CPTS 453 Graph Theory Week 11 Assessment

Charles Nguyen, 011606177

1. Incidence Matrices

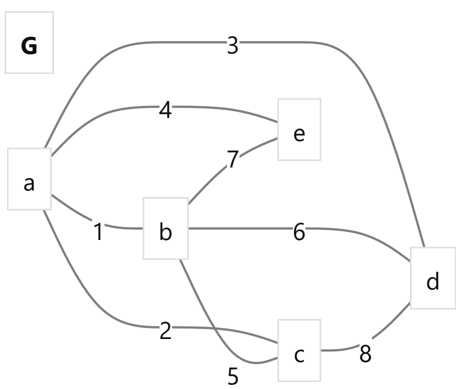
Given the matrices:

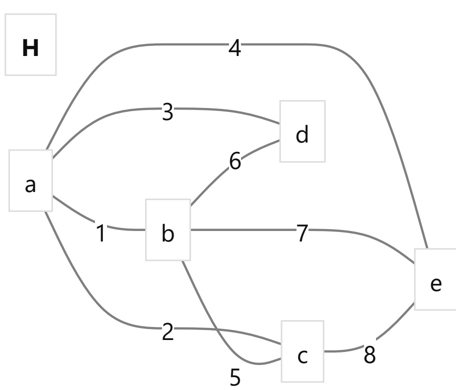
$$M_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad M_H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

A. Draw graphs from incidence matrix G and H.

Solution

Their graphs:





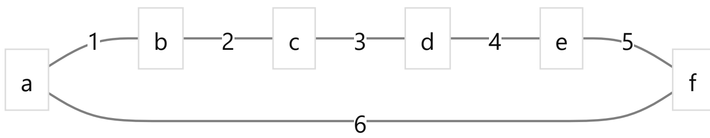
B.

Solution

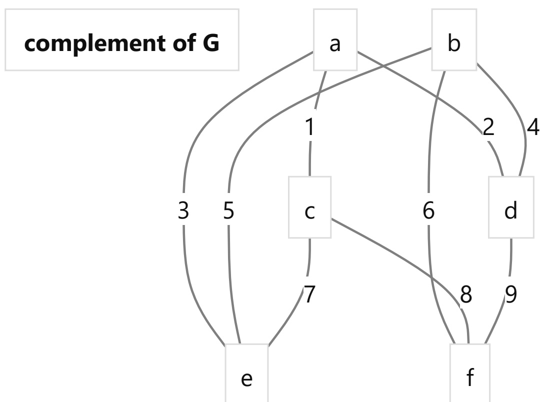
The two graph G and H are not isomorphic because there is no bijection between the vertex set of G and vertex set of H. Counter evident is the number of edges incident on $G.d = 3$ while $H.d = 2$.

2. Complement Graph

A. Suppose G is the six-cycle drawn below. Draw its complement $\overline{G}$.



✎ **Solution**



B. Suppose G is an r-regular simple graph of order n. Explain why $\overline{G}$ is an s-regular simple graph of order n and determine the value of s in terms of r.

✎ **Solution**

Given the simple graphs G where $|V_G| = n$ and $\overline{G} = H$ where $|V_H| = n$, by definition of regularity the amount of edges in G and H are:

$$|E_G| = \frac{n \cdot r}{2} = 6; \quad |E_H| = \frac{n \cdot s}{2} = 9$$
$$r = 2; \quad s = 3$$

We know that for a complete k-regular graph of n vertices, the maximal amount of edges allowed is given by:

$$|E|_{max} = \binom{n}{2} = \frac{n \cdot (n-1)}{2}$$

$$\text{for } n = 6 \quad \text{then } |E|_{max} = 15$$

We also know that for such maximal k-regular graph:

$$n = k + 1$$

Thus, because G and H share the same set of vertices, both graphs are totally disconnected (from each other) subgraphs of the maximal k-regular graph.

$$|E|_{max} = \frac{n \cdot k}{2} = \frac{n \cdot (r+s)}{2} = \frac{n \cdot r}{2} + \frac{n \cdot s}{2}$$

In order for the k-regular graph to be regular, its disconnected components has to be recursively regular:

$$k\text{-regular graph} = G \cup H$$

Thus, $\overline{G} = H$ is s-regular, and $s = n - 1 - r$.

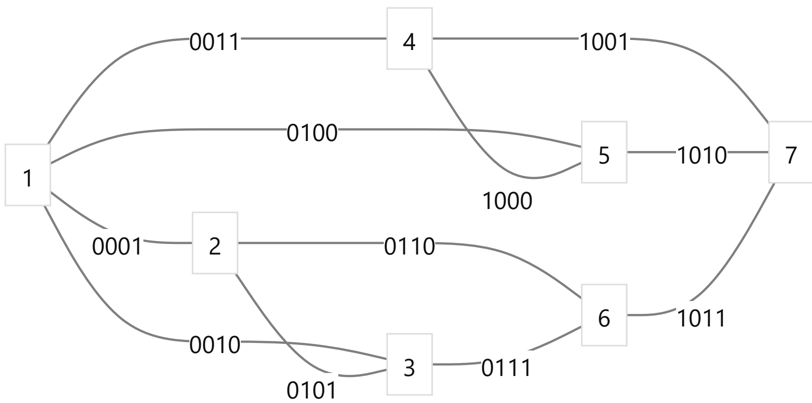
C. If $G = K_{p,q}$ where p and q are positive integers, describe $\overline{G}$.

Solution

G is a bipartite graph, but $\overline{G}$ is not guaranteed to be a bipartite graph. There is also not enough information for p and q about their complete boundaries in order to describe G's complement.

3. Adjacency Matrix & Eccentricities

Given G:



A. Write incidence matrix and adjacency matrix for G.

Solution

$$M_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The adjacency matrix of G is:

$$\begin{aligned} M_G \cdot M_G^T &= D + A \\ A &= M_G \cdot M_G^T - D \end{aligned}$$

$$\begin{aligned} &\text{where,} \\ &\qquad D : \text{diagonal matrix} \\ &\qquad A : \text{adjacency matrix} \end{aligned}$$

Thus,

$$M_G \cdot M_G^T = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & & & & & & \\ & 3 & & & & & \\ & & 3 & & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & 3 & \\ & & & & & & 3 \end{bmatrix}; A = \begin{bmatrix} & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & \end{bmatrix}$$

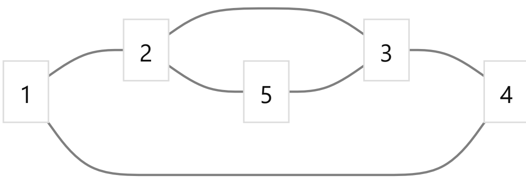
B.

Solution

G has diameter 6 (max eccentricity, between 1 and 2: 1-4-5-7-6-3-2) and radius 4 (min eccentricity, between 1 and 7: 1-2-3-6-7).

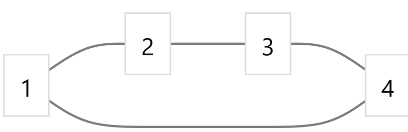
C. This is the graph from question 3 in homework 6. Explain why $(x - k)$ is a factor of its chromatic polynomial for every $k \in \{0, 1, 2, 3\}$. You are not required to find the chromatic polynomial of G.

Given the graph, we call it X:

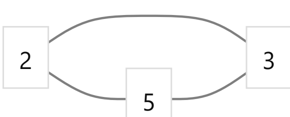


Solution

This graph is the disjoint union $X = G \sqcup H$ of the following two graphs:



G



H

where G is a 2-regular graph of order 4 (K_4), and H is a 2-regular graph of order 3 (K_3).

We then have,

$$p_{G \sqcup H}(k) = p_G(k) \cdot p_H(k)$$

where,

$$p(k) = \begin{cases} \frac{k!}{(k-n)!} & k \geq n \ (1) \\ 0 & k < n \ (2) \end{cases}$$

Beacuse $(k=3) < (n_G = 4)$ and $(k=3) \geq (n_H = 3)$, we have,

$$p_{G \sqcup H}(k) = 0 \cdot p_H(k)$$

It is required that a graph's $p(k) \neq 0$ in order for that graph to be k-colorable. Thus, this graph is not k-colorable, and for every $k \in \{0, 1, 2, 3\}$, any $(x - k)$ can be a factor of its chromatic polynomial.

4. Circuitry

Recall that a **circuit** is a closed walk (one in which the starting and ending vertices are the same) that does not repeat an edge. Explain why if G has a nontrivial circuit, then it must have a nontrivial cycle.

Solution

A trivial circuit of a graph is a circuit containing only one vertex. Thus, a nontrivial circuit is any circuit that is *not* that. This means a nontrivial circuit has $|V| > 1$. Because a nontrivial circuit is a closed walk, it must have $|V| > 2$. Thus, it also contains at least a nontrivial cycle.

5. Boundaries of n-ary tree

Let T be a full ternary (3-ary) tree of height 7.

A. Determine, with justification, a tight upper bound on the number of vertices T can have.

Solution

A full m-ary tree of height h has at most:

$$\sum_{i=0}^{h-1} m^i \quad \text{for} \quad \{m = 3, h = 7\} \quad |V|_{max} = 1093$$

B. Determine, with justification, a tight upper bound on the number of edges T can have.

Solution

a maximal tree (maximum number of vertices n) should have $n-1$ edges. thus,

$$|e|_{max} = 1092$$

C. Determine, with justification, a tight upper bound on the number of leaves T can have.

Solution

A full m-ary tree of height h has at most:

$$|L|_{max} = m^{(h-1)} = 729$$

D. Determine, with justification, whether it is possible for T to have exactly 100 leaves.

Solution

Because T is a **full** ternary tree, that means that every internal node must have exactly 3 child nodes. This means that:

$$|L| \not\equiv 100 \mod 3$$

6. Cartesian Product

Given the cylinder graph S defined as:

$$S_{q,r} = P_q \times C_r$$

A. Determine, in terms of q and r, the number of vertices in $S_{q,r}$

Solution

The number of vertices in S is:

$$|V_S| = |P_q \times C_r| = |P_q| \cdot |C_r| = q \cdot r$$

B. Determine, in terms of q and r, the number of edges in $S_{q,r}$

Solution

A *path graph* P_q has size of $(q-1)$ edges. A *cycle graph* C_r has size of r edges. Therefore,

$$|E_S| = q \cdot (q-1) + r \cdot r$$

C. Show that r is even if and only if $S_{q,r}$ is bipartite.

Solution

We assume that the subgraph C_r is an odd cycle, i.e. $(r \mod 2 \neq 0)$:

$$C_{2r+1} : v_0, e_1, e_1, \ldots, v_{2n}, e_{2n+1}, (v_{2n+1} = v_0)$$

If this was a bipartite graph, we could partition it into two parts V_0 and V_1 such that every edge incident on one vertex in V_0 and one vertex in V_1.

We assume that $v_0 \in V_0$. This means that we choose the even vertices to be in V_0 and odd vertices to be in V_1 as follows:

$$v_{2k} \in V_0; \quad v_{2k+1} \in V_1 \quad \forall k, 0 \leq k \leq n$$

Thus, $e_1 = v_0 v_1$ means $v_1 \in V_1$.

By induction,

$$e_{2k} \in V_0; \quad e_{2k+1} \in V_1$$
$$e_{2n} \in V_0; \quad e_{2n+1} \in V_1 \quad (1)$$

(1) is a contradiction because the original assumption is that $v_{2n+1} = v_0$ meaning $v_{2n+1} \in V_0$. Therefore, we have proven that C_r cannot be an odd cycle. In other words, C_r must be an even cycle and as a result r is even.