Graph Theory Fall 2022

Week 11 Assessment

1. Let G and H be graphs with incidence matrices

$$M_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}; \; M_H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

These matrices are the same except the last two entries in the final column are switched.

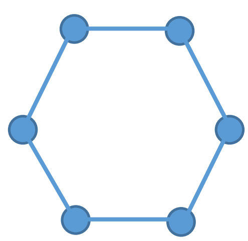
A. Draw G and H

B. Determine, with justification, whether G and H are isomorphic graphs.

2. Given a simple graph $G = (V, E)$, we define its **complement** $\overline{G} = (V, \overline{E})$ as the simple graph with the same vertex set and where two distinct vertices in $\overline{G}$ are joined by an edge in $\overline{G}$ if and only if they are not joined by an edge in G. Here, the set $\overline{E}$ is defined as

$$\overline{E} = \{uv\colon\ u \in V, v \in V, u \neq v \text{ and } uv \notin E\}.$$

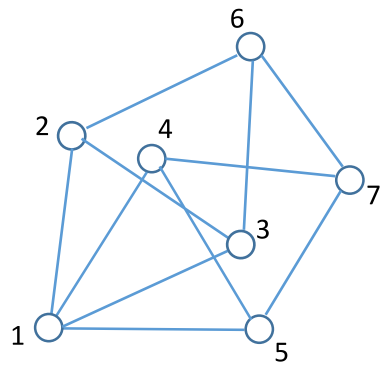
A. Suppose G is the six-cycle drawn below. Draw its complement $\overline{G}$.



B. Suppose G is an r-regular simple graph of order n. Explain why $\overline{G}$ is an s-regular simple graph of order n and determine the value of s in terms of r.

C. If $G = K_{p,q}$ where p and q are positive integers, describe $\overline{G}$.

3. Consider the graph G below

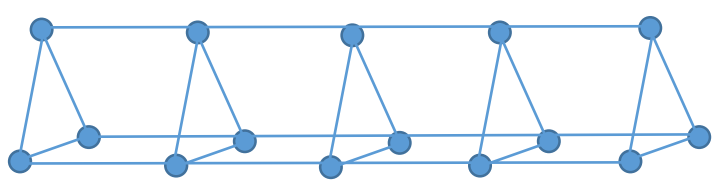


A. Write down an incidence matrix and an adjacency matrix for G.
B. Determine the radius and the diameter of G.
C. This is the graph from question 3 in homework 6. Explain why $(x - k)$ is a factor of its chromatic polynomial for every $k \in \{0,1,2,3\}$. You are not required to find the chromatic polynomial of G.

4. Recall that a **circuit** is a closed walk (one in which the starting and ending vertices are the same) that does not repeat an edge. Explain why if G has a nontrivial circuit, then it must have a nontrivial cycle.

5. Let T be a full ternary (3-ary) tree of height 7.
A. Determine, with justification, a tight upper bound on the number of vertices T can have.
B. Determine, with justification, a tight upper bound on the number of edges T can have.
C. Determine, with justification, a tight upper bound on the number of leaves T can have.
D. Determine, with justification, whether it is possible for T to have exactly 100 leaves.

6. For integers $q \geq 1$ and $r \geq 3$, the **cylinder graph** $\boldsymbol{S_{q,r}}$ is defined as $S_{q,r} = P_q \times C_r$. For instance, $S_{5,3}$ is drawn below:



A. Determine, in terms of q and r, the number of vertices in $S_{q,r}$
B. Determine, in terms of q and r, the number of edges in $S_{q,r}$
C. Show that r is even if and only if $S_{q,r}$ is bipartite.