Solution Set. HW #2

1) given $\eta = a e^{i(kx-\omega t)}$, Can find the (most general) Velocity Potential ϕ from our boundary conditions: $\begin{cases} \phi_t = -g\eta \\ \phi_y = \eta_t \end{cases}$ (at $y=0$)

at $y=0$,

$$\phi_t = -g\eta = -ga e^{i(kx-\omega t)}$$ (Note, this is not applicable at depth)

Integration gives:

$$\phi = \frac{-ga}{-i\omega} e^{i(kx-\omega t)} + F(x,y) \quad (\text{at } y=0)$$

$$= -i \frac{ga}{\omega} e^{i(kx-\omega t)} + F(x,y) \quad (\text{at } y=0) \qquad ----- \quad (1)$$

Can write the most general form of ϕ:

$$\phi = -i\frac{ga}{\omega} e^{i(kx-\omega t)} \frac{\cosh k(y+h)}{\cosh kh}$$

We can now find the Pressure at bottom from Bernoulli's eqn.:

$$P - P_0 = -\rho\phi_t - \rho g y \quad (\text{linearized})$$

$$\Rightarrow \boxed{P = P_0 + \rho g h + \frac{\rho g a}{\cosh(kh)} e^{i(kx-\omega t)}} \quad (\text{at } y=-h)$$

(Note for large depth, $P = P_0 + \rho g h + \rho g a e^{-kh} e^{i(kx-\omega t)}$)

2) Consider a two-liquid layer:

$$\phi_1 = A_1 e^{i(kx-\omega t)-ky} \quad \text{for } y>0 \quad (\phi_1 \to 0 \text{ as } y \to \infty \ \checkmark)$$

$$\phi_2 = A_2 e^{i(kx-\omega t)+ky} \quad \text{for } y<0 \quad (\phi_2 \to 0 \text{ as } y \to -\infty \ \checkmark)$$

Which satisfy $\nabla^2 \phi = 0$.

With interface $\eta = a\, e^{i(kx-\omega t)}$, $ka \ll 1$ (amplitude << Wavelength)

Interface Velocity, in linear approximation: $\dfrac{\partial \eta}{\partial t} = \left.\dfrac{\partial \phi_1}{\partial y}\right|_{y=0} = \left.\dfrac{\partial \phi_2}{\partial y}\right|_{y=0}$

(this is the Kinematic bc)

Substitute-in the η and ϕ's, find: $-i\omega a = -kA_1 = kA_2$ (1)

Pressure on each side of interface ($y=0^+$ and $y=0^-$) is:

$$P_1\big|_{y=0} = -\rho_1\left(\left.\frac{\partial \phi_1}{\partial t}\right|_{y=0} + g\eta\right) \quad \text{and} \quad P_2\big|_{y=0} = -\rho_2\left(\left.\frac{\partial \phi_2}{\partial t}\right|_{y=0} + g\eta\right)$$

Equating the Pressures $-\rho_2(-i\omega A_2 + ga) + \rho_1(-i\omega A_1 + ga) = 0$

Use Eqn. (1), get: $-\rho_2\left(-\dfrac{\omega^2}{k} + g\right) + \rho_1\left(\dfrac{\omega^2}{k} + g\right) = 0$

$$\Rightarrow \boxed{\omega^2 = gk\,\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

Note, if $\rho_1 = 0$, the dispersion relation reduces to that of a surface wave.

3) Since $\nabla^2\phi = 0$ it follows that $\nabla^2\phi' = 0$. Also, at $y=h$, $\frac{\partial\phi'}{\partial y} = 0$ as in the case of stationary fluid, hence $\boxed{F(y) \sim \cosh k(h-y)}$

and ϕ' has the form: $\phi' = B e^{i(kx-\omega t)} \cosh k(h-y)$. (Note: $h-y$ not $y-h$)

Bernoulli's Eqn., $P - P_0 = -\rho\phi_t - \rho\frac{|\nabla\phi|^2}{2} - \rho g y$, gives at the surface:

$$\frac{\partial\phi'}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - g\eta = \text{constant} \quad \text{or } f(t) \qquad (\text{at } y=\eta)$$

$$\frac{\partial\phi'}{\partial t} + \frac{V^2}{2} + uV + \frac{u^2}{2} - g\eta = \text{Constant} \quad \text{where } u \equiv \frac{\partial\phi'}{\partial x}$$

Note the disturbance due to the wave, $\frac{\partial\phi'}{\partial x}$, is much smaller than the mean flow, V. Thus $\frac{u^2}{2}$ is second order and can be neglected compared to uV. Also, since $\frac{V^2}{2}$ is a constant, it is convenient to choose the constant on the RHS of eqn. to be equal to $\frac{V^2}{2}$. Thus the dynamic free surface condition reduces to:

$$\boxed{\frac{\partial\phi'}{\partial t} - g\eta + V\frac{\partial\phi'}{\partial x} = 0} \qquad \text{at } y=\eta$$

The Kinematic Surface condition is:

$$\frac{\partial \phi'}{\partial y} = \frac{D\eta}{Dt}$$

$$= \frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x}$$

or $\boxed{\frac{\partial \phi'}{\partial y} - \frac{\partial \eta}{\partial t} - V\frac{\partial \eta}{\partial x} = 0}$ at $y = \eta$

(Combined BC becomes: $\phi'_{tt} - g\phi'_y + 2V\phi'_{xt} + V^2\phi'_{xx} = 0$)

Plug ϕ into our Combined BC:

$$(-i\omega)^2\phi' + gk\phi' \tanh kh + 2V\phi'(ik)(-i\omega) + V^2\phi'(ik)^2 = 0$$

dividing through by ϕ' and rearranging. Can write:

$$\boxed{\left(\frac{\omega}{k} - V\right)^2 = \frac{g}{k}\tanh(kh)}$$

which is identical to results obtained in class if we set $V = 0$

And the Phase Speed:

$$\boxed{c = \frac{\omega}{k} = V \pm \sqrt{\frac{g}{k}\tanh(kh)}}$$