t price of the maturity T_i discount bond is denoted by : $P(t, T_i)$.

The annuity stream is denoted by: $A(t) = \sum_{i=1}^{n} \delta_i P(t, T_i)$.

The swap rate is denoted by: $S(t) = \frac{P(t,T_0)-P(t,T_n)}{A(t)}$

The most basic CMS derivative is the CMS caplet whose payoff on the payment date T_p is given by:

$$Payoff\_Cap = P(T, T_p)(S(T) - K)^+$$

We would like to illustrate how to replicate the caplet payment on the payment swaptions with discrete strikes $K + m\Delta k$, $m = 0, 1, 2, ...$ and Δk representes a small increment on the strike rate starting from K.

Let W_m denote the notional amount of the payer swaption with strike $K + m\Delta k$, $m = 0, 1, 2, ...$ We illustrate how to determine W_i successively in order that the caplet payoff at T_p agrees with that of the replicating portfolio of payer swaptions under verious scenarios of the observed swap rate T_0, that is, realized value of $S(T)$.

When $S(t) \leq K$, the caplet has zero payoff and all the payer swaptions in the replicating portfolio are note in-the-money, so matching of the payoffs is achieved. Next we determine the notional amount of each of the payer swaptions successively by matching the payoffs of the caplet and replicating portfolio at various possible discrete values assumed by $S(T)$.

First, suppose $S(T) = K + \Delta k$, the caplet's payoff is Δk at T_p and the corresponding discounted value at T is $\Delta k P(T, T_p)|_{S(T)=K+\Delta k}$. On the other hand, only the payer swaptions with strike rate K is in-the-money while all the other swaptions with higher strike rate have zero payoff. The time-T payoff of the payer swaption with strike rate K when $S(T) = K + \Delta k$ is given in the form of an annuity $\Delta k \sum_{i=1}^{n} \delta_i P(T, T_i)|_{S(T)=K+\Delta k}$. To achieve matching of the payoffs of the caplet and replicating portfolio when $S(T) = K + \Delta k$, the notional amount W_0 just be set uniquely equal to the following bond-annuity ratio:

$$W_0 = \left. \frac{P(T, T_p)}{\sum_{i=1}^{n} \delta_i P(T, T_i)} \right|_{S(T)=K+\Delta k}$$

Naturally, the above bond-annuity ratio exhibits dependence of the swap rate $S(T)$. Similar to Hagan, we write formally the functional dependence of the bond annuity ratio on $S(T)$ in the form:

$$G(S(t)) = \frac{P(T, T_p)}{\sum_{i=1}^{n} \delta_i P(T, T_i)}$$

Accordingly, we may express the notional of the payer swpation with strike rate K in terms of G as follows:

$$W_0 = G(K + \Delta k)$$

Next, we determine the notional amount W_1 of the payer swaption with strike $K + \Delta k$ by matching the payoffs when $S(T)$ assumes the value $K + 2\Delta k$. Under such scenario, only two swaptions with respective strike K and $K + \Delta k$ are in-the-money so that:

$$(2W_o + W_1)\Delta k \sum_{i=1}^{n} \delta_i P(T, T_i)|_{S(T)=K+2\Delta k} = 2\Delta k P(T, T_p)|_{S(T)=K+2\Delta k}$$

giving

$$W_1 = 2[G(K + 2\Delta k) - G(K + \Delta k)]$$

In general, by matching the payoffs of the caplet and the replicating portfolio when $S(T) = K + (m + 1)\Delta k$, we have:

$$W_m = (m + 1)G(K + (m + 1)\Delta k) - 2mG(K + m\Delta k) + (m - 1)G(K + (m - 1)\Delta k)$$

Let $C_0(K)$ denote the time-0 value of the payer swaption with strike rate K and V_0^{caplet} denote the time-0 value of the CMS caplet. Since the payoff of the replication portfolio agrees with that of the caplet at discrete strikes according to this approximate static replication procedure, by applying the no arbitrage principle, the fair value of the CMS caplet is :

$$V_0^{caplet} = W_0 C_0(K) + \sum_{m>0} W_m C_0(K + m\Delta k)$$

2 Terminal Swap Rate Models

The TSR approach treats the swap rate $S(T)$ as the single fundamental state variable for the yield curve at time T. a TSR model specifies a map:

$$P(T, M) = \pi(S(T), M), \ \ M \geq T$$

In other words, each discount factor is assumed to be a deterministic , kwnon function of the swap rate.

In a proper term structure model, the relationship between the market rate $S(T)$ and the discount factor $P(T, M)$ emerges from the model itself, and is ultimately derived from no-arbitrage conditions. While now we seek to impose the function relationship exogenously, consideration of no-arbitrage must play a role.

Indeed, a first condition to be imposed "zero-coupon condtion" is :

$$P(0, M) = A(0)E^{A}\left(\frac{\pi(S(T), M)}{\sum_{i=1}^{n}\delta_i\pi(S(T), T_i)}\right)$$

the no-arbitrage condition by itself is not sufficient to obtain a workable model. Another restriction on the mapping functions is obtained by observating the swap rate $S(T)$ itself is a function of discount factors. This suggests the introduction of a "consistency condition", i.e. the requirement that the following holds for all x,

$$x = \frac{\pi(x, T_0) - \pi(x, T_n)}{\sum_{i=1}^{n}\delta_i\pi(x, T_i)}$$

The final condition that we impose on a TSR model is that the set of functions $\pi(., M)$ sould be reasonable. While somewhat harder to quantify that the other conditions, we shall mostly impose the following restrictions:

$$\begin{aligned} 0 &< \pi(x, M) \leq 1 \\ M_1 &< M_2 => \pi(x, M_1) \geq \pi(x, M_2) \end{aligned}$$

For example the different models of mapping present by Hagan(2003) does not respect the first condition (zero-coupon condition) and consequenlty he breaks the call-put parity.

Indeed :

$$CMS\_Cap = E[D(0,T)P(T,T_p)(S(T)-K)^+]$$

$$CMS\_Cap = A(0)E^A\left[\frac{P(T,T_p)(S(T)-K)^+}{A(T)}\right]$$

Moreover:

$$E^A\left[\frac{P(T,T_p)}{A(T)}\right] = \frac{P(0,T_p)}{A(0)}$$

Consequenlty:

$$\begin{aligned} CMS\_Cap &= P(0,T_p)E^A\left[\frac{P(T,T_p)(S(T)-K)^+}{A(T)}\frac{A(0)}{P(0,T_p)}\right] \\ &= P(0,T_p)E^A\left[(S(T)-K)^+\right] + P(0,T_p)E^A\left[(S(T)-K)^+\left(\frac{P(T,T_p)}{A(T)}\frac{A(0)}{P(0,T_p)}-1\right)\right] \end{aligned}$$

in a same way:

$$CMS\_Floor = P(0,T_p)E^A\left[(-S(T)+K)^+\right] + P(0,T_p)E^A\left[(-S(T)+K)^+\left(\frac{P(T,T_p)}{A(T)}\frac{A(0)}{P(0,T_p)}-1\right)\right]$$

Consequenlty :

$$\begin{aligned} CMS\_Cap - CMS\_Floor &= P(0,T_p)E^A[S(T)] - P(0,T_p)K \\ &\quad + P(0,T_p)E^A\left[S(T)\left(\frac{P(T,T_p)}{A(T)}\frac{A(0)}{P(0,T_p)}-1\right)\right] \\ &\quad + P(0,T_p)E^A\left[\left(\frac{P(T,T_p)}{A(T)}\frac{A(0)}{P(0,T_p)}-1\right)\right] \end{aligned}$$

However, as the model proposed by Hagan does not satisfy :

$$P(0,M) = A(0)E^A\left(\frac{\pi(S(T),M)}{\sum_{i=1}^n \delta_i \pi(S(T),T_i)}\right)$$

the term :

$$E^A\left[\left(\frac{P(T,T_p)}{A(T)}\frac{A(0)}{P(0,T_p)}-1\right)\right] \neq 0$$

and so:

$$CMS\_Cap - CMS\_Floor = CMS\_Swap - KX$$

$$X \neq P(0,T_p)$$

3 Linear TSR Model

The linear TSR model is obtained by specifying

$$\frac{\pi(x,M)}{\sum_{i=1}^{n}\delta_i\pi(x,T_i)} = a(M)x + b(M),\ \ M \geq T$$

for deterministic functions a(.) and b(.). The no-arbitrage condition requires

$$P(0,M) = A(0)E^{A}[a(M)S(T) + b(M)),$$

implying a condition on the free coefficient b(.),

$$b(M) = \frac{P(0,M)}{A(0)} - a(M)S(0)$$

The consistency condtion requires that:

$$\begin{aligned}
x &= \frac{\pi(x,T_0) - \pi(x,T_n)}{\sum_{i=1}^{n}\delta_i\pi(x,T_i)} = a(T_0)x + b(T_0) - a(T_n)x - b(T_n) \\
b(T_0) &= b(T_n) \\
a(T_0) &= 1 + a(T_n)
\end{aligned}$$

To connect a(.) to mean reversion, we interpret the linear TSR model as defining a(M) via:

$$a(M) = \frac{\partial}{\partial S(T)}\frac{P(T,M)}{\sum_{i=1}^{n}\delta_i P(T,T_i)}$$

which we rewrite, in the context of a Gaussian one-factor model as :

$$a(M) = \frac{\partial}{\partial x}\frac{P(T,M)}{\sum_{i=1}^{n}\delta_i P(T,T_i)}\bigg|_{S(T,x)=S(0)} * \left(\frac{\partial S(T,x)}{\partial x}\bigg|_{S(T,x)=S(0)}\right)^{-1}$$

where x is now the short rate in the Gaussian model on which all discount bonds and swap rates depend. We denote by $A(T,x)$ the annuity as the function of the short rate x,

$$\begin{aligned}
A(T,x) &= \sum_{i=1}^{n}\delta_i P(T,T_i,x) \\
S(T,x) &= \frac{P(T,T_0) - P(T,T_n)}{A(T,x)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{P(T,M,x)}{A(T,x)} &= \frac{-P(T,M,x)B(a,T,M)}{A(T,x)} - \frac{P(T,M,x)}{A(T,x)^2}\frac{\partial A(T,x)}{\partial x} \\
\frac{\partial}{\partial x}S(T,x) &= \frac{P(T,T_n)B(a,T,T_n) - P(T,T_0)B(a,T,T_0)}{A(T,x)} - \frac{S(T,x)}{A(T,x)}\frac{\partial A(T,x)}{\partial x} \\
B(a,t,T) &= \frac{1 - e^{-a(T-t)}}{a}
\end{aligned}$$

$$\begin{aligned}\gamma &= -\frac{1}{A(T,x)}\frac{\partial A(T,x)}{\partial x} \\ &= \frac{\sum_{i=1}^{n}\delta_i P(T,T_i,x)B(a,T,T_i)}{\sum_{i=1}^{n}\delta_i P(T,T_i,x)}\end{aligned}$$

We obtain:

$$\begin{aligned}a(M) &= \frac{P(0,M)(\gamma - B(a,T,M))}{P(0,T_n)B(a,T,T_n) - P(0,T_0)B(a,T,T_0) + A(0)S(0)\gamma} \\ B(a,t,T) &= \frac{1-e^{-a(T-t)}}{a} \\ \gamma &= \frac{\sum_{i=1}^{n}\delta_i P(0,T_i)B(a,T,T_i)}{\sum_{i=1}^{n}\delta_i P(0,T_i)}\end{aligned}$$

4 Static Replication with Linear TSR Model

We use the Linear TSR Model for the CMS static replication.

In a first time we compute a(.) for a the payment date of the fix leg $\{T_i, i = 1, ..., n\}$, b(.) is obtained by "zero-coupon condition"

$$b(M) = \frac{P(0,M)}{A(0)} - a(M)S(0)$$

We compute also :

$$\begin{aligned}b(T_0) &= b(T_n) \\ a(T_0) &= 1 + a(T_n)\end{aligned}$$

In the static replication we need to have:

$$G(S(t)) = \frac{P(T,T_p)}{\sum_{i=1}^{n}\delta_i P(T,T_i)} = a(T_p)S(t) + b(T_p)$$

$a(T_p)$ is obtained by the previous formula on $a(M)$ and

$$b(T_p) = \frac{P(0,T_p)}{A(0)} - a(T_p)S(0)$$

Finally, the weights W_i are immediatly for the CMS Cap in this model:

$$\begin{aligned}W_0 &= a(T_p)(K + \Delta k) + b(T_p) \\ W_m &= G'(S(t)) = 2a(T_p)\Delta k,\ m > 0\end{aligned}$$

With this model we have the call-put parity because all the condition are satisfied

5 MultiCurve Case

in the case of the Multicurve we have :

$$S(T) = \frac{P(T,T_0) - P(T,T_n)}{A(T)} + \frac{\displaystyle\sum_{i=1}^{n} c_i \delta_i P(T,T_i)}{A(T)}$$

New Consistency condition :

$$\begin{aligned}
x &= \frac{\pi(x,T_0) - \pi(x,T_n)}{\sum_{i=1}^{n} \delta_i \pi(x,T_i)} + \frac{\displaystyle\sum_{i=1}^{n} c_i \delta_i \pi(x,T_i)}{\sum_{i=1}^{n} \delta_i \pi(x,T_i)} = a(T_0)x + b(T_0) - a(T_n)x - b(T_n) + \sum_{i=1}^{n} c_i \delta_i (a(T_i)x + b(T_i)) \\
b(T_0) &= b(T_n) - \sum_{i=1}^{n} c_i \delta_i b(T_i) \\
a(T_0) &= 1 + a(T_n) - \sum_{i=1}^{n} c_i \delta_i a(T_i)
\end{aligned}$$

New value for $a(T)$:

$$\begin{aligned}
a(M) &= \frac{P(0,M)(\gamma - B(a,T,M))}{P(0,T_n)B(a,T,T_n) - P(0,T_0)B(a,T,T_0) - \displaystyle\sum_{i=1}^{n} c_i \delta_i P(0,T_i) B(a,T,T_i) + A(0)S(0)\gamma} \\
B(a,t,T) &= \frac{1 - e^{-a(T-t)}}{a} \\
\gamma &= \frac{\sum_{i=1}^{n} \delta_i P(0,T_i) B(a,T,T_i)}{\sum_{i=1}^{n} \delta_i P(0,T_i)}
\end{aligned}$$

References

[hagan] CONVEXITY CONUNDRUMS: PRICING CMS SWAPS, CAPS, AND FLOORS, P.Hagan

[AP] Interest rate modeling Volume III: Products and Risk Management