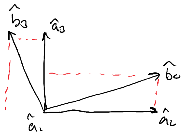
Reference frames
Basis / coordinate transformations
Standard Rotation Matrices:



$$\left[{}^{B}C^{A}\right]_1 = \begin{array}{c|ccc} & a_1 & a_2 & a_3 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & C_\theta & S_\theta \\ b_3 & 0 & -S_\theta & C_\theta \end{array}$$

$$\left[{}^{B}C^{A}\right]_2 = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \qquad \left[{}^{B}C^{A}\right]_3 = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- direct or Analytic derivative $\quad \frac{{}^{A}\partial \hat{a}_i}{\partial q_r} = 0$
- Notation $\left\{\frac{{}^{A}\partial}{\partial q}\vec{P}\right\}_B$
- Kinematic derivative:

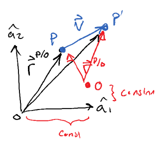
$$\frac{{}^{A}d}{dt}(\;) = \frac{{}^{B}d}{dt}(\;) + {}^{A}\vec{\omega}^{B}\times(\;)$$

- Cross Product must be evaluated in the same Basis
- Need angular velocity

$${}^{A}\vec{V}^{P/o} = \frac{{}^{A}d}{dt}\left(\vec{r}^{P/o}\right)$$

*Point "o" is fixed in A

$${}^{A}\vec{a}^{P/o} = \frac{{}^{A}d}{dt}\left({}^{A}\vec{V}^{P/o}\right)$$

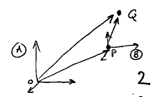


$$\boxed{{}^{A}\vec{V}^{Q} = {}^{A}\vec{V}^{P/o} + {}^{B}\vec{V}^{Q/P} + {}^{A}\vec{\omega}^{B}\times\vec{r}^{\,Q/P}}$$

o is fixed in A, P is fixed in B

Velocity of two points

"Velocity of Q through Point P"



P: Previous
Q: Next

2 points in a body:

$${}^{B}\vec{V}^{Q/P} = 0 \qquad {}^{B}\vec{a}^{Q/P} = 0$$

* acceleration of two Points

$$\boxed{{}^{A}\vec{a}^{Q} = {}^{A}\vec{a}^{P} + {}^{B}\vec{a}^{Q/P} + {}^{A}\vec{\alpha}^{B}\times\vec{r}^{\,Q/P} + 2\,{}^{A}\vec{\omega}^{B}\times{}^{B}\vec{V}^{Q/P} + {}^{A}\vec{\omega}^{B}\times\left({}^{A}\vec{\omega}^{B}\times\vec{r}^{\,Q/P}\right)}$$

- B projecte onto A (derive)
- take a vector in A (use) express it in B

$$\tilde{V}_x = \begin{bmatrix} 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix}_A$$

$$\left[{}^{A}C^{D}\right] = \left[{}^{A}C^{B}\right]\left[{}^{B}C^{C}\right]\left[{}^{C}C^{D}\right] \qquad \left[{}^{A}C^{B}\right] = \left[{}^{B}C^{A}\right]^T$$

Space 123 = Body 321

Rotation about a fixed axis:

$$\{\vec{r}\}_N = {}^{N}C^{A}\,{}^{A}C^{B}\,C_\beta\,{}^{A}C^{T}\,{}^{N}C^{T}\{\vec{r}\}_{N\,\text{Initial}}$$

- Angular velocity: if Simple: $\hat{a}_i = \hat{b}_i = \hat{\lambda}$ $\quad {}^{A}\vec{\omega}^{B} = \dot{\theta}\hat{\lambda}$

if Not $\;{}^{A}\vec{\omega}^{D} = {}^{A}\vec{\omega}^{C} + {}^{C}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{D}$

- Angular acceleration: definition $\quad {}^{A}\vec{\alpha}^{B} = \frac{{}^{A}d}{dt}\left({}^{A}\vec{\omega}^{B}\right)$

- Angular Acceleration:

if ${}^{N}\vec{\omega}^{B} = {}^{N}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B}$ then

$${}^{N}\vec{\alpha}^{B} = {}^{N}\vec{\alpha}^{A} + {}^{A}\vec{\alpha}^{B} + {}^{N}\vec{\omega}^{A}\times{}^{A}\vec{\omega}^{B}$$
$$= \;'' \;+\; '' \;+\; {}^{N}\vec{\omega}^{B}\times{}^{A}\vec{\omega}^{B}$$
$$= \;'' \;+\; '' \;+\; {}^{N}\vec{\omega}^{A}\times{}^{N}\vec{\omega}^{B}$$

Rolling:

Always Points to Contact Point

$${}^{N}\vec{V}^{A_B} = {}^{N}\vec{V}^{A_0} + {}^{N}\vec{\omega}^{A}\times\boxed{\vec{r}^{\,A_B/A_0}} \qquad \left({}^{A}\vec{V}^{A_B/A_0} = 0\right)$$

A_0 fixed in (A) at this instant

(A) is a frame attached to Body A

A_0 fixed in A

$${}^{N}\vec{V}^{B_A} = {}^{N}\vec{V}^{B_0} + {}^{N}\vec{\omega}^{B}\times\vec{r}^{\,B_A/B_0} \qquad \left({}^{B}\vec{V}^{B_A/B_0} = 0\right)$$

B_A fixed in (B) at this instant

(B) is a frame attached to Body B

B_0 fixed in B

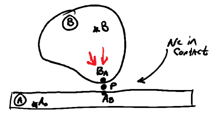
No slip:

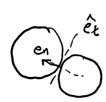
$${}^{N}\vec{V}^{B_A} = {}^{N}\vec{V}^{A_B}$$

with slip:

$${}^{N}\vec{V}^{B_A}\cdot\hat{e}_n = {}^{N}\vec{V}^{A_B}\cdot\hat{e}_n$$

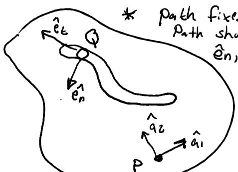
$${}^{N}\vec{a}^{B_A} \neq {}^{N}\vec{a}^{A_B} \text{ for Both}$$





Path variables:

* path fixed in A
Path shape is known/specified
$\hat{e}_n, \hat{e}_t, e$ defined

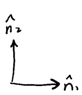


$${}^{A}\vec{V}^{Q} = v\hat{e}_t$$

$${}^{A}\vec{a}^{Q} = \dot{v}\hat{e}_t + \frac{v^2}{e}\hat{e}_n$$

Common Usage:

$${}^{N}\vec{V}^{Q} = {}^{N}\vec{V}^{P} + \underbrace{{}^{A}\vec{V}^{Q/P}}_{v\hat{e}_t} + {}^{N}\vec{\omega}^{A}\times\vec{r}^{\,Q/P}$$



$${}^{N}\vec{a}^{Q} = {}^{N}\vec{a}^{P} + \underbrace{{}^{A}\vec{a}^{Q/P}}_{\dot{v}e_t + \frac{v^2}{e}\hat{e}_n} + {}^{N}\vec{\alpha}^{A}\times\vec{r}^{\,Q/P} + 2\,{}^{N}\vec{\omega}^{A}\times{}^{A}\vec{V}^{Q/P} + {}^{N}\vec{\omega}^{A}\times\left({}^{N}\vec{\omega}^{A}\times\vec{r}^{\,Q/P}\right)$$

Lagranges Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} + \frac{\partial Q_d}{\partial \dot{q}_r} = Q_r^{n.c.}$$

$$L = T - V$$

$$Q_d = \sum_i \tfrac{1}{2} C_i \dot{q}_i^{\,2} \quad \text{dissapation function (damper)}$$

$$Q_r^{N.C} = \sum {}^{N}\vec{F}_i\cdot\frac{\partial \vec{r}_i}{\partial q_r} + \sum \vec{M}_j\cdot\frac{\partial {}^{N}\vec{\omega}_j}{\partial \dot{q}_r}$$

Sum of All Applied/External N.C. forces on the System

$$= mgh, \quad F = \frac{-\partial V}{\partial q_i} \qquad F_d = -\frac{\partial Q_d}{\partial \dot{q}_i}$$

$$s = \tfrac{1}{2}Kx^2, \quad F_s = \frac{-\partial V}{\partial q_i}$$

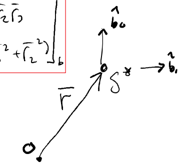
$$T^S = \tfrac{1}{2}m\,{}^{N}\vec{V}^{S^*}\cdot{}^{N}\vec{V}^{S^*} + \tfrac{1}{2}\,{}^{N}\vec{\omega}^{S}\cdot I^{S/S^*}\cdot{}^{N}\vec{\omega}^{S}$$

$r = 1, \dots, M$
$M = n - m$
n: total # of generalized coordinates
m = # of holonomic constraints
T = total Kinetic Energy (scalar)
V = total Potential Energy (scalar)

$$\boxed{\vec{H}^{S/o} = \int_V \vec{r}\times({}^{N}\vec{\omega}^{S}\times\vec{r})\,e\,dV} \qquad \left[\vec{H}^{S/o}\right]_a = \left[I^{S/o}\right]_a\left[{}^{N}\vec{\omega}^{S}\right]_a$$

$$\left[I^{S/o}\right]_b = \left[I^{S/S^*}\right]_b + m\begin{bmatrix} (\bar{r}_2^2+\bar{r}_3^2) & -\bar{r}_1\bar{r}_2 & -\bar{r}_1\bar{r}_3 \\ '' & (\bar{r}_1^2+\bar{r}_3^2) & -\bar{r}_2\bar{r}_3 \\ '' & '' & (\bar{r}_1^2+\bar{r}_2^2) \end{bmatrix}_b$$

Parallel axis theorem



Basis transformation

$$\left[I^{S/o}\right]_b = \left[{}^{B}C^{A}\right]\left[I^{S/o}\right]_a\left[{}^{A}C^{B}\right]$$

Newton Euler Equations

$$\sum \vec{M}^{S/o} = m\,\vec{r}^{\,S^*/o}\times{}^{N}\vec{a}^{S^*} + I^{S/S^*}\cdot{}^{N}\vec{\alpha}^{A} + {}^{N}\vec{\omega}^{A}\times\left(I^{S/S^*}\cdot{}^{N}\vec{\omega}^{A}\right)$$

$$\sum \vec{F}^{S} = m\,{}^{N}\vec{a}^{S^*} \qquad \text{Any Point O, basis Fixed in body}$$

Evaluation

$$\left[\sum \vec{F}^{S}\right]_{c'} = m\left[{}^{N}\vec{a}^{S^*}\right]_c \qquad c, c' \text{ any basis to equate } c' = c$$

$$\left[\sum M^{S/o}\right]_{b''} = \left[\vec{r}^{\,S/o}\,m\right]_{b'}\times\left[{}^{N}\vec{a}^{S^*}\right]_{b'} + \left[I^{S/S^*}\right]_b\left[{}^{N}\vec{\alpha}^{S}\right]_b + \left[{}^{N}\vec{\omega}^{S}\times\right]_b\left[I^{S/S^*}\right]_b\left[{}^{N}\vec{\omega}^{S}\right]_b$$

* $\boxed{b\text{ - body fixed basis}}$ *
b', b'' - any basis to equate $b = b' = b''$

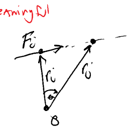
Example:

$$\left\{\left[\sum \vec{F}\right]_a\right\}_c = \left\{m\left[{}^{N}\vec{a}^{S^*}\right]_b\right\}_c$$

equate them in the same Basis to obtain meaningful equations

evaluate each term in any basis or even mixed basis.

$$M^{i/o} = \vec{r}^{\,i/o}\times\vec{F}_i$$



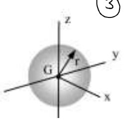
Eulers Equation (Alternate)

$$\sum \vec{M}^{S/o} = m\,\vec{r}^{\,S^*/o}\times{}^{N}\vec{a}^{o} + I^{S/o}\cdot{}^{N}\vec{\alpha}^{S} + {}^{N}\vec{\omega}^{S}\times\left(I^{S/o}\cdot{}^{N}\vec{\omega}^{S}\right)$$

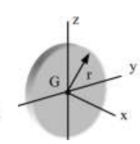
* Point o Fixed in Body *

General strategy:

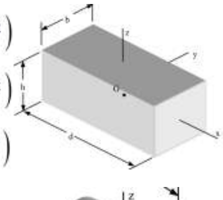
1. Find I^{S/S^*}
2. Line up Basis Vectors (Basis transformation)
3. move origin (Parall axis theorem)



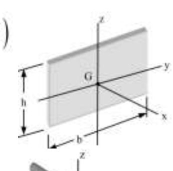
$$I_x = I_y = I_z = \frac{2}{5}mr^2$$



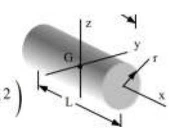
$$I_x = \frac{1}{2}mr^2 \qquad I_y = I_z = \frac{1}{4}mr^2$$



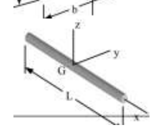
$$I_x = \frac{1}{12}m\left(b^2+h^2\right) \qquad I_y = \frac{1}{12}m\left(h^2+d^2\right) \qquad I_z = \frac{1}{12}m\left(b^2+d^2\right)$$



$$I_x = \frac{1}{12}m\left(b^2+h^2\right) \qquad I_y = \frac{1}{12}mh^2 \qquad I_z = \frac{1}{12}mb^2$$



$$I_x = \frac{1}{2}mr^2 \qquad I_y = I_z = \frac{1}{12}m\left(L^2+3r^2\right)$$



$$I_y = I_z = \frac{1}{12}mL^2$$