Graph Theory Fall 2022

Assignment 4

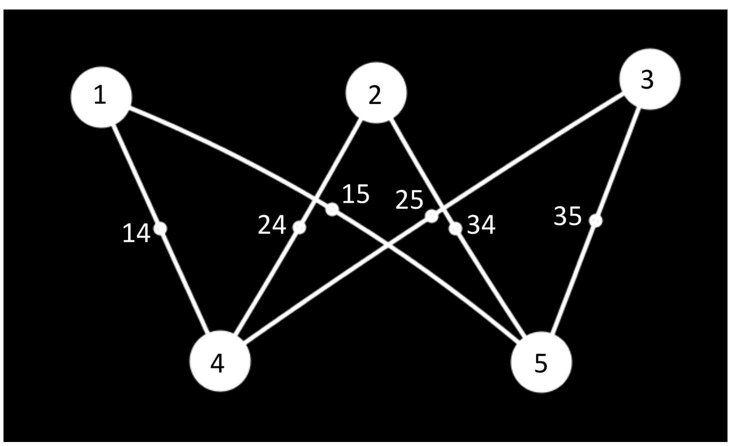
1. Let G be a graph without loops.
   We discussed the incidence matrix M that has n rows (a row for each vertex) and m columns (a column for each edge).

   Recall that the entry M_{ij} has the formula

   $$M_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is an endpoint of edge } j \\ 0 & \text{otherwise} \end{cases}$$

   Now consider the matrix MM^T. The results of parts A and B below let us write $MM^T = D + A$ where D is the diagonal matrix containing the degrees of the vertices and A is the **adjacency matrix** that encodes the numbers of edges joining pairs of vertices.

   A. Explain why $(MM^T)_{ii}$, the diagonal entry in position i, is the degree of vertex i.
   B. Explain why for $i \neq j$, the entry $(MM^T)_{ij}$ is the number of edges joining vertex i to vertex j.
   C. Determine the matrices M, MM^T, D and A for the graph drawn below (the vertices are labeled 1,2,3,4,5 and the edges are labeled 14,15,24,25,34,35 denoting the corresponding endpoints).

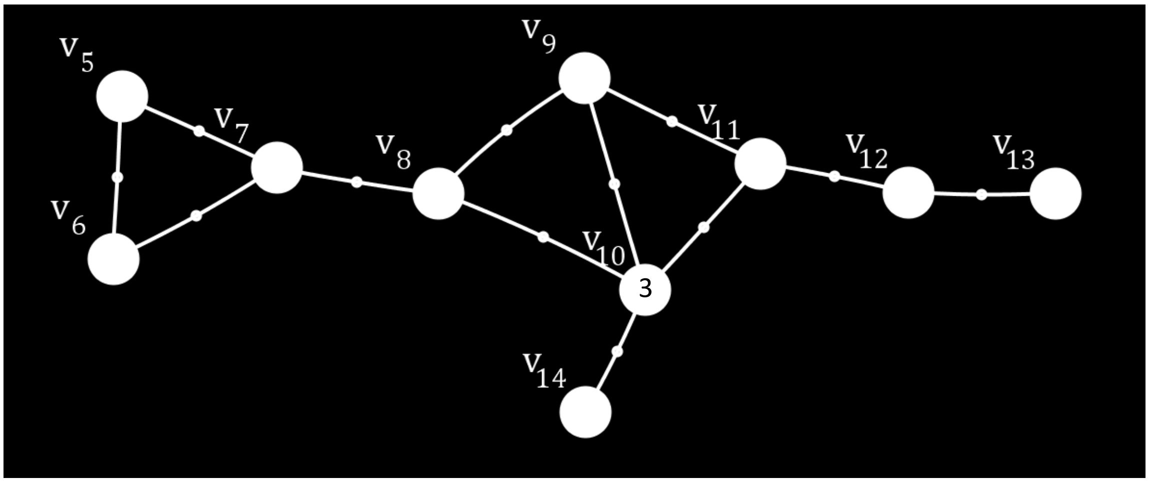


2. We let $G = (V, E)$ be a connected graph. For any vertex $v \in V$, define its **eccentricity** by the formula

$$\mathrm{ecc}(v) = \max\{d(u, v)\colon\ u \in V\}$$

where $d(u, v)$ is the distance between u and v.

A. Let G be the graph drawn below. Label each vertex with its eccentricity. This has already been done for v_{10}; here, $d(v_5, v_{10}) = d(v_6, v_{10}) = d(v_{13}, v_{10}) = 3$ and no vertex is further from v_{10}.



B. The **diameter** of a graph is the maximum among the eccentricities of its vertices and the **radius** of a graph is the minimum among the eccentricities of its vertices. What is the diameter and radius of the graph in part A?

A **central vertex** is a vertex v such that $\mathrm{ecc}(v) = \mathrm{radius}(G)$. Which of the vertices in the graph in part A are central vertices?

C. A **peripheral vertex** is a vertex v such that $\mathrm{ecc}(v) = \mathrm{diameter}(G)$. Which of the vertices in the graph in part A are peripheral vertices?

Let u and v be vertices such that $d(u, v) = \mathrm{diameter}(G)$. Explain why u and v must be peripheral vertices.

D. Suppose v_i and v_j are adjacent vertices. Explain why $\mathrm{ecc}(v_i)$ and $\mathrm{ecc}(v_j)$ differ by at most 1.

E. Explain why for any connected graph H, $\mathrm{radius}(H) \leq \mathrm{diameter}(H)$.

F. Let w be a central vertex for the graph H in part G. Use the fact that $d(u, v) \leq d(u, w) + d(w, v)$ for any vertices u and v to explain why $\mathrm{diameter}(H) \leq 2\ \mathrm{radius}(H)$.

3. Let G be a graph without loops and r be a vertex of G; we will call r the "root" vertex. We will say that $u \approx w$ if $d(r,u) = d(r,w)$. This means u and w are the same distance from r.
   A. Show that the relation $\approx$ is reflexive.
   B. Show that the relation $\approx$ is symmetric.
   C. Show that the relation $\approx$ is transitive.
   D. Describe $[r]$, the equivalence class of r under $\approx$.
   E. If ru is an edge, briefly describe $[u]$, the equivalence class of u under $\approx$.