Exercise: Fourier

1. Prove that if $x \in L^1$, then $\hat{x}$ is continuous.

2. (Riemann-Lebesgue thm) Show that if $x \in L^1$, then $\lim_{f\to\infty} \hat{x}(f) = 0$. (Hint: Prove is first for C^1 functions with compact support using integration-by-part, then use a density argument).

3. Let $F_s = \frac{1}{T_s}$, $x \in \mathcal{L}^2$ such that $\mathcal{F}x$ is supported on $[-F_s, F_s]$, and $m_j = \frac{1}{\sqrt{T_s}}\mathrm{sinc}\left(\frac{t-jT_s}{T_s}\right)$.

   (a) Show that $\langle m_j, m_i\rangle_{L^2} = 0$ and $\|m_j\|_{L^2} = 1$ for all $j \neq i$.

   (b) Why is $\{m_j\}$ an orthonormal basis for signals whose Fourier transform is supported on $[-F_s, F_s]$?

   (c) Show that $\langle x, m_j\rangle_{L^2} = x(jT_s)$.

4. Exo:

   (a) Show that if $\varphi_n \in \mathcal{D}$ converge to $\varphi \in \mathcal{D}$, then it also converges in $\mathcal{S}$.

   (b) Construct $\varphi_n \in \mathcal{S}$ that converges to 0 in $\mathcal{S}$, but not in $\mathcal{D}$.

5. Show that e^t is not a tempered distribution. (Hint: test it against a test function $\varphi(\cdot - a)$ for growing a, use condition for continuity)

6. Let a_k be complex numbers and $T = \sum_{k\geq 0} a_k\delta_k$. Show that T is a tempered distribution if and only if $|a_k| \leq C(1+k)^p$ for some C, p. (Hint: for the direct implication, use a test function with small compact support and shift it to express the a_k, then use continuity.)