The initial conditions must be Prescribed at the beginning $t = t_0$ for the dependent variable $\vec{u}$. Note however, that the other dependent variable P does not need to be prescribed since it can be determined (up to an additive constant) using the eqn. of motion with the given $\vec{u}$. (Lugt shows that:)

if we take the

$$\mathrm{div}\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}\right)$$

and using $\mathrm{div}\,\vec{u} = 0$, get a Poisson equation for P:

$$\nabla^2 p = -\rho\, \mathrm{div}(\vec{u}\cdot\nabla)\vec{u} \quad ------ \quad (3)$$

This eqn. can be solved (numerically) with the boundary condition ∇p obtained from the (initial) velocity field.

Similarly, the boundary condition for $\vec{u}$ must be Prescribed, while no boundary conditions are required for Pressure.

(another notation)

Problem formulation using (2) and (3) is generally referred to as "Pressure - Poisson Formulation"

The Specified Velocity BC must Satisfy Continuity for incompressible flo

(general Statement of Solenoidal Condition)

$$\iint_S \vec{n}\cdot\vec{u}\, ds = 0 \qquad \text{for } t \geq 0$$

(this becomes a complicated issue if the boundary is for entire flow field and extends to ∞)

What about Surface of a Solid body?
Of course, by definition, no fluid can cross the boundary, and the velocity of the boundary normal to itself must be equal to the normal component of fluid velocity

The solid boundary (which may be compliant, i.e. flexible) can be described by: $f(x,y,z,t)=0$, and normal velocity to itself:

$$-\frac{f_t}{\sqrt{f_x^2+f_y^2+f_z^2}} \quad (\partial f/\partial t)$$

must be equal to the fluid velocity normal to it

$$\frac{u f_x + v f_y + w f_z}{\sqrt{f_x^2+f_y^2+f_z^2}}.$$ Therefore,

$$\frac{df}{dt} = 0 = f_t + u f_x + v f_y + w f_z$$

$(\vec{U} = u\vec{i} + v\vec{j} + w\vec{k})$

which in a reference frame fixed to boundary can be written:

$$\boxed{\vec{n}\cdot\vec{U}=0}$$ (i.e., fluid velocity normal to a solid boundary is zero)

(Kinematic Condition)

Furthermore, we generally apply no-slip condition, i.e. tangential velocity on a (stationary) solid wall must vanish.

(Side note)
Note that fluid mechanicians rely on kinetic theory to furnish the no-slip condition, but from simple considerations in gas kinetics, find: $U_0 \approx \lambda \left.\frac{du}{dy}\right|_0$, U_0 is fluid velocity at wall, λ is mean-free path.

for air, $\lambda = O(10^{-8} - 10^{-7})\,m$ @ SL

(and) $= O(10^{-7} - 10^{-6})\,m$ @ flight altitude (say 30–40 k ft)

- For example, in the boundary layer on the wing of an aircraft, say at $x = 1\,m$, $U_\infty = 220\,m/s$ (500 mph), BL thickness (assuming laminar) $\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \Rightarrow \delta = 5x\sqrt{\frac{\nu}{x U_\infty}} = 3\times10^{-4}\,m$

$\Rightarrow \frac{du}{dy} \approx \frac{\Delta u}{\delta} = O(10^{6}\ s^{-1}) \Rightarrow U_0 = O(0.1\ m/s)$ (Negligible compared to 220 m/s!)

* For liquids, slip velocity is somewhat controversial & unless flow scales are very small, or fluid is non-Newtonian (e.g. polymer melt) under high shear, we can keep with no-slip B.C.
* Advent of microfluidics has not only brought this issue to forefront, but in case of gases in submicron channels, λ ($\approx 60\,nm$ @ 1 atm), it becomes important.

(the fact that tangential velocity must vanish) on a no-slip surface:

(can be expressed by:) $\boxed{\vec{n} \times \vec{u} = 0}$

- (beside the solid-wall boundary with its no-slip condition, we are interested in) Other boundary which we consider is that between two immiscible fluids A and B: (these result from the fact that velocity must be continuous across the interface and that a pressure jump occurs due to surface tension, as we saw in our wave work)

(for simplicity, we focus) In 2D, the stress components in the normal and tangential directions (i.e. the dynamic BC's):

$$[P] - \left[\frac{2\mu}{1+\eta_x^2}\left\{v_y + \eta_x^2 u_x - \eta_x(v_x + u_y)\right\}\right] = -\frac{\sigma}{R} - \frac{\sigma_y - \eta_x \sigma}{\sqrt{1+\eta_x^2}}$$

- Where $[\,]$ denotes the difference of a function between side A and side B of interface, Ex: $[P] = P_A - P_B$

and R is the radius of curvature $R = \frac{(1+\eta_x^2)^{3/2}}{\eta_{xx}}$ (as before) and tanential component:

$$\left[\mu\left\{2\eta_x(v_y - u_x) + (u_y + v_x)(1-\eta_x^2)\right]\right] = (\sigma_x + \eta_x \sigma_y)\sqrt{1+\eta_x^2}$$

(2.49 and 2.50 in Lugt)

here the intrinsic viscosity associated with the interface has been ignored (valid for clean interface; for a general description, see Aris, Vectors, Tensors and the Basic Eqns. of Fluid Mechanics)

The Kinematic BC for the interface is same as before (of course not same as solid wall)

For a "free surface", $\mu_B << \mu_A$ and the effects of B may be neglected. Furthermore, if interface is flat, $y = \eta = \text{const.}$ (0 if the coordinate system is placed on interface)

$\Rightarrow \eta_x = 0$, $v = 0$, and the normal & tangential stress boundary conditions become:

$$(P - P_0) = 0$$

i.e. Pressure on the liquid side of interface is same as atmospheric pressure

$$\mu u_y = \sigma_x$$

(These are eqns. 2.52 and 2.53 of Lugt as applied to a flat free surface)

The third type of formulation of eqns. of motion is the "Vorticity formulation" (which as we've seen already we get by taking the curl of the N-S eqn.)

in which we solve $\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega}$

along with a form of continuity. For example in 2D cases where a streamfunction can be defined, from the general expression $\vec{u} = \text{curl}\,\vec{A}$, we find for 2D flow $\vec{A} = \psi\vec{k}$, where ψ is the streamfunction. In Cartesian coordinates:

$$\frac{\partial \psi}{\partial y} = u \qquad \frac{\partial \psi}{\partial x} = -v$$

and the existence of the stream function ensures conservation of mass. ③

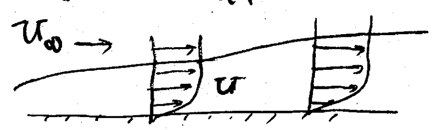
$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y\partial x} \Rightarrow -\frac{\partial V}{\partial y} = \frac{\partial u}{\partial x}$$

- and $\omega = (\omega_z) = \frac{\partial V}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2} = -\nabla^2\psi$

Therefore, by writing a single vorticity transport eqn., with $-\nabla^2\psi$ substituted in for ω, we can formulate the problem ("Vorticity-Streamfunction formulation")
(difficulty: can not be readily generalized for 3-D flow)

<u>Note</u> that the boundary conditions for vorticity can not be prescribed directly by vorticity, because vorticity on the boundary is usually part of the solution. We must therefore incorporate the boundary conditions for the velocity into a vorticity solution. For example, the tangential velocity

- component (such as no-slip, i.e. $U_t = 0$) can be prescribed and this boundary condition determines the amount of vorticity needed at the boundary (BL example



Vorticity at wall decreases w/ x)

Finally, (we should) consider Potential Flow (incompressible, irrot. flow) $\vec{u} = \nabla\phi$

since $\operatorname{div}\vec{u} = 0$ (Cont.)

$$\Rightarrow \nabla^2\phi = 0 \quad \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0\right) \text{ is governing eqn.}$$

which along with the BC $\frac{\partial\phi}{\partial n}$, the boundary value problem is uniquely defined for $\vec{u}$.

- Notice that this velocity field is completely described by the continuity eqn. and the mom. eqn. will only be needed to compute P.
(momentum is conserved by $\omega = 0$)
Note also that time derivatives do not appear in the Laplace eqn.

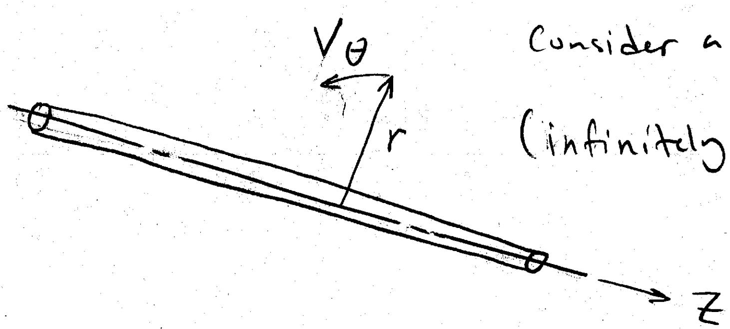
So that solutions of the Laplace eqn. can be steady or unsteady, determined solely by the velocity boundary conditions.

- This means that the history of the fluid motion is not involved and the solution is determined <u>only</u> by the instantaneous BCs.

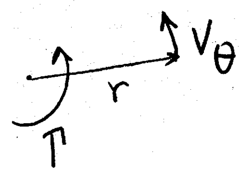
(before we begin our discussion of vorticity generation & decay, it's helpful to have seen how momentum and vorticity diffuse due to viscosity)
(A classic case which illustrates how the action of viscosity causes diffusion is:)

<u>Decaying Potential Vortex</u> ("Lamb Vortex" also "Oseen Vortex Soln.")

- Consider a potential vortex (infinitely thin, infinitely fast spinning vortex core)



Cross-Section:



$$V_\theta = \frac{\Gamma}{2\pi r}, \quad V_r = 0, \quad V_z = 0$$

(We want to know what would happen if we turn on viscosity) at $t=0$

Consider the vorticity transport eqn.

- $$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega}$$

In this case flow is 2D and vorticity only has one component. $\vec{\omega} = \omega_z \vec{k} = \omega \vec{k}$

and $\vec{u} = v_\theta \vec{e}_z$ (follows from Continuity eqn.: $\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$)

for axisymmetric flow: $\omega = \omega(r)$, $V_\theta = V_\theta(r)$

- and since $\omega_z = \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} \Rightarrow \omega = \frac{V_\theta}{r} + \frac{dV_\theta}{dr}$ ------ (1)

The z-component of the vorticity transport eqn. (Lugt P.93 gives $(\vec{\omega}\cdot\nabla)\vec{u}$ for Cartesian, cyln. & sphr. Coord.)
in cylindrical coordinates (r,θ,z)

$$\frac{\partial \omega_z}{\partial t} + \cancel{v_r}\frac{\partial \omega_z}{\partial r} + v_\theta \frac{1}{r}\cancel{\frac{\partial \omega_z}{\partial \theta}} + \cancel{v_z}\cancel{\frac{\partial \omega_z}{\partial z}} = \left(\cancel{\omega_r}\frac{\partial v_z}{\partial r} + \cancel{\omega_\theta}\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \omega_z \cancel{\frac{\partial v_z}{\partial z}}\right) +$$
$$\nu\left(\frac{\partial^2 \omega_z}{\partial r^2} + \frac{1}{r}\frac{\partial \omega_z}{\partial r} + \frac{1}{r^2}\cancel{\frac{\partial^2 \omega_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 \omega_z}{\partial z^2}}\right)$$

(Notice ω_r & ω_θ terms!)

w/BC:

$$\Rightarrow \boxed{\frac{\partial \omega}{\partial t} = \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right)} \quad \text{---(2)}$$

at $t \le 0$; $\Gamma = \oint \vec{u}\cdot d\vec{l} = \Gamma_0$

$t > 0$; $\omega_{(z)}$ and $V_{(\theta)}$ are non-singular

- To solve, we look for similarity solutions.

(by solving (2) we get ω_z and then we can use (1) to get V_θ)

(Since vorticity has units $\frac{1}{sec}$, we can look for ω such that:)

$$\omega = \frac{1}{t} f\left(\frac{r}{\sqrt{\nu t}}\right) \quad f \text{ is a dimensionless function}$$

$\frac{r}{\sqrt{\nu t}}$ is a similarity variable and is dimensionless; denote as η

viscous length scale — $\sqrt{\nu t}$

Substituting this form into the vorticity transport eqn. (2) (Pde), yields the following (ode):

- $$f'' + \left(\frac{1}{\eta} + \frac{\eta}{2}\right)f' + f = 0 \qquad \left(f' = \frac{df}{d\eta},\ f'' = \frac{d^2 f}{d\eta^2}\right)$$

This eqn. can be simplified by substituting $g(\eta) = f' + \frac{\eta}{2}f$
and becomes $\eta g' + g = 0$ which has the solution $g = \frac{A}{\eta}$ ← const

$$\Rightarrow \quad f' + \frac{\eta}{2} f = \frac{A}{\eta}$$

but since we can't have a singuler solution for ω as $\eta \to 0$, ($\eta = \frac{r}{\sqrt{\nu t}}$ and for finite ν and t approaching centerline we expect finite vorticity due to viscosity)

$\Rightarrow$ take (constant) $A = 0$

So $f' + \frac{\eta}{2} f = 0$ which has the solution: $f = \underset{\substack{\uparrow \\ \text{constant}}}{C} \exp\left(-\frac{\eta^2}{4}\right)$

thus, $\omega_z = \omega = \frac{C}{t} \exp\left(-\frac{r^2}{4\nu t}\right)$

(This eqn. describes how a point vortex diffuses out if we "turn on" the viscosity at $t=0$)

<u>to determine C:</u>

Consider Stokes theorem $\Gamma \equiv \oint \vec{u} \cdot d\vec{l} = \iint \vec{\omega} \cdot \vec{n}\, ds$

$$= \int_0^r \frac{C}{t} \exp\left(-\frac{r^2}{4\nu t}\right) 2\pi r\, dr$$

$$= 4\pi \nu C \left\{ 1 - \exp\left(-\frac{r^2}{4\nu t}\right) \right\}$$

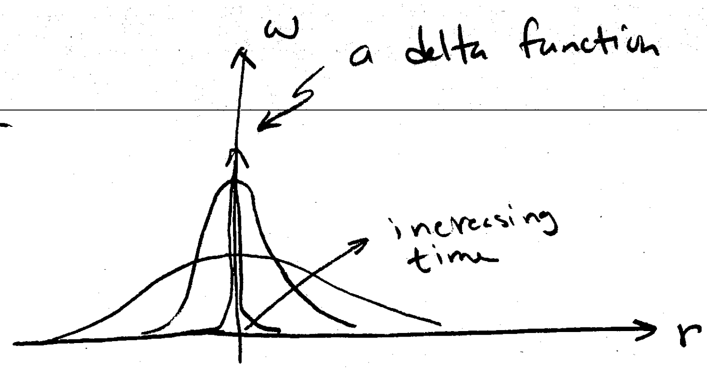
at $t=0$, $\Gamma = \Gamma_0 \Rightarrow C = \frac{\Gamma_0}{4\pi\nu}$

$$\therefore \; (\omega_z =) \boxed{\omega(r,t) = \frac{\Gamma_0}{4\pi\nu t} \exp\left\{-\frac{r^2}{4\nu t}\right\}}$$

[note that Lust (p.66) uses vortex strength $K = \frac{\Gamma}{2\pi}$ and gives $\omega = \frac{K_0}{2\nu t} e^{-r^2/4\nu t}$]

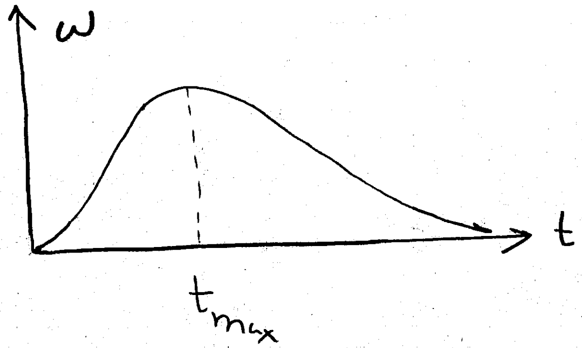
(notice that $\omega \to 0$ as $r \to \infty$)

a delta function for $t=0$ and/or $\nu=0$



(near the center, the behavior is like solid-body rotation)

at some $r_0 > 0$



$$t_{max} = \frac{r_0^2}{4\nu}$$

Now consider the velocity field $V_\theta(r,t)$
(it is possible to use our integral relation and find the velocity induced by this region of vorticity, but it is easier to just start from the definition of vorticity in terms of velocity, eqn. 1)

We have

$$\frac{V_\theta}{r} + \frac{dV_\theta}{dr} = \omega = \frac{\Gamma_0}{4\pi\nu t}\exp\left\{-\frac{r^2}{4\nu t}\right\}$$

Which we solve by seeking a solution of type

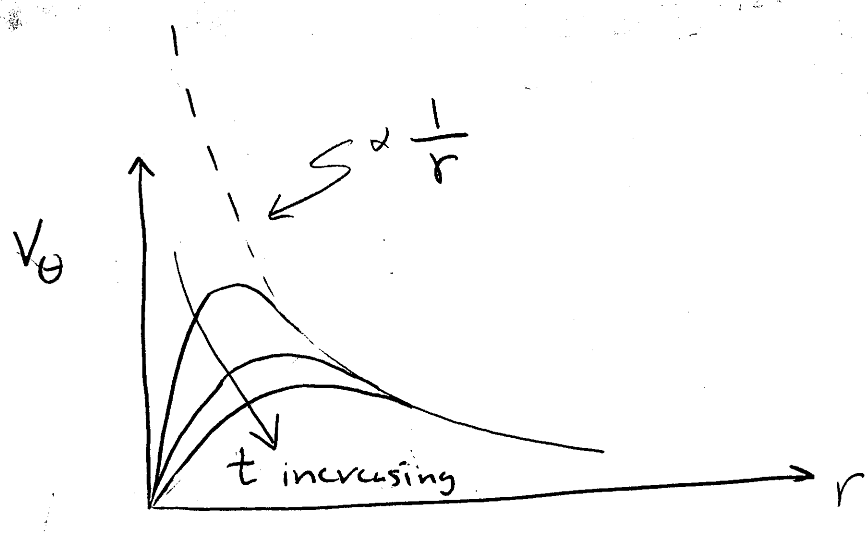
$$V_\theta = \frac{F(r,t)}{r}$$

which gives $F(r,t) = \frac{\Gamma_0}{2\pi}\left\{1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right\}$

$$\Rightarrow \boxed{V_\theta(r,t) = \frac{\Gamma_0}{2\pi r}\left\{1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right\}}$$

Notice that $V_\theta = \frac{\Gamma_0}{2\pi r}$ as $r \to \infty$
and $t \to 0$
and $\nu \to 0$

(which indeed gives the correct vorticity distribution $\frac{\Gamma_0}{4\pi\nu t}\exp(\,)$



Note: Can compute the pressure field using mom. eqn. in r-direction

$$\frac{dP}{dr} = \rho \frac{V_\theta^2}{r}$$

(Plug in the V_θ solution we got and integrate it)

* Show Vorticity Video

Vorticity Generation and Decay (Morton 1984)

(recall that the curl of N-S gave us the vorticity transport eqn.)

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\vec{u}^2\right) - \vec{u}\times\vec{\omega} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{u}$$

(when we took the curl we got:)

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{u} - \vec{\omega}\,\mathrm{div}\,\vec{u} + \left(\nabla P \times \nabla\left(\tfrac{1}{\rho}\right)\right)$$

(make the following observations:)

1) First note that this is valid in the absence of non-conservative forces

$$(N\text{-}S) \quad + \nabla f$$

(then curl $\nabla f \to 0$) ✓

So conservative forces, i.e. forces which are gradients of a potential, do not produce vorticity. e.g. gravity

2) In an incompressible flow ($\mathrm{div}\,\vec{u} = 0$) the only possible source of vorticity in the interior of fluid is the baroclinic term, i.e. must have inhomogeneous fluid (e.g. ocean)

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The Generation and Decay of Vorticity

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Vorticity, although not the primary variable of fluid dynamics, is an important derived variable playing both mathematical and physical roles in the solution and understanding of problems. The following treatment discusses the generation of vorticity at rigid boundaries and its subsequent decay. It is intended to provide a consistent and very broadly applicable framework within which a wide range of questions can be answered explicitly. The rate of generation of vorticity is shown to be the relative tangential acceleration of fluid and boundary without taking viscosity into account and the generating mechanism therefore involves the tangential pressure gradient within the fluid and the external acceleration of the boundary only. The mechanism is inviscid in nature and independent of the no-slip condition at the boundary, although viscous diffusion acts immediately after generation to spread vorticity outward from boundaries. Vorticity diffuses neither out of boundaries nor into them, and the only means of decay is by cross-diffusive annihilation within the fluid.

1. INTRODUCTION

The Helmholtz vorticity equation for an incompressible homogeneous fluid,

$$\partial \boldsymbol{\omega}/\partial t + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega},$$

---

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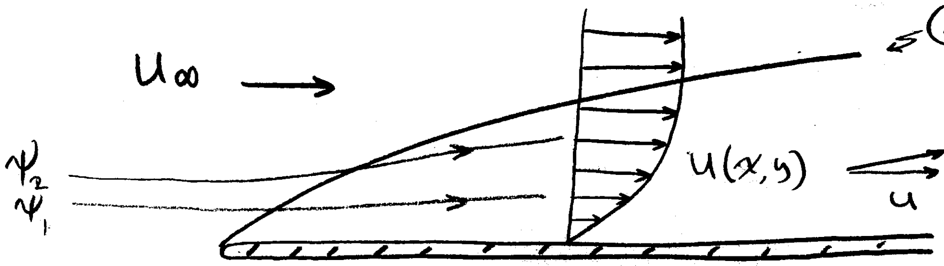
‡The National Center for Atmospheric Research is sponsored by the National Science Foundation.

(Since our primary interest is in fluids that are incompressible and homogeneous, we only look at the boundaries because it's the only source of vorticity) ①

- <u>Consider a flat-plate boundary-layer</u> (Blasius solution) of Prandtl's eqns.

(edge of BL, <u>not</u> a streamline)



(first, we ask ourselves, is there vorticity in the boundary layer?)

$$\omega = \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

(as we discussed last time, in a BL, since the flow is essentially parallel, v is small and changes along x are also very small, therefore, $\frac{\partial v}{\partial x}$ is a much smaller quantity than $\frac{\partial u}{\partial y}$. So it suffices to think of $-\frac{\partial u}{\partial y}$ as the vorticity)

$\vec{u}$ ↑v (Vertical velocity is due to the displacement effect is therefore small)

(Clearly) there is vorticity in the BL

a fluid element ⊠ later ⊠ deforms <u>and</u> (more importantly for us) rotates.

Note that vorticity is maximum near the wall and falls off to zero at its outer edge.

- (So the question is:) Is there vorticity generation at the wall?

(the answer is) <u>NO.</u> Although there is vorticity at the wall, there is no vorticity being generated at the wall; all the vorticity that exists was convected there

(Physically, I can explain this by noting the fact that the wall merely maintains the vorticity. Discuss the bowling ball analogy) (2)

(in fact, the vorticity at the wall decreases downstream, as we can see from the fact that $\left.\frac{\partial u}{\partial y}\right|_{y=0}$ decreases as we go downstream. The vorticity is neither being generated or diffused from the wall nor is diffused to the wall.)

$\nabla \omega' \to 0$ at wall

T.P. T.P.

(analogous to temperature: $T' \to 0$ at an insulated wall implies zero heat flux.)

How do we quantify vorticity generation?

(We could begin by writing the vorticity transport eqn. again:)

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{u} + \nu \nabla^2 \vec{\omega}$$

Since flow is 2-d:

$$\underbrace{\cancel{(\vec{\omega} \cdot \nabla)\vec{u}}}_{0}$$

(no stretching or turning of vorticity)

(and the only non-zero component of vorticity is normal to the plane of the flow, since 2-d)

$$\omega_3 \; (= \omega_z) = \omega$$

(in tensor notation:)

$$\underbrace{\cancel{\frac{\partial \omega_3}{\partial t}}}_{0} + u_j \frac{\partial \omega_3}{\partial x_j} = \nu \frac{\partial^2 \omega_3}{\partial x_j^2}$$

(w/) summation over repeated indices

0 steady state

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right)$$

(this eqn. tells us about the transport of ω but not necessarily about its generation at the wall. To see how vorticity is generated at wall)

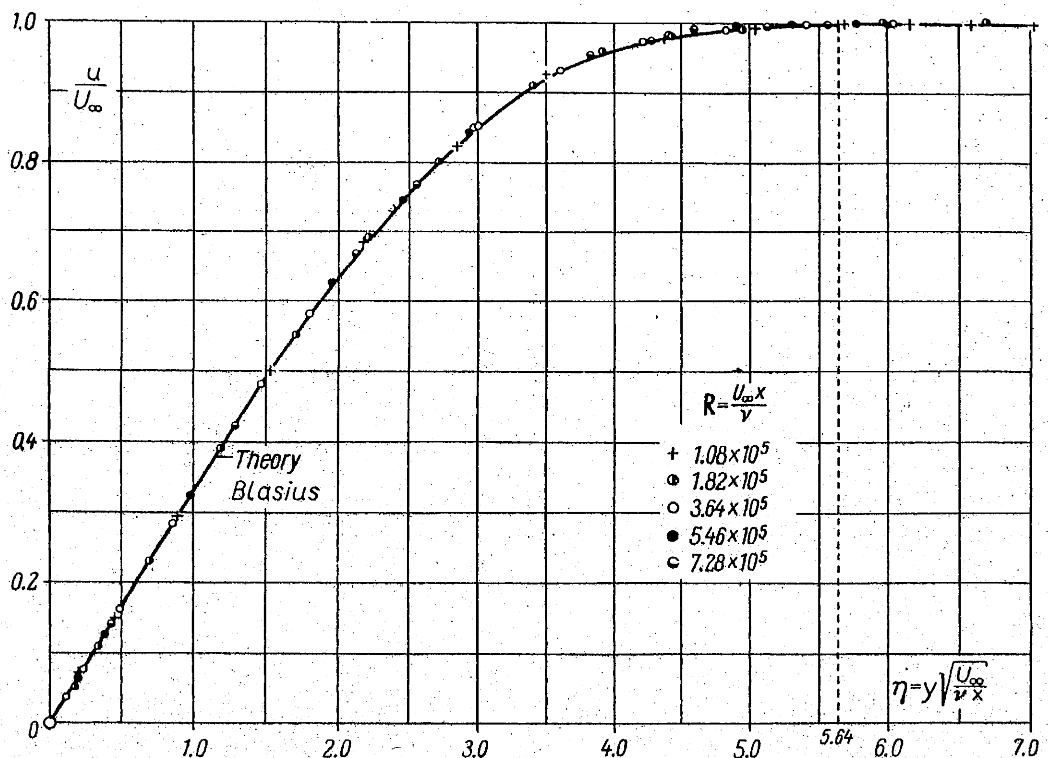


Fig. 7.9. Velocity distribution in the laminar boundary layer on a flat plate at zero incidence, as measured by Nikuradse [20]

Schlichting (7th ed.)

Schlichting (7th ed.)

Table 7.1. The function $f(\eta)$ for the boundary layer along a flat plate at zero incidence, after L. Howarth [16]

$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$	f	$f' = \frac{u}{U_\infty}$	f''
0	0	0	0·33206
0·2	0·00664	0·06641	0·33199
0·4	0·02656	0·13277	0·33147
0·6	0·05974	0·19894	0·33008
0·8	0·10611	0·26471	0·32739
1·0	0·16557	0·32979	0·32301
1·2	0·23795	0·39378	0·31659
1·4	0·32298	0·45627	0·30787
1·6	0·42032	0·51676	0·29667
1·8	0·52952	0·57477	0·28293
2·0	0·65003	0·62977	0·26675
2·2	0·78120	0·68132	0·24835
2·4	0·92230	0·72899	0·22809
2·6	1·07252	0·77246	0·20646
2·8	1·23099	0·81152	0·18401
3·0	1·39682	0·84605	0·16136
3·2	1·56911	0·87609	0·13913
3·4	1·74696	0·90177	0·11788
3·6	1·92954	0·92333	0·09809
3·8	2·11605	0·94112	0·08013
4·0	2·30576	0·95552	0·06424
4·2	2·49806	0·96696	0·05052
4·4	2·69238	0·97587	0·03897
4·6	2·88826	0·98269	0·02948
4·8	3·08534	0·98779	0·02187
5·0	3·28329	0·99155	0·01591
5·2	3·48189	0·99425	0·01134
5·4	3·68094	0·99616	0·00793
5·6	3·88031	0·99748	0·00543
5·8	4·07990	0·99838	0·00365
6·0	4·27964	0·99898	0·00240
6·2	4·47948	0·99937	0·00155
6·4	4·67938	0·99961	0·00098
6·6	4·87931	0·99977	0·00061
6·8	5·07928	0·99987	0·00037
7·0	5·27926	0·99992	0·00022
7·2	5·47925	0·99996	0·00013
7·4	5·67924	0·99998	0·00007
7·6	5·87924	0·99999	0·00004
7·8	6·07923	1·00000	0·00002
8·0	6·27923	1·00000	0·00001
8·2	6·47923	1·00000	0·00001
8·4	6·67923	1·00000	0·00000
8·6	6·87923	1·00000	0·00000
8·8	7·07923	1·00000	0·00000

Define vorticity flux vector $\vec{q}_{\omega} \equiv -\nu \nabla \omega$

(analogous to heat flux: $Q = -k \nabla T$)

(thus in our 2D flow:)

$$q_{\omega_3} = -\nu \nabla \omega_3$$

(to compute the vorticity flux, i.e. how much vorticity is crossing the boundary,) we start w/ mom. eqn. (N-S)

in tensor form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

With the BCs $u_1 = u_2 = u_3 = 0$ at $x_2 = 0$ (y)

at wall:

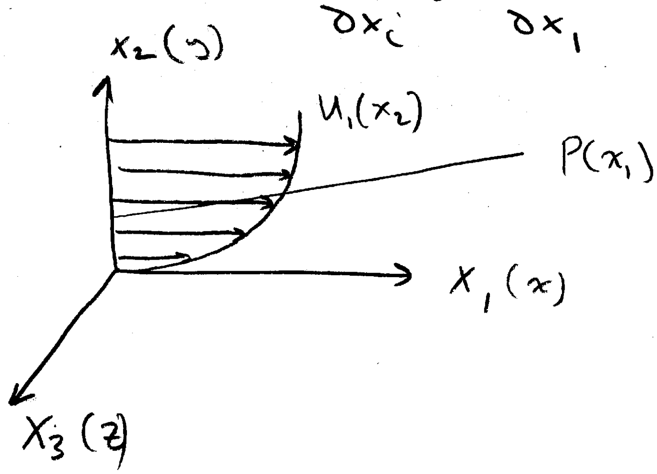
$$(u_i = 0 \quad u_j = 0)$$

$$+\nu\left(\underbrace{\cancel{\frac{\partial^2 u_i}{\partial x_1^2}}}_{0} + \frac{\partial^2 u_i}{\partial x_2^2} + \underbrace{\cancel{\frac{\partial^2 u_i}{\partial x_3^2}}}_{0}\right)$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x_i} = \nu \frac{\partial^2 u_i}{\partial x_2^2}$$

orient coordinates so $\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial x_1}$

i.e.



eqn. reduces to: $\frac{1}{\rho}\frac{\partial p}{\partial x_1} = \nu \frac{\partial^2 u_1}{\partial x_2^2}$

but since vorticity flux vector was defined $q_{\omega_3} \equiv -\nu \nabla \omega_3$

$$\omega_3 = \cancel{\frac{\partial u_2}{\partial x_1}}^{\,0 \text{ at boundary}} - \frac{\partial u_1}{\partial x_2}$$

$$\Rightarrow \quad q_{\omega_3} = -\nu \frac{\partial}{\partial x_2}\left(-\frac{\partial u_1}{\partial x_2}\right)$$

$$= \nu \frac{\partial^2 u_1}{\partial x_2^2}$$

$\therefore$ flux of vorticity

$$\boxed{q_{\omega_3} = \frac{1}{\rho}\frac{\partial p}{\partial x_1}}$$

"Like a ball on the wall rolling down $\frac{\partial p}{\partial x_1}$"

(i.e. Positive pressure gradient in x direction makes Positive or CCW rotation and Vice-Versa)

(and because in the flat-plate BL, i.e. Blasius Soln., there is no pressure gradient, $\frac{dp}{dx}=0$, there is no vorticity generation at the wall. All the vorticity that makes up the BL must have come from its leading edge where an infinite pressure gradient generate an infinite amount of vorticity which diffuses as it gets convected downstream)

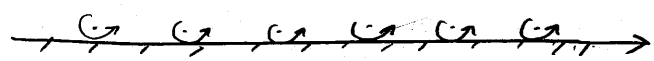
to summarize what we said:
(with homogeneous fluid, i.e. ρ=const, all the vorticity is generated at boundaries & not in the interior of fluid)

- (furthermore,) Vorticity is generated at boundaries either by pressure gradients (as we just saw) or by (tangential) acceleration of the wall.

Examples

1) Stokes' First Problem:
boundary impulsively set into motion (Parallel to itself)

$u=0$ at $t=0$



$u=U$ at $t>0$

Fluid next to the wall is set to rotational motion

- (All the vorticity is generated in the first instant and then viscosity diffuses it slowly to the interior of fluid)

(The problem is similar to the diffusion of heat. In fact the vorticity transport eqn. reduces to a 1D heat equation:)

$$\frac{\partial \omega}{\partial t} + \cancelto{0}{u\frac{\partial \omega}{\partial x}} + \cancelto{0}{v}\frac{\partial \omega}{\partial y} = \nu\left(\cancelto{0}{\frac{\partial^2 \omega}{\partial x^2}} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

(To see how all the vorticity is generated at the first instant, you can e.g. look at the similarity soln. for vorticity and compute its normal gradient)

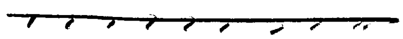
(As Morton points out) Shear stress at wall does not generate vorticity

because τ is non-zero even after the initial instant, but then
- there is no longer any vorticity being generated.

(However) The no-slip condition is essential for vorticity production.
(despite what Morton says)

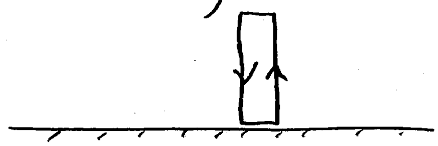
example

2) Plate accelerated uniformly into motion



$U = At$

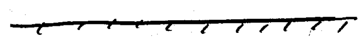
(if you drew a contour)



Circulation grows with t, vorticity is continuously generated at a fixed rate and continuously diffuses into interior by viscosity

3) Stokes Second Problem: (sinusoidal plate motion)

$U = 0$ at $y \to \infty$



$\leftrightarrow$ $U = U \cos \Omega t$

Alternating positive & negative vorticity is generated and viscosity diffuses it towards the interior, while positive & negative vorticity cross diffuse and annihilate each other.

So in all cases, vorticity is generated at wall due to accel. But vorticity can not be destroyed at wall, only opposite-signed vorticity can be generated. (this is unlike heat, where the wall can be a source or sink of heat; with vorticity, wall can only act as a source. One way of explaining this is to recall that vorticity is different from heat in that it can be positive or negative, whereas temp. is always positive)

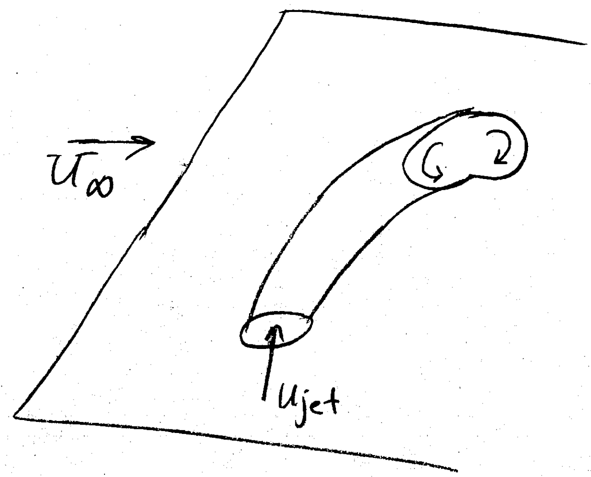
(Finally,)
What happens to vorticity once it is generated?
It must decay in the interior of fluid by cross-diffusion with opposite signed vorticity. (since it can not be destroyed at the boundary)

(Why all this concern with generation of vorticity?)

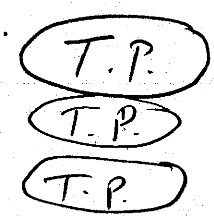
(oftentimes, if we don't understand where the vorticity comes from, we can totally misinterpret observations & make erroneous predictions)

(a nice example that illustrates the importance of vorticity dynamics is:)

<u>Transverse jet</u>:



Vortices are observed in the <u>wake</u> of the jet



Wake vortices are <u>not</u> generated the same way that wake vortices are produced by a solid body. (consistent w/ Morton's paper)

Origin of wake vortices behind transverse jet: BL on wall.

(The transverse jet can also demonstrate something more fundamental about the importance of vorticity in explaining fluid flow)

Consider the counter-rotating vortex pair structure observed in jet. (from velocity field it's not easy to see why they exist)

We can explain it by what happens to the primary structures, i.e. vortex rings in a jet: displacement of vortex rings and annihilation of opposite signed vorticity T.P. T.P.