Sparse Signal Processing
Lecture 4 — Fourier interpolation and the linear model

Nicolas Keriven (slides by Clément Elvira)

Outline

N. Keriven: CNRS researcher at IRISA, SIROCCO team.
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- introduce a linear target set ;
- derive a decoder achieving exact recovery ;

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- introduce a linear target set ;
- derive a decoder achieving exact recovery ;

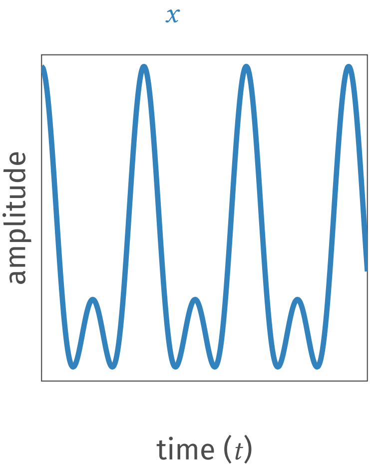
Material at `https://github.com/nkeriven/ENS-signal` (subject to changes, check every now and then)

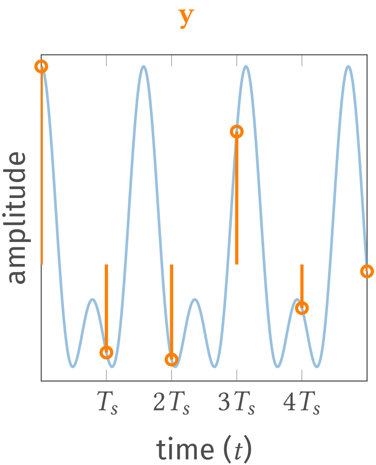
References

- Mallat, 2012. *A Wavelet Tour of Signal Processing: the Sparse Way*
- F Golse polycopié: `http://www.math.polytechnique.fr/~golse/MAT431-10/POLY431.pdf`
- Wikipédia !

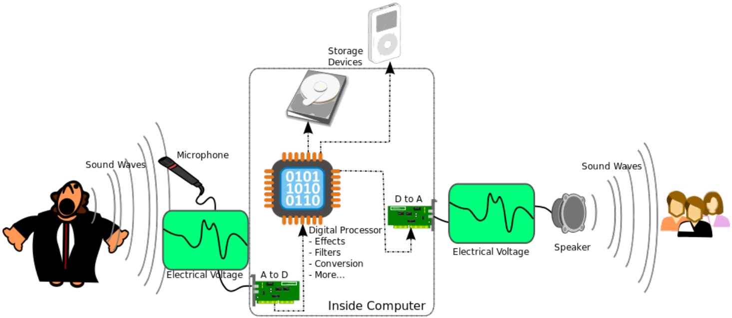
Considered Task

Recovering a signal x from **infinitely** many function evaluations



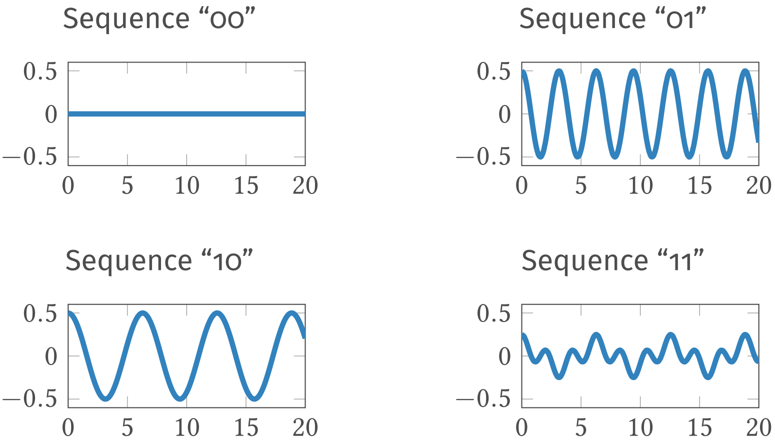


Application 1: sampling and reconstructing audio recording



Important component of sound perception: the frequency

Application 2: (simplified) transmitters



Support of information: electromagnetic (sinusoidal at a given frequency) waves

Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M \colon \mathcal{L}^2(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned}$$

2. ***The target set***: *To be defined*

   Related to some notion of frequency

3. ***The decoder***: *To be defined*

4. ***The accuracy criterion***: "noiseless exact recovery", *i.e.*

$$\forall x \in \mathcal{X}_{\texttt{target}}, \quad D\big(M(x)\big) = x$$

🤔 Which mathematical tool can be used to emphasize the frequencies characterizing a signal?

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☝️ The Fourier transform

Rappels: The Fourier transform

Definition

The Fourier transform is a linear operator defined by

$$\begin{aligned}\mathcal{F}\colon \mathcal{L}^1(\mathbf{R}) &\longrightarrow \mathbf{C}^{\mathbf{R}}\\ x &\longmapsto \left(f \mapsto \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t\right)\end{aligned}$$

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- Sometimes defined without the 2π (but less clean, multiplicative constants everywhere)
- $\mathcal{F}x$ is often written $\hat{x}$

An example

Exercice

Evaluate the Fourier transform of

$$\Pi_{[-\frac{a}{2},\frac{a}{2}]} : t \longmapsto \begin{cases} 1 & \text{if } t \in [-\frac{a}{2}, +\frac{a}{2}] \\ 0 & \text{otherwise} \end{cases}$$

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Answer

$$\mathcal{F}\Pi(f) = a\frac{\sin(\pi a f)}{\pi a f} \overset{\text{def.}}{=} a\,\mathrm{sinc}(af)$$

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Not in $\mathcal{L}^1$!

Rappels: Inverse Fourier transform

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The conjugate Fourier transform is the linear operator defined by

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Theorem: Inversion

If $x \in \mathcal{L}^1(\mathbf{R})$ is such that $\mathcal{F}x \in \mathcal{L}^1(\mathbf{R})$ then

$$\overline{\mathcal{F}}[\mathcal{F}x] \overset{\text{a.e.}}{=} x$$

with equality at every point of continuity.

Extension to L^2

Pbm: $x \in \mathcal{L}^1$ does **not** imply $\mathcal{F}x \in \mathcal{L}^1$! (ex: door function and cardinal sinus)

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$\rightarrow$ **Solution:** extension by *density*.

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If $f, g \in \mathcal{L}^1 \cap \mathcal{L}^2$,

$$\langle f, g \rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2}$$

(recall that $\langle f, g \rangle_{L^2} = \int f(t)\overline{g(t)}dt$)

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Consequence: $\|f\|_{L^2} = \|\mathcal{F}f\|_{L^2}$ and $f \in \mathcal{L}^1 \cap \mathcal{L}^2 \Rightarrow \mathcal{F}f \in \mathcal{L}^2$. Now we just need to extend the domain of definition of $\mathcal{F}$

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We are going to extend Fourier to $\mathcal{L}^2$ by *density* of $\mathcal{L}^1 \cap \mathcal{L}^2$ within $\mathcal{L}^2$

$\rightarrow$ i.e., $\forall \epsilon > 0, f \in \mathcal{L}^2, \exists g \in \mathcal{L}^2 \cap \mathcal{L}^1$ s.t. $\|f - g\|_{\mathcal{L}^2} \leq \epsilon$

Extension to $\mathcal{L}^2$

- Let $f \in \mathcal{L}^2$.
- By density of $\mathcal{L}^2 \cap \mathcal{L}^1$ in $\mathcal{L}^2$, there is a sequence $f_n \in \mathcal{L}^2 \cap \mathcal{L}^1$ such that $\|f_n - f\|_{\mathcal{L}^2} \to 0$

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- f_n is a Cauchy sequence, and by Parseval's identity $\|f_n - f_p\|_{\mathcal{L}^2} = \|\mathcal{F}(f_n - f_p)\|_{\mathcal{L}^2}$, hence $\mathcal{F}f_n \in \mathcal{L}^2$ is also a Cauchy sequence

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- Since $\mathcal{L}^2$ is a Hilbert space and **is therefore complete**, $\mathcal{F}f_n$ is a convergent sequence.
- We call its limit the "Fourier transform of f" denoted by $\mathcal{F}f$. It satisfies all the usual properties of the Fourier transform (see next slide)

Properties of Fourier Transform (...easy, or admitted)

- f symmetric $\Rightarrow \mathcal{F}f$ real
- f anti-symmetric $\Rightarrow \mathcal{F}f$ imaginary
- f real $\Rightarrow \mathcal{F}f$ Hermitian
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- $\mathcal{F}(f \star g)(t) = \mathcal{F}f(t)\mathcal{F}g(t)$ (convolution thm)
- $\langle f, g\rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2}$ (Parseval identity)
- $\mathcal{F}\frac{df}{dt}(t) = 2i\pi t\mathcal{F}f(t) \Rightarrow$ if f is k times differentiable, $\mathcal{F}f$ decrease faster than $1/t^k$

Properties (summary)

Property	Function	Fourier transform
	$x(t)$	$\hat{x}(f)$
Inverse	$\hat{x}(t)$	$x(-f)$
Convolution	$x_1 \star x_2(t)$	$\hat{x}_1\hat{x}_2(f)$
Multiplication	$x_1(t)x_2(t)$	$\hat{x}_1 \star \hat{x}_2(f)$
Translation	$x(t-u)$	$e^{-2i\pi uf}\hat{x}(f)$
Modulation	$e^{2i\pi\xi t}x(t)$	$\hat{x}(f-\xi)$
Scaling	$x(t/s)$	$\|s\|\hat{x}(sf)$
Time derivatives	$x^{(p)}(t)$	$(2i\pi f)^p\hat{x}(f)$
Freq. derivatives	$(-2i\pi t)^p x(t)$	$\hat{x}^{(p)}(f)$
Complex conjugate	$\overline{x}(t)$	$\overline{\hat{x}}(-f)$

Uncertainty principle

🤔 Can we construct a function that is **highly localised in space**, whose Fourier transform is **highly localised in frequency**?

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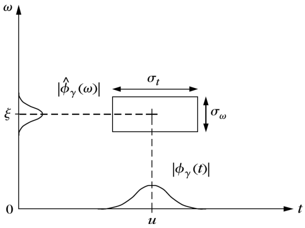
Heisenberg's uncertainty principle

For $x \in \mathcal{L}^2(\mathbf{R})$, define the two density functions $p_x = \frac{|x(t)|^2}{\|x\|_{L^2}}, \quad p_{\hat{x}} = \frac{|\hat{x}(f)|^2}{\|x\|_{L^2}}$ and their respective variance σ_x^2, $\sigma_{\hat{x}}^2$. Then

$$\sigma_x^2 \sigma_{\hat{x}}^2 \geq 1/4$$



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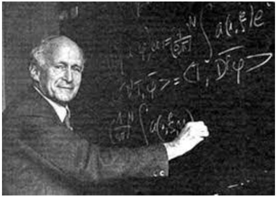
We need a “bigger” space to work with!

$\rightarrow$ Space of “distributions”, defined by *duality*.

We need the smallest dual space as possible, to make the biggest distribution space as possible!

Reminder (?) on distributions



Tests functions

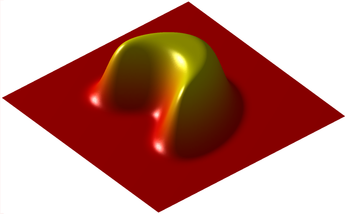
Definition

A test function refers to any infinitely differentiable function

$$\varphi\colon \mathbf{R} \longrightarrow \mathbf{C}$$

witch is compactly supported.

The set of test functions is denoted $\mathcal{D}(\mathbf{R})$.



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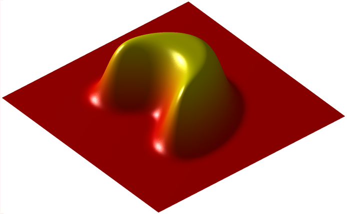
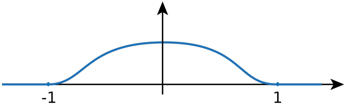


Example: *(standard one)* the function

$$t \longmapsto \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & \text{if } t \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$



can be proved to be a test function.

Topology on $\mathcal{D}(\mathbf{R})$

Definition: convergence in $\mathcal{D}(\mathbf{R})$

Let $\{\varphi_n\}_{n\in\mathbf{N}}$ be a family of test function.
We say that $\varphi_n \to \varphi \in \mathcal{D}(\mathbf{R})$ if and only if

- there exists an integer N and a compact set K such $n \geq N \Rightarrow \operatorname{support}(\varphi_n - \varphi) \subset K$
- for all $r \in \mathbf{N}$,
$$\left\| \varphi_n^{(r)} - \varphi^{(r)} \right\|_\infty \to 0$$
- i.e., supremum norm for φ and all its derivatives.

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Remarks

- Said differently, the space of distributions is the (topological) dual space of $\mathcal{D}(\mathbf{R})$, denoted by $\mathcal{D}'(\mathbf{R})$;
- When evaluating a distribution T at φ, we write

$$\langle T, \varphi \rangle$$

- By extension, we say that two distributions T, T' are equal iff

$$\langle T, \varphi \rangle = \langle T', \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbf{R})$$

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This can nevertheless be a misleading terminology, as distributions do not take scalar as inputs. (...but *test functions*)

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Besides mathematical benefits, there is a physical intuition behind this reference

A compactly supported test functions can be seen as a model of a physical system measuring some quantity

Regular distributions

Definition/Proposition

If g is any locally integrable function, then the linear form T_g defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T_g, \varphi \rangle \stackrel{\text{def.}}{=} \int g(t)\varphi(t)\mathrm{d}t$$

is a distribution.

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In short: Linearity is fine
Continuity: letting $\varphi_n \to \varphi$, there exists a compact K that contains all supports
Then $|\langle T_g, \varphi_n - \varphi \rangle| \leq \int g(t)\mathrm{d}t \|\varphi_n - \varphi\|_\infty$

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Remark

- Hence distributions is a "bigger" space...
- If $T_{g_1} = T_{g_2}$ then $g_1 \overset{\text{a.e.}}{=} g_2$

Singular distributions

The terminology “singular distributions” refers to any distribution that is not regular

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Example 1: The dirac function δ_t defined $\forall t \in \mathbf{R}$ as

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Example 2: The dirac comb $Ш_{T_s}$ defined as

$$\forall \varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle Ш_{T_s}, \varphi \rangle = \sum_{j \in \mathbf{Z}} \langle \delta_{jT_s}, \varphi \rangle = \sum_{j \in \mathbf{Z}} \varphi(jT_s)$$

Letting $\varphi_n \to \varphi$*, there exists a compact* K *that contains all supports, and* $|\langle \delta_t, \varphi_n - \varphi \rangle| \leq K_{T_s} \|\varphi_n - \varphi\|_\infty \to 0$*, where* K_{T_s} *is the number of* jT_s *contained in* K

🤔 **Regular distributions** provide intuition on how to define many concepts related to distributions

derivation, convolution, Fourier transform to name but a few

Derivation of a distribution

For regular distributions: let g be a differentiable locally integrable function. Then g' is also locally integrable and

$$\begin{aligned}\langle T_{g'}, \varphi\rangle &= \int g'(t)\varphi(t)\mathrm{d}t \\ &= -\int g(t)\varphi'(t)\mathrm{d}t \qquad \text{(IBP)} \\ &= -\langle T_g, \varphi'\rangle\end{aligned}$$

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Definition

Let T be a distribution. The derivative of T is the distribution $T^{(1)}$ defined by

$$\forall\varphi \in \mathcal{D}(\mathbf{R}), \qquad \langle T^{(1)}, \varphi\rangle \overset{\text{def.}}{=} -\langle T, \varphi'\rangle$$

A second example: Fourier transform

Let g be an integrable function (so that T_g is a regular distribution). Then $\mathcal{F}g$ is also integrable and

$$\begin{aligned}
\langle T_{\mathcal{F}g}, \varphi\rangle &= \int \mathcal{F}g(f)\varphi(f)\mathrm{d}f \\
&= \int \left(\int g(t)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t \right) \varphi(f)\mathrm{d}f \\
&= \int g(t) \left(\int \varphi(f)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}f \right) \mathrm{d}t \\
&\overset{(?)}{=} \langle T_g, \mathcal{F}\varphi\rangle
\end{aligned}$$

Fourier transform of a distribution (tentative)

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The set of distributions is too big a space!!

Tempered distribution

We will **extend** the set of test functions...

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☝️

... which will **reduce** the set of distributions (by duality).

Tempered distributions

Definition: Schwartz space

We say that a function g is a Schwartz function if it is infinitely differentiable and for all $k, q \in \mathbf{N}$

$$\lim_{t \longmapsto \pm\infty} |t^k \varphi^{(q)}(t)| = 0$$

The set of Schwartz functions is denoted $\mathcal{S}(\mathbf{R})$.

- Instead of compact support, we take **fast decrease** (faster than any polynomial).

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$$\lim_{t \longmapsto \pm\infty} |t^k \varphi^{(q)}(t)| = 0$$

The set of Schwartz functions is denoted $\mathcal{S}(\mathbf{R})$.

- Instead of compact support, we take **fast decrease** (faster than any polynomial).

Definition: Tempered distribution

We call "tempered distribution" any element of the (topological) dual space of the Schwartz space $\mathcal{S}'(\mathbf{R})$.

*"Topological": **continuous** linear functionals*

Properties

- Convergence in $\mathcal{S}(\mathbf{R})$ is defined through the family of semi-norms $\mathcal{N}_p(\varphi) = \sum_{k,q\leq p} \sup_t |t^q \varphi^{(q)}(t)|$, i.e. φ_n converge to φ if $\forall p,\ \mathcal{N}_p(\varphi_n - \varphi) \to 0$

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- Any test function is a Schwartz function: $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$
- (hence) any tempered distribution is a distribution: $\mathcal{S}'(\mathbf{R}) \subset \mathcal{D}'(\mathbf{R})$

Fourier and tempered distributions

Thm

- Thm: All tempered distribution **admit a Fourier transform** defined by
$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \mathcal{F}T, \varphi \rangle \overset{\text{def.}}{=} \langle T, \mathcal{F}\varphi \rangle$$
- The Fourier transform of a Schwartz function is **also** a Schwartz function

Hint: link between differentiability and polynomial decrease of the Fourier transform...

Classical tempered distribution

⚠️ Unlike regular distributions, all functions do not correspond to tempered distributions! Need all derivative to grows “as fast as” polynomials...

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$\rightarrow$ that's all we need in this course...?

⚠️ In signal processing, it is folklore to believe that distributions are always well-behaved

That it all operations are legitimate

It it nevertheless not always true

☝️ In this lecture (and companion lab) it will nevertheless be true

Some justifications will be out of scope and left in exercise

Some properties of (tempered) distributions

Some Fourier transform (1)

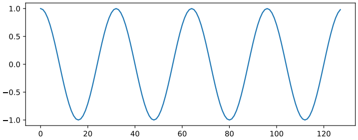
$$\begin{aligned}
\langle \mathcal{F}\delta_a, \varphi\rangle &= \langle \delta_a, \mathcal{F}\varphi\rangle \\
&= (\mathcal{F}\varphi)(a) \\
&= \int \varphi(t)\mathrm{e}^{-2\mathrm{i}\pi a t}\mathrm{d}t \\
&= \langle T_{t\mapsto \mathrm{e}^{-2\mathrm{i}\pi a t}}, \varphi\rangle
\end{aligned}$$

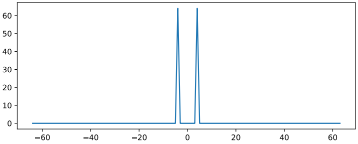
Some Fourier transform (2)

$$\begin{aligned}
\langle \mathcal{F} T_{t \mapsto \mathrm{e}^{2\mathrm{i}\pi a t}}, \varphi \rangle &\overset{\text{def.}}{=} \langle T_{t \mapsto \mathrm{e}^{2\mathrm{i}\pi a t}}, \mathcal{F}\varphi \rangle \\
&\overset{\text{def.}}{=} \int \mathrm{e}^{2\mathrm{i}\pi a f} \mathcal{F}\varphi(f) \mathrm{d}f \\
&\overset{\text{def.}}{=} \mathcal{F}^{-1}\mathcal{F}\varphi(a) \\
&= \langle \delta_a, \varphi \rangle
\end{aligned}$$

Some Fourier transform (3)

$$
\begin{aligned}
\langle \mathcal{F}T_{t\mapsto \cos(2\pi at)}, \varphi \rangle &= \langle \mathcal{F}T_{t\mapsto 0.5(\mathrm{e}^{2\pi at}+\mathrm{e}^{-2\pi at}}, \varphi \rangle \\
&= \frac{1}{2}\langle \mathcal{F}T_{t\mapsto \mathrm{e}^{2\pi at}}, \varphi \rangle + \frac{1}{2}\langle \mathcal{F}T_{t\mapsto \mathrm{e}^{-2\pi at}}, \varphi \rangle \\
&= \frac{1}{2}(\langle \delta_a, \varphi \rangle + \langle \delta_{-a}, \varphi \rangle)
\end{aligned}
$$





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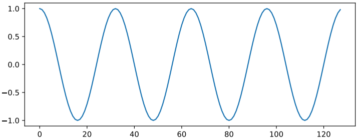

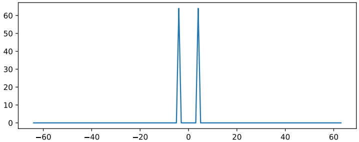

Recall that $\mathcal{F}[Re(f)](\xi) = \frac{1}{2}(\mathcal{F}f(\xi) + \mathcal{F}f(-\xi))$

🤔 Have we achieved our initial objective?

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☝️ It is folklore to provide a graphical representation of a Fourier transform

Another example: the "door" distribution

We define $\Pi_{[-\frac{a}{2},\frac{a}{2}]}$ such that

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \Pi_{[-\frac{a}{2},\frac{a}{2}]}, \varphi \rangle = \int_{-\frac{a}{2}}^{\frac{a}{2}} \varphi(t)\mathrm{d}t$$

Question: Evaluate the Fourier transform of $\Pi_{[-\frac{a}{2},\frac{a}{2}]}$.

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Question: Evaluate the Fourier transform of $\Pi_{[-\frac{a}{2},\frac{a}{2}]}$.

$$\begin{aligned}
\langle \mathcal{F}\Pi_{[-\frac{a}{2},\frac{a}{2}]}, \varphi \rangle &= \langle \Pi_{[-\frac{a}{2},\frac{a}{2}]}, \mathcal{F}\varphi \rangle \\
&\overset{\text{Fub.}}{=} \int \varphi(t) \int_{-\frac{a}{2}}^{\frac{a}{/}2} \mathrm{e}^{-2\mathrm{i}ft} \mathrm{d}f\,\mathrm{d}t \\
&= \langle T_{a\,\mathrm{sinc}(a\cdot)}, \varphi \rangle
\end{aligned}$$

🤔 Does it ring a bell?

Compatibility with the "standard" Fourier transform

Theorem (admitted)

If $x \in \mathcal{L}^1(\mathbf{R})$ (or $\mathcal{L}^2(\mathbf{R})$) then

$$\mathcal{F} T_x = T_{\mathcal{F} x}$$

Towards an inverse Fourier transform

Definition: Conjugate Fourier transform

Let T be a tempered distribution. The conjugate Fourier of T is the distribution $\overline{\mathcal{F}}T$ defined by

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \overline{\mathcal{F}}T, \varphi\rangle \overset{\text{def.}}{=} \langle T, \overline{\mathcal{F}}\varphi\rangle$$

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Theorem: Inversion formula

Let T be a tempered distribution.
Then, its Fourier transform satisfies

$$T = \overline{\mathcal{F}}[\mathcal{F}T]$$

In other words, the standard inversion formula holds in the sense of tempered distributions

Convolution

Definition

Let T_1, T_2.
Their convolution (if it exists) is the distribution $T_1 * T_2$ defined as

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle T_1 * T_2, \varphi \rangle \stackrel{\text{def.}}{=} \langle T_1, \varphi_{T_2} \rangle$$

where

$$\forall t \in \mathbf{R}, \qquad \varphi_{T_2}(t) = \langle T_2, \varphi(\cdot + t) \rangle$$

(T_1, T_2 can be exchanged in the definition)

Example: of $T_1 = \delta_a$

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle \delta_a * T_2, \varphi \rangle = \langle T_2, \varphi(\cdot - a) \rangle$$

$\rightarrow$ Translation of the distribution

Support of a distribution

Definition

Let $T \in \mathcal{S}(\mathbf{R})'$.
Define Ω as the largest open set of $\mathbf{R}$ such that

$$\operatorname{support}(\varphi) \subseteq \Omega \implies \langle T, \varphi \rangle = 0$$

By extension, the (closed) set $\operatorname{support}(T) \overset{\text{def.}}{=} \Omega^c$ is called the **support** of the distribution

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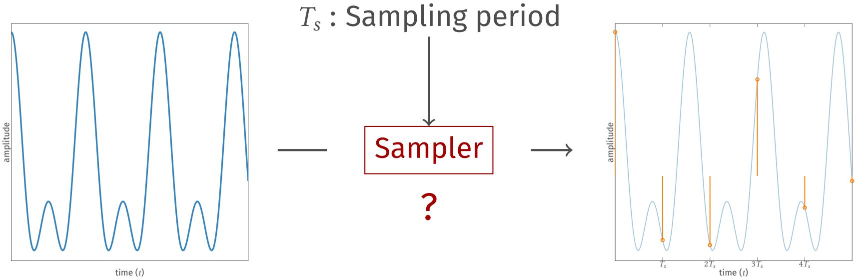
Examples:

- $\mathrm{support}(\delta_a) = \{a\}$
- $\mathrm{support}(\delta_a + \delta_{-a}) = \{-a, a\}$

Back to our sampling problem

(with new ingredients)

Ideal Sampling



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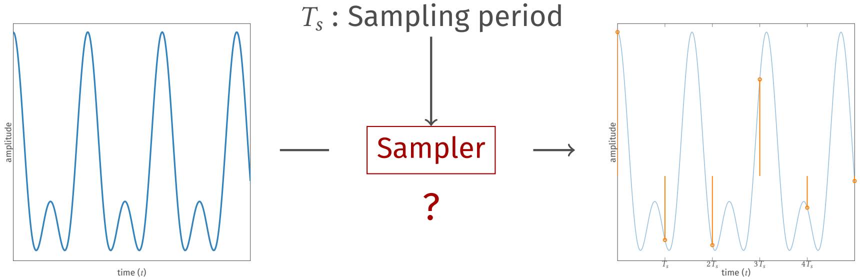

Observation model: $\equiv$ $\mathbf{x}$ $\times$ Dirac comb with period T_s:

$$x_e \overset{\text{def.}}{=} \sum_{j \in \mathbf{Z}} x(jT_s)\delta_{jT_s} \overset{?}{=} Ш_{T_s} x$$

Such a definition is sounded

Is the distribution $\sum_{j\in\mathbf{Z}} x(jT_s)\delta_{jT_s}$ sounded?

Said differently, does the “distribution-function product” $Ш_{T_s}x$ mean something?

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Is the distribution $\sum_{j\in\mathbf{Z}} x(jT_s)\delta_{jT_s}$ sounded?

Said differently, does the “distribution-function product” $\text{Ш}_{T_s}x$ mean something?

Theorem

Let $T \in \mathcal{S}(\mathbf{R})'$ and $x \in \mathcal{C}^\infty(\mathbf{R})$ be such that

$$\forall r \geq 0, \exists C_r > 0 \text{ and } n_r \in \mathbf{N}: \quad |x^{(r)}(t)| \leq C_r(1+|t|)^{n_r}.$$

Then the product Tx defined as

$$\forall \varphi \in \mathcal{S}(\mathbf{R}), \qquad \langle Tx, \varphi\rangle \stackrel{\text{def.}}{=} \langle T, x\varphi\rangle$$

is a tempered distribution

Since $x\varphi \in \mathcal{S}(\mathbf{R})$.
The set of such functions is denoted $\mathcal{O}(\mathbf{R})$

Wrapping up: main ingredients

1. ***The sensing operator***: The (linear) operator defined by

$$\begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathbf{R}^{\mathbf{Z}} \\ x &\longmapsto \{x(jT_s)\}_{j\in\mathbf{Z}} \end{aligned} \qquad \text{or} \qquad \begin{aligned} M\colon \mathcal{O}(\mathbf{R}) &\longrightarrow \mathcal{S}(\mathbf{R})' \\ x &\longmapsto \text{Ш}_{T_s} x \end{aligned}$$

2. ***The target set***:

3. ***The decoder***:

4. ***The accuracy criterion***: “noiseless exact recovery”, *i.e.*

$$\forall x \in \mathcal{X}_{\mathrm{target}}, \quad D\big(M(x)\big) = x$$

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$\mathcal{F}[Ш_{T_s} x] \quad =$$

Two (rather technical) admitted results

Fourier transform of a product *(admitted)*

Let x be such that T_x has a compactly supported distribution.
Then

$$\mathcal{F}[Ш_{T_s} x] = \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x]$$

(The result holds under broader assumptions)

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Fourier transform of a dirac comb *(admitted)*

$$\mathcal{F}[\text{Ш}_{T_s}] = \frac{1}{T_s} \text{Ш}_{\frac{1}{T_s}}$$

Spectrum of a sampled signal (following)

Question: Does the sampling process affect the spectrum?

$$\mathcal{F}[Ш_{T_s} x] \overset{\text{Result 1.}}{=} \mathcal{F}[Ш_{T_s}] * \mathcal{F}[T_x]$$

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where $F_s = \frac{1}{T_s}$ is the sampling frequency

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&\overset{\text{Result 2.}}{=} F_s \text{Ш}_{F_s} * \mathcal{F}[T_x] \\
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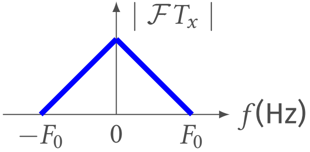
Sampling in the time domain $\Longrightarrow$ Periodic spectrum

Spectrum of band-limited signals

Band-limited signals

A band-limited signal x is a signal whose Fourier transform density has **bounded support**

Let $\mathbf{x}$ be a band-limited signal whose TF has support $[-F_0, F_0]$

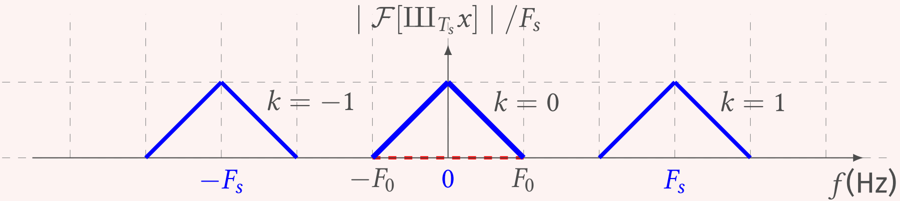


Question : Spectrum of $\text{Ш}_{T_s}\mathbf{x}$?

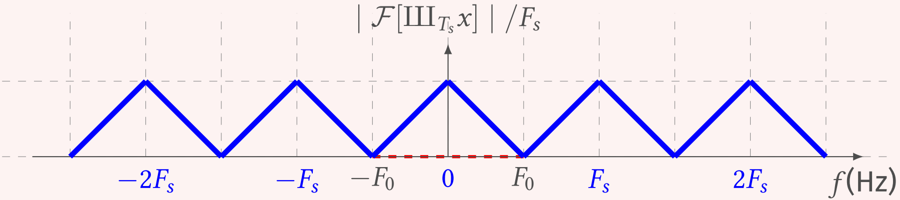
$$\mathcal{F}[\text{Ш}_{T_s}\mathbf{x}] = F_s \sum_{j=-\infty}^{+\infty} \delta_{jF_s} * \mathcal{F}T_x$$

Spectrum of a band-limited signal

Case 1: $F_s > 2F_{\max}$

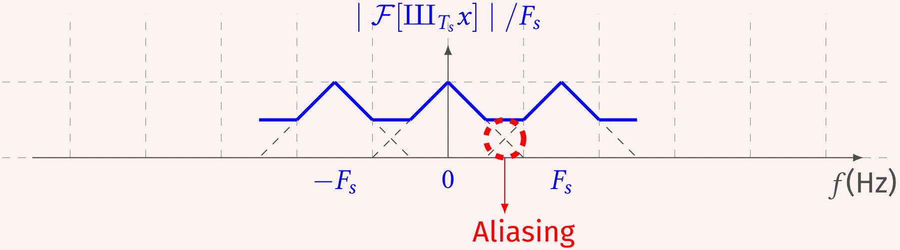


Case 2: $F_s = 2F_{\max}$



Spectrum of a band-limited signal

Case 3: $F_s < 2F_{\max}$



The signal is **deteriorated** $\implies$ **Loss** of information

Shannon-Nyquist sampling theorem

Question: Necessary condition on F_s such that no information is lost?

Nyquist–Shannon sampling theorem

If a function $x(t)$ contains no frequencies higher than $F_{\max}$ hertz, it is **completely determined** by giving its ordinates at a series of points spaced $T_s \geq \frac{1}{2F_{\max}}$ seconds apart.

$\longrightarrow$ **Sufficient condition**

$F_s = 2F_{\max}$ is dubbed "**Nyquist frequency**"

Interpolation formula

Interpolation formula

If $F_s \geq 2F_{\max}$, then T_x is equal to the regular distribution spanned by

$$t \longmapsto \sum_{j \in \mathbf{Z}} x(jT_s) \operatorname{sinc}(F_s(t - jT_s))$$

(Informal) proof: if $F_s > 2F_{\max}$ the spectrum of x can be recovered with an ideal **low pass filter**

☝️ This operation (operating on the spectrum) is common in signal processing and often referred to as “filtering”

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$$\mathcal{F}T_x \overset{\mathcal{S}(\mathbf{R})'}{=} \mathcal{F}[\text{Ш}_{T_s} x] \cdot \frac{1}{F_s} \Pi_{\frac{F_s}{2}}(f)$$

$$\Rightarrow$$

$$T_x \overset{\mathcal{S}(\mathbf{R})'}{=} \text{Ш}_{T_s} x * T_{\operatorname{sinc}(2\pi F_s \cdot)}$$

$$\overset{\mathcal{S}(\mathbf{R})'}{=} \sum_{j \in \mathbf{Z}} x[jT_s] \delta_{jT_s} * T_{\operatorname{sinc}(2\pi F_s \cdot)}$$

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2. ***The target set***: The set of functions which admit a Fourier transform which is compactly supported in $[-F_s, F_s]$

3. ***The decoder***:

$$\begin{aligned} M\colon \qquad \mathbf{R}^{\mathbf{Z}} &\longrightarrow \mathcal{S}' \\ \{y_j\}_{j\in\mathbf{Z}} &\longmapsto \textstyle\sum_{j\in\mathbf{Z}} y_j \operatorname{sinc}(F_s(\cdot - jT_s)) \end{aligned} \quad \text{if} \quad F_s \geq 2F_{\max}$$

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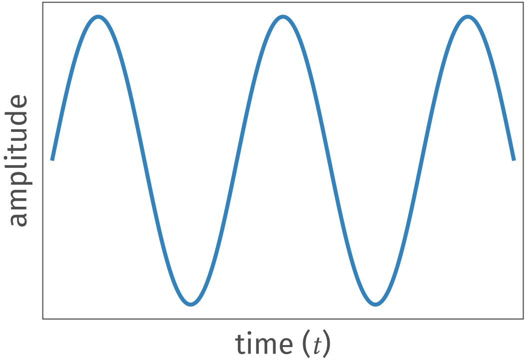
Conclusion

Numerical illustration



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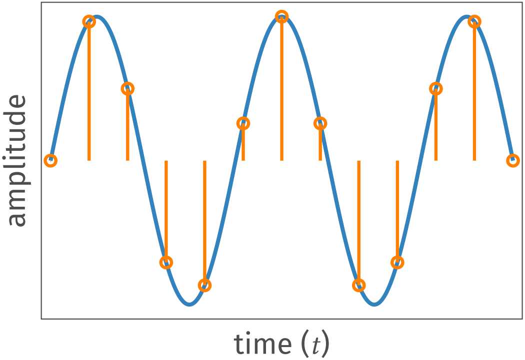
Numerical illustration



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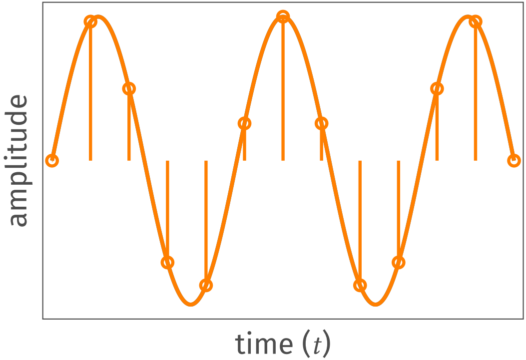

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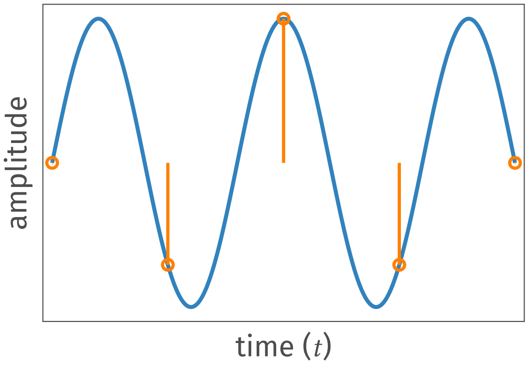

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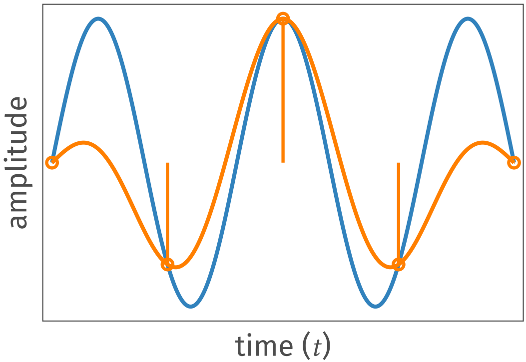

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☝️ **Rappels:** a linear model is characterized by

- ✓ a linear observation operator operator M ✓; *Already done*
- ✓ a vector space as choices for $\mathcal{X}_{\mathtt{target}}$; *By linearity of the Fourier transform*
- ▶ (an orthogonal projection as “best” decoder) *In the finite-dimensional case*

A concluding remark

If we restrict ourself to signals $x \in \mathcal{L}^2(\mathbf{R})$

- $\mathcal{L}^2(\mathbf{R})$ is a Hilbert space equipped with the inner product

$$\langle x_1, x_2 \rangle = \int x_1(t)\overline{x_2}(t)\mathrm{d}t$$

- The family of function

$$\forall j, \quad m_j : t \mapsto \frac{1}{\sqrt{T_s}} \operatorname{sinc}\left(\frac{t - jT_s}{T_s}\right)$$

  is an orthonormal basis of the set of functions with Fourier transform supported in $[-F_s, F_s]$

- For all signals x with Fourier transform supported in $[-F_s, F_s]$,

$$\langle m_j, x \rangle = x(jT_s)$$

- The proposed decoder thus rewrites

$$t \mapsto \sum_{j \in \mathbf{Z}} \langle m_j, x \rangle m_j(t)$$

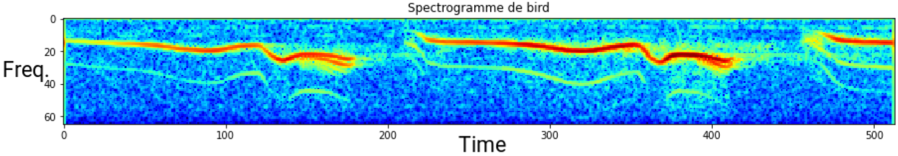
☝️ It is an orthogonal projection!

Bonus: Time-Frequency analysis, wavelets

Time-frequency analysis

- In most domains (audio processing...), the frequency spectrum *itself* evolves in time.

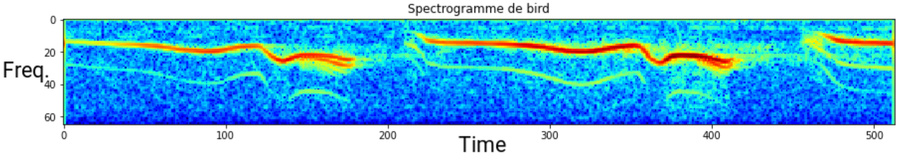
Example: audio *spectrogram*



Time-frequency analysis

- In most domains (audio processing...), the frequency spectrum *itself* evolves in time.
- We want to analyze the frequency component *localized in time*, hence “time-frequency analysis”

Example: audio *spectrogram*



Windowed Fourier transform

- Popular tool in audio: **Windowed Fourier Transform**, also called **Short Time Fourier Transform** (STFT) in its discrete version.

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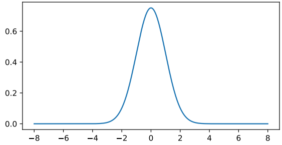
Define a symmetric window $g(t) = g(-t)$ with $\|g\|_{L^2} = 1$



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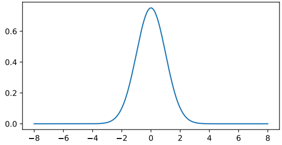
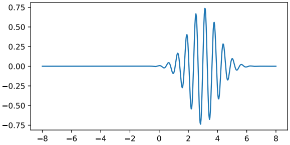


Time-frequency atoms are localized in time and frequency:

$$g_{u,\xi}(t) = \underbrace{e^{2i\pi\xi t}}_{\text{Freq. } \xi} \underbrace{g(t-u)}_{\text{Time } u}$$



Windowed Fourier transform

The **Windowed Fourier Transform** is defined as

$$Sx(u, \xi) = \langle x, g_{u,\xi} \rangle_{L^2} = \int x(t) g(t-u) e^{-2i\pi\xi t} \mathrm{d}t$$

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- Analyze the signal x around time u and frequency f

By Parseval's identity,

$$Sx(u,\xi) = \langle \hat{x}, \hat{g}_{u,\xi} \rangle_{L^2}$$

and by the shift formula

$$\hat{g}_{u,\xi}(f) = \hat{g}(f-\xi) e^{-2i\pi u(f-\xi)}$$

- since g is real symmetric, $\hat{g}$ is real symmetric (ex: for g Gaussian, $\hat{g}$ is Gaussian!)
- Also performs a windowed transform of $\hat{x}$ around ξ (and "inverse frequency" u)

Heisenberg boxes

$$Sx(u,\xi) = \langle x, g_{u,\xi} \rangle_{L^2} = \langle \hat{x}, \hat{g}_{u,\xi} \rangle_{L^2}$$

Heisenberg boxes

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Considering $\mathbb{P} = |g_{u,\xi}|^2$ and $\hat{\mathbb{P}} = |\hat{g}_{u,\xi}|^2$ as **probabilities densities** (since $\|g_{u,\xi}\|_{L^2} = \|g_{u,\xi}\|_{L^2} = 1$):

$$\mathbb{E}(\mathbb{P}) = u \qquad \text{Var}(\mathbb{P}) = \sigma_{\mathbb{P}}^2$$

$$\mathbb{E}(\hat{\mathbb{P}}) = \xi \qquad \text{Var}(\mathbb{P}) = \sigma_{\hat{\mathbb{P}}}^2$$

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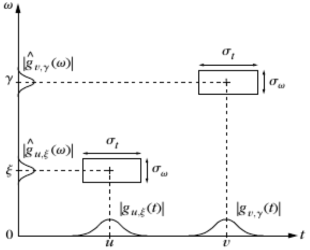
$$\mathbb{E}(\hat{\mathbb{P}}) = \xi \qquad \mathrm{Var}(\mathbb{P}) = \sigma_{\hat{\mathbb{P}}}^2$$

and by Heisenberg's thm:

$$\sigma_{\mathbb{P}}^2 \sigma_{\hat{\mathbb{P}}}^2 \geq 1/4$$

with equality if and only if g is (any) **Gaussian**.



Reconstruction

Thm: inversion formula

If $x \in \mathcal{L}^2$,

$$x(t) = \iint Sx(u,\xi) g(t-u) e^{2i\pi\xi t} \mathrm{d}\xi \mathrm{d}u$$

and

$$\|x\|_{L^2}^2 = \iint |Sx(u,\xi)|^2 \mathrm{d}\xi \mathrm{d}u$$

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Thm: inversion formula

If $x \in \mathcal{L}^2$,

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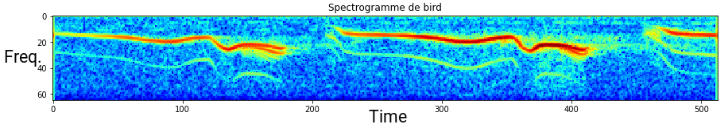
and

$$\|x\|_{L^2}^2 = \iint |Sx(u,\xi)|^2 \mathrm{d}\xi \mathrm{d}u$$

- no information is lost (true ”transform”)
- $Sx \in \mathcal{L}^2(\mathbf{R}^2)$

Example of applications

Spectrogram $|Sx(u,\xi)|^2$ used a lot in audio.



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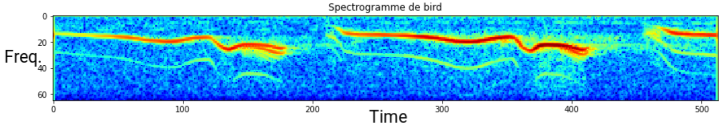


Ex: wav2midi with sparse matrix factorization.

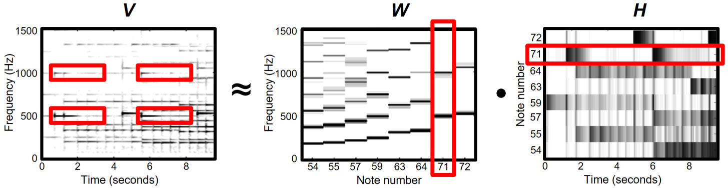
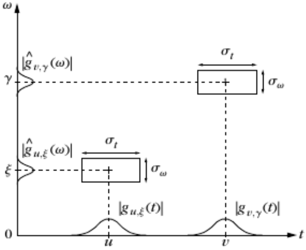


Figure 8.21c-d from [Müller, FMP, Springer 2015]

More precisely, "Non-negative matrix factorization" (NMF), a low-rank factorization of a non-neg. matrix (the spectrogram) into non-neg. components, frequency/activation. See `https://www.audiolabs-erlangen.de/resources/MIR/FMP/C8/C8S3_NMFbasic.html`

Problem with WFT

- The WFT analyzes content with the same window length
- Element can have different durations... (especially low-frequency)
- Human hearing is spread "logarithmically" in frequency



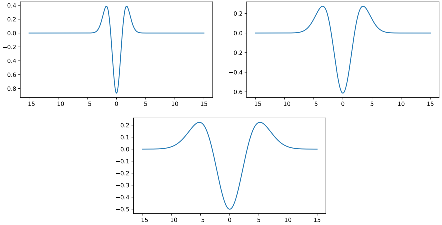
Wavelet transform

A wavelet is a normalized function $\psi \in \mathcal{L}^2(\mathbf{R})$ with a zero average:

$$\int \psi(t)\,dt = 0, \quad \|\psi\| = 1$$

Time-frequency atoms are obtained by *scaling* the wavelet:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$



Wavelet transform:

$$Wx(u,s) = \langle x, \psi_{u,s} \rangle$$

Heisenberg boxes

Assume ψ is analytic for simplicity (i.e. zero negative frequency, see Mallat book).

Heisenberg boxes

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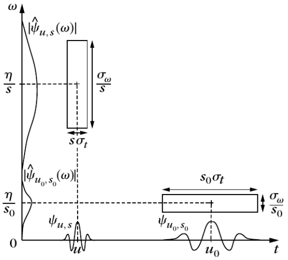
Considering $\mathbb{P} = |\psi_{u,s}|^2$ and $\hat{\mathbb{P}} = |\hat{\psi}_{u,s}|^2$ as **probabilities densities**, and $\mathbb{P}_1, \hat{\mathbb{P}}_1$ the same for reference $s = 1$ (u does not matter). Then:

$$\mathbb{E}(\mathbb{P}) = u \qquad \mathrm{Var}(\mathbb{P}) = s^2 \mathrm{Var}(\mathbb{P}_1)$$

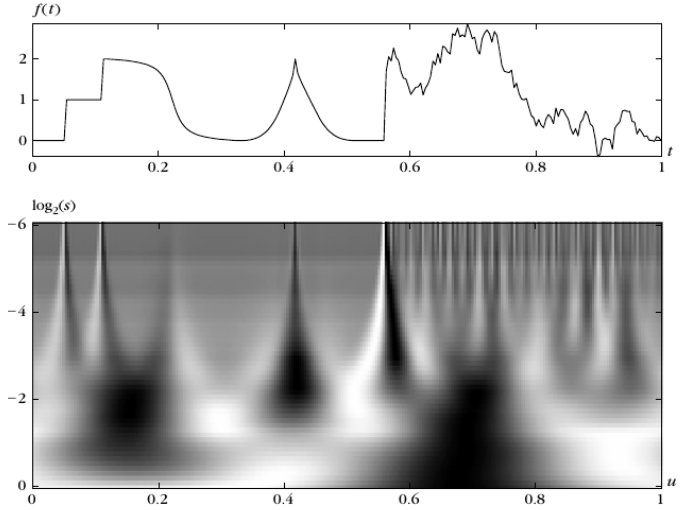
$$\mathbb{E}(\hat{\mathbb{P}}) = \mathbb{E}(\hat{\mathbb{P}}_1)/s \qquad \mathrm{Var}(\hat{\mathbb{P}}) = \mathrm{Var}(\hat{\mathbb{P}}_1)/s^2$$

The "right" multiplicative scale! (for human ears)



Scalogram

$|Wx|^2$ is called a scalogram.



Beyond...

- Reconstruction theorem for Wavelets
- Discrete Wavelets
- Fast computation (FFT, aka "**the most important algorithm of the XXth century**", fast hierarchical wavelets)
- multi-D wavelets (especially 2D!)
- etc.