r-momentum transport eqn.

$$\cancelto{0(SS)}{\frac{\partial u_r}{\partial t}} + \cancelto{0}{u_r}\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\cancel{\frac{\partial u_r}{\partial \theta}} - \frac{u_\theta^2}{r} + \cancelto{0}{u_z}\frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\cancelto{0}{\nabla^2 u_r} - \cancelto{0}{\frac{u_r}{r^2}} - \frac{2}{r^2}\cancel{\frac{\partial u_\theta}{\partial \theta}}\right)$$

Where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

(Laplace operator in cylin. coord.; for ref., see Kaplan)

in our notation:

$$-\frac{V^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\Downarrow$$

$$\boxed{\frac{dp}{dr} = \rho\frac{V^2}{r}}$$

Pressure gradient = Centrifugal force

(since we don't know $\frac{dp}{dr}$ a priori, we can't use this to solve for V, so use θ-momentum eqn.)

θ-momentum:

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} = 0$$

(d.e. for V — these are terms inside $\mu(\ldots)$ — states that there is no net viscous force on fluid element)

Solution of this o.d.e.:

$$V = Ar + \frac{B}{r}$$

for the tangential velocity (u_r and $u_z = 0$)

(important observation:)

Note that this solution tells us that all axisymmetric, rotating flows are made up of 2 components:

* Solid-body rotation, Ar (constant times radius)
* Potential (or "Free") vortex, $\frac{B}{r}$

Recall: $u_\theta = \frac{\Gamma}{2\pi r}$ and $u_r = 0$ define a vortex in potential flow. In our notation this is $V = \frac{\text{Const}}{r}$

(true, no matter what the BCs are)

B.C. (Viscous flow) $V = \Omega_1 R_1$ at $r = R_1$

$V = \Omega_2 R_2$ at $r = R_2$

Hence, A & B are found:

$$V = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2} \frac{1}{r}$$

Special Cases:

1) $\Omega_2 = \Omega_1 = \Omega$ gives $V = \Omega r$ Solid-body rotation throughout the gap.
No deformation, i.e. no strain

2) $R_2 \to \infty$, $\Omega_2 = 0$ gives $V = \frac{1}{r} \Omega_1 R_1^2$ outer cylinder removed

Using definition of Circulation $\Gamma \equiv \oint \vec{u} \cdot d\vec{l}$ and applying it on surface of inner cylinder, $\Gamma = (\Omega_1 R_1) 2\pi R_1 = 2\pi \Omega R_1^2$

$\Rightarrow V = \frac{\Gamma}{2\pi r}$ (i.e. removal of outer cylinder) results in a Potential Vortex flow

---

We introduce small disturbances in the following form –

Velocity components $\underset{u_r}{u}, \underset{u_\theta}{v}, \underset{u_z}{w}$; consider only axisymmetric disturbances which are periodic in z :

(why this form? it is observed as first bifurcation vs. other forms such as rotating waves)

$$u = u_1(r)\, e^{\sigma t} \cos \alpha z$$

$$v = v_0(r) + v_1(r)\, e^{\sigma t} \cos \alpha z$$

$$w = w_1(r)\, e^{\sigma t} \sin \alpha z$$

$$p = p_0(r) + p_1(r)\, e^{\sigma t} \cos \alpha z$$

where α is wave number (in water waves, k was denoting wavenumber)

σ is frequency (if real, then disturbances grow, if imaginary, then motion is oscillatory)

(Note that $p_0(r)$ is obtained from r-momentum eqn. for base flow)
↑ base flow pressure distribution

B.C. $u_1 = v_1 = w_1 = 0$ at $r = R_1$ and $r = R_2$

$V_0 = Ar + \frac{B}{r}$ (A and B are known, as shown above)

$\frac{dP_0}{dr} = \rho \frac{V_0^2}{r}$

(4 unknown functions, u_1, v_1, w_1, P_1 ; 4 eqns., 3 momentum, 1 continu

Substitute in (full) Navier-Stokes eqns., neglect products of small quantities, eliminate w_1 and P_1:

(Come up w/ 2 ode's)

$$\nu\left(L-\alpha^2-\frac{\sigma}{\nu}\right)\left(L-\alpha^2\right) u_1(r) = 2\alpha^2 \frac{V_0(r)}{r} v_1(r)$$

and
$$\nu\left(L-\alpha^2-\frac{\sigma}{\nu}\right) v_1(r) = 2A\, u_1(r)$$

where an operator L has been defined:

$$L(\;) \equiv \frac{d^2(\;)}{dr^2} + \frac{1}{r}\frac{d(\;)}{dr} - \frac{(\;)}{r^2}$$

(looks similar to Laplacian in cylindrical coordinates) except that it's missing the z-term

B.C.'s $u_1 = v_1 = \frac{du_1}{dr} = 0$ at $r = R_1$ and $r = R_2$

This is a complicated eigenvalue problem for σ in terms of other parameters

(i.e. it has a solution for u_1 & v_1 for each value of σ)

Can rewrite in nondimensional form.

For a reference length can use the gap size:

$d = R_2 - R_1$

Reference velocity for V $(= u_\theta)$ is $\Omega_1 R_1$

tangential velocity

" " " $u(=u_r)$ is $\Omega_1, R_1, \dfrac{\nu}{A d^2}$

radial velocity

$\underbrace{A}_{[1/\text{time}]}$

- If equations are nondimensionalized in this way, there will appear one dimensionless parameter:

$$T = -\frac{4A\Omega_1 d^4}{\nu^2}$$ "Taylor no." (Show it's dimensionless)

(Watch the notation, we had used T for temperature, in Bénard convection)

We will usually be considering $A<0$ (will see soon that $A>0$ is stable) when $V = \underset{\uparrow}{A}r + \frac{B}{r}$

<u>Special cases</u>: "Narrow gap" approximation $\frac{d}{r} \ll 1$, $\frac{\Omega_2}{\Omega_1} \approx 1$ and negligible ν (i.e. inviscid)

$\Rightarrow L(\,) \approx \frac{d^2(\,)}{dr^2}$ $\Rightarrow V_0 \approx \Omega r$

- Differential equations give

(dimensional form)
$$\frac{d^2 V_1}{dr^2} - \left(1 + \frac{4A\Omega_1}{\sigma^2}\right)\alpha^2 V_1 = 0$$
(α is wavenumber in z-dir., and σ is frequency)

B.C. $V_1 = 0$ at $r = R_1$ and $r = R_2$

Solution is $V_1 = (\text{Constant}) \sin \dfrac{n\pi(r-R_1)}{d}$

and substitution into differential equation gives:

$$-\frac{n^2\pi^2}{d^2} - \left(1 + \frac{4A\Omega_1}{\sigma^2}\right)\alpha^2 = 0$$

or
$$\sigma^2 = \frac{-4A\Omega_1}{1 + \frac{n^2\pi^2}{\alpha^2 d^2}}$$

Taking $\Omega_1 > 0$ (no loss of generality)

- σ is real if $A < 0$, giving 1 root which grows with time ($\Rightarrow$ unstable)

  σ " imaginary " $A > 0$ giving oscillatory motion ($\Rightarrow$ neutral stability), i.e. disturbances neither grow or decay

Recall $A = \dfrac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$

$$\therefore \begin{cases} \Omega_2 R_2^2 < \Omega_1 R_1^2 & \text{get instability} \\ \Omega_2 R_2^2 > \Omega_1 R_1^2 & \text{get stability} \end{cases}$$

$\therefore$ For inviscid flow, circulation must increase radially, otherwise unstab

(recall) $\Gamma = V \cdot 2\pi R$ (for axisymmetric flow)

$= (\Omega R)\, 2\pi R = (\text{const.})\, \Omega R^2$

If $\frac{d}{R}$ is not small and Ω_2 is not close to Ω_1, same conclusion can be reached.

The general result: "Rayleigh criteria" (for inviscid flow)

- States that flow is stable if and only if $\frac{d}{dr}(r V_0)^2 > 0$ (for $R_1 < r < R_2$)

  i.e. square of the circulation should not decrease as you go radially out

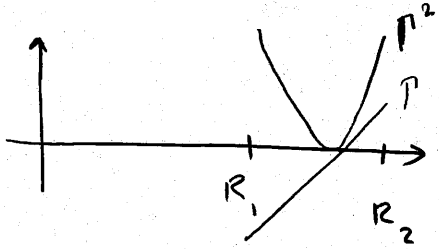
  when $V_0 = Ar + \frac{B}{r}$ is the velocity of the base flow

⇒ That is, the flow is stable if the square of the circulation increases with r, unstable if it decreases with r, at any r between R_1 and R_2.

* For cylinders rotating in the same direction $(\Omega_2 R_2^2)^2 > (\Omega_1 R_1^2)$ for stability, consistent with what was found earlier (ΩR^2 must increase)

* For cylinders rotating in opposite direction, $\frac{d}{dr}(r_1^2 V_0^2)$ necesscrily changes sign, flow is unstable.

to see this



thus $\frac{d\Gamma^2}{dr}$ changes sign

---

e.g. outer cylinder rotating, inner cylinder fixed ⇒ stable

" " fixed " " rotating ⇒ unstable

This criterion was obtained when viscosity is negligible.

(Physical significance:)

Von Karman interpreted this result by considering a fluid element initially at radius r_a with velocity V_a, and displaced to radius r_b with angular momentum unchanged, so that its new velocity is $\frac{1}{r_b}(V_a r_a)$.

If element were to be at equilibrium at r_b, a pressure gradient $\left(\frac{dp}{dr} = \rho \frac{V^2}{r}\right)$ equal to $\rho \frac{1}{r_b}\left(\frac{V_a r_a}{r_b}\right)^2$ would be needed. But actual pressure gradient is $\rho \frac{1}{r_b}(V_b)^2$. Thus, if $V_b^2 < \left(\frac{r_a V_a}{r_b}\right)^2$, element would continue to move outward; if $V_b^2 > \left(\frac{r_a V_a}{r_b}\right)^2$ element would move back toward r_a.

This is really a necessary condition for stability; the condition is also sufficient, as noted above

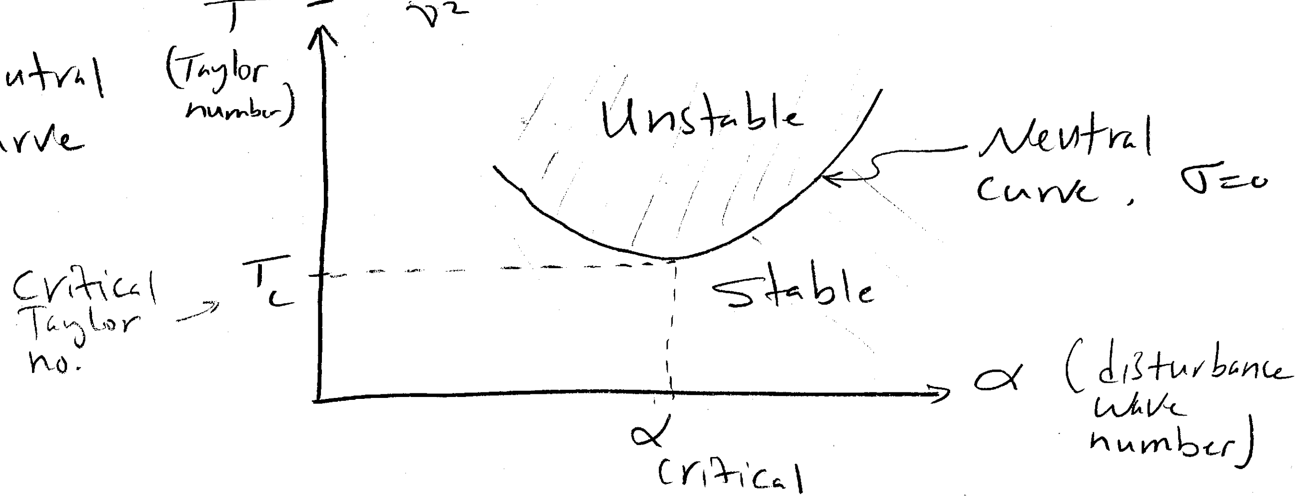
You can think of this as "static" stability, because it can not say that the restoring force does not cause an oscillation which grows in time.

If viscosity is not negligible, the Rayleigh condition remains a sufficient condition for stability, but it is no longer necessary, because viscosity has a stabilizing effect in this flow.

A neutral condition is found by setting $\sigma = 0$ in disturbance equations.

(get) Neutral Curve

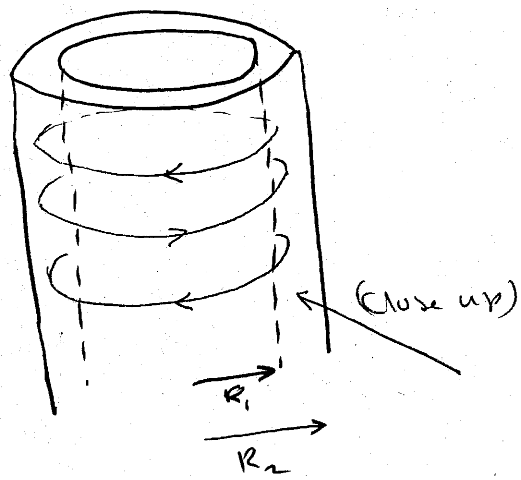
$$T \equiv \frac{-4A\Omega_1 d^4}{\nu^2}$$

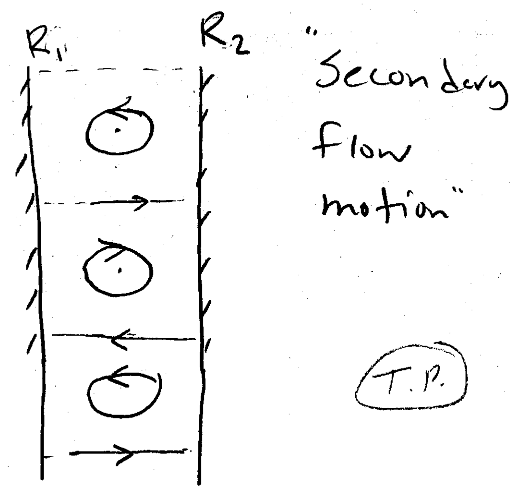


For $T < T_c$ all initially infintesimal, axisymmetric disturbances that are Periodic in z are damped and decay to zero with increasing t.

For $T > T_c$, a range of disturbances grow as t increas.

The instability leads to a new steady secondary axisymmetric flow in the form of regularly spaced toroidal vortices (i.e. vortex rings) "Taylor vortices"





Increasing T leads to wavy Taylor Vortices (azimuthal waves) and ultimately to turbulence.

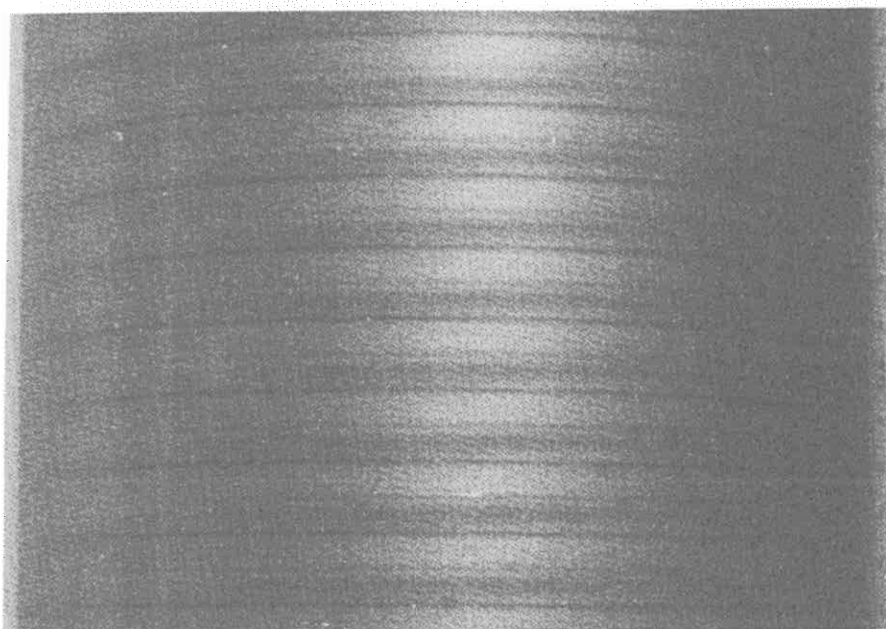
Finally, note that in the small gap approximation and $\Omega_2 \approx \Omega_1$, disturbance eqns. give (with $L \approx \frac{d^2}{dr^2}$ and eliminating u_1)

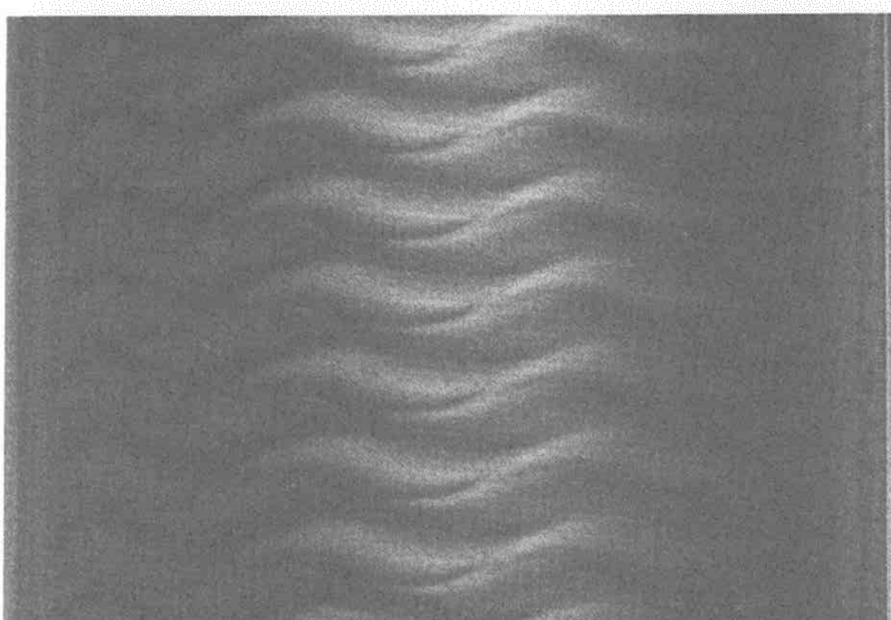
$$\left(\frac{d^2}{dr^2} - \alpha^2\right)^3 V_1 = \frac{4A\Omega_1}{\nu^2}\alpha^2 V_1 \qquad \text{for } \sigma = 0 \text{ (i.e. neutral stability)}$$

$$= -\frac{T}{d^4}\alpha^2 V_1$$

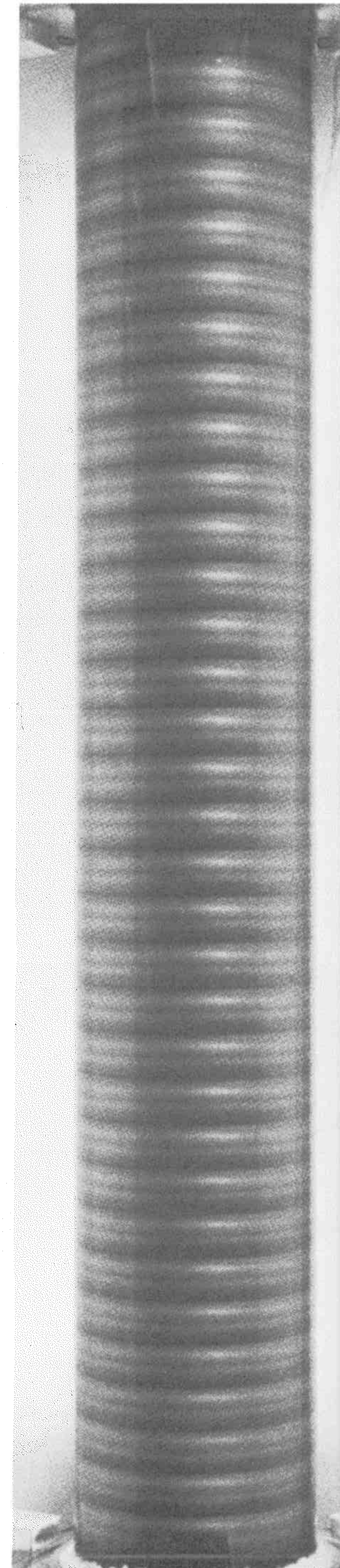
(identical in form to eqn. found for Bénard Convection at neutral stability)

127. Axisymmetric laminar Taylor vortices. Machine oil containing aluminum powder fills the gap between a fixed outer glass cylinder and a rotating inner metal one, of relative radius 0.727. The top and bottom plates are fixed. The rotation speed is 9.1 times that at which Taylor predicts the onset of the regularly spaced toroidal vortices seen here. The flow is radially inward on the heavier dark horizontal rings and outward on the finer ones. The motion was started impulsively, giving narrower vortices than would result from a smooth start. *Burkhalter & Koschmieder 1974*





128. Laminar Taylor vortices in a narrow gap. A larger inner cylinder in the apparatus to the right gives a radius ratio of 0.896. Again only the inner cylinder rotates. The upper photograph shows the center section of axisymmetric vortices at 1.16 times the critical speed. In the lower, at 8.5 times the critical speed, the flow is doubly periodic, with six waves around the circumference, drifting with the rotation. *Koschmieder 1979*



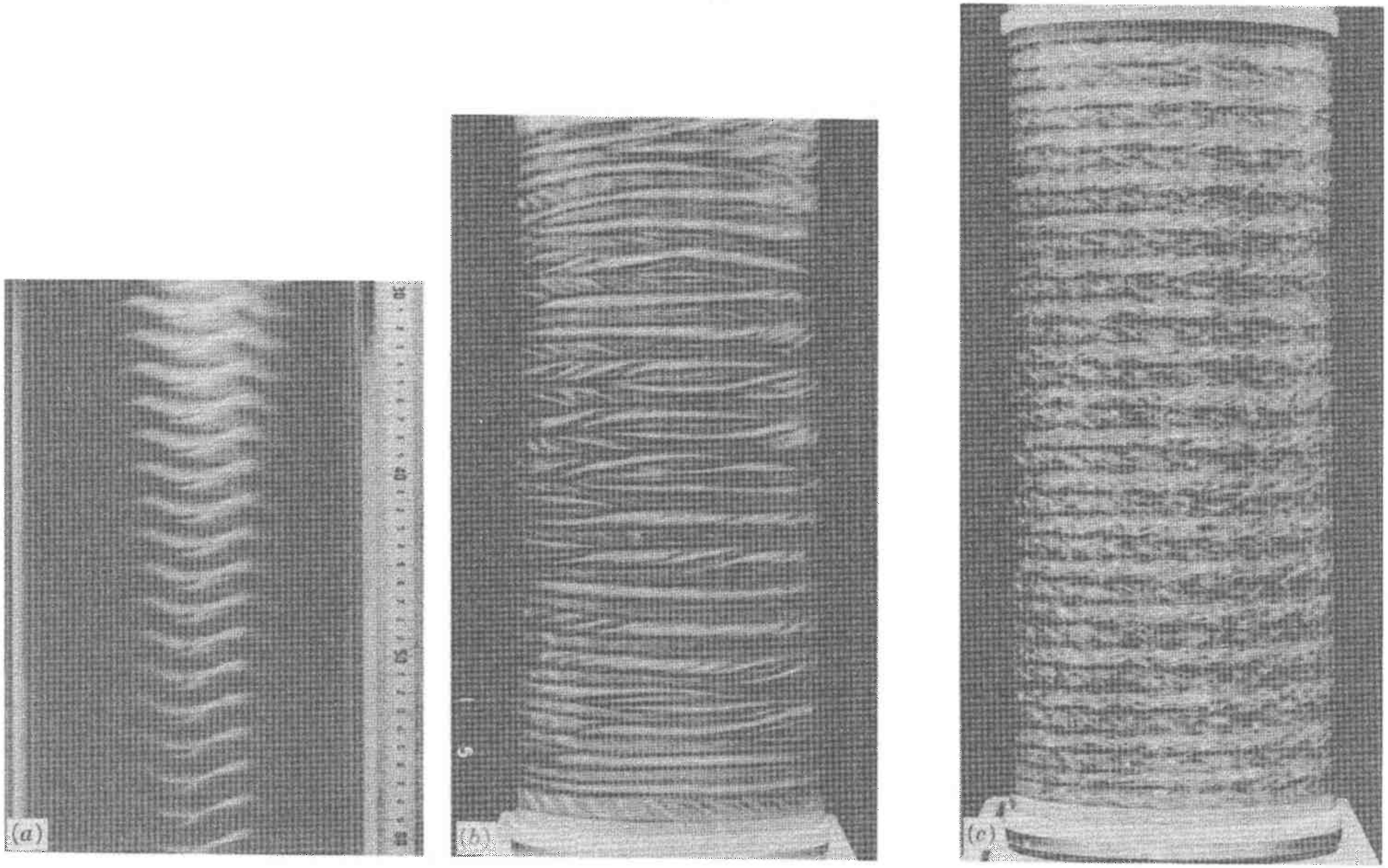
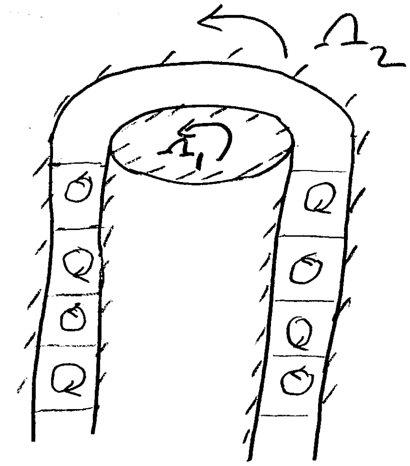


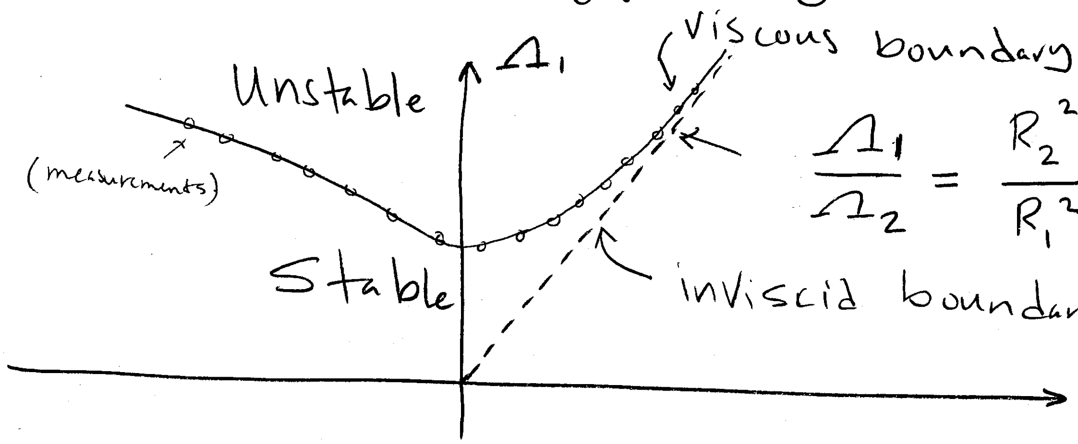
Figure 25.16 (*a*) Wavy Taylor vortices. Reprinted with permission from Koschmieder (1979). (*b*) Braided Taylor vortices. From Andereck et al. (1983). (*c*) Turbulent Taylor vortices. Courtesy of Zhang and Swinney (1985), University of Texas. Reprinted with permission.

(Final comments about Taylor-Couette instability)



(A convenient way to summarize our results is the following plot:)

Taylor (1923) plotted Ω_2 vs. Ω_1 and showed extremely close agreement between narrow-gap theory & experiments ($\frac{d}{R_1} \approx 1/7$)



$$\frac{\Omega_1}{\Omega_2} = \frac{R_2^2}{R_1^2} \; (>1)$$

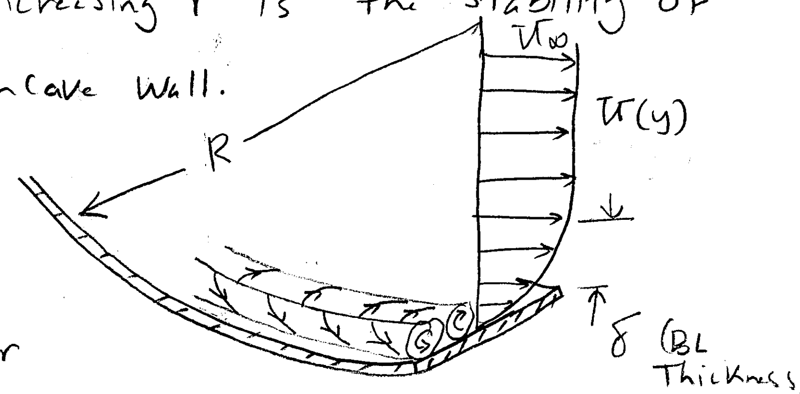
(which implies that $\Omega_1 = \Omega_2$ line is always stable, as expected)

The negative Ω_2 side has a smaller instability limit for Ω_1 (as expected).

(much has been learned about TC instability. Annual Conf. dedicated to it; "cottage industry")

(A of forward 60 years)
T.P. Psalm 25:15

Another application of the general result that for stability, circulation (squared) must increase with increasing r is the stability of the boundary layer on a concave wall.



Since in the boundary layer velocity quickly decreases with increasing r, the circulation (Ur) decreases close to wall → Flow is unstable

25.13 TAYLOR INSTABILITY OF COUETTE FLOW

Viscosity plays only its stabilizing role in Taylor–Couette flows. A chart of the stability characteristics is given as Fig. 25.15. The viscous stability of these flows was first determined by Taylor (1921, 1923) both experimentally and theoretically. The theoretical problem is quite difficult, and most work is done using a thin-gap assumption. This assumption takes centrifugal effects out of the main flow but retains them partially in the disturbance equations. The problem, simplified for axisymmetric disturbances, contains a parameter called the *Taylor* (Ta) *number*. Several definitions are in use. A typical one is

$$\mathrm{Ta} \equiv \frac{r_i(r_o - r_i)^3(\Omega_i^2 - \Omega_o^2)}{\nu^2} \tag{25.13.1}$$

In Eq. 25.13.1, r is the cylinder radius, Ω the angular velocity, and the subscripts i and o refer to inner and outer, respectively.

Essentially, Ta represents the centrifugal effect divided by the viscous effect. Upon crossing Taylor's first stability boundary, one encounters a second stable laminar flow pattern with toroidal vortices. For inner rotation only, $\Omega_0 = 0$, this boundary is Ta = 1708. The new flow pattern is an example of the principle of *exchange of stabilities.* Taylor vortices and Couette flow are both stable laminar flow patterns.

Taylor vortices themselves become unstable at higher rotation rates, where they give way to wavy Taylor vortices as shown in Fig. 25.16*a*. Many states of different mode numbers can be attained in the wavy patterns. Moreover, when the outer cylinder is

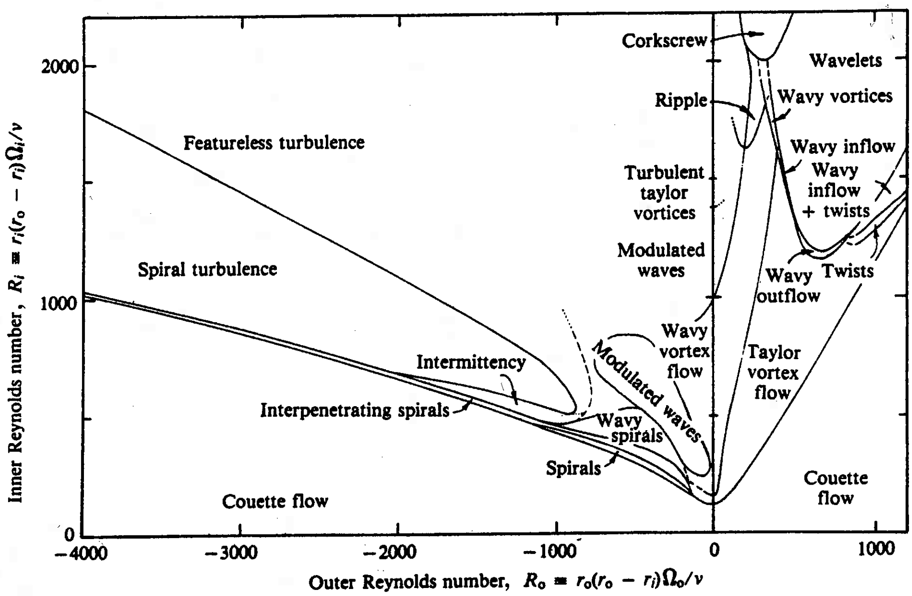


Figure 25.15 Stability chart for Taylor vortex behavior. Reprinted with permission from Andereck et al. (1986).

Panton, Incompressible Flow, 3rd edn. (2005)

The instability manifests as streamwise counter-rotating vortices, "Goertler vortices."

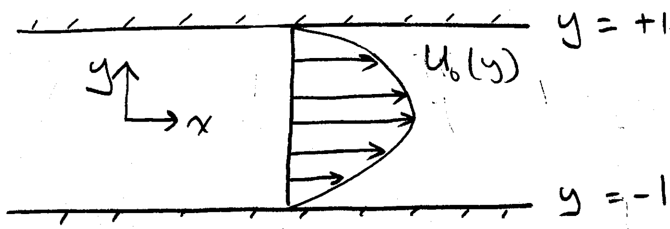
(This problem has important implications including the design of nozzles (for everything ranging from wind tunnels to intake duct of a jet engine nacelle)

(The last topics in stability analysis :)

D) Poiseuille Flow ("PWAH-ZOI")

(i.e. flow in a channel, which has important implications to turbulence in general)

Consider the planar case, in nondimensional variables



(dimensionless notation)

(When fully developed)

(Base flow is $U_0(y) = 1 - y^2$, $P_0(x) = -\frac{2}{Re}x$

(In case you've forgotten)

(this is easy to show, since for) fully developed flow $\frac{\partial (\text{vel.})}{\partial x} \equiv 0$

(and from) continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ & bc, find $v=0$; from y-momentum find $P = P(x)$; and from x-momentum, in dimensional form

$$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0\ \text{ss}} + u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{v}}_{0} \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \frac{\partial^2 u}{\partial y^2}\right)$$

, which for a given pressure gradient gives $U = U_{max}\left(1 - \left(\frac{y}{h}\right)^2\right)$ when you apply $u=0$ at $y = \pm h$

centerline speed

(hence the result above)

(R_e is based on max base flow velocity and channel half width. ($R_e = \frac{h \cdot U_{max}}{\nu}$; here, $R_e = \frac{1}{\nu}$)