http://users.ics.forth.gr/~lourakis/sba/PRCV_colloq.pdf
lecture by Lourakis “Bundle adjustment gone public”

- Bundle Adjustment (BA) is a key ingredient of SaM, almost always used as its last step
    - • It is an optimization problem over the 3D structure and viewing parameters (camera pose, intrinsic calibration, & radial distortion parameters), which are simultaneously refined for minimizing reprojection error
    - very large nonlinear least squares problem, typically solved with the Levenberg-Marquardt (LM) algorithm
    - Std LM involves the repetitive solution of linear systems, each with O(N^3) time and O(N^2) storage complexity, resp.
        - • Example: for 54 cameras and 5207 3D points, N = 15945. ==> N^3 = 1e12
    - Sparse LM is a better solution.
    - Example:
        - M images
        - N features
        - **x_i_j** = measured feature “i” on image “j”
        - **a_j** = vector of parameters for camera “j”
        - **b_i** = vectors of parameters for point “i”
        - Q(aj , bi) = the predicted projection of point i on image j, <== needs to be in camera ref frame
        - d(., .) the Euclidean distance between image points
        - vij = 1 iff point i is visible in image j
        - minimize reprojection error over a_j, b_i: min_aj, bi (summation_i=1_to_N(summation_j=1_to_M((v_i_j * d(Q(aj,bi), x_i_j))^2)))
            - ==> total number of parameters is M*(camera parameters) + N*(point parameters)
        - let **P** = parameter vector of camera then point parameters = [P_C P_P]
        - let **X_hat** = [(x_hat_1_1)^T x_hat_1_2)^T … x_hat_1_M)^T x_hat_2_1)^T … x_hat_N_M)^T]
        - where ***x_hat_i_j = Q(a_j, b_i)*** is the projection onto camera plane
        - let error **eps** = [(eps_1_1)^T eps_1_2)^T … eps_1_M)^T eps_2_1)^T … eps_N_M)^T]
        - where ***eps = x_i_j - x_hat_i_j***

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Bundle Adjustment (BA) becomes

min (summation_i=1_to_N(summation_j=1_to_M((eps_i_j)^2))) over P

Jacobian J = d(X_hat) / d(P) which has a block structure because of P being [camera parameters   point parameters]

J = [A | B] where A = d(X_hat) / d(a) and B = A = d(X_hat) / d(b)

The LM updating vector delta = [(delta(a))^T (delta(b))^T

The normal equations:

| A^T * A   A^T * B | * | delta(a)) | = | A^T * eps |
| B^T * A   B^T * B | | delta(b)) | | B^T * eps |

The lhs matrix above is sparse due to A and B being sparse:

$\partial \hat{x}_{ij} \partial a_k = 0, \forall j \neq k$ and

$\partial \hat{x}_{ij} \partial b_k = 0, \forall i \neq k$

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<u>(example cont.) M images = 3, N features = 4</u>

$\mathbf{J} = \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}}$ has a block structure $[\,\mathbf{A}|\mathbf{B}\,]$,

Let $\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

- The Jacobian J in block form:

$$\frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \begin{array}{c} \\ \mathbf{x}_{11} \\ \mathbf{x}_{12} \\ \mathbf{x}_{13} \\ \mathbf{x}_{21} \\ \mathbf{x}_{22} \\ \mathbf{x}_{23} \\ \mathbf{x}_{31} \\ \mathbf{x}_{32} \\ \mathbf{x}_{33} \\ \mathbf{x}_{41} \\ \mathbf{x}_{42} \\ \mathbf{x}_{43} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T & \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T \end{array} \\ \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{B}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{13} & \mathbf{B}_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{23} & \mathbf{0} & \mathbf{B}_{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{32} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{33} & \mathbf{0} \\ \mathbf{A}_{41} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{41} \\ \mathbf{0} & \mathbf{A}_{42} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{42} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{43} \end{pmatrix} \end{array}$$

(1)

This is the so-called *primary structure* of BA

- Approximate Hessian in block form:

$$\mathbf{J}^T\mathbf{J} = \begin{array}{c} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T & \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T \end{array} \\ \begin{pmatrix} \mathbf{U}_1 & 0 & 0 & \mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} \\ 0 & \mathbf{U}_2 & 0 & \mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} \\ 0 & 0 & \mathbf{U}_3 & \mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} \\ \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T & \mathbf{V}_1 & 0 & 0 & 0 \\ \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T & 0 & \mathbf{V}_2 & 0 & 0 \\ \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T & 0 & 0 & \mathbf{V}_3 & 0 \\ \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T & 0 & 0 & 0 & \mathbf{V}_4 \end{pmatrix} \end{array} \equiv$$

(2)

$$\equiv \begin{pmatrix} \mathbf{U} & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V} \end{pmatrix},$$

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij}$, <u>**for 1 image, summing over all features**</u>

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij}$, <u>**for 1 feature, summing over all images**</u>

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$

(example cont.) M images = 3, N features = 4

Bundle Adjustment Revisited

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memory limits and efficiency consideration.

For convenience, we use (18) to show how to solve the bundle adjustment problem. set $\hat{u}_{ij} = \pi(C_j, X_i)$, and we order the parameter x into camera block c and structure block p:

$$x = [c,\ p] \tag{20}$$

it's easily to realize that:

$$J_{ij} = \frac{\partial r_{ij}}{\partial x_k} = \frac{\partial \hat{u}_{ij}}{\partial x_k}, \frac{\partial \hat{u}_{ij}}{\partial c_k} \underline{= 0, \forall j \neq k}, \frac{\partial \hat{u}_{ij}}{\partial p_k} \underline{= 0, \forall i \neq k} \tag{21}$$

Consider now, that we have $m = 3$ cameras and $n = 4$ 3D points. Set $A_{ij} = \frac{\partial \hat{u_{ij}}}{\partial c_j}, B_{ij} = \frac{\partial \hat{u_{ij}}}{\partial p_i}$, we can obtain the Jacobi:

k dimension is row number w.r.t. the block of i where i is the feature number

$$J = \frac{\partial \hat{u}}{\partial x} = \begin{array}{c} \\ k = 1 \\ \\ \\ \\ \\ \\ \\ \\ \\ k = 4 \\ \\ \end{array} \begin{bmatrix} A_{11} & 0 & 0 & B_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & B_{12} & 0 & 0 & 0 \\ 0 & 0 & A_{13} & B_{13} & 0 & 0 & 0 \\ A_{21} & 0 & 0 & 0 & B_{21} & 0 & 0 \\ 0 & A_{22} & 0 & 0 & B_{22} & 0 & 0 \\ 0 & 0 & A_{23} & 0 & B_{23} & 0 & 0 \\ A_{31} & 0 & 0 & 0 & 0 & B_{31} & 0 \\ 0 & A_{32} & 0 & 0 & 0 & B_{32} & 0 \\ 0 & 0 & A_{33} & 0 & 0 & B_{33} & 0 \\ A_{41} & 0 & 0 & 0 & 0 & 0 & B_{41} \\ 0 & A_{42} & 0 & 0 & 0 & 0 & B_{42} \\ 0 & 0 & A_{43} & 0 & 0 & 0 & B_{43} \end{bmatrix} \tag{22}$$

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(example cont.) M images = 3, N features = 4

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NOTE: eps_a = A^T * eps, eps_b =  B^T * eps
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- The augmented normal equations $(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\delta_\mathbf{p} = \mathbf{J}^T\epsilon$ take the form

where μ > 0

(3) $$\begin{pmatrix} \mathbf{U}^* & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V}^* \end{pmatrix} \begin{pmatrix} \delta_\mathbf{a} \\ \delta_\mathbf{b} \end{pmatrix} = \begin{pmatrix} \epsilon_\mathbf{a} \\ \epsilon_\mathbf{b} \end{pmatrix}$$

- Performing block Gaussian elimination in the lhs matrix, $\delta_\mathbf{a}$ is determined with Cholesky from $\mathbf{V}^*$'s Schur complement:

(4) $$(\mathbf{U}^* - \mathbf{W}\,\mathbf{V}^{*-1}\,\mathbf{W}^T)\,\delta_\mathbf{a} = \epsilon_\mathbf{a} - \mathbf{W}\,\mathbf{V}^{*-1}\,\epsilon_\mathbf{b}$$

Schur complement: multiply 1st matrix on res by
| I 0 |
| (((-V*)^-1)W^T I |
resulting in:
| (U*) - W(((V*)^-1)W^T W | * | delta_a | = | eps_a |
| 0 (V*) | | delta_b | | eps_b |

$$\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

note (V*) is invertible and only the block diagonals are populated, so each V_i is inverted.

separate delta_b: 0*delta_a + (V*) * delta_b = eps_b ==> delta_b = eps_b * ((V*)^-1)
solving for delta_a (typically M images << N features) after substitute delta_b:
((U*) - W(((V*)^-1)W^T) * delta_a + W * delta_b = eps_a
((U*) - W(((V*)^-1)W^T) * delta_a = eps_a - W((V*)^-1)eps_b

NOTE: (U*) - W(((V*)^-1)W^T is called the reduced camera matrix (because delta_a is camera parameters)

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RCM (reduced camera matrix) is sparse because not all features appear in all cameras.
this is known as ***secondary structure*.**

For very large datasets, RCM tends to be in one of two classes:
(1) ○ visual mapping: extended areas are traversed, limited image overlap (sparse RCM)
(2) ○ centered-object: a large number of overlapping images taken in a small area (dense RCM)

Solving for delta_a in the equation containing RCM. several ways:
(1) Store as dense, decompose with **ordinary linear algebra** ○[M. Lourakis, A. Argyros: SBA: A Software Package For Generic Sparse Bundle Adjustment. ACM Trans. Math. Softw. 36(1): (2009) ○ C. Engels, H. Stewenius, D. Nister: Bundle Adjustment Rules. Photogrammetric Computer Vision (PCV), 2006.
(2) • Store as sparse, factorize with **sparse direct solvers** ○ K. Konolige: Sparse Sparse Bundle Adjustment. BMVC 2010: 1-11
(3) Store as sparse, use **conjugate gradient methods** memory efficient, iterative, precoditioners necessary! ○ S. Agarwal, N. Snavely, S.M. Seitz, R. Szeliski: Bundle Adjustment in the Large. ECCV (2) 2010: 29-42 ○ M. Byrod, K. Astrom: Conjugate Gradient Bundle Adjustment. ECCV (2) 2010: 114-127
(4) • Avoid storing altogether ○ C. Wu, S. Agarwal, B. Curless, S.M. Seitz: Multicore Bundle Adjustment. CVPR 2011: 30 57-3064 ○ M. Lourakis: Sparse Non-linear Least Squares Optimization for Geometric Vision. ECCV (2) 2010: 43-56

Engels, Stewenius, Nister 2006, “Bundle Adjustment Rules”

m images (= video frames from same calibrated camera)
? features
each feature **x** has **M** dimensions
n iterations of bundle adjustment over the last m video frames

Engels “RCM” is formed from a jacobian which places point parameters before camera parameters,
so is different than that in the Lourakis notes.

$$J_f = \begin{bmatrix} J_P & J_C \end{bmatrix}, \tag{15}$$

$$H = \begin{bmatrix} J_P^\top J_P & J_P^\top J_C \\ J_C^\top J_P & J_C^\top J_C \end{bmatrix}, \tag{16}$$

$$\begin{bmatrix} H_{PP} & H_{PC} \\ H_{PC}^\top & H_{CC} \end{bmatrix} \begin{bmatrix} dP \\ dC \end{bmatrix} = \begin{bmatrix} b_P \\ b_C \end{bmatrix}, \tag{17}$$

where we have defined $H_{PP} = J_P^\top J_P$, $H_{PC} = J_P^\top J_C$, $H_{CC} = J_C^\top J_C$, $b_P = -J_P^\top f$,$b_C = -J_C^\top f$ to simpify the notation, and dP and dC represent the update of the point parameters and the camera parameters, respectively. Note that the matrices H_{PP} and H_{CC} are block-diagonal, where the blocks correspond to

of as multiplying by

$$\begin{bmatrix} I & 0 \\ -H_{PC}^\top & I \end{bmatrix} \tag{20}$$

from the left on both sides, resulting in the smaller equation system (from the lower part)

$$\underbrace{\left(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC}\right)}_{A} dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B} \tag{21}$$

for the camera parameter update dC. For very large systems,

We use straightforward Cholesky factorization.

Engels, Stewenius, Nister 2006, “Bundle Adjustment Rules”

M images = 3, j
N features = 4, i

	J_P is [2*n*m X 3*m]	**J_C is [2n*m X 9*n]**	**J is [2nm X (3m + 9n)]**
blocks:	**J_P_i is [2*n*m X 3]**	**J_C_i is [2n*m X 9]**	
sparse blocks along diagonal:	**J_P_i is [3 X 3]**	**J_C_i is [9 X 9]**	

their “RCM” is formed from a jacobian which places point parameters before camera parameters, so is different than that in the Lourakis notes.

Lourakis

Let $\underset{\boxed{2X9}}{\mathbf{A}_{ij}} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ **C** and $\underset{\boxed{2X3}}{\mathbf{B}_{ij}} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$ **P**

$$\mathbf{J} = \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \left[\frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{a}} \; \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{b}} \right] = \left[\; J_C \quad J_P \; \right]$$

- The Jacobian J in block form:

$$\frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \begin{array}{c} \\ \mathbf{x}_{11} \\ \mathbf{x}_{12} \\ \mathbf{x}_{13} \\ \mathbf{x}_{21} \\ \mathbf{x}_{22} \\ \mathbf{x}_{23} \\ \mathbf{x}_{31} \\ \mathbf{x}_{32} \\ \mathbf{x}_{33} \\ \mathbf{x}_{41} \\ \mathbf{x}_{42} \\ \mathbf{x}_{43} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T & \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T \end{array} \\ \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{B}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{13} & \mathbf{B}_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{23} & \mathbf{0} & \mathbf{B}_{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{32} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{33} & \mathbf{0} \\ \mathbf{A}_{41} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{41} \\ \mathbf{0} & \mathbf{A}_{42} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{42} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{43} \end{pmatrix} \end{array} \quad (1)$$

Engels

Let $\underset{\boxed{2X9}}{\mathbf{A}_{ij}} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ **C** and $\underset{\boxed{2X3}}{\mathbf{B}_{ij}} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$ **P**

$$J_f = \left[\; J_P \quad J_C \; \right],$$

BundleAdjustment.java code is using Engels pattern

- The Jacobian J in block form:

$$\frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \begin{array}{c} \\ \mathbf{x}_{11} \\ \mathbf{x}_{12} \\ \mathbf{x}_{13} \\ \mathbf{x}_{21} \\ \mathbf{x}_{22} \\ \mathbf{x}_{23} \\ \mathbf{x}_{31} \\ \mathbf{x}_{32} \\ \mathbf{x}_{33} \\ \mathbf{x}_{41} \\ \mathbf{x}_{42} \\ \mathbf{x}_{43} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \end{array} \\ \begin{pmatrix} \mathbf{B}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} \\ \mathbf{B}_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{B}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{31} & \mathbf{0} & \mathbf{A}_{31} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{32} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{32} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{41} & \mathbf{A}_{41} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{42} & \mathbf{0} & \mathbf{A}_{42} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{43} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} \end{pmatrix} \end{array} \quad (1)$$

Engels, Stewenius, Nister 2006, “Bundle Adjustment Rules”

J is [2nm X (3n + 9m)]

J^T * J is [(3n+9m) X (3n+9m)]

M images = 3, j

N features = 4, i

their “RCM” is formed from a jacobian which places point parameters before camera parameters, so is different than that in the Lourakis notes.

Lourakis

Let $\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

C **P**

$$\mathbf{J} = \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \left[\frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{a}} \; \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{b}} \right] = [\; J_C \quad J_P \;]$$

$\mathbf{J}^T\mathbf{J} =$

J^T * J is [(3*n + 9*m)] X (3*n + 9*m)]

$$\begin{array}{c|c}
 & \begin{array}{ccccccc} \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T & \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T \end{array} \\
\begin{array}{c} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \end{array} &
\begin{pmatrix}
\mathbf{U}_1 & 0 & 0 & \mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} \\
0 & \mathbf{U}_2 & 0 & \mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} \\
0 & 0 & \mathbf{U}_3 & \mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} \\
\mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T & \mathbf{V}_1 & 0 & 0 & 0 \\
\mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T & 0 & \mathbf{V}_2 & 0 & 0 \\
\mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T & 0 & 0 & \mathbf{V}_3 & 0 \\
\mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T & 0 & 0 & 0 & \mathbf{V}_4
\end{pmatrix}
\end{array}$$

Engels

Let $\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

2X9 **C** 2X3 **P**

$$\mathbf{J} = \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{P}} = \left[\frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{b}} \; \frac{\partial \hat{\mathbf{X}}}{\partial \mathbf{a}} \right] = [\; J_P \quad J_C \;],$$

$\mathbf{J}^T\mathbf{J} =$

J^T * J is [(3*n + 9*m)] X (3*n + 9*m)]

$$\begin{array}{c|c}
 & \begin{array}{ccccccc} \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \end{array} \\
\begin{array}{c} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{array} &
\begin{pmatrix}
\mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\
0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\
0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\
0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\
\mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\
\mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\
\mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3
\end{pmatrix}
\end{array}$$

where

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij}$, <u>**for 1 image, sum over features**</u> 9X9

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij}$, <u>**for 1 feature, sum over images**</u> 3X3 $\equiv \sum_{j=1}^{3} \left(\frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}\right)^T \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ 9X3

Engels, Stewenius, Nister 2006, “Bundle Adjustment Rules”

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$\begin{aligned} H_{PP} &= J_P^\top J_P &\equiv \mathbf{V}^* \\ H_{PC} &= J_P^\top J_C, &\equiv \mathbf{W}^T \\ H_{CC} &= J_C^\top J_C, &\equiv \mathbf{U}^* \\ b_P &= -J_P^\top f, &\equiv \epsilon_{\mathbf{b}} \\ b_C &= -J_C^\top f &\equiv \epsilon_{\mathbf{a}} \end{aligned}$$

$\delta_{\mathbf{b}}$ is dP is [3*n_features X 1]

$\epsilon_{\mathbf{b}}$ is b_P is [3*n_features X 1]

$\delta_{\mathbf{a}}$ is dC is [9*m_images X 1]

$\epsilon_{\mathbf{a}}$ is b_C is [9*m_images X 1]

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

Lourakis

The augmented normal equations $(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\delta_{\mathbf{p}} = \mathbf{J}^T\epsilon$ take the form

$$\begin{pmatrix} \mathbf{U}^* & \mathbf{W} \\ \mathbf{W}^T & \mathbf{V}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{a}} \\ \delta_{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{a}} \\ \epsilon_{\mathbf{b}} \end{pmatrix} \tag{3}$$

$$\begin{bmatrix} U - WV^{-1}W^T & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathbf{a}} \\ \delta_{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} I & -WV^{-1} \end{bmatrix} \begin{bmatrix} \epsilon_{\mathbf{a}} \\ \epsilon_{\mathbf{b}} \end{bmatrix}$$

(solve delta a first because typically m_images << n_features)

determine $\delta_{\mathbf{a}}$ with Cholesky **(or other method)**

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$\delta_{\mathbf{b}}$ can be computed by back substitution into

$$\mathbf{V}^*\ \delta_{\mathbf{b}} = \epsilon_{\mathbf{b}} - \mathbf{W}^T\ \delta_{\mathbf{a}}$$

$$\delta_{\mathbf{b}} = \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}} - \mathbf{V}^{*-1}\ \mathbf{W}^T\ \delta_{\mathbf{a}}$$

Engels

```
JPTJP JPTJC =  |HPP  HPC|
JCTJP JCTJC    |HPCT HCC|
|HPP  HPC|*|dP|=|bP|
|HPCT HCC| |dC| |bC|

|HPP-HPC*HCC^-1*HPCT|*dP = bP -HPC*HCC^-1*bC
and
(HPC-HPP*HPCT^-1*HCC)*dC = bP -HPP*HPCT^-1*bC
or multiply both sides on left by
|HPP^-1         0|
|-HPCT*HPP^-1   I|
=> |dP + (HPP^-1*HPC)*dC      |=|HPP^-1*bP           |
   |(-HPCT*HPP^-1*HPC+HCC)*dC| |-HPCT*HPP^-1*bP + bC|
```

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$$\delta_{\mathbf{b}} = \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}} - \mathbf{V}^{*-1}\ \mathbf{W}^T\ \delta_{\mathbf{a}}$$

$$dP = H_{PP}^{-1} b_P - H_{PP}^{-1} H_{PC}\, dC.$$

Qu

$$\begin{pmatrix} \boldsymbol{B} & \boldsymbol{E} \\ \boldsymbol{E}^T & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_c \\ \boldsymbol{p}_p \end{pmatrix} = - \begin{pmatrix} \boldsymbol{g}_c \\ \boldsymbol{g}_p \end{pmatrix} \tag{3.61}$$

see eqn (3.70) too

$$-\boldsymbol{g}^k = -\boldsymbol{J}^{k^T} \boldsymbol{F}^k$$

where k is a feature block summed over all images?

$$(\boldsymbol{B} - \boldsymbol{E}\boldsymbol{C}^{-1}\boldsymbol{E}^T)\boldsymbol{p}_c = -\boldsymbol{g}_c + \boldsymbol{E}\boldsymbol{C}^{-1}\boldsymbol{g}_p$$

$$\boldsymbol{p}_p = \boldsymbol{C}^{-1}(-\boldsymbol{g}_p - \boldsymbol{E}^T\boldsymbol{p}_c)$$

Engels, et al 2006

M images = 3, j

N features = 4, i

$$
\begin{aligned}
H_{PP} &= J_P^\top J_P & \equiv \mathbf{V}^* \\
H_{PC} &= J_P^\top J_C, & \equiv \mathbf{W}^T \\
H_{CC} &= J_C^\top J_C, & \equiv \mathbf{U}^* \\
b_P &= -J_P^\top f, & \equiv \epsilon_\mathbf{b} \\
b_C &= -J_C^\top f & \equiv \epsilon_\mathbf{a}
\end{aligned}
$$

$$
(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_\mathbf{a} = \epsilon_\mathbf{a} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_\mathbf{b}
$$

$$
\underbrace{(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}
$$

$$
dP = H_{PP}^{-1} b_P - H_{PP}^{-1} H_{PC} dC.
$$

= tP - tPC^T * dC

[3nX1] [3nX9m] [9mX1]

$$
\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} \text{ and } \mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}
$$

$\boxed{2X9}$ **C** $\boxed{2X3}$ **P**

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$
\mathbf{J}^T\mathbf{J} =
\begin{array}{c}
\\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \end{array} \\
\begin{pmatrix}
\mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\
0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\
0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\
0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\
\mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\
\mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\
\mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3
\end{pmatrix}
\end{array}
$$

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$
\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix}
\begin{pmatrix} \delta_\mathbf{b} \\ \delta_\mathbf{a} \end{pmatrix}
=
\begin{pmatrix} \epsilon_\mathbf{b} \\ \epsilon_\mathbf{a} \end{pmatrix}
$$

$$
\mathbf{V}^* = \text{n blocks}\left\{
\underbrace{\begin{matrix}
\underset{[3X3]}{\mathbf{V}_1} & 0 & 0 & 0 \\
0 & \underset{[3X3]}{\mathbf{V}_2} & 0 & 0 \\
0 & 0 & \underset{[3X3]}{\mathbf{V}_3} & 0 \\
0 & 0 & 0 & \underset{[3X3]}{\mathbf{V}_4}
\end{matrix}}_{\text{n blocks}}\right.
\qquad
\delta_\mathbf{b} = \begin{matrix} [x0,y0,z0]^T \\ [x1,y1,z1]^T \\ [x2,y2,z2]^T \\ [x3,y3,z3]^T \end{matrix}
$$

[3*nX1]

$$
\mathbf{U}^* = \text{m blocks}\left\{
\underbrace{\begin{matrix}
\underset{[9X9]}{\mathbf{U}_1} & 0 & 0 \\
0 & \underset{[9X9]}{\mathbf{U}_2} & 0 \\
0 & 0 & \underset{[9X9]}{\mathbf{U}_3}
\end{matrix}}_{\text{m blocks}}\right.
\qquad
\delta_\mathbf{a} = \begin{matrix} [9X1]^T \\ [9X1]^T \\ [9X1]^T \end{matrix}
$$

[9*mX1]

Engels, et al 2006

M images = 3, j

N features = 4, i

$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$
$$H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$$
$$H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$$
$$b_P = -J_P^\top f, \equiv \epsilon_{\mathbf{b}}$$
$$b_C = -J_C^\top f \equiv \epsilon_{\mathbf{a}}$$

$$\underbrace{\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}}_{\boxed{2X9}\ \mathbf{C}} \text{ and } \underbrace{\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}}_{\boxed{2X3}\ \mathbf{P}}$$

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$

```
* calculate bC = -JC^T*F  [9m X 1]
JC^T[9m X 2mn]                                                          F [2mn X 1]]
A11T 0    0    | A21T 0    0    | A31T 0    0    | A41T 0    0      *   F11 is feature1, img1
0    A12T 0    | 0    A22T 0    | 0    A32T 0    | 0    A42T 0          F12
0    0    A13T | 0    0    A23T | 0    0    A33T | 0    0    A43T       F13 is feature1, img3
  [9m X 2m]        [9m X 2m]        [9m X 2m]        [9m X 2m]           [2m X 1]
                                                                         ——
                                                                         F21 is feature2, img1
                                                                         F22
                                                                         F23
                                                                         [2m X 1]
                                                                         ——
                                                                         F31
                                                                         F32
                                                                         F33
                                                                         [2m X 1]
                                                                         ——
                                                                         F41
                                                                         F42
                                                                         F43
                                                                         [2m X 1]

bC [9m X 1]
-A11T*F11-A21T*F21-A31T*F31-A41T*F41
-A12T*F12-A22T*F22-A32T*F32-A42T*F42
-A13T*F13-A23T*F23-A33T*F33-A43T*F43
each block is [9X1]

     i=1:nFeatures
        j = 1:mImages
           AIJ, FIJ

   [i=1:n,j=1] row=j, col=1 : Σ_i(-AIJT*F1J)
   [i=1:n,j=2] row=j, col=1 : Σ_i(-AIJT*F1J)
   [i=1:n,j=3] row=j, col=1 : Σ_i(-AIJT*F1J)
```

Engels, et al 2006

M images = 3, j

N features = 4, i

$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$
$$H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$$
$$H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$$
$$b_P = -J_P^\top f, \equiv \epsilon_\mathbf{b}$$
$$b_C = -J_C^\top f \equiv \epsilon_\mathbf{a}$$

$$\underbrace{\mathbf{A}_{ij}}_{2X9} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$$ **C** and $$\underbrace{\mathbf{B}_{ij}}_{2X3} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$$ **P**

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$\mathbf{U}_j \equiv \sum_{i=1}^4 \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_\mathbf{b} \\ \delta_\mathbf{a} \end{pmatrix} = \begin{pmatrix} \epsilon_\mathbf{b} \\ \epsilon_\mathbf{a} \end{pmatrix}$$

```
*calculate bP = -JP^T*F  [3n X 1]
JP^T [3n X 2mn]                                                                  F [2mn X 1]
B11^T B12^T B13^T | 0     0     0     | 0     0     0     | 0     0     0      *  F11
0     0     0     | B21^T B22^T B23^T | 0     0     0     | 0     0     0         F12
0     0     0     | 0     0     0     | B31^T B32^T B33^T | 0     0     0         F13
0     0     0     | 0     0     0     | 0     0     0     | B41^T B42^T  B43^T    [2m X 1]
   [3n X 2m]               [3n X 2m]          [3n X 2m]           [3n X 2m]       —
                                                                                 F21
                                                                                 F22
                                                                                 F23
                                                                                 [2m X 1]
                                                                                 —
                                                                                 F31
                                                                                 F32
                                                                                 F33
                                                                                 [2m X 1]
                                                                                 —
                                                                                 F41
                                                                                 F42
                                                                                 F43
                                                                                 [2m X 1]
-JP^T*F  [3n X 1]
-B11T*F11 - B12T*F12 - B13T*F13
-B21T*F21 - B22T*F22 - B23T*F23
-B31T*F31 - B32T*F32 - B33T*F33
-B41T*F41 - B42T*F42 - B43T*F43
each block is [3X1]

     i=1:nFeatures
        j = 1:mImages
           BIJ, FIJ

   [i=1,j=1:m] row=i, col=1 : Σ_j(-BIJT*F1J)
   [i=2,j=1:m] row=i, col=1 : Σ_j(-BIJT*F1J)
   [i=3,j=1:m] row=i, col=1 : Σ_j(-BIJT*F1J)
   [i=4,j=1:m] row=i, col=1 : Σ_j(-BIJT*F1J)
```

Engels, et al 2006

M images = 3, j

N features = 4, i

$$
\begin{aligned}
H_{PP} &= J_P^\top J_P &\equiv \mathbf{V}^* \\
H_{PC} &= J_P^\top J_C, &\equiv \mathbf{W}^T \\
H_{CC} &= J_C^\top J_C, &\equiv \mathbf{U}^* \\
b_P &= -J_P^\top f, &\equiv \epsilon_{\mathbf{b}} \\
b_C &= -J_C^\top f &\equiv \epsilon_{\mathbf{a}}
\end{aligned}
$$

$$\left(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T\right)\delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$$\underbrace{\left(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC}\right)}_{A} dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$$\underset{\boxed{2X9}}{\mathbf{A}_{ij}} = \underset{\mathbf{C}}{\frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}} \text{ and } \underset{\boxed{2X3}}{\mathbf{B}_{ij}} = \underset{\mathbf{P}}{\frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}}$$

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$
\mathbf{J}^T\mathbf{J} =
\begin{array}{c}
 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \end{array} \\
\begin{pmatrix}
\mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\
0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\
0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\
0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\
\mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\
\mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\
\mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3
\end{pmatrix}
\end{array}
$$

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$

```
*calculate tP = HPP^-1*bP = V^-1*bP [3n X 1] or [1n X 3] row format

HPP^-1 = V^-1  [3nX3n]      | bP = -JP^T * F  [3n X 1]
V1^-1 0     0     0         | -B11T*F11-B12T*F12-B13T*F13
0     V2^-1 0     0         | -B21T*F21-B22T*F22-B23T*F23
0     0     V3^-1 0         | -B31T*F31-B32T*F32-B33T*F33
0     0     0     V4^-1     | -B41T*F41-B42T*F42-B43T*F43
where each block is [3X3]     each row is [3X2]*[2X1]=[3X1]


tP = V^-1*bP [3nX1]
V1^-1*(-B11T*F11-B12T*F12-B13T*F13)
V2^-1*(-B21T*F21-B22T*F22-B23T*F23)
V3^-1*(-B31T*F31-B32T*F32-B33T*F33)
V4^-1*(-B41T*F41-B42T*F42-B43T*F43)
   each block is [3X1], or rather [1X3] row format

     i=1:nFeatures
        j = 1:mImages
           invVI, BIJ
     [row I, col 0] = invVI*(Σ_j(-BIJT*F1J))
```

Engels, et al 2006

M images = 3, j

N features = 4, i

$$
\begin{aligned}
H_{PP} &= J_P^\top J_P &&\equiv \mathbf{V}^* \\
H_{PC} &= J_P^\top J_C, &&\equiv \mathbf{W}^T \\
H_{CC} &= J_C^\top J_C, &&\equiv \mathbf{U}^* \\
b_P &= -J_P^\top f, &&\equiv \epsilon_{\mathbf{b}} \\
b_C &= -J_C^\top f &&\equiv \epsilon_{\mathbf{a}}
\end{aligned}
$$

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j}$ and $\mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$

$\boxed{\text{2X9}}$ **C** $\boxed{\text{2X3}}$ **P**

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$\mathbf{J}^T\mathbf{J} =$$

$$
\begin{array}{c|ccccccc}
 & \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \\
\hline
\mathbf{b}_1 & \mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\
\mathbf{b}_2 & 0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\
\mathbf{b}_3 & 0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\
\mathbf{b}_4 & 0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\
\mathbf{a}_1 & \mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\
\mathbf{a}_2 & \mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\
\mathbf{a}_3 & \mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3
\end{array}
$$

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$

```
*calculate tPC = HPC^T*HPP^-1 = W*V^-1 [9m X 3n]

[9m X 3n]           [3n X 3n]
W11 W21 W31 W41  * V1^-1 0     0     0
W12 W22 W32 W42    0     V2^-1 0     0
W13 W23 W33 W43    0     0     V3^-1 0
                   0     0     0     V4^-1

W*V^-1=       [9mX3n]
W11*V1^-1   W21*V2^-1   W31*V3^-1   W41*V4^-1
W12*V1^-1   W22*V2^-1   W32*V3^-1   W42*V4^-1
W13*V1^-1   W23*V2^-1   W33*V3^-1   W43*V4^-1
each block is [9X3]

     i=1:nFeatures
        invVI from hPPIInv
        j = 1:mImages
            WIJT from HPC transpose -> WIJ

     [row J, col I] = WIJ*invVI
```

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M images = 3, j
N features = 4, i

$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$
$$H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$$
$$H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$$
$$b_P = -J_P^\top f, \equiv \epsilon_\mathbf{b}$$
$$b_C = -J_C^\top f \equiv \epsilon_\mathbf{a}$$

$$\left(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T\right)\delta_\mathbf{a} = \epsilon_\mathbf{a} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_\mathbf{b}$$
$$\underbrace{\left(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC}\right)}_{A} dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} \text{ and } \mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$$

2X9 **C** 2X3 **P**

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$\mathbf{J}^T\mathbf{J} =$$
$$\begin{array}{c|c} & \begin{array}{ccccccc} \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \end{array} \\ \begin{array}{c} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{array} & \begin{pmatrix} \mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\ 0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\ 0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\ 0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\ \mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\ \mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\ \mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3 \end{pmatrix} \end{array}$$

$\mathbf{U}_j \equiv \sum_{i=1}^4 \mathbf{A}_{ij}^T \mathbf{A}_{ij}$, **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij}$, **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_\mathbf{b} \\ \delta_\mathbf{a} \end{pmatrix} = \begin{pmatrix} \epsilon_\mathbf{b} \\ \epsilon_\mathbf{a} \end{pmatrix}$$

```
*calculate rightside of mA: tPC*HPC2 = W*V^-1*W^T [9m X 9m]

W*V^-1=       [9mX3n]
W11*V1^-1   W21*V2^-1   W31*V3^-1   W41*V4^-1 * W11T W12T W13T
W12*V1^-1   W22*V2^-1   W32*V3^-1   W42*V4^-1   W21T W22T W23T
W13*V1^-1   W23*V2^-1   W33*V3^-1   W43*V4^-1   W31T W32T W33T
                                                W41T W42T W43T
each block is [9X3]                              each block is [3X9]

W11*V1^-1*W11T + W21*V2^-1*W21T + W31*V3^-1*W31T + W41*V4^-1*W41T  W11*V1^-1*W12T + W21*V2^-1*W22T + W31*V3^-1*W32T + W41*V4^-1*W42T  W11*V1^-1*W13T + W21*V2^-1*W23T + W31*V3^-1*W33T +
W41*V4^-1*W43T
W12*V1^-1*W11T + W22*V2^-1*W21T + W32*V3^-1*W31T + W42*V4^-1*W41T  W12*V1^-1*W12T + W22*V2^-1*W22T + W32*V3^-1*W32T + W42*V4^-1*W42T  W12*V1^-1*W13T + W22*V2^-1*W23T + W32*V3^-1*W33T +
W42*V4^-1*W43T
W13*V1^-1*W11T + W23*V2^-1*W21T + W33*V3^-1*W31T + W43*V4^-1*W41T  W13*V1^-1*W12T + W23*V2^-1*W22T + W33*V3^-1*W32T + W43*V4^-1*W42T  W13*V1^-1*W13T + W23*V2^-1*W23T + W33*V3^-1*W33T +
W43*V4^-1*W43T

      i = 1:nFeatures
        WIJ
        j = 1:mImages
           calc tPC = HPC^T * invHPP
           j2 = 1:mImages
               WIJ2T
               Subtract tPC*HPC2 (=HPC^T*invHPP*HPC2 = W*inv(V)*W2^T) from block (c, c2)
        i=1,j=1,j2=1: block(1,1)-= (W11 * inv(V1) * W11^T)
        i=2,j=1,j2=1: block(1,1)-= (W21 * inv(V2) * W21^T)
        i=1,j=1,j2=2: block(1,2)-= (W11 * inv(V1) * W12^T)
        i=1,j=1,j2=3: block(1,3)-= (W11 * inv(V1) * W13^T)
        i=1,j=2,j2=1: block(2,1)-= (W12 * inv(V1) * W11^T)
        i=1,j=2,j2=2: block(2,2)-= (W12 * inv(V1) * W12^T)
```

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M images = 3, j

N features = 4, i

$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$
$$H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$$
$$H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$$
$$b_P = -J_P^\top f, \equiv \epsilon_\mathbf{b}$$
$$b_C = -J_C^\top f \equiv \epsilon_\mathbf{a}$$

$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} \text{ and } \mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$$

2X9 **C** 2X3 **P**

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_\mathbf{a} = \epsilon_\mathbf{a} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_\mathbf{b}$$

$$\underbrace{(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$$\mathbf{J}^T\mathbf{J} = \begin{array}{c c} & \begin{array}{ccccccc} \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \end{array} \\ \begin{array}{c} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{array} & \begin{pmatrix} \mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\ 0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\ 0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\ 0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\ \mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\ \mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\ \mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3 \end{pmatrix} \end{array}$$

$\mathbf{U}_j \equiv \sum_{i=1}^4 \mathbf{A}_{ij}^T \mathbf{A}_{ij}$, **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij}$, **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_\mathbf{b} \\ \delta_\mathbf{a} \end{pmatrix} = \begin{pmatrix} \epsilon_\mathbf{b} \\ \epsilon_\mathbf{a} \end{pmatrix}$$

```
*calculate rightside of vB: HPC^T*tP = HPC^T*HPP^-1*bP = W*V^-1*bP [9m X 1]

HPC^T=W [9m X 3n]   tP = V^-1*bP [3nX1]
W11 W21 W31 W41 |  |V1^-1*(-B11T*F11-B12T*F12-B13T*F13)|
W12 W22 W32 W42 |  |V2^-1*(-B21T*F21-B22T*F22-B23T*F23)|
W13 W23 W33 W43 |  |V3^-1*(-B31T*F31-B32T*F32-B33T*F33)|
                   |V4^-1*(-B41T*F41-B42T*F42-B43T*F43)|
=
W11 W21 W31 W41 |  |tP_1|
W12 W22 W32 W42 |  |tP_2|
W13 W23 W33 W43 |  |tP_3|
                   |tP_4|
block is [9X3]        each block is [3X1]

W11*tP_1 + W21*tP_2 + W31*tP_3 + W41*tP_4
W12*tP_1 + W22*tP_2 + W32*tP_3 + W42*tP_4
W13*tP_1 + W23*tP_2 + W33*tP_3 + W43*tP_4
each block is [9X1]

     i=1:nFeatures
        tPI
        j = 1:mImages
           WIJT from HPC transpose -> WIJ

     [row j, col 0] += WIJ*tPI
```

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M images = 3, j

N features = 4, i

$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$
$$H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$$
$$H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$$
$$b_P = -J_P^\top f, \equiv \epsilon_{\mathbf{b}}$$
$$b_C = -J_C^\top f \equiv \epsilon_{\mathbf{a}}$$

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$$\mathbf{A}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} \text{ and } \mathbf{B}_{ij} = \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i}$$

2X9 **C** 2X3 **P**

Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$\mathbf{J}^T\mathbf{J} =$$

$$\begin{array}{c|ccccccc} & \mathbf{b}_1{}^T & \mathbf{b}_2{}^T & \mathbf{b}_3{}^T & \mathbf{b}_4{}^T & \mathbf{a}_1{}^T & \mathbf{a}_2{}^T & \mathbf{a}_3{}^T \\ \hline \mathbf{b}_1 & \mathbf{V}_1 & 0 & 0 & 0 & \mathbf{W}_{11}^T & \mathbf{W}_{12}^T & \mathbf{W}_{13}^T \\ \mathbf{b}_2 & 0 & \mathbf{V}_2 & 0 & 0 & \mathbf{W}_{21}^T & \mathbf{W}_{22}^T & \mathbf{W}_{23}^T \\ \mathbf{b}_3 & 0 & 0 & \mathbf{V}_3 & 0 & \mathbf{W}_{31}^T & \mathbf{W}_{32}^T & \mathbf{W}_{33}^T \\ \mathbf{b}_4 & 0 & 0 & 0 & \mathbf{V}_4 & \mathbf{W}_{41}^T & \mathbf{W}_{42}^T & \mathbf{W}_{43}^T \\ \mathbf{a}_1 & \mathbf{W}_{11} & \mathbf{W}_{21} & \mathbf{W}_{31} & \mathbf{W}_{41} & \mathbf{U}_1 & 0 & 0 \\ \mathbf{a}_2 & \mathbf{W}_{12} & \mathbf{W}_{22} & \mathbf{W}_{32} & \mathbf{W}_{42} & 0 & \mathbf{U}_2 & 0 \\ \mathbf{a}_3 & \mathbf{W}_{13} & \mathbf{W}_{23} & \mathbf{W}_{33} & \mathbf{W}_{43} & 0 & 0 & \mathbf{U}_3 \end{array}$$

$\mathbf{U}_j \equiv \sum_{i=1}^4 \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **[9X9], for 1 image, sum over features**

$\mathbf{V}_i \equiv \sum_{j=1}^3 \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **[3X3], for 1 feature, sum over images**

$\mathbf{W}_{ij} = \mathbf{A}_{ij}^T \mathbf{B}_{ij}$ **[9X3]**

V* is [3*n X 3*n]

W is [9*m X 3*n]

U* is [9*m X 9*m]

$$\begin{pmatrix} \mathbf{V}^* & \mathbf{W}^T \\ \mathbf{W} & \mathbf{U}^* \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{b}} \\ \delta_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{b}} \\ \epsilon_{\mathbf{a}} \end{pmatrix}$$

1 Initialize λ.

2 **Compute cost function** at initial camera and point configuration.

3 Clear the left hand side matrix A and right hand side vector B.

4 For each track p **(p is feature i of N)**

{

$\mathbf{V}^*$ $\mathbf{V}^*$

Clear a variable H_{pp} to represent block p of H_{PP} (in our case a symmetric 3×3 matrix) and a variable b_p to represent part p of b_P (in our case a 3-vector). $\epsilon_{\mathbf{b}}$ $\epsilon_{\mathbf{b}}$

(Compute derivatives) For each camera c on track p **(c is image j of M)**

{ Compute error vector f of reprojection in camera c of point p and its Jacobians J_p and J_c with respect to the

$\mathbf{B}$ $\mathbf{A}$

point parameters (in our case a 2×3 matrix) and the camera parameters (in our case a 2×6 matrix), respectively.

$\mathbf{B}^T\mathbf{B}$ $\mathbf{V}^*$

Add $J_p^\top J_p$ to the upper triangular part of H_{pp}.

Subtract $J_p^\top f$ from b_p.

$\mathbf{B}^T$ $\epsilon_{\mathbf{b}}$

If camera c is free

{ $\mathbf{A}^T\mathbf{A}$

Add $J_c^\top J_c$ (optionally with an augmented diagonal) to upper triangular part of block (c, c) of left hand side matrix A (in our case a 6×6 matrix).

Compute block (p, c) of H_{PC} as $H_{pc} = J_p^\top J_c$ (in our case a 3×6 matrix) and store it until track is done.

Subtract $J_c^\top f$ from part c of right hand side vector B (related to b_C).

} }

Augment diagonal of H_{pp}, which is now accumulated and ready. Invert H_{pp}, taking advantage of the fact that it is a symmetric matrix.

Compute $H_{pp}^{-1} b_p$ and store it in a variable t_p.

(Outer product of track) For each free camera c on track p

{

Subtract $H_{pc}^\top t_p = H_{pc}^\top H_{pp}^{-1} b_p$ from part c of right hand side vector B.

Compute the matrix $H_{pc}^\top H_{pp}^{-1}$ and store it in a variable T_{pc}

For each free camera $c2 \geq c$ on track p

{

Subtract $T_{pc} H_{pc2} = H_{pc}^\top H_{pp}^{-1} H_{pc2}$ from block $(c, c2)$ of left hand side matrix A.

} }

}

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Note that $\mathbf{V}^{*-1} = \begin{pmatrix} \mathbf{V}_1^{*-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{V}_2^{*-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$\begin{aligned} H_{PP} &= J_P^\top J_P &\equiv \mathbf{V}^* \\ H_{PC} &= J_P^\top J_C, &\equiv \mathbf{W}^T \\ H_{CC} &= J_C^\top J_C, &\equiv \mathbf{U}^* \\ b_P &= -J_P^\top f, &\equiv \epsilon_{\mathbf{b}} \\ b_C &= -J_C^\top f &\equiv \epsilon_{\mathbf{a}} \end{aligned}$$

gradients

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ <u>**for 1 feature, sum over images**</u>

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ <u>**for 1 image, sum over features**</u>

$\delta_{\mathbf{b}}$ is dP is [3*n_features X 1]

$\epsilon_{\mathbf{b}}$ is b_P is [3*n_features X 1]

$\delta_{\mathbf{a}}$ is dC is [9*m_images X 1]

$\epsilon_{\mathbf{a}}$ is b_C is [9*m_images X 1]

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^\top H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

$$\mathbf{V}^*\ \delta_{\mathbf{b}} = \epsilon_{\mathbf{b}} - \mathbf{W}^T\ \delta_{\mathbf{a}}$$

$$dP = H_{PP}^{-1} b_P - H_{PP}^{-1} H_{PC} dC.$$

$$= \underset{[3nX1]}{tP} - \underset{[3nX9m]}{tPC^\wedge T} * \underset{[9mX1]}{dC}$$

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5 (Optional) Fix gauge by freezing appropriate coordinates and thereby reducing the linear system with a few dimensions.

6 **(Linear Solving)** Cholesky factor the left hand side matrix B and solve for dC. Add frozen coordinates back in.

7 **(Back-substitution)** For each track p

{

  Start with point update for this track $dp = t_p$.

  For each camera c on track p

  {

    Subtract $T_{pc}^\top dc$ from dp (where dc is the update for camera c).

  }

  Compute updated point.

}

8 **Compute the cost function** for the updated camera and point configuration.

9 If cost function has improved, accept the update step, decrease λ and go to Step 3 (unless converged, in which case quit).

10 Otherwise, increase λ and go to Step 3 (unless exceeded the maximum number of iterations, in which case quit).

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N features = 4, i

$$H_{PP} = J_P^\top J_P \equiv \mathbf{V}^*$$
$$H_{PC} = J_P^\top J_C, \equiv \mathbf{W}^T$$
$$H_{CC} = J_C^\top J_C, \equiv \mathbf{U}^*$$
$$b_P = -J_P^\top f, \equiv \epsilon_{\mathbf{b}}$$
$$b_C = -J_C^\top f \equiv \epsilon_{\mathbf{a}}$$

gradients

$\mathbf{V}_i \equiv \sum_{j=1}^{3} \mathbf{B}_{ij}^T \mathbf{B}_{ij},$ **for 1 feature, sum over images**

$\mathbf{U}_j \equiv \sum_{i=1}^{4} \mathbf{A}_{ij}^T \mathbf{A}_{ij},$ **for 1 image, sum over features**

$\delta_{\mathbf{b}}$ is dP is [3*n_features X 1]

$\epsilon_{\mathbf{b}}$ is b_P is [3*n_features X 1]

$\delta_{\mathbf{a}}$ is dC is [9*m_images X 1]

$\epsilon_{\mathbf{a}}$ is b_C is [9*m_images X 1]

$$dP = H_{PP}^{-1} b_P - H_{PP}^{-1} H_{PC} dC.$$

= tP - tPC^T * dC

[3nX1] [3nX9m] [9mX1]

```
tPC [9m X 3n]

tPC[1,1]   tPC[1,2]  tPC[1,3]  tPC[1,4]
tPC[2,1]   tPC[2,2]  tPC[2,3]  tPC[2,4]
tPC[3,1]   tPC[3,2]  tPC[3,3]  tPC[3,4]
each block is [9X3]

tPC^T
tPC[1,1]^T tPC[2,1]^T tPC[3,1]^T
tPC[1,2]^T tPC[2,2]^T tPC[3,2]^T
tPC[1,3]^T tPC[2,3]^T tPC[3,3]^T
tPC[1,4]^T tPC[2,4]^T tPC[3,4]^T
each block is [3X9]
```

```
* calculate dP = tP - (tPC)^T * dC   [3nX1]-[3nX9m][9mX1] = [3nX1]
|tP_1|          -    tPC[1,1]^T tPC[2,1]^T tPC[3,1]^T * dC[0:9]  dC[9:2*9]  dC[2*9:3*9]
|tP_2|               tPC[1,2]^T tPC[2,2]^T tPC[3,2]^T
|tP_3|               tPC[1,3]^T tPC[2,3]^T tPC[3,3]^T
|tP_4|               tPC[1,4]^T tPC[2,4]^T tPC[3,4]^T
 block is [3X1]                each block is [3X9]           each block is [9X1]

    i=1:nFeatures
       tPI
       j = 1:mImages
          dCJ=dC[j*9:j*9+1]
          tPC[J,I] transpose -> tPC[J,I]^T

    [row i, col 0] = tPI - (Σ_j(tPC[J,I]^T*dCJ))
))
```

```
double[] bC = new double[9*mImages];
double[] bP = new double[3*nFeatures];
BlockMatrixIsometric hPCBlocks /*W^T*/= new BlockMatrixIsometric(MatrixUtil.zeros(3*nFeatures, 9*mImages), 3, 9);
BlockMatrixIsometric hPPIInvBlocks /*VI^-1*/= new BlockMatrixIsometric(MatrixUtil.zeros(3*nFeatures, 3), 3, 3);
BlockMatrixIsometric mA = new BlockMatrixIsometric(MatrixUtil.zeros(9*mImages, 9*mImages), 9, 9);
// storing hCCJBlocks in mA, while populating mA with only HCC.  later will add the negative rightsize of mA to mA
//BlockMatrixIsometric hCCJBlocks /*UJ*/= new BlockMatrixIsometric(MatrixUtil.zeros(9*mImages, 1), 9, 9);

for (i = 0; i < nFeatures; ++i) {
    for (j = 0; j < mImages; ++j) { // this is camera c in Engels pseudocode
        //aIJ is [2X9].  a is partial derivative of measurement vector X w.r.t. camera portion of parameter vector P
        //bIJ is [2X3]
        // populate aIJ and bIJ as output of method:
        aIJBIJ(xWI, xWCI, auxIntr, k1, k2, extrRotThetas[j], rotM, extrTrans[j], aa, aIJ, bIJ);
        calc FIJ
        outFSqSum[0] += MatrixUtil.innerProduct(fIJ, fIJ);

        //V_i = Σ_j(BIJT*B1J) // HPP=V
        calc BIJT*BIJ and add it to hPPI. later invert it, and add to hPPIInvBlocks for block(i,0)
        //U_j = Σ_i(AIJT*A1J) // HCC=U
        calc AIJT*AIJ.  add it to mA block(j,j)
        //W_i_j^T = (AIJT*BIJ)^T = BIJ^T*AIJ
        calc BIJ^T*AIJ and store in hPCBlocks for block(i,j)
        //calc bC: row=j, col=0 : Σ_i(-AIJT*F1J)
        //calc bP: row=i, col=0 : Σ_j(-BIJT*F1J)
    } // end j loop over images
    // augment hPPI by damping term.  invert it and add it to hPPIInvBlocks for block(i,0)
} // end i loop over features
// loop over mA to augment the diagonal by damping term.
BlockMatrixIsometric tPBlocks = new BlockMatrixIsometric(MatrixUtil.zeros(3*nFeatures, 1), 3, 1);
BlockMatrixIsometric tPCBlocks = new BlockMatrixIsometric(MatrixUtil.zeros(9*mImages, 3*nFeatures), 9, 3);
for (i = 0; i < nFeatures; ++i) {
    //calc tP = HPP^-1*bP = V^-1*bP and set into tPBlocks(i,0) += invVI*(Σ_j(-BIJT*F1J))
    for (j = 0; j < mImages; ++j) { // this is camera c in Engels pseudocode
       //calc tPC = HPC^T*HPP^-1 = W*V^-1 and set into tPCBlocks(j,i)=WIJ*invVI
    }
}
double[][] vB = MatrixUtil.zeros(mImages, 9);
//calc rightside of mA: tPC*HPC2 = W*V^-1*W^T [9m X 9m] and subtract it from mA
//calc rightside of vB: HPC^T*tP = HPC^T*HPP^-1*bP = W*V^-1*bP and set vB = bC - rightside

//then Engels steps 5-10
```

M images = 3, j

N features = 4, i

$$(\mathbf{U}^* - \mathbf{W}\ \mathbf{V}^{*-1}\ \mathbf{W}^T)\ \delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\ \mathbf{V}^{*-1}\ \epsilon_{\mathbf{b}}$$

$$\underbrace{(H_{CC} - H_{PC}^{\top} H_{PP}^{-1} H_{PC})}_{A}\, dC = \underbrace{b_C - H_{PC}^{\top} H_{PP}^{-1} b_P}_{B}$$

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see Appendix B, and page 23…

$$(\mathbf{U}^* - \mathbf{W}\boxed{\mathbf{V}^{*-1}}\mathbf{W}^T)\,\delta_{\mathbf{a}} = \epsilon_{\mathbf{a}} - \mathbf{W}\,\mathbf{V}^{*-1}\,\epsilon_{\mathbf{b}}$$

$$\underbrace{\underbrace{(H_{CC} - H_{PC}^\top \boxed{H_{PP}^{-1}} H_{PC})}_{A}}_{\overline{\mathsf{D}}}\,dC = \underbrace{b_C - H_{PC}^\top H_{PP}^{-1} b_P}_{B}$$

6.1 The Schur Complement and the Reduced Bundle System

Schur complement: Consider the following block triangular matrix factorization:

$$\mathsf{M} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathsf{C}\,\mathsf{A}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \overline{\mathsf{D}} \end{pmatrix} \begin{pmatrix} 1 & \mathsf{A}^{-1}\mathsf{B} \\ 0 & 1 \end{pmatrix}, \qquad \overline{\mathsf{D}} \equiv \mathsf{D} - \mathsf{C}\,\mathsf{A}^{-1}\mathsf{B} \tag{16}$$

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\mathsf{A}^{-1}\mathsf{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{A}^{-1} & 0 \\ 0 & \overline{\mathsf{D}}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathsf{C}\,\mathsf{A}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} \mathsf{A}^{-1} + \mathsf{A}^{-1}\mathsf{B}\,\overline{\mathsf{D}}^{-1}\mathsf{C}\,\mathsf{A}^{-1} & -\mathsf{A}^{-1}\mathsf{B}\,\overline{\mathsf{D}}^{-1} \\ -\overline{\mathsf{D}}^{-1}\mathsf{C}\,\mathsf{A}^{-1} & \overline{\mathsf{D}}^{-1} \end{pmatrix} \tag{17}$$

Here A must be square and invertible, and for (17), the whole matrix must also be square and invertible. $\overline{\mathsf{D}}$ is called the **Schur complement** of A in M. If both A and D are invertible, complementing on D rather than A gives $\overline{\mathsf{D}}$, the Schur complement, is HPP is V*.

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \overline{\mathsf{A}}^{-1} & -\overline{\mathsf{A}}^{-1}\mathsf{B}\,\mathsf{D}^{-1} \\ -\mathsf{D}\,\mathsf{C}\,\overline{\mathsf{A}}^{-1} & \mathsf{D}^{-1} + \mathsf{D}^{-1}\mathsf{C}\,\overline{\mathsf{A}}^{-1}\mathsf{B}\,\mathsf{D}^{-1} \end{pmatrix}, \qquad \overline{\mathsf{A}} = \mathsf{A} - \mathsf{B}\boxed{\mathsf{D}^{-1}}\mathsf{C}$$

Equating upper left blocks gives the **Woodbury formula**:

$$(\mathsf{A} \pm \mathsf{B}\,\mathsf{D}^{-1}\mathsf{C})^{-1} = \mathsf{A}^{-1} \mp \mathsf{A}^{-1}\mathsf{B}\,(\mathsf{D} \pm \mathsf{C}\,\mathsf{A}^{-1}\mathsf{B})^{-1}\,\mathsf{C}\,\mathsf{A}^{-1} \tag{18}$$

This is the usual method of updating the inverse of a nonsingular matrix A after an update (especially a low rank one) $\mathsf{A} \to \mathsf{A} \pm \mathsf{B}\,\mathsf{D}^{-1}\mathsf{C}$. (See §8.1).

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see Appendix B, and page 23…

$\mathsf{L} = \textbf{profile\_cholesky\_decomp}(\mathsf{A})$
  for $i = 1$ **to** n **do**
    for $j = \text{first}(i)$ **to** i **do**

$$a = \mathsf{A}_{ij} - \sum_{k=\max(\text{first}(i),\text{first}(j))}^{j-1} \mathsf{L}_{ik}\,\mathsf{L}_{jk}$$

$$\mathsf{L}_{ij} = (j < i)\ ?\ a \,/\, \mathsf{L}_{jj}\ :\ \sqrt{a}$$

$\mathsf{x} = \textbf{profile\_cholesky\_forward\_subs}(\mathsf{A}, \mathsf{b})$
  for $i = \text{first}(\mathsf{b})$ **to** n **do**

$$\mathsf{x}_i = \left(\mathsf{b}_i - \sum_{k=\max(\text{first}(i),\text{first}(\mathsf{b}))}^{i-1} \mathsf{L}_{ik}\,\mathsf{x}_k\right) / \,\mathsf{L}ii$$

$\mathsf{y} = \textbf{profile\_cholesky\_back\_subs}(\mathsf{A}, \mathsf{x})$
  $\mathsf{y} = \mathsf{x}$
  for $i = \text{last}(\mathsf{b})$ **to** 1 **step** -1 **do**
    for $k = \max(\text{first}(i), \text{first}(\mathsf{y}))$ **to** i **do**
      $\mathsf{y}_k = \mathsf{y}_k - \mathsf{y}_i\,\mathsf{L}_{ik}$
    $\mathsf{y}_i = \mathsf{y}_i \,/\, \mathsf{L}_{ii}$

Figure 10: A complete implementation of profile Cholesky decomposition.

cholesky:
_A_ = L ∗ D ∗ L^T
= L ∗ √D ∗ √D ∗ L^T
let C = L ∗ √D
then **_A_** = **C ∗ C^T**
aside: L is invertible if none of its diagonal elements are 0.

for _A_=L ∗ L*
(_A_)^-1 = (L*) ∗ (L ∗ L*)^-1

but usually, for A ∗ x= b:
(1) A=L ∗ L*
(2) L ∗ y = b ==> y via forward subst
(3) L* ∗ x = y ==> x via backward subst

http://users.ics.forth.gr/~argyros/mypapers/2004_08_tr340_forth_sba.pdf

The Design and Implementation of a Generic Sparse Bundle Adjustment Software Package Based on the Levenberg-Marquardt Algorithm†

Manolis I.A. Lourakis and Antonis A. Argyros

In all cases, the function pointed to by `proj` is assumed to estimate in `xij` the projection in image `j` of the point `i`. Arguments `aj` and `bi` are respectively the parameters of the `j`-th camera and `i`-th point. In other words, `proj` implements the parameterizing function $\mathbf{Q}()$. Similarly, `projac` is assumed to compute in `Aij` and `Bij` the functions $\frac{\partial \mathbf{Q}(\mathbf{a}_j, \mathbf{b}_i)}{\partial \mathbf{a}_j}$ and $\frac{\partial \mathbf{Q}(\mathbf{a}_j, \mathbf{b}_i)}{\partial \mathbf{b}_i}$, i.e. the jacobians with respect to `aj` and `bi` of the projection of point `i` in image `j`. If `projac` is NULL, the jacobians are

The employed world coordinate frame is taken to be aligned with the initial camera location. All subsequent camera motions are defined relative to the initial location, through the combination of a 3D rotation and a 3D translation. A 3D rotation by an angle θ about a unit vector $\mathbf{u} = (u_1, u_2, u_3)^T$ is represented by the quaternion $\mathbf{R} = (\cos(\frac{\theta}{2}),\ u_1 \sin(\frac{\theta}{2}),\ u_2 \sin(\frac{\theta}{2}),\ u_3 \sin(\frac{\theta}{2}))$ [26]. A 3D translation is defined by a vector $\mathbf{t}$. A 3D point is represented by its Euclidean coordinate vector $\mathbf{M}$. Thus, the parameters of each camera j and point i are $\mathbf{a}_j = (\mathbf{R}_j, \mathbf{t}_j{}^T)^T$ and $\mathbf{b}_i = \mathbf{M}_i$, respectively. With the previous definitions, the predicted projection of point i on image j is

$$\mathbf{Q}(\mathbf{a}_j, \mathbf{b}_i) = \mathbf{K}\ (\mathbf{R}_j\ \mathbf{N}_i\ \mathbf{R}_j^{-1}\ +\ \mathbf{t}_j), \tag{28}$$

where $\mathbf{K}$ is the 3×3 intrinsic camera calibration matrix and $\mathbf{N}_i = (0,\ \mathbf{M}_i{}^T)$ is the vector quaternion corresponding to the 3D point $\mathbf{M}_i$. The expression $\mathbf{R}_j\ \mathbf{N}_i\ \mathbf{R}_j^{-1}$ corresponds to point $\mathbf{M}_i$ rotated by an angle θ_j about unit vector $\mathbf{u}_j$, as specified by the quaternion $\mathbf{R}_j$. Source file `eucsbademo.c` accompanying the `sba` package imhttp://users.ics.forth.gr/~argyros/mypapers/2004_08_tr340_forth_sba.pdf

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algorithm[3]. This procedure can be embedded into the LM algorithm of section 2 at the point indicated by the rectangular box in Fig. 1, leading to a sparse bundle adjustment algorithm.

Figure 2: Algorithm for solving the sparse normal equations arising in generic bundle adjustment; see text for details.

Input: The current parameter vector partitioned into m camera parameter vectors $\mathbf{a}_j$ and n 3D point parameter vectors $\mathbf{b}_i$, a function $\mathbf{Q}$ employing the $\mathbf{a}_j$ and $\mathbf{b}_i$ to compute the predicted projections $\hat{\mathbf{x}}_{ij}$ of the i-th point on the j-th image, the observed image point locations $\mathbf{x}_{ij}$ and a damping term μ for LM.
Output: The solution δ to the normal equations involved in LM-based bundle adjustment.
Algorithm:

Compute the derivative matrices $\mathbf{A}_{ij} := \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{a}_j} = \frac{\partial \mathbf{Q}(\mathbf{a}_j,\mathbf{b}_i)}{\partial \mathbf{a}_j}$, $\mathbf{B}_{ij} := \frac{\partial \hat{\mathbf{x}}_{ij}}{\partial \mathbf{b}_i} = \frac{\partial \mathbf{Q}(\mathbf{a}_j,\mathbf{b}_i)}{\partial \mathbf{b}_i}$
and the error vectors $\epsilon_{ij} := \mathbf{x}_{ij} - \hat{\mathbf{x}}_{ij}$,
where i and j assume values in $\{1,\dots,n\}$ and $\{1,\dots,m\}$ respectively.

Compute the following auxiliary variables:

$$\mathbf{U}_j := \sum_i \mathbf{A}_{ij}{}^T \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \mathbf{A}_{ij} \quad \mathbf{V}_i := \sum_j \mathbf{B}_{ij}{}^T \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \mathbf{B}_{ij} \quad \mathbf{W}_{ij} := \mathbf{A}_{ij}{}^T \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \mathbf{B}_{ij}$$

$$\epsilon_{\mathbf{a}_j} := \sum_i \mathbf{A}_{ij}{}^T \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \epsilon_{ij} \qquad \epsilon_{\mathbf{b}_i} := \sum_j \mathbf{B}_{ij}{}^T \mathbf{\Sigma}_{\mathbf{x}_{ij}}^{-1} \epsilon_{ij}$$

Augment $\mathbf{U}_j$ and $\mathbf{V}_i$ by adding μ to their diagonals to yield $\mathbf{U}_j^*$ and $\mathbf{V}_i^*$.

Compute $\mathbf{Y}_{ij} := \mathbf{W}_{ij} \mathbf{V}_i^{*-1}$.

Compute $\delta_{\mathbf{a}}$ from $\mathbf{S}\,(\delta_{\mathbf{a}_1}{}^T, \delta_{\mathbf{a}_2}{}^T, \dots, \delta_{\mathbf{a}_m}{}^T)^T = (\mathbf{e}_1{}^T, \mathbf{e}_2{}^T, \dots, \mathbf{e}_m{}^T)^T$,
where $\mathbf{S}$ is a matrix consisting of $m \times m$ blocks; block jk is defined by
$\mathbf{S}_{jk} = \delta_{jk} \mathbf{U}_j^* - \sum_i \mathbf{Y}_{ij} \mathbf{W}_{ik}{}^T$, where δ_{jk} is Kronecker's delta
and
$\mathbf{e}_j = \epsilon_{\mathbf{a}_j} - \sum_i \mathbf{Y}_{ij}\, \epsilon_{\mathbf{b}_i}$.

Compute each $\delta_{\mathbf{b}_i}$ from the equation $\delta_{\mathbf{b}_i} = \mathbf{V}_i^{*-1} \left(\epsilon_{\mathbf{b}_i} - \sum_j \mathbf{W}_{ij}{}^T\, \delta_{\mathbf{a}_j}\right)$.

Form δ as $(\delta_{\mathbf{a}}{}^T,\ \delta_{\mathbf{b}}{}^T)^T$.

http://users.ics.forth.gr/~argyros/mypapers/2004_08_tr340_forth_sba.pdf

The Design and Implementation of a Generic Sparse Bundle Adjustment Software Package Based on the Levenberg-Marquardt Algorithm†

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Figure 1: Levenberg-Marquardt non-linear least squares algorithm; see text and [16, 20] for details. The reason for enclosing a statement in a rectangular box will be explained in section 3.

$\mathbf{p}_{new} := \mathbf{p} + \delta_{\mathbf{p}};$
$\rho := (||\epsilon_{\mathbf{p}}||^2 - ||\mathbf{x} - f(\mathbf{p}_{new})||^2)/(\delta_{\mathbf{p}}^T(\mu\delta_{\mathbf{p}} + \mathbf{g}));$

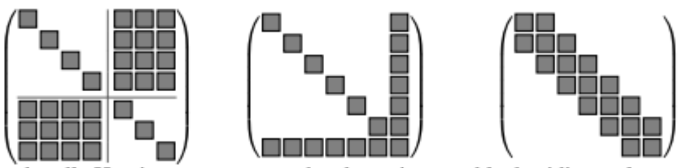
gain ratio

Input: A vector function $f : \mathcal{R}^m \to \mathcal{R}^n$ with $n \geq m$, a measurement vector $\mathbf{x} \in \mathcal{R}^n$ and an initial parameters estimate $\mathbf{p}_0 \in \mathcal{R}^m$.
Output: A vector $\mathbf{p}^+ \in \mathcal{R}^m$ minimizing $||\mathbf{x} - f(\mathbf{p})||^2$.
Algorithm:
$k := 0;\ \nu := 2;\ \mathbf{p} := \mathbf{p}_0;$
$\mathbf{A} := \mathbf{J}^T\mathbf{J};\ \epsilon_{\mathbf{p}} := \mathbf{x} - f(\mathbf{p});\ \mathbf{g} := \mathbf{J}^T\epsilon_{\mathbf{p}};$
stop:=($||\mathbf{g}||_\infty \leq \varepsilon_1$); $\mu := \tau * \max_{i=1,\ldots,m}(A_{ii});$
while (not stop) and ($k < k_{max}$)
    $k := k + 1;$
    repeat
        $\boxed{\text{Solve } (\mathbf{A} + \mu\mathbf{I})\delta_{\mathbf{p}} = \mathbf{g};}$
        if ($||\delta_{\mathbf{p}}|| \leq \varepsilon_2||\mathbf{p}||$)
            stop:=true;
        else
            $\mathbf{p}_{new} := \mathbf{p} + \delta_{\mathbf{p}};$
            $\rho := (||\epsilon_{\mathbf{p}}||^2 - ||\mathbf{x} - f(\mathbf{p}_{new})||^2)/(\delta_{\mathbf{p}}^T(\mu\delta_{\mathbf{p}} + \mathbf{g}));$
            if $\rho > 0$
                $\mathbf{p} = \mathbf{p}_{new};$
                [3n X 1]
                $\mathbf{A} := \mathbf{J}^T\mathbf{J};\ \epsilon_{\mathbf{p}} := \mathbf{x} - f(\mathbf{p});\ \mathbf{g} := \mathbf{J}^T\epsilon_{\mathbf{p}};$
                stop:=($||\mathbf{g}||_\infty \leq \varepsilon_1$);
                $\mu := \mu * \max(\frac{1}{3}, 1 - (2\rho - 1)^3);\ \nu := 2;$
            else
                $\mu := \mu * \nu;\ \nu := 2 * \nu;$
            endif
        endif
    until ($\rho > 0$) or (stop)
endwhile

factoring an arrowhead Hessian matrix to get a reduced camera system or reduced structure system.

Bill Triggs, Philip Mclauchlan, Richard Hartley, Andrew Fitzgibbon.
Bundle Adjustment – A Modern Synthesis. International Workshop on Vision Algorithms, Sep 2000, Corfu, Greece. pp.298–372, 10.1007/3-540-44480-7_21 . inria-00548290

We seek variable orderings that approximately minimize the total operation count or fill-in over the whole elimination chain. For many problems a suitable ordering can be fixed in advance, typically giving one of a few standard pattern matrices such as band or arrowhead matrices, perhaps with such structure at several levels.



bundle Hessian   arrowhead matrix   block tridiagonal matrix   (25)

The most prominent pattern structure in bundle adjustment is the primary subdivision of the Hessian into structure and camera blocks. To get the reduced camera system (19), we treat the Hessian as an arrowhead matrix with a broad final column containing all of the camera parameters. Arrowhead matrices are trivial to factor or reduce by block 2×2 Schur complementation, *c.f.* (16, 19). For bundle problems with many independent images and only a few features, one can also complement on the image parameter block to get a reduced *structure* system.

http://users.ics.forth.gr/~lourakis/sba/PRCV_colloq.pdf

Bundle adjustment gone public
Manolis Lourakis

- A few interesting practical situations violate its underlying assumption regarding the problem's sparsity pattern, rendering it inapplicable. E.g., fixed but unknown intrinsics shared by all cameras
- Example: Hessians corresponding to BA for motion and structure (left) and BA for motion, structure and shared intrinsics (right)

applying local updates rather than global

How to Avoid Singularity For Euler Angle Set?

Puneet Singla*, Daniele Mortari†, and John L. Junkins‡

Since all rotations are performed about the principal axes of the reference frame, we define $\mathbf{M}_i = \exp(-[\tilde{e}_i]\theta_i)$ as an elementary rotation matrix about the $\mathbf{e}_i$-body axis. Here, $[\tilde{e}_i]$ represents the skew-symmetric cross product matrix given by the following expression:

$$[\tilde{\mathbf{a}}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{1}$$

From the expression of $\mathbf{M}_i$, we can construct the following three elementary rotation matrices:

$$\mathbf{M}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

$$\mathbf{M}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3}$$

$$\mathbf{M}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

if i,j, and k, indicate the coordinate axes about which each subsequent rotation is performed, that is, they can be any integer from 1-3, provided that $i \neq j$ and $j \neq k$, are satisfied then the resultant direction cosine matrix can be written as

$$\mathbf{C}_{ijk}(\theta_1, \theta_2, \theta_3) = \mathbf{M}_k(\theta_3)\,\mathbf{M}_j(\theta_2)\,\mathbf{M}_i(\theta_1) \tag{5}$$

applying local updates rather than global

How to Avoid Singularity For Euler Angle Set?

Puneet Singla*, Daniele Mortari†, and John L. Junkins‡

It is a fundamental topological fact that singularities can never be eliminated in any 3-dimensional representation of orientation. But we can avoid this singularity by describing the attitude at a particular instant by the Euler angle set *which is farthest away from singularity.* In this paper, we present an algorithm to switch between different sets of Euler angles to avoid this singularity.

Table 1: Classical Parameterizations of Attitude Rotation Matrix

Parametrization	Dimension	Attitude Matrix	Kinematic Equations	Singularities	Constraints
DCM, (C_{ij})	9	$\mathbf{C} = [C_{ij}]$	$\dot{\mathbf{C}} = -[\tilde{\boldsymbol{\omega}}]\mathbf{C}$	None	$\mathbf{C}^T\mathbf{C} = \mathbf{I}$
Euler Angles EA (θ_i)	3	$\mathbf{C} = \left[\text{transcendental functions of } \theta_i's\right]$	$\dot{\boldsymbol{\theta}} = \left[\text{transcendental functions of } \theta_i's\right]\boldsymbol{\omega}$	$\theta_2 = \pm\frac{\pi}{2}$	None
, Euler-Rodrigues Symmetric Parameters ERSP (q_i)	4	$\mathbf{C} = \left[\text{algebraic functions of } q_i's\right]$	$\dot{\mathbf{q}} = \left[\text{linear functions of } q_i's\right]\begin{Bmatrix} 0 \\ \boldsymbol{\omega} \end{Bmatrix}$	None	$\mathbf{q}^T\mathbf{q} = 1$
RP (r_i)	3	$\mathbf{C} = \left[\text{quadratic functions of } r_i's\right]$	$\dot{\mathbf{r}} = \left[\text{non-linear functions of } r_i's\right]\boldsymbol{\omega}$	$\phi = \pm\pi$	None
MRP (σ_i)	3	$\mathbf{C} = \left[\text{quartic functions of } \sigma_i's\right]$	$\dot{\boldsymbol{\sigma}} = \left[\text{non-linear functions of } \sigma_i's\right]\boldsymbol{\omega}$	$\phi = \pm 2\pi$	None

<u>applying local updates rather than global</u>

How to Avoid Singularity For Euler Angle Set?

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second reason comes out from the capability of the Shuster's Method of Sequential Rotations[4] (MSR) to avoid the *un-avoidable* singularity affecting all the minimum attitude parameter. Not only the original QUEST[4] algorithm has taken advantage from the MSR technique, but also some recent attitude determination approaches, like ESOQ2[5] and OLAE[6].

Snavely's GitHub bundler_sfm/lib/sba-1.5/sba_levmar.c