Sparse Signal Processing
Lecture 4 — Fourier interpolation and the linear model

Nicolas Keriven (slides by Clément Elvira)

Outline

N. Keriven: CNRS researcher at IRISA, SIROCCO team.
https://nkeriven.github.io

Outline

N. Keriven: CNRS researcher at IRISA, SIROCCO team.
`https://nkeriven.github.io`

In today's class we will

- particularize the linear model to an infinite-dimensional setting ;
- introduce a linear target set ;
- derive a decoder achieving exact recovery ;

Outline

N. Keriven: CNRS researcher at IRISA, SIROCCO team.
`https://nkeriven.github.io`

In today's class we will

- particularize the linear model to an infinite-dimensional setting ;
- introduce a linear target set ;
- derive a decoder achieving exact recovery ;

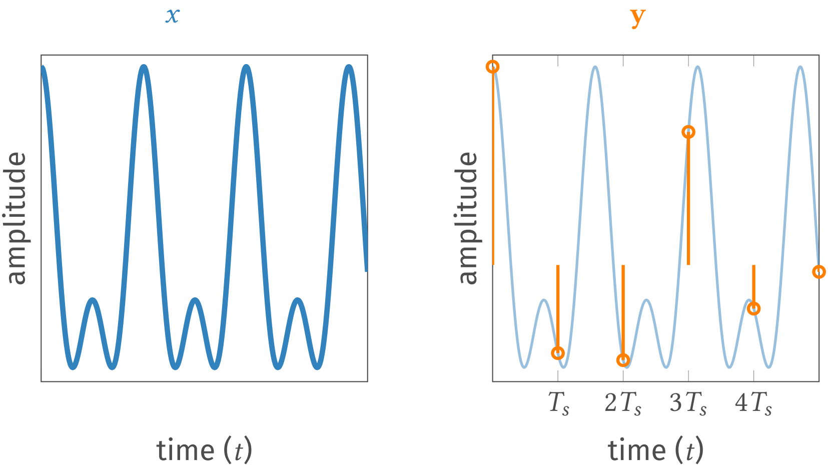
Material at `https://github.com/nkeriven/ENS-signal` (subject to changes, check every now and then)

References

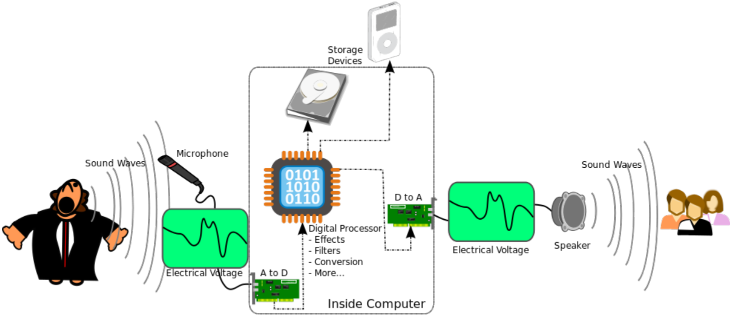
- Mallat, 2012. *A Wavelet Tour of Signal Processing: the Sparse Way*
- F Golse polycopié: `http://www.math.polytechnique.fr/golse/MAT431-10/POLY431.pdf`
- Wikipédia !

Considered Task

Recovering a signal x from **infinitely** many function evaluations

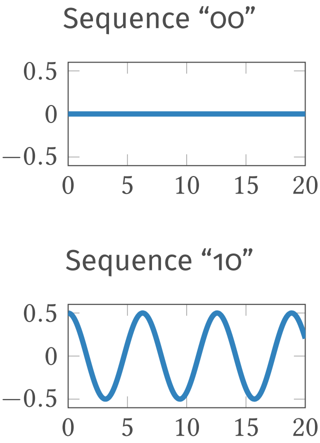


Application 1: sampling and reconstructing audio recording



Important component of sound perception: the frequency

Application 2: (simplified) transmitters



Support of information: electromagnetic (sinusoidal at a given frequency) waves

Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{L}^2(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned}$$

2. ***The target set***: *To be defined*

   Related to some notion of frequency

3. ***The decoder***: *To be defined*

4. ***The accuracy criterion***: “noiseless exact recovery”, *i.e.*

$$\forall x \in \mathcal{X}_{\mathtt{target}}, \quad D\big(M(x)\big) = x$$

🤔 Which mathematical tool can be used to emphasize the frequencies characterizing a signal?

🤔 Which mathematical tool can be used to emphasize the frequencies characterizing a signal?

☝️ The Fourier transform

Rappels: The Fourier transform

Definition

The Fourier transform is a linear operator defined by

$$\begin{aligned}\mathcal{F}\colon \mathcal{L}^1(\mathbf{R}) &\longrightarrow \mathcal{L}^1(\mathbf{R})\\ x &\longmapsto \left(f \mapsto \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t\right)\end{aligned}$$

Rappels: The Fourier transform

Definition

The Fourier transform is a linear operator defined by

$$\begin{aligned}\mathcal{F}\colon \mathcal{L}^1(\mathbf{R}) &\longrightarrow \mathcal{L}^1(\mathbf{R})\\ x &\longmapsto \left(f \mapsto \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t\right)\end{aligned}$$

- Sometimes defined without the 2π (but less clean, multiplicative constants everywhere)
- $\mathcal{F}x$ is often written $\hat{x}$

An example

Exercice

Evaluate the Fourier transform of

$$\Pi_{[-a/2,a/2]} : t \longmapsto \begin{cases} 1 & \text{if } t \in [-\frac{a}{2}, +\frac{a}{2}] \\ 0 & \text{otherwise} \end{cases}$$

An example

Exercice

Evaluate the Fourier transform of

$$\Pi_{[-a/2, a/2]} : t \longmapsto \begin{cases} 1 & \text{if } t \in [-\frac{a}{2}, +\frac{a}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Answer

$$\mathcal{F}\Pi(f) = a\frac{\sin(\pi a f)}{\pi a f} \overset{\text{def.}}{=} a \operatorname{sinc}(af)$$

Rappels: Inverse Fourier transform

Definition

The conjugate Fourier transform is the linear operator defined by

$$\begin{aligned}\overline{\mathcal{F}}\colon \mathcal{L}^1(\mathbf{R}) &\longrightarrow \mathcal{L}^1(\mathbf{R})\\ x &\longmapsto \left(f \mapsto \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{\mathrm{i}2\pi f t}\mathrm{d}t\right)\end{aligned}$$

Rappels: Inverse Fourier transform

Definition

The conjugate Fourier transform is the linear operator defined by

$$\begin{aligned}\overline{\mathcal{F}}\colon \mathcal{L}^1(\mathbf{R}) &\longrightarrow \mathcal{L}^1(\mathbf{R})\\ x &\longmapsto \left(f \mapsto \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{\mathrm{i}2\pi f t}\mathrm{d}t\right)\end{aligned}$$

Theorem: Inversion

If $x \in \mathcal{L}^1(\mathbf{R})$ is such that $\mathcal{F}x \in \mathcal{L}^1(\mathbf{R})$ then

$$\overline{\mathcal{F}}[\mathcal{F}x] \overset{\text{a.e.}}{=} x$$

with equality at every point of continuity.

Extension to L^2

Pbm: $x \in \mathcal{L}^1$ does **not** imply $\mathcal{F}x \in \mathcal{L}^1$! (ex: door function and cardinal sinus)

Extension to L^2

Pbm: $x \in \mathcal{L}^1$ does **not** imply $\mathcal{F}x \in \mathcal{L}^1$! (ex: door function and cardinal sinus)

$\rightarrow$ **Solution:** extension to $\mathcal{L}^2$ and Parseval identity.

Extension to L^2

Pbm: $x \in \mathcal{L}^1$ does **not** imply $\mathcal{F}x \in \mathcal{L}^1$! (ex: door function and cardinal sinus)

$\rightarrow$ Solution: extension to $\mathcal{L}^2$ and Parseval identity.

Pbm: the classical Fourier integral is not defined for $\mathcal{L}^2$...

Extension to L^2

Pbm: $x \in \mathcal{L}^1$ does **not** imply $\mathcal{F}x \in \mathcal{L}^1$! (ex: door function and cardinal sinus)

$\rightarrow$ Solution: extension to $\mathcal{L}^2$ and Parseval identity.

Pbm: the classical Fourier integral is not defined for $\mathcal{L}^2$...

$\rightarrow$ Solution: extension by *density*.

Parseval's identity

Parseval's identity

If $f, g \in \mathcal{L}^1 \cap \mathcal{L}^2$,

$$\langle f, g \rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}$$

(recall that $\langle f, g \rangle_{L^2} = \int f(t)\overline{g(t)}dt$)

Parseval's identity

Parseval's identity

If $f, g \in \mathcal{L}^1 \cap \mathcal{L}^2$,

$$\langle f, g \rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}$$

(recall that $\langle f, g \rangle_{L^2} = \int f(t)\overline{g(t)}dt$)

Consequence: $\|f\|_{L^2} = \|\mathcal{F}f\|_{L^2}$ and $f \in \mathcal{L}^1 \cap \mathcal{L}^2 \Rightarrow \mathcal{F}f \in \mathcal{L}^2$. Now we just need to extend the domain of definition of $\mathcal{F}$

Parseval's identity

Parseval's identity

If $f, g \in \mathcal{L}^1 \cap \mathcal{L}^2$,

$$\langle f, g \rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}$$

(recall that $\langle f, g \rangle_{L^2} = \int f(t)\overline{g(t)}dt$)

Consequence: $\|f\|_{L^2} = \|\mathcal{F}f\|_{L^2}$ and $f \in \mathcal{L}^1 \cap \mathcal{L}^2 \Rightarrow \mathcal{F}f \in \mathcal{L}^2$. Now we just need to extend the domain of definition of $\mathcal{F}$

We are going to extend Fourier to $\mathcal{L}^2$ by *density* of $\mathcal{L}^1 \cap \mathcal{L}^2$ within $\mathcal{L}^2$

$\rightarrow$ i.e., $\forall \epsilon > 0, f \in \mathcal{L}^2, \exists g \in \mathcal{L}^2 \cap \mathcal{L}^1$ s.t. $\|f - g\|_{\mathcal{L}^2} \leq \epsilon$

Extension to $\mathcal{L}^2$

- Let $f \in \mathcal{L}^2$.
- By density of $\mathcal{L}^2 \cap \mathcal{L}^1$ in $\mathcal{L}^2$, there is a sequence $f_n \in \mathcal{L}^2 \cap \mathcal{L}^1$ such that $\|f_n - f\|_{\mathcal{L}^2} \to 0$

Extension to $\mathcal{L}^2$

- Let $f \in \mathcal{L}^2$.
- By density of $\mathcal{L}^2 \cap \mathcal{L}^1$ in $\mathcal{L}^2$, there is a sequence $f_n \in \mathcal{L}^2 \cap \mathcal{L}^1$ such that $\|f_n - f\|_{\mathcal{L}^2} \to 0$
- f_n is a Cauchy sequence, and by Parseval's identity $\|f_n - f_p\|_{\mathcal{L}^2} = \|\mathcal{F}(f_n - f_p)\|_{\mathcal{L}^2}$, hence $\mathcal{F}f_n \in \mathcal{L}^2$ is also a Cauchy sequence

Extension to $\mathcal{L}^2$

- Let $f \in \mathcal{L}^2$.
- By density of $\mathcal{L}^2 \cap \mathcal{L}^1$ in $\mathcal{L}^2$, there is a sequence $f_n \in \mathcal{L}^2 \cap \mathcal{L}^1$ such that $\|f_n - f\|_{\mathcal{L}^2} \to 0$
- f_n is a Cauchy sequence, and by Parseval's identity $\|f_n - f_p\|_{\mathcal{L}^2} = \|\mathcal{F}(f_n - f_p)\|_{\mathcal{L}^2}$, hence $\mathcal{F}f_n \in \mathcal{L}^2$ is also a Cauchy sequence
- Since $\mathcal{L}^2$ is a Hilbert space and **is therefore complete**, $\mathcal{F}f_n$ is a convergent sequence.

Extension to $\mathcal{L}^2$

- Let $f \in \mathcal{L}^2$.
- By density of $\mathcal{L}^2 \cap \mathcal{L}^1$ in $\mathcal{L}^2$, there is a sequence $f_n \in \mathcal{L}^2 \cap \mathcal{L}^1$ such that $\|f_n - f\|_{\mathcal{L}^2} \to 0$
- f_n is a Cauchy sequence, and by Parseval's identity $\|f_n - f_p\|_{\mathcal{L}^2} = \|\mathcal{F}(f_n - f_p)\|_{\mathcal{L}^2}$, hence $\mathcal{F}f_n \in \mathcal{L}^2$ is also a Cauchy sequence
- Since $\mathcal{L}^2$ is a Hilbert space and **is therefore complete**, $\mathcal{F}f_n$ is a convergent sequence.
- We call its limit the "Fourier transform of f" denoted by $\mathcal{F}f$. It satisfies all the usual properties of the Fourier transform (see next slide)

Properties of Fourier Transform (...easy, or admitted)

- f symmetric $\Rightarrow \mathcal{F}f$ real
- f anti-symmetric $\Rightarrow \mathcal{F}f$ imaginary
- f real $\Rightarrow \mathcal{F}f$ Hermitian
- f imaginary $\Rightarrow \mathcal{F}f$ (conjugate) anti-symmetric

Properties of Fourier Transform (...easy, or admitted)

- f symmetric $\Rightarrow \mathcal{F}f$ real
- f anti-symmetric $\Rightarrow \mathcal{F}f$ imaginary
- f real $\Rightarrow \mathcal{F}f$ Hermitian
- f imaginary $\Rightarrow \mathcal{F}f$ (conjugate) anti-symmetric

- $\mathcal{F}(f \star g)(t) = \mathcal{F}f(t)\mathcal{F}g(t)$ (convolution thm)
- $\langle f, g \rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}$ (Parseval identity)
- $\mathcal{F}\frac{df}{dt}(t) = 2i\pi t \mathcal{F}f(t) \Rightarrow$ if f is k times differentiable, $\mathcal{F}f$ decrease faster than $1/t^k$

Properties (summary)

Property	Function	Fourier transform
	$x(t)$	$\hat{x}(f)$
Inverse	$\hat{x}(t)$	$x(-f)$
Convolution	$x_1 \star x_2(t)$	$\hat{x}_1 \hat{x}_2(f)$
Multiplication	$x_1(t)x_2(t)$	$\hat{x}_1 \star \hat{x}_2(f)$
Translation	$x(t-u)$	$e^{-2i\pi uf}\hat{x}(f)$
Modulation	$e^{2i\pi\xi t}x(t)$	$\hat{x}(f-\xi)$
Scaling	$x(t/s)$	$\|s\|\hat{x}(sf)$
Time derivatives	$x^{(p)}(t)$	$(2i\pi f)^p\hat{x}(f)$
Freq. derivatives	$(-2i\pi t)^p x(t)$	$\hat{x}^{(p)}(f)$
Complex conjugate	$\overline{x}(t)$	$\overline{\hat{x}}(-f)$

Uncertainty principle

Can we construct a function that is **highly localised in space**, whose Fourier transform is **highly localised in frequency**?

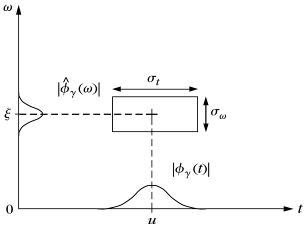
Uncertainty principle

Can we construct a function that is **highly localised in space**, whose Fourier transform is **highly localised in frequency**?

Heisenberg's uncertainty principle

For $x \in \mathcal{L}^2(\mathbf{R})$, define the two density functions $p_x = \frac{|x(t)|^2}{\|x\|_{L^2}}, \quad p_{\hat{x}} = \frac{|\hat{x}(f)|^2}{\|x\|_{L^2}}$ and their respective variance $\sigma_x^2, \sigma_{\hat{x}}^2$. Then

$$\sigma_x^2 \sigma_{\hat{x}}^2 \geq 1/4$$



🤔 Are we done?

🤔 Are we done?

⚠️ The function of interest, namely $t \longmapsto \cos(2\pi f t)$ does not belong to $\mathcal{L}^1(\mathbf{R})$!

nor $\mathcal{L}^2(\mathbf{R})$

Are we done?

The function of interest, namely $t \longmapsto \cos(2\pi f t)$ does not belong to $\mathcal{L}^1(\mathbf{R})$!

nor $\mathcal{L}^2(\mathbf{R})$

We need a “bigger” space to work with!

Are we done?

The function of interest, namely $t \longmapsto \cos(2\pi f t)$ does not belong to $\mathcal{L}^1(\mathbf{R})$!

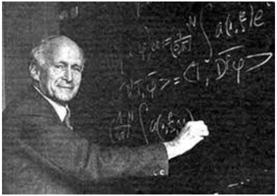
nor $\mathcal{L}^2(\mathbf{R})$

We need a “bigger” space to work with!

$\rightarrow$ Space of “distributions”, defined by *duality*.

We need the smallest dual space as possible, to make the biggest distribution space as possible!

Reminder (?) on distributions



Tests functions

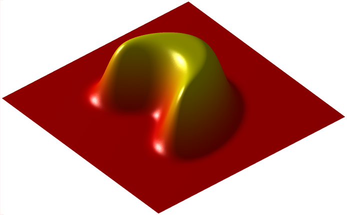
Definition

A test function refers to any infinitely differentiable function

$$\varphi\colon \mathbf{R} \longrightarrow \mathbf{C}$$

witch is compactly supported.

The set of test functions is denoted $\mathcal{D}(\mathbf{R})$.



Tests functions

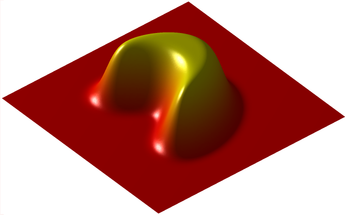
Definition

A test function refers to any infinitely differentiable function

$$\varphi\colon \mathbf{R} \longrightarrow \mathbf{C}$$

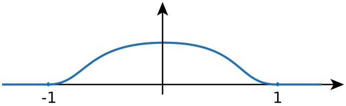
witch is compactly supported.

The set of test functions is denoted $\mathcal{D}(\mathbf{R})$.



Example: *(standard one)* the function

$$t \longmapsto \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & \text{if } t \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$



can be proved to be a test function.

Topology on $\mathcal{D}(\mathbf{R})$

Definition: convergence in $\mathcal{D}(\mathbf{R})$

Let $\{\varphi_n\}_{n\in\mathbf{N}}$ be a family of test function.
We say that $\varphi_n \to \varphi \in \mathcal{D}(\mathbf{R})$ if and only if

- there exists an integer N and a compact set K such $n \geq N \Rightarrow \mathrm{support}(\varphi_n - \varphi) \subset K$
- for all $r \in \mathbf{N}$,
$$\left\| \varphi_n^{(r)} - \varphi^{(r)} \right\|_\infty \to 0$$
- i.e., supremum norm for φ and all its derivatives.

Distributions

Definition

We refer to "distribution" as any continuous linear functional on $\mathcal{D}(\mathbf{R})$.

Distributions

Definition

We refer to “distribution” as any continuous linear functional on $\mathcal{D}(\mathbf{R})$.

Remarks

- Said differently, the space of distributions is the (topological) dual space of $\mathcal{D}(\mathbf{R})$, denoted by $\mathcal{D}'(\mathbf{R})$;

Distributions

Definition

We refer to “distribution” as any continuous linear functional on $\mathcal{D}(\mathbf{R})$.

Remarks

- Said differently, the space of distributions is the (topological) dual space of $\mathcal{D}(\mathbf{R})$, denoted by $\mathcal{D}'(\mathbf{R})$;
- When evaluating a distribution T at φ, we write

$$\langle T, \varphi \rangle$$

- By extension, we say that two distributions T, T' are equal iff

$$\langle T, \varphi \rangle = \langle T', \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbf{R})$$

Distribution are often referred to as "generalized functions".

This can nevertheless be a misleading terminology, as distributions do not take scalar as inputs. (...but *test functions*)

⚠️

Distribution are often referred to as "generalized functions".

This can nevertheless be a misleading terminology, as distributions do not take scalar as inputs. (...but *test functions*)

☝️

Besides mathematical benefits, there is a physical intuition behind this reference

A compactly supported test functions can be seen as a model of a physical system measuring some quantity

Regular distributions

Definition/Proposition

If g is any locally integrable function, then the linear form T_g defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T_g, \varphi \rangle \stackrel{\text{def.}}{=} \int g(t)\varphi(t)\mathrm{d}t$$

is a distribution.

Such a distribution is referred to as "regular distribution".

Regular distributions

Definition/Proposition

If g is any locally integrable function, then the linear form T_g defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T_g, \varphi \rangle \overset{\text{def.}}{=} \int g(t)\varphi(t)\mathrm{d}t$$

is a distribution.

Such a distribution is referred to as "regular distribution".

In short: Linearity is fine
Continuity: letting $\varphi_n \to \varphi$, there exists a compact K that contains all supports
Then $|\langle T_g, \varphi_n - \varphi \rangle| \leq \int g(t)\mathrm{d}t \|\varphi_n - \varphi\|_\infty$

Regular distributions

Definition/Proposition

If g is any locally integrable function, then the linear form T_g defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T_g, \varphi \rangle \stackrel{\text{def.}}{=} \int g(t)\varphi(t)\mathrm{d}t$$

is a distribution.

Such a distribution is referred to as "regular distribution".

In short: Linearity is fine
Continuity: letting $\varphi_n \to \varphi$, there exists a compact K that contains all supports
Then $|\langle T_g, \varphi_n - \varphi \rangle| \leq \int g(t)\mathrm{d}t \|\varphi_n - \varphi\|_\infty$

Remark

- Hence distributions is a "bigger" space...
- If $T_{g_1} = T_{g_2}$ then $g_1 \stackrel{\text{a.e.}}{=} g_2$

Singular distributions

The terminology “singular distributions” refers to any distribution that is not regular

Singular distributions

The terminology “singular distributions” refers to any distribution that is not regular

Example 1: The dirac function δ_t defined $\forall t \in \mathbf{R}$ as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \delta_t, \varphi \rangle = \varphi(t)$$

since $|\langle \delta_t, \varphi_n - \varphi \rangle| \leq \|\varphi_n - \varphi\|_\infty \to 0$

Singular distributions

The terminology “singular distributions” refers to any distribution that is not regular

Example 1: The dirac function δ_t defined $\forall t \in \mathbf{R}$ as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \delta_t, \varphi \rangle = \varphi(t)$$

since $|\langle \delta_t, \varphi_n - \varphi \rangle| \leq \|\varphi_n - \varphi\|_\infty \to 0$

Example 2: The dirac comb Ш_{T_s} defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \text{Ш}_{T_s}, \varphi \rangle = \sum_{j \in \mathbf{Z}} \langle \delta_{jT_s}, \varphi \rangle = \sum_{j \in \mathbf{Z}} \varphi(jT_s)$$

Letting $\varphi_n \to \varphi$*, there exists a compact* K *that contains all supports, and* $|\langle \delta_t, \varphi_n - \varphi \rangle| \leq K_{T_s} \|\varphi_n - \varphi\|_\infty \to 0$*, where* K_{T_s} *is the number of* jT_s *contained in* K

🤔 **Regular distributions** provide intuition on how to define many concepts related to distributions

derivation, convolution, Fourier transform to name but a few

Derivation of a distribution

For regular distributions: let g be a differentiable locally integrable function. Then g' is also locally integrable and

$$\begin{aligned} \langle T_{g'}, \varphi \rangle &= \int g'(t)\varphi(t)\mathrm{d}t \\ &= -\int g(t)\varphi'(t)\mathrm{d}t \qquad \text{(IBP)} \\ &= -\langle T_g, \varphi' \rangle \end{aligned}$$

Derivation of a distribution

For regular distributions: let g be a differentiable locally integrable function. Then g' is also locally integrable and

$$\begin{aligned}\langle T_{g'}, \varphi\rangle &= \int g'(t)\varphi(t)\mathrm{d}t \\ &= -\int g(t)\varphi'(t)\mathrm{d}t \qquad \text{(IBP)} \\ &= -\langle T_g, \varphi'\rangle\end{aligned}$$

Definition

Let T be a distribution. The derivative of T is the distribution $T^{(1)}$ defined by

$$\forall\varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T^{(1)}, \varphi\rangle \overset{\text{def.}}{=} -\langle T, \varphi'\rangle$$

A second example: Fourier transform

Let g be an integrable function (so that T_g is a regular distribution). Then $\mathcal{F}g$ is also integrable and

$$\begin{aligned}
\langle T_{\mathcal{F}g}, \varphi\rangle &= \int \mathcal{F}g(f)\varphi(f)\mathrm{d}f \\
&= \int \left(\int g(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}t \right) \varphi(f)\mathrm{d}f \\
&= \int g(t) \left(\int \varphi(f)\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}f \right) \mathrm{d}t \\
&\overset{(?)}{=} \langle T_g, \mathcal{F}\varphi\rangle
\end{aligned}$$

Fourier transform of a distribution (tentative)

Definition

Let T be a distribution. The Fourier of T is the distribution $\mathcal{F}T$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \stackrel{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

Fourier transform of a distribution (tentative)

Definition

Let T be a distribution. The Fourier of T is the distribution $\mathcal{F}T$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

- (need complex-valued test functions to technically make sense)

Fourier transform of a distribution (tentative)

Definition

Let T be a distribution. The Fourier of T is the distribution $\mathcal{F}T$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

- (need complex-valued test functions to technically make sense)

Unfortunately, in general $\mathcal{F}\varphi$ is not compactly supported, it may not be properly defined

Fourier transform of a distribution (tentative)

Definition

Let T be a distribution. The Fourier of T is the distribution $\mathcal{F}T$ defined by

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \stackrel{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$

- (need complex-valued test functions to technically make sense)

⚠️ Unfortunately, in general $\mathcal{F}\varphi$ is not compactly supported, it may not be properly defined

The set of distributions is too big a space!!

Tempered distribution

☝️ We will **extend** the set of test functions...

Tempered distribution

We will **extend** the set of test functions…

☝️

… which will **reduce** the set of distributions (by duality).

Tempered distributions

Definition: Schwartz space

We say that a function g is a Schwartz function if it is infinitely differentiable and for all $k, q \in \mathbf{N}$

$$\lim_{t \longmapsto \pm\infty} |t^k \varphi^{(q)}(t)| = 0$$

The set of Schwartz functions is denoted $\mathcal{S}(\mathbf{R})$.

- Instead of compact support, we take **fast decrease** (faster than any polynomial).

Tempered distributions

Definition: Schwartz space

We say that a function g is a Schwartz function if it is infinitely differentiable and for all $k, q \in \mathbf{N}$

$$\lim_{t \longmapsto \pm\infty} |t^k \varphi^{(q)}(t)| = 0$$

The set of Schwartz functions is denoted $\mathcal{S}(\mathbf{R})$.

- Instead of compact support, we take **fast decrease** (faster than any polynomial).

Definition: Tempered distribution

We call "tempered distribution" any element of the (topological) dual space of the Schwartz space $\mathcal{S}'(\mathbf{R})$.

*"Topological": **continuous** linear functionals*

Properties

- Convergence in $\mathcal{S}(\mathbf{R})$ is defined through the family of semi-norms $\mathcal{N}_p(\varphi) = \sum_{k,q \leq p} \sup_t |t^q \varphi^{(q)}(t)|$, i.e. φ_n converge to φ if $\forall p,\ \mathcal{N}_p(\varphi_n - \varphi) \to 0$

Properties

- Convergence in $\mathcal{S}(\mathbf{R})$ is defined through the family of semi-norms $\mathcal{N}_p(\varphi) = \sum_{k,q\leq p} \sup_t |t^q \varphi^{(q)}(t)|$, i.e. φ_n converge to φ if $\forall p,\ \mathcal{N}_p(\varphi_n - \varphi) \to 0$
- For a tempered distribution T, there is p, C such that $\forall \varphi \in \mathcal{S}(\mathbf{R})$, $\langle T, \varphi \rangle \leq C\mathcal{N}_p(\varphi)$ (continuity)

Properties

- Convergence in $\mathcal{S}(\mathbf{R})$ is defined through the family of semi-norms $\mathcal{N}_p(\varphi) = \sum_{k,q \leq p} \sup_t |t^q \varphi^{(q)}(t)|$, i.e. φ_n converge to φ if $\forall p,\ \mathcal{N}_p(\varphi_n - \varphi) \to 0$
- For a tempered distribution T, there is p, C such that $\forall \varphi \in \mathcal{S}(\mathbf{R})$, $\langle T, \varphi \rangle \leq C\mathcal{N}_p(\varphi)$ (continuity)
- Any test function is a Schwartz function: $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$

Properties

- Convergence in $\mathcal{S}(\mathbf{R})$ is defined through the family of semi-norms $\mathcal{N}_p(\varphi) = \sum_{k,q\leq p} \sup_t |t^q \varphi^{(q)}(t)|$, i.e. φ_n converge to φ if $\forall p,\ \mathcal{N}_p(\varphi_n - \varphi) \to 0$
- For a tempered distribution T, there is p, C such that $\forall \varphi \in \mathcal{S}(\mathbf{R})$, $\langle T, \varphi \rangle \leq C\mathcal{N}_p(\varphi)$ (continuity)
- Any test function is a Schwartz function: $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$
- (hence) any tempered distribution is a distribution: $\mathcal{S}'(\mathbf{R}) \subset \mathcal{D}'(\mathbf{R})$

Fourier and tempered distributions

Thm

- Thm: All tempered distribution **admit a Fourier transform** defined by
$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$
- The Fourier transform of a Schwartz function is **also** a Schwartz function

Hint: link between differentiability and polynomial decrease of the Fourier transform...

Classical tempered distribution

Unlike regular distributions, all functions do not correspond to tempered distributions! Need all derivative to grows “as fast as” polynomials...

Classical tempered distribution

Unlike regular distributions, all functions do not correspond to tempered distributions! Need all derivative to grows “as fast as” polynomials...

Still:

- The Dirac function δ_t is a tempered distribution
- The Dirac comb is a tempered distribution
- the function $t \longmapsto \cos(2\pi ft)$ is a tempered distribution (as all bounded functions)

Classical tempered distribution

⚠️ Unlike regular distributions, all functions do not correspond to tempered distributions! Need all derivative to grows “as fast as” polynomials...

Still:

- The Dirac function δ_t is a tempered distribution
- The Dirac comb is a tempered distribution
- the function $t \longmapsto \cos(2\pi f t)$ is a tempered distribution (as all bounded functions)

$\rightarrow$ that’s all we need in this course...? 🧑‍🏫

⚠️ In signal processing, it is folklore to believe that distributions are always well-behaved

That it all operations are legitimate

It it nevertheless not always true

☝️ In this lecture (and companion lab) it will nevertheless be true

Some justifications will be out of scope and left in exercise

Some properties of (tempered) distributions

Some Fourier transform (1)

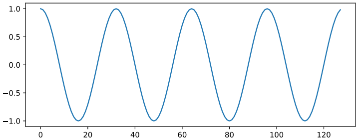
$$
\begin{aligned}
\langle \mathcal{F}\delta_a, \varphi \rangle &= \langle \delta_a, \mathcal{F}\varphi \rangle \\
&= (\mathcal{F}\varphi)(a) \\
&= \int \varphi(t) \mathrm{e}^{-2\mathrm{i}\pi a t} \mathrm{d}t \\
&= \langle T_{t \mapsto \mathrm{e}^{-2\mathrm{i}\pi a t}}, \varphi \rangle
\end{aligned}
$$

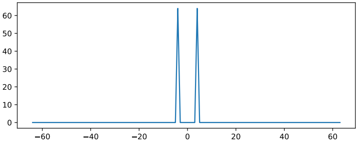
Some Fourier transform (2)

$$
\begin{aligned}
\langle \mathcal{F} T_{t \mapsto \mathrm{e}^{2\mathrm{i}\pi a t}}, \varphi \rangle &\overset{\text{def.}}{=} \langle T_{t \mapsto \mathrm{e}^{2\mathrm{i}\pi a t}}, \mathcal{F}\varphi \rangle \\
&\overset{\text{def.}}{=} \int \mathrm{e}^{2\mathrm{i}\pi a f} \mathcal{F}\varphi(f) \mathrm{d}f \\
&\overset{\text{def.}}{=} \mathcal{F}^{-1}\mathcal{F}\varphi(a) \\
&= \langle \delta_a, \varphi \rangle
\end{aligned}
$$

Some Fourier transform (3)

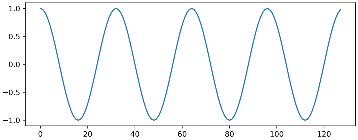
$$
\begin{aligned}
\langle \mathcal{F}T_{t\mapsto\cos(2\pi at)}, \varphi\rangle &= \langle \mathcal{F}T_{t\mapsto 0.5(\mathrm{e}^{2\pi at}+\mathrm{e}^{-2\pi at}}, \varphi\rangle \\
&= \frac{1}{2}\langle \mathcal{F}T_{t\mapsto \mathrm{e}^{2\pi at}}, \varphi\rangle + \frac{1}{2}\langle \mathcal{F}T_{t\mapsto \mathrm{e}^{-2\pi at}}, \varphi\rangle \\
&= \frac{1}{2}(\langle \delta_a, \varphi\rangle + \langle \delta_{-a}, \varphi\rangle)
\end{aligned}
$$

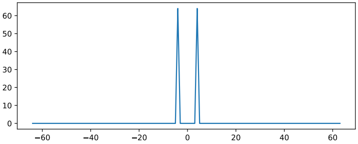




Some Fourier transform (3)

$$\begin{aligned}
\langle \mathcal{F} T_{t\mapsto\cos(2\pi a t)}, \varphi\rangle &= \langle \mathcal{F} T_{t\mapsto 0.5(\mathrm{e}^{2\pi a t}+\mathrm{e}^{-2\pi a t}}, \varphi\rangle \\
&= \frac{1}{2}\langle \mathcal{F} T_{t\mapsto \mathrm{e}^{2\pi a t}}, \varphi\rangle + \frac{1}{2}\langle \mathcal{F} T_{t\mapsto \mathrm{e}^{-2\pi a t}}, \varphi\rangle \\
&= \frac{1}{2}(\langle \delta_a, \varphi\rangle + \langle \delta_{-a}, \varphi\rangle)
\end{aligned}$$





Recall that $\mathcal{F}[Re(f)](\xi) = \frac{1}{2}(\mathcal{F}f(\xi) + \mathcal{F}f(-\xi))$

🤔 Have we achieved our initial objective?

🤔 Have we achieved our initial objective?

☝️ It is folklore to provide a graphical representation of a Fourier transform

Another example: the "door" distribution

We define $\Pi_{[-a/2,a/2]}$ such that

$$\forall\varphi\in\mathcal{S}(\mathbf{R}),\qquad \langle\Pi_{[-a/2,a/2]},\varphi\rangle=\int_{-a/2}^{a/2}\varphi(t)\mathrm{d}t$$

Question: Evaluate the Fourier transform of $\Pi_{[-a/2,a/2]}$.

Another example: the "door" distribution

We define $\Pi_{[-a/2, a/2]}$ such that

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \Pi_{[-a/2, a/2]}, \varphi \rangle = \int_{-a/2}^{a/2} \varphi(t) \mathrm{d}t$$

Question: Evaluate the Fourier transform of $\Pi_{[-a/2, a/2]}$.

$$\begin{aligned} \langle \mathcal{F}\Pi_{[-a/2, a/2]}, \varphi \rangle &= \langle \Pi_{[-a/2, a/2]}, \mathcal{F}\varphi \rangle \\ &\overset{\text{Fub.}}{=} \int \varphi(t) \int_{-a/2}^{a//2} \mathrm{e}^{-2\mathrm{i}ft} \mathrm{d}f \mathrm{d}t \\ &= \langle T_{a\,\mathrm{sinc}(a\cdot)}, \varphi \rangle \end{aligned}$$

🤔 Does it ring a bell?

Compatibility with the "standard" Fourier transform

Theorem (admitted)

If $x \in \mathcal{L}^1(\mathbf{R})$ (or $\mathcal{L}^2(\mathbf{R})$) then

$$\mathcal{F} T_x = T_{\mathcal{F} x}$$

Towards an inverse Fourier transform

Definition: Conjugate Fourier transform

Let T be a tempered distribution. The conjugate Fourier of T is the distribution $\overline{\mathcal{F}}T$ defined by

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \overline{\mathcal{F}}T, \varphi\rangle \overset{\text{def.}}{=} \langle T, \overline{\mathcal{F}}\varphi\rangle$$

Towards an inverse Fourier transform

Definition: Conjugate Fourier transform

Let T be a tempered distribution. The conjugate Fourier of T is the distribution $\overline{\mathcal{F}}T$ defined by

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \overline{\mathcal{F}}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \overline{\mathcal{F}}\varphi \rangle$$

Theorem: Inversion formula

Let T be a tempered distribution.
Then, its Fourier transform satisfies

$$T = \overline{\mathcal{F}}[\mathcal{F}T]$$

In other words, the standard inversion formula holds in the sense of tempered distributions

Convolution

Definition

Let T_1, T_2.
Their convolution (if it exists) is the distribution $T_1 * T_2$ defined as

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle T_1 * T_2, \varphi \rangle \overset{\text{def.}}{=} \langle T_1, \varphi_{T_2} \rangle$$

where

$$\forall t \in \mathbf{R}, \qquad \varphi_{T_2}(t) = \langle T_2, \varphi(\cdot + t) \rangle$$

(T_1, T_2 can be exchanged in the definition)

Example: of $T_1 = \delta_a$

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \delta_a * T_2, \varphi \rangle = \langle T_2, \varphi(\cdot - a) \rangle$$

$\rightarrow$ Translation of the distribution

Support of a distribution

Definition

Let $T \in \mathcal{S}(\mathbf{R})'$.
Define Ω as the largest open set of $\mathbf{R}$ such that

$$\operatorname{support}(\varphi) \subseteq \Omega \implies \langle T, \varphi \rangle = 0$$

By extension, the (closed) set $\operatorname{support}(T) \overset{\text{def.}}{=} \Omega^c$ is called the **support** of the distribution

Support of a distribution

Definition

Let $T \in \mathcal{S}(\mathbf{R})'$.
Define Ω as the largest open set of $\mathbf{R}$ such that

$$\mathrm{support}(\varphi) \subseteq \Omega \implies \langle T, \varphi \rangle = 0$$

By extension, the (closed) set $\mathrm{support}(T) \stackrel{\text{def.}}{=} \Omega^c$ is called the **support** of the distribution

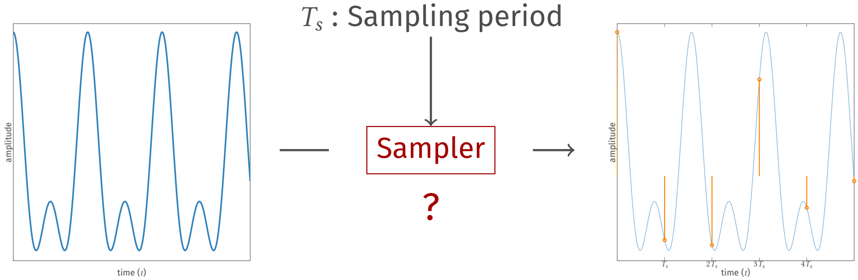
Examples:

- $\mathrm{support}(\delta_a) = \{a\}$
- $\mathrm{support}(\delta_a + \delta_{-a}) = \{-a, a\}$

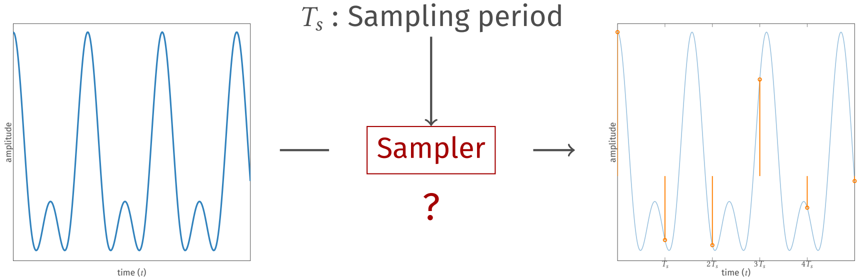
Back to our sampling problem

(with new ingredients)

Ideal Sampling



Ideal Sampling



Observation model: $\equiv$ $\mathbf{x}$ $\times$ Dirac comb with period T_s:

$$x_e \stackrel{\text{def.}}{=} \sum_{j \in \mathbf{Z}} x(jT_s)\delta_{jT_s} \stackrel{?}{=} \text{Ш}_{T_s} x$$

Such a definition is sounded

Is the distribution $\sum_{j \in \mathbf{Z}} x(jT_s)\delta_{jT_s}$ sounded?

Said differently, does the “distribution-function product” $\text{Ш}_{T_s} x$ mean something?

Such a definition is sounded

Is the distribution $\sum_{j\in\mathbf{Z}} x(jT_s)\delta_{jT_s}$ sounded?

Said differently, does the "distribution-function product" $Ш_{T_s}x$ mean something?

Theorem

Let $T \in \mathcal{S}(\mathbf{R})'$ and $x \in \mathcal{C}^\infty(\mathbf{R})$ be such that

$$\forall r \geq 0, \exists C_r > 0 \text{ and } n_r \in \mathbf{N} : \quad |x^{(r)}(t)| \leq C_r(1+|t|)^{n_r}.$$

Then the product Tx defined as

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle Tx, \varphi\rangle \overset{\text{def.}}{=} \langle T, x\varphi\rangle$$

is a tempered distribution

Since $x\varphi \in \mathcal{S}(\mathbf{R})$.
The set of such functions is denoted $\mathcal{O}(\mathbf{R})$

Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned} \qquad \text{or} \qquad \begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathcal{S}(\mathbf{R})' \\ x &\longmapsto Ш_{T_s} x \end{aligned}$$

2. ***The target set***:

3. ***The decoder***:

4. ***The accuracy criterion***: "noiseless exact recovery", *i.e.*

$$\forall x \in \mathcal{X}_{\mathtt{target}}, \quad D\big(M(x)\big) = x$$

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$\mathcal{F}[Ш_{T_s} x] \quad =$$

Two (rather technical) admitted results

Fourier transform of a product *(admitted)*

Let x be such that T_x has a compactly supported distribution.
Then

$$\mathcal{F}[Ш_{T_s} x] = \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x]$$

(The result holds under broader assumptions)

Two (rather technical) admitted results

Fourier transform of a product *(admitted)*

Let x be such that T_x has a compactly supported distribution.
Then

$$\mathcal{F}[Ш_{T_s} x] = \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x]$$

(The result holds under broader assumptions)

Fourier transform of a dirac comb *(admitted)*

$$\mathcal{F}[Ш_{T_s}] = \frac{1}{T_s} Ш_{1/T_s}$$

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$\mathcal{F}[\text{Ш}_{T_s} x] \overset{\text{Result 1.}}{=} \mathcal{F}[\text{Ш}_{T_s}] * \mathcal{F}[T_x]$$

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$\mathcal{F}[Ш_{T_s} x] \overset{\text{Result 1.}}{=} \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x]$$
$$\overset{\text{Result 2.}}{=} F_s Ш_{F_s} * \mathcal{F}[T_x]$$

where $F_s = \frac{1}{T_s}$ is the sampling frequency

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$
\begin{aligned}
\mathcal{F}[Ш_{T_s} x] &\overset{\text{Result 1.}}{=} \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x] \\
&\overset{\text{Result 2.}}{=} F_s Ш_{F_s} * \mathcal{F}[T_x] \\
&\overset{\text{Distribu.}}{=} F_s \sum_{j \in \mathbf{Z}} (\delta_{jF_s} * \mathcal{F}[T_x])
\end{aligned}
$$

where $F_s = \frac{1}{T_s}$ is the sampling frequency

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$
\begin{aligned}
\mathcal{F}[\text{Ш}_{T_s} x] &\overset{\text{Result 1.}}{=} \mathcal{F}[\text{Ш}_{T_s}] * \mathcal{F}[T_x] \\
&\overset{\text{Result 2.}}{=} F_s \text{Ш}_{F_s} * \mathcal{F}[T_x] \\
&\overset{\text{Distribu.}}{=} F_s \sum_{j \in \mathbf{Z}} (\delta_{jF_s} * \mathcal{F}[T_x])
\end{aligned}
$$

where $F_s = \frac{1}{T_s}$ is the sampling frequency

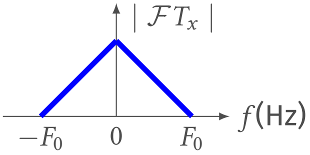
Sampling in the time domain $\Longrightarrow$ Periodic spectrum

Spectrum of band-limited signals

Band-limited signals

A band-limited signal x is a signal whose Fourier transform density has **bounded support**

Let $\mathbf{x}$ be a band-limited signal whose TF has support $[-F_0, F_0]$

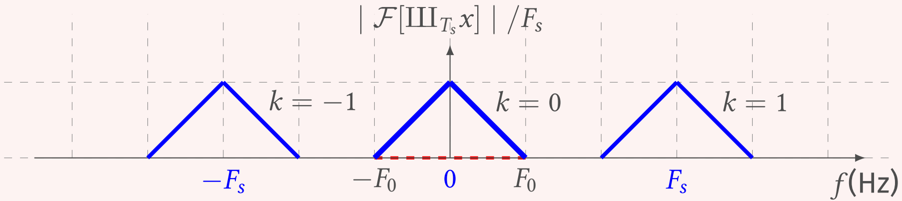


Question : Spectrum of $Ш_{T_s}\mathbf{x}$?

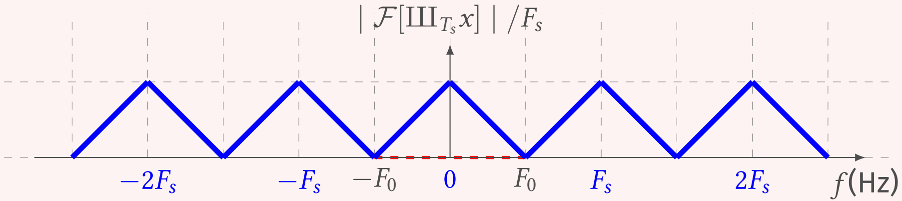
$$\mathcal{F}[Ш_{T_s}\mathbf{x}] = F_s \sum_{j=-\infty}^{+\infty} \delta_{jF_s} * \mathcal{F}T_x$$

Spectrum of a band-limited signal

Case 1: $F_s > 2F_{\max}$

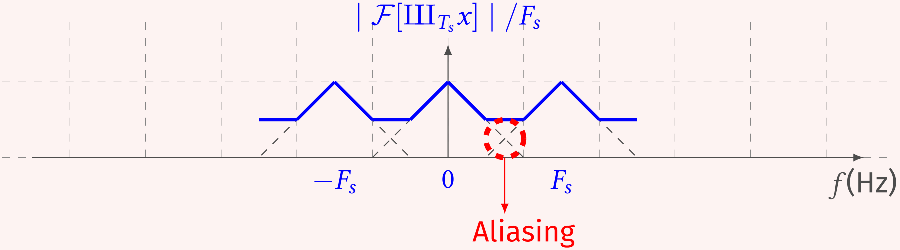


Case 2: $F_s = 2F_{\max}$



Spectrum of a band-limited signal

Case 3: $F_s < 2F_{\max}$



The signal is **deteriorated** $\Longrightarrow$ **Loss** of information

Shannon-Nyquist sampling theorem

Question: Necessary condition on F_s such that no information is lost?

Nyquist–Shannon sampling theorem

If a function $x(t)$ contains no frequencies higher than $F_{\max}$ hertz, it is **completely determined** by giving its ordinates at a series of points spaced $T_s \geq \frac{1}{2F_{\max}}$ seconds apart.

$\longrightarrow$ **Sufficient condition**

$F_s = 2F_{\max}$ is dubbed "**Nyquist frequency**"

Interpolation formula

Interpolation formula

If $F_s \geq 2F_{\max}$, then T_x is equal to the regular distribution spanned by

$$t \longmapsto \sum_{j \in \mathbf{Z}} x(jT_s)\,\mathrm{sinc}(F_s(t - jT_s))$$

(Informal) proof: if $F_s > 2F_{\max}$ the spectrum of x can be recovered with an ideal **low pass filter**

☝️ This operation (operating on the spectrum) is common in signal processing and often referred to as “filtering”

Interpolation formula

Interpolation formula

If $F_s \geq 2F_{\max}$, then T_x is equal to the regular distribution spanned by

$$t \longmapsto \sum_{j \in \mathbf{Z}} x(jT_s)\,\mathrm{sinc}(F_s(t - jT_s))$$

(Informal) proof: if $F_s > 2F_{\max}$ the spectrum of x can be recovered with an ideal **low pass filter**

$$\mathcal{F}T_x \overset{\mathcal{S}(\mathbf{R})'}{=} \mathcal{F}[\text{Ш}_{T_s}x] \cdot \frac{1}{F_s}\Pi_{F_s/2}(f)$$

$$\Rightarrow$$

$$T_x \overset{\mathcal{S}(\mathbf{R})'}{=} \text{Ш}_{T_s}x * T_{\mathrm{sinc}(2\pi F_s \cdot)}$$

$$\overset{\mathcal{S}(\mathbf{R})'}{=} \sum_{j \in \mathbf{Z}} x[jT_s]\delta_{jT_s} * T_{\mathrm{sinc}(2\pi F_s \cdot)}$$

Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned} \qquad \text{or} \qquad \begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathcal{S}(\mathbf{R})' \\ x &\longmapsto \text{Ш}_{T_s} x \end{aligned}$$

2. ***The target set***: The set of functions which admit a Fourier transform which is compactly supported in $[-F_s, F_s]$

3. ***The decoder***:

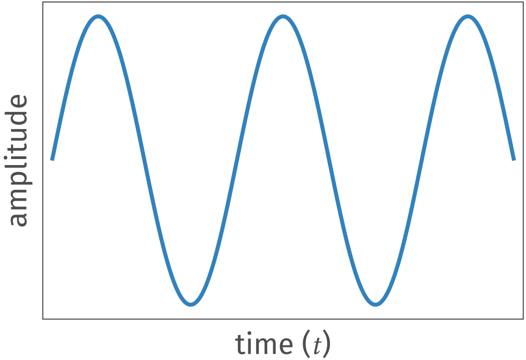
$$\begin{aligned} M\colon \quad \mathbf{R}^{\mathbf{Z}} &\longrightarrow \mathcal{S}' \\ \{y_j\}_{j\in\mathbf{Z}} &\longmapsto \textstyle\sum_{j\in\mathbf{Z}} y_j \operatorname{sinc}(F_s(\cdot - jT_s)) \end{aligned} \quad \text{if} \quad F_s \geq 2F_{\max}$$

4. ***The accuracy criterion***: “noiseless exact recovery”, *i.e.*

$$\forall x \in \mathcal{X}_{\mathrm{target}}, \quad D\big(M(x)\big) \overset{\mathcal{D}(\mathbf{R})'}{=} T_x$$

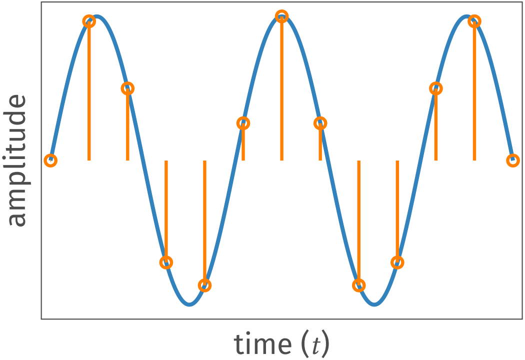
Conclusion

Numerical illustration



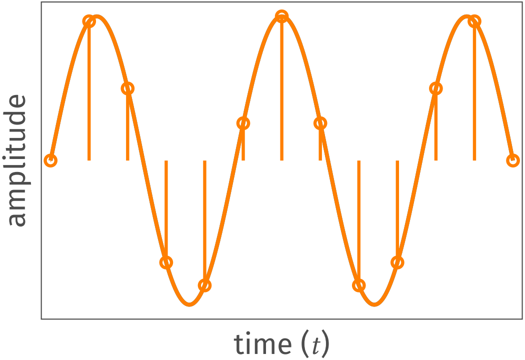
- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$

Numerical illustration



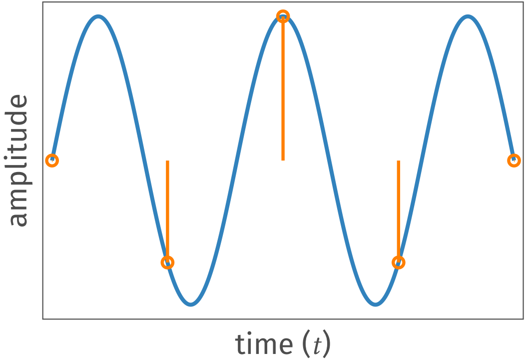
- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$
- $T_s = 0.4 \implies F_s = 2.5 > 2F_{\max}$

Numerical illustration



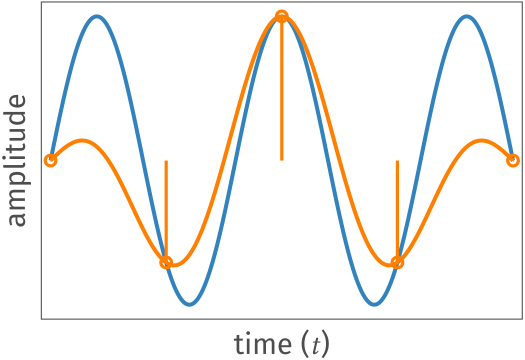
- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$
- $T_s = 0.4 \implies F_s = 2.5 > 2F_{\max}$

Numerical illustration



- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$
- $T_s = 0.4 \implies F_s = 2.5 > 2F_{\max}$
- $T_s = 1.2 \implies F_s = .8 < 2F_{\max}$ ⚠

Numerical illustration



- x : $F_{\max} = 0.5$ and $x(t) = \cos(2\pi F_{\max} t)$
- $T_s = 0.4 \implies F_s = 2.5 > 2F_{\max}$
- $T_s = 1.2 \implies F_s = .8 < 2F_{\max}$ ⚠

🤔 Are we really particularizing the linear model?

This is a legitimate question

🤔 Are we really particularizing the linear model?

This is a legitimate question

☝️ **Rappels:** a linear model is characterized by

✓ a linear observation operator operator M ✓; *Already done*

Are we really particularizing the linear model?

This is a legitimate question

Rappels: a linear model is characterized by

- ✓ a linear observation operator operator M ✓; *Already done*
- ✓ a vector space as choices for $\mathcal{X}_{\mathrm{target}}$; *By linearity of the Fourier transform*

Are we really particularizing the linear model?

This is a legitimate question

Rappels: a linear model is characterized by

- ✓ a linear observation operator operator M ✓; *Already done*
- ✓ a vector space as choices for $\mathcal{X}_{\mathtt{target}}$; *By linearity of the Fourier transform*
- ▶ (an orthogonal projection as “best” decoder) *In the finite-dimensional case*

A concluding remark

If we restrict ourself to signals $x \in \mathcal{L}^2(\mathbf{R})$

- $\mathcal{L}^2(\mathbf{R})$ is a Hilbert space equipped with the inner product

$$\langle x_1, x_2 \rangle = \int x_1(t)\overline{x_2}(t)\mathrm{d}t$$

- The family of function

$$\forall j, \quad m_j : t \mapsto \frac{1}{\sqrt{T_s}} \operatorname{sinc}\left(\frac{t - jT_s}{T_s}\right)$$

  is an orthonormal basis of the set of functions with Fourier transform supported in $[-F_s, F_s]$

- For all signals x with Fourier transform supported in $[-F_s, F_s]$,

$$\langle m_j, x \rangle = x(jT_s)$$

- The proposed decoder thus rewrites

$$t \mapsto \sum_{j \in \mathbf{Z}} \langle m_j, x \rangle m_j(t)$$

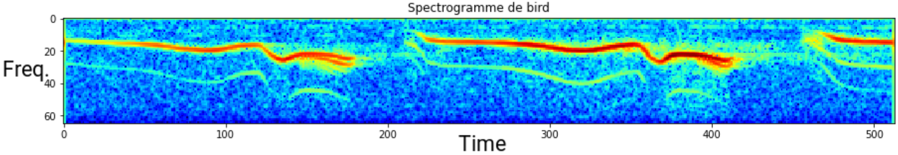
It is an orthogonal projection!

Bonus: Time-Frequency analysis, wavelets

Time-frequency analysis

- In most domains (audio processing...), the frequency spectrum *itself* evolves in time.

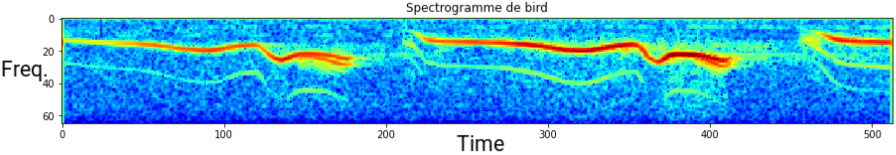
Example: audio *spectrogram*



Time-frequency analysis

- In most domains (audio processing...), the frequency spectrum *itself* evolves in time.
- We want to analyze the frequency component *localized in time*, hence "time-frequency analysis"

Example: audio *spectrogram*



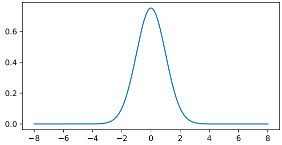
Windowed Fourier transform

- Popular tool in audio: **Windowed Fourier Transform**, also called **Short Time Fourier Transform** (STFT) in its discrete version.

Windowed Fourier transform

- Popular tool in audio: **Windowed Fourier Transform**, also called **Short Time Fourier Transform** (STFT) in its discrete version.

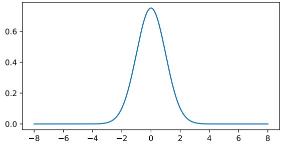
Define a symmetric window $g(t) = g(-t)$ with $\|g\|_{L^2} = 1$



Windowed Fourier transform

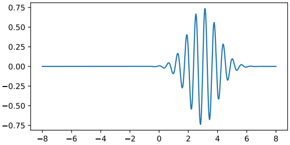
- Popular tool in audio: **Windowed Fourier Transform**, also called **Short Time Fourier Transform** (STFT) in its discrete version.

Define a symmetric window $g(t) = g(-t)$ with $\|g\|_{L^2} = 1$



Time-frequency atoms are localized in time and frequency:

$$g_{u,\xi}(t) = \underbrace{e^{2i\pi\xi t}}_{\text{Freq. } \xi} \underbrace{g(t-u)}_{\text{Time } u}$$



Windowed Fourier transform

The **Windowed Fourier Transform** is defined as

$$Sx(u,\xi) = \langle x, g_{u,\xi} \rangle_{L^2} = \int x(t) g(t-u) e^{-2i\pi\xi t} \mathrm{d}t$$

Windowed Fourier transform

The **Windowed Fourier Transform** is defined as

$$Sx(u,\xi) = \langle x, g_{u,\xi} \rangle_{L^2} = \int x(t) g(t-u) e^{-2i\pi\xi t} \mathrm{d}t$$

- Analyze the signal x around time u and frequency f

Windowed Fourier transform

The **Windowed Fourier Transform** is defined as

$$Sx(u,\xi) = \langle x, g_{u,\xi}\rangle_{L^2} = \int x(t)g(t-u)e^{-2i\pi\xi t}\mathrm{d}t$$

- Analyze the signal x around time u and frequency f

By Parseval's identity,

$$Sx(u,\xi) = \langle \hat{x}, \hat{g}_{u,\xi}\rangle_{L^2}$$

and by the shift formula

$$\hat{g}_{u,\xi}(f) = \hat{g}(f-\xi)e^{-2i\pi u(f-\xi)}$$

- since g is real symmetric, $\hat{g}$ is real symmetric (ex: for g Gaussian, $\hat{g}$ is Gaussian!)
- Also performs a windowed transform of $\hat{x}$ around ξ (and "inverse frequency" u)

Heisenberg boxes

$$Sx(u,\xi) = \langle x, g_{u,\xi} \rangle_{L^2} = \langle \hat{x}, \hat{g}_{u,\xi} \rangle_{L^2}$$

Heisenberg boxes

$$Sx(u, \xi) = \langle x, g_{u,\xi} \rangle_{L^2} = \langle \hat{x}, \hat{g}_{u,\xi} \rangle_{L^2}$$

Considering $\mathbb{P} = |g_{u,\xi}|^2$ and $\hat{\mathbb{P}} = |\hat{g}_{u,\xi}|^2$ as **probabilities densities** (since $\|g_{u,\xi}\|_{L^2} = \|g_{u,\xi}\|_{L^2} = 1$):

$$\mathbb{E}(\mathbb{P}) = u \qquad \text{Var}(\mathbb{P}) = \sigma^2_{\mathbb{P}}$$

$$\mathbb{E}(\hat{\mathbb{P}}) = \xi \qquad \text{Var}(\mathbb{P}) = \sigma^2_{\hat{\mathbb{P}}}$$

Heisenberg boxes

$$Sx(u,\xi) = \langle x, g_{u,\xi}\rangle_{L^2} = \langle \hat{x}, \hat{g}_{u,\xi}\rangle_{L^2}$$

Considering $\mathbb{P} = |g_{u,\xi}|^2$ and $\hat{\mathbb{P}} = |\hat{g}_{u,\xi}|^2$ as **probabilities densities** (since $\|g_{u,\xi}\|_{L^2} = \|g_{u,\xi}\|_{L^2} = 1$):

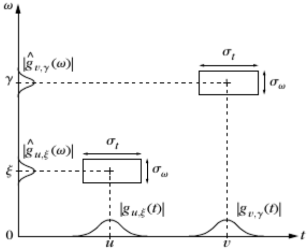
$$\mathbb{E}(\mathbb{P}) = u \qquad \mathrm{Var}(\mathbb{P}) = \sigma_{\mathbb{P}}^2$$

$$\mathbb{E}(\hat{\mathbb{P}}) = \xi \qquad \mathrm{Var}(\mathbb{P}) = \sigma_{\hat{\mathbb{P}}}^2$$

and by Heisenberg's thm:

$$\sigma_{\mathbb{P}}^2 \sigma_{\hat{\mathbb{P}}}^2 \geq 1/4$$

with equality if and only if g is (any) **Gaussian**.



Reconstruction

Thm: inversion formula

If $x \in \mathcal{L}^2$,

$$x(t) = \iint Sx(u,\xi) g(t-u) e^{2i\pi\xi t} \mathrm{d}\xi \mathrm{d}u$$

and

$$\|x\|_{L^2}^2 = \iint |Sx(u,\xi)|^2 \mathrm{d}\xi \mathrm{d}u$$

Reconstruction

Thm: inversion formula

If $x \in \mathcal{L}^2$,

$$x(t) = \iint Sx(u,\xi) g(t-u) e^{2i\pi\xi t} \mathrm{d}\xi \mathrm{d}u$$

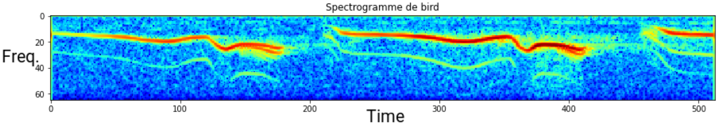
and

$$\|x\|_{L^2}^2 = \iint |Sx(u,\xi)|^2 \mathrm{d}\xi \mathrm{d}u$$

- no information is lost (true ”transform”)
- $Sx \in \mathcal{L}^2(\mathbf{R}^2)$

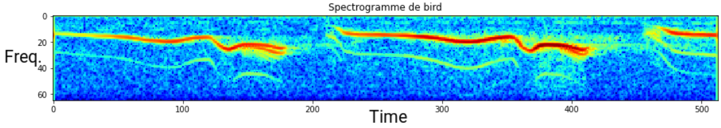
Example of applications

Spectrogram $|Sx(u,\xi)|^2$ used a lot in audio.



Example of applications

Spectrogram $|Sx(u,\xi)|^2$ used a lot in audio.



Ex: wav2midi with sparse matrix factorization.

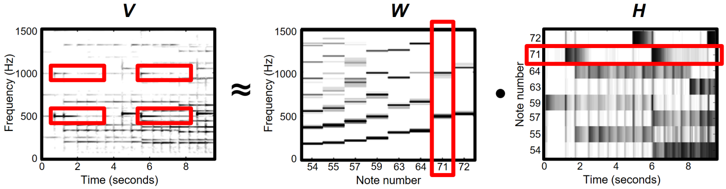
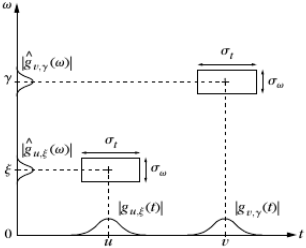


Figure 8.21c-d from [Müller, FMP, Springer 2015]

More precisely, "Non-negative matrix factorization" (NMF), a low-rank factorization of a non-neg. matrix (the spectrogram) into non-neg. components, frequency/activation. See `https://www.audiolabs-erlangen.de/resources/MIR/FMP/C8/C8S3NMFbasic.html`

Problem with WFT

- The WFT analyzes content with the same window length
- Element can have different durations... (especially low-frequency)
- Human hearing is spread “logarithmically” in frequency



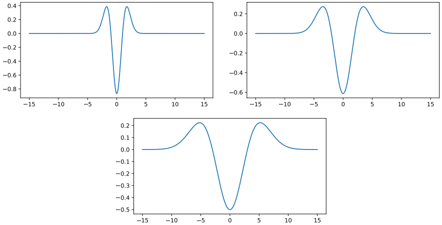
Wavelet transform

A wavelet is a normalized function $\psi \in \mathcal{L}^2(\mathbf{R})$ with a zero average:

$$\int \psi(t)dt = 0, \quad \|\psi\| = 1$$

Time-frequency atoms are obtained by *scaling* the wavelet:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$



Wavelet transform:

$$Wx(u, s) = \langle x, \psi_{u,s} \rangle$$

Heisenberg boxes

Assume ψ is analytic for simplicity (i.e. zero negative frequency, see Mallat book).

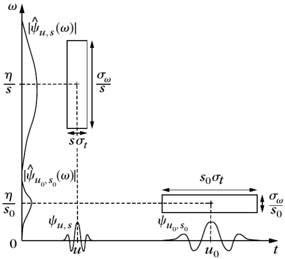
Heisenberg boxes

Assume ψ is analytic for simplicity (i.e. zero negative frequency, see Mallat book).

Considering $\mathbb{P} = |\psi_{u,s}|^2$ and $\hat{\mathbb{P}} = |\hat{\psi}_{u,s}|^2$ as **probabilities densities**, and $\mathbb{P}_1, \hat{\mathbb{P}}_1$ the same for reference $s = 1$ (u does not matter). Then:

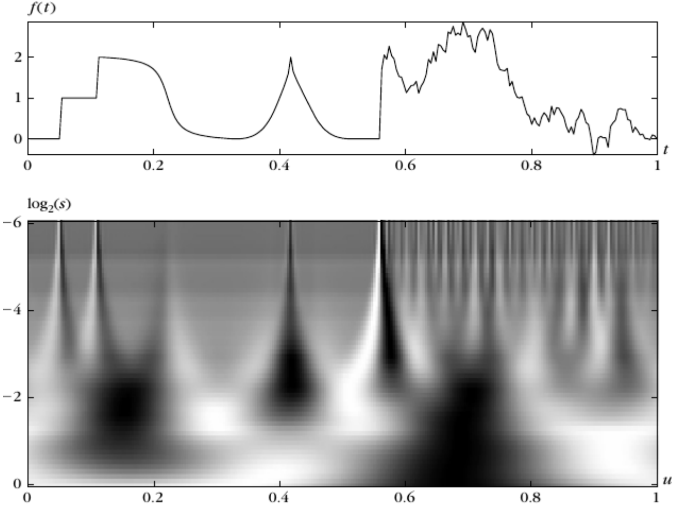
$$\mathbb{E}(\mathbb{P}) = u \qquad \mathrm{Var}(\mathbb{P}) = s^2 \mathrm{Var}(\mathbb{P}_1)$$
$$\mathbb{E}(\hat{\mathbb{P}}) = \mathbb{E}(\hat{\mathbb{P}}_1)/s \qquad \mathrm{Var}(\mathbb{P}) = \mathrm{Var}(\mathbb{P}_1)/s^2$$

The "right" multiplicative scale! (for human ears)



Scalogram

$|Wx|^2$ is called a scalogram.



Beyond...

- Reconstruction theorem for Wavelets
- Discrete Wavelets
- Fast computation (FFT, fast hierarchical wavelets)
- multi-D wavelets (especially 2D!)
- etc.