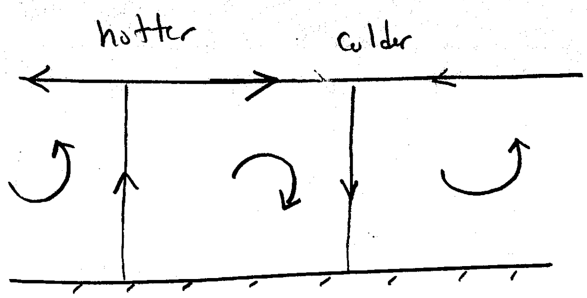


Viscous forces then give motion as shown
(i.e. Still unstable at $Ra = 0$)
i.e. in absence of gravity

This is a form of "Marangoni" convection (in particular, this is called thermo-capillary motion; a.k.a. thermo-Marangoni)

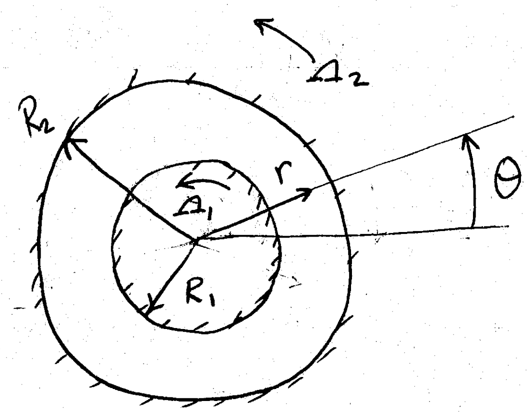
Buoyancy forces are also present if $Ra > 0$

→ Other patterns have been document, some due to imperfection (T.P.)

C) Taylor-Couette Instability: Flow between rotating cylinders

"Centrifugal Instability" (cf ~~gravitational~~, capillary & thermal)

Consider viscous flow between concentric circular cylinders



N-S eqn. for incomp. flow (dimensional)

$$\operatorname{div} \vec{u} = 0 \qquad \text{Continuity}$$

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{u} \qquad \text{momentum}$$

Write N-S eqns in cylindrical coordinates (r, θ, z). For basic state (unperturbed)

$$\frac{\partial}{\partial t} = 0 \qquad \frac{\partial}{\partial z} = 0 \qquad \frac{\partial}{\partial \theta} = 0 \quad ; \quad u_1 = u_r = 0 \ , \ u_2 = u_\theta = V \ , \ u_3 = u_z = 0$$

Velocity in r-direction (not to be mistaken as a partial derivative)

(See Schlichting P. 66)

Continuity: $\frac{1}{r}\frac{\partial}{\partial r}(r \cancel{u_r}^{\,0}) + \frac{1}{r}\frac{\cancel{\partial u_\theta}^{\,0}}{\partial \theta} + \frac{\cancel{\partial u_z}^{\,0}}{\partial z} = 0$

∴ every term in continuity is zero
(this would've told us that $u_r = 0$ if we didn't know already)

r-momentum transport eqn.

$$\cancelto{0\,(SS)}{\frac{\partial u_r}{\partial t}} + \cancelto{0}{u_r}\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\cancelto{0}{\frac{\partial u_r}{\partial \theta}} - \frac{u_\theta^2}{r} + \cancelto{0}{u_z}\frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 \cancelto{0}{u_r} - \cancelto{0}{\frac{u_r}{r^2}} - \frac{2}{r^2}\cancelto{0}{\frac{\partial u_\theta}{\partial \theta}}\right)$$

Where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

(Laplace operator in cylin. coord.; for ref., see Kaplan)

in our notation:

$$-\frac{V^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$\Downarrow$

$$\boxed{\frac{dp}{dr} = \rho\frac{V^2}{r}}$$

Pressure gradient = Centrifugal force

(since we don't know $\frac{dp}{dr}$ a priori, we can't use this to solve for V, so use θ-momentum eqn.)

θ-momentum:

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} = 0$$

(d.e. for V states that there is no net viscous force on fluid element)

these are terms inside $\mu(\ldots)$

Solution of this o.d.e.:

$$V = Ar + \frac{B}{r}$$

for the tangential velocity (u_r and $u_z = 0$)

(important observation:)

Note that this solution tells us that all axisymmetric, rotating flows are made up of 2 components:

* Solid-body rotation, Ar (constant times radius)
* Potential (or "Free") vortex, $\frac{B}{r}$

Recall: $u_\theta = \frac{\Gamma}{2\pi r}$, and $u_r = 0$ define a vortex in potential flow. In our notation this is $V = \frac{\text{Consta}}{r}$

(true, no matter what the BCs are)

B.C. (Viscous flow) $\quad V = \Omega_1 R_1 \quad$ at $\quad r = R_1$

$V = \Omega_2 R_2 \quad$ at $\quad r = R_2$

Hence, A & B are found:

$$V = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2} \frac{1}{r}$$

Special Cases:

1) $\Omega_2 = \Omega_1 = \Omega \quad$ gives $\quad V = \Omega r \quad$ Solid-body rotation throughout the gap.
No deformation, i.e. no strain

2) $R_2 \to \infty, \ \Omega_2 = 0 \quad$ gives $\quad V = \frac{1}{r} \Omega_1 R_1^2 \quad$ outer cylinder removed

Using definition of Circulation $\Gamma \equiv \oint \vec{u} \cdot d\vec{l}$ and applying it on surface of inner cylinder, $\Gamma = (\Omega_1 R_1)\, 2\pi R_1 = 2\pi \Omega R_1^2$

$\Rightarrow V = \frac{\Gamma}{2\pi r}$ (i.e. removal of outer cylinder) results in a Potential Vortex flow

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We introduce small disturbances in the following form –

Velocity components $\overset{u_r}{u}, \overset{u_\theta}{v}, \overset{u_z}{w}$; consider only axisymmetric disturbances which are periodic in z:

(why this form? it is observed as first bifurcation vs. other forms such as rotating waves)

$$u = u_1(r)\, e^{\sigma t} \cos \alpha z$$

$$v = V_0(r) + V_1(r)\, e^{\sigma t} \cos \alpha z$$

$$w = w_1(r)\, e^{\sigma t} \sin \alpha z$$

$$p = P_0(r) + P_1(r)\, e^{\sigma t} \cos \alpha z$$

where α is wavenumber (in water waves, k was denoting wavenumber)

σ is frequency (if real, then disturbances grow, if imaginary, then motion is oscillatory)

(Note that $P_0(r)$ is obtained from r-momentum eqn. for base flow)
↖ base flow pressure distribution

B.C. $u_1 = v_1 = w_1 = 0$ at $r = R_1$ and $r = R_2$

$V_0 = Ar + \frac{B}{r}$ (A and B are known, as shown above)

$\frac{dp_0}{dr} = \rho \frac{V_0^2}{r}$

(4 unknown functions, u_1, v_1, w_1, P_1 ; 4 eqns., 3 momentum, 1 continu

Substitute in (full) Navier-Stokes eqns., neglect products of small quantities, eliminate w_1 and P_1 :

(Come up w/ 2 ode's)

$$\nu\left(L - \alpha^2 - \frac{\sigma}{\nu}\right)\left(L - \alpha^2\right) u_1(r) = 2\alpha^2 \frac{V_0(r)}{r} v_1(r)$$

and
$$\nu\left(L - \alpha^2 - \frac{\sigma}{\nu}\right) v_1(r) = 2A\, u_1(r)$$

where an operator L has been defined:

$$L(\;) \equiv \frac{d^2(\;)}{dr^2} + \frac{1}{r}\frac{d(\;)}{dr} - \frac{(\;)}{r^2}$$ (looks similar to Laplacian in cylindrical coordinates) except that it's missing the z-term

B.C.'s $u_1 = v_1 = \frac{du_1}{dr} = 0$ at $r = R_1$ and $r = R_2$

This is a complicated eigenvalue problem for σ in terms of other parameters

(i.e. it has a solution for u_1 & v_1 for each value of σ)

Can rewrite in nondimensional form.

For a reference length can use the gap size:

$d = R_2 - R_1$

Reference velocity for V $(= u_\theta)$ is $\Omega_1 R_1$
tangential velocity

" " " $u(=u_r)$ is $\Omega_1, R_1 \frac{\nu}{Ad^2}$
radial Velocity [1/time]

- If equations are nondimensionalized in this way, there will appear one dimensionless parameter:

$$T = -\frac{4A\Omega_1 d^4}{\nu^2}$$ "Taylor no." (Show it's dimensionless)

(Watch the notation, we had used T for temperature, in Bénard convection)

We will usually be considering $A < 0$ (will see soon that $A > 0$ is stable) when $V = Ar + \frac{B}{r}$

Special cases: "Narrow gap" approximation $\frac{d}{r} \ll 1$, $\frac{\Omega_2}{\Omega_1} \approx 1$ and negligible ν (i.e. inviscid)

$\Rightarrow L(\,) \approx \frac{d^2(\,)}{dr^2}$ $\Rightarrow V_0 \approx \Omega r$

- Differential equations give

(dimensional form) $$\frac{d^2 V_1}{dr^2} - \left(1 + \frac{4A\Omega_1}{\sigma^2}\right)\alpha^2 V_1 = 0$$ (α is wavenumber in z-dir., and σ is frequency)

B.C. $V_1 = 0$ at $r = R_1$ and $r = R_2$

Solution is $V_1 = (\text{constant}) \sin \frac{n\pi(r-R_1)}{d}$

and substitution into differential equation gives:

- $$-\frac{n^2\pi^2}{d^2} - \left(1 + \frac{4A\Omega_1}{\sigma^2}\right)\alpha^2 = 0$$

or $$\sigma^2 = \frac{-4A\Omega_1}{1 + \frac{n^2\pi^2}{\alpha^2 d^2}}$$

Taking $\Omega_1 > 0$ (no loss of generality)

- σ is real if $A < 0$, giving 1 root which grows with time ($\Rightarrow$ unstable)

  σ " imaginary " $A > 0$ giving oscillatory motion ($\Rightarrow$ neutral stability), i.e. disturbances neither grow or decay

Recall $A = \dfrac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$

- $\therefore \begin{cases} \Omega_2 R_2^2 < \Omega_1 R_1^2 & \text{get instability} \\ \Omega_2 R_2^2 > \Omega_1 R_1^2 & \text{get stability} \end{cases}$

$\therefore$ For inviscid flow, circulation must increase radially, otherwise unstab

(recall) $\Gamma = V \cdot 2\pi R$ (for axisymmetric flow)
$= (\Omega R)\, 2\pi R = (\text{const.})\, \Omega R^2$

If $\frac{d}{R}$ is not small and Ω_2 is not close to Ω_1, same conclusion can be reached.

The general result: "Rayleigh criteria" (for inviscid flow)

- States that flow is stable if and only if $\frac{d}{dr}(r V_0)^2 > 0$ (for $R_1 < r < R_2$)
  i.e. square of the circulation should not decrease as you go out radially
  where $V_0 = Ar + \frac{B}{r}$ is the velocity of the base flow