The Vertical Waist in the Ratio Method Analysis

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Abstract

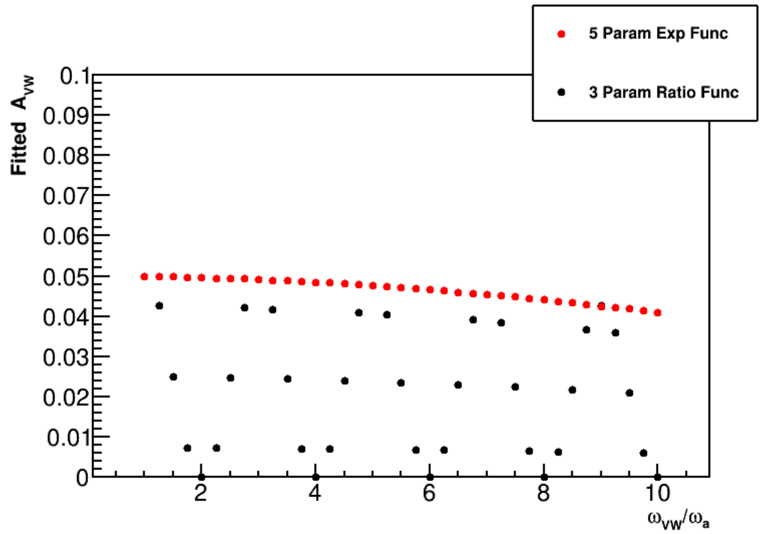
This note details the handling of the vertical waist (VW) effect in the Ratio Method analysis. The 60h and Endgame datasets lie on resonances where $\omega_{VW} \approx 10 \cdot \omega_a$, which in combination with the fast rotation (FR) effect leads to inflated VW amplitudes in the Ratio Method, whereas one might originally assume that the VW effect can be neglected from the analysis. In the 9d dataset (not on a resonance) it was found that the Ratio Method flattened out the VW amplitudes as a function of calorimeter, leading to greater cancellation and a systematically lower calorimeter sum VW amplitude as compared to the T-Method. The solution used to eliminate these problems was to randomize out the VW (in tandem with the FR) in the data so that the effect can be acceptably ommitted from the fit.

1 Introduction

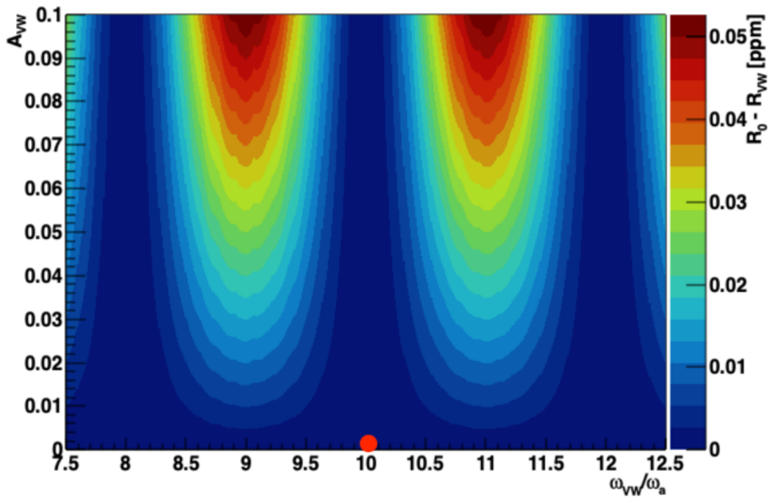
In the Ratio Method, or R-Method, effects with frequencies which are an even multiple of ω_a divide out completely, as shown in Figure 1. The VW frequency is given by

$$\begin{aligned} f_{VW} &= f_c - 2 \cdot f_y, && (1) \\ &= f_c - 2 \cdot \boldsymbol{\kappa_{VW}} \cdot f_{cbo} \sqrt{2 f_c / (\boldsymbol{\kappa_{VW}} \cdot f_{cbo}) - 1}, && (2) \end{aligned}$$

where f_c is the cyclotron frequency, $f_y = \sqrt{n} f_c$ is the vertical betatron frequency with n the quad n value, and f_{cbo} is the CBO frequency. The time-dependence of the VW effect is excluded and the fit parameter is $\boldsymbol{\kappa_{VW}}$, a percent level adjustment factor to the theoretical frequency as found by both the tracking and ω_a analysis [1]. Table 1 gives the VW frequencies for the Run 1 datasets. As shown the VW frequency for both the 60h and Endgame datasets are very close to an even multiple of ω_a. This implies that the VW effect should be divided out in the data, unobservable, and therefore unecessary in the fit function. However, when looking at the FFT of ratio fit residuals in Figure 2, a VW peak can be seen at early times in the 60h dataset. Figure 3 shows a fit start scan for the p value of the fit for the Endgame dataset, where the p value rises rapidly at early times. These two pieces of evidence point to the necessary inclusion of the VW in the ratio fit, even for even multiple frequency datasets.

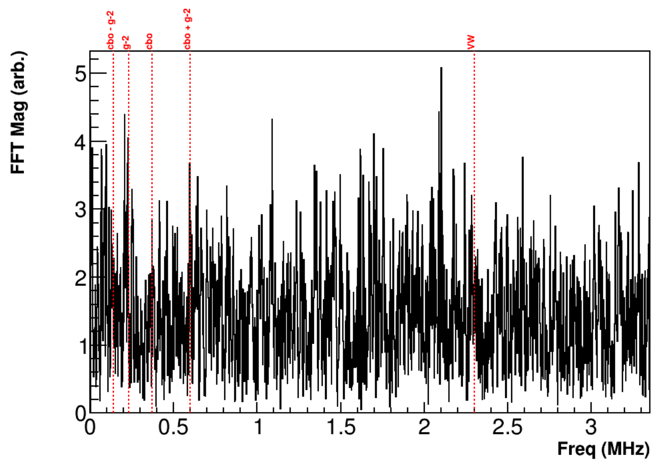


(a) Fitted VW amplitude with a five parameter function in red and a three parameter ratio function in black. The input amplitude was 0.05. The slight fall off of the red points is due to the high frequencies interacting with the bin widths; performing an integral fit removes this trend.

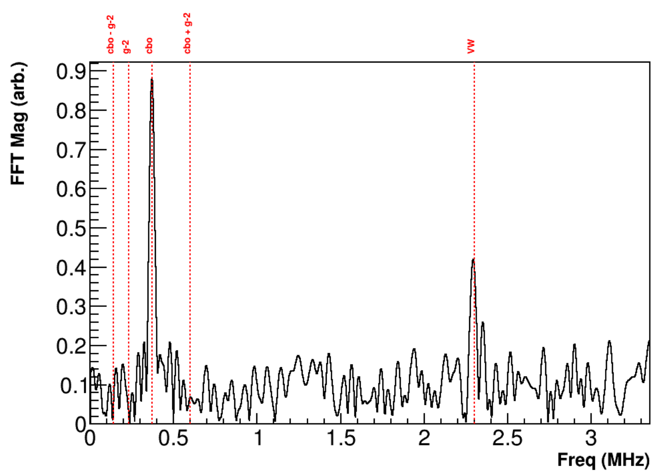


(b) The difference in the maximum value of the ratio (not the fit parameter **R**), as a function of VW frequency and amplitude.The difference in the ratio is approximately 5×10^{-6} at 30 µs.

Figure 1: Two separate Toy MC simulations showing the division of a VW effect as a function of it's frequency in units of ω_a. For even multiple frequencies the effect dies away while for odd multiples it is preserved.



(a) All times, 30.2 µs to 650 µs.



(b) Early times, 30.2 µs to 60.2 µs.

Figure 2: FFT of fit residuals using a 3 parameter ratio function to fit the 60h dataset. Peaks can be seen when looking at fit residuals over early times as opposed to all times. The size of the VW peak greatly depends on the choice of random seed. The CBO peak can also be seen.

Run 1 Dataset VW Frequencies			
Name	n Value	f_{VW} (MHz)	Multiple of ω_a
60h	0.108	2.3	10.04
9d	0.120	2.04	8.87
Endgame	0.108	2.3	10.04

Table 1: n values and VW frequencies for the three Run 1 datasets analyzed for this note. (The Highkick dataset has the same parameters as the 9d.) While technically the VW frequency is changing, the frequencies given here are those determined from the peak of the VW peak in the FFT of the residuals from a five parameter fit to the data, so the numbers are close enough.

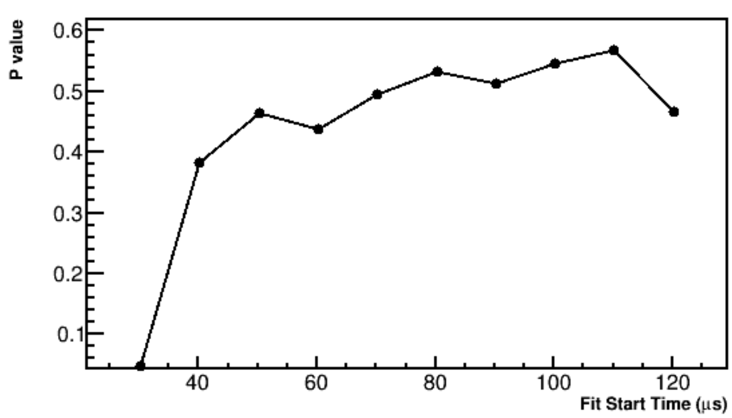


Figure 3: P value vs fit start time for the Endgame dataset. A sharp rise can be seen at early times. A similar trend, though less severe, can be seen for the 60h dataset.

2 On a resonance in the 60h and Endgame datasets

When the VW is included in the ratio fit for the 60h and Endgame results, the resulting fitted VW amplitude is signifcantly larger than that in the T-Method, $A_{VW-R} \sim 0.2 \pm 0.04$ vs $A_{VW-T} \sim 0.003 \pm 0.003$ vs. See Figure 4. This amplitude rises to values 10 to 100 times larger than that in the T-Method (depending on random seed) to levels which would naively seem to be directly observable by eye in the data. Interestingly enough, it is only when VW amplitude in the ratio fit is allowed to float that the VW FFT peak disappears and the fit start scans are repaired. The fit does truly seem to prefer such a large value, however this was an anomaly which needed to be understood.

One possibility considered was whether the ratio fit artificially raises the VW amplitude, and whether a different fit function should be used. Fits were done with the three parameter ratio function,

$$R(t) \approx A\cos(\omega_a t), \tag{3}$$

as well as the full fit function,

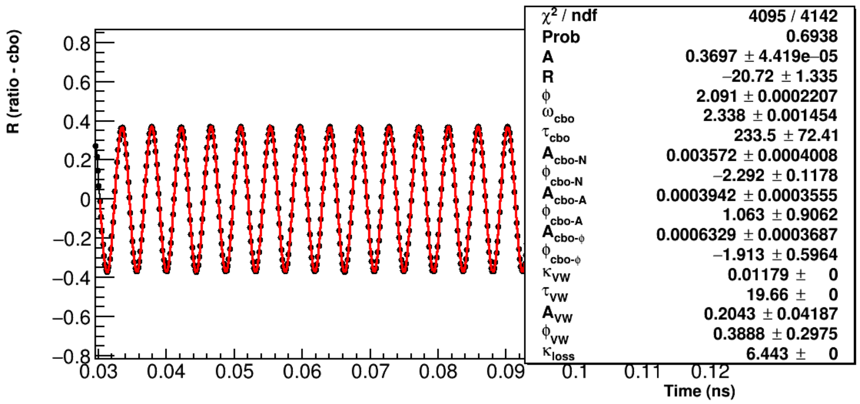
$$R(t) = \frac{2f(t) - f_+(t) - f_-(t)}{2f(t) + f_+(t) + f_-(t)}, \tag{4}$$

$$f_\pm(t) = f(t \pm T_a/2), \tag{5}$$

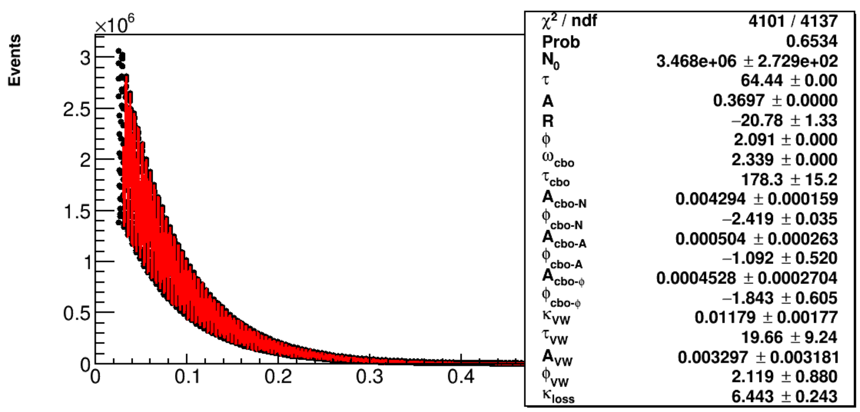
$$f(t) = V(t) \cdot (1 + A \cdot \cos(\omega_a t + \phi)), \tag{6}$$

where $V(t)$ is the vertical waist effect. In the latter, $f(t)$ represents the same function used in a 5 parameter or T-Method fit function, barring the N_0 and τ_μ terms which divide out. Fitting the Toy MC simulated data with these functions yielded the results shown in Figure 5. For the three parameter function fits, it can be seen that the VW effect dies away as the frequency approaches 10 times ω_a. For the full ratio function results however, it can be seen that the fitted VW amplitudes start to vary with large errors, typically consistent with zero. This effect makes sense, as while the VW effect has been removed from the ratio data, but the fit parameter gets divided out in the fit function, and so can be any value while still giving the same goodness of fit. The instability of this fit parameter (and it's demotivation), was originally thought to be the source of the large amplitudes in the fit to the data. However, regardless of the way the dataset was sliced, and regardless of whether the 60h or Endgame was fit, the fits always preferred a large amplitude with a relatively small error.

Therefore a more data-like Toy MC simulation was built which included the FR effect and separate calorimeters, with the ability to turn off various parts of the simulation. It was found that when the FR was turned off the ratio data was consistent with 0, while when the FR was turned on, the VW effect reappeared with a larger amplitude and a strange beating structure, Figure 6. A simpler Mathematica simulation revealed the same behavior with the same beating structure, Figure 7. So something in the way that the R-Method was interacting with the FR was causing these large

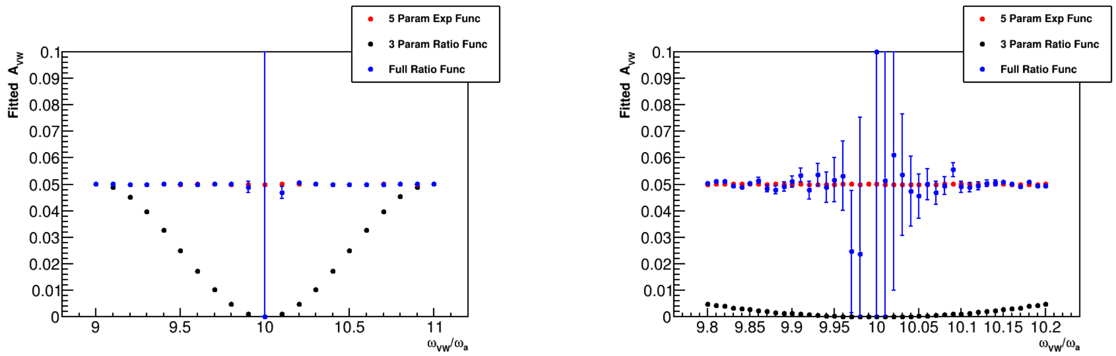


(a) R-Method fit results with the VW included.



(b) T-Method fit results.

Figure 4: Fits to the 60h dataset with the VW included in the ratio fit. The VW frequency and lifetime in the ratio fit are fixed to those determined from the T-Method fit, as it can struggle with those parameters, while the amplitude and phase are allowed to float. Comparing the VW fitted amplitudes, the amplitude in the ratio fit is significantly larger.

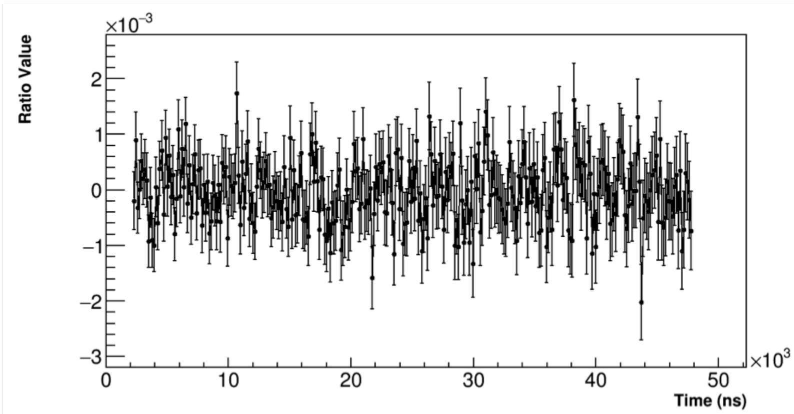


(a) 9× to 11× ω_a.

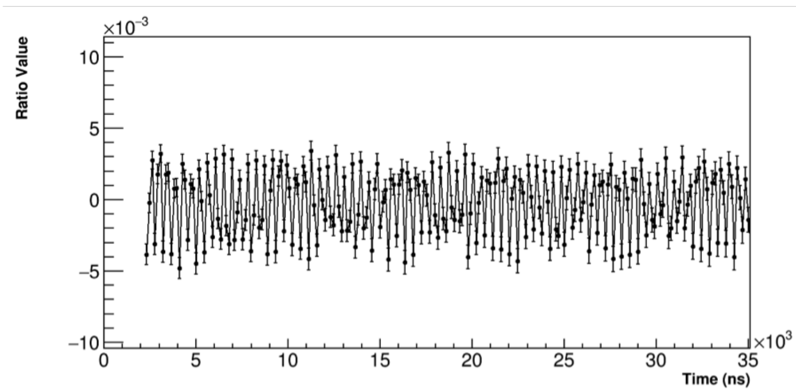
(b) 9.8× to 10.2× ω_a.

Figure 5: Fitted VW amplitude as a function of frequency and with different types of fit functions.

amplitudes. Performing the R-Method analysis, but this time with varying time-shifts, revealed the resonance very clearly in both the data and the Mathematica simulation, Figure 8. As shown the running point for 60h and Endgame datasets sits on this resonance, stemming from the fact that $f_{VW} \approx 10 \cdot f_a$. This begs the questions as to what exactly is the source of this resonance, and how exactly does the R-Method introduce it in combination with the FR.

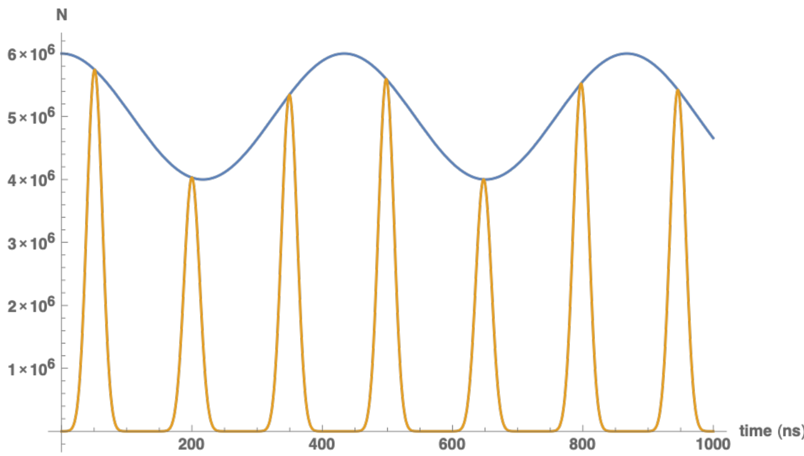


(a) Without the FR effect included.

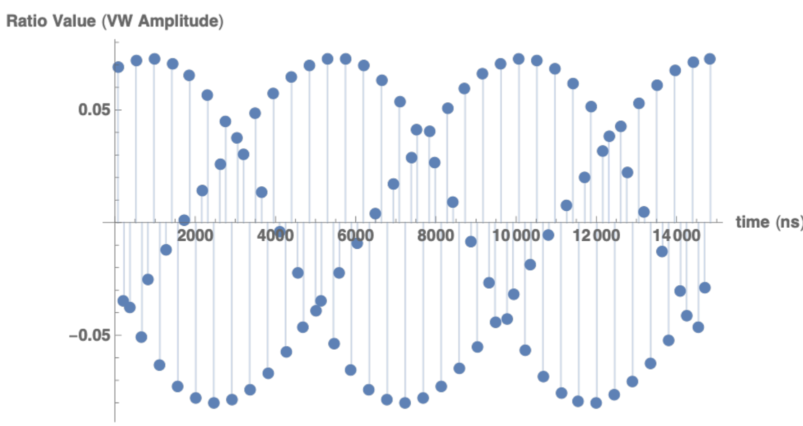


(b) With the FR effect included.

Figure 6: Ratio data with and without the FR effect from a Toy MC simulation, with a VW effect with a frequency $f_{VW} = 10 \cdot \omega_a$. The $g-2$ wiggle has been removed, and the lifetime of the VW was set to a large number. The top plot shows ratio data which is consistent with 0 after all effects have been removed and the VW has divided out. The bottom plot shows ratio data inconsistent with 0, with oscillations at the VW frequency, and an interesting beating structure.

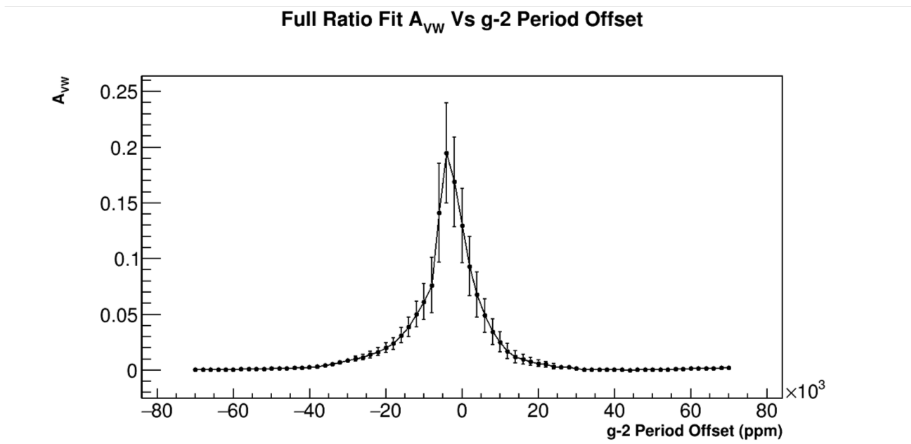


(a) The input functions used, blue for the VW and orange for a FR effect.

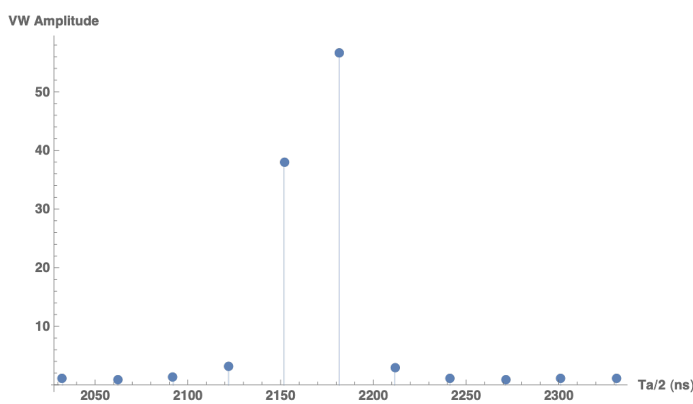


(b) The ratio value with the FR effect include. Oscillations with large amplitudes at the VW frequency can be seen with the same strange beating structure.

Figure 7: Results from a simple Mathematica simulation with a VW and FR effect included.



(a) Fitted VW amplitude as a function of the choice of $g-2$ period offset in units of thousands of ppm for the 60h dataset.



(b) Fitted VW amplitude as a function of the choice of $g-2$ period in the Mathematica simulation.

Figure 8: The amplitude of the fitted VW, in both the data (top) and Mathematic simulation (bottom), as a function of the time-shift or $g-2$ period over 2, $T_a/2$. In both the resonace can be clearly seen where the VW amplitude blows up from it's real value.

With yet another Toy MC, the resonances in the VW amplitude can be seen in Figure 9 as a function of the time shift over the VW period, Δ/T_{VW}. As shown the resonances appear at integer steps in Δ/T_{VW}. Let's determine why these resonances appear by looking at the ratio fit function more carefully as a function of the time shift. The explicit ratio function is given as [2]

$$R(t) = \frac{2A\cos(\omega_a t)(1 - \cos(\omega_a \Delta))}{4 + 2A\cos(\omega_a t)(1 + \cos(\omega_a \Delta))}, \tag{7}$$

where Δ is the time shift, typically

$$\Delta \approx T_a/2. \tag{8}$$

In the Toy MC the ω_a oscillation was removed and replaced with a VW oscillation, so the ratio function instead goes as

$$R(t) = \frac{2A\cos(\omega_{VW} t)(1 - \cos(\omega_{VW} \Delta))}{4 + 2A\cos(\omega_{VW} t)(1 + \cos(\omega_{VW} \Delta))}, \tag{9}$$

where ω_a has simply been replaced by ω_{VW}. Looking at the numerator of this function, it can be seen that the numerator goes to zero when

$$\Delta = 2n\pi/\omega_{VW} = n \cdot T_{VW}, \tag{10}$$

at integer steps in Δ/T_{VW}. This can be seen in Figure 10a. When looking at the real simulated data however, including the effects of the FR, as shown in Figure 10b, it can be seen that the real ratio data does not in fact go to zero. This is the root cause for the VW amplitude resonances. Because the ratio function is going to zero at points where the real data is non-zero, the VW amplitude in the fit function explodes to compensate for the wrong fit function form. The 60h and Endgame datasets sit very near the point $\Delta = T_a/2 \approx 5 \cdot T_{VW}$, or $\omega_{VW} \approx 10 \cdot \omega_a$, and thus on one of these resonances.

Note that the large resonant amplitude does not mean that there is a correspondingly large wiggle in the data, as it will divide out in the full ratio function. It simply produces greater agreement between the data and the fit function. Technically the VW envelope used in the fit function is wrong for fitting the data and the Toy MC, however the VW effect is small enough in the data and the frequency is close enough to an even multiple of ω_a that fits are still good. To solve this resonance and wrong envelope issue, either the real envelope should be incorporated in the fit, or the VW should be eliminated. The former is not so straightforward, as the envelope shown in Figure 6 is complicated, though it is probably doable. The latter approach was chosen, at the disadvantage of a small hit to the statistical error, as will be described in Section 4.

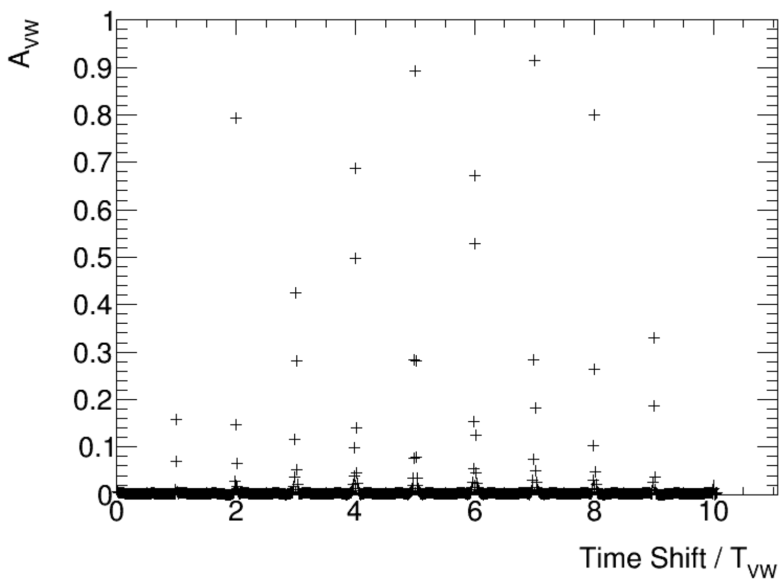
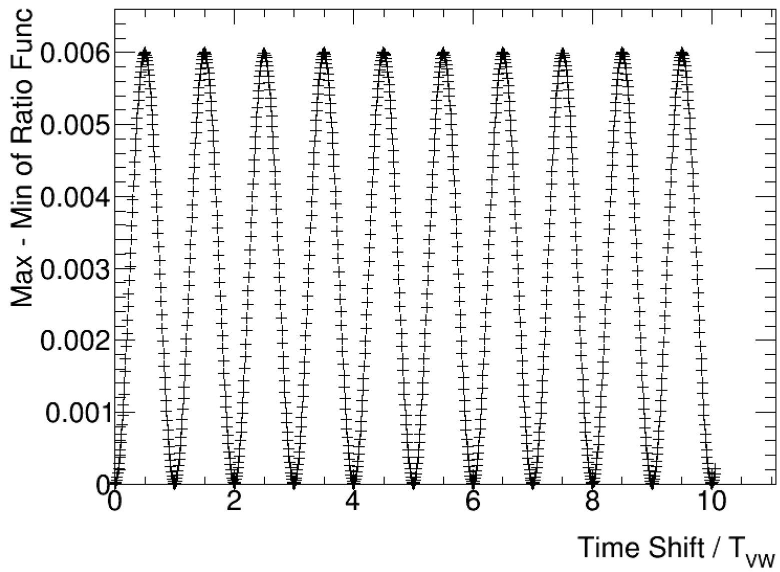
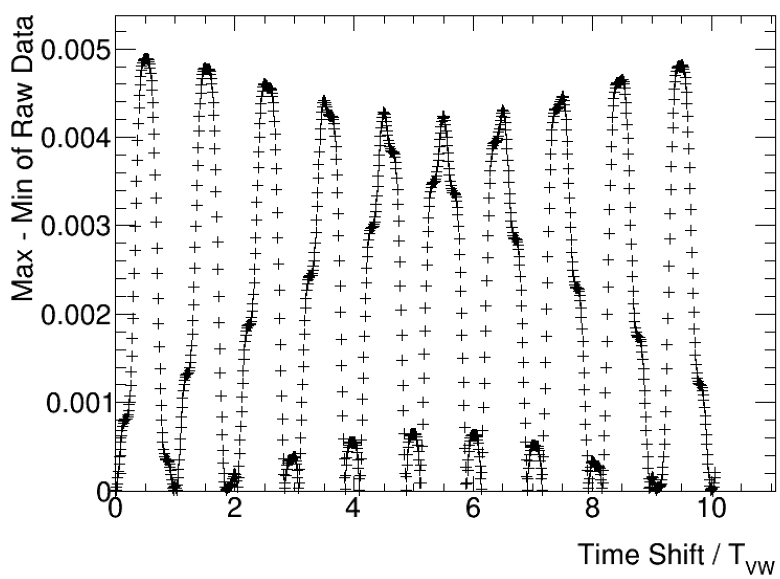


Figure 9: VW amplitude resonances as a function of the time shift in the R-Method divided by the VW period, T_{VW}.



(a) At integer steps in the time shift over the VW period the amplitude of the ratio fit function goes to zero.



(b) At integer steps in the time shift over the VW period the amplitude of the ratio data does not go to zero, due to the existence of the FR effect.

Figure 10: The maximum amplitude of the ratio function (top) and Toy MC ratio data (bottom), as a function of the time shift divided by the VW period.

3 Reduction in calorimeter sum VW amplitude in the 9d Dataset

While in the 60h and Endgame datasets the VW frequency is nearly 10 times ω_a leading to the afore-discussed resonance, the 9d dataset VW frequency is closer to an odd multiple of ω_a. This implies that the fitted VW amplitude should be the same between the T-Method and R-Method results. When looking at the VW amplitudes in the R-Method however, I see a systematically smaller amplitude compared to the T-Method, as shown in Figure 11. When looking at per calorimeter fits there is no immediately obvious difference in the results, as shown in Figure 12. When removing the time randomization of the cyclotron period however (used by default in order to reduce the FR effect in the data, in the T-Method fit as well as the R-Method fit), then a very clear difference in both the VW amplitudes and phases can be seen as shown in Figure 13. Going back to the Toy MC with the separate calorimeters, something similar can be seen, Figure 14. While the phase difference plots between the two show something different, the amplitude difference plots show a slightly similar shape. In both cases, the R-Method fits tend to flatten out the per calorimeter VW amplitudes as a function of calorimeter. Since the VW phases in general run from 0 to 2π, the VW amplitude of the calorimeter sum data is less than any single calorimeter. The flattening out of the VW amplitudes in the R-Method as a function of calorimeter then means that an even greater cancellation is done, leading to the systematically smaller amplitude.

Currently the best guess for the source of this flattening out is that

-source of this flattening out exactly??

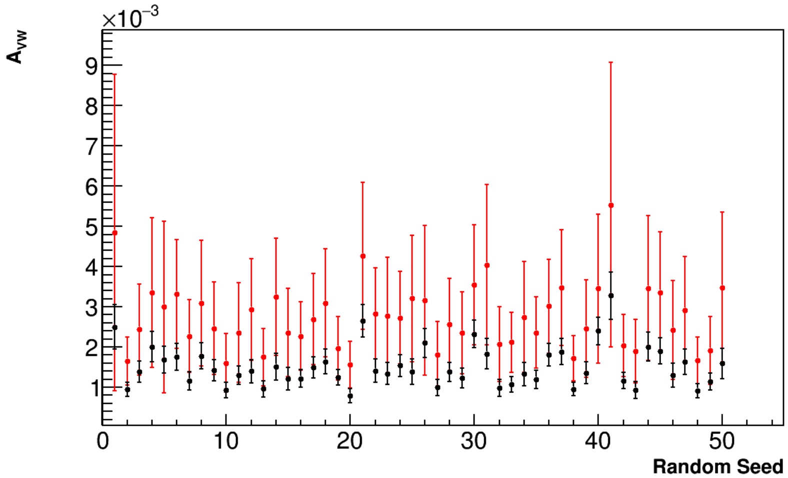
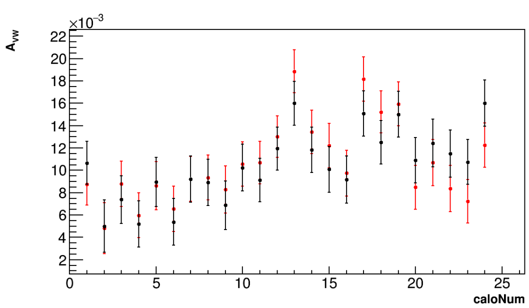
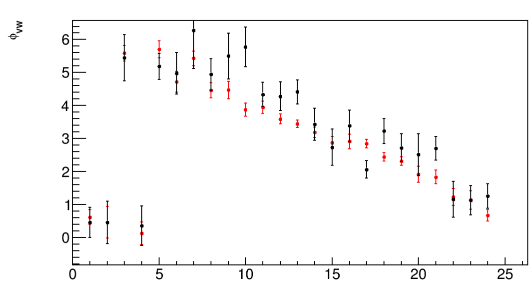


Figure 11: VW amplitudes as a function of random seed for T-Method fits compared to R-Method fits. T-Method fits are in red and ratio fits are in black. The error bars are smaller on the ratio fits because the VW frequencies and lifetimes are fixed to those from the T-Method fits. There is a consistently smaller VW amplitude in the ratio fits as compared to the T-Method fits.

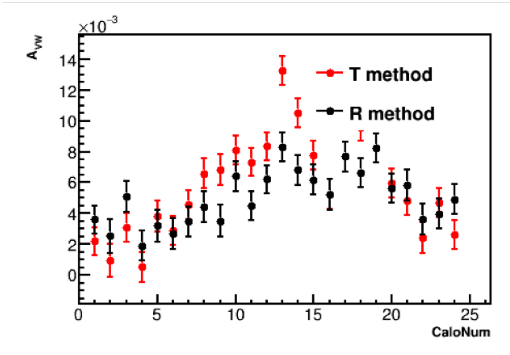


(a) VW amplitudes.

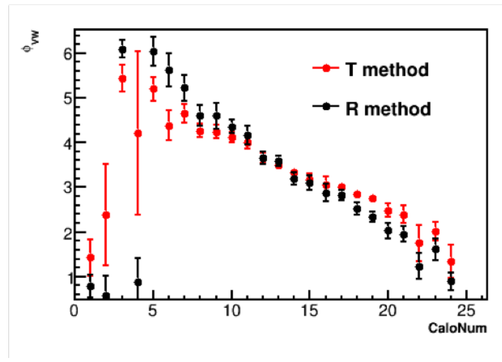


(b) VW phases.

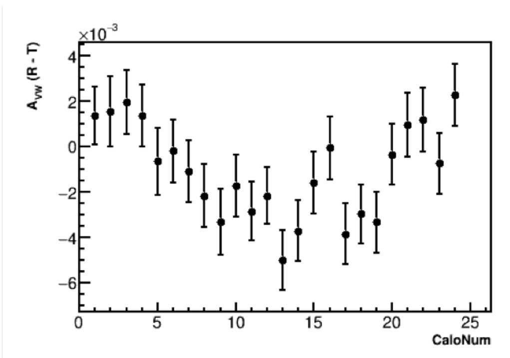
Figure 12: VW amplitudes (left) and phases (right) per calorimeter in the 9d dataset with T_c randomization included. Red points are T-Method results and black points are R-Method results. In the ratio fits, the VW frequencies and lifetimes are fixed to those from the T-Method results.



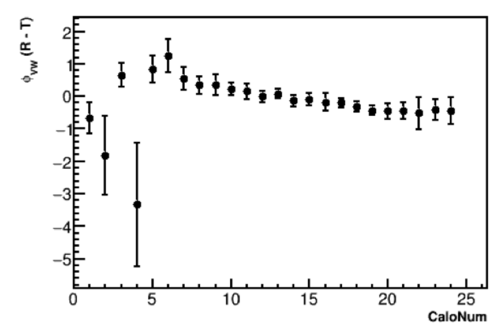
(a) VW amplitudes.



(b) VW Phases

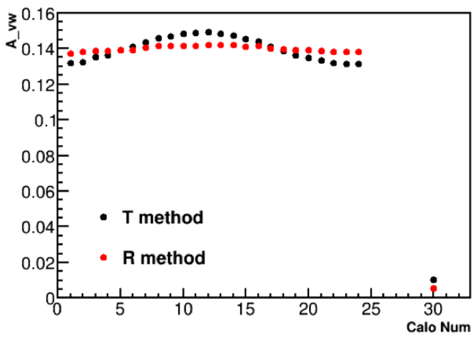


(c) VW amplitude differences.

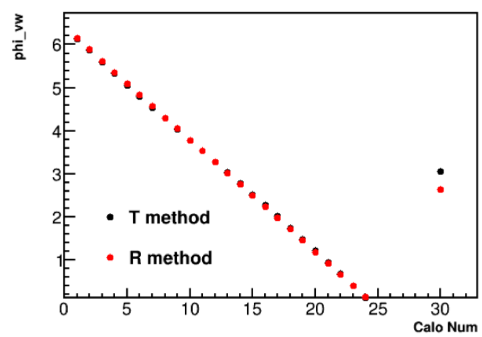


(d) VW phase differences.

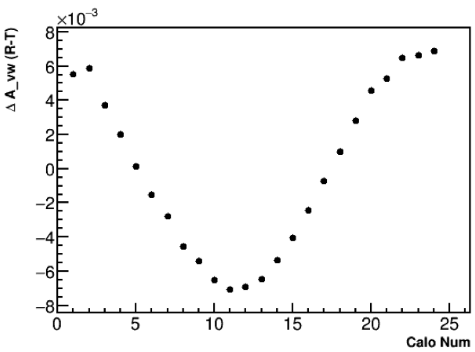
Figure 13: 9d dataset per calorimeter VW amplitudes (top left) and phases (top right) for T-Method and R-Method fits, and their differences (bottom). Both the amplitudes and phases are systematically different between the two fit types, which is more readily apparent in the difference plots.



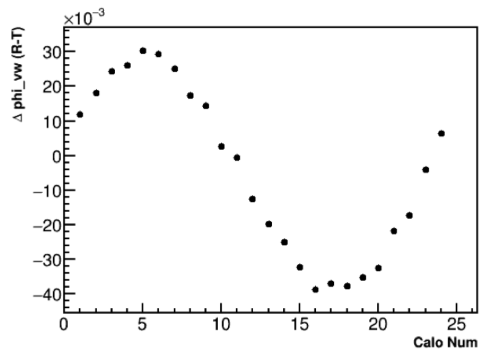
(a) VW amplitudes.



(b) VW phases.



(c) VW amplitude differences.



(d) VW phase differences.

Figure 14: Toy MC per calorimeter VW amplitudes (top left) and phases (top right) for T-Method and R-Method fits, and their differences (bottom). In all cases the points at calorimter number 30 represent the sum of all calorimeter data. Notice the flattened out red points in the amplitude plot, leading to the lower red point for the sum of calorimeter data.

Change in Asymmetry due to Randomization				
Dataset	A no randomzation	A with randomization	ΔA	$\Delta\sigma_R$ (ppb)
60H	0.3697	0.3637	-0.0060	22.7
9d	0.3714	0.3639	-0.0075	18.1
Endgame	0.3714	0.3639	-0.0061	10.7

Table 2: Asymmetry values in the Run 1 datasets with and without the VW randomization, and the corresponding change in the statistical error on R. An energy cut of 1700 MeV was applied to the data.

4 Randomizing out the VW

The standard technique to remove the FR from the data as much as possible, is to not only set the bin width of the histograms as close to T_c as possible, at 149.2 ns, but the times of the hits are randomized by $\pm T_c/2$. In order to remove the effects of the VW described in this document, the same strategy is used, where the times are also randomized by $\pm T_{VW}/2$. In doing so, the VW is effectively removed from the data as shown in Figure 15. Any attempts to fit VW parameters are unsuccessful as the VW has been successfully removed from the data. Fit start scans are also repaired as in Figure 16. The randomization of the data by $\pm T_{VW}/2$ decreases the asymmetry A of the $g-2$ wiggle by 0.006 in the 60h and Endgame datasets, and 0.0075 in the 9d dataset, corresponding to increases in the statistical error on R by $10.7 - 22.7$ ppb, where the increase can be determined as[1]

$$\sigma_R \to \frac{A_{\text{randomized}}}{A}\sigma_R. \tag{11}$$

See Table 2. For the Run 1 datasets this statistical error increase was considered negligible and thus acceptable. For the forthcoming datasets it may be desirable to find an alternative solution. Attempts to randomize out the VW by randomizing out the $2f_y$ component from Equation 1 were made (before randomizing out VW directly in fact) [3]. The statistical error increase was reduced to 6 ppb and the VW was eliminated from the 60h and Endgame datasets, however there was a residual VW signal in the 9d dataset. For this reason this solution was discounted in favor of randomizing out the VW directly.

For final checks...

References

[1] J. Mott et al. *CBO Frequency Change*. Muon g-2 DocDB 14208. 2018.

[2] N. Kinnaird. *Analysis Note for 60H Dataset Relative Unblinding*. Muon $g-2$ Note 172 DocDB 16270. 2019.

[1]The statistical error increase does not always exactly correspond to the change in A, but it's close, presumably do a change in statistics and fit parameter correlations.

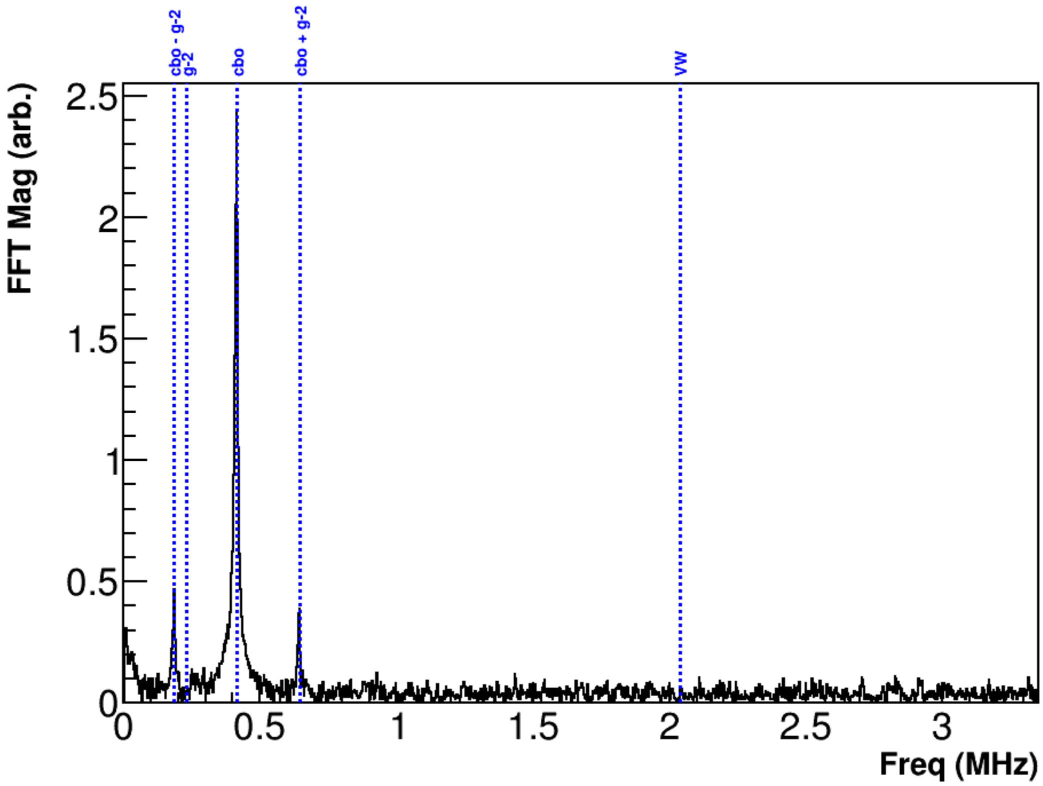


Figure 15: FFT of residuals to a five parameter fit of the 9d dataset, with the VW randomization included. There is no observable VW signal remaining in the dataset.

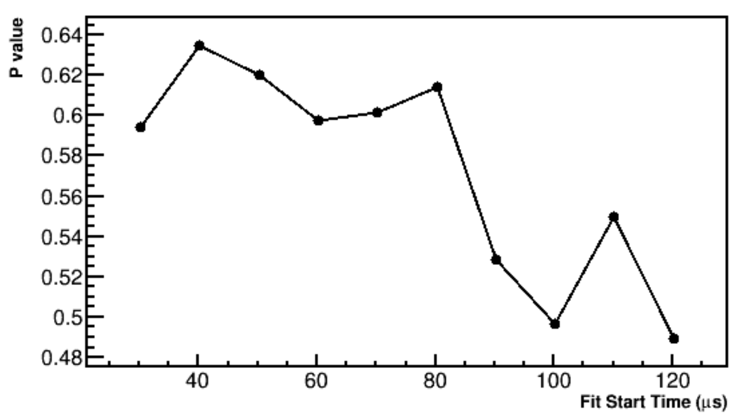


Figure 16: P value fit start scan for the Endgame dataset including the VW randomization. There is no longer a sharp rise at early times.

R Comparison with and without Randomization						
		With VW No Randomization		Without VW With Randomization		
Dataset	Fit Method	Mean	RMS	Mean	RMS	ΔR
60H	T	−20.345	0.143	−20.294	0.298	0.051
	R	−20.527	0.216	−20.480	0.357	0.047
9d	T	−17.278	0.065	−17.280	0.126	−0.002
	R	−17.355	0.087	−17.374	0.133	−0.019
Endgame	T	−17.743	0.096	−17.684	0.186	0.059
	R	−17.747	0.125	−17.688	0.192	0.059

Table 3: Means and RMS' of R values for 50 different random seeds for three of the Run 1 datasets, without the T_{VW} randomization and including the VW in the fit, versus with the T_{VW} randomization and the VW terms excluded from the fit.

[3] N. Kinnaird. *Vertical Waist and Fast Rotation in the Ratio Method*. Muon g-2 DocDB 19210. 2019.