# Sparse Signal Processing
# Lecture 4 — Fourier interpolation and the linear model

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# Outline

N. Keriven: CNRS researcher at IRISA, SIROCCO team.
https://nkeriven.github.io

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- introduce a linear target set ;
- derive a decoder achieving exact recovery ;

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- introduce a linear target set ;
- derive a decoder achieving exact recovery ;

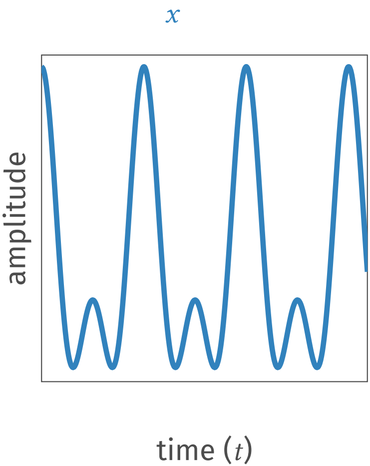
Material at `https://github.com/nkeriven/ENS-signal` (subject to changes, check every now and then)

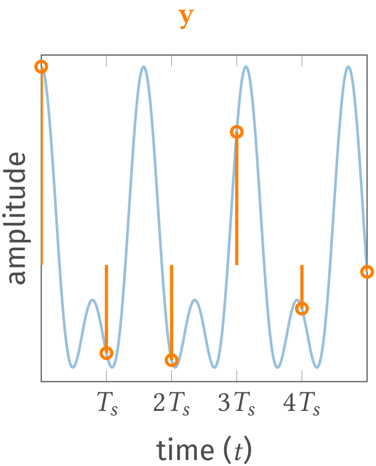
# References

- Mallat, 2012. *A Wavelet Tour of Signal Processing: the Sparse Way*
- F Golse polycopié: `http://www.math.polytechnique.fr/golse/MAT431-10/POLY431.pdf`
- Wikipédia !

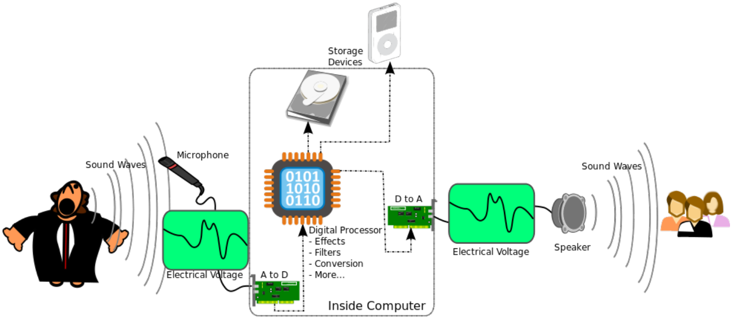
# Considered Task

Recovering a signal $x$ from **infinitely** many function evaluations



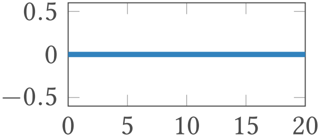
# Application 1: sampling and reconstructing audio recording



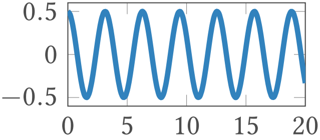

**Important component of sound perception:** the frequency

# Application 2: (simplified) transmitters

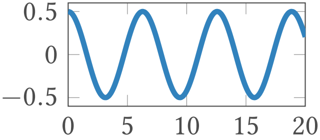
Sequence “00”



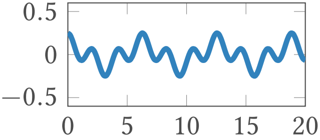

Sequence “01”




Sequence “10”




Sequence “11”




**Support of information:** electromagnetic (sinusoidal at a given frequency) waves

# Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{L}^2(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned}$$

2. ***The target set***: *To be defined*

   *Related to some notion of frequency*

3. ***The decoder***: *To be defined*

4. ***The accuracy criterion***: “noiseless exact recovery”, *i.e.*

$$\forall x \in \mathcal{X}_{\mathtt{target}}, \quad D\big(M(x)\big) = x$$

🤔 Which mathematical tool can be used to emphasize the frequencies characterizing a signal?

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☝️ The Fourier transform

# Rappels: The Fourier transform

## Definition

**The Fourier transform** is a linear operator defined by

$$\begin{aligned} \mathcal{F}\colon \mathcal{L}^1(\mathbf{R}) &\longrightarrow \mathcal{L}^1(\mathbf{R}) \\ x &\longmapsto \left(f \mapsto \int_{-\infty}^{+\infty} x(t) \mathrm{e}^{-\mathrm{i}2\pi f t} \mathrm{d}t\right) \end{aligned}$$

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- Sometimes defined without the $2\pi$ (but less clean, multiplicative constants everywhere)
- $\mathcal{F}x$ is often written $\hat{x}$

# An example

**Exercice**

Evaluate the Fourier transform of

$$\Pi_{[-a/2,a/2]} : t \longmapsto \begin{cases} 1 & \text{if } t \in [-\frac{a}{2}, +\frac{a}{2}] \\ 0 & \text{otherwise} \end{cases}$$

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**Answer**

$$\mathcal{F}\Pi(f) = a\frac{\sin(\pi a f)}{\pi a f} \overset{\text{def.}}{=} a \operatorname{sinc}(af)$$

# Rappels: Inverse Fourier transform

**Definition**

**The conjugate Fourier transform** is the linear operator defined by

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## Theorem: Inversion

If $x \in \mathcal{L}^1(\mathbf{R})$ is such that $\mathcal{F}x \in \mathcal{L}^1(\mathbf{R})$ then

$$\overline{\mathcal{F}}[\mathcal{F}x] \overset{\text{a.e.}}{=} x$$

with equality at every point of continuity.

# Extension to $L^2$

Pbm: $x \in \mathcal{L}^1$ does **not** imply $\mathcal{F}x \in \mathcal{L}^1$! (ex: door function and cardinal sinus)

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$\rightarrow$ **Solution:** extension by *density*.

# Parseval's identity

## Parseval's identity

If $f, g \in \mathcal{L}^1 \cap \mathcal{L}^2$,

$$\langle f, g \rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}$$

(recall that $\langle f, g \rangle_{L^2} = \int f(t)\overline{g(t)}dt$)

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We are going to extend Fourier to $\mathcal{L}^2$ by *density* of $\mathcal{L}^1 \cap \mathcal{L}^2$ within $\mathcal{L}^2$

$\rightarrow$ i.e., $\forall \epsilon > 0, f \in \mathcal{L}^2, \exists g \in \mathcal{L}^2 \cap \mathcal{L}^1$ s.t. $\|f - g\|_{\mathcal{L}^2} \leq \epsilon$

# Extension to $\mathcal{L}^2$

- Let $f \in \mathcal{L}^2$.
- By density of $\mathcal{L}^2 \cap \mathcal{L}^1$ in $\mathcal{L}^2$, there is a sequence $f_n \in \mathcal{L}^2 \cap \mathcal{L}^1$ such that $\|f_n - f\|_{\mathcal{L}^2} \to 0$

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- $f_n$ is a Cauchy sequence, and by Parseval's identity $\|f_n - f_p\|_{\mathcal{L}^2} = \|\mathcal{F}(f_n - f_p)\|_{\mathcal{L}^2}$, hence $\mathcal{F}f_n \in \mathcal{L}^2$ is also a Cauchy sequence

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- Since $\mathcal{L}^2$ is a Hilbert space and **is therefore complete**, $\mathcal{F}f_n$ is a convergent sequence.
- We call its limit the “Fourier transform of $f$” denoted by $\mathcal{F}f$. It satisfies all the usual properties of the Fourier transform (see next slide)

# Properties of Fourier Transform (…easy, or admitted)

- $f$ symmetric $\Rightarrow \mathcal{F}f$ real
- $f$ anti-symmetric $\Rightarrow \mathcal{F}f$ imaginary
- $f$ real $\Rightarrow \mathcal{F}f$ Hermitian
- $f$ imaginary $\Rightarrow \mathcal{F}f$ (conjugate) anti-symmetric

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- $\mathcal{F}(f \star g)(t) = \mathcal{F}f(t)\mathcal{F}g(t)$ (convolution thm)
- $\langle f, g\rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2}$ (Parseval identity)
- $\mathcal{F}\frac{df}{dt}(t) = 2i\pi t\mathcal{F}f(t) \Rightarrow$ if $f$ is $k$ times differentiable, $\mathcal{F}f$ decrease faster than $1/t^k$

# Properties (summary)

| Property | Function | Fourier transform |
|---|---|---|
| | $x(t)$ | $\hat{x}(f)$ |
| Inverse | $\hat{x}(t)$ | $x(-f)$ |
| Convolution | $x_1 \star x_2(t)$ | $\hat{x}_1 \hat{x}_2(f)$ |
| Multiplication | $x_1(t)x_2(t)$ | $\hat{x}_1 \star \hat{x}_2(f)$ |
| Translation | $x(t-u)$ | $e^{-2i\pi uf}\hat{x}(f)$ |
| Modulation | $e^{2i\pi\xi t}x(t)$ | $\hat{x}(f-\xi)$ |
| Scaling | $x(t/s)$ | $\|s\|\hat{x}(sf)$ |
| Time derivatives | $x^{(p)}(t)$ | $(2i\pi f)^p\hat{x}(f)$ |
| Freq. derivatives | $(-2i\pi t)^p x(t)$ | $\hat{x}^{(p)}(f)$ |
| Complex conjugate | $\overline{x}(t)$ | $\overline{\hat{x}}(-f)$ |

# Uncertainty principle

Can we construct a function that is **highly localised in space**, whose Fourier transform is **highly localised in frequency**?

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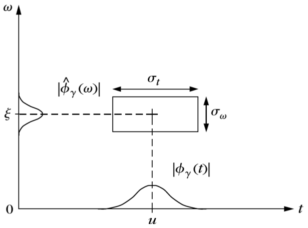
## Heisenberg's uncertainty principle

For $x \in \mathcal{L}^2(\mathbf{R})$, define the two density functions

$p_x = \frac{|x(t)|^2}{\|x\|_{L^2}}, \quad p_{\hat{x}} = \frac{|\hat{x}(f)|^2}{\|x\|_{L^2}}$ and their respective variance $\sigma_x^2$, $\sigma_{\hat{x}}^2$. Then

$$\sigma_x^2 \sigma_{\hat{x}}^2 \geq 1/4$$



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The function of interest, namely $t \longmapsto \cos(2\pi f t)$ does not belong to $\mathcal{L}^1(\mathbf{R})$!

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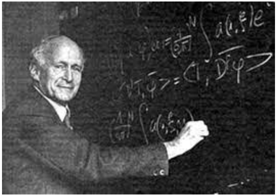
We need a “bigger” space to work with!

$\rightarrow$ Space of “distributions”, defined by *duality*.

We need the smallest dual space as possible, to make the biggest distribution space as possible!

# Reminder (?) on distributions



# Tests functions

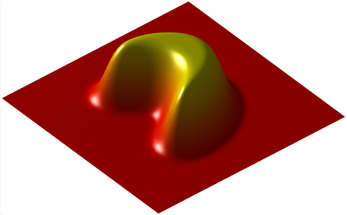
**Definition**

A test function refers to any infinitely differentiable function

$$\varphi \colon \mathbf{R} \longrightarrow \mathbf{C}$$

witch is compactly supported.

The set of test functions is denoted $\mathcal{D}(\mathbf{R})$.



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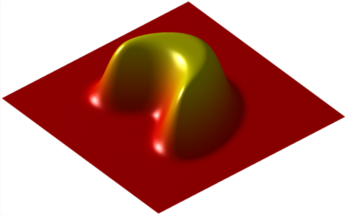
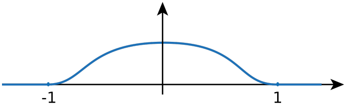


**Example:** *(standard one)* the function

$$t \longmapsto \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & \text{if } t \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$




can be proved to be a test function.

# Topology on $\mathcal{D}(\mathbf{R})$

**Definition: convergence in $\mathcal{D}(\mathbf{R})$**

Let $\{\varphi_n\}_{n\in\mathbf{N}}$ be a family of test function.
We say that $\varphi_n \to \varphi \in \mathcal{D}(\mathbf{R})$ if and only if

- there exists an integer $N$ and a compact set $K$ such $n \geq N \Rightarrow \text{support}(\varphi_n - \varphi) \subset K$
- for all $r \in \mathbf{N}$,
$$\left\|\varphi_n^{(r)} - \varphi^{(r)}\right\|_\infty \to 0$$
- i.e., supremum norm for $\varphi$ and all its derivatives.

# Distributions

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**Remarks**

- Said differently, the space of distributions is the (topological) dual space of $\mathcal{D}(\mathbf{R})$, denoted by $\mathcal{D}'(\mathbf{R})$;
- When evaluating a distribution $T$ at $\varphi$, we write

$$\langle T, \varphi \rangle$$

- By extension, we say that two distributions $T, T'$ are equal iff

$$\langle T, \varphi \rangle = \langle T', \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbf{R})$$

⚠️

Distribution are often referred to as “generalized functions”.

This can nevertheless be a misleading terminology, as distributions do not take scalar as inputs. (...but *test functions*)

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Besides mathematical benefits, there is a physical intuition behind this reference

A compactly supported test functions can be seen as a model of a physical system measuring some quantity

# Regular distributions

**Definition/Proposition**

If $g$ is any locally integrable function, then the linear form $T_g$ defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T_g, \varphi \rangle \stackrel{\text{def.}}{=} \int g(t)\varphi(t)\mathrm{d}t$$

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*In short: Linearity is fine*
*Continuity: letting $\varphi_n \to \varphi$, there exists a compact $K$ that contains all supports*
*Then $|\langle T_g, \varphi_n - \varphi \rangle| \leq \int g(t)\mathrm{d}t \|\varphi_n - \varphi\|_\infty$*

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**Remark**

- Hence distributions is a "bigger" space...
- If $T_{g_1} = T_{g_2}$ then $g_1 \stackrel{\text{a.e.}}{=} g_2$

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**Example 1:** The dirac function $\delta_t$ defined $\forall t \in \mathbf{R}$ as

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**Example 2:** The dirac comb $\text{Ш}_{T_s}$ defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \text{Ш}_{T_s}, \varphi \rangle = \sum_{j \in \mathbf{Z}} \langle \delta_{jT_s}, \varphi \rangle = \sum_{j \in \mathbf{Z}} \varphi(jT_s)$$

*Letting* $\varphi_n \to \varphi$*, there exists a compact* $K$ *that contains all supports, and* $|\langle \delta_t, \varphi_n - \varphi \rangle| \leq K_{T_s} \|\varphi_n - \varphi\|_\infty \to 0$*, where* $K_{T_s}$ *is the number of* $jT_s$ *contained in* $K$

🤔 **Regular distributions** provide intuition on how to define many concepts related to distributions

*derivation, convolution, Fourier transform to name but a few*

# Derivation of a distribution

**For regular distributions**: let $g$ be a differentiable locally integrable function. Then $g'$ is also locally integrable and

$$\begin{aligned}
\langle T_{g'}, \varphi \rangle &= \int g'(t)\varphi(t)\mathrm{d}t \\
&= -\int g(t)\varphi'(t)\mathrm{d}t \qquad \text{(IBP)} \\
&= -\langle T_g, \varphi' \rangle
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**Definition**

Let $T$ be a distribution. The derivative of $T$ is the distribution $T^{(1)}$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T^{(1)}, \varphi \rangle \overset{\text{def.}}{=} -\langle T, \varphi' \rangle$$

# A second example: Fourier transform

Let $g$ be an integrable function (so that $T_g$ is a regular distribution). Then $\mathcal{F}g$ is also integrable and

$$\begin{aligned}
\langle T_{\mathcal{F}\mathrm{g}}, \varphi \rangle &= \int \mathcal{F}\mathrm{g}(f)\varphi(f)\mathrm{d}f \\
&= \int \left( \int \mathrm{g}(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}t \right) \varphi(f)\mathrm{d}f \\
&= \int \mathrm{g}(t) \left( \int \varphi(f)\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}f \right) \mathrm{d}t \\
&\overset{(?)}{=} \langle T_{\mathrm{g}}, \mathcal{F}\varphi \rangle
\end{aligned}$$

# Fourier transform of a distribution (tentative)

**Definition**

Let $T$ be a distribution. The Fourier of $T$ is the distribution $\mathcal{F}T$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \stackrel{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

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- (need complex-valued test functions to technically make sense)

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Let $T$ be a distribution. The Fourier of $T$ is the distribution $\mathcal{F}T$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

- (need complex-valued test functions to technically make sense)

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The set of distributions is too big a space!!

# Tempered distribution

We will **extend** the set of test functions...

☝

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☝️

... which will **reduce** the set of distributions (by duality).

# Tempered distributions

**Definition: Schwartz space**

We say that a function $g$ is a Schwartz function if it is infinitely differentiable and for all $k, q \in \mathbf{N}$

$$\lim_{t \longmapsto \pm\infty} |t^k \varphi^{(q)}(t)| = 0$$

The set of Schwartz functions is denoted $\mathcal{S}(\mathbf{R})$.

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**Definition: Tempered distribution**

We call "tempered distribution" any element of the (topological) dual space of the Schwartz space $\mathcal{S}'(\mathbf{R})$.

*"Topological": **continuous** linear functionals*

# Properties

- Convergence in $\mathcal{S}(\mathbf{R})$ is defined through the family of semi-norms $\mathcal{N}_p(\varphi) = \sum_{k,q\leq p} \sup_t |t^q \varphi^{(q)}(t)|$, i.e. $\varphi_n$ converge to $\varphi$ if $\forall p,\ \mathcal{N}_p(\varphi_n - \varphi) \to 0$

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- For a tempered distribution $T$, there is $p, C$ such that $\forall \varphi \in \mathcal{S}(\mathbf{R})$, $\langle T, \varphi \rangle \leq C\mathcal{N}_p(\varphi)$ (continuity)
- Any test function is a Schwartz function: $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$
- (hence) any tempered distribution is a distribution: $\mathcal{S}'(\mathbf{R}) \subset \mathcal{D}'(\mathbf{R})$

# Fourier and tempered distributions

**Thm**

- Thm: All tempered distribution **admit a Fourier transform** defined by

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

- The Fourier transform of a Schwartz function is **also** a Schwartz function

*Hint: link between differentiability and polynomial decrease of the Fourier transform...*

# Classical tempered distribution

Unlike regular distributions, all functions do not correspond to tempered distributions! Need all derivative to grows “as fast as” polynomials...

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$\rightarrow$ that’s all we need in this course...?

⚠️

In signal processing, it is folklore to believe that distributions are always well-behaved

*That it all operations are legitimate*

It it nevertheless not always true

☝️

In this lecture (and companion lab) it will nevertheless be true

*Some justifications will be out of scope and left in exercise*

# Some properties of (tempered) distributions

# Some Fourier transform (1)

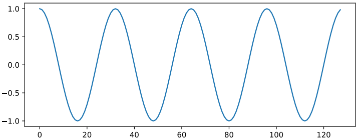
$$
\begin{aligned}
\langle \mathcal{F}\delta_a, \varphi \rangle &= \langle \delta_a, \mathcal{F}\varphi \rangle \\
&= (\mathcal{F}\varphi)(a) \\
&= \int \varphi(t) \mathrm{e}^{-2\mathrm{i}\pi a t} \mathrm{d}t \\
&= \langle T_{t \mapsto \mathrm{e}^{-2\mathrm{i}\pi a t}}, \varphi \rangle
\end{aligned}
$$

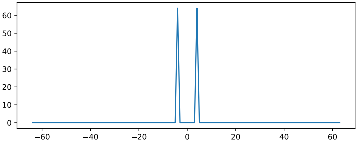
# Some Fourier transform (2)

$$
\begin{aligned}
\langle \mathcal{F} T_{t \mapsto e^{2i\pi at}}, \varphi \rangle &\overset{\text{def.}}{=} \langle T_{t \mapsto e^{2i\pi at}}, \mathcal{F}\varphi \rangle \\
&\overset{\text{def.}}{=} \int e^{2i\pi af} \mathcal{F}\varphi(f) \mathrm{d}f \\
&\overset{\text{def.}}{=} \mathcal{F}^{-1}\mathcal{F}\varphi(a) \\
&= \langle \delta_a, \varphi \rangle
\end{aligned}
$$

# Some Fourier transform (3)

$$
\begin{aligned}
\langle \mathcal{F}T_{t\mapsto\cos(2\pi at)}, \varphi\rangle &= \langle \mathcal{F}T_{t\mapsto 0.5(\mathrm{e}^{2\pi at}+\mathrm{e}^{-2\pi at}}, \varphi\rangle \\
&= \frac{1}{2}\langle \mathcal{F}T_{t\mapsto \mathrm{e}^{2\pi at}}, \varphi\rangle + \frac{1}{2}\langle \mathcal{F}T_{t\mapsto \mathrm{e}^{-2\pi at}}, \varphi\rangle \\
&= \frac{1}{2}(\langle \delta_a, \varphi\rangle + \langle \delta_{-a}, \varphi\rangle)
\end{aligned}
$$



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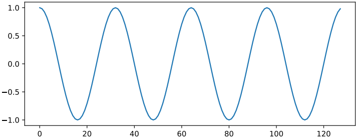


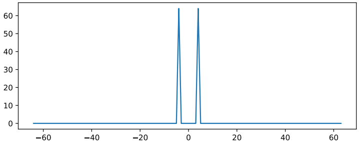


Recall that $\mathcal{F}[Re(f)](\xi) = \frac{1}{2}(\mathcal{F}f(\xi) + \mathcal{F}f(-\xi))$

Have we achieved our initial objective?

🤔 Have we achieved our initial objective?

☝️ It is folklore to provide a graphical representation of a Fourier transform

# Another example: the “door” distribution

We define $\Pi_{[-a/2, a/2]}$ such that

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \Pi_{[-a/2, a/2]}, \varphi \rangle = \int_{-a/2}^{a/2} \varphi(t) \mathrm{d}t$$

**Question:** Evaluate the Fourier transform of $\Pi_{[-a/2, a/2]}$.

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**Question:** Evaluate the Fourier transform of $\Pi_{[-a/2,a/2]}$.

$$\begin{aligned}
\langle \mathcal{F}\Pi_{[-a/2,a/2]}, \varphi\rangle &= \langle \Pi_{[-a/2,a/2]}, \mathcal{F}\varphi\rangle \\
&\overset{\text{Fub.}}{=} \int \varphi(t) \int_{-a/2}^{a//2} \mathrm{e}^{-2\mathrm{i}ft}\mathrm{d}f\,\mathrm{d}t \\
&= \langle T_{a\,\mathrm{sinc}(a\cdot)}, \varphi\rangle
\end{aligned}$$

🤔 Does it ring a bell?

# Compatibility with the "standard" Fourier transform

**Theorem (admitted)**

If $x \in \mathcal{L}^1(\mathbf{R})$ (or $\mathcal{L}^2(\mathbf{R})$) then

$$\mathcal{F} T_x = T_{\mathcal{F} x}$$

# Towards an inverse Fourier transform

**Definition: Conjugate Fourier transform**

Let $T$ be a tempered distribution. The conjugate Fourier of $T$ is the distribution $\overline{\mathcal{F}}T$ defined by

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \overline{\mathcal{F}}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \overline{\mathcal{F}}\varphi \rangle$$

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**Theorem: Inversion formula**

Let $T$ be a tempered distribution.
Then, its Fourier transform satisfies

$$T = \overline{\mathcal{F}}[\mathcal{F}T]$$

In other words, the standard inversion formula holds in the sense of tempered distributions

# Convolution

**Definition**

Let $T_1, T_2$.
Their convolution (if it exists) is the distribution $T_1 * T_2$ defined as

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle T_1 * T_2, \varphi \rangle \stackrel{\text{def.}}{=} \langle T_1, \varphi_{T_2} \rangle$$

where

$$\forall t \in \mathbf{R}, \qquad \varphi_{T_2}(t) = \langle T_2, \varphi(\cdot + t) \rangle$$

*($T_1, T_2$ can be exchanged in the definition)*

**Example:** of $T_1 = \delta_a$

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \delta_a * T_2, \varphi \rangle = \langle T_2, \varphi(\cdot - a) \rangle$$

$\rightarrow$ Translation of the distribution

# Support of a distribution

**Definition**

Let $T \in \mathcal{S}(\mathbf{R})'$.
Define $\Omega$ as the largest open set of $\mathbf{R}$ such that

$$\text{support}(\varphi) \subseteq \Omega \implies \langle T, \varphi \rangle = 0$$

By extension, the (closed) set $\text{support}(T) \overset{\text{def.}}{=} \Omega^c$ is called the **support** of the distribution

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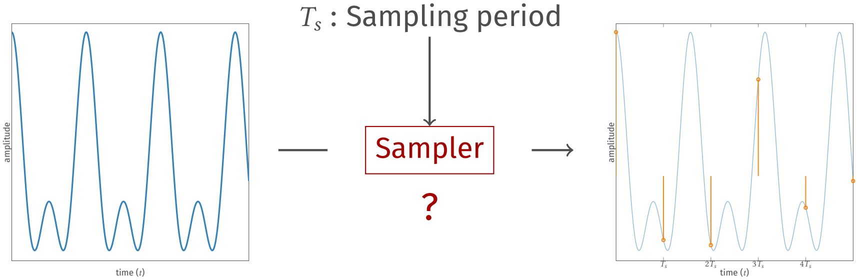
**Examples:**

- $\text{support}(\delta_a) = \{a\}$
- $\text{support}(\delta_a + \delta_{-a}) = \{-a, a\}$

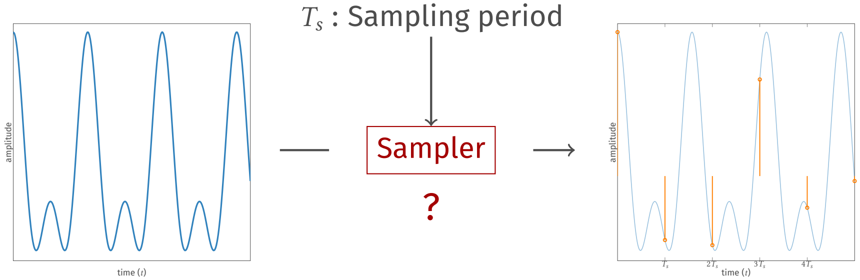
# Back to our sampling problem

(with new ingredients)

# Ideal Sampling



# Ideal Sampling




**Observation model:** $\equiv$ $\mathbf{x}$ $\times$ Dirac comb with period $T_s$:

$$x_e \stackrel{\text{def.}}{=} \sum_{j \in \mathbf{Z}} x(jT_s)\delta_{jT_s} \stackrel{?}{=} \text{Ш}_{T_s} x$$

# Such a definition is sounded

Is the distribution $\sum_{j \in \mathbf{Z}} x(jT_s)\delta_{jT_s}$ sounded?

Said differently, does the “distribution-function product” $Ш_{T_s} x$ mean something?

# Such a definition is sounded

Is the distribution $\sum_{j\in\mathbf{Z}} x(jT_s)\delta_{jT_s}$ sounded?

Said differently, does the "distribution-function product" $Ш_{T_s} x$ mean something?

**Theorem**

Let $T \in \mathcal{S}(\mathbf{R})'$ and $x \in \mathcal{C}^\infty(\mathbf{R})$ be such that

$$\forall r \geq 0, \exists C_r > 0 \text{ and } n_r \in \mathbf{N} : \quad |x^{(r)}(t)| \leq C_r(1+|t|)^{n_r}.$$

Then the product $Tx$ defined as

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle Tx, \varphi \rangle \overset{\text{def.}}{=} \langle T, x\varphi \rangle$$

is a tempered distribution

*Since $x\varphi \in \mathcal{S}(\mathbf{R})$.*
*The set of such functions is denoted $\mathcal{O}(\mathbf{R})$*

# Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned} \qquad \text{or} \qquad \begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathcal{S}(\mathbf{R})' \\ x &\longmapsto Ш_{T_s} x \end{aligned}$$

2. ***The target set***:

3. ***The decoder***:

4. ***The accuracy criterion***: “noiseless exact recovery”, *i.e.*

$$\forall x \in \mathcal{X}_{\mathtt{target}}, \quad D\big(M(x)\big) = x$$

# Spectrum of a sampled signal (following)

**Question:** Does the sampling process affect the spectrum?

$$\mathcal{F}[Ш_{T_s} x] \quad =$$

# Two (rather technical) admitted results

**Fourier transform of a product *(admitted)***

Let $x$ be such that $T_x$ has a compactly supported distribution.
Then

$$\mathcal{F}[Ш_{T_s} x] = \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x]$$

(The result holds under broader assumptions)

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(The result holds under broader assumptions)

**Fourier transform of a dirac comb *(admitted)***

$$\mathcal{F}[\text{Ш}_{T_s}] = \frac{1}{T_s} \text{Ш}_{1/T_s}$$

# Spectrum of a sampled signal (following)

**Question:** Does the sampling process affect the spectrum?

$$\mathcal{F}[\text{Ш}_{T_s} x] \overset{\text{Result 1.}}{=} \mathcal{F}[\text{Ш}_{T_s}] * \mathcal{F}[T_x]$$

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where $F_s = \frac{1}{T_s}$ is the sampling frequency

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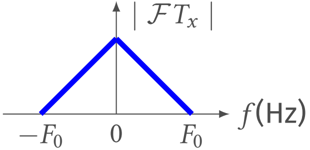
Sampling in the time domain $\implies$ Periodic spectrum

# Spectrum of band-limited signals

**Band-limited signals**

A band-limited signal $x$ is a signal whose Fourier transform density has **bounded support**

Let $\mathbf{x}$ be a band-limited signal whose TF has support $[-F_0, F_0]$

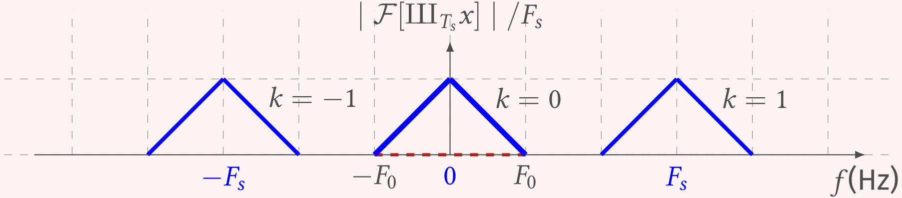



**Question** : Spectrum of $\text{Ш}_{T_s}\mathbf{x}$?

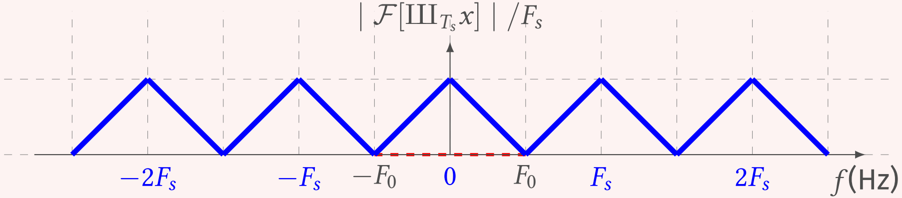
$$\mathcal{F}[\text{Ш}_{T_s}\mathbf{x}] = F_s \sum_{j=-\infty}^{+\infty} \delta_{jF_s} * \mathcal{F}T_x$$

# Spectrum of a band-limited signal

**Case 1:** $F_s > 2F_{\max}$

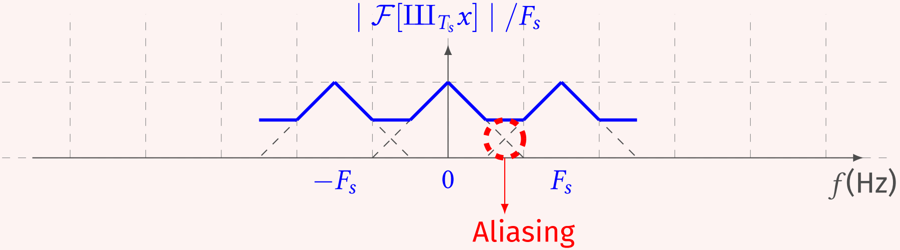



**Case 2:** $F_s = 2F_{\max}$



# Spectrum of a band-limited signal

**Case 3:** $F_s < 2F_{\max}$




The signal is **deteriorated** $\implies$ **Loss** of information

# Shannon-Nyquist sampling theorem

**Question:** Necessary condition on $F_s$ such that no information is lost?

**Nyquist–Shannon sampling theorem**

If a function $x(t)$ contains no frequencies higher than $F_{\max}$ hertz, it is **completely determined** by giving its ordinates at a series of points spaced $T_s \geq \frac{1}{2F_{\max}}$ seconds apart.

$\longrightarrow$ **Sufficient condition**

$F_s = 2F_{\max}$ is dubbed "**Nyquist frequency**"

# Interpolation formula

## Interpolation formula

If $F_s \geq 2F_{\max}$, then $T_x$ is equal to the regular distribution spanned by

$$t \longmapsto \sum_{j \in \mathbf{Z}} x(jT_s)\,\mathrm{sinc}(F_s(t - jT_s))$$

**(Informal) proof:** if $F_s > 2F_{\max}$ the spectrum of $x$ can be recovered with an ideal **low pass filter**

☝️ This operation (operating on the spectrum) is common in signal processing and often referred to as “filtering”

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$$\mathcal{F}T_x \overset{\mathcal{S}(\mathbf{R})'}{=} \mathcal{F}[\text{Ш}_{T_s}x] \cdot \frac{1}{F_s}\Pi_{F_s/2}(f)$$

$$\Rightarrow$$

$$T_x \overset{\mathcal{S}(\mathbf{R})'}{=} \text{Ш}_{T_s}x * T_{\mathrm{sinc}(2\pi F_s\cdot)}$$

$$\overset{\mathcal{S}(\mathbf{R})'}{=} \sum_{j \in \mathbf{Z}} x[jT_s]\delta_{jT_s} * T_{\mathrm{sinc}(2\pi F_s\cdot)}$$

# Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned} \qquad \text{or} \qquad \begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathcal{S}(\mathbf{R})' \\ x &\longmapsto Ш_{T_s} x \end{aligned}$$

2. ***The target set***: The set of functions which admit a Fourier transform which is compactly supported in $[-F_s, F_s]$

3. ***The decoder***:

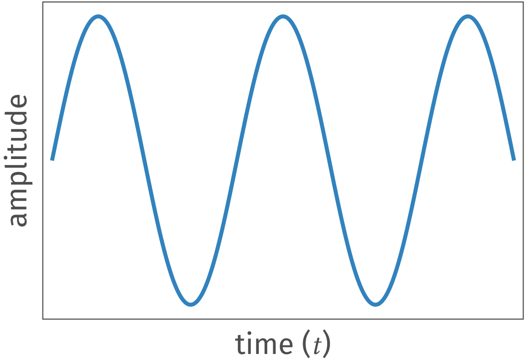
$$\begin{aligned} M\colon \qquad \mathbf{R}^{\mathbf{Z}} &\longrightarrow \mathcal{S}' \\ \{y_j\}_{j\in\mathbf{Z}} &\longmapsto \textstyle\sum_{j\in\mathbf{Z}} y_j \operatorname{sinc}(F_s(\cdot - jT_s)) \end{aligned} \qquad \text{if} \qquad F_s \geq 2F_{\max}$$

4. ***The accuracy criterion***: "noiseless exact recovery", *i.e.*

$$\forall x \in \mathcal{X}_{\text{target}}, \qquad D\big(M(x)\big) \overset{\mathcal{D}(\mathbf{R})'}{=} T_x$$

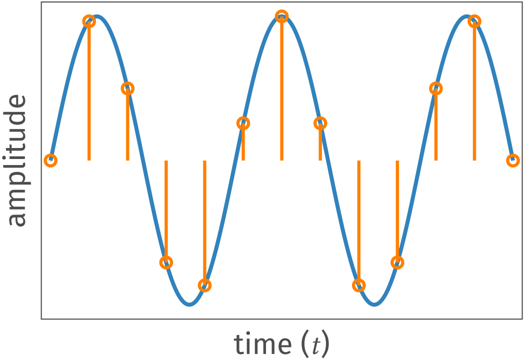
# Conclusion

# Numerical illustration



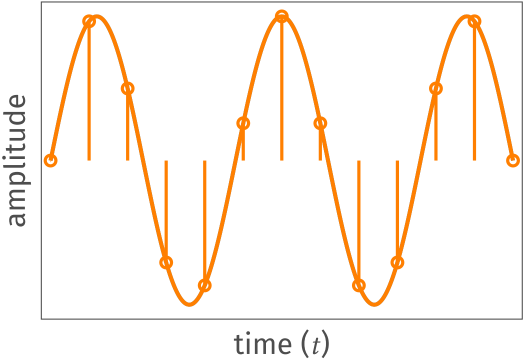

- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$

# Numerical illustration




- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$
- $T_s = 0.4 \Longrightarrow F_s = 2.5 > 2F_{\max}$

# Numerical illustration

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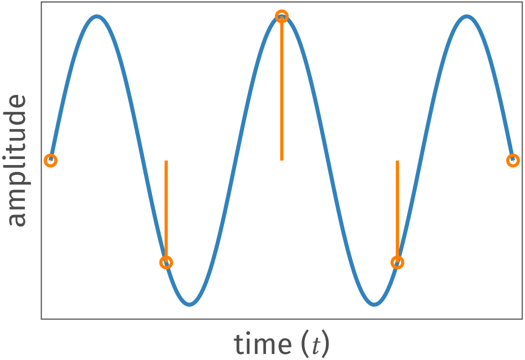
# Numerical illustration




- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$
- $T_s = 0.4 \implies F_s = 2.5 > 2F_{\max}$
- $T_s = 1.2 \implies F_s = .8 < 2F_{\max}$ ⚠

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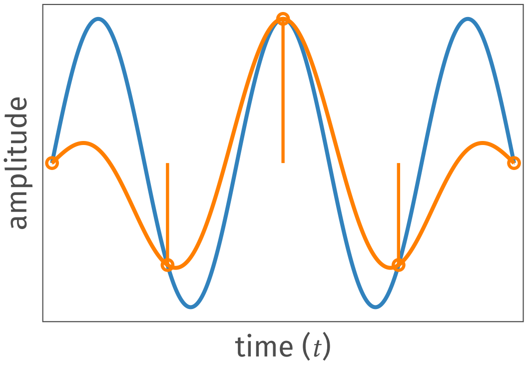


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- ► (an orthogonal projection as “best” decoder) *In the finite-dimensional case*

# A concluding remark

If we restrict ourself to signals $x \in \mathcal{L}^2(\mathbf{R})$

- $\mathcal{L}^2(\mathbf{R})$ is a Hilbert space equipped with the inner product

$$\langle x_1, x_2 \rangle = \int x_1(t)\overline{x_2}(t)\mathrm{d}t$$

- The family of function

$$\forall j, \quad m_j : t \mapsto \frac{1}{\sqrt{T_s}} \operatorname{sinc}\left(\frac{t - jT_s}{T_s}\right)$$

  is an orthonormal basis of the set of functions with Fourier transform supported in $[-F_s, F_s]$

- For all signals $x$ with Fourier transform supported in $[-F_s, F_s]$,

$$\langle m_j, x \rangle = x(jT_s)$$

- The proposed decoder thus rewrites

$$t \mapsto \sum_{j \in \mathbf{Z}} \langle m_j, x \rangle m_j(t)$$

☝️ It is an orthogonal projection!

# Bonus: Discrete Fourier, FFT

# Bonus: Time-Frequency analysis, towards wavelets