lecture by Lourakis "Bundle adjustment gone public"

- Bundle Adjustment (BA) is a key ingredient of SaM, almost always used as its last step
 - It is an optimization problem over the 3D structure and viewing parameters (camera pose, intrinsic calibration, & radial distortion parameters), which are simultaneously refined for minimizing reprojection error
 - very large nonlinear least squares problem, typically solved with the Levenberg-Marquardt (LM) algorithm
 - Std LM involves the repetitive solution of linear systems, each with $O(N^3)$ time and $O(N^2)$ storage complexity, resp.
 - Example: for 54 cameras and 5207 3D points, N = 15945. ==> $N^3 = 1e12$
 - Sparse LM is a better solution.
 - Example:
 - M images
 - N features
 - **x_i_j** = measured feature "i" on image "j"
 - **a_j** = vector of parameters for camera "j"
 - **b_i** = vectors of parameters for point "i"
 - Q(aj, bi) = the predicted projection of point i on image j,
 - d(., .) the Euclidean distance between image points
 - vij = 1 iff point i is visible in image j
 - minimize reprojection error over a_j, b_i: min_aj, bi (summation_i=1_to_N(summation_j=1_to_M((v_i_j * d(Q(aj,bi), x_i_j))^2)))
 - ==> total number of parameters is M^* (camera parameters) + N^* (point parameters)
 - let $P = \text{parameter vector of camera then point parameters} = [P_C P_P]$
 - let $X_hat = [(x_hat_1_1)^T x_hat_1_2)^T ... x_hat_1_M)^T x_hat_2_1)^T ... x_hat_N_M)^T]$
 - where $x_hat_i = Q(a_j, b_i)$ is the projection onto camera plane
 - let error **eps** = $[(eps_1_1)^T eps_1_2)^T ... eps_1_M)^T eps_2_1)^T ... eps_N_M)^T$
 - where eps = x i j x hat i j

Bundle Adjustment (BA) becomes

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min (summation_i=1_to_N(summation_j=1_to_M( (eps_i_j)^2 ))) over P
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Jacobian $J = d(X_hat) / d(P)$ which has a block structure because of P being [camera parameters point parameters] $J = [A \mid B]$ where $A = d(X_hat) / d(a)$ and $B = A = d(X_hat) / d(b)$

The LM updating vector delta = $[(delta(a))^T (delta(b))^T$

The normal equations:

The lhs matrix above is sparse due to A and B being sparse:

$$\partial x^{ij} \partial a_k = 0, \forall j != k \text{ and } \partial x^{ij} \partial b k = 0, \forall i != k$$

(example cont.) M images = 3, N features = 4

$$\mathbf{J} = rac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}}$$
 has a block structure $[\,\mathbf{A}|\mathbf{B}\,]$,

Let
$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$$
 and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

The Jacobian J in block form:

$$\frac{\mathbf{a_1}^1}{\mathbf{x_{12}}} \begin{pmatrix} \mathbf{a_2}^1 & \mathbf{a_3}^1 & \mathbf{b_1}^1 & \mathbf{b_2}^1 & \mathbf{b_3}^1 & \mathbf{b_4}^1 \\ \mathbf{x_{12}} & \mathbf{A_{11}} & \mathbf{0} & \mathbf{0} & \mathbf{B_{11}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_{12}} & \mathbf{0} & \mathbf{B_{12}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A_{13}} & \mathbf{B_{13}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A_{13}} & \mathbf{B_{13}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A_{21}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{21}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_{22}} & \mathbf{0} & \mathbf{0} & \mathbf{B_{22}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A_{23}} & \mathbf{0} & \mathbf{B_{23}} & \mathbf{0} & \mathbf{0} \\ \mathbf{A_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{31}} & \mathbf{0} \\ \mathbf{x_{32}} & \mathbf{x_{33}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{32}} & \mathbf{0} \\ \mathbf{x_{33}} & \mathbf{x_{41}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{33}} & \mathbf{0} \\ \mathbf{x_{41}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{41}} \\ \mathbf{x_{42}} & \mathbf{0} & \mathbf{A_{42}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{42}} \\ \mathbf{x_{43}} & \mathbf{0} & \mathbf{0} & \mathbf{A_{43}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B_{43}} \end{pmatrix}$$

$$(1)$$

This is the so-called primary structure of BA

Approximate Hessian in block form:

$$\equiv \left(\begin{array}{cc} \mathbf{U} & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V} \end{array} \right),$$

$$\mathbf{U}_{j} \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^{T} \mathbf{A}_{ij}$$
, for 1 image, summing over all features $\mathbf{V}_{i} \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^{T} \mathbf{B}_{ij}$, for 1 feature, summing over all images $\mathbf{W}_{ij} = \mathbf{A}_{ij}^{T} \mathbf{B}_{ij}$

(example cont.) M images = 3, N features = 4 **Bundle Adjustment Revisited**

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For convenience, we use (18) to show how to solve the bundle adjustment problem. set $\hat{u}_{ij} = \pi(C_j, X_i)$, and we order the parameter x into camera block c and structure block p:

$$x = [c, p]$$
 (20)

it's easily to realize that:

$$J_{ij} = \frac{\partial r_{ij}}{\partial x_k} = \frac{\partial \hat{u}_{ij}}{\partial x_k}, \frac{\partial \hat{u}_{ij}}{\partial c_k} = 0, \forall j \neq k, \frac{\partial \hat{u}_{ij}}{\partial p_k} = 0, \forall i \neq k$$
(21)

Consider now, that we have m=3 cameras and n=4 3D points. Set $A_{ij} = \frac{\partial u_{ij}}{\partial e_i}$, $B_{ij} = \frac{\partial u_{ij}}{\partial p_i}$, we can obtain the Jacobi:

(example cont.) M images = 3, N features = 4

NOTE:
$$eps_a = A^T * eps_b = B^T * eps_b$$

• The augmented normal equations $(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\delta_{\mathbf{p}} = \mathbf{J}^T\epsilon$ take the form

(3)
$$\begin{pmatrix} \mathbf{U}^* & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{a}} \\ \delta_{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{a}} \\ \epsilon_{\mathbf{b}} \end{pmatrix}$$

Performing block Gaussian elimination in the lhs matrix, δ_a is determined with Cholesky from V*'s Schur complement:

$$(\mathbf{U}^* - \mathbf{W} \mathbf{V}^{*-1} \mathbf{W}^T) \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \mathbf{V}^{*-1} \epsilon_{\mathbf{b}}$$

note (V*) is invertible and only the block diagonals are populated, so each V_i is inverted.

$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

separate delta_b: 0*delta_a + (V*)* delta_b = eps_b ==> delta_b = eps_b * $((V*)^{-1})$ solving for delta_a (typically M images << N features) after substitute delta_b: $((U*) - W(((V*)^{-1})W^{T})*$ delta a + W * delta b = eps_a

$$((U^*) - W(((V^*)^*-1)W^*T) * delta_a = eps_a - W((V^*)^*-1)eps_b$$

NOTE: (U^*) - $W(((V^*)^*-1)W^*T)$ is called the reduced camera matrix (because delta_a is camera parameters)

RCM (reduced camera matrix) is sparse because not all features appear in all cameras. this is known as **secondary structure**.

For very large datasets, RCM tends to be in one of two classes:

- (1) visual mapping: extended areas are traversed, limited image overlap (sparse RCM)
- (2) centered-object: a large number of overlapping images taken in a small area (dense RCM)

Solving for delta_a in the equation containing RCM. several ways:

- (1) Store as dense, decompose with **ordinary linear algebra** ○[M. Lourakis, A. Argyros: SBA: A Software Package For Generic Sparse Bundle Adjustment. ACM Trans. Math. Softw. 36(1): (2009) C. Engels, H. Stewenius, D. Nister: Bundle Adjustment Rules. Photogrammetric Computer Vision (PCV), 2006.
- (2) Store as sparse, factorize with sparse direct solvers K. Konolige: Sparse Sparse Bundle Adjustment. BMVC 2010: 1-11
- (3) Store as sparse, use **conjugate gradient methods** memory efficient, iterative, precoditioners necessary! S. Agarwal, N. Snavely, S.M. Seitz, R. Szeliski: Bundle Adjustment in the Large. ECCV (2) 2010: 29-42 M. Byrod, K. Astrom: Conjugate Gradient Bundle Adjustment. ECCV (2) 2010: 114-127
- (4) Avoid storing altogether C. Wu, S. Agarwal, B. Curless, S.M. Seitz: Multicore Bundle Adjustment. CVPR 2011: 30 57-3064 M. Lourakis: Sparse Non-linear Least Squares Optimization for Geometric Vision. ECCV (2) 2010: 43-56

Engels, Stewenius, Nister 2006, "Bundle Adjustment Rules"

<u>m</u>images (= video frames from same calibrated camera)

? features

each feature x has M dimensions

n iterations of bundle adjustment over the last m video frames

Engels "RCM" is formed from a jacobian which places point parameters before camera parameters, so is different than that in the Lourakis notes.

$$J_f = \left[\begin{array}{cc} J_P & J_C \end{array} \right], \tag{15}$$

$$H = \begin{bmatrix} J_P^\top J_P & J_P^\top J_C \\ J_C^\top J_P & J_C^\top J_C \end{bmatrix}, \tag{16}$$

$$\begin{bmatrix} H_{PP} & H_{PC} \\ H_{PC}^{\top} & H_{CC} \end{bmatrix} \begin{bmatrix} dP \\ dC \end{bmatrix} = \begin{bmatrix} b_P \\ b_C \end{bmatrix}, \tag{17}$$

where we have defined $H_{PP} = J_P^\top J_P$, $H_{PC} = J_P^\top J_C$, $H_{CC} = J_C^\top J_C$, $b_P = -J_P^\top f$, $b_C = -J_C^\top f$ to simplify the notation, and dP and dC represent the update of the point parameters and the camera parameters, respectively. Note that the matrices H_{PP} and H_{CC} are block-diagonal, where the blocks correspond to

of as multiplying by

$$\begin{bmatrix} I & 0 \\ -H_{PC}^{\top} & I \end{bmatrix}$$
 (20)

from the left on both sides, resulting in the smaller equation system (from the lower part)

$$\underbrace{(H_{CC} - H_{PC}^{\top} H_{PP}^{-1} H_{PC})}_{A} dC = \underbrace{b_{C} - H_{PC}^{\top} H_{PP}^{-1} b_{P}}_{B}$$
 (21)

for the camera parameter update dC. For very large systems,

We use straightforward Cholesky factorization.

Engels, Stewenius, Nister 2006, "Bundle Adjustment Rules"

M images = 3, j N features = 4, i **sparse:** J_P is [2*n*m X 3] J_C is [2n*m X 9]

their "RCM" is formed from a jacobian which places point parameters before camera parameters, so is different than that in the Lourakis notes.

Lourakis Let $\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

$$\mathbf{J} = rac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \begin{bmatrix} rac{\partial \hat{\mathbf{X}}}{\partial \mathbf{a}} & rac{\partial \hat{\mathbf{X}}}{\partial \mathbf{b}} \end{bmatrix} = \begin{bmatrix} J_C & J_P \end{bmatrix}$$

. The Jacobian J in block form:

$$\frac{\mathbf{x}_{11}}{\mathbf{x}_{12}} \begin{pmatrix} \mathbf{a}_{1}^{T} & \mathbf{a}_{2}^{T} & \mathbf{a}_{3}^{T} & \mathbf{b}_{1}^{T} & \mathbf{b}_{2}^{T} & \mathbf{b}_{3}^{T} & \mathbf{b}_{4}^{T} \\ \mathbf{x}_{12} & \mathbf{x}_{13} & \mathbf{x}_{21} & \mathbf{x}_{13} & \mathbf{x}_{21} & \mathbf{x}_{22} & \mathbf{x}_{23} \\ \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \frac{\mathbf{x}_{23}}{\mathbf{x}_{31}} & \mathbf{x}_{22} & \mathbf{x}_{23} & \mathbf{x}_{31} & \mathbf{x}_{22} & \mathbf{x}_{23} & \mathbf{x}_{31} & \mathbf{x}_{32} & \mathbf{x}_{32} & \mathbf{x}_{33} & \mathbf{x}_{41} & \mathbf{x}_{42} & \mathbf{x}_{41} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{41} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}_{42} & \mathbf{x}_{43} & \mathbf{x}_{44} & \mathbf{x}$$

Engels Let
$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$$
 and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

$$J_f = \begin{bmatrix} J_P & J_C \end{bmatrix}$$
,

The Jacobian J in block form:

$$\frac{\partial \hat{X}}{\partial P} = \begin{pmatrix} \mathbf{b_1}^T & \mathbf{b_2}^T & \mathbf{b_3}^T & \mathbf{b_4}^T & \mathbf{a_1}^T & \mathbf{a_2}^T & \mathbf{a_3}^T \\ \mathbf{B}_{11} & 0 & 0 & 0 & (\mathbf{A}_{11} & 0 & 0) \\ \mathbf{B}_{12} & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{12} & 0 & 1 \\ \mathbf{B}_{13} & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{13} & 1 \\ 0 & \mathbf{B}_{21} & 0 & 0 & \mathbf{A}_{21} & 0 & 0 \\ \mathbf{0} & \mathbf{B}_{22} & 0 & 0 & 0 & \mathbf{A}_{22} & 0 \\ 0 & \mathbf{B}_{22} & 0 & 0 & 0 & \mathbf{A}_{22} & 0 \\ 0 & \mathbf{B}_{23} & 0 & 0 & 0 & \mathbf{A}_{22} & 0 \\ \mathbf{0} & \mathbf{B}_{23} & 0 & 0 & 0 & \mathbf{A}_{23} \\ \mathbf{0} & 0 & \mathbf{B}_{31} & 0 & \mathbf{A}_{31} & 0 & 0 \\ \mathbf{x}_{33} & \mathbf{x}_{31} & 0 & 0 & \mathbf{B}_{32} & 0 & 0 & \mathbf{A}_{32} & 0 \\ \mathbf{x}_{33} & 0 & 0 & \mathbf{B}_{33} & 0 & 0 & 0 & \mathbf{A}_{33} \\ \mathbf{x}_{41} & \mathbf{x}_{42} & 0 & 0 & \mathbf{B}_{41} & \mathbf{A}_{41} & 0 & 0 \\ \mathbf{x}_{42} & \mathbf{x}_{43} & 0 & 0 & \mathbf{B}_{42} & 0 & \mathbf{A}_{42} & 0 \\ \mathbf{0} & 0 & 0 & \mathbf{B}_{43} & 0 & 0 & \mathbf{A}_{43} \end{pmatrix}$$

J is [2nm X (3n + 9m)]

J^T * J is [(3n+9m) X (3n+9m)]

Engels, Stewenius, Nister 2006, "Bundle Adjustment Rules"

M images = 3, j

N features = 4, i

their "RCM" is formed from a jacobian which places point parameters before camera parameters, so is different than that in the Lourakis notes.

 $\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$

Lourakis Let
$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$$
 and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$ C P
$$\mathbf{J} = \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \begin{bmatrix} \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{a}} & \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{b}} \end{bmatrix} = \begin{bmatrix} Jc & JP \end{bmatrix}$$

$$\mathbf{J}^T \mathbf{J} = \begin{bmatrix} J^T * J \text{ is } [(3*n + 9*m)] \times (3*n + 9*m)]}$$

Engels Let
$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_{j}}$$
 and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_{i}}$

$$\mathbf{J} = \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \begin{bmatrix} \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{b}} & \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{a}} \end{bmatrix} = \begin{bmatrix} J_{P} & J_{C} \end{bmatrix},$$

$$\mathbf{J}^{T} \mathbf{J} = \begin{bmatrix} \mathbf{J}^{T} & \mathbf{J} & \mathbf{J}^{T} & \mathbf{J} & \mathbf{J}^{T} & \mathbf{J}$$

where
$$\mathbf{U}_{j} \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^{T} \mathbf{A}_{ij}$$
, for 1 image, sum over features $\mathbf{V}_{i} \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^{T} \mathbf{B}_{ij}$, for 1 feature, sum over images $\mathbf{X}_{i} \equiv \mathbf{X}_{j}^{3} \equiv \mathbf{X}_{j}^{3} = \mathbf{X}_$

3X3
$$\equiv \sum_{j=1}^{3} \left(\frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_{i}} \right)^{\mathsf{T}} \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_{i}}$$

Engels, Stewenius, Nister 2006, "Bundle Adjustment Rules"

Note that
$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

U* is [9*m X 9*m]

Lourakis

The augmented normal equations $(\mathbf{J}^T\mathbf{J} + \mu \mathbf{I})\delta_{\mathbf{p}} = \mathbf{J}^T\epsilon$ take the form

(3)
$$\begin{pmatrix} \mathbf{U}^* & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{a}} \\ \delta_{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{a}} \\ \epsilon_{\mathbf{b}} \end{pmatrix}$$

$$\left[\begin{array}{cc} U - W V^{-1} W^T & 0 \end{array}\right] \left[\begin{array}{c} \delta_{\mathbf{a}} \\ \delta_{\mathbf{b}} \end{array}\right] = \left[\begin{array}{cc} I & -W V^{-1} \end{array}\right] \left[\begin{array}{c} \epsilon_{\mathbf{a}} \\ \epsilon_{\mathbf{b}} \end{array}\right]$$

(solve delta a first because typically m images << n features) determine δ_a with Cholesky (or other method)

$$(\mathbf{U}^* - \mathbf{W} \mathbf{V}^{*-1} \mathbf{W}^T) \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \mathbf{V}^{*-1} \epsilon_{\mathbf{b}}$$

 $\delta_{\rm b}$ can be computed by back substitution into

$$\mathbf{V}^* \ \delta_{\mathbf{b}} = \epsilon_{\mathbf{b}} - \mathbf{W}^T \ \delta_{\mathbf{a}}$$

$$\delta_{\mathbf{b}} = \mathbf{V}^{*-1} \; \boldsymbol{\epsilon}_{\mathbf{b}} \; - \; \mathbf{V}^{*-1} \; \mathbf{w}^{T} \; \delta_{\mathbf{a}}$$

Engels

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$

$$H_{PP} = J_P^\top J_P$$

$$\begin{bmatrix} H_{PP} & H_{PC} \\ H_{PC}^\top & H_{CC} \end{bmatrix} \begin{bmatrix} dP \\ dC \end{bmatrix} = \begin{bmatrix} b_P \\ b_C \end{bmatrix}, \quad H_{PC} = J_P^\top J_C,$$

$$b_P = -J_P^\top f,$$

$$b_P = -J_P^\top f,$$

$$\underbrace{(\mathbf{U}^* - \mathbf{W} \ \mathbf{V}^{*-1} \ \mathbf{W}^T)}_{A} \ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \ \mathbf{V}^{*-1} \ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^{\perp} H_{PP}^{-1} H_{PC})}_{A} dC = \underbrace{b_C - H_{PC}^{\perp} H_{PP}^{-1} b_P}_{B}$$

 $\delta_{\mathbf{b}}$ can be computed by back substitution into

$$\mathbf{V}^* \ \delta_{\mathbf{b}} = \epsilon_{\mathbf{b}} - \mathbf{W}^T \ \delta_{\mathbf{a}}$$

$$H_{PP} \ dP = b_P - H_{PC} \ dC$$

$$dP = H_{PP}^{-1} b_P - H_{PP}^{-1} H_{PC} dC.$$

Qu

$$egin{pmatrix} m{B} & m{E} \ m{E}^T & m{C} \end{pmatrix} \, egin{pmatrix} m{p_c} \ m{p_p} \end{pmatrix} = - \, egin{pmatrix} m{g_c} \ m{g_p} \end{pmatrix}$$

see eqn (3.70) too

see eqn (3.70) to
$$-g^k = -J^{kT}F^k$$
 where k is a feature block summed over all images?

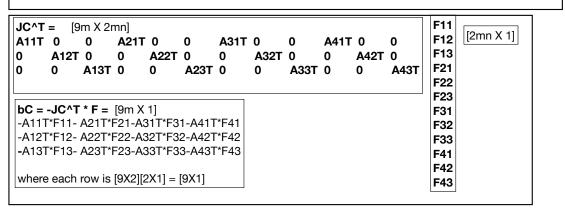
$$({m B} - {m E} {m C}^{-1} {m E}^T) {m p}_c = -{m g}_c + {m E} {m C}^{-1} {m g}_{m p}$$

$$\boldsymbol{p}_{\boldsymbol{p}} = \boldsymbol{C}^{-1}(-\boldsymbol{g}_{\boldsymbol{p}} - \boldsymbol{E}^T \boldsymbol{p}_{\boldsymbol{c}})$$

Engels, et al 2006
$$H_{PP} = J_P^{\mathsf{T}} J_P \equiv \mathbf{V}^*$$
 $H_{PC} = J_P^{\mathsf{T}} J_C, \equiv \mathbf{W}^{\mathsf{T}}$ $\mathbf{W}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} + \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} + \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} + \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} + \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} = \mathbf{J}^{\mathsf{T}} + \mathbf{J}^{\mathsf{T}} = \mathbf$

-B41T*F41-B42T*F42-B43T*F43

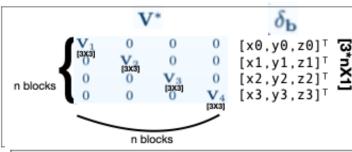
where each row is [3X2]*[2X1] = [3X1]

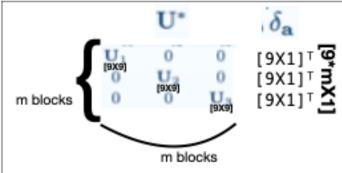


Note that
$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

F42 F43 $\mathbf{U}_{j} \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^{T} \mathbf{A}_{ij}$, [9X9], for 1 image, sum over features $\mathbf{V}_{i} \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^{T} \mathbf{B}_{ij}$, [3X3], for 1 feature, sum over images [9X3]

$$\begin{array}{c} \begin{bmatrix} \mathbf{V}^{\star} \text{ is } [\mathbf{3}^{\star}\mathbf{n} \ \mathbf{X} \ \mathbf{3}^{\star}\mathbf{n}] \end{bmatrix} & \begin{pmatrix} \mathbf{V}^{*} & \mathbf{W}^{T} \\ \mathbf{W} & \mathbf{U}^{*} \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix} \\ \begin{bmatrix} \mathbf{U}^{\star} \text{ is } [\mathbf{9}^{\star}\mathbf{m} \ \mathbf{X} \ \mathbf{9}^{\star}\mathbf{m}] \end{bmatrix} & \begin{pmatrix} \mathbf{V}^{*} & \mathbf{W}^{T} \\ \mathbf{V}^{*} & \mathbf{V}^{*} \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$





Engels, et al 2006
$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$

 $H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$
M images = 3, j $H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$
N features = 4, i $b_P = -J_P^\top f_C \equiv \mathbf{E}_{\mathbf{b}}$

M images = 3, j

N features = 4, i

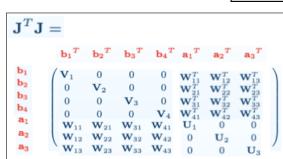
$$H_{PC} = J_P^{\top} J_C, \equiv \mathbf{W}^T$$
 $H_{CC} = J_C^{\top} J_C, \equiv \mathbf{U}^*$
 $b_P = -J_P^{\top} f, \equiv \epsilon_{\mathbf{b}}$
 $b_C = -J_C^{\top} f \equiv \epsilon_{\mathbf{a}}$

$$\underbrace{(\mathbf{U}^* - \mathbf{W} \ \mathbf{V}^{*-1} \ \mathbf{W}^T)}_{A} \ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \ \mathbf{V}^{*-1} \ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^{\top} H_{PP}^{-1} H_{PC})}_{A} dC = \underbrace{b_C - H_{PC}^{\top} H_{PP}^{-1} b_P}_{B}$$

[9mX3n] the V's are inverses W11*V1 W21*V2 W31*V3 W41*V4 W12*V1 W22*V2 W32*V3 W42*V4 W13*V1 W23*V2 W33*V3 W43*V4 where each block is [9X3]*[3X3] = [9X3]

$$\mathbf{A}_{ij} = rac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} ext{ and } \mathbf{B}_{ij} = rac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$$



W*V^-1*W^T= [9mX9m] the V's are inverses W11*V1*W11T + W21*V2*W21T + W31*V3*W31T + W41*V4*W41T W12*V1*W12T + W22*V2*W22T + W32*V3*W32T + W42*V4*W42T W13*V1*W13T + W23*V2*W23T + W33*V3*W33T + W43*V4*W43T where each block is [9X3]*[3X9] = [9X9]

```
U - W*V^-1*W^T= [9mX9m]
                     the V's are inverses
U2 0 | | W12*V1*W12T + W22*V2*W22T + W32*V3*W32T + W42*V4*W42T
  0 U3 | | W13*V1*W13T + W23*V2*W23T + W33*V3*W33T + W43*V4*W43T |
```

bC - (W*V^-1)*bP

-A11T*F11- A21T*F21-A31T*F31-A41T*F41 -A12T*F12- A22T*F22-A32T*F32-A42T*F42 -A13T*F13- A23T*F23-A33T*F33-A43T*F43

where each row is [9X2][2X1] = [9X1]

 $bC = JC^T * F = [9m X 1]$

W*V^-1= [9mX3n] the V's are inverses W11*V1 W21*V2 W31*V3 W41*V4

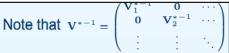
W12*V1 W22*V2 W32*V3 W42*V4 W13*V1 W23*V2 W33*V3 W43*V4

where each block is [9X3]*[3X3] = [9X3]

 $bP = JP^T * F [3n X 1]$

-B11T*F11-B12T*F12-B13T*F13 -B21T*F21-B22T*F22-B23T*F23 -B31T*F31-B32T*F32-B33T*F33 -B41T*F41-B42T*F42-B43T*F43

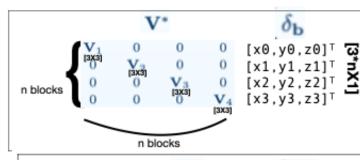
where each row is [3X2]*[2X1] = [3X1]

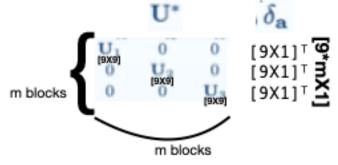


 $\mathbf{U}_{j} \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^{T} \mathbf{A}_{ij}$, [9X9], for 1 image, sum over features

 $\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij}$, [3X3], for 1 feature, sum over images $\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ [9X3]

V* is [3*n X 3*n] W is [9*m X 3*n] U* is [9*m X 9*m] $\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$





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$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$

 $H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$
M images = 3, j $H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$
N features = 4, i $b_P = -J_P^\top f, \equiv \boldsymbol{\epsilon}_{\mathbf{b}}$
 $b_C = -J_C^\top f \equiv \boldsymbol{\epsilon}_{\mathbf{a}}$

$$\underbrace{(\mathbf{U}^* - \mathbf{W} \ \mathbf{V}^{*-1} \ \mathbf{W}^T)}_{A} \ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \ \mathbf{V}^{*-1} \ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^{\top} H_{PP}^{-1} H_{PC})}_{A} dC = \underbrace{b_C - H_{PC}^{\top} H_{PP}^{-1} b_P}_{B}$$

$$dP = H_{PP}^{-1}b_P - H_{PP}^{-1}H_{PC}dC.$$

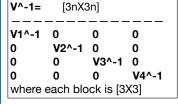
$$\mathbf{A}_{ij} = rac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$$
 and $\mathbf{B}_{ij} = rac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

Note that
$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{array}{l} \mathbf{U}_{j} \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^{T} \mathbf{A}_{ij}, \\ \mathbf{V}_{i} \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^{T} \mathbf{B}_{ij}, \\ \mathbf{W}_{ij} = \mathbf{A}_{ij}^{T} \mathbf{B}_{ij} \end{array} \begin{array}{l} \textbf{[9X9], for 1 image, sum over features} \\ \textbf{[3X3], for 1 feature, sum over images} \\ \textbf{[9X3]} \end{array}$$

$$\begin{array}{l} \mathbf{V}^{\star} \text{ is } \textbf{[3*n X 3*n]} \\ \textbf{W is } \textbf{[9*m X 3*n]} \\ \textbf{U* is } \textbf{[9*m X 9*m]} \end{array} \begin{pmatrix} \mathbf{V}^{\star} & \mathbf{W}^{T} \\ \mathbf{W} & \mathbf{U}^{\star} \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$

$$tP = hPPI^-1 * bPI is V^-1*bPI // [3X3][3X1] = [3X1]$$



bP = JP^T * F [3n X 1]

-B11T*F11-B12T*F12-B13T*F13
-B21T*F21-B22T*F22-B23T*F23
-B31T*F31-B32T*F32-B33T*F33
-B41T*F41-B42T*F42-B43T*F43

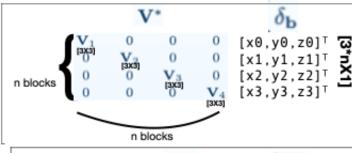
where each row is [3X2]*[2X1]=[3X1]

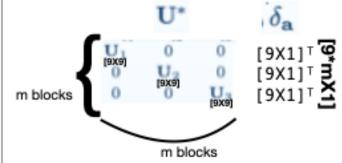
 $H_{PP}^{-1}b_{P}$ [3n X 1]

-(V1^-1)*(B11T*F11-B12T*F12-B13T*F13) -(V2^-1)*(B21T*F21-B22T*F22-B23T*F23) -(V3^-1)*(B31T*F31-B32T*F32-B33T*F33) -(V4^-1)*(B41T*F41-B42T*F42-B43T*F43)

where each row is [3X1]

tPC = hPCIJT * invHPPI is W*V^-1// [9X3][3X3]=[9X3]
$$\frac{H_{PC}^{T}H_{PP}^{-1}}{H_{PP}^{-1}} [9mX3n] \\ \hline \frac{W*V^-1}{W11*V1^-1} [9mX3n] \\ \hline W11*V1^-1 W21*V2^-1 W31*V3^-1 W41*V4^-1 \\ W12*V1^-1 W22*V2^-1 W32*V3^-1 W42*V4^-1 \\ W13*V1^-1 W23*V2^-1 W33*V3^-1 W43*V4^-1 \\ where each block is [9X3]*[3X3] = [9X3] \\ \hline$$





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$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$

 $H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$
M images = 3, j $H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$
N features = 4, i $b_P = -J_P^\top f, \equiv \epsilon_{\mathbf{b}}$
 $b_C = -J_C^\top f \equiv \epsilon_{\mathbf{a}}$

$$\underbrace{(\mathbf{U}^* - \mathbf{W} \mathbf{V}^{*-1} \mathbf{W}^T)}_{A} \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \mathbf{V}^{*-1} \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^{\top} H_{PP}^{-1} H_{PC})}_{A} dC = \underbrace{b_C - H_{PC}^{\top} H_{PP}^{-1} b_P}_{B}$$

- 1 Initialize λ .
- 2 Compute cost function at initial camera and point configuration.
- 3 Clear the left hand side matrix A and right hand side vector B.
- 4 For each track p (p is feature i of N)

$$\mathbf{V}^*$$

Clear a variable H_{pp} to represent block p of H_{PP} (in our case a symmetric 3×3 matrix) and a variable b_p to represent part p of b_P (in our case a 3-vector).

 $\epsilon_{\mathbf{b}}$

(Compute derivatives) For each camera c on track p (c is image j of M)

Compute error vector f of reprojection in camera c of point p and its Jacobians J_p and J_c with respect to the

point parameters (in our case a 2×3 matrix) and the camera parameters (in our case a 2×6 matrix), respectively.

$$\mathbf{B}^T \mathbf{B}$$

Add $J_p^{\top} J_p$ to the upper triangular part of H_{pp} . Subtract $J_n^{\top} f$ from b_p .

 \mathbf{B}^T

$$\mathbf{A}_{ij} = rac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} ext{ and } \mathbf{B}_{ij} = rac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$$

Note that
$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

 $\mathbf{U}_{i} \equiv \sum_{i=1}^{4} \mathbf{A}_{ii}^{T} \mathbf{A}_{ij}$ $\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij}$, $\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$

[9X9], for 1 image, sum over features [3X3], for 1 feature, sum over images [9X3]

V* is [3*n X 3*n] W is [9*m X 3*n] U* is [9*m X 9*m]

 $\left(egin{array}{cc} \mathbf{V}^* & \mathbf{W}^T \ \mathbf{W} & \mathbf{U}^* \end{array}
ight) \left(egin{array}{cc} \delta_{\mathbf{b}} \ \delta_{-} \end{array}
ight) = \left(egin{array}{cc} \epsilon_{\mathbf{b}} \ \epsilon_{\mathbf{a}} \end{array}
ight)$

If camera c is free $\mathbf{A}^T \mathbf{A}$

Add $J_c^{\top} J_c$ (optionally with an augmented diagonal) to upper triangular part of block (c, c) of left hand side (matrix A) (in our case a 6×6 matrix).

Compute block (p, c) of H_{PC} as $H_{pc} = J_p^{\top} J_c$ (in our case a 3 × 6 matrix) and store it until track is done. Subtract $J_c^{\top} f$ from part c of right hand side vector B (related to b_C).

// end c loop

Augment diagonal of H_{pp} , which is now accumulated and ready. Invert H_{pp} , taking advantage of the fact that it is a symmetric matrix.

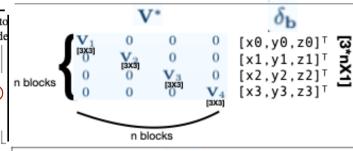
Compute $H_{pp}^{-1}b_p$ and store it in a variable t_p .

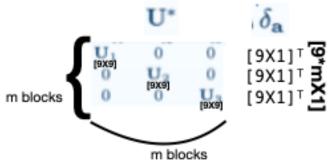
(Outer product of track) For each free camera c on track p

Subtract $H_{pc}^{\top}t_p = H_{pc}^{\top}H_{pp}^{-1}b_p$ from part c of right hand Compute the matrix $H_{pc}^{\top}H_{pp}^{-1}$ and store it in a variable T_{pc} For each free camera $c2 \ge c$ on track p

Subtract $T_{pc}H_{pc2}=H_{pc}^{\top}H_{pp}^{-1}H_{pc2}$ from block (c,c2) of left hand side matrix A

} // end p loop





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Note that
$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$H_{PP} = J_P^\top J_P \quad \equiv \quad \mathbf{V}^* \\ H_{PC} = J_P^\top J_C, \quad \equiv \quad \mathbf{W}^T \\ H_{CC} = J_C^\top J_C, \quad \equiv \quad \mathbf{U}^* \\ b_P = -J_P^\top f, \quad \equiv \quad \mathbf{E}_b \\ b_C = -J_C^\top f \quad \equiv \quad \mathbf{E}_a$$

$$\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij}, \quad \text{for 1 feature, sum over images} \\ \mathbf{U}_j \equiv \sum_{i=1}^4 \mathbf{A}_{ij}^T \mathbf{A}_{ij}, \quad \text{for 1 image, sum over features}$$

$$\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij}$$
, for 1 feature, sum over images

$$\mathbf{U}_{j} \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^{T} \mathbf{A}_{ij}$$
, for 1 image, sum over features

δ is dP is [3*n_features X 1] бы bp is [3*n features X 1] δ_a is dC is [9*m images X 1] ϵ_n is b_c is [9*m_images X 1]

$$(\mathbf{U}^* - \mathbf{W} \mathbf{V}^{*-1} \mathbf{W}^T) \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W} \mathbf{V}^{*-1} \epsilon_{\mathbf{b}}$$
$$(H_{CC} - H_{PC}^{\top} H_{PP}^{-1} H_{PC}) dC = \underbrace{b_C - H_{PC}^{\top} H_{PP}^{-1} b_P}_{B}$$

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- 5 (Optional) Fix gauge by freezing appropriate coordinates and thereby reducing the linear system with a few dimensions.
- 6 (Linear Solving) Cholesky factor the left hand side matrix B and solve for dC. Add frozen coordinates back in.

```
7 (Back-substitution) For each track p
     Start with point update for this track dp = t_p.
     For each camera c on track p
       Subtract T_{pc}^{\top}dc from dp (where dc is the update for camera
       c).
     Compute updated point.
```

- 8 Compute the cost function for the updated camera and point configuration.
- 9 If cost function has improved, accept the update step, decrease λ and go to Step 3 (unless converged, in which case quit).
- 10 Otherwise, increase λ and go to Step 3 (unless exceeded the maximum number of iterations, in which case quit).

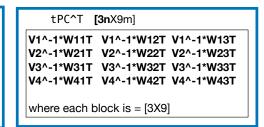
$$\mathbf{V}^* \ \boldsymbol{\delta_b} = \boldsymbol{\epsilon_b} - \mathbf{W}^T \ \boldsymbol{\delta_a}$$

$$dP = H_{PP}^{-1} b_P - H_{PP}^{-1} H_{PC} dC.$$

$$= tP - tPC^T * dC$$

$$[3nX1] [3nX9m] [9mX1]$$

```
tP = hPPI^-1 * bPI is V^-1*bPI [3nX1]
-(V1^-1)*(B11T*F11-B12T*F12-B13T*F13)
-(V2^-1)*(B21T*F21-B22T*F22-B23T*F23)
-(V3^-1)*(B31T*F31-B32T*F32-B33T*F33)
-(V4^-1)*(B41T*F41-B42T*F42-B43T*F43)
where each row is [3X1]
```



```
dC [9mX1]
each block is = [9X1]
```

Bill Triggs, Philip Mclauchlan, Richard Hartley, Andrew Fitzgibbon.

Bundle Adjustment - A Modern Synthesis.

International Workshop on Vision Algorithms,

Sep 2000, Corfu, Greece. pp.298-372,

10.1007/3-540-44480-7 21 inria-00548290 see Appendix B, and page 23...

6.1 The Schur Complement and the Reduced Bundle System

Schur complement: Consider the following block triangular matrix factorization:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C\,A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \overline{D} \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix} \,, \qquad \overline{D} \equiv \, D - C\,A^{-1}B \qquad (16)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & \overline{D}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -CA^{-1} & 1 \end{pmatrix} = \begin{pmatrix} A^{-1}+A^{-1}B\overline{D}^{-1}CA^{-1} & -A^{-1}B\overline{D}^{-1} \\ -\overline{D}^{-1}CA^{-1} & \overline{D}^{-1} \end{pmatrix}$$
 (17)

Here \underline{A} must be square and invertible, and for (17), the whole matrix must also be square and invertible. \overline{D} is called the **Schur complement** of \underline{A} in \underline{M} . If both \underline{A} and \underline{D} are invertible, complementing on \underline{D} rather than \underline{A} gives \overline{D} , the Schur complement, is HPP is \underline{V}^* .

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)^{-1} \; = \; \left(\begin{smallmatrix} \overline{A}^{-1} & -\overline{A}^{-1}B \, D^{-1} \\ -D \, C \, \overline{A}^{-1} & D^{-1} + D^{-1} \, C \, \overline{A}^{-1}B \, D^{-1} \end{smallmatrix} \right), \qquad \overline{A} = A - B \boxed{D^{-1}} C$$

Equating upper left blocks gives the Woodbury formula:

$$(A \pm B D^{-1}C)^{-1} = A^{-1} \mp A^{-1}B (D \pm C A^{-1}B)^{-1} C A^{-1}$$
(18)

This is the usual method of updating the inverse of a nonsingular matrix A after an update (especially a low rank one) $A \rightarrow A \pm B D^{-1}C$. (See §8.1).



Bill **Triggs**, Philip Mclauchlan, Richard Hartley, Andrew Fitzgibbon.

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see Appendix B, and page 23...

$$\begin{aligned} \mathsf{L} &= \mathbf{profile_cholesky_decomp}(\mathsf{A}) \\ & \textbf{for } i = 1 \textbf{ to } n \textbf{ do} \\ & \textbf{for } j = \mathsf{first}(i) \textbf{ to } i \textbf{ do} \\ & a &= \mathsf{A}_{ij} - \sum_{k=\max(\mathsf{first}(i),\mathsf{first}(j))}^{j-1} \mathsf{L}_{ik} \mathsf{L}_{jk} \\ & \mathsf{L}_{ij} = (j < i) \ ? \ a / \mathsf{L}_{jj} \ : \ \sqrt{a} \end{aligned}$$

$$\begin{split} \mathbf{x} &= \mathbf{profile_cholesky_forward_subs}(\mathsf{A}, \mathsf{b}) \\ \mathbf{for} \ i &= \mathrm{first}(\mathsf{b}) \ \mathbf{to} \ n \ \mathbf{do} \\ \mathbf{x}_i &= \left(\mathsf{b}_i - \sum_{k=\max(\mathrm{first}(i),\mathrm{first}(\mathsf{b}))}^{i-1} \mathsf{L}_{ik} \, \mathbf{x}_k \right) / \mathsf{L}ii \\ \\ \mathbf{y} &= \mathbf{profile_cholesky_back_subs}(\mathsf{A}, \mathbf{x}) \\ \mathbf{y} &= \mathbf{x} \\ \mathbf{for} \ i &= \mathrm{last}(\mathsf{b}) \ \mathbf{to} \ 1 \ \mathbf{step} - 1 \ \mathbf{do} \end{split}$$

for $k = \max(\text{first}(i), \text{first}(y))$ to i do

 $y_k = y_k - y_i L_{ik}$

 $v_i = v_i / L_{ii}$

but usually, for A * x= b:

- (1) A=L*L*
- (2) L * y = b ==> y via forward subst
- (3) $L^* * x = y ==> x$ via backward subst

http://users.ics.forth.gr/~argyros/mypapers/2004_08_tr340_forth_sba.pdf

The Design and Implementation of a Generic Sparse Bundle Adjustment Software Package Based on the Levenberg-Marquardt Algorithm†

Manolis I.A. Lourakis and Antonis A. Argyros

In all cases, the function pointed to by proj is assumed to estimate in xij the projection in image j of the point i. Arguments aj and bi are respectively the parameters of the j-th camera and i-th point. In other words, proj implements the parameterizing function $\mathbf{Q}()$. Similarly, project is assumed to compute in Aij and Bij the functions $\frac{\partial \mathbf{Q}(\mathbf{a}_j,\mathbf{b}_i)}{\partial \mathbf{a}_j}$ and $\frac{\partial \mathbf{Q}(\mathbf{a}_j,\mathbf{b}_i)}{\partial \mathbf{b}_i}$, i.e. the jacobians with respect to aj and bi of the projection of point i in image j. If project is NULL, the jacobians are

The employed world coordinate frame is taken to be aligned with the initial camera location. All subsequent camera motions are defined relative to the initial location, through the combination of a 3D rotation and a 3D translation. A 3D rotation by an angle θ about a unit vector $\mathbf{u} = (u_1, u_2, u_3)^T$ is represented by the quaternion $\mathbf{R} = (\cos(\frac{\theta}{2}), u_1 \sin(\frac{\theta}{2}), u_2 \sin(\frac{\theta}{2}), u_3 \sin(\frac{\theta}{2}))$ [26]. A 3D translation is defined by a vector \mathbf{t} . A 3D point is represented by its Euclidean coordinate vector \mathbf{M} . Thus, the parameters of each camera j and point i are $\mathbf{a}_j = (\mathbf{R}_j, \mathbf{t}_j^T)^T$ and $\mathbf{b}_i = \mathbf{M}_i$, respectively. With the previous definitions, the predicted projection of point i on image j is

$$Q(\mathbf{a}_j, \mathbf{b}_i) = \mathbf{K} (\mathbf{R}_j \mathbf{N}_i \mathbf{R}_i^{-1} + \mathbf{t}_j),$$
 (28)

where **K** is the 3 × 3 intrinsic camera calibration matrix and $\mathbf{N}_i = (0, \mathbf{M}_i^T)$ is the vector quaternion corresponding to the 3D point \mathbf{M}_i . The expression \mathbf{R}_j \mathbf{N}_i \mathbf{R}_j^{-1} corresponds to point \mathbf{M}_i rotated by an angle θ_j about unit vector \mathbf{u}_j , as specified by the quaternion \mathbf{R}_j . Source file eucsbademo.c accompanying the sba package im-

http://users.ics.forth.gr/~argyros/mypapers/2004_08_tr340_forth_sba.pdf

The Design and Implementation of a Generic Sparse Bundle Adjustment Software Package Based on the Levenberg-Marquardt Algorithm[†]

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algorithm³. This procedure can be embedded into the LM algorithm of section 2 at the point indicated by the rectangular box in Fig. 1, leading to a sparse bundle adjustment algorithm.

Figure 2: Algorithm for solving the sparse normal equations arising in generic bundle adjustment; see text for details.

Input: The current parameter vector partitioned into m camera parameter vectors \mathbf{a}_j and n 3D point parameter vectors \mathbf{b}_i , a function \mathbf{Q} employing the \mathbf{a}_j and \mathbf{b}_i to compute the predicted projections $\hat{\mathbf{x}}_{ij}$ of the i-th point on the j-th image, the observed image point locations \mathbf{x}_{ij} and a damping term μ for L.M.

Output: The solution δ to the normal equations involved in LM-based bundle adjustment.

Algorithm:

Compute the derivative matrices $\mathbf{A}_{ij} := \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} = \frac{\partial \mathbf{Q}(\mathbf{a}_j, \mathbf{b}_i)}{\partial \mathbf{a}_j}$, $\mathbf{B}_{ij} := \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i} = \frac{\partial \mathbf{Q}(\mathbf{a}_j, \mathbf{b}_i)}{\partial \mathbf{b}_i}$ and the error vectors $\epsilon_{ij} := \mathbf{x}_{ij} - \hat{\mathbf{x}}_{ij}$, where i and j assume values in $\{1, \dots, n\}$ and $\{1, \dots, m\}$ respectively.

Compute the following auxiliary variables:

$$\begin{aligned} \mathbf{U}_{j} &:= \sum_{i} \mathbf{A}_{ij}^{T} \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \mathbf{A}_{ij} \quad \mathbf{V}_{i} &:= \sum_{j} \mathbf{B}_{ij}^{T} \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \mathbf{B}_{ij} \quad \mathbf{W}_{ij} &:= \mathbf{A}_{ij}^{T} \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \mathbf{B}_{ij} \\ \epsilon_{\mathbf{a}_{j}} &:= \sum_{i} \mathbf{A}_{ij}^{T} \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \epsilon_{ij} \quad \epsilon_{\mathbf{b}_{i}} &:= \sum_{j} \mathbf{B}_{ij}^{T} \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \epsilon_{ij} \end{aligned}$$

Augment U_j and V_i by adding μ to their diagonals to yield U_i^* and V_i^* .

Compute
$$\mathbf{Y}_{ij} := \mathbf{W}_{ij} \mathbf{V}_i^{*-1}$$
.

Compute $\delta_{\mathbf{a}}$ from \mathbf{S} $(\delta_{\mathbf{a_1}}{}^T, \delta_{\mathbf{a_2}}{}^T, \dots, \delta_{\mathbf{a_m}}{}^T)^T = (\mathbf{e_1}^T, \mathbf{e_2}^T, \dots, \mathbf{e_m}^T)^T$, where \mathbf{S} is a matrix consisting of $m \times m$ blocks; block jk is defined by $\mathbf{S}_{jk} = \delta_{jk} \mathbf{U}_j^* - \sum_i \mathbf{Y}_{ij} \mathbf{W}_{ik}^T$, where δ_{jk} is Kronecker's delta and

$$\mathbf{e}_{j} = \epsilon_{\mathbf{a}_{j}} - \sum_{i} \mathbf{Y}_{ij} \epsilon_{\mathbf{b}_{i}}.$$

Compute each $\delta_{\mathbf{b}_i}$ from the equation $\delta_{\mathbf{b}_i} = \mathbf{V}_i^{*-1} \left(\epsilon_{\mathbf{b}_i} - \sum_j \mathbf{W}_{ij}^T \ \delta_{\mathbf{a}_j} \right)$.

Form δ as $(\delta_{\mathbf{a}}^T, \delta_{\mathbf{b}}^T)^T$.

http://users.ics.forth.gr/~argyros/mypapers/2004_08_tr340_forth_sba.pdf

The Design and Implementation of a Generic Sparse Bundle Adjustment Software Package Based on the Levenberg-Marquardt Algorithm†

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Figure 1: Levenberg-Marquardt non-linear least squares algorithm; see text and [16, 20] for details. The reason for enclosing a statement in a rectangular box will be explained in section 3.

```
Input: A vector function f: \mathbb{R}^m \to \mathbb{R}^n with n \geq m, a measurement vector \mathbf{x} \in \mathbb{R}^n
and an initial parameters estimate \mathbf{p}_0 \in \mathbb{R}^m.
Output: A vector \mathbf{p}^+ \in \mathcal{R}^m minimizing ||\mathbf{x} - f(\mathbf{p})||^2.
Algorithm:
k := 0; \ \nu := 2; \ \mathbf{p} := \mathbf{p}_0;
\mathbf{A} := \mathbf{J}^T \mathbf{J}; \ \epsilon_{\mathbf{p}} := \mathbf{x} - f(\mathbf{p}); \ \mathbf{g} := \mathbf{J}^T \epsilon_{\mathbf{p}};
stop:=(||g||_{\infty} \le \varepsilon_1); \mu := \tau * \max_{i=1,...,m}(A_{ii});
while (not stop) and (k < k_{max})
        k := k + 1;
       repeat
               Solve (\mathbf{A} + \mu \mathbf{I})\delta_{\mathbf{p}} = \mathbf{g};
              if (\|\delta_{\mathbf{p}}\| \le \varepsilon_2 \|\mathbf{p}\|)
                    stop:=true;
              else
                    \mathbf{p}_{new} := \mathbf{p} + \delta_{\mathbf{p}};
                   \rho := (||\epsilon_{\mathbf{p}}||^2 - ||\mathbf{x} - f(\mathbf{p}_{new})||^2) / (\delta_{\mathbf{p}}^T(\mu \delta_{\mathbf{p}} + \mathbf{g}));
                    if \rho > 0
                          \mathbf{p} = \mathbf{p}_{new};
                          \mathbf{A} := \mathbf{J}^T \mathbf{J}; \ \epsilon_{\mathbf{p}}^{[3]} \stackrel{\mathsf{X}}{=} \stackrel{\mathsf{J}}{\times} -1 \}(\mathbf{p}); \ \mathbf{g} := \mathbf{J}^T \epsilon_{\mathbf{p}};
                          \text{stop}:=(\|\mathbf{g}\|_{\infty} \leq \varepsilon_1);
                          \mu := \mu * \max(\frac{1}{3}, 1 - (2\rho - 1)^3); \nu := 2;
                    else
                          \mu := \mu * \nu; \nu := 2 * \nu;
                    endif
              endif
       until (\rho > 0) or (\text{stop})
end while
```