Other Kinematic Significance of group Velocity:

- Assume that a collection of waves of different wave numbers has sufficiently dispersed.

We can then consider $k \left(= \frac{2\pi}{\lambda}\right)$ and $\omega \left(= \frac{2\pi}{T}\right)$ as functions of x and t.

If we think about the Phase function Θ, where

$$\frac{\partial \Theta}{\partial x} \equiv k \quad \text{and} \quad \frac{\partial \Theta}{\partial t} \equiv -\omega$$

we notice that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad \left(\text{because } \frac{\partial^2 \Theta}{\partial x \partial t} = \frac{\partial^2 \Theta}{\partial t \partial x}\right)$$

- If ω is only a function of k $\left(\text{e.g. } \omega^2 = k\left(g + \frac{\sigma}{\rho} k^2\right)\right)$

then

$$\frac{\partial k}{\partial t} + \underbrace{\frac{d\omega}{dk}}_{\equiv\, C_g} \frac{\partial k}{\partial x} = 0$$

$$\therefore \quad \boxed{\frac{\partial k}{\partial t} + C_g \frac{\partial k}{\partial x} = 0}$$

-

group velocity is the speed with which (waves of a given) wavenumber travel.

(a consequence of this is that for example if you travel over waves in ocean (at a fixed speed) you would see only waves of a fixed wavelength beneath you.)

also, since $\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$

$$\Downarrow$$

$$c_g \frac{\partial k}{\partial t} + c_g \frac{\partial \omega}{\partial x} = 0$$

$$\frac{\partial \omega}{\partial k}\frac{\partial k}{\partial t} + c_g \frac{\partial \omega}{\partial x} = 0$$

or $$\boxed{\frac{\partial \omega}{\partial t} + c_g \frac{\partial \omega}{\partial x} = 0}$$

i.e. frequency also propagates at group velocity.

<u>Summary</u>: Average wave energy, wavenumber, frequency, wave packets (and nulls, due to destructive interference) all travel at group velocity

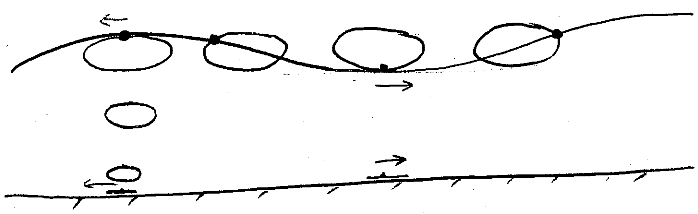
Viscous Effects

3/4

In general, Viscosity dampens waves through three mechanisms.

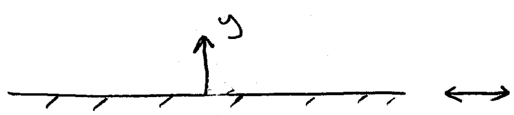
1) Bottom friction
2) Internal dissipation
3) Surface dissipation

1) Bottom friction is caused by the oscillatory motion produced at the bottom when Wavelength (λ) is large compared to depth (h)



Note

The problem is analogeous to "Stokes' Second-Problem" (See e.g. Schlichting BL theory or Lighthill) (usually studied in BL theory classes)



$$u(0,t) = U_0 \cos \omega t$$ (no x-dependence)

which has the Self-similar solution:

$$u(y,t) = U_0 e^{-\eta} \cos(\omega t - \eta)$$

where $\eta = y\sqrt{\frac{\omega}{2\nu}}$

T.P.

Notice that the layer has a maximum physical thickness δ corresponding to $\eta \approx 5$ (i.e. by $\eta = 5$, Velocity decreases to less than 1% wall value)

$$\Rightarrow \quad 5 = \delta\sqrt{\frac{\omega}{2\nu}} \quad \Rightarrow \quad \delta \approx 7\sqrt{\frac{\nu}{\omega}} \quad \text{or} \quad \delta \propto \sqrt{\frac{\nu}{\omega}}$$

end Note

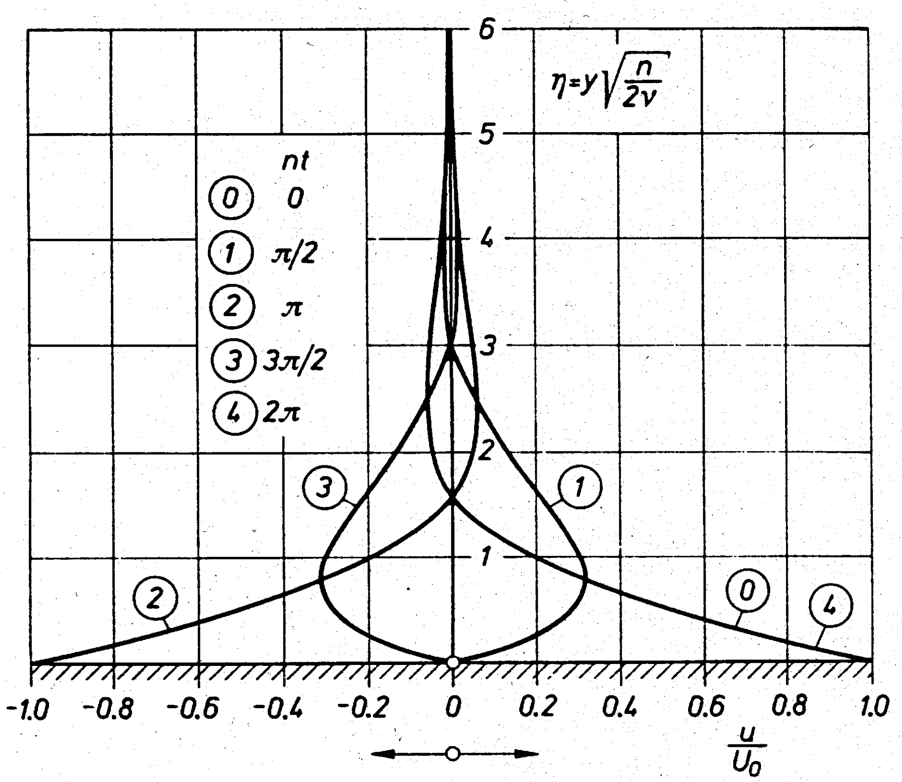


Fig. 5.9. Velocity distribution in the neighbourhood of an oscillating wall (Stokes's second problem)

(schlichting)

Lighthill shows that for the wave problem, bottom boundary layer is approximately:

$$\delta \simeq 2\pi\sqrt{\frac{\nu}{2\omega}} \approx 4.4\sqrt{\frac{\nu}{\omega}} \quad \text{i.e. } \delta \propto \sqrt{\frac{\nu}{\omega}}$$

also

And the energy loss per period $\left(\frac{2\pi}{\omega}\right)$:

$$= \frac{2\delta k}{\sinh(2kh)} \left(\rightarrow \frac{2\pi}{h}\sqrt{\frac{\nu}{2\omega}} \quad \text{for small } h\right)$$

Note that for a given k (or wavelength) energy loss per period exponentially decreases with increasing h. Ex.: at $\frac{h}{\lambda} = 0.5$, energy loss is 166 times smaller than it would be at $\frac{h}{\lambda} = 0.1$ (for a given λ). Thus, if h is at least $\frac{1}{2}\lambda$, bottom losses may be negligible. T.P.

(The second mechanism for viscous damping is)

2) <u>Internal Dissipation</u> : Viscous dissipation in the fluid outside the boundary layer that may exist at the free surface
(We'll discuss the FS BL lastly)

Use an argument due to Stokes:

Potential flow solution would remain correct right up to the free surface if stresses equal to the normal and tangential viscous stresses were applied at surface, (i.e add an anti-viscous stress tensor) where stress tensor τ_{ij} is:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{and in 2-D flow:}$$

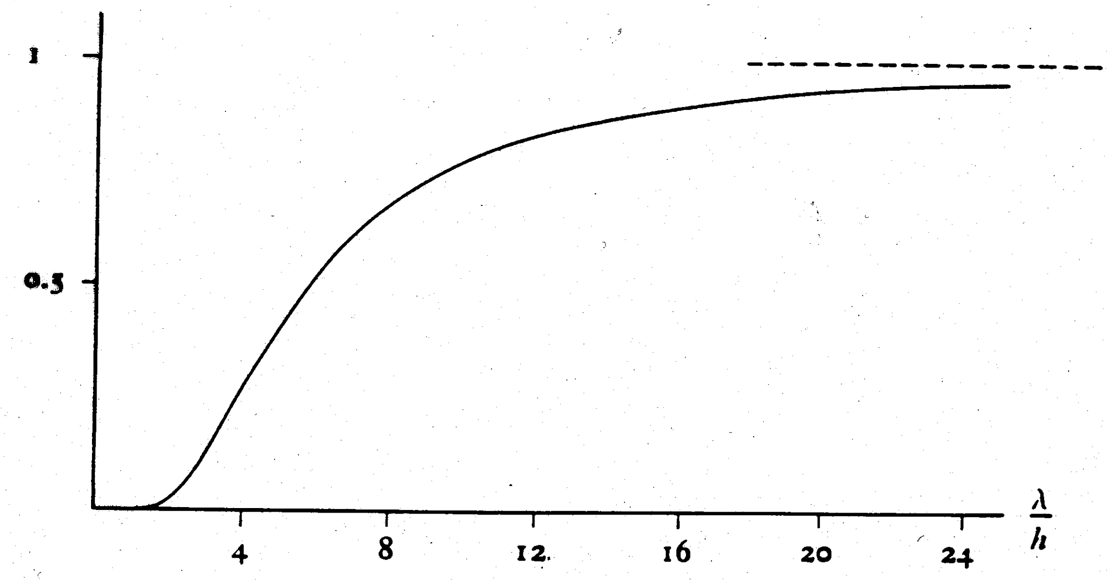


Figure 58. Ratio of energy loss by bottom friction to its value in the long-wave limit, plotted against the wavelength-depth ratio λ/h. (This is a graph of the modifying factor in curly brackets in (74).)

Lighthill (Waves in Fluids)

$$\tau_{ij} = \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\mu \frac{\partial v}{\partial y} \end{pmatrix}$$

i.e. We must apply an external force / Unit area equal to

$-2\mu \frac{\partial v}{\partial y}$ in the vertical direction and

$$-\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -2\mu \frac{\partial u}{\partial y} = -2\mu \frac{\partial v}{\partial x}$$

in horiz. dir.

since $\omega_z = 0$ irrotational flow.

Vorticity $\vec{\omega} = \text{curl}\,\vec{u} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{pmatrix}$

and thus the only component of Vorticity in 2-D Flow is:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

So irrotational flow is: $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

(alternatively, $\phi_{xy} = \phi_{yx}$)

Therefore stresses in the x- and y-direction equal to

$-2\mu\phi_{xy}$ and $-2\mu\phi_{yy}$ must be applied at the free surface (x & y dir. respectively). These stresses do work at rate:

$$\frac{\text{Force}}{\text{Area}} \times \text{Velocity}$$

$$W = -2\mu\,\phi_{yy}\,\phi_y - 2\mu\,\phi_{xy}\,\phi_x$$

if $\phi = a\,c\,\cos k(x-ct)\,e^{ky}$

then average work done per unit time per unit area becomes: $-2\mu k^3 a^2 c^2$. Recall that the energy per unit surface area ($KE+PE=2PE$) is $\frac{1}{2}\rho a^2 c^2 k$.

If we take $a=a(t)$, i.e. variable amplitude, equating work done and rate of change of energy:

$$-2\mu k^3 a^2 c^2 = \frac{d}{dt}\left(\frac{1}{2}\rho a^2 c^2 k\right)$$

(Soln of this ode is:)

$$\boxed{a(t) = a_0 e^{-2\nu k^2 t}}$$

(this is a measure of how fast the wave amplitude decays)

For water (@ 15°C, $\nu = .0114$ cm/s²) the time constant, $\frac{1}{2\nu k^2} = 1.11\,\lambda^2$ second, with λ in cm.

(cf. 0.712 in Lamb) ⇒ 0°C!

For example, $\lambda = 1.7$ cm (wavelength corresponding to minimum wavespeed on water)

this takes ≈ 3.18 seconds for amplitude to decrease by e.

This is a long time, considering at $\lambda = 1.7$ cm, wave frequency is: $\omega = \sqrt{k\left(g+\frac{\sigma}{\rho}k^2\right)} = 85$ radians/sec and wave period is $T = \frac{2\pi}{\omega} = 0.074$ seconds (≈ 43 periods)

Internal dissipation produces a proportional loss of wave PE $2\nu k^2$ and loss of total (KE+PE) wave energy $4\nu k^2$, per unit time, and $8\pi\nu\frac{k^2}{\omega}$ per period $\left(\frac{2\pi}{\omega}\right)$. T.P.

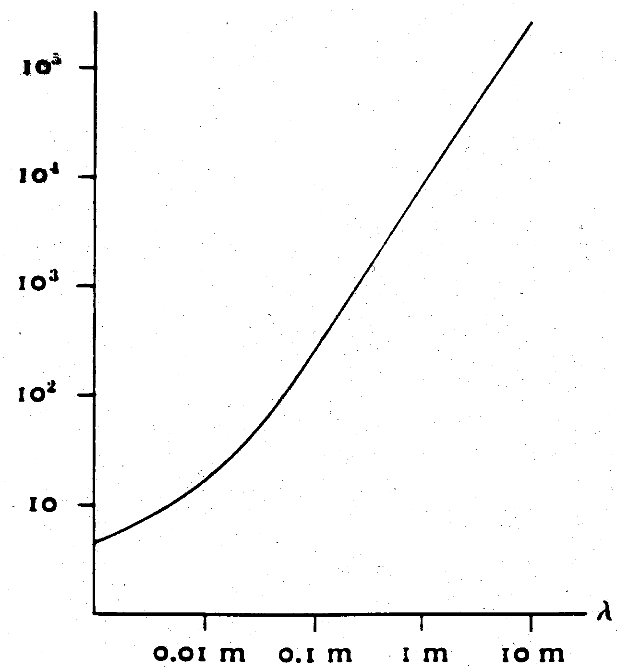


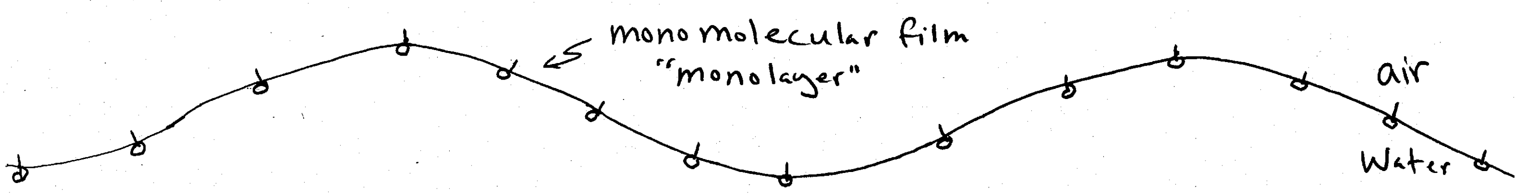
Figure 59. Number of periods required for the energy of sinusoidal waves of length λ on deep water to be reduced by a factor e through internal viscous dissipation.

(The third mechanism for wave dampening:) ✓(Show Movie on σ)

3) <u>Surface dissipation</u> Refs. Adamson, <u>Physical Chemistry of Surfaces</u>

Edwards, Brenner & Wasan <u>Interfacial Transport Processes & Rheology</u>

In the presence of surface-active materials (surfactants), wave motion causes concentration gradients, hence surface tension gradients, requiring added terms in shear stress BC. This can lead to greatly increased dissipation in surface boundary layer & much more rapid viscous damping. (basis of the proverbial "oil on troubled waters")



(why do we care about surfactant monolayers?)
It is rare for the air-water interface to be completely free of surfactants. Countless materials are active on the air-water interface.
Any molecule that is "amphiphilic", i.e. has a polar head (hydrophilic) and hydrophobic tail group will be energetically favored to reside at the air-water interface, where it reduces the surface tension (thus making the interface <u>elastic</u> due to the concentration dependence of surface tension).
It takes as little as 1 milligram of any number of long-carbon-chain molecules to decrease the surface tension of a 1 m^2 water surface by as much as 50%. In addition to their strong influence on wave damping, surfactants play an important role in two-phase flow, in the case of water particularly, e.g. boiling; also, transport of mass, momentum & energy between ocean and atmosphere, among other things. Also in respiration, material processing involving film drainage (e.g. foam manufacturing), etc. (e.g., it has been estimated that $\sim\frac{1}{2}$ of our basic metabolic rate would have been spent on respiration action - i.e. additional work of the diaphragm - if our lungs were not covered with surface-tension reducing material) (also note surfactant repla