MATH 4600: ADVANCED CALCULUS
Spring 2019

TEST I

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NOTES

1. Please make sure that your answer book has 8 pages. The worksheets at the end are extra pages should you need them. If you continue a problem onto a worksheet, indicate the page number of the worksheet at the point of continuation.
2. Attempt all four problems; these are equally weighted.
3. **Read the questions carefully before answering.**
4. If you would like full credit, then **justify your answers with appropriate, but brief, reasoning.**
5. Books, notes, crib sheets and calculators are not to be used.
6. **Put your mobile devices away.**
7. Best wishes.

1	11
2	18
3	14
4	18
TOTAL	

61

1. Consider the planes

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$$\begin{aligned} 3x + 2y - z - 3 &= 0, \\ x - z - 1 &= 0. \end{aligned}$$

(a) Find a parametric equation for L, the line of intersection of the planes.

(b) Find the coordinates of P, the point of intersection of L and the surface $S : z = x^2 - y^2$.

(c) Find an equation for the tangent plane to the surface S at P.

{Note: Sorry, my hands are cold; will warm up on later pages.}

ok.

Problem #1 : Surfaces & Vectors in $\mathbb{R}^3$

(a) Parametric Equation for "L" : Intersection of two planes.

$S_1 : 3x + 2y - z - 3 = 0$

$S_2 : x - z - 1 = 0$

$3x + 2y - z - 3 = x - z + 1 \to 2x + 2y - z + \cancel{z} - 3 + 1 = 0$

$(ax + by + cz = 0)$ but $c = 0$

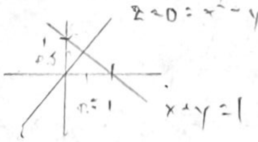
$L : 2x + 2y - 2 = 0 \to \underline{x + y = 1} \to$ Need to parametrize in $\mathbb{R}^3$ $\begin{cases} x(t) = t \\ y(x) = 1 - x(t) \end{cases}$ & $z(t) = 0$

• Made mistake going into $\mathbb{R}^2$. Erased & second attempt; Hands still cold.

$\boxed{L = \langle t, 1-t, 0 \rangle}$ or $\underline{(0, 1, 0) + t\langle 1, -1, 0 \rangle}$ ✗ −6

(b) Coordinates of point P

$z = 0 = x^2 - y^2 \to y^2 = x^2 \to y = x$



{Haha, now gotta prove it's (0.5, 0.5)}

• Graph in plane xy & $z = 0$:

$\cdot \left[t\langle 1, -1, 0 \rangle + (0, 1, 0) \right] = (x, y, z) = \boxed{(0.5, 0.5, 0)}$

(c) Equation for tangent plane @ P of S

$\nabla S : \langle 2x, -2y, \frac{\partial(-z)}{\partial z} = -1 \rangle = \langle 2x, -2y, -1 \rangle |_P$

$L_{tangent} : \vec{P} + (\nabla S) t = \boxed{\frac{1}{2}(1, 1, 0) + t\langle 1, -1, -1 \rangle}$

✗ −8

* So, so cold...

18

2. Let $\boldsymbol{F}(u,v) : \mathcal{R}^2 \to \mathcal{R}^3$ be defined by (a)

Index Reference

$$\boldsymbol{F} = \begin{bmatrix} uv^2 \\ v^3 - u \\ v \sin u \end{bmatrix}, \quad \begin{matrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{matrix} \qquad \begin{matrix} F: 2 \to i: 2 \\ G: 3 \to j: 3 \end{matrix}$$

and $\boldsymbol{G}(x,y,z) : \mathcal{R}^3 \to \mathcal{R}^2$ by (c)

$$\boldsymbol{G} = \begin{bmatrix} yze^x \\ y^3 \cos(xz) \end{bmatrix}. \quad \begin{matrix} u(x,y,z) \\ v(x,y,z) \end{matrix}$$

Perfect Connection

(a) Find $D(\boldsymbol{F} \circ \boldsymbol{G})(x,y,z)$ at $(x,y,z) = \cancel{(0,1,10)}$ → $(0,1,1) : (1,1,1)$

(b) Find $D(\boldsymbol{G} \circ \boldsymbol{F})(u,v)$ at $(u,v) = (0,1)$. $(0,1) : (1,-1,0)$

Problem # 2

$$F_{i,j} = \begin{bmatrix} v^2 & 2uv \\ -1 & 3v^2 \\ v\cos u & \sin u \end{bmatrix} \checkmark \; ; \; G_{j,i} = \begin{bmatrix} e^x yz & ze^x & ye^x \\ -zy^3 \sin(xz) & 3y^2\cos(xz) & -y^3 \sin(xz) \end{bmatrix}$$ Jacobian Matrix

$$D(\underset{\sim}{F} \circ \underset{\sim}{G}) = D\underset{\sim}{F} \cdot D\underset{\sim}{G} =$$

Only these keys

$$\begin{bmatrix} \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} & \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} \\ \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} & \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} & \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} \\ \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} & \frac{\partial F}{\partial v}\frac{\partial v}{\partial z} & \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{(0,1,1)} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 1 & \sin(1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 7 & 1 \\ -1 & 8 & -1 \\ 1 & 1+3\sin(1) & 1 \end{bmatrix}}$$

?

-7

$$D(\underset{\sim}{G} \circ \underset{\sim}{F}) = D\underset{\sim}{G} \cdot D\underset{\sim}{F}$$

$$\begin{bmatrix} \frac{\partial G}{\partial x}\frac{\partial x}{\partial u} & \frac{\partial G}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial G}{\partial x}\frac{\partial x}{\partial u} & \frac{\partial G}{\partial y}\frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(0,1)} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 1 & \sin \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2+3\sin(1) \\ -3 & 3 \end{bmatrix}}$$

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$\int u\,dv = uv - \int v\,du$ 14

$e^x \cos y$

3. (a) Determine if

$$\boldsymbol{F} = \langle e^x \cos y + e^{-x} \sin z,\ -e^x \sin y + yz^2,\ -e^{-x} \cos z + y^2 z + e^z \rangle$$

is a gradient field, and if so, find its potential.

(b) Show that for a general, twice differentiable vector field $\boldsymbol{F}$, divergence of the curl of $\boldsymbol{F}$ is zero.

(c) Suppose that a particle of mass m travels along a path $\boldsymbol{r}(t)$ according to Newton's law,

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F},$$

where $\boldsymbol{v} = d\boldsymbol{r}/dt$ is the velocity of the particle and the force $\boldsymbol{F}$ is a gradient vector field. If the particle is also constrained to lie on an equipotential surface of $\boldsymbol{F}$, then show that it must have a constant speed.

Note: parts (a), (b) and (c) are unrelated.

Problem # 3a

a) Gradient Field: Appropriate if some "$\nabla f = \underset{\sim}{F}$"

- $F_x = e^x \cos y + e^{-x} \sin z \to f(x,y,z) = A(y,z) + \cos y\, e^x + (-x) e^{-x} \sin z$ (#1) (#2)
- $F_y = -e^x \sin y + yz^2 \to f(x,y,z) = B(x,z) + (-e^x)(-\cos y) + \frac{1}{2}(yz)^2$ (#3)
- $F_z = (-e^{-x}) \cos z + y^2 z + e^z \to f(x,y,z) = C(x,y) + (-e^{-x}) \sin z + \frac{1}{2}(yz)^2 + e^z + G(z)$

By inspection, f can be constructed

$$\boxed{f(x,y,z) = e^x \cos y + (1+x) e^{-x} \sin z + \frac{(yz)^2}{2} + K, \quad K = \text{Constant}}$$ -4

b) Show that $\operatorname{div}(\operatorname{curl}(\underset{\sim}{F})) = 0$: ① $= \partial_i \epsilon_{ijk} \partial_j F_k = \epsilon_{ijk} \partial_i \partial_j F_k = \epsilon_{jik} \partial_j \partial_i F_k$ ② "$i \to j$; $j \to i$" ③

Mixed Partials

② $\Rightarrow \epsilon_{ijk} \partial_i \partial_j F_k = \epsilon_{jik} \partial_j \partial_i F_k = \epsilon_{jik} \partial_i \partial_j F_k$, odd permutation

$$\boxed{= -\epsilon_{ijk} \partial_i \partial_j F_k \overset{?}{=} 0, \quad \text{QED}}$$

You need to say a bit more -

c) Newton's 2nd Law: $m\frac{dv}{dt} = \underset{\sim}{F} \to m v_{i;i} = F_i \to m \dot{v} = F_i \to$ if $\nabla F = 0$ for an equipotential field, like gravity at constant elevation $m\frac{dv}{dt} = F \to m \frac{d}{dt}(v_0 + t\dot{v}) = \underset{\sim}{F} = 0$

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$m\frac{d(\dot{v}t)}{dt} = m\dot{v} = 0$, $\boxed{\dot{v} = 0, \text{QED}}$

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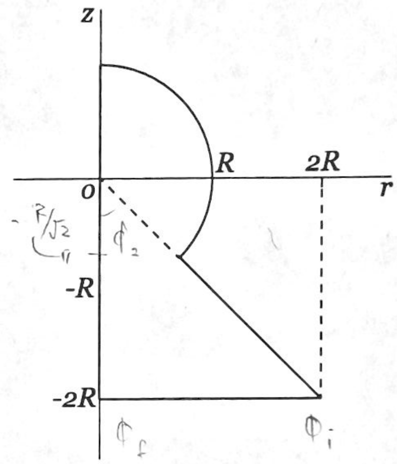
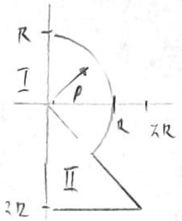
4. **The plane shape shown in solid lines in the figure below consists of a triangle inserted into a circle.** When rotated about its vertical axis of symmetry it generates a solid in the shape of a cone thrust into a sphere. Describe the solid by means of appropriate inequalities in (i) spherical coordinates (ρ, θ, ϕ) and (ii) cylindrical coordinates (r, θ, z). Use the point O as the origin. Be sure to provide three inequalities, one for every coordinate, in both cases. Feel free to divide the solid region into subregions, if need be.



Problem #4: Coordinate Systems

(i) Spherical Coordinates (ρ, θ, ϕ)



Region I: $\rho : [0, R]$
$\theta : [0, 2\pi)$
$\phi : [0, 3\pi/4)$

Spherical Shape

Region II: $\rho : [0, f(\phi)]$ indep. of θ
$\theta : [0, 2\pi)$
$\phi = [0, f^{-1}(\rho)]$

Need Relationship b/w ρ & ϕ, $\rho(\phi = \pi) = 2R$ & $\rho(\phi = 3\pi/4) = 2\sqrt{2}R$ ∴ Linear!

so that $\rho(\phi) =$

$\rho : [0, R\cos\cancel{\phi}]$
$\theta : [0, 2\pi)$
$\phi : [3\pi/4, \pi]$

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(ii) Cylindrical Coordinates: (r, θ, z)

-2

Region I: $r : [\cancel{0}, \sqrt{R^2 - z^2}]$
$\theta : (0, 2\pi]$
$z : [-R/\sqrt{2}, R)$

Region II: $r : [0, z]$
$\theta : [0, 2\pi)$
$z : (-R/\sqrt{2}, -2R]$