ADVANCED CALCULUS
SPRING 2019

LESSON 4: Line and Surface Integrals

This lesson considers the following topics.

1. Line integrals.
2. Parametric description of surfaces.
3. Surface integrals.
4. Vector integral theorems.

Supplements Chapters 7 and 8 of the Marsden-Tromba text.

1 Introduction

As we have already observed, integration is a summation process, designed to compute the full amount of a quantity in a given region when its distribution is known. Thus the integral

$$\int_a^b f(x)\, dx$$

in R^1 computes the total amount of a quantity over the interval $[a, b]$ whose distribution (*i.e.*, the amount per unit length) is $f(x)$. The quantity is area if $f(x)$ is the height, mass if $f(x)$ is the linear density, and electric charge if $f(x)$ is the charge density per unit length. We saw in the previous lesson how the concept, defined for a straight-line segment, can be extended to planar and spatial regions in the form of double and triple integrals.

We now turn to summation over curves and surfaces by defining integrals on curves and surfaces. First, a brief review of the line integral.

2 The line integral

2.1 Line or path integral of a scalar function

Suppose one wishes to construct a fence of varying height on the bounding curve of an estate that lies in a rolling countryside. Then the boundary is, in general, not a planar curve, but rather a curve in R^3. The area of the fence is the sum of elementary contributions of the form $f\ ds$, where f is the height function and s is the arc length along the boundary. The result, a generalization of the standard 1-d integral, is defined as the *line integral* or *path integral* of the height function along the curve.

Def. The line integral of a scalar function $f(x, y, z)$ on a path C parametrized as

$$C : \boldsymbol{r} = \boldsymbol{r}(t) =< x(t), y(t), z(t) >, \quad a \leq t \leq b,$$

is written as

$$\int_C f\ ds,$$

and evaluated as the 1-d integral

$$\int_a^b f(x(t), y(t), z(t)) \left\| < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > \right\| dt.$$

A shorthand notation for the above integral is

$$\int_a^b f(\boldsymbol{r}(t))\ \|\boldsymbol{r}'(t)\|\ dt.$$

For $f \equiv 1$ the line integral simply gives the arc length.

Example 2.1. Area of the wall along a helical stairway. Consider the integral

$$I = \int_C (x^2 + y^2 + \sin z)\ ds,$$

where the curve C is a helix, parametrized as

$$\boldsymbol{r}(\theta) =< 3\cos\theta, 3\sin\theta, 5\theta >, \quad 0 \leq \theta \leq 6\pi.$$

We have

$$\left\| \frac{d\boldsymbol{r}}{d\theta} \right\| = \| < -3\sin\theta, 3\cos\theta, 5 > \| = \sqrt{34}.$$

Then,

$$\begin{aligned} I &= \int_0^{6\pi} (9\cos^2\theta + 9\sin^2\theta + \sin 5\theta)\sqrt{34}\, d\theta \\ &= \left[9\theta - \frac{1}{5}\cos 5\theta \right]_0^{6\pi} \sqrt{34} \\ &= 54\sqrt{34}\pi. \end{aligned}$$

2.2 Line Integral of a vector function

The work done by a force field $\boldsymbol{F}(x, y, z)$ in moving a particle along a curve is the summation of elementary contributions of the form $\boldsymbol{F} \cdot \boldsymbol{T}\, ds$. Here $\boldsymbol{T}$ is the unit tangent vector along the path and therefore, $\boldsymbol{F} \cdot \boldsymbol{T}$ is the component of the force along the path. The result is defined as the line integral of the force field along the curve.

Def. The line integral of a vector function $\boldsymbol{F}(x, y, z)$ on a path parametrized as

$$C : \boldsymbol{r} = \boldsymbol{r}(t) =< x(t), y(t), z(t) >, \quad a \leq t \leq b,$$

is denoted by

$$\int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds,$$

and is evaluated as the line integral of the scalar field $\boldsymbol{F} \cdot \boldsymbol{T}$ according to the prescription given earlier in terms of the parameter t describing the path. Thus,

$$\int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_a^b \boldsymbol{F} \cdot \boldsymbol{T}\, \|d\boldsymbol{r}/dt\|\, dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt$$

where in the last step above we have made use of the expression

$$\boldsymbol{T} = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}.$$

A shorthand notation for the line integral is

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r},$$

or in component form,

$$\int_C F_1\, dx + F_2\, dy + F_3\, dz, \quad \text{where} \quad \boldsymbol{F} =< F_1, F_2, F3 > .$$

Example 2.2. Work done by the force field $\boldsymbol{F} =< y, -x, z^2 >$ along the helical path

$$\boldsymbol{r}(\theta) =< 3\cos\theta, 3\sin\theta, 5\theta >, \quad 0 \leq \theta \leq 6\pi.$$

We have

$$\begin{aligned} I &= \int_C \boldsymbol{F}(\boldsymbol{r}(\theta)) \cdot \boldsymbol{r}'(\theta)\, d\theta \\ &= \int_0^{6\pi} < 3\sin\theta, -3\cos\theta, 25\theta^2 > \cdot < -3\sin\theta, 3\cos\theta, 5 > \, d\theta \\ &= \int_0^{6\pi} (-9 + 125\theta^2)\, d\theta = 9000\pi^3 - 54\pi. \end{aligned}$$

Remarks

- The scalar line integral is independent of the orientation of the path, while the vector line integral changes sign when the orientation is reversed (as a result of the tangent vector $\boldsymbol{T}$ changing sign).
- The integrals are unchanged under a reparametrization of the path. The reparametrization must be orientation-preserving for this result to hold for the line integral of a vector function.
- The line integral of a vector field around a closed curve is known as the circulation of the field.

Example 2.3. Let $\boldsymbol{F} =< -y, x >$ be a vector field in R^2 and C the positively oriented unit circle $C :< \cos\theta, \sin\theta >,\ 0 \le \theta \le 2\pi$. Then the circulation of $\boldsymbol{F}$ around C is

$$\begin{aligned} I &= \oint_C \boldsymbol{F}(\boldsymbol{r}(\theta)) \cdot \boldsymbol{r}'(\theta)\, d\theta \\ &= \int_0^{2\pi} < -\sin\theta, \cos\theta > \cdot < -\sin\theta, \cos\theta > \ d\theta = 2\pi. \end{aligned}$$

For the vector field $\boldsymbol{G} =< 0, -1 >$, the circulation is

$$\begin{aligned} I &= \int_0^{2\pi} \boldsymbol{G}(\boldsymbol{r}(\theta)) \cdot \boldsymbol{r}'(\theta)\, d\theta \\ &= \int_0^{2\pi} < 0, -1 > \cdot < -\sin\theta, \cos\theta > \ d\theta = 0. \end{aligned}$$

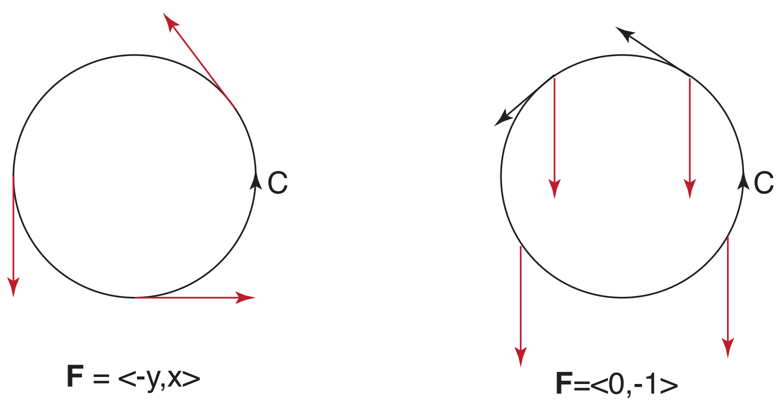

Figure 1: Circulation. The vector field is in red.

Figure 1 shows that the vector field $\boldsymbol{F} =< -y, x >$ is oriented in such a way that at every point on C the field has a positive component $\boldsymbol{F} \cdot \boldsymbol{T}$ along the tangent vector. This alignment generates significant circulation. The field $\boldsymbol{F} =< 0, -1 >$, on the other hand, is directed so that its component along the tangent vector changes sign as C is traversed. As a result circulation is poor, and in this case is in fact zero. Thus, circulation is a measure of the rotational tendency of the field. As we shall see later, it is related to the curl.

3 The surface integral

3.1 Parametric representation of a surface

To this point we have generally regarded a surface as the graph of a two-variable function, $z = f(x, y)$, or as the level set of a three-variable function, $F(x, y, z) = c$. An example of the former is the paraboloid

$z = x^2 + y^2$, and of the latter the sphere $x^2 + y^2 + z^2 = 4$. Neither of these representations, however, is adequate for a description of, say, a helicoid or a torous. A much more general description of a surface is by means of a parametrization, which is a representation of the form

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),$$

or equivalently,

$$\boldsymbol{r} = \boldsymbol{r}(u, v) ==< x(u, v), y(u, v), z(u, v) >,$$

where u and v range over a given region D in the $uv-$ plane. The dependence on two parameters highlights the two-dimensional nature of the surface.

Examples 3.1.

- Plane, $x = u,\ y = v,\ z = au + bv + c,\ -\infty < u, v < \infty.$
- Sphere, $x = R\sin\phi\cos\theta,\ y = R\sin\phi\sin\theta,\ z = R\cos\phi,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$
- Cylinder $x^2 + y^2 = 4$, or $x = 2\cos u,\ y = \sin u$, and $z = v$, i.e., $\underline{r} =< 2\cos u, 2\sin u, v >$.
- Cone $x^2 + y^2 = z^2$, or $\underline{r} =< v\cos u, v\sin u, v >, 0 \le u \le 2\pi,\ -\infty < v < \infty.$
- Paraboloid $z = x^2 + y^2$, or $\underline{r} =< \sqrt{u}\cos v, \sqrt{u}\sin v, u >, 0 \le u < \infty,\ 0 \le v \le 2\pi.$
- The torus $\underline{r} =< (R + a\cos\phi)\cos\theta, (R + a\cos\phi)\sin\theta, a\sin\phi >,\quad 0 \le \theta, \phi \le 2\pi.$
- The helicoid $x = r\cos\theta,\ y = r\sin\theta,\ z = \theta,\quad 0 \le r \le 1,\ 0 \le \theta \le 4\pi.$

Example 3.2. Cylindrical can, $x^2+y^2 = 4$, capped below by $z = 0$ and above by the surface $z = 2+y^2-x$. Describe all of its bounding surfaces parametrically in cylindrical coordinates.

The lower boundary is the plane circular disk, described parametrically as

$$S_1 :< x, y, z >= \boldsymbol{r}(r, \theta) =< r\cos\theta, r\sin\theta, 0 >,\quad 0 \le r \le 1,\ 0 \le \theta < 2\pi.$$

The lateral boundary is

$$S_2 :< x, y, z >= \boldsymbol{r}(\theta, z) =< \cos\theta, \sin\theta, z >,\quad 0 \le \theta < 2\pi,\ 0 \le z \le 2 + \sin^2\theta - \cos\theta.$$

The upper boundary is

$$S_3 :< x, y, z >= \boldsymbol{r}(r, \theta) =< r\cos\theta, r\sin\theta, 2 + \sin^2\theta - \cos\theta >,\quad 0 \le r \le 1,\ 0 \le \theta < 2\pi.$$

Remarks.

- For the surface $z = f(x, y)$, x and y can be considered parameters, so that $\underline{r} =< x, y, f(x, y) >$.

  Example 3.3. The upper half of the unit sphere has the parametric form

  $$\underline{r} =< x, y, \sqrt{1 - x^2 - y^2} >,$$

  with the region D given by the unit circle

  $$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2},\ -1 \le x \le 1.$$

- The parametric representation

  $$S :\ \underline{r} =< x(u, v), y(u, v), z(u, v) >,\quad (u, v) \in D,$$

  can be thought of as a transformation or mapping from $D \subset R^2$ to R^3, with the surface S the 3-d image of the planar region D. Curves in the surface corresponding to u constant and v constant form two families of grid lines. The vectors $\boldsymbol{r}_u = \partial\boldsymbol{r}/\partial u$ and $\boldsymbol{r}_v = \partial\boldsymbol{r}/\partial v$ are tangents to these grid lines, and therefore, $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$ is the normal vector to the surface. Surface is smooth if the normal is well defined ($\boldsymbol{r}_u \times \boldsymbol{r}_v$ should exist, and be nonvanishing). With the normal known, equation of the tangent plane can be written down.

Example 3.4. For the ellipsoidal patch

$$S: \ \boldsymbol{r} =< 2\cos\theta\sin\phi, \ \ 2\sin\theta\sin\phi, \ \cos\phi >, \ \ 0 \le \theta \le \pi, \ 0 \le \phi \le \pi/2,$$

find the normal vector and the tangent plane at $\theta = \phi = \pi/4$.

The normal vector is

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{r}_\theta \times \boldsymbol{r}_\phi \\ &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\sin\phi & 2\sin\theta\cos\phi & -\sin\phi \end{bmatrix} \\ &= < -2\cos\theta\sin^2\phi, -2\sin\theta\sin^2\phi, -4\sin\phi\cos\phi > . \end{aligned}$$

At the point $\theta = \phi = \pi/4$, $\boldsymbol{r}_0 =< 1, 1, 1/\sqrt{2} >$ and $\boldsymbol{N} =< -1/\sqrt{2}, -1/\sqrt{2}, -2 >$. Therefore the equation of the tangent plane is

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{N} = 0,$$

leading to

$$-\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) - 2\left(z - \frac{1}{\sqrt{2}}\right) = 0,$$

which simplifies to

$$x + y + 2\sqrt{2}z = 4.$$

3.2 Surface area

An element of area on the surface is $dS = \|\boldsymbol{r}_u \times \boldsymbol{r}_v\| \, dudv$, so that area of the surface is the integral

$$A(S) = \iint_S dS = \iint_D \|\boldsymbol{r}_u \times \boldsymbol{r}_v\| \, dudv.$$

Example 3.5. Area of the sphere of radius R. The sphere is parametrized as

$$\boldsymbol{r}(\theta, \phi) =< R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi >, \quad 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi.$$

We compute the scale factor

$$\begin{aligned} \|\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi\| &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \end{vmatrix} \\ &= \| < -R^2\sin^2\phi\cos\theta, -R^2\sin^2\phi\sin\theta, -R^2\sin\phi\cos\phi > \| \\ &= R^2\sin\phi. \end{aligned}$$

Therefore the area is given by

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} R^2[-\cos\phi]_0^{\pi} \, d\theta \\ &= \int_0^{2\pi} 2R^2 \, d\theta = 4\pi R^2. \end{aligned}$$

Example 3.6. Area of the hemisphere of radius 2, centered at the origin and cut by the cylinder $x^2 + (y-1)^2 = 1$.

We parametrize the hemisphere using cylindrical coordinates r and θ. On the hemisphere, $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$. The circular boundary of the cylinder $x^2 + (y-1)^2 = 1$ has the polar equivalent $r^2 \cos^2\theta + r^2 \sin^2\theta + 1 - 2r\sin\theta = 1$, or $r = 2\sin\theta$. Therefore the patch of interest on the hemisphere is parametrized as

$$\boldsymbol{r}(r,\theta) =< r\cos\theta, r\sin\theta, \sqrt{4-r^2} >, \quad 0 \le r \le 2\sin\theta,\ 0 \le \theta \le \pi.$$

The scale factor for the integral is

$$\begin{aligned}
\|\boldsymbol{r}_r \times \boldsymbol{r}_\theta\| &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & -\frac{r}{\sqrt{4-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= \left\| \left\langle \frac{r^2\cos\theta}{\sqrt{4-r^2}}, \frac{r^2\sin\theta}{\sqrt{4-r^2}}, r \right\rangle \right\| \\
&= \frac{2r}{\sqrt{4-r^2}}.
\end{aligned}$$

Therefore the area is given by

$$\begin{aligned}
A &= \int_0^\pi \int_0^{2\sin\theta} \frac{2r}{\sqrt{4-r^2}}\, dr\, d\theta \\
&= \int_0^\pi \left\{ -2\sqrt{4-r^2} \right\}_0^{2\sin\theta} d\theta \\
&= 2\int_0^\pi [2 - 2|\cos\theta|]\, d\theta = \pi - 2.
\end{aligned}$$

3.3 Surface integral of a scalar function

Def. The surface integral of a scalar function $f(x,y,z)$ over the surface $S:\ \boldsymbol{r} = \boldsymbol{r}(u,v),\ (u,v) \in D,$ is

$$\int\int_S f\, dS = \int\int_D f(x(u,v), y(u,v), z(u,v)\ ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, du\, dv.$$

The surface integral yields the total amount on the surface of the quantity whose distribution on the surface is the function f. For example, if f is the density per unit area of the material of the surface, then the surface integral provides the mass of the surface.

Recall the case of the line integral, where a parametric representation of the path allowed the line integral to be evaluated as a 1-d integral. Here, parametrization of the surface allows the surface integral to be evaluated as a double integral over the region D.

Example 3.7. Consider a thin film made in the shape of a helicoid, which is parametrized by

$$S:\ \boldsymbol{r}(s,t) =< s\cos t, s\sin t, t >, \quad 0 \le s \le 1,\ 0 \le t \le 4\pi.$$

Let $\delta = \sqrt{x^2 + y^2}$ be the mass density per unit area of the film. We seek to find the total mass of the film.

The total mass is the surface integral of the density,

$$M = \int\int_S \delta(x,y)\, dS$$

For the given parametrization the scale factor is

$$\begin{aligned}
||\boldsymbol{r}_s \times \boldsymbol{r}_t|| &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{bmatrix} \\
&= ||< \sin t, -\cos t, s >|| \\
&= \sqrt{1+s^2}.
\end{aligned}$$

Also, $\delta = \sqrt{s^2\cos^2 t + s^2\sin^2 t} = s$. Therefore,

$$\begin{aligned} M &= \int_0^{4\pi}\int_0^1 s\sqrt{1+s^2}\,ds\,dt \\ &= \int_0^{4\pi}\frac{1}{3}[2^{3/2}-1]\,dt \\ &= \frac{4\pi}{3}[2^{3/2}-1]. \end{aligned}$$

Example 3.8. Let us find the mass of a thin hemispherical shell of radius R whose density δ is constant along circles parallel to its base but decreases linearly with latitude, from $2\rho_0$ at the base to ρ_0 at the apex.

We shall use spherical coordinates, whereby the hemispherical surface is parametrized as

$$S:\ \boldsymbol{r}(\theta,\phi) = < R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi >, \quad D:\ 0\le\theta<2\pi,\ 0\le\phi\le\pi/2.$$

The density distribution in spherical coordinates is

$$\delta = \rho_0\left(1+\frac{2\phi}{\pi}\right).$$

Therefore the mass is given by the integral

$$M = \iint_S \delta\,dS.$$

The scale factor was computed in Example 3.5 to be $R^2\sin\phi$. Therefore,

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^{\pi/2}\left(1+\frac{2\phi}{\pi}\right)R^2\sin\phi\,d\phi\,d\theta \\ &= R^2\int_0^{2\pi}\left[\left(1+\frac{2\phi}{\pi}\right)(-\cos\phi)-\frac{2}{\pi}(-\sin\phi)\right]_0^{\pi/2}d\theta \\ &= R^2\int_0^{2\pi}\left(\frac{2}{\pi}+1\right)d\theta = 2\pi\left(\frac{2}{\pi}+1\right)R^2. \end{aligned}$$

Remark. When a surafce S is given in the explicit form $z=z(x,y)$, $(x,y)\in D$, then a natural choice for the parameters is just the pair (x,y). Thus S can be parametrized as

$$S:\ x=x,\ y=y,\ z=z(x,y),\ \ (x,y)\in D.$$

We then have

$$\boldsymbol{r}_x\times\boldsymbol{r}_y = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{bmatrix} = < -z_x, -z_y, 1 >,$$

so that the scale factor is

$$\|\boldsymbol{r}_x\times\boldsymbol{r}_y\| = \sqrt{1+z_x^2+z_y^2}.$$

Recall also that $\boldsymbol{r}_x\times\boldsymbol{r}_y = \boldsymbol{N}$, the normal vector to the surface. Normalization produces the unit normal vector

$$\boldsymbol{n} = \frac{\boldsymbol{r}_x\times\boldsymbol{r}_y}{\|\boldsymbol{r}_x\times\boldsymbol{r}_y\|} = \frac{1}{\sqrt{1+z_x^2+z_y^2}} < -z_x, -z_y, 1 > . \tag{3.1}$$

Since

$$\boldsymbol{k}\cdot\boldsymbol{n} = \frac{1}{\sqrt{1+z_x^2+z_y^2}},$$

the scale factor can also be written as

$$\sqrt{1+z_x^2+z_y^2}=\frac{1}{\boldsymbol{k}\cdot\boldsymbol{n}}.$$

Thus we can write

$$\iint_S f(x,y,z)\,dS=\iint_D f(x,y,z(x,y))\,\sqrt{1+z_x^2+z_y^2}\,dx\,dy,$$

or equivalently,

$$\iint_S f(x,y,z)\,dS=\iint_D f(x,y,z(x,y))\frac{1}{\boldsymbol{k}\cdot\boldsymbol{n}}\,dx\,dy.$$

3.4 Surface integral of a vector function

Consider a fluid flowing with a uniform velocity $\boldsymbol{v}$ across a cross section of area A normal to the direction of flow. Then the flow rate across the cross section is $\|\boldsymbol{v}\|A$ in units of volume per unit time. If the unit normal $\boldsymbol{n}$ to the area A is inclined to the velocity field $\boldsymbol{v}$ at an angle α, then the flow rate through the area A is $\boldsymbol{v}\cdot\boldsymbol{n}\,A$. In other words, it is the component of the field along the normal to the area that counts.

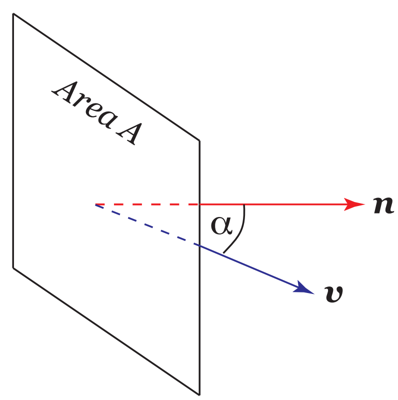

Figure 2: Flow rate across area A.

For a general curved surface S, the flow rate [1] of a vector field $\boldsymbol{F}$ across S can be computed as the surface integral

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS.$$

For the parametrized surface

$$S:\ \boldsymbol{r}=\boldsymbol{r}(u,v),\ (u,v)\in D,$$

we have

$$\boldsymbol{n}=\frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{||\boldsymbol{r}_u\times\boldsymbol{r}_v||}.$$

Then we can write

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS &= \iint_D \boldsymbol{F}\cdot\boldsymbol{n}\,||\boldsymbol{r}_u\times\boldsymbol{r}_v||\,du\,dv &(3.2)\\ &= \iint_D \boldsymbol{F}\cdot\frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{||\boldsymbol{r}_u\times\boldsymbol{r}_v||}\,||\boldsymbol{r}_u\times\boldsymbol{r}_v||\,du\,dv\\ &= \iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,du\,dv. &(3.3)\end{aligned}$$

We can evaluate the integral by using either of the results (3.2) or (3.3).

[1] The term *flux* is often used instead of *flow rate*. This is abuse of term, as flux is reserved for flow rate *per unit area*. However, the abuse is now too common to enforce and these notes will succumb to it on occasion.

Example 3.8. Consider the flow rate of a vector field $\boldsymbol{F} =< x^2y, -3xy^2, 4y^3 >$ across the surface S of the paraboloid $z = 1 + x^2 + y^2$ that lies above the rectangle $D : 0 \le x \le 2,\ 0 \le y \le 1$ in the xy-plane. We parametrize the surface by x and y so that

$$\boldsymbol{r}(x, y) =< x, y, 1 + x^2 + y^2 >, \quad (x, y) \in D.$$

The unit vector to the surface, oriented so that its $\boldsymbol{k}$ component is positive, is, according to (3.1),

$$\begin{aligned} \boldsymbol{n} &= \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|} = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}} < -z_x, -z_y, 1 > \\ &= \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} < -2x, -2y, 1 > . \end{aligned}$$

Then,

$$\boldsymbol{F} \cdot \boldsymbol{n} = \frac{-2x^3y + 6xy^3 + 4y^3}{\sqrt{1 + 4x^2 + 4y^2}}.$$

Use of (3.2) yields

$$\begin{aligned} I &= \iint_D \boldsymbol{F} \cdot \boldsymbol{n}\ ||\boldsymbol{r}_x \times \boldsymbol{r}_y||\ dy\ dx \\ &= \int_0^2 \int_0^1 \left(\frac{-2x^3y + 6xy^3 + 4y^3}{\sqrt{1 + 4x^2 + 4y^2}} \right) \sqrt{1 + 4x^2 + 4y^2}\ dy\ dx \\ &= \int_0^2 \int_0^1 (-2x^3y + 6xy^3 + 4y^3)\ dy\ dx. \end{aligned}$$

(Equation (3.3) would have led to the above expression directly.) The double integral evaluates to

$$\begin{aligned} I &= \int_0^2 \left[-x^3y^2 + \frac{6xy^4}{4} + y^4 \right]_0^1 dx \\ &= \int_0^2 \left(-x^3 + \frac{3x}{2} + 1 \right) dx = \frac{3}{2}. \end{aligned}$$