A Quick Tour of the Fourier Series

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1 Introduction

The Fourier transform is one of mathematics's most successful exports to the natural sciences. Joseph Fourier's method of understanding functions threw open doors for scientists investigating every corner of reality. The triumphs that followed are too great in number to list. They include the discovery of DNA, modern sound engineering, quantum mechanics, and the data compression methods that allow for the internet as we know it. And the Fourier transform remains relevant for answering modern research questions:

> Scientists [are using] the Fourier transform to study the vibrations of submersible structures interacting with fluids, to try to predict upcoming earthquakes, to identify the ingredients of very distant galaxies, to search for new physics in the heat remnants of the Big Bang, to uncover the structure of proteins from X-ray diffraction patterns, to analyze digital signals for NASA, to study the acoustics of musical instruments, to refine models of the water cycle, to search for pulsars (spinning neutron stars), and to understand the structure of molecules using nuclear magnetic resonance. [2]

At the heart of this outstanding idea and tool is the Fourier series, the topic of this paper. The next section defines the Fourier series. The third section lays out conditions under which Fourier series converge. And the final section examines a notable Fourier series.

2 Definitions

To motivate the definition of a Fourier series, we turn to a key result in linear algebra. Let V be a finite-dimensional vector space over $\mathbb{R}$. On V, define an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Then, for any $v \in V$,

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n. \tag{1}$$

To apprehend the truth of (1), note that since $\{e_1, \ldots, e_n\}$ is a basis of V, we can write

$$v = a_1 e_1 + \cdots + a_n e_n$$

where a_i is a scalar for $i \in \{1, \ldots, n\}$. Now, taking the inner product of v with e_i yields $\langle v, e_i\rangle = a_i$ where we have used additivity and homogeneity of the inner product as well as the fact that $\langle e_i, e_j\rangle = 1$ for $i = j$ and $\langle e_i, e_j\rangle = 0$ for $i \neq j$. Since any basis can be made into an orthonormal basis (simply apply the Gram-Schmidt procedure to it), we can express vectors in finite-dimensional vector spaces in this elegant way [1].

The Fourier series generalizes this idea to vectors in infinite dimensional vector spaces. This paper is concerned specifically with the Fourier series of a 2π-periodic function. For such a function $f : \mathbb{R} \to \mathbb{R}$ in the set of Riemann integrable functions on $[-\pi, \pi]$,

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

is called its Fourier series where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy, \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy.$$

Further, let $\{S_N(x)\}_{N=1}^{\infty}$ be the sequence of partial sums of the Fourier series. For $N \in \mathbb{P}$,

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n\cos(nx) + b_n\sin(nx).$$

How do these definitions relate to the discussion at the beginning of this section? Let V be the vector space of continuous real-valued functions on $[-\pi, \pi]$ over $\mathbb{R}$. Define an inner-product on V such that for $f, g \in V$, $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. Now consider the set

$$X = \left\{\frac{\cos(nx)}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{2\pi}} \mid n \in \mathbb{N}\right\}.$$

X is orthonormal with respect to this inner product. (I omit the proof of this fact as it is routine but tedious.) Now, imagine that X is more than an infinite orthonormal set. Imagine that it is an infinite-dimensional orthonormal *basis*. If this were true, we could represent an f in V using (1). Following this idea, observe that

$$\langle f, \frac{\cos(0x)}{\sqrt{2\pi}}\rangle\frac{\cos(0x)}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = a_0, \quad \text{and} \quad \langle f, \frac{\sin(0x)}{\sqrt{2\pi}}\rangle\frac{\sin(0x)}{\sqrt{2\pi}} = 0.$$

Moreover, for $n > 0$,

$$\langle f, \frac{cos(nx)}{\sqrt{\pi}}\rangle\frac{cos(nx)}{\sqrt{\pi}} = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cos(nx)dx\right) cos(nx) = a_n\cos(nx),$$

$$\text{and} \quad \langle f, \frac{sin(nx)}{\sqrt{\pi}}\rangle\frac{sin(nx)}{\sqrt{\pi}} = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\sin(nx)dx\right)\sin(nx) = b_n\sin(nx).$$

Thus, if we take X to be a basis and represent a vector in V with respect to it, we reach the Fourier series definition! The next section asks: when are we justified in making this imaginative leap?

3 Convergence of Fourier Series

We have established that we can think of the Fourier series for a 2π-periodic function f as an attempt at expressing f as a linear combination of an infinite orthonormal basis. Now we investigate when that attempt will succeed. As a first step, we define the Dirichlet Kernel

$$D_N(x) = \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\right),$$

and consider the following four lemmas.

Lemma 1. $\int_{-\pi}^{\pi} D_N(x)dx = 1.$

Proof.

$$\begin{aligned}
\int_{-\pi}^{\pi}\frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\right) &= \int_{-\pi}^{\pi}\frac{1}{2\pi} + \int_{-\pi}^{\pi}\sum_{n=1}^{N}\cos(nx) \\
&= \left[\frac{x}{2\pi}\right]_{-\pi}^{\pi} + \left[\sum_{n=1}^{N}\frac{\sin(nx)}{\pi n}\right]_{-\pi}^{\pi} \\
&= 1 + 0 \\
&= 1.
\end{aligned}$$

$\square$

Lemma 2. *$S_N(x) = \int_{-\pi}^{\pi} f(y)D_N(x-y)dy.$*

Proof.

$$\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \sum_{n=1}^{N}\left[\frac{\cos(nx)}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy + \frac{\sin(nx)}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{N}\left[\int_{-\pi}^{\pi} f(y)\left(\cos(nx)\cos(ny)+\sin(nx)\sin(ny)\right)dy\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy + \frac{1}{\pi}\sum_{n=1}^{N}\left[\int_{-\pi}^{\pi} f(y)\cos(n(x-y))dy\right].
\end{aligned}$$

To reach the last equality note that $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Along with the fact that $\cos(-x) = x$ and $\sin(-x) = -x$, this implies that $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$. To complete the proof, we factor and rearrange terms to get that

$$\begin{aligned}
S_N(x) &= \int_{-\pi}^{\pi} f(y)\frac{1}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{N}\cos(n(x-y))\right)dy \\
&= \int_{-\pi}^{\pi} f(y)D_N(x-y)dy.
\end{aligned}$$

$\square$

Lemma 3. *$D_N(x) = \left(\frac{1}{2\pi}\right)\left(\frac{\sin\left[(N+\frac{1}{2})x\right]}{\sin(\frac{x}{2})}\right)$ for $\sin(\frac{x}{2}) \neq 0$.*

Proof. First, we rewrite $D_N(x)$:

$$\left(\frac{1}{2\pi}\right)\left(\frac{\sin\left[(N+\frac{1}{2})x\right]}{\sin(\frac{x}{2})}\right) = \left(\frac{1}{2\pi}\right)\left(\frac{\sin\left[(N+\frac{1}{2})x\right] - \sin(\frac{x}{2}) + \sin(\frac{x}{2})}{\sin(\frac{x}{2})}\right).$$

Moreover,

$$\sin\left[(N+\frac{1}{2})x\right] - \sin(\frac{x}{2}) = \sum_{n=1}^{N}\left(-\sin\left[(n-1+\frac{1}{2})x\right] + \sin\left[(n+\frac{1}{2})x\right]\right).$$

Now, using the identity $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin(\beta)\cos(\alpha)$,

$$\sum_{n=1}^{N}\left(-\sin\left[(n-1+\frac{1}{2})x\right] + \sin\left[(n+\frac{1}{2})x\right]\right) = \sum_{n=1}^{N} 2\sin(\frac{x}{2})\cos(nx) = 2\sin(\frac{x}{2})\sum_{n=1}^{N}\cos(nx).$$

Using this result,

$$\begin{aligned}
\left(\frac{1}{2\pi}\right)\left(\frac{\sin\left[(N+\frac{1}{2})x\right] - \sin(\frac{x}{2}) + \sin(\frac{x}{2})}{\sin(\frac{x}{2})}\right) &= \left(\frac{1}{2\pi}\right)\left(\frac{2\sin(\frac{x}{2})\sum_{n=1}^{N}\cos(nx) + \sin(\frac{x}{2})}{\sin(\frac{x}{2})}\right) \\
&= \frac{1}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{N}\cos(nx)\right).
\end{aligned}$$

$\square$

Lemma 4 (Riemann-Lebesgue). *If f is 2π-periodic and is continuous on $[-\pi,\pi]$, then*

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}|b_n| = 0.$$

This paper's first substantial result is now at hand.

Theorem 1. *If f is continuously differentiable on $[-\pi,\pi]$, then $\{S_N(x)\}$ converges pointwise to f on $[-\pi,\pi]$.*

Proof. We begin by using Lemma 2 to write $S_N(x) = \int_{-\pi}^{\pi} f(y)D_N(x-y)dy$. Now,

$$\int_{-\pi}^{\pi} f(y)D_N(x-y)dy = \int_{-\pi}^{\pi} f(y)D_N(y-x)dy$$

since $\cos(x-y) = \cos(y-x)$, and with a change of variables,

$$\int_{-\pi}^{\pi} f(y)D_N(y-x)dy = \int_{-\pi}^{\pi} f(x+y)D_N(y)dy.$$

For $x \in [-\pi,\pi]$,

$$\begin{aligned}
|S_N(x) - f(x)| &= \left|\int_{-\pi}^{\pi} f(x+y)D_N(y)dy - f(x)\right| \\
&= \left|\int_{-\pi}^{\pi} f(x+y)D_N(y)dy - f(x)\int_{-\pi}^{\pi} D_N(y)dy\right| && \text{(by Lemma 1)} \\
&= \left|\int_{-\pi}^{\pi} \left(f(x+y)D_N(y) - f(x)D_N(y)\right)dy\right| \\
&= \left|\int_{-\pi}^{\pi} \left(f(x+y) - f(x)\right)D_N(y)dy\right|.
\end{aligned}$$

Define

$$g(x;y) = \begin{cases} f'(x) & \text{for } y = 0 \\ \dfrac{f(x+y)-f(x)}{2\sin(y/2)} & \text{otherwise.} \end{cases}$$

$g(y;x)$ is continuous on $[-\pi,\pi]$, since

$$\begin{aligned}
\lim_{y\to 0}\frac{f(x+y)-f(x)}{2\sin(y/2)} &= \lim_{y\to 0}\left(\frac{f(x+y)-f(x)}{y}\right)\left(\frac{y/2}{\sin(y/2)}\right) \\
&= \left(\lim_{y\to 0}\frac{f(x+y)-f(x)}{y}\right)\left(\lim_{y\to 0}\frac{y/2}{\sin(y/2)}\right) \\
&= f'(x). && \text{(by l'Hôpital's rule)}
\end{aligned}$$

We can then write

$$|S_N(x) - f(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} g(y;x)\sin\left[(N+\frac{1}{2})y\right]dy\right|. \qquad \text{(using Lemma 3)}$$

We are unconcerned with what value $g(y;x)\sin\left[(N+\frac{1}{2})y\right]$ takes at 0 since the integrand, as a continuous function, is Riemann integrable, so it can differ in value from $(f(x+y)-f(x))D_N(y)$ at a set of points that is of measure zero. Now, with the help of the fact that $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$,

$$\begin{aligned}
|S_N(x) - f(x)| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} g(y;x)\left[\sin(Ny)\cos(\frac{y}{2}) + \cos(Ny)\sin(\frac{y}{2})\right]dy\right| \\
&= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} g(y;x)\cos(\frac{y}{2})\sin(Ny)dy + \frac{1}{\pi}\int_{-\pi}^{\pi} g(y;x)\sin(\frac{y}{2})\cos(Ny)dy\right|.
\end{aligned}$$

We notice that $\frac{1}{\pi}\int_{-\pi}^{\pi} g(y;x)\cos(\frac{y}{2})\sin(Ny)dy$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} g(y;x)\sin(\frac{y}{2})\cos(Ny)dy$ are Fourier coefficients, so

$$|S_N(x) - f(x)| = |b_N + a_N|$$

Now define $f^{(1)}(y) = g(y;x)\cos(\frac{y}{2})$ and $f^{(2)}(y) = g(y;x)\sin(\frac{y}{2})$. Then

$$\begin{aligned} f^{(1)}(y+2\pi) &= \frac{f(x+y+2\pi)-f(x)}{2\sin(\frac{y+2\pi}{2})}\cos(\frac{y+2\pi}{2}) \\ &= \frac{f(x+y+2\pi)-f(x)}{2}\cot(\frac{y+2\pi}{2}) \\ &= \frac{f(x+y)-f(x)}{2\sin(\frac{y}{2})}\cos(\frac{y}{2}) \\ &= f^{(1)}(y), \end{aligned}$$

where in second to last equality we have used that f is 2π-periodic and the cotangent function is π-periodic. Moreover,

$$\begin{aligned} f^{(2)}(y+2\pi) &= \frac{f(x+y+2\pi)-f(x)}{2\sin(\frac{y+2\pi}{2})}\sin(\frac{y+2\pi}{2}) \\ &= \frac{f(x+y+2\pi)-f(x)}{2} \\ &= \frac{f(x+y)-f(x)}{2} && \text{(since } f \text{ is } 2\pi\text{-periodic)} \\ &= \frac{f(x+y)-f(x)}{2\sin(\frac{y}{2})}2\sin(\frac{y}{2}) \\ &= f^{(2)}(y). \end{aligned}$$

In addition to being 2π-periodic, $f^{(1)}$ and $f^{(2)}$ are also continuous on $[-\pi,\pi]$ since they are the product of continuous functions on $[-\pi,-\pi]$. We are now prepared to complete the proof.

$$\begin{aligned} |S_N(x)-f(x)| &= |b_N + a_N| \\ &\le |b_N| + |a_N| \\ &< \epsilon. && \text{(for some } K \text{ and all } N \ge K) \end{aligned}$$

The final inequality follows by the Riemann-Lebesgue Lemma. Since $|b_N|$ and $|a_N|$ each converges to 0, their sum converges to 0. Thus, for all $\epsilon > 0$, there exists K such that for all $N \ge K$, $||b_N| + |a_N| - 0| = |b_N| + |a_N| < \epsilon$.[1] □

So, we have seen that we can find a sequence that convergence pointwise to f. Can we do better? It turns out that we can! In the second part of this section, we show that if f is continuous on $[-\pi,\pi]$, there exists a sequence that converges uniformly to f. On the way to that result is a definition and four lemmas. Define the Fejer kernel

$$K_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x).$$

[1] In this proof, I have followed the strategy in [4].

Lemma 5. $\int_{-\pi}^{\pi} K_N(x)dx = 1$.

Proof.

$$\begin{aligned}
\int_{-\pi}^{\pi} K_N(x)dx &= \int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{k=0}^{N} D_k(x)dx \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\int_{-\pi}^{\pi} D_k(x)dx \\
&= \frac{1}{N+1}\sum_{k=0}^{N} 1 && \text{(by Lemma 1)} \\
&= 1.
\end{aligned}$$

□

Lemma 6. $K_N(x) = \dfrac{1}{2\pi(N+1)}\dfrac{\sin^2\left[(N+1)\frac{x}{2}\right]}{\sin^2(\frac{x}{2})}$ *for* $\sin(\frac{x}{2}) \neq 0$.

Proof. By Lemma 3, for all x such that $\sin(\frac{x}{2}) \neq 0$,

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1}\sum_{k=0}^{N}\left(\frac{1}{2\pi}\right)\left(\frac{\sin\left[(k+\frac{1}{2})x\right]}{\sin(\frac{x}{2})}\right) && (2) \\
&= \frac{1}{2\pi(N+1)}\sum_{k=0}^{N}\frac{\sin\left[(k+\frac{1}{2})x\right]}{\sin(\frac{x}{2})} && (3) \\
&= \frac{1}{2\pi(N+1)\sin^2(\frac{x}{2})}\sum_{k=0}^{N}\sin\left[(k+\frac{1}{2})x\right]\sin(\frac{x}{2}). && (4)
\end{aligned}$$

Now we apply the fact that $\sin(\alpha)\sin(\beta) = \dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$ to write

$$\sin\left[(k+\frac{1}{2})x\right]\sin(\frac{x}{2}) = \frac{\cos(kx)-\cos\left[(k+1)x\right]}{2}.$$

We observe that

$$\sum_{k=0}^{N}\frac{\cos(kx)-\cos\left[(k+1)x\right]}{2} = \frac{1-\cos\left[(N+1)x\right]}{2},$$

since the sum is telescoping. Finally, since $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$,

$$\frac{1-\cos\left[(N+1)x\right]}{2} = \sin^2\left[(N+1)\frac{x}{2}\right].$$

Substituting into (4), for $\sin(\frac{x}{2}) \neq 0$,

$$K_N(x) = \frac{1}{2\pi(N+1)}\frac{\sin^2\left[(N+1)\frac{x}{2}\right]}{\sin^2(\frac{x}{2})}.$$

□

Lemma 7. $K_N(x) \geq 0$.

Proof. This follows immediately from Lemma 6. N and $\sin^2(\theta)$ are nonnegative. $\square$

Lemma 8. $K_N(x) \leq \dfrac{1}{2\pi(N+1)\sin^2(\frac{\delta}{2})}$ *for* $0 < \delta \leq |x| \leq \pi$.

Proof. Let $0 < \delta \leq |x| \leq \pi$. For $0 < |x| < \pi$, $\sin(\frac{x}{2}) \neq 0$, so we can apply Lemma 6:

$$\begin{aligned} K_N(x) &= \frac{1}{2\pi(N+1)} \frac{\sin^2\left[(N+1)\frac{x}{2}\right]}{\sin^2(\frac{x}{2})} \\ &\leq \frac{1}{2\pi(N+1)\sin^2(\frac{x}{2})} \\ &\leq \frac{1}{2\pi(N+1)\sin^2(\frac{\delta}{2})}. \end{aligned}$$

$\square$

We're now ready for the second significant result of this section. Define

$$\sigma_N(x) = \int_{-\pi}^{\pi} f(y) K_N(x-y) dy.$$

$\sigma_N(x)$ is a trigonometric polynomial, which is a function of the form

$$F(x) = \alpha_0 + \sum_{n=1}^{N} \alpha_n \cos(nx) + \beta_n \sin(nx).$$

To see this, we observe that

$$\begin{aligned} \sigma_N(x) &= \int_{-\pi}^{\pi} f(y) K_N(x-y) dy \\ &= \frac{1}{N+1} \int_{-\pi}^{\pi} f(y) \sum_{k=0}^{N} D_k(x-y) dy \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \int_{-\pi}^{\pi} f(y) D_k(x-y) dy \\ &= \frac{1}{N+1} \sum_{k=0}^{N} S_k \end{aligned}$$

where we have applied Lemma 2 to reach the final equality. Comparing the definitions of S_k with that of a trigonometric polynomial, it is apparent that for $k \geq 0$, S_k is a trigonometric polynomial. Moreover, the trigonometric polynomials are closed under addition and scalar multiplication, so $\frac{1}{N+1}\sum_{k=0}^{N} S_k$ is also a trigonometric polynomial. In light of this, the next result should seem plausible.

Theorem 2. *If f is 2π-periodic and is continuous on $[-\pi, \pi]$, then the sequence σ_N converges uniformly to f on $[-\pi, \pi]$.*

Proof. First, we rewrite σ_N with a change of variables:

$$\begin{aligned}
\sigma_N(x) &= \int_{-\pi}^{\pi} f(y)K_N(x-y)dy \\
&= \int_{x+\pi}^{x-\pi} (-1)f(x-t)K_N(t)dt \\
&= \int_{x-\pi}^{x+\pi} f(x-t)K_N(t)dt \\
&= \int_{-\pi}^{\pi} f(x-t)K_N(t)dt.
\end{aligned}$$

The final equality follows since for a periodic function f with period p, $\int_a^{a+p} f = \int_c^{c+p} f$.

Now pick $\epsilon > 0$. Since f is continuous on $[-\pi, \pi]$, it is bounded on $[-\pi, \pi]$. Choose M such that $|f(x)| \leq M$. f is also uniformly continuous on $[-\pi, \pi]$ (since it is continuous over a closed interval), so we can pick $\delta > 0$ such that

$$0 < \delta \leq \pi \quad \text{and} \quad |x-y| \leq \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\epsilon}{4}. \tag{5}$$

Moreover, from Lemma 8, $K_N(t)$ tends to 0 as $N \to \infty$, so we can pick N such that

$$\delta \leq |t| \leq \pi \quad \text{and} \quad n \geq N \quad \text{implies} \quad K_n(t) \leq \frac{\epsilon}{16M\pi}. \tag{6}$$

Using our rewritten σ_N,

$$\begin{aligned}
|\sigma_N(x) - f(x)| &= \left| \int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x) \right| \\
&= \left| \int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \int_{-\pi}^{\pi} f(x)K_N(t)dt \right| && \text{(by Lemma 5)} \\
&= \left| \int_{-\pi}^{\pi} \left(f(x-t) - f(x)\right) K_N(t)dt \right| \\
&\leq \int_{-\pi}^{\pi} \left| \left(f(x-t) - f(x)\right) K_N(t) \right| dt = \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t)dt. \\
& && \text{(by Lemma 7)}
\end{aligned}$$

For any N, if $|t| \leq \delta$, then $|t| = |(x-t) - x| \leq \delta$, and

$$\begin{aligned}
\int_{-\delta}^{\delta} \left| [f(x-t) - f(x)] \right| K_N(t)dt &\leq \frac{\epsilon}{4} \int_{-\delta}^{\delta} K_N(t)dt && \text{(using (5))} \\
&\leq \frac{\epsilon}{4}. && \text{(by Lemma 5 and 7)}
\end{aligned}$$

If $\delta \leq |t| \leq \pi$, from (6) we can pick $n \geq N$ such that

$$\begin{aligned}
&\int_{-\pi}^{-\delta} |f(x-t) - f(x)| K_N(t)dt + \int_{\delta}^{\pi} |f(x-t) - f(x)| K_N(t)dt \\
&\qquad \leq \frac{\epsilon}{16M\pi} \left[\int_{-\pi}^{-\delta} |f(x-t) - f(x)|dt + \int_{\delta}^{\pi} |f(x-t) - f(x)|dt \right].
\end{aligned}$$

Further, with our choice of M at the beginning of the proof,

$$\frac{\epsilon}{16M\pi}\left[\int_{-\pi}^{-\delta}|f(x-t)-f(x)|dt+\int_{\delta}^{\pi}|f(x-t)-f(x)|dt\right]\leq\frac{\epsilon}{16M\pi}\int_{-\pi}^{\pi}|f(x-t)-f(x)|dt$$
$$\leq\frac{\epsilon}{16M\pi}(4M\pi)=\frac{\epsilon}{4}.$$

Thus, we may conclude that

$$\int_{-\pi}^{-\delta}|f(x-t)-f(x)|K_N(t)dt+$$
$$\int_{-\delta}^{\delta}|f(x-t)-f(x)|K_N(t)dt+\int_{\delta}^{\pi}|f(x-t)-f(x)|K_N(t)dt\leq\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2}<\epsilon.$$ [2]

□

The success of this argument relies on the surprising fact hiding in Lemma 8 that we can bound $K_N(x)$ without appealing to x. We are not so lucky with $D_N(x)$. As a result, S_N only converges pointwise to f while σ_N converges to it uniformly.

4 Exploring Weierstrass's Monster Fourier Series

Until the late nineteenth century, many if not most mathematicians believed that if a function is continuous, it is differentiable at all but a countable number of points. In fact, André-Marie Ampère had a "proof" of this claim that by the 1850s had found inclusion in nearly every calculus textbook. The intuitive credentials of this idea are quite strong. However, in 1872, Karl Weierstrass announced, with proof, that he had identified a function that was continuous everywhere and differentiable nowhere. Weierstrass's discovery shook the world of mathematics. His concern for rigor, something that the the early developers of calculus had neglected in favor of physical intuitions, could be ignored no longer. Many of Weierstrass's contemporaries were dismayed by his finding, with Henri Poincaré going so far as to call the function a "monster" [6]. But by drawing attention to the insufficiency of geometric intuitions and offering a more technical, precise method for reasoning about these ideas, Weierstrass's work enabled the development of a better mathematics of the infinite—modern analysis.

When Weierstrass discovered the monster, he expressed it as a Fourier series. In this section, we define the Weierstrass function and show that it is indeed continuous everywhere and differentiable nowhere.

For $x\in\mathbb{R}$, $a\in(0,1)$, and b, an odd integer such that $ab>1+\frac{3\pi}{2}$, define the Weierstrass function

$$\omega(x)=\sum_{n=0}^{\infty}a^n\cos(b^n\pi x).$$

To demonstrate that ω is continuous everywhere, we need three lemmas. The first, I state without proof for a shortage of space in this paper.

Lemma 9 (The Weierstrass M-Test)**.** *Let f_n be a sequence of functions on a set X. If there exists a sequence M_n of positive numbers such that $\sum_{n=1}^{\infty}M_n$ converges and $|f_n(x)|\leq M_n$ for all $x\in X$ and each $n\in\mathbb{P}$, then $\sum_{n=1}^{\infty}f_n$ converges uniformly on X.*

[2] In this proof, I have followed the strategy in [5]

Lemma 10. *Let f_n be a sequence of functions which converges uniformly to f on $\mathbb{R}$. If each f_n is continuous at a point x_0 in $\mathbb{R}$, then f is continuous at x_0.*

Proof. Let $\epsilon > 0$ and denote the function to which f_n converges uniformly f. By uniform convergence, for any $x \in \mathbb{R}$ we can find N such that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3},$$

and

$$|f_N(x_0) - f(x_0)| < \frac{\epsilon}{3}.$$

By continuity of each f_n, we can find $\delta > 0$ for which $|x - x_0| < \delta$ implies that

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}.$$

Now $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Lemma 11. $\left|\frac{\sin(x)}{x}\right| \leq 1$ *for all* $x \in \mathbb{R} \setminus \{0\}$.

Proof. Let $f(x) = |x| - |\sin(x)|$ for $x \in \mathbb{R}$. Applying the chain rule to the absolute value function,

$$f'(x) = \begin{cases} -1 - \cos(x) & \text{if } -\pi < x \mod 2\pi < 0 \\ -1 + \cos(x) & \text{if } -2\pi < x \mod 2\pi < -\pi \\ 1 - \cos(x) & \text{if } 0 < x \mod 2\pi < \pi \\ 1 + \cos(x) & \text{if } \pi < x \mod 2\pi < 2\pi \end{cases}.$$

Thus, for $x > 0$, f is non-decreasing everywhere it is differentiable. Now suppose that at some point $x_1 > 0$ where f is not differentiable, $0 < x_0 < x_1$ and $f(x_0) > f(x_1)$. We can construct a non-decreasing sequence s_n with $s_n \in [x_0, x_1)$ and $s_0 = x_0$ that converges to x_1. By non-negativity of f', $f(s_n)$ is also non-decreasing. By continuity of f, $f(s_n)$ must converge to $f(x_1)$. Moreover, since $f(s_n)$ is non-decreasing, for each $n \in \mathbb{N}$, $f(s_n) \leq f(x_1)$. However, this is a contradiction with $f(x_0) > f(x_1)$. A similar argument establishes that $x_1 < x_2$ also implies $f(x_1) < f(x_2)$. We have established that for $x > 0$, f is non-decreasing everywhere. A parallel argument establishes that for $x < 0$, f is non-increasing everywhere. This implies that f attains a global minimum at 0. Since $f(0) = 0$, for $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &\geq 0 \\ |x| - |\sin(x)| &\geq 0 \\ 1 &\geq \left|\frac{\sin(x)}{x}\right|. \end{aligned}$$

□

We're now ready for the proof of Weierstrass's famous result.

Theorem 3. *For each $x \in \mathbb{R}$, ω is continuous but not differentiable.*

Proof. We address continuity first. Clearly, for each $x \in \mathbb{R}$, $\sum_{n=0}^{\infty} |a^n \cos(b^n \pi x)|$ is bounded above by $\sum_{n=0}^{\infty} a^n$. Since $a \in (0,1)$, $\sum_{n=0}^{\infty} a^n$ converges and each term in the sum is positive. Consequently, we can apply the Weierstrass M-Test to conclude that $\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ converges uniformly on $\mathbb{R}$. Now, each term in $\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ is continuous on $\mathbb{R}$, and since a sum of continuous functions on $\mathbb{R}$ is continuous on $\mathbb{R}$, by Lemma 10, $\omega(x)$ is continuous on $\mathbb{R}$.

We now address differentiability. We will show that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

does not exist. To this end, fix $x_0 \in \mathbb{R}$ and define $c_m \in \mathbb{Z}$ such that for each $m \in \mathbb{N}$

$$b^m x_0 - c_m \in \left(-\frac{1}{2}, \frac{1}{2} \right].$$

Furthermore, define

$$x_m = b^m x_0 - c_m, \quad y_m = \frac{c_m - 1}{b^m}.$$

We notice that

$$y_m - x_0 = -\frac{x_m + 1}{b^m} < 0, \tag{7}$$

implying that $y_m < x_0$. With this fact in hand, observe that

$$\lim_{m \to \infty} |y_m - x_0| = \lim_{m \to \infty} (x_0 - y_m) = \lim_{m \to \infty} \frac{x_m + 1}{b^m} = 0.$$

Thus, we have identified a sequence that converges to x_0. To complete the proof, we will show that

$$\lim_{m \to \infty} \frac{f(y_m) - f(x_0)}{y_m - x_0}$$

does not exist. This is sufficient to show that the limit of the difference quotient at x_0 does not exist. If this limit did exist, then for every sequence converging to x_0, the difference quotient would converge to the same value. To begin this argument, observe that

$$\begin{aligned}
\frac{f(y_m) - f(x_0)}{y_m - x_0} &= \frac{\sum_{n=0}^{\infty} a^n \cos(b^n \pi y_m) - \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_0)}{y_m - x_0} \\
&= \frac{\sum_{n=0}^{\infty} [a^n \cos(b^n \pi y_m) - a^n \cos(b^n \pi x_0)]}{y_m - x_0} \\
&= \sum_{n=0}^{\infty} \frac{a^n [\cos(b^n \pi y_m) - \cos(b^n \pi x_0)]}{y_m - x_0} \\
&= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)} + \sum_{n=0}^{\infty} a^{n+m} \frac{\cos(b^{n+m} \pi y_m) - \cos(b^{n+m} \pi x_0)}{y_m - x_0}.
\end{aligned}$$

Let S_1 represent the leftmost summation in the previous expression and S_2 the rightmost. Applying the trigonometric identity $\cos x - \cos y = -2 \sin(\frac{x+y}{2}) \sin(\frac{x-y}{2})$,

$$\begin{aligned}
S_1 &= \sum_{n=0}^{m-1} \frac{-2(ab)^n}{b^n (y_m - x_0)} \sin\left(\frac{b^n \pi (y_m + x_0)}{2} \right) \sin\left(\frac{b^n \pi (y_m - x_0)}{2} \right) \\
&= \sum_{n=0}^{m-1} -\pi (ab)^n \sin\left(\frac{b^n \pi (y_m + x_0)}{2} \right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2} \right)}{\frac{b^n \pi (y_m - x_0)}{2}}.
\end{aligned}$$

Applying the triangle inequality, the fact that $|\sin(x)| \leq 1$, and Lemma 11,

$$|S_1| \leq \sum_{n=0}^{m-1} \pi (ab)^n.$$

The sum is a finite geometric series. Using the fact that for $r \neq 1$, $\sum_{k=0}^{n-1} ar^k = a(\frac{1-r^n}{1-r})$,

$$\sum_{n=0}^{m-1} \pi(ab)^n = \pi \frac{1-(ab)^m}{1-ab} = \pi \frac{(ab)^m - 1}{ab-1} < \pi \frac{(ab)^m}{ab-1}.$$

This implies that there exists $\epsilon_1 \in (-1, 1)$ such that $S_1 = \epsilon_1 \pi \frac{(ab)^m}{ab-1}$. Now, we turn our attention to S_2. We can simplify $\cos(b^{n+m}\pi y_m)$ by noticing that

$$\begin{aligned}
\cos(b^{n+m}\pi y_m) &= \cos(b^n \pi (c_m - 1)) && \text{(by the definition of } y_m\text{)} \\
&= (-1)^{b^n(c_m-1)} && \text{(since } b \text{ and } c_m \text{ are integers)} \\
&= (-1)^{(c_m-1)} && \text{(since } b \text{ is an odd integer)} \\
&= -(-1)^{c_m}.
\end{aligned}$$

We can also simplify $\cos(b^{n+m}\pi x_0)$ by observing that

$$\cos(b^{n+m}\pi x_0) = \cos(b^n \pi (x_m + c_m)). \qquad \text{(by the definition of } x_m\text{)}$$

Using that $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$,

$$\begin{aligned}
\cos(b^n\pi(x_m + c_m)) &= \cos(b^n\pi x_m)\cos(b^n\pi c_m) - \sin(b^n\pi x_m)\sin(b^n\pi c_m) \\
&= \cos(b^n\pi x_m)(-1)^{b^n c_m} && \text{(since } b \text{ and } c_m \text{ are integers)} \\
&= (-1)^{c_m}\cos(b^n\pi x_m). && \text{(since } b \text{ is odd)}
\end{aligned}$$

With these simplifications,

$$\begin{aligned}
S_2 &= \sum_{n=0}^{\infty} a^{n+m} \frac{-(-1)^{c_m} - (-1)^{c_m}\cos(b^n\pi x_m)}{y_m - x_0} \\
&= \sum_{n=0}^{\infty} a^{n+m}(-1)(-1)^{c_m} \frac{1 + \cos(b^n\pi x_m)}{y_m - x_0} \\
&= \sum_{n=0}^{\infty} a^{n+m}(-1)^{c_m} \frac{1 + \cos(b^n\pi x_m)}{\frac{x_m+1}{b^m}} && \text{(with (7))} \\
&= (ab)^m(-1)^{c_m} \sum_{n=0}^{\infty} a^n \frac{1 + \cos(b^n\pi x_m)}{x_m + 1}
\end{aligned}$$

We have chosen c_m so that $x_m \in (-\frac{1}{2}, \frac{1}{2}]$. Consequently, each term in $\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n\pi x_m)}{x_m+1}$ is non-negative. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} a^n \frac{1 + \cos(b^n\pi x_m)}{x_m + 1} &\geq \frac{1 + \cos(\pi x_m)}{x_m + 1} \\
&\geq \frac{1}{\frac{1}{2} + 1} = \frac{2}{3}. && (x_m \in (-\tfrac{1}{2}, \tfrac{1}{2}])
\end{aligned}$$

This implies that there exists $\epsilon_2 \geq 1$ such that $S_2 = (ab)^m(-1)^{c_m}\epsilon_2 \frac{2}{3}$. Joining these strands of thought, we obtain

$$\begin{aligned}
\frac{f(y_m) - f(x_0)}{y_m - x_0} &= \epsilon_1 \pi \frac{(ab)^m}{ab-1} + (ab)^m(-1)^{c_m}\epsilon_2 \frac{2}{3} \\
&= (ab)^m(-1)^{c_m}\epsilon_2 \left(\frac{\epsilon_1}{\epsilon_2} \frac{\pi}{(ab-1)}(-1)^{c_m} + \frac{2}{3} \right).
\end{aligned}$$

Restating the definition, $ab > 1 + \frac{3\pi}{2}$, so $\frac{\pi}{ab-1} < \frac{2}{3}$. Thus,

$$\frac{\epsilon_1}{\epsilon_2}\frac{\pi}{(ab-1)}(-1)^{c_m} + \frac{2}{3} > -\frac{\pi}{(ab-1)} + \frac{2}{3} > 0.$$

And,

$$\left|\frac{f(y_m) - f(x_0)}{y_m - x_0}\right| > (ab)^m \left(-\frac{\pi}{(ab-1)} + \frac{2}{3}\right).$$

As $m \to \infty$, $\left|\frac{f(y_m)-f(x_0)}{y_m-x_0}\right|$ grows without bound. Thus, as promised, $\lim_{m\to\infty} \frac{f(y_m)-f(x_0)}{y_m-x_0}$ does not exist.[3] $\square$

[3] In this proof, I have followed the strategy in [3].

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