Graph Theory Fall 2022

Assignment 7

1. We consider trying to draw Q_k graphs on surfaces.
   A. Show that Q_3 can be drawn on the plane without edges crossing. The quickest way is to actually draw it.
   B. Use the fact that Q_4 has no triangles as subgraphs and an edge counting argument similar to that used for $K_{3,3}$ to show that Q_4 cannot be drawn on the plane without edges crossing. Some data to recall: Q_4 has $n = 16$ vertices and $m = 32$ edges. What would r have to be in Euler's equation?
   C. The graph Q_4 is isomorphic to $C_4 \times C_4$. Use this fact to draw Q_4 on the torus, using the representation in Figure 1.

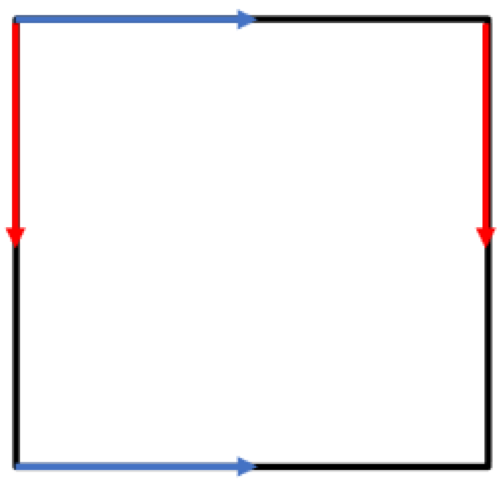


Figure 1. A representation of the one-holed torus.

   D. Recall that the graph Q_5 has $n = 32$ vertices and $m = 80$ edges. Since Q_5 is bipartite, there are no triangles as subgraphs. Use a total edge count argument to show that $r \leq 40$. If you feed this information into Euler's equation $n - m + r = 2 - 2h$ for the h-holed torus, find a lower bound on h.

2. Draw $K_{4,4}$ on a torus without the edges crossing. Here's a suggested layout for the parts; join every black vertex to every white vertex.

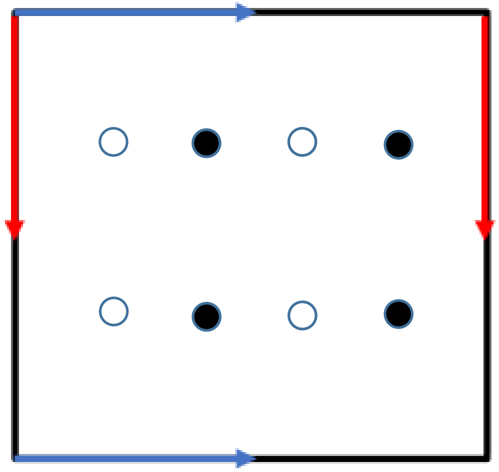


Figure 2. Starting layout for $K_{4,4}$ on a torus.

3. The tournament in Figure 3 shows the outcome of a round-robin event among five competitors. Recall that an arc from u to w means u defeated w in their match.

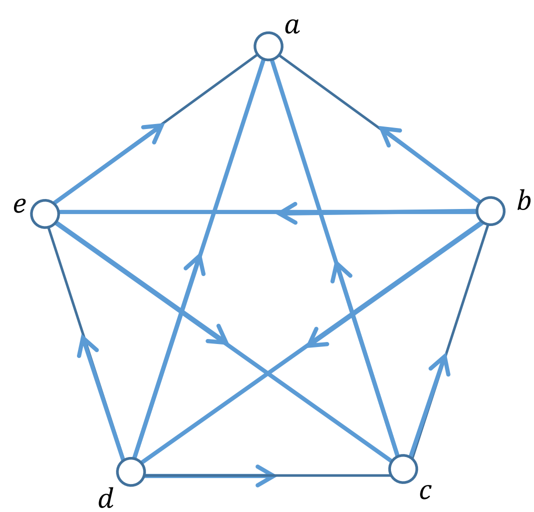
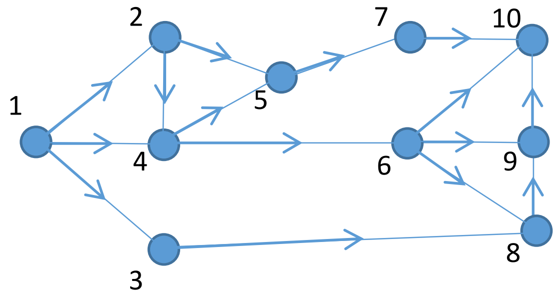


Figure 3. Round-robin tournament among five competitors a, b, c, d, e.

A. Form the victory matrix A for this tournament.
B. Let $\mathbf{w}_0 = \mathbf{1}$ be the all-ones vector and define $\mathbf{w}_{i+1} = A\mathbf{w}_i$. Compute $\mathbf{w}_i$ for enough values of i to see a ranking stabilize. Suggestion: Automate this process.
C. Use an online matrix calculator (e.g. https://matrixcalc.org) to find the eigenvector for the principal eigenvalue (this eigenvalue is about 1.395.) This eigenvector should provide the same ranking as your answer in part B.

4. Consider the digraph D depicted in figure 4.



A. Compute its incidence matrix M

B. Compute its Laplacian matrix L