MATH 4600: ADVANCED CALCULUS
Spring 2017

TEST II

NAME (Please print) ____________________________

NOTES

1. Please make sure that your answer book has 8 pages.
2. Attempt all four problems; these are not equally weighted.
3. **Read the questions carefully before answering**.
4. If you would like full credit, then **justify your answers with appropriate, but brief, reasoning**.
5. Books, notes, crib sheets and calculators are not to be used.
6. Put your mobile devices away.
7. Best wishes.

1	
2	
3	
4	
TOTAL	

1. (25 points) Figure shows a solid body in the first octant bounded by the three coordinate planes and a plane roof. The curved wall of the body is the surface $x^2 + y = 4$. Some additional information is provided in the sketch. Write down an integral for the volume of the body, and evaluate it.

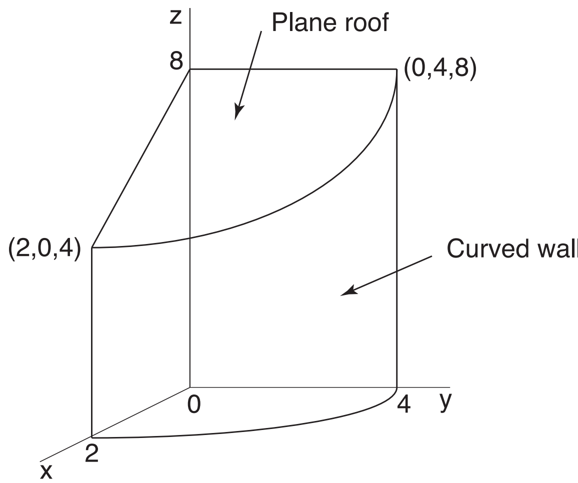


Figure 1: Problem 1

We shall need the equation of the planar roof. The plane is parallel to the y-axis, and therefore has an equation of the form $z = ax + b$. Passage through the points $(0, 0, 8)$ and $(2, 0, 4)$ yields the equations

$$8 = b,\ 4 = 2a + b, \quad \text{so that} \quad a = -2,\ b = 8.$$

Thus $z = -2x + 8$ on the roof. The solid is now bounded as follows.

$$0 \le z \le -2x + 8, \quad 0 \le y \le 4 - x^2, \quad 0 \le x \le 2.$$

Therefore the volume is given by

$$\begin{aligned} V &= \int_0^2 \int_0^{4-x^2} \int_0^{-2x+8} dz\, dy\, dx \\ &= \int_0^2 \int_0^{4-x^2} (8 - 2x)\, dy\, dx \\ &= \int_0^2 (4 - x^2)(8 - 2x)\, dx \\ &= \int_0^2 (32 - 8x - 8x^2 + 2x^3)\, dx \\ &= \left[32x - 4x^2 - \frac{8x^3}{3} + \frac{x^4}{2} \right]_0^2 = \frac{104}{3}. \end{aligned}$$

2. (30 points)

   (a) A uniform fluid that flows vertically downwards (heavy rain) is described by the vector field $\boldsymbol{F}(x,y,z) =< 0,0,-1 >$. Find the total flux through the cone $z^2 = x^2 + y^2,\ x^2 + y^2 \leq 1$.

   (b) Suppose that the rain is driven sideways by a strong wind, so that the velocity field is now $\boldsymbol{F}(x,y,z) = -\left\langle 1/\sqrt{2}, 0, 1/\sqrt{2}\right\rangle$. Now what is the flux through the cone?

The flux of a vector field $\boldsymbol{F}$ across a surface S parametrized by $S:\ \boldsymbol{r}(u,v),\ (u,v) \in D,$ is

$$\mathcal{F} = \int\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS.$$

This can also be expressed as

$$\mathcal{F} = \int\int_D \boldsymbol{F} \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)\, dA,$$

provided $(\boldsymbol{r}_u \times \boldsymbol{r}_v)$, a normal to S, points outwards. Here, the surface of the cone can be parametrized as

$$S:\ \boldsymbol{r} = (r\cos\theta, r\sin\theta, r), \quad 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi.$$

Also, then,

$$\boldsymbol{r}_\theta \times \boldsymbol{r}_r = r(\cos\theta, \sin\theta, -1 >,$$

and the negative sign of the $\boldsymbol{k}$ component indicates outward normal.

   (a) Here, $\boldsymbol{F} =< 0,0,-1 >$ so that

   $$\boldsymbol{F} \cdot (\boldsymbol{r}_\theta \times \boldsymbol{r}_r) = r.$$

   Then

   $$\mathcal{F} = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \pi.$$

   (b) With $\boldsymbol{F} = -\left\langle 1/\sqrt{2}, 0, 1/\sqrt{2}\right\rangle$,

   $$\boldsymbol{F} \cdot (\boldsymbol{r}_\theta \times \boldsymbol{r}_r) = \frac{1}{\sqrt{2}} r(1 - \cos\theta),$$

   so that

   $$\begin{aligned} \mathcal{F} &= \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{2}} r(1-\cos\theta)\, dr\, d\theta \\ &= \frac{\pi}{\sqrt{2}}. \end{aligned}$$

3. (25 points)

(a) Determine the value of the integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r},$$

where

$$\boldsymbol{F} = (e^{-y} - ze^{-x})\, \boldsymbol{i} + (e^{-z} - xe^{-y})\, \boldsymbol{j} + (e^{-x} - ye^{-z})\, \boldsymbol{k},$$

and C is the path

$$\boldsymbol{r} = \frac{\ln(1+t)}{\ln 2}\, \boldsymbol{i} + \sin(\pi t/2)\, \boldsymbol{j} + \frac{1-e^t}{1-e}\, \boldsymbol{k}, \quad 0 \le t \le 1.$$

Think before you compute.

(b) Let C be the unit circle centered at the origin, traversed counterclockwise. Let $\boldsymbol{F}$ be a vector field of magnitude M inclined at an angle of 45° to the tangent to C at every point of C. Draw a relevant sketch and find the circulation of $\boldsymbol{F}$ around C.

(a) Let us assume that $\boldsymbol{F}$ is a gradient field with potential f so that $\boldsymbol{F} = \nabla f$. The assumption will be confirmed if f can be successfully constructed. We have

$$f_x = e^{-y} - ze^{-x} \quad \text{so that} \quad f = xe^{-y} + ze^{-x} + g(y,z).$$

Then

$$f_y = -xe^{-y} + g_y = e^{-z} - xe^{-y} \quad \text{so that} \quad g_y = e^{-z}, \quad \text{or,} \quad g = ye^{-z} + h(z).$$

At this stage, $f = xe^{-y} + ze^{-x} + ye^{-z} + h(z)$. It remains to satisfy

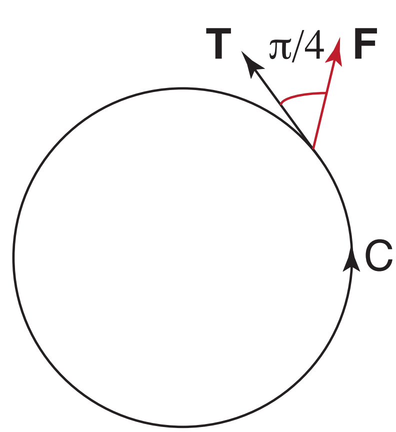
$$f_z = e^{-x} - ye^{-z} + h'(z) = e^{-x} - ye^{-z}.$$

Therefore $h'(z) = 0$, which we satisfy by letting $h(z) = 0$. Thus $f = xe^{-y} + ze^{-x} + ye^{-z}$. Therefore the line integral is the potential difference between the end point $A(0,0,0)$ and the starting point $B(1,1,1)$. The result is

$$I = 3e^{-1} - 0 = 3e^{-1}.$$

(b) The tangential component of the vector field is $M/\sqrt{2}$, and the length of the curve is 2π. Therefore the circulation is

$$\Gamma = \frac{2\pi M}{\sqrt{2}}.$$



4. (20 points) Use the method of Lagrange multipliers to show that the rectangular box with fixed surface area and maximum volume is a cube.

Let the sides of the rectangular box be x, y and z, the volume V and the surface area $2S$. Then

$$g(x, y, z) = xy + yz + zx - S = 0 \tag{1}$$

is the constraint under which

$$V(x, y, z) = xyz$$

is to be maximized. We expect that x, y and z will all be positive.

The necessary conditions are $\nabla V = \lambda \nabla f$, leading to

$$\begin{aligned} yz - \lambda(y + z) &= 0 && (2) \\ zx - \lambda(z + x) &= 0 && (3) \\ xy - \lambda(x + y) &= 0. && (4) \end{aligned}$$

Eliminate λ from (2) and (3) by means of the operation $(z + x) \times$ (2) - $(y + z) \times$ (3). The result is

$$yz(z + x) - zx(y + z) = 0.$$

The above simplifies to

$$(y - x)z^2 = 0.$$

As $z \neq 0$, the only solution is $y = x$. A similar operation with (3) and (4) leads to $y = z$. Then (1) yields the result

$$x = y = z = \sqrt{\frac{s}{3}}.$$

Thus there box is a cube. The result does maximize V. Minimizing V would require admitting the possibility that one of the sides is zero, leading to zero as the minimum of V. Incidentally, once x, y and z are determined, λ is given by

$$\lambda = \frac{xy}{x + y} = \frac{x}{2} = \sqrt{\frac{s}{12}}.$$