CMS Replication - Linear swap rate model

Details of the new resolution method and implementation choices

Macs team – 06/05/2015

Table of contents

I. Recall: CMS replication principle

We consider the pricing of a CMS option (caplet, floorlet, digital cap, digital floor) with strike K, maturity T and payment date T_p. We will denote by:

- F_t : the swap rate seen at time t, fixed at time T
- S_T : the swap rate seen at time T, fixed at time T
- $A(t)$: the annuity seen at time t, for a swap starting at time T
- $Z\left(t, T_p\right)$: the value at time t of a stochastic bond with maturity date T_p

The payoff of a CMS caplet with strike K can be replicated with a basket of European payer swaptions for different strikes chosen in a range $[K\ ;\ K_{max}]$ where K_{max} is the maximum strike.

Similarly the payoff of a CMS floorlet with strike K can be replicated with a basket of European receiver swaptions for different strikes chosen in a range $[K_{min}; K]$ where K_{min} is the minimum strike.

The way the strikes K_{min} and K_{max} are computed and the way the set of strikes used in the replication basket are chosen is described in section "A ***Replication basket"***.

The value of a CMS option is computed by pricing the equivalent replication portfolio of swaptions. The weights of the swaptions with different strikes are computed such that at option maturity, the CMS option's intrinsic value matches the replication portfolio's intrinsic value.

We showed in [1] that the analytical formula standing for the weights independently of the swap rate model chosen is, for a CMS caplet[1]:

$$\omega_i = \frac{G(K_{i+1})(K_{i+1} - K)_+ - G(K_i)(K_i - K)_+}{K_{i+1} - K_i} - \frac{G(K_i)(K_i - K)_+ - G(K_{i-1})(K_{i-1} - K)_+}{K_i - K_{i-1}}$$

Where $G(S_T) = \frac{Z(S_T)}{A(S_T)} \approx \frac{Z\left(T, T_p\right)}{A(T)}$ depends on the swap rate model chosen.

[1] When payer swaptions are used to replicate the payoff

II. Linear swap rate model

A. Motivations

At maturity we wish to have for a CMS caplet replicated with payer swaptions:

$$Z(T,T_p)(S_T-K)_+ = \sum_{i=0}^{N-1} \omega_i A(T)(S_T-K_i)_+ \qquad ,S_T \in (K_i)_{1\le i\le N}$$

Which may be written:

$$\frac{Z(T,T_p)}{A(T)}(S_T-K)_+ = \sum_{i=0}^{N-1} \omega_i (S_T-K_i)_+ \qquad ,S_T \in (K_i)_{1\le i\le N}$$

The caplet valuation under the annuity measure gives:

$$Caplet(t) = A(t)\mathbb{E}^A\left[\frac{Z(T,T_p)}{A(T)}(S_T-K)_+\right] = A(t)\mathbb{E}^A\left[\sum_{i=0}^{N-1} \omega_i (S_T-K_i)_+\right]$$

In order to compute the weights $(\omega_i)_{0\le i\le N-1}$ we approximate the quantity $\frac{Z(T,T_p)}{A(T)}$ by the function of the swap rate $G(S_T) \approx \frac{Z(T,T_p)}{A(T)}$, where $G(S_T)$ is given by a swap rate model.

The CMS caplet value becomes:

$$Caplet(t) = A(t)\mathbb{E}^A[G(S_T)(S_T-K)_+]$$

Similarly, the CMS floorlet value becomes:

$$Floorlet(t) = A(t)\mathbb{E}^A[G(S_T)(K-S_T)_+]$$

The CMS swaplet value is then:

$$Swaplet(t) = A(t)\mathbb{E}^A[G(S_T)(S_T-K)]$$
$$Swaplet(t) = A(t)\mathbb{E}^A[G(S_T)S_T] - KA(t)\mathbb{E}^A[G(S_T)]$$

Under the $Z(t,T_p)$ numeraire, the swaplet value is:

$$Swaplet(t) = Z(t,T_p)\mathbb{E}^{T_p}[(S_T-K)]$$
$$Swaplet(t) = Z(t,T_p)\mathbb{E}^{T_p}[(S_T)] - Z(t,T_p)K$$

Thus in order to make sure that call-put parity is respected, we need to use a swap rate model where:

$$KA(t)\mathbb{E}^A[G(S_T)] = Z(t,T_p)K$$
$$\mathbb{E}^A[G(S_T)] = \frac{Z(t,T_p)}{A(t)}$$

B. Linear swap rate model

The swap rate model implemented in MACS and satisfying this requirement is the linear swap rate model described in [2].

Its expression is:

$$G(S_T) = aS_T + b$$

Where a and b are chosen such that $\mathbb{E}[aS_T + b] = \frac{Z(t,T_p)}{A(t)}$

More details on the estimation of a and b can be found in [3].

III. Implementation choices: caplets and floorlets

A. Replication basket

1. Choice of bounds for the strikes

A caplet with strike K is priced by using a portfolio of payer swaptions with strikes contained in the range $[K; K_{max}]$ replicating the payoff when the swap rate is contained in the range $[K; S_{max}]$.

A floorlet with strike K is priced by using a portfolio of receiver swaptions with strikes contained in the range $[K_{min}; K]$ replicating the payoff when the swap rate is contained in the range $[S_{min}; K]$.

The lower and upper bounds are defined for a CMS option (floorlet and caplet) as:

$$S_{min} = (F_t + \Delta)e^{-\mathcal{N}^{-1}(\alpha)\sigma_K\sqrt{T}-\frac{1}{2}\sigma_K^2 T} - \Delta$$
$$S_{max} = (F_t + \Delta)e^{\mathcal{N}^{-1}(\alpha)\sigma_K\sqrt{T}-\frac{1}{2}\sigma_K^2 T} - \Delta$$

When F_t follows a shifted log-normal distribution (the log-normal distribution corresponds to the specific case $\Delta = 0$), or when F_t follows a normal distribution:

$$S_{min} = F_t - \mathcal{N}^{-1}(\alpha)\sigma_K\sqrt{T}$$
$$S_{max} = F_t + \mathcal{N}^{-1}(\alpha)\sigma_K\sqrt{T}$$

With $\alpha = 0.999999995$, which corresponds to approximately 5.73039 standard deviations.

These choices of $S_{min}$and S_{max} enable the replicated CMS floorlet and caplet values to converge reasonably well towards their expected analytical value in the Black Scholes world (the analytical values of caplets and floorlets in the BS world with the linear swap rate model are detailed in [4]).

2. Choice of strikes between the bounds

Let N be the number of replication points inserted by the user to replicate a CMS option.

In the shifted log-normal volatility case, a fixed grid of strikes spread log-uniformly between $S_{min} + \Delta$ and $S_{max} + \Delta$ is built, with step $K_{step} = \frac{\log(S_{max}+\Delta)-\log(S_{min}+\Delta)}{N}$. K_{min} and K_{max} are respectively the first strike after $S_{min} + \Delta$ and the last strike before $S_{max} + \Delta$ (so we also have: $K_{step} = \frac{\log(K_{max}+\Delta)-\log(K_{min}+\Delta)}{N-2}$).

The real strikes used for the replication are then $K_{min} + i * K_{step}$, with $i \in [\![0, N-2]\!]$.

In the normal volatility case, the grid of strikes is spread uniformly between S_{min} and S_{max}, with step $K_{step} = \frac{S_{max}-S_{min}}{N}$.

The strikes used to replicate a CMS option are the strike of the option and all the grid's strikes above for a caplet (respectively the strike and all the grid's strikes below for a floorlet).

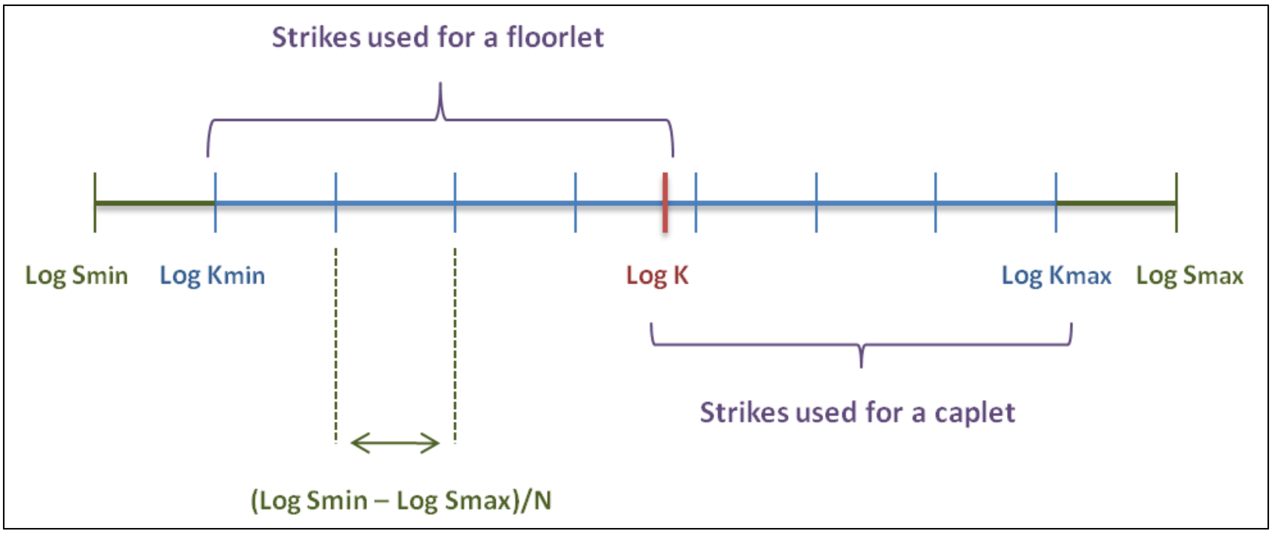

Figure 1 Strike discretization in the log-normal case

The effective number of steps used to spread the strikes in the range of strikes chosen is thus equal to:

- $N_{eff}^{caplet} = \left[\frac{\log(K_{max}+\Delta)-\log(K+\Delta)}{K_{step}}\right]_{+} + 1$ for caplets
- $N_{eff}^{floor} = \left[\frac{\log(K+\Delta)-\log(K_{min}+\Delta)}{K_{step}}\right]_{-} + 1$ for floorlets

Spreading the strikes log-uniformly enables to use more swaptions around the strikes which are closer to the money and gives better convergence results towards the theoretical BS value of CMS options than when the strikes are spread uniformly.

The effective replication number N_{eff} enables to value floorlets and caplets with the same level of accuracy for a given user inserted replication number N. Indeed, more points are required to value a caplet with the same level of accuracy as a floorlet because the upper bound S_{max} is a proxy for $+\infty$ (and therefore $S_{max} - K \gg K - S_{min}$). It makes therefore more sense to modify N_{eff} such that the log-strikes are equally spaced when a floorlet or a caplet is priced.

The choice of a fixed grid enables to obtain smooth prices when pricing several options where only the deal strike changes.

3. Special cases: strikes outstide the K_{min} and K_{max} bounds

Options with strike K contained in the range $[S_{min}; K_{min}]$ (respectively $[K_{max}; S_{max}]$) are valued by replacing the strike K_{min}(respectively K_{max}) by the strike K (and consequently only one replication point is used).

An option with a strike above S_{max} is valued as:

- An option with strike S_{max} if it is out of the money (caplet)
- The sum of an option with strike S_{max} and $K - S_{max}$ if it is in the money (floorlet)

These approximations enable to have call-put parity respected for strikes above S_{max}:

$$Caplet(K) - Floorlet(K) = Caplet(S_{max}) - Floorlet(S_{max}) - (K - S_{max}) * D(t,T)$$
$$= (F - S_{max}) * D(t,T) - (K - S_{max}) * D(t,T)$$

⇨ $Caplet(K) - Floorlet(K) = (F - K) * D(t,T)$

An option with a strike below S_{min} is valued as:

- An option with strike S_{min} if it is out of the money (floorlet)
- The sum of an option with strike S_{min} and $K - S_{min}$ if it is in the money (caplet)

Similarly, these approximations respect call-put parity for strikes below S_{min}.

B. Analytical formulae for the weights

1. Caplet

Assuming the strikes K_i are sorted in increasing order:

- $\omega_0 = a * K_1 + b$
- $\omega_i = a * (K_{i+1} - K_{i-1}) > 0 \quad , 1 \leq i \leq N$

2. Floorlet

Assuming the strikes K_i are sorted in decreasing order:

- $\omega_0 = a * K_1 + b$
- $\omega_i = a * (K_{i+1} - K_{i-1}) < 0 \quad , 1 \leq i \leq N$

IV. Implementation choices: digital options

A. Replication basket

Digital options are valued using spread of vanilla options.

The strike spread used to value them is set equal to $\epsilon = \max\left(\frac{1}{1000}; \frac{F_0+\Delta}{1000}\right)$, where Δ corresponds to the shift used with shifted log-normal volatilities. The normal and log-normal cases are dealt with using $\Delta = 0$.

A digital cap is valued as a spread of caplets: $D_{cap}(K) = \frac{1}{2\epsilon}\big(Caplet(K-\epsilon) - Caplet(K+\epsilon)\big)$.

A digital floor is valued as a spread of floorlets: $D_{floor}(K) = \frac{1}{2\epsilon}\big(Floorlet(K+\epsilon) - Floorlet(K - \epsilon)\big)$.

With the linear swap rate model, if the replication portfolios of the two vanilla options share the same set of strikes, the weights for the strikes above $K+\epsilon$ for caplets (respectively below $K-\epsilon$ for floorlets) cancel.

As a result, digital options may be replicated with only two european swaptions. The figure below illustrates the fact that weights cancel in the case of a digital caplet for strikes above $K+\epsilon$.

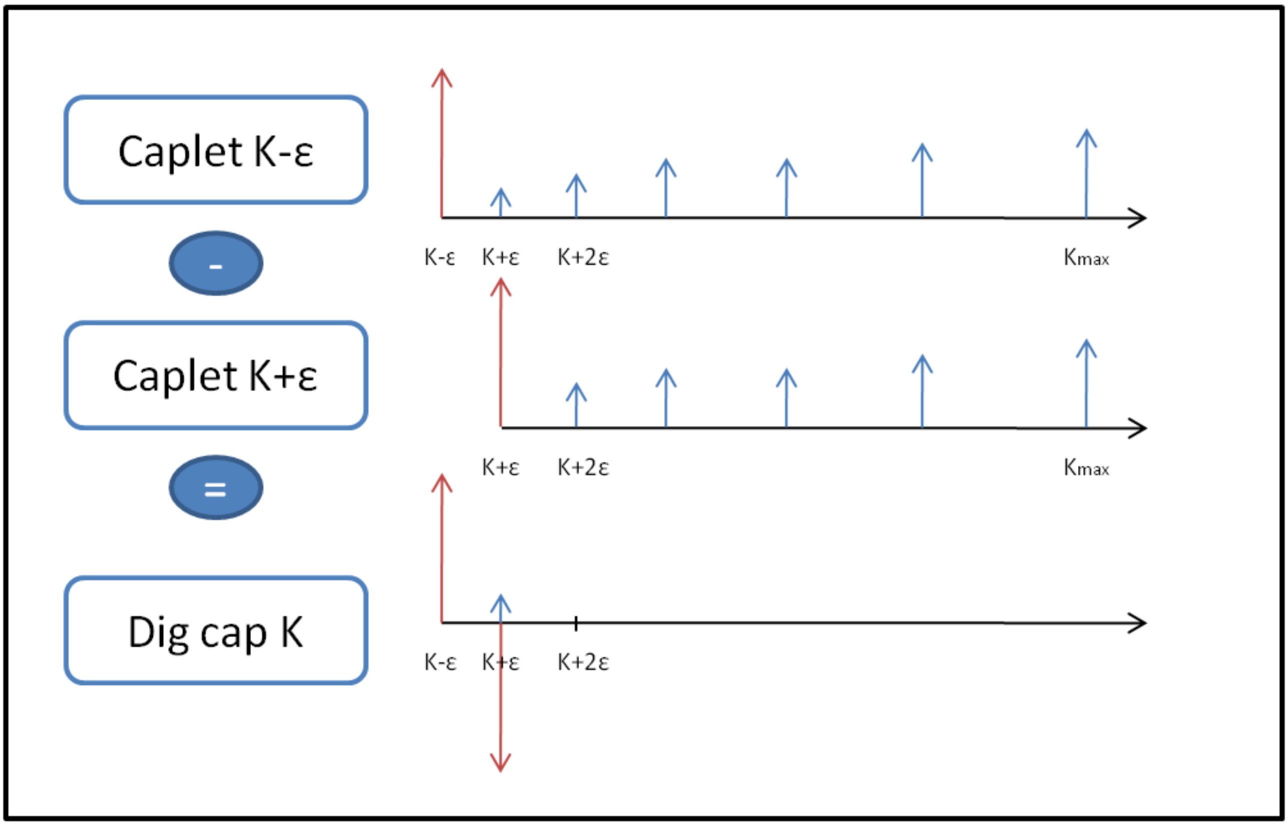

Figure 2 Weights of a digital caplet

- **Special cases : very small strikes**

We need to have $\mathrm{K} > 2\epsilon$ in order to replicate a digital option with the methodology described above.

We treat digital options with strikes $K < 2\epsilon$ as follows:

- If $K < 2\epsilon_{critic}$ where $\epsilon_{critic} = 10^{-6}$ → the value of the digital option is its intrinsic value
- If $2\epsilon_{critic} < K < 2\epsilon$, we replicate the digital option using a smaller spread : $\epsilon_{new} = \frac{K}{2}$

B. Analytical formulae for the weights

1. Digital cap

The weights simplify to:

- $\omega_0 = \frac{1}{2\epsilon}[a * (K + \epsilon) + b]$
- $\omega_1 = \frac{1}{2\epsilon}[-[a * (K + 2\epsilon) + b] + 3a\epsilon] = -\frac{1}{2\epsilon}[a(K - \epsilon) + b]$
- $\omega_i = 0 \quad , i \geq 2$

2. Digital floor

The weights simplify to:

- $\omega_0 = \frac{1}{2\epsilon}[a * (K - \epsilon) + b]$
- $\omega_1 = \frac{1}{2\epsilon}[-[a * (K - 2\epsilon) + b] - 3a\epsilon] = -\frac{1}{2\epsilon}[a(K - \epsilon) + b]$
- $\omega_i = 0 \quad , i \geq 2$

3. Digital cap + digital floor

We can check that the sum of a digital cap and a digital floor converges to one with the linear swap rate model. At maturity:

$$\frac{\mathbb{I}_{S_T > K}}{A(T)} = \frac{1}{2\epsilon}[a * (K + \epsilon) + b]\big(S_T - (K - \epsilon)\big)_+ - \frac{1}{2\epsilon}[a(K - \epsilon) + b]\big(S_T - (K + \epsilon)\big)_+$$

$$\frac{\mathbb{I}_{S_T < K}}{A(T)} = \frac{1}{2\epsilon}[a * (K - \epsilon) + b]\big((K + \epsilon) - S_T\big)_+ - \frac{1}{2\epsilon}[a(K - \epsilon) + b]\big((K - \epsilon) - S_T\big)_+$$

$$\frac{\mathbb{I}_{S_T > K} + \mathbb{I}_{S_T < K}}{A(T)} = \frac{1}{2\epsilon}\big[(aK + b)\big(S_t - (K - \epsilon)\big)\big] - \frac{1}{2\epsilon}\big[(aK + b)\big(S_t - (K + \epsilon)\big)\big] + \frac{1}{2\epsilon}\big[a\epsilon\big(S_T - (K - \epsilon)\big) - a\epsilon\big((K + \epsilon) - S_T\big)\big]$$

$$\frac{\mathbb{I}_{S_T > K} + \mathbb{I}_{S_T < K}}{A(T)} = aK + b + \frac{a}{2}[2(S_T - K)]$$

$$\mathbb{I}_{S_T > K} + \mathbb{I}_{S_T < K} = (aS_T + b)A(T)$$

Hence, the expected value of the sum of the digitals is:

$$Dig_{cap}(t) + Dig_{floor}(t) = Z\left(t, T_p\right)\mathbb{E}^{T_p}\left[\mathbb{I}_{S_T>K} + \mathbb{I}_{S_T<K}\right]$$
$$Dig_{cap}(t) + Dig_{floor}(t) = Z\left(t, T_p\right)\mathbb{E}^{T_p}[(aS_T + b)A(T)]$$
$$Dig_{cap}(t) + Dig_{floor}(t) = A(t)\mathbb{E}^{A}[(aS_T + b)]$$
$$Dig_{cap}(t) + Dig_{floor}(t) = A(t)\frac{Z\left(t,T_p\right)}{A(t)}$$
$$Dig_{cap}(t) + Dig_{floor}(t) = Z\left(t, T_p\right)$$

V. Multi curves

We describe below how we can adapt the results obtained for the CMS replication when using a specific **CMS discount rate curve** to compute the CMS option prices in MACS multicurve framework.

A. Rate curves definition

As stated previously, at maturity T we wish to have, for a CMS caplet replicated with payer swaptions, the following equality respected:

$$\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})(S_T-K)_+ = \sum_{i=0}^{N-1} \omega_i \boldsymbol{A}(\boldsymbol{T})(S_T-K_i)_+ \qquad ,S_T \in (K_i)_{1\le i\le N}$$

Several rate curves are used to estimate the different quantities above:

- The **evaluation rate curve** is used to estimate the forward swap rate S_T.
- The **CMS discount rate curve** is used to discount the swaptions prices.
- The **discount rate curve** is used to discount the value of the caplet (it may take into account a **credit spread**).

The **evaluation rate curve** is taken into account when pricing the swaptions in the optimal basket and when modeling $\frac{\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})}{\boldsymbol{A}(\boldsymbol{T})}$ as a function of the swap rate $G(S_T)$.

More details concerning the multi-curve framework in MACS may be found in [5]

B. Implementation choices

We can rewrite the target equality as:

$$\frac{\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})}{\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})} * \boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p}) * (S_T-K)_+ = \sum_{i=0}^{N-1} \omega_i \boldsymbol{A}(\boldsymbol{T})(S_T-K_i)_+ \qquad ,S_T \in (K_i)_{1\le i\le N}$$

If we assume that the spread between the **CMS discount rate curve** and **discount rate curve** is constant, the quantity $\frac{\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})}{\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})}$ is nothing but a constant multiplicative factor.

Consequently, solving

$$\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})(S_T-K)_+ = \sum_{i=0}^{N-1} \omega_i \boldsymbol{A}(\boldsymbol{T})(S_T-K_i)_+ \qquad ,S_T \in (K_i)_{1\le i\le N}$$

Is equivalent to solving

$$\boldsymbol{Z}(\boldsymbol{T},\boldsymbol{T_p})(S_T-K)_+ = \sum_{i=0}^{N-1} \widetilde{\omega}_i \boldsymbol{A}(\boldsymbol{T})(S_T-K_i)_+ \qquad ,S_T \in (K_i)_{1\le i\le N}$$

And multiplying all the weights obtained by the constant multiplicative factor $\frac{Z(T,T_p)}{Z(T,T_p)}$:

$$\omega_i = \widetilde{\omega}_i * \frac{Z(T,T_p)}{Z(T,T_p)}$$

Similarly, the parameters a and b computed to model $\frac{Z(T,T_p)}{A(T)} \approx G(S_T) = aS_T + b$ can be found by adapting the parameters computed to model $\frac{Z(T,T_p)}{A(T)} \approx \tilde{G}(S_T) = \tilde{a}S_T + \tilde{b}$:

$$a = \tilde{a} * \frac{Z(T,T_p)}{Z(T,T_p)} \text{ and } b = \tilde{b} * \frac{Z(T,T_p)}{Z(T,T_p)}$$

It is straightforward to see that computing the optimal weights ω_i with these new parameters would lead to the same results.

⇨ In order to take into account the **discount rate curve** when replicating a CMS option with the linear swap rate model:
- The linear model parameters are computed in the multi-curve framework[2] with an **evaluation rate curve** and a **CMS discount rate curve,**
- the linear model parameters are then adapted to take into account the **discount rate curve,**
- the optimal weights are computed using the adapted linear model parameters,
- the option value at fixing date is computed by pricing the optimal basket of swaptions in the multi-curve framework with an **evaluation rate curve** and a **CMS discount rate curve** curve (the weights contain the information on the **discount rate curve**),
- the option value at pricing date is obtained by discounting the option value at fixing date using the **discount rate curve.**

[2] More details on its implementation for the CMS replication may be found in [3]

References

[1] MACS team (16/02/11). CMS Replication - Hagan swap rate model.

[2] Interest rate modeling Volume III: Products and Risk Management

[3] Murex, J. Lellouche (31/01/11). CMS Replication.

[4] Hagan, P. (2003). Convexity Conundrums: Pricing CMS Swaps, Caps, and Floors. *Wilmott*, (pp. 38-44).

[5] Murex, Murex Analytics (04/02/11). Multicurve framework in Macs.