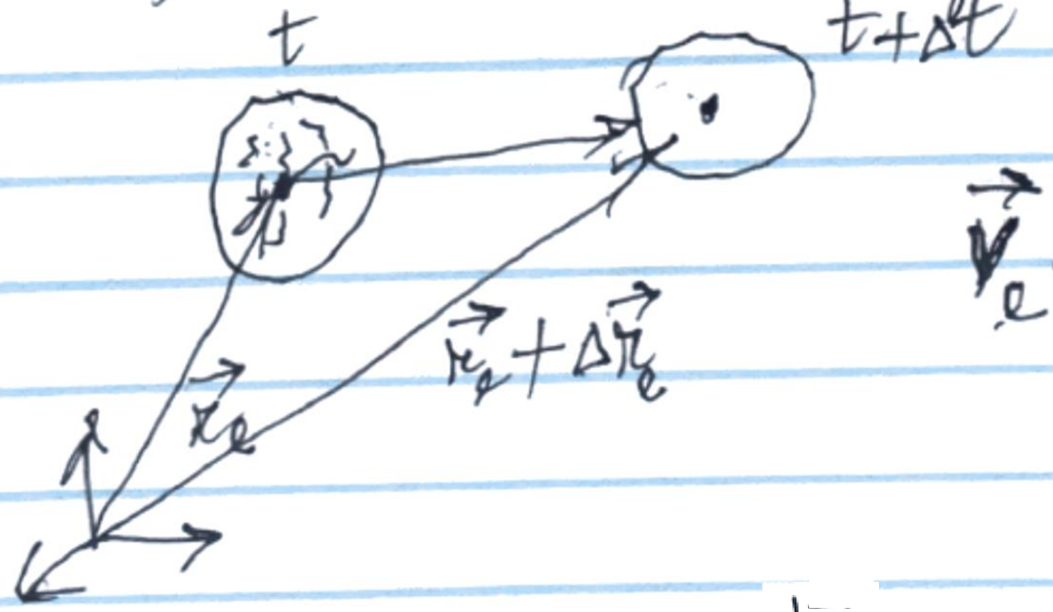
Fluid element kinematics

The velocity vector of a fluid element is the rate of change in time of the element center of gravity

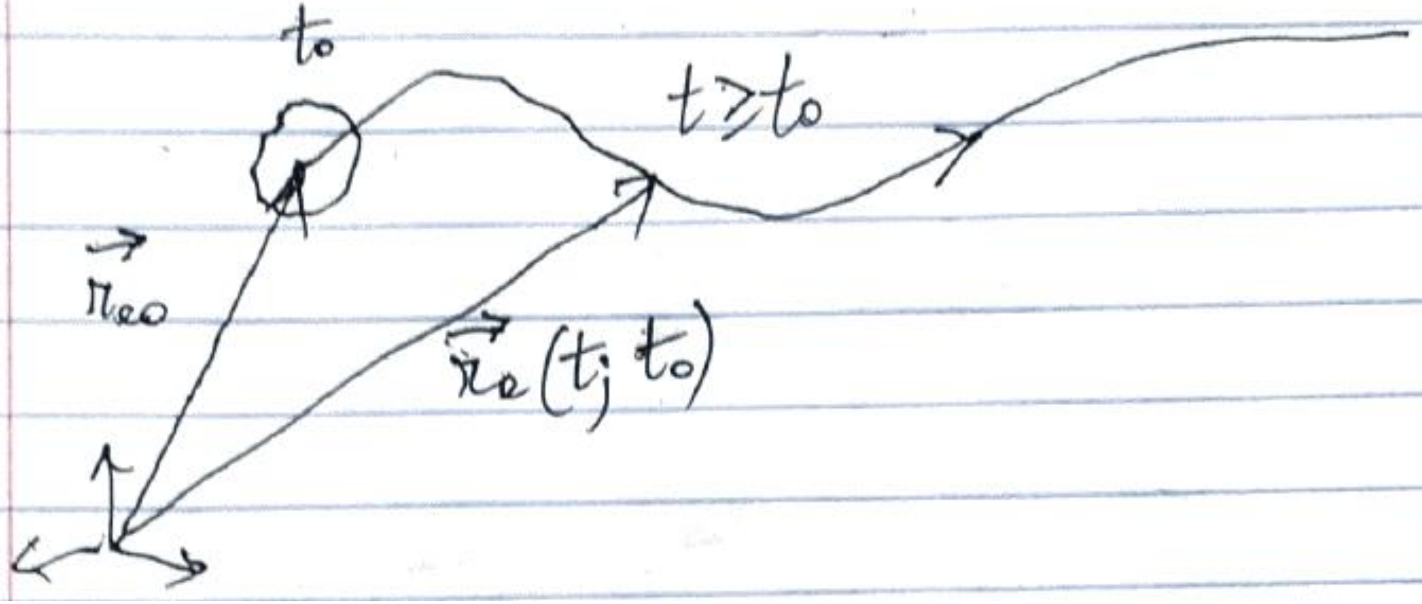


$$\vec{V}_e = \frac{d\vec{r}_e}{dt} = \lim_{\Delta t \to (\Delta t)_{min}} \frac{\Delta \vec{r}_e}{\Delta t}$$

a scalar $P_e = P(t; t_0) \rightarrow \frac{dP_e}{dt}$

a path line of a fluid element, $\vec{r}_e(t; t_0)$

= the trajectory of an element



$$\frac{d\vec{r}_e}{dt} = \vec{V}_e(t) \quad \text{with initial condition } \vec{r}_e(t=t_0; t_0) = \vec{r}_{e0}$$

Let $\vec{r}_e = x_e \vec{e}_x + y_e \vec{e}_y + z_e \vec{e}_z$ we have

$$\begin{cases} \dfrac{dx_e}{dt} = (\vec{V}_e)_x(t) \,, & x_e(t_0) = x_{e0} \\ \dfrac{dy_e}{dt} = (\vec{V}_e)_y(t) \,, & y_e(t_0) = y_{e0} \\ \dfrac{dz_e}{dt} = (\vec{V}_e)_z(t) \,, & z_e(t_0) = z_{e0} \end{cases}$$

<u>A Langrangian approach</u>: follow elements.

<u>The Eulerian approach</u>: fields of properties $p, \rho, T, \vec{V}$

= a <u>mapping</u> of the property in the flow as a function of space and time $p(x,y,z,t)$, $\rho(x,y,z,t)$ $T(x,y,z,t)$, $\vec{V}(x,y,z,t)$

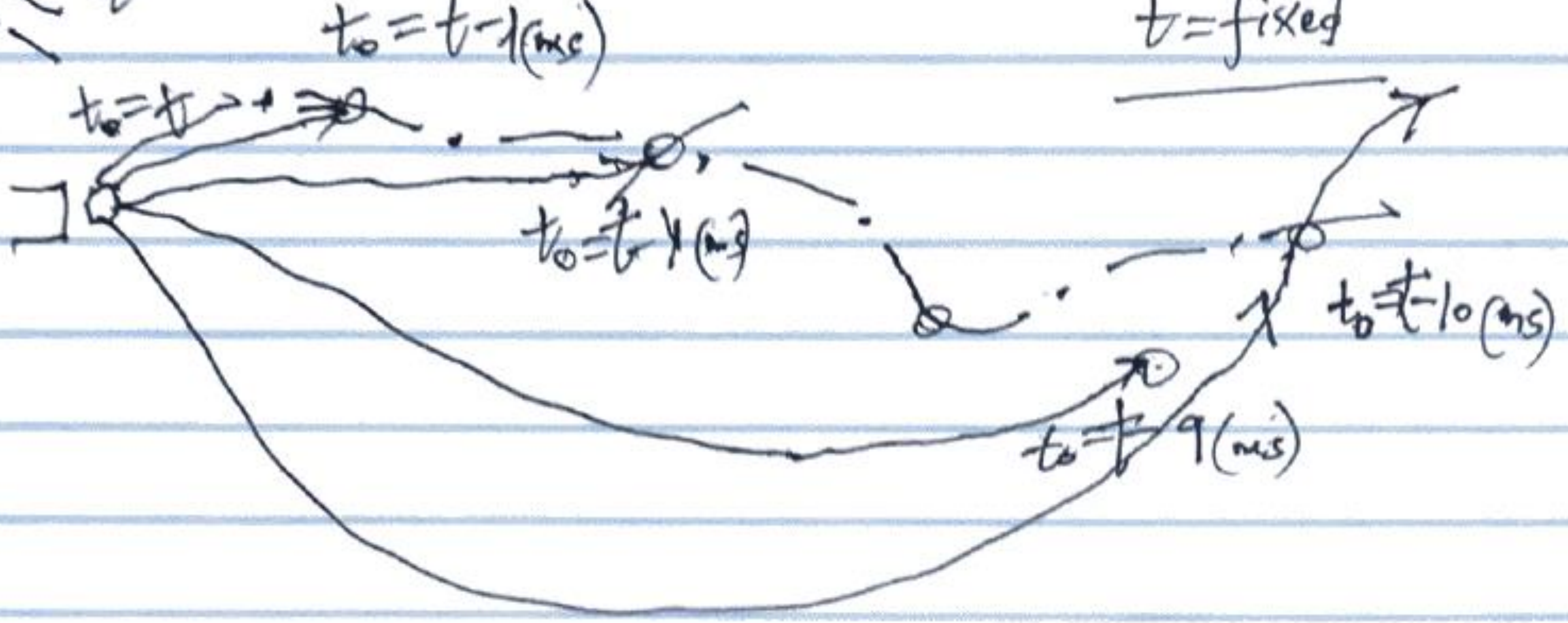
In the Eulerian approach: $\vec{V}_e(t) = \vec{V}(x=x_e(t), y=y_e(t), z=z_e(t), t)$

path line eqs of a fluid element:

$$\begin{cases} \dfrac{d\vec{r}_e}{dt} = \vec{V}_e(t) = \vec{V}(x=x_e(t), y=y_e(t), z=z_e(t), t) \\ \text{with } \vec{r}_e(t_0; t_0) = \vec{r}_{e0} \end{cases}$$

$$\therefore \begin{cases} \dfrac{dx_e}{dt} = V_x(x_e(t), y_e(t), z_e(t), t), \quad x_e(t_0; t_0) = x_{e0} \\ \dfrac{dy_e}{dt} = V_y(x_e(t), y_e(t), z_e(t), t), \quad y_e(t_0; t_0) = y_{e0} \\ \dfrac{dz_e}{dt} = V_z(x_e(t), y_e(t), z_e(t), t), \quad z_e(t_0; t_0) = z_{e0} \end{cases}$$

A streakline is the collection of all fluid elements at ~~a fixed~~ time t that were crossing the point $\vec{r}_{e0}$ at some time $t_0 < t$



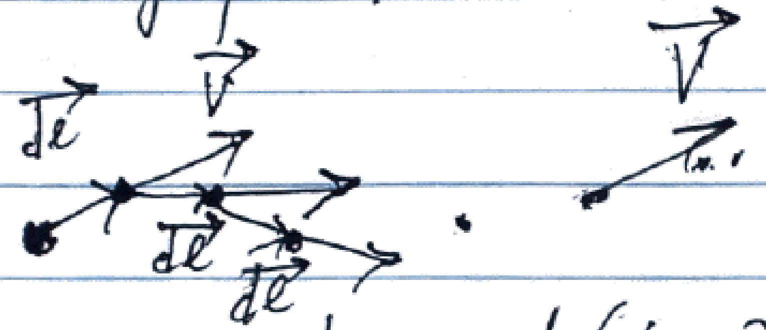
The eqs of a streakline

solve $\frac{d\vec{r}_e}{dt} = \vec{V}(x_e(t), y_e(t), z_e(t), t)$ with a generic $t_0 \leq t$

$\Rightarrow \vec{r}_e(t; t_0)$ for $t \geq t_0$

we then fix t and $\vec{r}_e(t=\text{fixed}; t_0)$ is a function of t_0 for $t_0 \leq t = \text{fixed}$

The streamline is a line that is parallel to $\vec{V}$ at every point at a fixed time t



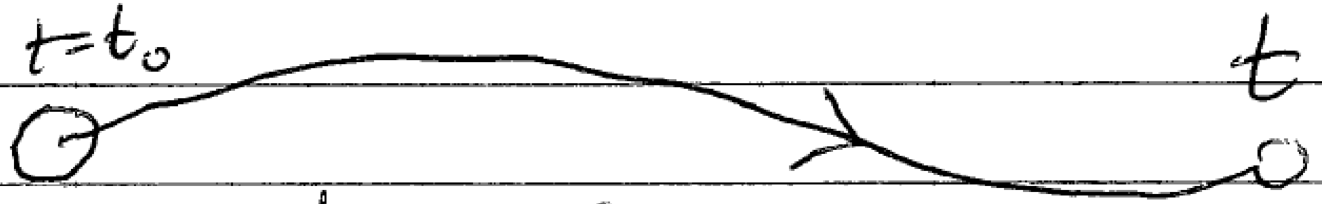
t=fixed

$$\vec{dl} \times \vec{V} = 0$$

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{V_y(x,y,z,t=\text{fixed})}{V_x(x,y,z,t=\text{fixed})} & \text{with } y(x_0)=y_0 \\[2ex] \dfrac{dz}{dx} = \dfrac{V_z(x,y,z,t=\text{fixed})}{V_x(x,y,z,t=\text{fixed})} & \text{with } z(x_0)=z_0 \end{cases}$$

where $\vec{dl} = dx\,\vec{e_x} + dy\,\vec{e_y} + dz\,\vec{e_z}$

In the <u>Lagrangian</u> approach we follow elements of the fluid in time



All properties of the fluid element depend only on time $t \geq t_0$ i.e. if F_e is a property of the element, $F_e\ (\equiv \vec{r}_e, \vec{V}_e, p_e, \rho_e, T_e, u_e, h_e, s_e)$,

$F_e(t; t_0)$ where $\vec{r}_e(t_0) = \vec{r}_{e0}$

$\frac{dF_e}{dt}$ = rate of change in time of F_e

= material or substantial derivate

In the Eulerian approach we construct fields (mappings in space and time) of all properties F $(= \vec{V}, \vec{a}, p, \rho, T, u, h, S, \ldots)$, i.e $F(x, y, z, t)$

an increment of F:

$$dF = \left(\frac{\partial F}{\partial x}\right)_{y,z,t} dx + \left(\frac{\partial F}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial F}{\partial z}\right)_{x,y,t} dz + \left(\frac{\partial F}{\partial t}\right)_{x,y,z} dt$$

$$\frac{dF}{dt} = \left(\frac{\partial F}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial F}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial F}{\partial z}\right)\frac{dz}{dt} + \left(\frac{\partial F}{\partial t}\right)$$

$$= \left(\frac{\partial F}{\partial t}\right) + \vec{V}_o \cdot \vec{\nabla} F$$

where $\vec{V}_o = \frac{dx}{dt}\vec{e}_x + \frac{dy}{dt}\vec{e}_y + \frac{dz}{dt}\vec{e}_z$

$\vec{V}_o$ is the observer velocity vector which is independent of $\vec{V}$.

to match between the approaches, we need that

$\vec{V}_\theta \equiv \vec{V}_e$ for all time $t \geq t_0$, where $\vec{r}_e(t_0) = \vec{r}_{e0}$,

to follow an element

Then, $F(x = x_e(t), y = y_e(t), z = z_e(t), t) = F_e(t)$

and $$\frac{dF}{dt} = \left(\frac{\partial F}{\partial x}\right)_e \frac{dx_e}{dt} + \left(\frac{\partial F}{\partial y}\right)_e \frac{dy_e}{dt} + \left(\frac{\partial F}{\partial z}\right)_e \frac{dz_e}{dt} + \left(\frac{\partial F}{\partial t}\right)_e = \frac{dF_e}{dt}$$

Here $\left(\frac{\partial F}{\partial i}\right)_e = \frac{\partial F}{\partial i}(x = x_e(t), y = y_e(t), z = z_e(t), t)$

for $i = x, y, z, t$

$$\frac{DF_e}{Dt} = \frac{dF_e}{dt} = \left(\frac{\partial F}{\partial t}\right)_e + \vec{V}_e(t) \cdot \left(\vec{\nabla} F\right)_e$$

specific interest: $F = \vec{V}$

$$\vec{a}_e = \frac{D\vec{V}_e}{Dt} = \frac{d\vec{V}_e}{dt} = \left(\frac{\partial \vec{V}}{\partial t}\right)_e + \vec{V}_e(t) \cdot \left(\underline{\underline{\vec{\nabla}\vec{V}}}\right)_e$$