MATH 4600: ADVANCED CALCULUS
Fall 2017

TEST I SOLUTIONS

NAME (Please print) ____________________________

NOTES

1. Please make sure that your answer book has 8 pages. The worksheets at the end are extra pages should you need them.
2. Attempt all four problems; these are equally weighted.
3. **Read the questions carefully before answering**.
4. If you would like full credit, then **justify your answers with appropriate, but brief, reasoning**.
5. Books, notes, crib sheets and calculators are not to be used.
6. Put your mobile devices away.
7. Best wishes.

1	
2	
3	
4	
TOTAL	

1. (a) Show that the curve described parametrically by

$$\boldsymbol{r}(t) = \left\langle \cos(t-1), t^3 - 1, \frac{1}{t} - 2 \right\rangle$$

is tangent to the graph of the surface $x^3 + y^3 + z^3 - xyz = 0$ when $t = 1$. Find the tangent plane to the surface at the point of contact with the curve.

(b) Consider two skew (non-parallel, non-intersecting) lines L_1 and L_2. Let $\boldsymbol{r}_1$ be the position vector of a point on L_1 and $\boldsymbol{v}_1$ a vector along the direction of L_1. Similarly, let $\boldsymbol{r}_2$ be the position vector of a point on L_2 and $\boldsymbol{v}_2$ a vector along the direction of L_2. Find an expression for the shortest distance between L_1 and L_2 in terms of the quantities given above. Be sure to offer adequate justification.

(a) The point on the curve corresponding to $t = 1$ is $P(1, 0, -1)$. The general tangent vector to the curve is

$$\boldsymbol{r}'(t) = \left\langle -\sin(t-1), 3t^2, -\frac{1}{t^2} \right\rangle.$$

When specialized to $P(t = 1)$, the tangent vector becomes

$$\boldsymbol{r}'(1) =< 0, 3 - 1 > .$$

The normal to the surface $F(x, y, z) = x^3 + y^3 + z^3 - xyz = 0$ is the gradient

$$\nabla F =< 3x^2 - yz, 3y^2 - zx, 3z^2 - xy >,$$

which at P is given by $\boldsymbol{n} =< 3, 1, 3 >$. Note that

$$\boldsymbol{n} \cdot \boldsymbol{r}'(1) = 0 + 3 - 3 = 0,$$

indicating that the tangent to the curve is orthogonal to the normal to the surface at P. Therefore the curve touches the surface tangentially at P. The equation of the tangent plane at P is

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}(1)) = 0, \quad \text{or} \quad < 3, 1, 3 > \cdot < x - 1, y, z + 1 >= 0,$$

or

$$3x + y + 3z = 0.$$

(b) The lines being skew lie in parallel planes. The unit normal to the planes is a vector perpendicular to both lines, and is given by

$$\boldsymbol{n} = \frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{||\boldsymbol{v}_1 \times \boldsymbol{v}_2||}.$$

A vector connecting two points, one on each line, is $\boldsymbol{r}_2 - \boldsymbol{r}_1$. The desired distance is the magnitude of the orthogonal projection of this vector along the normal, *i.e.*,

$$d = \left| (\boldsymbol{r}_2 - \boldsymbol{r}_1) \cdot \frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{||\boldsymbol{v}_1 \times \boldsymbol{v}_2||} \right|.$$

2. (a) Use the chain rule to find $D(\boldsymbol{f} \circ \boldsymbol{g})(-1,2)$ for

$$\begin{aligned} \boldsymbol{f}(u,v,w) &= \; < v^2+w^2, u^3-vw, u^2v+w >, \\ \boldsymbol{g}(x,y) &= \; < 3x+2y, x^3y, y^2-x^2 > . \end{aligned}$$

(b) Let $f(x,y)$ and $g(x,y)$ be functions such that $\nabla f = \lambda \nabla g$ for some function $\lambda(x,y)$. What is the relation between the level curves of f and g? Explain why there might be a function F such that $g(x,y) = F(f(x,y))$.

(a) Note that the function $\boldsymbol{f}$ is $R^3 \to R^3$ while $\boldsymbol{g}$ is $R^2 \to R^3$. Therefore the composite function $\boldsymbol{f} \circ \boldsymbol{g}$ is $R^2 \to R^3$. We have

$$D\boldsymbol{f}(u,v,w) = \begin{bmatrix} f_{1_u} & f_{1_v} & f_{1_w} \\ f_{2_u} & f_{2_v} & f_{2_w} \\ f_{3_u} & f_{3_v} & f_{3_w} \end{bmatrix} = \begin{bmatrix} 0 & 2v & 2w \\ 3u^2 & -w & -v \\ 2uv & u^2 & 1 \end{bmatrix},$$

$$D\boldsymbol{g}(x,y) = \begin{bmatrix} g_{1_x} & g_{1_y} \\ g_{2_x} & g_{2_y} \\ g_{3_x} & g_{3_y} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3x^2y & x^3 \\ -2x & 2y \end{bmatrix}.$$

At $x=-1$ and $y=2$, $\boldsymbol{g} =< u,v,w >=< 1,-2,3 >$. Then,

$$\begin{aligned} D(\boldsymbol{f} \circ \boldsymbol{g})(-1,2) &= D\boldsymbol{f}(1,-2,3)\; D\boldsymbol{g}(-1,2) \\ &= \begin{bmatrix} 0 & -4 & 6 \\ 3 & -3 & 2 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 6 & -1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -12 & 28 \\ -5 & 17 \\ -4 & -5 \end{bmatrix}. \end{aligned}$$

(b) Consider a level curve C of $g(x,y)$, given by $g(x,y) = K$. At every point $(x,y) \in C$, ∇g is a vector normal to C. Since $\nabla f = \lambda(x,y)\nabla g$, ∇f is a multiple of ∇g so that ∇f is also a normal vector to C. Hence C is a level curve of f as well.

The relation $g(x,y) = F(f(x,y))$ makes sense because upon differentiation it yields $\nabla g = F'(f)\nabla f$ or $\nabla f = (1/F'(f))\nabla g$, thereby identifying $\lambda(x,y) = 1/F'(f)$.

3. (a) Verify that

$$\boldsymbol{r}(t) =< \sin t, \cos t, e^{2t} >$$

is a flow line of the vector field $\boldsymbol{F} =< y, -x, 2z >$.

(b) Determine if

$$\boldsymbol{F} = \left\langle yze^{xy}, xze^{xy} + 2y\cos z, e^{xy} - y^2 \sin z \right\rangle$$

is a gradient field, and if so, find its potential.

(c) Show that the curl of a vector field is incompressible.

(a) We have

$$\boldsymbol{r}'(t) =< \cos t, -\sin t, 2e^{2t} >=< y, -x, 2z >= \boldsymbol{F}.$$

Hence the result.

(b) Let $\boldsymbol{F} = \nabla f$. Then,

$$f_x = yze^{xy} \quad \text{so that} \quad f = ze^{xy} + F(y, z).$$

Continuing,

$$f_y = xze^{xy} + 2y\cos z = xze^{xy} + F_y.$$

Therefore

$$F_y = 2y\cos z \quad \text{so that} \quad F = y^2 \cos z + G(z).$$

Thus

$$f = ze^{xy} + F(y, z) = ze^{xy} + y^2 \cos z + G(z).$$

Finally,

$$f_z = e^{xy} - y^2 \sin z = e^{xy} - y^2 \sin z + G'(z).$$

Therefore $G'(z) = 0$, or G is a constant, say K. Then

$$f = ze^{xy} + y^2 \cos z + K.$$

As we have succeeded in finding the potential, $\boldsymbol{F}$ is a gradient field.

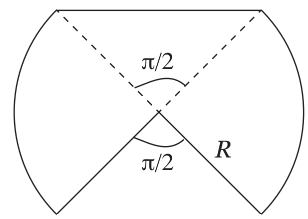
(c)

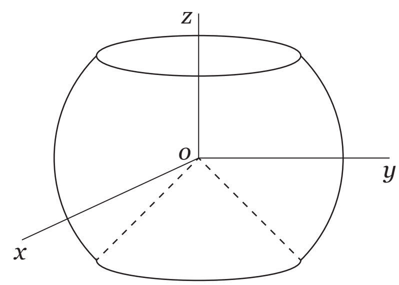
$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{bmatrix} \\
&= < F_{3_y} - F_{2_z}, F_{1_z} - F_{3_x}, F_{2_x} - F_{1_y} > .
\end{aligned}$$

For incompressiblity, divergence of $\nabla \times \boldsymbol{F}$ must vanish. We see below that it is indeed so.

$$\nabla \cdot \nabla \times \boldsymbol{F} = (F_{3_y} - F_{2_z})_x + (F_{1_z} - F_{3_x})_y + (F_{2_x} - F_{1_y})_z = 0.$$

4. The plane shape shown on the top in the figure below is rotated about its vertical axis of symmetry to generate the solid shown at the bottom; a sphere whose top has been flattened and from which a cone-shaped section has been removed at the bottom. Describe the solid by means of appropriate inequalities in (i) cartesian coordinates, (ii) cylindrical coordinates and (iii) spherical coordinates.





In cartesian coordinates we divide the solid into two parts; an inner part S_1 which is a cylinder of radius $R/\sqrt{2}$ with a flat top and a conical bottom, and an outer part which is a sphere of radius R from which a cylinder of radius $R/\sqrt{2}$ has been removed. The projection of S_1 in the xy-plane is a circle of radius $R/\sqrt{2}$ centered at the origin, and that of S_2 is the annulus centered at the origin with inner radius $R/\sqrt{2}$ and outer radius R. The two parts have the following descriptions:

$$\begin{aligned} S_1 &: \quad x^2+y^2 \le R^2/2, \quad -\sqrt{x^2+y^2} \le z \le R/\sqrt{2}, \\ S_2 &: \quad R^2/2 \le x^2+y^2 \le R^2, \quad -\sqrt{R^2-x^2-y^2} \le z \le \sqrt{R^2-x^2-y^2}. \end{aligned}$$

In cylindrical coordinates the same two parts are described as follows.

$$\begin{aligned} S_1 &: \quad r \le R/\sqrt{2}, \quad -r \le z \le R/\sqrt{2}, \quad 0 \le \theta < 2\pi, \\ S_2 &: \quad R/\sqrt{2} \le r \le R, \quad -\sqrt{R^2-r^2} \le z \le \sqrt{R^2-r^2}, \quad 0 \le \theta < 2\pi. \end{aligned}$$

In spherical coordinates we divide the solid into two different parts, a cone B_1 with vertex at the origin, a vertex angle of $\pi/2$ and located above the xy-plane, and the remainder of the solid B_2. The descriptions are as follows.

$$\begin{aligned} B_1 &: \quad 0 \le \phi \le \pi/4, \quad 0 \le \rho \le \frac{R}{\sqrt{2}\cos\phi}, \quad 0 \le \theta < 2\pi, \\ B_2 &: \quad \pi/4 \le \phi \le 3\pi/4, \quad 0 \le \rho \le R, \quad 0 \le \theta < 2\pi. \end{aligned}$$