ADVANCED CALCULUS
SPRING 2019

LESSON 5: Calculus of Variations

This lesson considers Calculus of Variations, which is concerned with optimization of functions of functions, or functionals. This material is not covered in the Marsden-Tromba text. Some inexpensive but good references are:

- *Calculus of Variations with Applications to Physics and Engineering,* Robert Weinstock, Dover.
- *Calculus of Variations with Applications,* George M. Ewing, Dover.
- *Introduction to the Calculus of Variations,* Hans Sagan, Dover.

1 Introduction

Calculus of variations draws problems from fields ranging from geometry, mechanics and physics to economics. Here we are interested in optimizing not functions but *functions of functions*, also known as *functionals*. In the simplest case a functional is an object that takes a function as input and generates a number as output. For example, the distance between Troy, NY and Washington, DC depends upon which path we take. The path, upon parametrization, can be described by a vector-valued function of the parameter, and the distance is the number that depends upon the path.

To motivate the study of the subject, let us examine three canonical problems which were among the earliest considered in the field.

The brachistochrone problem. We seek a smooth curve along which a point mass, starting from rest, descends under gravity from an initial point (a, y_i) to the final point (b, y_f) in the shortest possible time. The velocity v of the particle can be obtained from conservation of energy as

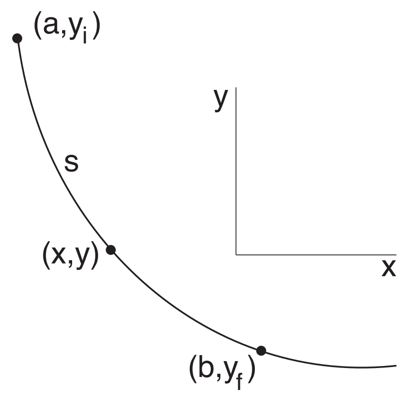

Figure 1: The brachistochrone problem.

$$\frac{1}{2}mv^2 = mg(y_i - y), \quad \text{so that} \quad v = \frac{ds}{dt} = \sqrt{2g(y_i - y)}. \tag{1.1}$$

Then the time of travel is

$$T = \int_0^{s(T)} \frac{1}{v}\, ds = \int_a^b \frac{1}{v}\frac{ds}{dx}\, dx = \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_i - y)}}\, dx.$$

We note that the time of travel, a number, is a function of the path $y(x)$, *i.e.*, $T = T[y]$. (We shall use the square-brackets notation to denote the dependence of the functional T upon the function $y(x)$.) Determining the function $y(x)$ that minimizes $T[y]$ is the subject of Calculus of Variations.

The geodesic problem. A geodesic is the curve of shortest length connecting two points. We seek the geodesic connecting $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on the surface defined implicitly by $F(x, y, z) = 0$.

Let $C : \boldsymbol{r}(t) =< x(t), y(t), z(t) >$, $a \le t \le b$, be a parametric description of the geodesic. Then the quantity to be minimized is the length of the curve,

$$L = \int_a^b ||d\boldsymbol{r}/dt||\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

subject to the constraint that the curve lies on the surface, *i.e.*,

$$F(x, y, z) = 0.$$

We note that the functional L depends upon three functions $x(t)$, $y(t)$ and $z(t)$, which are not independent but are constrained by the above equation. This problem is akin to locating constrained extrema of a function.

The isoperimetric problem. Consider the problem of finding the planar region of maximum area A bounded by a curve C of given length L. Green's theorem allows us to write the area as the line integral

$$A = \oint_C x \, dy.$$

Upon parametrizing the curve as

$$C : \boldsymbol{r}(t) =< x(t), y(t) >, \ a \leq t \leq b,$$

we can write

$$A = \int_a^b x(t) \frac{dy}{dt} \, dt,$$

subject to the constraint

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

Again, we are faced with a constrained optimization problem but now the constraint is in the form of an integral.

2 A general problem

We note that the brachistochrone problem is the simplest of the three problems described above as it is not subject to a constraint. It is a special case of the following problem: extremalize[1] the functional $I[y]$ defined by the integral

$$I[y] = \int_a^b f\big(x, y(x), y'(x)\big) \, dx \tag{2.1}$$

among all sufficiently smooth functions that satisfy the boundary conditions

$$y(a) = A, \quad y(b) = B. \tag{2.2}$$

By sufficient smoothness we mean that $y(x)$ has enough derivatives for $x \in [a, b]$ so that the integral in (2.1) exists and the manipulations we are about to perform are valid.

Our approach is to transform the problem from one that extremalizes a functional to one that extremalizes a function. Suppose that the extremalizing function is $y(x)$. Consider *nearby* functions $\tilde{y}(x)$ defined by

$$\tilde{y}(x) = y(x) + \epsilon\eta(x) \tag{2.3}$$

where ϵ is sufficiently small, and $\eta(x)$, regarded as given, is sufficiently smooth and satisfies the boundary conditions

$$\eta(a) = \eta(b) = 0. \tag{2.4}$$

For definiteness let us assume that $y(x)$ maximizes I. (Analogous arguments apply if $y(x)$ is the minimizer.) Then

$$I[\tilde{y}] = I[y + \epsilon\eta] \leq I[y].$$

We can now treat $I[y+\epsilon\eta]$ as a function of ϵ, which we shall denote by $\mathcal{I}(\epsilon)$, and which is maximized when $\epsilon = 0$. From our earlier study of local maxima/minima we know that the necessary condition is

$$\mathcal{I}'(\epsilon) = 0 \quad \text{when} \quad \epsilon = 0. \tag{2.5}$$

Now we can write $\mathcal{I}(\epsilon)$ as

$$\mathcal{I}(\epsilon) = I[y + \epsilon\eta] = \int_a^b f\left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right) \, dx.$$

[1] We use the term *extremalize*, or *render stationary*, which includes both *maximize* and *minimize*.

Upon differentiating under the integral sign we get

$$\mathcal{I}'(\epsilon) = \int_a^b \frac{d}{d\epsilon} f\left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right)\, dx.$$

Let us set

$$u = y + \epsilon\eta,\ \ v = y' + \epsilon\eta'.$$

Then

$$\begin{aligned}\frac{d}{d\epsilon} f(x,u,v) &= \frac{\partial f}{\partial u}(x,u,v)\frac{du}{d\epsilon} + \frac{\partial f}{\partial v}(x,u,v)\frac{dv}{d\epsilon} \\ &= \frac{\partial f}{\partial u}(x,u,v)\eta + \frac{\partial f}{\partial v}(x,u,v)\eta'.\end{aligned}$$

When evaluated at $\epsilon = 0$, *i.e.*, at $u = y$ and $v = y'$,

$$\frac{d}{d\epsilon} f(x,u,v)\Big|_{\epsilon=0} = \frac{\partial f}{\partial y}(x,y,y')\eta + \frac{\partial f}{\partial y'}(x,y,y')\eta'.$$

Therefore (4.10) leads to

$$\int_a^b \left[\frac{\partial f}{\partial y}(x,y,y')\eta + \frac{\partial f}{\partial y'}(x,y,y')\eta'\right] dx = 0. \tag{2.6}$$

We integrate the second term in the integral by parts to get

$$\int_a^b \left[\frac{\partial f}{\partial y}(x,y,y')\eta(x) - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x,y,y')\right\}\eta(x)\right] dx + \frac{\partial f}{\partial y'}(x,y,y')\eta(x)\Big|_a^b = 0. \tag{2.7}$$

Application of the boundary conditions (4.9) causes the last term above to vanish, so that

$$\int_a^b \left[\frac{\partial f}{\partial y}(x,y,y') - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x,y,y')\right\}\right]\eta(x)\, dx = 0. \tag{2.8}$$

Continuity of the integrand coupled with arbitrariness of $\eta(x)$ requires that the integrand must vanish, *i.e.*,

$$\frac{\partial f}{\partial y}(x,y,y') - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x,y,y')\right\} = 0. \tag{2.9}$$

The above equation, known as the *Euler-Lagrange equation*, is the *necessary condition* that the extremalizing function $y(x)$ must satisfy. When the given function $f(x,y,y')$ is substituted into it the result is a second-order ODE for $y(x)$ whose solution is determined subject to the boundary conditions (2.2). We note that only the *necessary condition* for extremalizing has been satisfied. Additional consideration, usually based on application-dependent physical arguments, is needed to ascertain whether the result is a maximizer or a minimizer.

2.1 Integration of the Euler-Lagrange equation in special cases

The E-L equation (4.14) may be expanded to read

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial y'^2}y'' - \frac{\partial^2 f}{\partial y'\partial y}y' - \frac{\partial^2 f}{\partial y'\partial x} = 0.$$

As already mentioned, this is a second-order ODE for the extremalizing function $y(x)$. Some special cases emerge.

1. f **does not depend explicitly on** y'. Then the equation reduces to the algebraic equation

   $$\frac{\partial f}{\partial y} = 0,$$

   which determines $y(x)$ implicitly and does not accept arbitrary boundary conditions.

2. f **does not depend explicitly on** y. Then the equation reduces to

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0,$$

which has the first integral

$$\frac{\partial f}{\partial y'} = C, \tag{2.10}$$

where C is a constant.

3. f **does not depend explicitly on** x. To see the resulting simplification consider

$$\begin{aligned} \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) &= \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\frac{\partial f}{\partial y'} \\ &= y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right). \end{aligned}$$

The RHS vanishes as per the E-L equation. Therefore, so does the LHS, yielding the first integral

$$f - y'\frac{\partial f}{\partial y'} = C, \tag{2.11}$$

where C is a constant.

Remark. In 1744 the French scientist Maupertuis proposed the idea that nature is parsimonious; it operates with the greatest economy. He enunciated what came to be known as the Principle of Least Action: if a change occurs in nature, then the amount of action needed to produce the change must be as small as possible. This rather vague statement of Maupertuis was elaborated upon and perfected into the modern tool of variational calculus by several contributors including the Bernoulli brothers, Jacob and Johann, Johann's student Euler, Lagrange, Hamilton, and Jacobi, among others. Variational calculus allows one to pose problems involving deformation, motion, pattern formation, etc in the *variational form*, *i.e.*, in terms of rendering stationary functionals defined by integrals. The variational form has some advantages, including the fact that many modern computational algorithms are based on it.

We begin with a simple example that illustrates the idea, before considering other problems.

Example 2.1. Hamilton's Principle of Stationary Action asserts that the motion of a dynamical system under the action of conservative forces is such as to render stationary the integral of the Lagrangian, where the Lagrangian is the difference between the kinetic and potential energies of the system. We now show, for a single particle, that the resulting equation of motion is the same as that given by Newton's law.

Consider a point mass m attached to a string with stiffness k and moving horizontally without friction. Let $y(t)$ be the displacement of the mass from the rest position of zero extension of the spring, Figure 2. Then the kinetic energy of the mass is

$$K = \frac{1}{2}m\{y'(t)\}^2$$

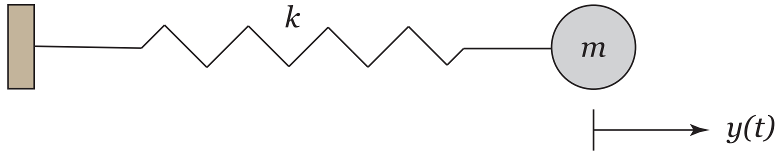

Figure 2: Spring-mass system.

and the potential energy (*i.e.*, the energy stored in the spring due to the work done in extending it through the amount y) is

$$P = \int_0^y ky\, dy = \frac{1}{2}ky^2.$$

Hamilton's Principle asserts that the motion of the system, $y(t)$, over the time duration of interest, $[0, T]$, say, is such as to rendered stationary the integral

$$I[y] = \int_0^T (K - P)\, dt = \int_0^T \left(\frac{1}{2}m\{y'(t)\}^2 - \frac{1}{2}ky^2 \right) dt.$$

The integrand $K - P$ is also known as the Lagrangian, L, where

$$L = \frac{1}{2}m\{y'(t)\}^2 - \frac{1}{2}ky^2.$$

The Euler-Lagrange equation for the optimal solution is

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y'} = 0,$$

or

$$-ky - \frac{d}{dt}(my') = 0,$$

leading to the familiar equation

$$y'' + \frac{k}{m}y = 0$$

for the spring-mass system, usually derived by using Newton's law of motion. Thus we have shown that Newton's law, based upon balancing forces (requiring vector considerations in more complicated systems), is equivalent to Hamilton's principle, which minimizes (or more precisely, renders stationary), the *action* which in this case is the integral of the Lagrangian, a scalar quantity. Thus Hamilton's principle is the preferred mode of deriving equations of motion in complex systems.

Example 2.2. We return to the brachistochrone problem for which the functional to be minimized is the time of travel,

$$T[y] = \frac{1}{\sqrt{2g}} \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{(y_i - y)}}\, dx$$

subject to the boundary conditions

$$y(a) = y_i, \quad y(b) = y_f.$$

Here the function f is given by

$$f(x, y, y') = \frac{1}{\sqrt{2g}}(1 + y'^2)^{1/2}(y_i - y)^{-1/2}.$$

Upon substituting into the Euler-Lagrange equation (4.14), followed by algebraic manipulations, we arrive at the differential equation

$$y'' = \frac{1 + y'^2}{2(y_i - y)}.$$

The substitution $z = y'$ generates the system

$$\begin{aligned} y' &= z, \\ z' &= \frac{1 + z^2}{2(y_i - y)} \end{aligned}$$

which can be written as

$$\frac{dz}{dy} = \frac{1 + z^2}{2z(y_i - y)}$$

or, upon separating variables, as

$$\frac{2z\,dz}{1+z^2} = \frac{dy}{y_i - y}$$

which integrates to

$$\ln(1+z^2) = -\ln(y_i - y) + \ln 2C,$$

or equivalently, to

$$1 + z^2 = \frac{2C}{y_i - y}.$$

Upon solving for z, setting $z = y'$ and noting that y' is negative for a descent path,

$$y' = -\sqrt{\frac{2C}{y_i - y} - 1}.$$

Further simplification results from the substitution

$$y_i - y = C(1 - \cos\theta). \tag{2.12}$$

Then $-y' = C\sin\theta\,\theta'$ and the ODE reduces to

$$C\sin\theta\,\theta' = \sqrt{\frac{2}{1-\cos\theta} - 1} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta},$$

and then to $C(1-\cos\theta)\theta' = 1$ which integrates to

$$x - a = C(\theta - \sin\theta). \tag{2.13}$$

Equations (2.12) - (2.13), taken together, provide the parametric description of the extremalizing curve, a cycloid. This curve is generated by the motion of a fixed point on the rim of a circle of radius C, which rolls without slipping on the underside of the line $y = y_i$. The initial point corresponds to $\theta = 0$ while the final point (b, y_f) determines C and θ_f uniquely. Also, the minimum time is given by

$$T = \sqrt{\frac{1}{2g}}\int_a^b \frac{\sqrt{1+z^2}}{\sqrt{y_i - y}}\,dx = \sqrt{\frac{C}{g}}\int_a^b \frac{1}{y_i - y}\,dx = \sqrt{\frac{C}{g}}\int_0^{\theta_f} d\theta = \sqrt{\frac{C}{g}}\theta_f.$$

Example 2.3. We consider the problem of finding the minimal surface of revolution. Given two points $(0, A)$ and (b, B), see Figure 3, we seek the smooth curve $y(x)$ connecting them such that rotation of the curve about the x-axis generates a surface with the smallest area. (The restriction of smoothness is important otherwise a discontinuous function satisfying $y(0) = A$, $y(b) = B$ and $y(x) = 0$ for $0 < x < b$ would create a surface of revolution with zero area.)

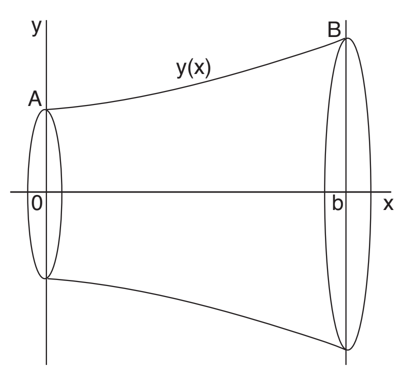

Figure 3: Surface of rotation of least area.

The area is given by the functional

$$I[y] = 2\pi \int_0^b y\sqrt{1+y'^2}\, dx.$$

Here $f = y\sqrt{1+y'^2}$ is independent of x. Therefore Case 3 in section 2.1 applies and equation (2.11) then provides the first integral

$$f - y'\frac{\partial f}{\partial y'} = y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = C,$$

which simplifies to

$$\frac{y}{\sqrt{1+y'^2}} = C.$$

On squaring and separating variables,

$$\frac{dy}{\sqrt{y^2/C^2 - 1}} = dx,$$

which integrates to yield

$$y = C\cosh(x/C + C_1).$$

The graph of the hyperbolic cosine is called a *catenary.* The constants C and C_1 are determined by the boundary conditions.

Example 2.4. Consider the problem of determining a geodesic curve, *i.e.*, the curve of minimum arc length connecting two points on a surface. Let the surface S be parametrized as $\boldsymbol{r} = \boldsymbol{r}(\alpha, \beta)$, and let the two points be $A(\alpha_0, \beta_0)$, $B(\alpha_1, \beta_1)$.

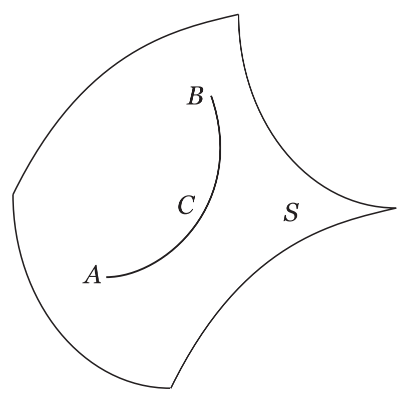

Figure 4: Geodesic on surface S.

Let a curve C connecting the points and lying on S be $\beta = \beta(\alpha)$. Then C can be parametrized as

$$C:\ \boldsymbol{r} = \boldsymbol{r}\big(\alpha, \beta(\alpha)\big), \quad \alpha_0 \le \alpha \le \alpha_1.$$

The arc length of C is a functional of $\beta(\alpha)$, given by

$$I[\beta] = \int_C ds = \int_{\alpha_0}^{\alpha_1} \frac{ds}{d\alpha}\, d\alpha = \int_{\alpha_0}^{\alpha_1} \left\| \frac{d\boldsymbol{r}}{d\alpha} \right\| d\alpha.$$

As an example, let S be a right-circular cylinder of radius R parametrized as

$$S:\ \boldsymbol{r} = < R\cos\theta, R\sin\theta, z > .$$

Let a curve C joining points $A(\theta_0, z_0)$ and $B(\theta_1, z_1)$ be parametrized as

$$C:\ \boldsymbol{r} = < R\cos\theta, R\sin\theta, z(\theta) >, \quad \theta_0 \le \theta \le \theta_1.$$

Then the arc length functional is

$$I[z] = \int_{\theta_0}^{\theta_1} \left|\left|\frac{d\boldsymbol{r}}{d\theta}\right|\right| \, d\theta = \int_{\theta_0}^{\theta_1} \sqrt{R^2 + z'^2}\, d\theta.$$

Now,

$$f(\theta, z, z') = \sqrt{R^2 + z'^2}.$$

Since f is independent of z, a first integral of the E-L equation is $\partial f/\partial z' =$ a constant, which leads to

$$\frac{z'}{\sqrt{R^2 + z'^2}} = \text{ a constant},$$

and therefore, to

$$z' = a,$$

which integrates to

$$z = a\theta + b,$$

where a and b are determined by the boundary conditions to be

$$a = \frac{z_1 - z_0}{\theta_1 - \theta_0}, \quad b = \frac{z_0\theta_1 - z_1\theta_0}{\theta_1 - \theta_0}.$$

The curve C is the helix

$$C: \, \boldsymbol{r} =< R\cos\theta, R\sin\theta, a\theta + b >, \quad \theta_0 \le \theta \le \theta_1.$$

The minimum arc length is given by

$$I = \int_{\theta_0}^{\theta_1} \sqrt{R^2 + a^2}\, d\theta = \sqrt{R^2 + a^2}\, (\theta_1 - \theta_0).$$

3 Boundary conditions

In the discussion so far boundary conditions on the extremalizing function were specified. In many problems there are either no boundary conditions or one or more boundary conditions correspond to the end points moving on prescribed curves. Thus determination of the boundary conditions is a part of the problem.

3.1 Natural boundary condition

We consider again the functional $I[y]$ defined in (2.1) and reproduced below.

$$I[y] = \int_a^b f\big(x, y(x), y'(x)\big)\, dx,$$

subject to the boundary condition $y(a) = A$. No boundary condition at the end $x = b$ is specified. Again we consider nearby functions $\tilde{y}(x) = y(x) + \epsilon\eta(x)$ where now, $\eta(a) = 0$ but $\eta(b)$ is left unspecified. We expect that $\mathcal{I}(\epsilon) = I(y + \epsilon\eta)$ will be optimized at $\epsilon = 0$ independent of what value the optimal function $y(x)$ (and hence the perturbation $\eta(x)$) takes at $x = b$. Setting $\mathcal{I}'(\epsilon) = 0$ at $\epsilon = 0$ followed by an integration by parts leads, as before, to (4.12), reproduced below.

$$\int_a^b \left[\frac{\partial f}{\partial y}(x, y, y')\eta(x) - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x, y, y')\right\}\eta(x)\right] dx + \frac{\partial f}{\partial y'}(x, y, y')\eta(x)\Big|_a^b = 0.$$

Since $\eta(a) = 0$, the last term on the LHS can be simplified and we get

$$\int_a^b \left[\frac{\partial f}{\partial y}(x, y, y')\eta(x) - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x, y, y')\right\}\eta(x)\right] dx + \frac{\partial f}{\partial y'}\big(b, y(b), y'(b)\big)\eta(b) = 0. \tag{3.1}$$

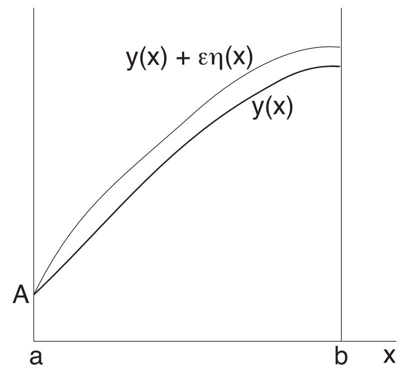

Figure 5: Extremalizing function $y(x)$ and nearby function $y(x) + \epsilon\eta(x)$ with $\eta(b)$ arbitrary.

The above equation holds for arbitrary $\eta(b)$, and certainly holds when $\eta(b) = 0$. Then the boundary term in the above equation vanishes and we are back to (4.13), and hence to the Euler-Lagrange equation (4.14), reproduced below.

$$\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x, y, y')\right\} = 0.$$

Substitution of the above equation back into (3.1) leads to the requirement

$$\frac{\partial f}{\partial y'}\big(b, y(b), y'(b)\big)\eta(b) = 0.$$

which is satisfied for arbitrary $\eta(b)$ only if

$$\frac{\partial f}{\partial y'}\big(b, y(b), y'(b)\big) = 0.$$

This now is the boundary condition at $x = b$ that the extremalizing function $y(x)$ must satisfy, in addition to the condition $y(a) = A$. It is called the *natural boundary condition.*

Example 3.1. To see how boundary conditions affect the solution, consider the functional

$$I[y] = \int_0^\pi (2y\sin x + y'^2)\, dx,$$

with $y(0) = 0$ as the left boundary condition. At the right boundary we apply the natural condition. Here,

$$f(x, y, y') = 2y\sin x + y'^2$$

with

$$\frac{\partial f}{\partial y'} = 2y'.$$

Therefore the natural condition at $x = \pi$ reduces to $y'(\pi) = 0$. The E-L equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

simplifies to the ODE

$$\sin x - y'' = 0$$

which has the general solution

$$y = ax + b - \sin x.$$

The boundary conditions $y(0) = 0$ and $y'(\pi) = 0$ find $a = -1$, $b = 0$, leading to the solution $y = -x - \sin x$.

If instead of the natural boundary condition we had applied a different condition, $y(\pi) = 0$, then the solution would have been $y = -\sin x$.

Example 3.2. We consider the boat transit problem. A boat is to cross a river with parallel banks a distance b across. We assume that the left bank coincides with the y-axis and the right bank with the line $x = b$, and that the starting point is the origin. The boat travels at a natural speed v relative to the surrounding water while the river flows in the positive y-direction with speed $w(x)$. It is of interest to determine the path of the boat corresponding to the shortest transit time. Note that the landing point on the bank $x = b$ is not specified. Thus the natural boundary condition applies there.

Let $\theta(t)$ be the steering angle of the boat, measured between the x-axis and the axis of the boat. Then the trajectory of the boat, $\boldsymbol{r} =< x(t), y(t) >$, satisfies the equations

$$\frac{dx}{dt} = v\cos\theta, \quad \frac{dy}{dt} = v\sin\theta + w(x).$$

If the function $y(x)$ defines the path of the boat, then

$$y'(x) = \frac{dy/dt}{dx/dt} = \frac{v\sin\theta + w}{v\cos\theta}. \tag{3.2}$$

The time of transit can be expressed as

$$T = \int_0^T dt = \int_0^b \frac{dt}{dx}\,dx = \int_0^b \frac{dx}{v\cos\theta}. \tag{3.3}$$

We can solve for $\cos\theta$ from equation (3.2) as follows. On squaring the equation and replacing $\sin^2\theta$ by $1-\cos^2\theta$ the following quadratic for $\cos\theta$ emerges.

$$(1 + y'^2)\cos^2\theta - 2\alpha y'\cos\theta + \alpha^2 - 1 = 0.$$

Here,

$$\alpha(x) = \frac{w(x)}{v} \tag{3.4}$$

is the ratio of the speed of the river to the natural speed of the boat. We assume that the boat can travel faster than the river, so that $\alpha(x) < 1$. The solution of the quadratic is

$$\cos\theta = \frac{\alpha y' \pm [\alpha^2 y'^2 + (1-\alpha^2)(1+y'^2)]^{1/2}}{1+y'^2}.$$

We select the upper sign which ensures that $\cos\theta > 0$, consistent with our expectation that $dx/dt = v\cos\theta > 0$, *i.e.*, the boat is aimed at the far bank. Simple algebraic manipulations allow the above equation to be rewritten as

$$\cos\theta = \frac{1-\alpha^2}{\sqrt{1+y'^2-\alpha^2} - \alpha y'}.$$

The equation (3.3) for the transit time becomes

$$T[y] = \int_0^b \frac{\sqrt{1+y'^2-\alpha^2} - \alpha y'}{v(1-\alpha^2)}\,dx. \tag{3.5}$$

Consider first the case when the speed of the river is independent of x, so that w and hence α is a constant. Then, the transit time is

$$T[y] = \frac{1}{v(1-\alpha^2)} \int_0^b f(x, y, y')\,dx,$$

where

$$f(x, y, y') = \sqrt{1+y'^2-\alpha^2} - \alpha y'.$$

Since f does not explicitly depend upon x or y the E-L equation has the first integral

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2-\alpha^2}} - \alpha = C,$$

where C is a constant. The natural boundary condition $\partial f/\partial y' = 0$ at $x = b$ selects the constant to be zero. Then above equation can be solved for y' to yield

$$y' = \alpha.$$

The solution subject to $y(0) = 0$ is the straight line

$$y = \alpha x = \frac{w}{v} x.$$

Now consider the case when $w(x)$, the speed of the river, is not a constant. Then $\alpha(x)$, defined by (3.4), is a function of x. Equation (3.5) for the transit time can now be written as

$$T[y] = \frac{1}{v} \int_0^b f(x, y, y')\, dx,$$

where now,

$$f(x, y, y') = \frac{\sqrt{1 + y'^2 - \alpha^2} - \alpha y'}{1 - \alpha^2}.$$

Since f does not explicitly depend upon y the E-L equation again has the first integral

$$\frac{\partial f}{\partial y'} = \frac{1}{1 - \alpha^2} \left(\frac{y'}{\sqrt{1 + y'^2 - \alpha^2}} - \alpha \right) = C,$$

where the constant C is selected to be zero, as above, by the natural boundary condition $\partial f/\partial y' = 0$ at $x = b$. Then above equation can be solved for y' to yield

$$y' = \alpha(x).$$

The solution subject to $y(0) = 0$ is

$$y = \int_0^x \alpha(s)\, ds.$$

For the special case $\alpha(x) = b - x$,

$$y(x) = bx - \frac{x^2}{2}.$$

For $b = 1$ the boat trajectory is shown below. The boat approaches the far bank $x = 1$ head on because

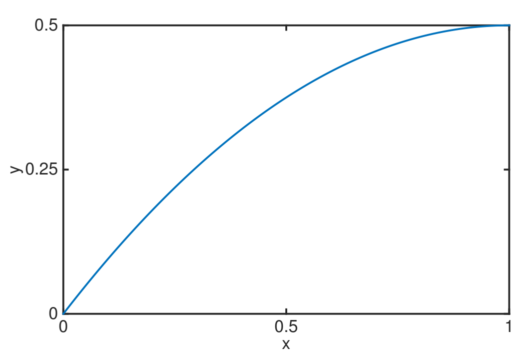

Figure 6: Boat trajectory.

the river speed at that location is zero.

3.2 Transversality condition

Let us now consider the case when the left end of the extremalizing function remains fixed at $y(a) = A$ but the right end is allowed to float on the curve

$$T(x, y) = 0. \tag{3.6}$$

Proceeding as before, we start with the functional

$$I[y] = \int_a^b f\big(x, y(x), y'(x)\big)\, dx,$$

with b, the right end of the extremalizing function y, unknown, but subject to $T(b, y(b)) = 0$. Determining b will be a part of the solution. We consider nearby functions $\tilde{y} = y + \epsilon\eta$ and nearby locations of the right boundary, $\tilde{b} = b + \epsilon\beta$. Note that $\eta(a) = 0$. Now,

$$\mathcal{I}(\epsilon) = I[y + \epsilon\eta] = \int_a^{b+\epsilon\beta} f\left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right)\, dx.$$

Then,

$$\begin{aligned}\mathcal{I}'(\epsilon) \quad &= \quad \beta f\big(b + \epsilon\beta, y(b + \epsilon\beta) + \epsilon\eta(b + \epsilon\beta), y'(b + \epsilon\beta) + \epsilon\eta'(b + \epsilon\beta)\big) \\ &\quad + \int_a^{b+\epsilon\beta} \left[\frac{\partial f}{\partial \tilde{y}}(x, y + \epsilon\eta, y' + \epsilon\eta')\eta + \frac{\partial f}{\partial \tilde{y}'}(x, y + \epsilon\eta, y' + \epsilon\eta')\eta'\right] dx.\end{aligned}$$

The necessary condition for extremalization is $\mathcal{I}'(\epsilon) = 0$ at $\epsilon = 0$, leading to

$$\beta f\big(b, y(b), y'(b)\big) + \int_a^b \left[\frac{\partial f}{\partial y}(x, y, y')\eta + \frac{\partial f}{\partial y'}(x, y, y')\eta'\right] dx = 0.$$

The by-now-usual integration by parts yields

$$\beta f\big(b, y(b), y'(b)\big) + \frac{\partial f}{\partial y'}(b, y(b), y'(b))\eta(b) + \int_a^b \left[\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x, y, y')\right\}\right] \eta(x)\, dx = 0,$$

where we have used the fact that $\eta(a) = 0$. The argument employed in deriving the natural boundary condition applies again, leading to the same E-L equation,

$$\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}(x, y, y')\right\} = 0,$$

as before, along with the condition

$$\beta f\big(b, y(b), y'(b)\big) + \frac{\partial f}{\partial y'}(b, y(b), y'(b))\eta(b) = 0. \tag{3.7}$$

Since the point $(b + \epsilon\beta, y(b + \epsilon\beta) + \epsilon\eta(b + \epsilon\beta))$ lies on $T(x, y) = 0$, we have

$$T\big(b + \epsilon\beta, y(b + \epsilon\beta) + \epsilon\eta(b + \epsilon\beta)\big) = 0.$$

On differentiation with respect to ϵ, the above yields

$$\beta T_x\big(b+\epsilon\beta, y(b+\epsilon\beta)+\epsilon\eta(b+\epsilon\beta)\big)+[\beta y'(b+\epsilon\beta)+\eta(b+\epsilon\beta)+\epsilon\beta\eta'(b+\epsilon\beta)]T_y\big(b+\epsilon\beta, y(b+\epsilon\beta)+\epsilon\eta(b+\epsilon\beta)\big) = 0.$$

On setting $\epsilon = 0$ we obtain

$$\beta T_x(b, y(b)) + [\beta y'(b) + \eta(b)]T_y(b, y(b)) = 0. \tag{3.8}$$

Elimination of $\eta(b)/\beta$ from (3.7) and (3.8) leads to the condition

$$fT_y - f_{y'}[T_x + y'T_y] = 0, \tag{3.9}$$

where $f(x, y, y')$ and $T(x, y)$ in the above equation are evaluated at $x = b$, $y = y(b)$. This condition, known as the *transversality condition,* is now imposed on the right end of the extremalizing function $y(x)$, along with the constraint (3.6).

Example 3.3. Consider the functional

$$I[y] = \int_0^b (1 + y'^2)\, dx.$$

We seek the extremalizing function $y(x)$ which satisfies the boundary condition $y(0) = 0$ on the left and whose right end must lie on the curve $xy = 1$. Here, $f = 1 + y'^2$ and right end floats on the curve $T(x, y) = xy - 1 = 0$.

The E-L equation is

$$f_y - \frac{d}{dx} f_{y'} = 0,$$

which simplyfies to $y'' = 0$ and has the general solution $y = px + q$. The condition $y(0) = 0$ finds $q = 0$ so that $y = px$. Since the right end $x = b$ lies on $xy - 1 = 0$, we have $by(b) - 1 = 0$, or

$$pb^2 = 1. \tag{3.10}$$

The transversality condition (3.9) becomes

$$(1 + y'^2)x - 2y'(y + y'x) = 0, \quad \text{or} \quad x - xy'^2 - 2yy' = 0.$$

It must be satisfied by $y = px$ at $x = b$. With $y = pb$ and $y' = p$, we get $b - 3bp^2 = 0$. Since $b = 0$ corresponds to the left end point, we ignore that solution to get $p^2 = 1/3$ or $p = \pm 1/\sqrt{3}$. Equation (3.10) requires p to be positive, so we choose $p = 1/\sqrt{3}$. Then (3.10) yields $b = 1/\sqrt{p} = 3^{1/4}$ as the right end of the extremalizing curve, which now becomes $y = x/\sqrt{3}$.

4 Generalizations

The procedure outline above can be extended with ease to more general classes of functionals, and we present some of the extensions below.

4.1 Several dependent variables

Let

$$I[\boldsymbol{y}] = \int_a^b f(x, \boldsymbol{y}, \boldsymbol{y}')\, dx$$

where $\boldsymbol{y}(x)$ is now a vector-valued function in R^n, with

$$\boldsymbol{y}(x) =< y_1(x), y_2(x), \cdots, y_n(x)).$$

Proceeding much in the same way as for the scalar case, we now find that there are n E-L equations,

$$\frac{\partial f}{\partial y_j} - \frac{d}{dx}\frac{\partial f}{\partial y_j'} = 0, \quad j = 1, 2, \cdots, n,$$

with two boundary conditions for each.

4.2 Functionals with higher-order derivatives

Let

$$I[y] = \int_a^b f(x, y, y', y'', \cdots, y^{(n)})\, dx$$

At each boundary, $y, y', \cdots, y^{(n-1)}$ are prescribed. Proceeding as before, we now have

$$\mathcal{I}(\epsilon) = \int_a^b f\big(x, y + \epsilon\eta, y' + \epsilon\eta', \cdots, y^{n)} + \eta^{(n)}\big)\, dx.$$

On setting $\mathcal{I}'(\epsilon) = 0$ at $\epsilon = 0$ we get

$$\int_a^b \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \frac{\partial f}{\partial y''}\eta'' + \cdots + \frac{\partial f}{\partial y^{(n)}}\eta^{(n)} \right] dx = 0.$$

Assuming that $\eta,\ \eta',\ \cdots, \eta^{(n-1)}$ all vanish at $x = a$ and $x = b$, integration by parts can be performed (as many times as needed so that no derivatives of η remain) to yield

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} \right] \eta \, dx.$$

The usual argument about the arbitrariness of η and the continuity of the integrand leads to the E-L equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} = 0.$$

This, in general, is an ODE of order $2n$, subject to n boundary conditions at either end.

4.3 Several independent variables

Let us consider the case where the extremalizing function u depends upon three independent variables, x, y and z, *i.e.*, $u = u(x, y, z)$, and the functional $I(u)$ is defined by the integral

$$I[u] = \iiint_{\mathcal{B}} f(x, y, z, u, u_x, u_y, u_z)\, dV.$$

Here $\mathcal{B}$ is a fixed region in R^3 with smooth boundary S and outward unit normal $\boldsymbol{n}$. We seek amongst the class of twice continuously differentiable functions the extremalizing function u that satisfies the boundary condition

$$u = g(x, y, z) \quad \text{on} \quad S.$$

Consider nearby functions

$$\tilde{u}(x, y, z) = u(x, y, z) + \eta(x, y, z),$$

where $\eta = 0$ on S. Define $\mathcal{I}(\epsilon)$ by

$$\mathcal{I}(\epsilon) = I[u + \epsilon\eta] = \iiint_{\mathcal{B}} f(x, y, z, u + \epsilon\eta, u_x + \epsilon\eta_x, u_y + \epsilon\eta_y, u_z + \epsilon\eta_z)\, dV.$$

Upon setting $\mathcal{I}'(\epsilon) = 0$ at $\epsilon = 0$ we obtain

$$\mathcal{I}(\epsilon) = I[u + \epsilon\eta] = \iiint_{\mathcal{B}} \left[\frac{\partial f}{\partial u}\eta + \frac{\partial f}{\partial u_x}\eta_x + \frac{\partial f}{\partial u_y}\eta_y + \frac{\partial f}{\partial u_z}\eta_z \right] dV = 0.$$

The above takes the succinct form

$$\iiint_{\mathcal{B}} \left(\frac{\partial f}{\partial u}\eta + \boldsymbol{F} \cdot \nabla\eta \right) dV = 0, \tag{4.1}$$

where

$$\boldsymbol{F} \equiv \left\langle \frac{\partial f}{\partial u_x}, \frac{\partial f}{\partial u_y}, \frac{\partial f}{\partial u_z} \right\rangle.$$

Now, according to the Divergence Theorem,

$$\begin{aligned} \iint_S \eta\boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iiint_{\mathcal{B}} \nabla \cdot (\eta\boldsymbol{F})\, dV \\ &= \iiint_{\mathcal{B}} \eta\nabla \cdot \boldsymbol{F}\, dV + \iiint_{\mathcal{B}} \boldsymbol{F} \cdot \nabla\eta\, dV. \end{aligned}$$

The LHS of the above equation is zero as $\eta = 0$ on S. Therefore

$$\iiint_{\mathcal{B}} \boldsymbol{F} \cdot \nabla \eta \, dV = -\iiint_{\mathcal{B}} \eta \nabla \cdot \boldsymbol{F} \, dV,$$

which allows (4.1) to be written as

$$\iiint_{\mathcal{B}} \left(\frac{\partial f}{\partial u} - \nabla \cdot \boldsymbol{F} \right) \eta \, dV = 0.$$

The arbitrariness of η and the continuity of the integrand leads to the E-L equation,

$$\frac{\partial f}{\partial u} - \nabla \cdot \boldsymbol{F} = 0,$$

or in expanded form,

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial f}{\partial u_z} = 0.$$

This a PDE of second order, to be solved subject to the prescribed condition $u = g$ on S.

Example 4.1. Consider small transverse vibrations of a stretched elastic membrane which when at rest occupies the planar region D with boundary C in the xy-plane. Let the transverse displacement be $z = u(x, y, t)$. The stretched surface has the parametric description

$$\boldsymbol{r}(x, y, t) =< x, y, u(x, y, t) >, \quad (x, y) \in D.$$

Note that

$$\boldsymbol{r}_x \times \boldsymbol{r}_y =< -u_x, -u_y, 1 > \quad \text{so that} \quad ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{1 + u_x^2 + u_y^2}.$$

Then the increase in surface area of the stretched membrane from the unstretched position is

$$\begin{aligned} S(t) - S_0 &= \iint_S dS - S_0 \\ &= \iint_D ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| \, dA - \iint_D dA \\ &= \iint_D \left(\sqrt{1 + u_x^2 + u_y^2} - 1 \right) \, dA. \end{aligned}$$

The work done in stretching the membrane is stored in it as its potential energy, and is given by

$$P = \mu \iint_D \left(\sqrt{1 + u_x^2 + u_y^2} - 1 \right) \, dA,$$

where the constant μ is a material property of the membrane. If $\rho(x, y)$ is the density of the membrane material per unit area, then the kinetic energy of the membrane is

$$K = \iint_D \frac{1}{2} \rho(x, y) u_t^2 \, dA.$$

According to Hamilton's Principle of Stationary Action, introduced in Example 2.1, the motion of the membrane between a time interval $[t_1, t_2]$ renders stationary the functional

$$\begin{aligned} I[u] &= \int_{t_1}^{t_2} (K - P) \, dt \\ &= \int_{t_1}^{t_2} \iint_D \left[\frac{1}{2} \rho(x, y) u_t^2 - \mu \left(\sqrt{1 + u_x^2 + u_y^2} - 1 \right) \right] \, dA \, dt. \end{aligned}$$

The integrand, also known as the Lagrangian, is

$$L(x,y,t,u,u_x,u_y,u_t) = \frac{1}{2}\rho(x,y)u_t^2 - \mu\left(\sqrt{1+u_x^2+u_y^2}-1\right).$$

This problem has one dependent variable u and three independent variables x, y and t. The procedure of section 5.3 applies and leads to the Euler-Lagrange equation

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} - \frac{\partial}{\partial t}\frac{\partial f}{\partial u_t} = 0,$$

or,

$$-\frac{\partial}{\partial x}\left(\frac{-\mu u_x}{\sqrt{1+u_x^2+u_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{-\mu u_y}{\sqrt{1+u_x^2+u_y^2}}\right) - \frac{\partial}{\partial t}(\rho(x,y)u_t) = 0.$$

The above equation can be rewritten as

$$\rho(x,y)u_{tt} = \mu\left[\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right)\right].$$

For small transverse displacements u, and under the assumption that u_x and u_y are small as well, nonlinear terms can be neglected and the above PDE reduces to the familiar linear wave equation,

$$\rho u_{tt} = \mu(u_{xx}+u_{yy}).$$

Example 4.2. Consider the oscillations of a pendulum consisting of a mass m attached to a spring of stiffness k. The configuration is shown in Figure 7.

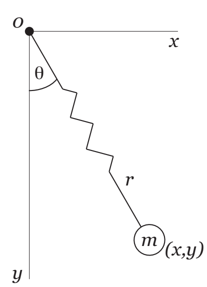

Figure 7: Pendulum with spring as string.

We again use Hamilton's principle to derive the equations of motion. Let $\theta(t)$ be the inclination of the pendulum to the vertical and $r(t)$ its length at time t. Then the coordinates of the mass are

$$x = r\sin\theta, \quad y = r\cos\theta.$$

The velocity of the mass is given by

$$\boldsymbol{v} =< dx/dt, dy/dt >=< r'\sin\theta + r\cos\theta\,\theta', r'\cos\theta - r\sin\theta\,\theta' >,$$

so that the kinetic energy is

$$K = \frac{1}{2}m||\boldsymbol{v}||^2 = \frac{1}{2}m[r'^2 + r^2\theta'^2].$$

Let L be the *unstretched* length of the spring. Then the potential energy P of the system is the sum of the potential energy of the mass, $-mgr\cos\theta$, and the stored energy in the spring, $(1/2)k(r-L)^2$. Therefore the Lagrangian is

$$\mathcal{L} = K - P = \frac{1}{2}m[r'^2 + r^2\theta'^2] + mgr\cos\theta - (1/2)k(r-L)^2.$$

Hamilton's Principle finds the equations of motion for r and θ by rendering stationary the integral

$$I[r, \theta] = \int_0^T \mathcal{L}\, dt.$$

As there are two dependent variables, r and θ, the treatment of section 4.1 applies and we get the two E-L equations as follows. For r the equation is

$$\mathcal{L}_r - \frac{d}{dt}\mathcal{L}_{r'} = 0,$$

which becomes

$$mr{\theta'}^2 + mg\cos\theta - k(r - L) - \frac{d}{dt}(mr') = 0,$$

or,

$$r'' + \frac{k}{m}(r - L) = g\cos\theta + r{\theta'}^2. \tag{4.2}$$

For θ the equation is

$$\mathcal{L}_\theta - \frac{d}{dt}\mathcal{L}_{\theta'} = 0,$$

which becomes

$$-mgr\sin\theta - \frac{d}{dt}(mr^2\theta') = 0,$$

or

$$r^2\theta'' + 2rr'\theta' + gr\sin\theta = 0. \tag{4.3}$$

We note that equations (4.2) and (4.3) are coupled differential equations for the evolution of r and θ. Let us consider two special cases. When the oscillations are vertical, $\theta = 0$. Then (4.3) is satisfied identically while (4.2) simplifies to

$$r'' + \frac{k}{m}(r - L) = g.$$

When the system is in equilibrium, $r' = r'' = 0$ and r is a constant given by $k(r - L) = mg$. Thus $r - L$ is the extension needed in the spring to balance gravity.

When the spring is replaced by an inextensible string, r equals L and $k = \infty$. Equation (4.3) then reduces to the usual pendulum equation,

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

What do you think happens to equation (4.2)?

EXTRA PRACTICE PROBLEMS

1. Find the Euler-Lagrange equations for the following functionals.

   (a)
$$I[y] = \int_a^b (y^2 - y'^2 - 2y\cos x)\, dx$$

   (b)
$$I[y] = \int_a^b \frac{y'^2}{x^2}\, dx$$

   (a) Here we have
$$f(x, y, y') = y^2 - y'^2 - 2y\cos x.$$
   The Euler-Lagrange equation is
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \tag{4.4}$$
   which becomes
$$2y - 2\cos x - (-2y')' = 0, \quad \text{or} \quad y'' = \cos x - y.$$

   (b) Here,
$$f(x, y, y') = \frac{y'^2}{x^2}.$$
   Since there is no explicit dependence upon y the Euler-Lagrange equation reduces to
$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \quad \text{or} \quad \frac{\partial f}{\partial y'} = \text{ a constant.}$$
   Thus we have
$$\frac{y'}{x^2} = C.$$

2. Show that, if y satisfies the Euler-Lagrange equation associated with the integral
$$I[y] = \int_a^b (p^2 y'^2 + q^2 y^2)\, dx,$$
   where $p(x)$ and $q(x)$ are known functions, then $I[y]$ has the value
$$(p^2 y y')\Big|_a^b.$$

Here, $f(x, y, y') = p^2 y'^2 + q^2 y^2$. Then the Euler-Lagrange equation (4.14) becomes
$$2q^2 y - (2p^2 y')' = 0, \quad \text{or} \quad q^2 y = (p^2 y')'. \tag{4.5}$$
We rewrite $I[y]$ by applying by-parts integration to the first term, to get
$$I[y] \;=\; p^2 y' y\Big|_a^b - \int_a^b y(p^2 y')'\, dx + \int_a^b q^2 y^2\, dx$$
We use (4.5) to replace $(p^2 y')'$ in the above equation by $q^2 y$, to get
$$\begin{aligned} I[y] &= p^2 y' y\Big|_a^b - \int_a^b y q^2 y'\, dx + \int_a^b q^2 y^2\, dx \\ &= p^2 y y'\Big|_a^b. \end{aligned}$$

3. Find the curve $y(x)$ that minimizes

$$I[y] = \int_0^1 \left(\frac{1}{2} y'^2 + yy' + y' + y \right) \, dx$$

when $y(0)$ and $y(1)$ are not specified.

Here,

$$f(x, y, y') = \frac{1}{2} y'^2 + yy' + y' + y.$$

The Euler-Lagrange equation (4.14) becomes

$$y' + 1 - (y' + y + 1)' = 0, \quad \text{or} \quad y'' = 1.$$

The general solution is

$$y = Cx + D + \frac{x^2}{2}.$$

The boundary condition at each end is the natural boundary condition $\partial f / \partial y' = 0$, or $y' + y + 1 = 0$. Here, $y' = C + x$. Thus

$$y' + y + 1 = C + x + Cx + D + \frac{x^2}{2} + 1 = 0 \quad \text{at} \quad x = 0 \text{ and } x = 1.$$

The two boundary conditions lead to

$$\begin{aligned} C + D &= -1, \\ 2C + D &= -\frac{5}{2}, \end{aligned}$$

whence

$$C = -\frac{3}{4}, \ D = -\frac{1}{4}.$$

Then the solution becomes

$$y = \frac{x^2}{2} - \frac{3x}{4} - \frac{1}{4}.$$

4. Find the function $y(x)$ that renders stationary the integral

$$I[y] = \int_0^{\pi/2} (y''^2 - y^2 + x^2) \, dx$$

subject to the boundary conditions $y(0) = y(\pi/2) = 0$, $y'(0) = y'(\pi/2) = 1$.

Here, the Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0.$$

For

$$f(x, y, y', y'') = y''^2 - y^2 + x^2$$

it becomes

$$-2y + 2y^{iv} = 0, \quad \text{or} \quad y^{iv} - y = 0.$$

The general solution is

$$y = C_1 \cos x + C_2 \sin x + C_3 \cosh x + C_4 \sinh x,$$

with

$$y' = -C_1 \sin x + C_2 \cos x + C_3 \sinh x + C_4 \cosh x.$$

The boundary conditions lead to

$$\begin{aligned} C_1 + C_3 &= 0, \\ C_2 + C_4 &= 1, \\ C_2 + C_3 \cosh \pi/2 + C_4 \sinh \pi/2 &= 0, \\ -C_1 + C_3 \sinh \pi/2 + C_4 \cosh \pi/2 &= 1. \end{aligned}$$

The solution of this linear system is found to be

$$\begin{aligned} C_1 &= -\frac{1 - \sinh \pi/2 - \cosh \pi/2}{2}, \\ C_2 &= \frac{1 - \sinh \pi/2 - \cosh \pi/2}{2}, \\ C_3 &= \frac{1 - \sinh \pi/2 - \cosh \pi/2}{2}, \\ C_4 &= \frac{1 + \sinh \pi/2 + \cosh \pi/2}{2}. \end{aligned}$$

5. Find the extremals of the functional

$$I[y, z] = \int_0^{\pi/2} ({y'}^2 + {z'}^2 + 2yz)\, dx$$

subject to $y(0) = 0,\ y(\pi/2) = 1,\ z(0) = 0,\ z(\pi/2) = 1.$

Here, $f(x, y, z, y', z') = {y'}^2 + {z'}^2 + 2yz$. The two Euler-Lagrange equations,

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 0, \\ \frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) &= 0, \end{aligned}$$

become

$$\begin{aligned} y'' &= z, \\ z'' &= y. \end{aligned}$$

Elimination of z leads to the single 4-th order equation

$$y^{iv} - y = 0.$$

This is the same ODE as in the preceding problem, with general solution

$$y = C_1 \cos x + C_2 \sin x + C_3 \cosh x + C_4 \sinh x.$$

Then

$$z = y'' = -C_1 \cos x - C_2 \sin x + C_3 \cosh x + C_4 \sinh x.$$

The boundary conditions lead to

$$\begin{aligned} C_1 + C_3 &= 0, \\ -C_1 + C_3 &= 0, \\ C_2 + C_3 \cosh \pi/2 + C_4 \sinh \pi/2 &= 1, \\ -C_2 + C_3 \cosh \pi/2 + C_4 \sinh \pi/2 &= 1. \end{aligned}$$

The solution of this linear system is found to be

$$C_1 = C_2 = C_3 = 0,\ C_4 = \frac{1}{\sinh \pi/2}.$$

Therefore,

$$y = z = \frac{\sinh x}{\sinh \pi/2}.$$

6. Consider the problem of finding a geodesic on the surface of a sphere, joining two given points A and B. We shall use spherical coordinates θ and ϕ on the surface of the sphere, and let $\theta = \theta_1,\ \phi = \phi_1$ at A and $\theta = \theta_2,\ \phi = \phi_2$ at B. We shall show that the geodesic curve is in fact the great circle passing through A and B.

Let the curve C joining A and B be parametrized, using spherical coordinates, as

$$C:\ \boldsymbol{r} = R < \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >, \quad \theta = \theta(\phi), \quad \theta(\phi_1) = \theta_1,\ \theta(\phi_2) = \theta_2. \tag{4.6}$$

Then

$$\frac{d\boldsymbol{r}}{d\phi} = R < \cos\phi\cos\theta - \sin\phi\sin\theta\ \theta'(\phi), \cos\phi\sin\theta + \sin\phi\cos\theta\ \theta'(\phi), -\sin\phi > .$$

After some simplification we get

$$\left|\left|\frac{d\boldsymbol{r}}{d\phi}\right|\right| = R(1 + \sin^2\phi\ \theta'^2)^{1/2}$$

Therefore the length of C is the functional

$$\begin{aligned} L &= \int_C ds \\ &= \int_{\phi_1}^{\phi_2} \left|\left|\frac{d\boldsymbol{r}}{d\phi}\right|\right|\ d\phi \\ &= R\int_{\phi_1}^{\phi_2} (1 + \sin^2\phi\ \theta'^2)^{1/2}\ d\phi. \end{aligned}$$

The integrand is $f(\phi, \theta, \theta') = (1 + \sin^2\phi\ \theta'^2)^{1/2}$. The E-L equation is

$$\frac{\partial f}{\partial \theta} - \frac{d}{d\phi}\frac{\partial f}{\partial \theta'} = 0.$$

As f does not depend explicitly on θ, the E-L equation has the first integral

$$\frac{\partial f}{\partial \theta'} = c_1, \quad \text{or,} \quad \frac{\theta'\sin^2\phi}{(1 + \sin^2\phi\ \theta'^2)^{1/2}} = c_1.$$

This can be solved for θ' to get

$$\frac{d\theta}{d\phi} = \frac{C}{\sin\phi(\sin^2\phi - C^2)^{1/2}}.$$

Integration leads to

$$\theta - c_2 = \int \frac{c_1}{\sin\phi(\sin^2\phi - c_1^2)^{1/2}}\ d\phi.$$

This integral looks daunting but can be evaluated by using a trigonometric substitution. First we write it as

$$\begin{aligned} \theta - c_2 &= \int \frac{c_1}{\sin^2\phi\left(1 - c_1^2/\sin^2\phi\right)^{1/2}}\ d\phi \\ &= \int \frac{c_1\csc^2\phi\ d\phi}{(1 - c_1^2\csc^2\phi)^{1/2}}. \end{aligned}$$

Now let

$$\cot\phi = u, \ -\csc^2\phi \, d\phi = du.$$

(Recall that $\csc\phi = 1/\sin\phi$, $\csc^2\phi = 1 + \cot^2\phi$.) Then

$$\theta - c_2 = -\int \frac{c_1 \, du}{\left[1 - c_1^2(1+u^2)\right]^{1/2}} = -\int \frac{du}{\sqrt{c^2 - u^2}} = \cos^{-1}\left(\frac{u}{c}\right),$$

where

$$c = \sqrt{\frac{1-c^2}{c^2}}.$$

Therefore the solution is $u = c\cos(\theta - c_2)$, or

$$\cot\phi = c\cos(\theta - c_2) = c\cos c_2 \cos\theta + c\sin c_2 \sin\theta.$$

Note that c and c_2 are just two arbitrary constants. Let us define two new constants by $c\cos c_2 = a$ and $c\sin c_2 = b$. Then the solution for the curve of shortest length is

$$\cot\phi = a\cos\theta + b\sin\theta. \tag{4.7}$$

The constants a, b can be found from the requirement that the curve passes through the points $A(\theta_1, \phi_1)$ and $B(\theta_2, \phi_2)$. Equation (4.7) for the curve can be rewritten as

$$R\cos\phi = aR\sin\phi\cos\theta + bR\sin\phi\sin\theta,$$

or, in view of the parametrization (4.6), as

$$z = ax + by.$$

Thus every point on the curve also lies on the plane $z = ax + by$, a plane that passes through the origin. The intersection of such a plane with the sphere is a great circle.

7. Consider the functional

$$I[y] = \int_a^b f(x, y, y', y'') \, dx.$$

(a) Derive the differential equation satisfied by the four-times differentiable function $y(x)$ which renders $I[y]$ stationary under the condition that both y and y' are specified at $x = a$ and $x = b$.

(b) Show that if neither y nor y' are specified at either end point, then the following conditions must be met at each end point.

$$\frac{\partial f}{\partial y''} = 0, \quad \frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

(a) Suppose that the extremalizing function is $y(x)$. Consider *nearby* functions $\tilde{y}(x)$ defined by

$$\tilde{y}(x) = y(x) + \epsilon\eta(x) \tag{4.8}$$

where ϵ is sufficiently small, and $\eta(x)$, regarded as given, is sufficiently smooth and satisfies the boundary conditions

$$\eta(a) = \eta(b) = \eta'(a) = \eta'(b) = 0. \tag{4.9}$$

We can now treat $I[y + \epsilon\eta]$ as a function of ϵ, which we shall denote by $\mathcal{I}(\epsilon)$, and which is extremalized when $\epsilon = 0$. The necessary condition is

$$\mathcal{I}'(\epsilon) = 0 \quad \text{when} \quad \epsilon = 0. \tag{4.10}$$

Now we can write $\mathcal{I}(\epsilon)$ as

$$\mathcal{I}(\epsilon) = I[y+\epsilon\eta] = \int_a^b f\left(x, y(x)+\epsilon\eta(x), y'(x)+\epsilon\eta'(x), y''(x)+\epsilon\eta''(x)\right)\, dx.$$

Upon differentiating under the integral sign we get

$$\mathcal{I}'(\epsilon) = \int_a^b \frac{d}{d\epsilon} f\left(x, y(x)+\epsilon\eta(x), y'(x)+\epsilon\eta'(x), y'(x)+\epsilon\eta'(x)\right)\, dx.$$

Let us set

$$u = y+\epsilon\eta,\ \ v = y'+\epsilon\eta',\ \ w = y''+\eta''.$$

Then

$$\begin{aligned} \frac{d}{d\epsilon} f(x,u,v,w) &= \frac{\partial f}{\partial u}(x,u,v,w)\frac{du}{d\epsilon} + \frac{\partial f}{\partial v}(x,u,v,w)\frac{dv}{d\epsilon} + \frac{\partial f}{\partial w}(x,u,v,w)\frac{dv}{d\epsilon} \\ &= \frac{\partial f}{\partial u}(x,u,v,w)\eta + \frac{\partial f}{\partial v}(x,u,v,w)\eta' + \frac{\partial f}{\partial v}(x,u,v,w)\eta''. \end{aligned}$$

When evaluated at $\epsilon = 0$, *i.e.*, at $u = y$, $v = y'$ and $w = y''$,

$$\frac{d}{d\epsilon} f(x,u,v)\Big|_{\epsilon=0} = \frac{\partial f}{\partial y}(x,y,y',y'')\eta + \frac{\partial f}{\partial y'}(x,y,y',y'')\eta' + \frac{\partial f}{\partial y''}(x,y,y',y'')\eta''.$$

Therefore (4.10) leads to

$$\int_a^b \left[\frac{\partial f}{\partial y}(x,y,y')\eta + \frac{\partial f}{\partial y'}(x,y,y')\eta' + \frac{\partial f}{\partial y''}(x,y,y',y'')\eta''\right] dx = 0. \tag{4.11}$$

We integrate the second term in the integral by parts once and the third term twice to get

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}\right\} + \frac{d^2}{dx^2}\left\{\frac{\partial f}{\partial y''}\right\}\right]\eta(x)\, dx + \left\{\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right\}\eta(x)\Big|_a^b + \frac{\partial f}{\partial y''}\eta'(x)\Big|_a^b = 0. \tag{4.12}$$

Application of the boundary conditions (4.9) causes the last two terms above to vanish, so that

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}\right\} + \frac{d^2}{dx^2}\left\{\frac{\partial f}{\partial y''}\right\}\right]\eta(x)\, dx = 0. \tag{4.13}$$

Continuity of the integrand coupled with arbitrariness of $\eta(x)$ requires that the integrand must vanish, *i.e.*,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left\{\frac{\partial f}{\partial y'}\right\} + \frac{d^2}{dx^2}\left\{\frac{\partial f}{\partial y''}\right\} = 0. \tag{4.14}$$

This is the (Euler-Lagrange) equation that the extremalizing function $y(x)$ must satisfy.

(b) If the boundary conditions at the end points are not satisfied, then $\eta(x)$ and $\eta'(x)$ are arbitrary at $x = a$ and $x = b$. Focusing on those $\eta(x)$ that do satisfy $\eta = \eta' = 0$ at $x = a$ and $x = b$, the E-L equation (4.14) emerges again, and then, (4.12) reduces for arbitrary $\eta(x)$ to

$$\left\{\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right\}\eta(x)\Big|_a^b + \frac{\partial f}{\partial y''}\eta'(x)\Big|_a^b = 0.$$

The arbitrariness of $\eta(x)$ then forces the following *natural* boundary conditions on $y(x)$:

$$\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) = 0, \quad \frac{\partial f}{\partial y''} = 0,$$

at $x = a$ and $x = b$.

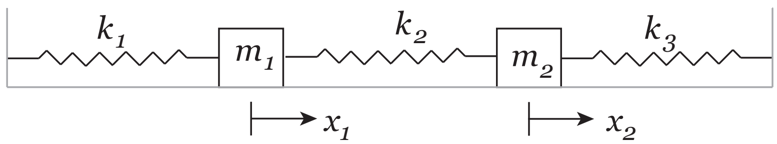

Figure 8: Spring-mass system.

8. Consider a system of two masses connected to three springs as shown in Figure 8. Let $x_1(t)$ and $x_2(t)$ be the displacements of the masses from the rest position, when the springs are unstretched, and k_1, k_2 and k_3 the stiffness constants for the linear springs.

   (a) Write down expressions for the kinetic energy K and the potential energy P of the system at time t.

   (b) Find the equations of motion of the system, *i.e.*, the Euler-Lagrange equations satisfied by x_1 and x_2 which correspond to the extremalizing of the functional

$$I[x_1, x_2] = \int_0^T (K - P)\, dt.$$

(a) The kinetic and potential energies are

$$\begin{aligned} K &= \frac{1}{2}(m_1 x_1'^2 + m_2 x_2'^2), \\ P &= \frac{1}{2}\left\{k_1 x_1^2 + k_2(x_2 - x_1)^2 + k_3 x_2^2\right\}. \end{aligned}$$

Here the prime denotes d/dt.

(b) Hamilton's Principle states that the motion renders stationary the integral

$$I[x_1, x_2] = \int_a^b L\, dt$$

where $t = a$ and $t = b$ are the end points of the time interval of interest and the integrand, known as the Lagrangian, is

$$L = K - P = \frac{1}{2}\left\{m_1 x_1'^2 + m_2 x_2'^2 - k_1 x_1^2 - k_2(x_2 - x_1)^2 - k_3 x_2^2\right\}.$$

The E-L equations are

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_i'}\right) = 0, \quad i = 1, 2.$$

These reduce to

$$\begin{aligned} m_1 x_1'' + k_1 x_1 + k_2(x_1 - x_2) &= 0, \\ m_2 x_2'' + k_2(x_2 - x_1) + k_3 x_2 &= 0. \end{aligned}$$