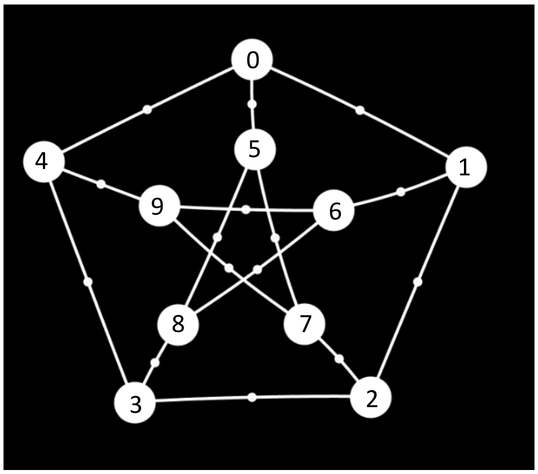
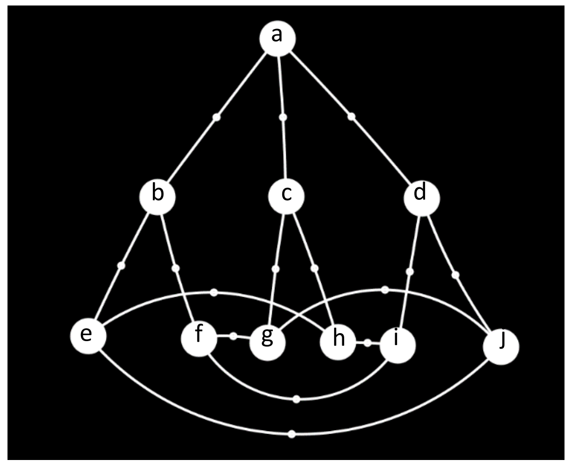
Graph Theory Fall 2022

Assignment 3

1. Show that these two graphs are isomorphic by finding a bijection $\psi: \{0,1,2,3,4,5,6,7,8,9\} \rightarrow \{a, b, c, d, e, f, g, h, i, j\}$ that preserves adjacencies.





2. Let a and b be integers at least 2. Find the maximum value among distances between vertices in each of the following graphs:
   a. P_a
   b. C_a
   c. K_a
   d. $K_{a,b}$
   e. Q_a

3. We investigate using the phrase "distance between vertices" to denote the length of a shortest path between two vertices in a graph G. We define $d(u, v)$ as the length of a shortest u, v-path in G. If there is no such path, then we define $d(u, v) = \infty$. Notice that if $d(u, v) < \infty$, then there must be a u, v-path in G.
   A. Explain why $0 \leq d(u, v)$ for all vertices u and v.
   B. Under what conditions on u and v would the equation $d(u, v) = 0$ be true?
   C. Explain why $d(u, v) = d(v, u)$ for all vertices u and v.
   D. Suppose $d(u, v) < \infty$ and $d(v, w) < \infty$. Why must $d(u, w) < \infty$ hold? Here, you'll want to consider a shortest u, v-path P and a shortest v, w-path Q. What can you do with P and Q?
   E. Suppose $d(u, w) = \infty$. What can you say about $d(u, v)$ or $d(v, w)$?
   F. Show that if $d(u, v) < \infty$ and $d(v, w) < \infty$, then $d(u, w) \leq d(u, v) + d(v, w)$.
   G. What can you say about $d(u, w)$ if $d(u, v) < \infty$ and $d(v, w) = \infty$?
   H. What can you say about $d(u, w)$ if $d(u, v) = \infty$ and $d(v, w) = \infty$?

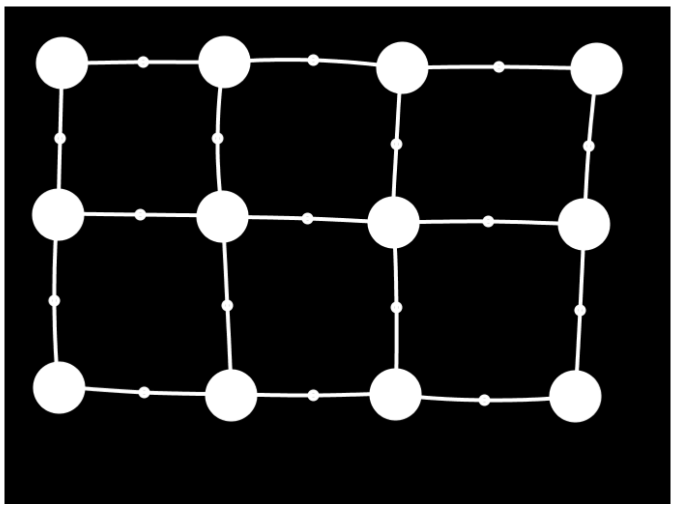
4. We discuss the Cartesian product of two simple graphs in this question; particularly, we avoid discussing parallel edges here to make things easier. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. Their **Cartesian product** is the graph $G \times H$ whose vertex set is

$$V_G \times V_H = \{(v, w)\colon\ v \in V_G \text{ and } w \in V_H\}$$

and where vertices (v_1, w_1) and (v_2, w_2) are joined by an edge if and only if one of the following conditions is met:

a) $v_1 = v_2$ and $w_1 w_2 \in E_H$,
b) $v_1 v_2 \in E_G$ and $w_1 = w_2$.

As an example, the Cartesian product of P_a and P_b is the "grid graph" $P_a \times P_b$. We show $P_4 \times P_3$ below:



A) In terms of $|V_G|$ and $|V_H|$, what is $|V_{G\times H}|$?
B) In terms of $|V_G|, |V_H|, |E_G|$, and $|E_H|$, what is $|E_{G\times H}|$?
C) Suppose G is a-regular and H is b-regular. Show that $G \times H$ is $(a+b)$-regular.

Figure 1. A sketch of G_3

5. Recall that a graph G is "cubic" if and only if it is 3-regular (i.e., every vertex has degree 3.)
   A. Show that there exists no cubic graph with an odd number of vertices.
   B. Show that there exists a simple cubic graph with 4 vertices.
   C. For every integer $n \geq 3$, show how to construct a simple cubic graph with $2n$ vertices. You can do this using a sketch of small examples with an obvious generalization, or by specifying a set V with $2n$ elements and a recipe for joining them that produces the desired graph.