ADVANCED CALCULUS
SPRING 2019

LESSON 1: Vector-valued Functions

This lesson considers the following topics.

1. Functions $\boldsymbol{f} : \mathcal{R} \to \mathcal{R}^3$. Paths, curves and their geometry.
2. Functions $\boldsymbol{f} : \mathcal{R}^3 \to \mathcal{R}^3$. The del operator, gradient, divergence and curl.

It supplements Chapter 4 of the Marsden-Tromba text.

1 Introduction

Quantities such as electrical and magnetic fields in electromagnetics, velocity and displacement fields in mechanics and gravitational fields in celestial mechanics are vector-valued quantities. Typically these quantities are functions of spatial location and/or time. This lesson is concerned with the differential calculus of such functions. However, we begin with a simpler situation, that of motion along a path.

2 Paths and curves

A curve is a basic geometric object. In $\mathcal{R}^2$ it is the figure one obtains by putting pen to paper and moving it from point A to point B. Mathematically it could be described as the graph of $y = f(x)$ or $F(x, y) = 0$. This is a *static* description of the curve as a collection of points, and has some disadvantages. Consider, for example, the points $A(-2, 0)$ and $B(0, 2)$ on a circle of radius 2 centered at the origin. These points are the end points of two segments of the circle, the shorter segment C_1 and the longer segment C_2. The segment C_1 can be described as the graph of $y = \sqrt{4 - x^2}$, $-2 \leq x \leq 0$. The segment C_2 cannot be described by the function $\sqrt{4 - x^2}$ alone. We could employ the equation $x^2 + y^2 = 4$, but it would be awkward to specify restrictions on x and y that correspond to segment C_2. Extension of such a description to curves in $\mathcal{R}^3$ is also problematic.

An alternative is the parametric description of the circle, $\boldsymbol{r}(t) =< x(t), y(t) >=< 2\cos t, 2\sin t >$. Then $t \in [\pi/2, \pi]$ on C_1 and $t \in [-\pi, \pi/2]$ on C_2. We call such a parametric description a *path*, and distinguish it from *curve*, reserving the latter term for the *image* of the path, which is the collection of points on the path, either C_1 or C_2 in this case. Extension to $\mathcal{R}^3$ is straightforward.

> A path in $\mathcal{R}^n$ is the vector-valued function or map $\boldsymbol{r}(t) : [a, b] \to \mathcal{R}^n$, a path in the plane if $n = 2$ and in space if $n = 3$. The collection C of points $\boldsymbol{r}(t)$ is a curve with end points $\boldsymbol{r}(a)$ and $\boldsymbol{r}(b)$.

Remarks.

- $\boldsymbol{r}(t)$ is the position vector, $< x(t), y(t) >$ in $\mathcal{R}^2$ and $< x(t), y(t), z(t) >$ in $\mathcal{R}^3$, of a generic point on the path. We shall often refer to such a point as the point $\boldsymbol{r}(t)$ or simply as the point t.
- The path is a *dynamic* and *directed* description of the curve. The point $\boldsymbol{r}(t)$ traces the curve from $\boldsymbol{r}(a)$ to $\boldsymbol{r}(b)$ as t increases from a to b.
- The path is continuous if $\boldsymbol{r}(t)$ is continuous, and smooth or C^1 if $\boldsymbol{r}(t)$ is continuously differentiable and $\boldsymbol{r}'(t) \neq 0$.

Examples.

Straight line, $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$, $t \in [0, \infty)$.

Semicircle $\boldsymbol{r} =< a\cos t, a\sin t >$, $t \in [0, \pi]$.

Ellipse $\boldsymbol{r} =< a\cos t, b\sin t >$, $t \in [0, 2\pi]$.

One arch of a cycloid $\boldsymbol{r} =< t - \sin t, 1 - \cos t >$, $t \in [0, 2\pi]$.

Three turns of a helix, $\boldsymbol{r} =< a\cos t, a\sin t, bt >$, $t \in [0, 6\pi]$.

A three-dimensional astroid, $\boldsymbol{r} =< \cos^3 t, \sin^3 t, \cos 2t >$, $t \in [0, 2\pi]$.

2.1 Motion

If the parameter t is time, then it is natural to think of the path as the trajectory of motion of a particle. Three kinds of questions may arise in connection with motion.

(i) Given the equation of the path, $\boldsymbol{r} = \boldsymbol{r}(t)$, determine the velocity, speed and acceleration.

(ii) Given the force acting on the particle, determine its trajectory.

(iii) Given the path, determine the intrinsic properties (geometric invariants) of the curve, such as curvature, arc length, torsion, etc.

2.1.1 Motion of a particle, given its path

Velocity, being the rate of change of position with time, is given by

$$\begin{aligned}\boldsymbol{v}(t) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} \\ &= \frac{d\boldsymbol{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle .\end{aligned}$$

By geometric construction, we see from Figure 1 that as $\Delta t \to 0$, the direction of the vector $\vec{PQ} =$

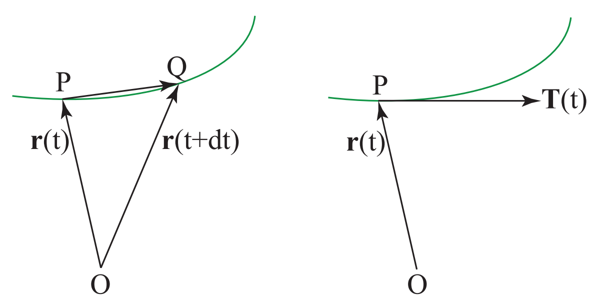

Figure 1: Velocity as a tangent vector, by construction.

$\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)$ limits to the direction of the tangent at P. Thus the velocity vector $\boldsymbol{v}(t)$ is tangent to the path $\boldsymbol{r}(t)$ at time t, provided $\boldsymbol{v}(t) \neq \mathbf{0}$. The speed is the magnitude of the velocity, and also the rate of change of the arc length $s(t)$, so that

$$\frac{ds}{dt} = \left\| \frac{d\boldsymbol{r}}{dt} \right\| . \tag{2.1}$$

The arc length, as measured from the initial point $t = 0$ on the path, is then given by

$$s(t) = \int_0^t \| \boldsymbol{r}'(\tau) \| \; d\tau . \tag{2.2}$$

The unit tangent vector to the path at the point t is

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} . \tag{2.3}$$

The tangent line to the path at time t_0 is given by

$$\boldsymbol{\ell}(t) = \boldsymbol{r}(t_0) + (t - t_0)\boldsymbol{r}'(t_0),$$

where $\boldsymbol{\ell}(t)$ is the general point on the line.

Example 2.1. The path of a particle attached to the rim of a bicycle wheel of radius R traveling with speed v_0 is given by the cycloid

$$\boldsymbol{r}(t) = \left\langle v_0 t - R\sin\frac{v_0 t}{R},\ R - R\cos\frac{v_0 t}{R} \right\rangle .$$

Find the velocity.
The velocity is

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \left\langle v_0 - v_0\cos\frac{v_0 t}{R},\ v_0\sin\frac{v_0 t}{R} \right\rangle .$$

Note that the horizontal component of the velocity is zero when $v_0 t/R = 2k\pi,$ where k is an integer. Note also that at these times the vertical component of the velocity is zero as well. Furthermore,

$$\boldsymbol{r}(2k\pi R/v_0) =< 2k\pi R, 0),$$

i.e., the y-component of the position of the particle is zero, so that the particle is touching the ground.

Example 2.2. A particle follows the path $\boldsymbol{r}(t) =< e^t, e^{-t}, \cos t >$ until it flies off on a tangent at $t = 1$. Where is the particle at $t = 3$?

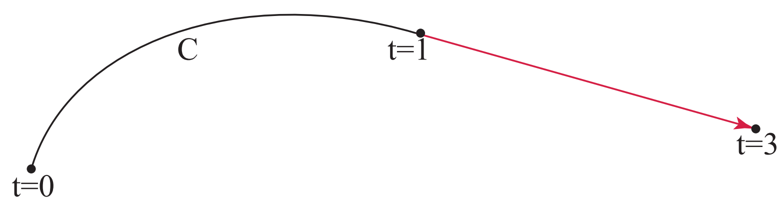

Figure 2: Flyoff.

The velocity at t is

$$\boldsymbol{r}'(t) =< e^t - e^{-t}, -\sin t > .$$

At $t = 1$ the position of the particle is

$$\boldsymbol{r}(1) =< e, e^{-1}, \cos 1 >,$$

and its velocity,

$$\boldsymbol{r}'(1) =< e, -e^{-1}, -\sin 1 > .$$

The tangent line at $t = 1$ is directed along the above velocity vector. Therefore the position of the particle at time $t > 1$ is given by

$$\boldsymbol{r}(1) + (t-1)\boldsymbol{r}'(1) =< e, e^{-1}, \cos 1 > +(t-1) < e, -e^{-1}, -\sin 1 > .$$

At $t = 3$ the position is

$$< e, e^{-1}, \cos 1 > +(3-1) < e, -e^{-1}, -\sin 1 >=< 3e, -e^{-1}, \cos 1 - 2\sin 1 > .$$

2.1.2 Motion of a particle under a prescribed force

Given the force, we illustrate the computation of the trajectory by means of the following example.

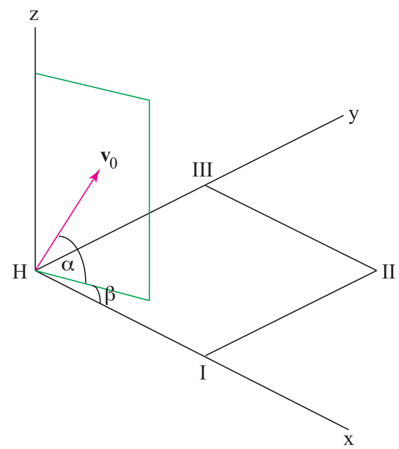

Figure 3: Hit.

Example 2.3. A ball is hit at home plate with speed v_0 such that the initial-velocity vector, inclined at an angle α to the horizontal, lies in a vertical plane that makes an angle β with the first base line. The ball is subject to gravity and an aerodynamic force due to wind (blowing from the third base towards the home plate) that causes the ball to curve towards the first-base foul line. Find the landing point of the ball. All other quantities being given, what must the angle β be so that the ball lands just inside the foul line?

Let the origin be at the home plate (Figure 3), the first base along the x-axis and the third base along the y-axis.Then the initial position and velocity are given by

$$\boldsymbol{r}(0) =< 0,0,0 >, \quad \boldsymbol{r}'(0) = v_0 < \cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha > .$$

The ball of mass m is subject to the force of gravity acting vertically downwards, $-mg\boldsymbol{k}$, and the force due to the wind acting in the direction from third base to home plate, $-ma\boldsymbol{j}$, where we have assumed that a is the wind-generated acceleration. Then according to Newton's law,

$$m\boldsymbol{r}''(t) = m < 0, -a, -g > .$$

One integration leads to

$$\boldsymbol{r}'(t) = \boldsymbol{r}'(0) + t < 0, -a, -g >,$$

and a second integration yields

$$\begin{aligned} \boldsymbol{r}(t) &= \boldsymbol{r}(0) + t\boldsymbol{r}'(0) + \frac{t^2}{2} < 0, -a, -g > \\ &= \left\langle v_0 t\cos\alpha\cos\beta,\ v_0 t\cos\alpha\sin\beta - \frac{at^2}{2},\ v_0 t\sin\alpha - \frac{gt^2}{2} \right\rangle. \end{aligned}$$

Let the ball hit the ground at $t = t_0$. Then at the landing point $\boldsymbol{r}(t_0) =< x_0, y_0, z_0 >$, $z_0 = 0$ so that

$$v_0 t_0 \sin\alpha - \frac{gt_0^2}{2} = 0, \quad \text{or,} \quad t_0 = \frac{2v_0\sin\alpha}{g}.$$

Then,

$$\begin{aligned} x_0 &= v_0 t_0 \cos\alpha\cos\beta = \frac{2v_0^2\sin\alpha\cos\alpha\cos\beta}{g}, \\ y_0 &= v_0 t_0 \cos\alpha\sin\beta - \frac{at^2}{2} = \frac{2v_0^2\sin\alpha\cos\alpha\sin\beta}{g} - \frac{2av_0^2\sin^2\alpha}{g^2}. \end{aligned}$$

If the ball just lands inside the foul line, then $y_0 = 0$, leading to

$$\sin\beta = \frac{a}{g}\tan\alpha.$$

3 Vector fields

The term field is generally reserved for functions of position in $\mathcal{R}^n$, where $n = 2$ or $3.$. Fields may be scalar functions, $f : \mathcal{R}^n \to \mathcal{R}$, or vector functions, $\boldsymbol{F} : \mathcal{R}^n \to \mathcal{R}^n$. We shall typically write $f(\boldsymbol{r})$ and $\boldsymbol{F}(\boldsymbol{r})$, with $\boldsymbol{r} =< x, y >$ in $\mathcal{R}^2$ and $< x, y, z >$ in $\mathcal{R}^3$. If time is also an independent variable, then we shall use the notation $f(\boldsymbol{r}, t)$ and $\boldsymbol{F}(\boldsymbol{r}, t)$. We may replace $\boldsymbol{r}$ by $\boldsymbol{x}$ on occasion.

A vector field can be visualized by superposing a grid (cartesian, cylindrical or spherical, for example) on the domain and drawing a directed line segment at each point on the grid, with the arrow indicating the direction, and the length of the segment the magnitude, of the field.

3.1 Gradient fields

In many applications the vector field is derived from an associated scalar field. For example, the flow of heat by conduction (a vector field) is determined by the temperature distribution in the domain (a scalar field). Specifically, when the vector field $\boldsymbol{F}$ is derived from the scalar field f according to the prescription

$$\boldsymbol{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle,$$

we say that $\boldsymbol{F}$ is a *gradient field* or a *conservative field* with potential f. Electrostatic and gravitational fields are examples of gradient fields.

Examples 3.1.

(i) $\boldsymbol{r} =< x, y, z >$ is a gradient field with potential $f = (x^2 + y^2 + z^2)/2$, so that $\boldsymbol{r} = \nabla f$.

(ii) The gravitational field due to a mass M located at the origin is given by

$$\boldsymbol{F} = -\frac{GM}{r^3}\boldsymbol{r} = -GM\frac{< x, y, z >}{(x^2 + y^2 + z^2)^{3/2}}.$$

This is just the inverse square law, with G the Gravitational Constant. Here $\boldsymbol{F}$ is the gravitational force of attraction due to the mass M felt by a unit mass located at $\boldsymbol{r}$, and is directed towards the origin. Simple calculation shows that $\boldsymbol{F}$ is a gradient field with potential

$$f = \frac{GM}{r}.$$

Check: First, note that

$$\frac{\partial x}{\partial r} = \frac{x}{r}, \quad \frac{\partial y}{\partial r} = \frac{y}{r}, \quad \frac{\partial z}{\partial r} = \frac{z}{r}.$$

Then,

$$\frac{\partial f}{\partial x} = -\frac{GM}{r^2}\frac{\partial x}{\partial r} = -\frac{GMx}{r^3}.$$

Similarly,

$$\frac{\partial f}{\partial y} = -\frac{GMy}{r^3}, \quad \frac{\partial f}{\partial z} = -\frac{GMz}{r^3}.$$

Hence

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = -\frac{GM}{r^3} < x, y, z >= -\frac{GM}{r^3}\boldsymbol{r}.$$

(iii) Often one is faced with determining whether a given field is a gradient field, and if so, with finding its potential. Consider the vector field

$$\boldsymbol{F} =< 2xe^z, z\cos y, x^2e^z + z\sin y > .$$

If $\boldsymbol{F} = \nabla f$ then

$$f_x = 2xe^z, \; f_y = z\cos y, \; f_z = x^2e^z + \sin y.$$

To see whether f exists we integrate the first of the above differential equations with respect to x to get

$$f = x^2e^z + g(y,z),$$

where g is arbitrary. The second differential equation then requires that

$$g_y = z\cos y,$$

which integrates to $g(y,z) = z\sin y + h(z)$, and yields

$$f = x^2e^z + z\sin y + h(z).$$

The third differential equation provides the constraint

$$x^2e^z + \sin y + h'(z) = x^2e^z + \sin y,$$

and requires that h be a constant, say C. The success of these calculations indicates that $\boldsymbol{F}$ is indeed a gradient field, with potential

$$f = x^2e^z + z\sin y + C.$$

Had we reached a contradiction we would have concluded that $\boldsymbol{F}$ is not a gradient field.

(iv) Motion of a particle under the action of a gradient force field. Let the position of the particle be $\boldsymbol{r}(t)$, and the applied force $\boldsymbol{F} = -\nabla u$, where $u(x,y,z)$ is the potential of the gradient force. According to Newton's Law, the equation of motion is

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} = -\nabla u,$$

where the velocity $\boldsymbol{v} = d\boldsymbol{r}/dt$. On taking a dot product of both sides with $\boldsymbol{v}$,

$$m\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} + \frac{d\boldsymbol{r}}{dt}\cdot\nabla u = 0.$$

Note that

$$m\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v}\right) = \frac{d}{dt}\left(\frac{1}{2}m\|\boldsymbol{v}\|^2\right),$$

and that

$$\frac{d\boldsymbol{r}}{dt}\cdot\nabla u = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\rangle = \frac{du}{dt}.$$

Therefore the equation of motion becomes

$$\frac{d}{dt}\left(\frac{1}{2}m\|\boldsymbol{v}\|^2 + u\right) = 0,$$

or,

$$\frac{1}{2}m\|\boldsymbol{v}\|^2 + u = C,$$

where C is a constant. *This is the law of conservation of energy which states that the sum of the kinetic and potential energies is conserved if the force field is a gradient field..*

Remark. The surfaces $f(\boldsymbol{r}) = C$ are called the *equipotential surfaces* of the gradient field $\boldsymbol{F} = \nabla f$. Recalling that the projection of ∇f in any direction $\boldsymbol{u}$, given by $\boldsymbol{u}\cdot\nabla f$ (here $\boldsymbol{u}$ is a unit vector) is the directional derivative of f in that direction, we note that ∇f must be the normal vector to the surface $f = C$, on which the rate of change of f is zero. Thus the gradient vector field $\boldsymbol{F}$ is normal to its equipotential surfaces. For a gravitational field due to a single mass, equipotential surfaces are spheres.

3.1.1 Flow lines

A path is a flow line of a vector field if at every point on the path the vector field points along the tangent to the path. For the vector field $\boldsymbol{F}(\boldsymbol{r})$, the flow line $\boldsymbol{r}(t)$ satisfies the vector differential equation

$$\frac{d\boldsymbol{r}(t)}{dt} = \boldsymbol{F}\big(\boldsymbol{r}(t)\big),$$

or,

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = < F_1, F_2, F_3 > .$$

Examples 3.2.

(i) Let $\boldsymbol{F} = \boldsymbol{i} + y\boldsymbol{j}$. Then the flow lines satisfy

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = y.$$

The above differential equations have the general solutions

$$x = t + A, \quad y = Be^t.$$

The constants of integration A and B can be found by specifying one point on the flow line. If $x = x_0$ and $y = y_0$ at $t = 0$, then

$$x_0 = A, \quad y_0 = B,$$

so the parametric description of the flow line is

$$x = x_0 + t, \; y = y_0 e^t.$$

The trace, or curve, corresponding to the path can be obtained by elimination t, to get

$$y = y_0 e^{(x - x_0)}.$$

Alternatively, we could have written the expression for the slope of the curve as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = y,$$

which under the initial condition $y = y_0$ at $x = x_0$ generates the same solution as above.

(ii) Let

$$\boldsymbol{F} = \frac{1}{\sqrt{x^2 + y^2}} < -y, x > .$$

The parametric form of the flow lines satisfies the differential equations

$$\frac{dx}{dt} = -\frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{dy}{dt} = \frac{x}{\sqrt{x^2 + y^2}}.$$

These are nonlinear coupled ODEs for $x(t)$ and $y(t)$. We proceed instead with the slope of the flow lines in the form

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x}{y},$$

which integrates to

$$x^2 + y^2 = C.$$

Thus the flow lines are circles. The initial condition $y = y_0$ when $x = x_0$ finds $C = x_0^2 + y_0^2$.

3.2 Divergence and curl

We have already been introduced to the vector differential operator ∇, called the *del* operator or the gradient operator, and defined by

$$\nabla \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle .$$

In algebraic operations such as dot or cross products the operator acts like a vector, but its components operate upon, rather than multiply, the scalar or vector field to which the operator is applied. Thus, application to a scalar field yields the gradient, a vector field, whose components are the partial derivatives of the scalar field,

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle .$$

The operator can be used to define several other scalar and vector fields of interest, consisting of arrangements of partial derivatives in various ways. Thus, with the vector field $\boldsymbol{F}(x,y,z)$ one can associate a scalar field denoted by $\operatorname{div} \boldsymbol{F}$ or $\nabla \cdot \boldsymbol{F}$, called the *divergence* of $\boldsymbol{F}$ and defined as

$$\begin{aligned} \nabla \cdot \boldsymbol{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot < F_1, F_2, F_3 > \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \end{aligned}$$

Similarly, one can associate with $\boldsymbol{F}$ another vector field called the *curl* of $\boldsymbol{F}$ and defined as

$$\begin{aligned} \operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{bmatrix} \\ &= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle . \end{aligned}$$

The fields defined above by means of the del operator, and others yet to be defined, have physical meanings that do not depend upon the coordinate system of choice, and to which we shall turn in due course. For now we note that $\boldsymbol{F}$ is called a *solenoidal* or an *incompressible* field if $\nabla \cdot \boldsymbol{F} = 0$ and an *irrotational* field if $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

Example 3.3. Compute the divergence and curl of the gravitational field

$$\boldsymbol{F} = -\frac{GM}{r^3}\boldsymbol{r}, \quad \boldsymbol{r} =< x, y, z >, \quad r = \sqrt{x^2+y^2+z^2}.$$

We begin by noting that since $r^2 = x^2 + y^2 + z^2$,

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Now,

$$\boldsymbol{F} =< F_1, F_2, F_3 >= -GM \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle .$$

First, consider the divergence. We have

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= -GM\left(\frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}\right) \\ &= -GM\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) \\ &= -GM\frac{r^2-3x^2}{r^5} \\ &= -GM\frac{y^2+z^2-2x^2}{r^5}. \end{aligned}$$

Similarly,

$$\frac{\partial F_2}{\partial y} = -GM\frac{z^2+x^2-2y^2}{r^5}, \quad \frac{\partial F_3}{\partial z} = -GM\frac{x^2+y^2-2z^2}{r^5}.$$

Therefore,

$$\nabla\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = -GM\frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{r^5} = 0.$$

Thus $\boldsymbol{F}$ is solenoidal.

Now consider the curl.

$$\begin{aligned}\nabla\times\boldsymbol{F} &= \begin{bmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_1 & F_2 & F_3\end{bmatrix}\\ &= \left\langle \frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right\rangle.\end{aligned}$$

Note that

$$\frac{\partial F_3}{\partial y} = -GM\frac{-3z}{r^4}\frac{\partial r}{\partial y} = 3GM\frac{yz}{r^5}$$

Similarly,

$$\frac{\partial F_2}{\partial z} = -GM\frac{-3y}{r^4}\frac{\partial r}{\partial z} = 3GM\frac{yz}{r^5}.$$

Therefore,

$$\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z} = 3GM\frac{yz}{r^5}-3GM\frac{yz}{r^5} = 0,$$

so that the first component of the curl is zero. Similar arguments show that the remaining two components of the curl vanish as well, so that $\nabla\times\boldsymbol{F} = \boldsymbol{0}$. Thus $\boldsymbol{F}$ is also irrotational.

The physical meaning of divergence. Consider the time-dependent flow of a fluid of density $\rho(x,y,z,t)$ and velocity $\boldsymbol{v}(x,y,z,t)$ in a domain $\mathcal{D}$ of $\mathcal{R}^3$. We take ρ and $\boldsymbol{v} =< v_1, v_2, v_3 >$ to be continuously differentiable functions of their arguments. The vector $\boldsymbol{m} =< m_1, m_2, m_3 >$, defined as

$$\boldsymbol{m} = \rho\boldsymbol{v} =< \rho v_1, \rho v_2, \rho v_3 >, \tag{3.1}$$

is known as the mass flux vector, with units of mass per unit area per unit time. If $\boldsymbol{m}$ is constant in space, then the mass flow rate across any flat surface immersed in the fluid is $(\boldsymbol{m}\cdot\boldsymbol{n})A$, where A is the area of the surface and $\boldsymbol{n}$ the unit normal to the surface.

In the domain $\mathcal{D}$ consider a rectangular box of vanishingly small dimensions $(\Delta x, \Delta y, \Delta z)$ with one corner at $P(x_0, y_0, z_0)$ as shown in Figure 4. Of interest is the rate at which the mass of the fluid inside the box is changing with time as a result of flow across the boundaries of the box. Mass crosses the front face of the box, located at $x = x_0 + \Delta x$ and with area $\Delta y\Delta z$, at rate

$$-m_1(x_0+\Delta x, y_0, z_0)\Delta y\Delta z,$$

where the negative sign indicates that the flow is directed out of the box.[1] The corresponding expression for the flow across the rear face is

$$m_1(x_0, y_0, z_0)\Delta y\Delta z,$$

[1]The above expression assumes that the mass flux across the face is constant and can therefore be evaluated at the corner $(x_0+\Delta x, y_0, z_0)$. This is an excellent approximation for the small dimensions of the face. A precise expression for the mass flow rate would be $-m_1(x_0+\Delta x, y^*, z^*)\Delta y\Delta z$, where $y_0 \le y^* \le y_0+\Delta y$ and $z_0 \le z^* \le z_0+\Delta z$. In either case the limiting process $\Delta x, \Delta y, \Delta z \to 0$ applied at the end leads to the same result.

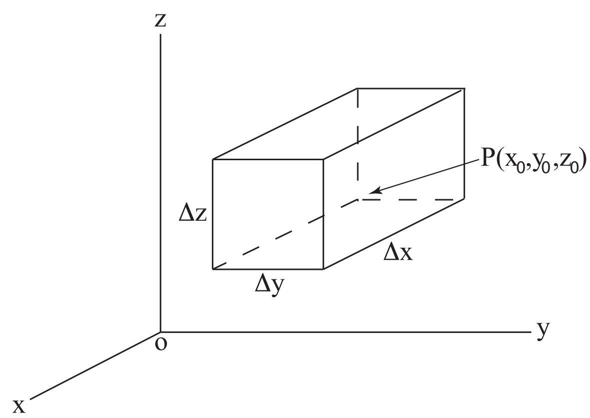

Figure 4: Meaning of divergence.

with the positive sign indicating flow into the box. Contribution of the front and rear faces to the rate of accumulation of mass in the box is the sum

$$
\begin{aligned}
& [-m_1(x_0+\Delta x, y_0, z_0) + m_1(x_0, y_0, z_0)]\Delta y \Delta z \\
= & -\frac{m_1(x_0+\Delta x, y_0, z_0) - m_1(x_0, y_0, z_0)}{\Delta x}\Delta x \Delta y \Delta z \\
= & -\frac{\Delta m_1}{\Delta x}\Delta x \Delta y \Delta z.
\end{aligned}
$$

Here Δm_1 is the increment in m_1 as x is increased from x_0 to $x_0 + \Delta x$. (Analogous definitions apply to increments Δm_2 and Δm_3 below.) Similar contributions arise from the two remaining pairs of faces, leading to the following expression for the total rate of mass accumulation within the box per unit time,

$$-\left(\frac{\Delta m_1}{\Delta x} + \frac{\Delta m_2}{\Delta y} + \frac{\Delta m_3}{\Delta z}\right)\Delta x \Delta y \Delta z.$$

Since the volume of the box is fixed, the above accumulation of mass causes a change in the density of the box, yielding yet another expression for the mass accumulation rate within the box,

$$\frac{\rho(x_0, y_0, z_0, t+\Delta t) - \rho(x_0, y_0, z_0, t)}{\Delta t}\Delta x \Delta y \Delta z.$$

On equating the two expressions one obtains

$$\frac{\rho(x_0, y_0, z_0, t+\Delta t) - \rho(x_0, y_0, z_0, t)}{\Delta t} = -\left(\frac{\Delta m_1}{\Delta x} + \frac{\Delta m_2}{\Delta y} + \frac{\Delta m_3}{\Delta z}\right).$$

In the limit as $\Delta x, \Delta y, \Delta z$ and $\Delta t \to 0$, the above reduces to

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \boldsymbol{m} \\
&= -\nabla \cdot (\rho \boldsymbol{v}),
\end{aligned}
$$

or to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

This equation is the mathematical statement of mass conservation of a compressible fluid, and states that the divergence of the mass flux vector $\rho \boldsymbol{v}$ is a measure of the rate of mass accumulation per unit volume per unit time. When the motion is steady, $\partial \rho / \partial t = 0$ and the mass balance reduces to $\nabla \cdot (\rho \boldsymbol{v}) = 0$. When the fluid is incompressible, *i.e.*, has a constant density, then there is no mass accumulation and the mass balance reduces to $\nabla \cdot \boldsymbol{v} = 0$.

The above interpretation of divergence applies more generally, and indicates that the divergence of a vector field is a measure of the tendency of the field to expand or contract. It is important to recognize that the concept is independent of the coordinate frame in which the vector field is viewed or represented, even though we defined divergence in terms of partial derivatives in the cartesian frame. Equivalent expressions can be derived in any other coordinate system, cylindrical and spherical for example, but for the sake of brevity we shall not pursue that line of inquiry here.

The physical meaning of curl. Consider a rigid-body rotating with angular velocity $\boldsymbol{\omega}$, Figure 5. The vector $\boldsymbol{\omega}$ points along the axis of rotation such that the sense of rotation is given by the right-hand rule. The magnitude $\omega = \|\boldsymbol{\omega}\|$ yields the angular speed. If the origin of coordinates is taken to lie on the axis of

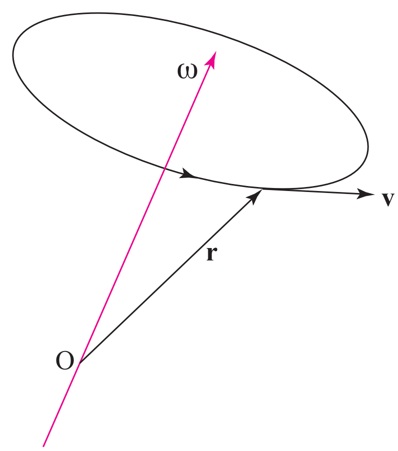

Figure 5: Rigid body rotation with angular velocity $\boldsymbol{\omega}$. The linear velocity is $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$.

rotation and $\boldsymbol{r}$ is the position vector of a point P on the rigid body, then the linear velocity of P is given by $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ (check this.) With $\boldsymbol{\omega} =< \omega_1, \omega_2, \omega_3 >$ and $\boldsymbol{r} =< x, y, z >$, we have

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} =< \omega_2 z - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x > .$$

Let us compute the curl of $\boldsymbol{v}$.

$$\begin{aligned} \nabla \times \boldsymbol{v} &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{bmatrix} \\ &= 2\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k} = 2\boldsymbol{\omega}. \end{aligned}$$

This calculation shows that

$$\boldsymbol{\omega} = \frac{1}{2} \nabla \times \boldsymbol{v},$$

i.e., one-half of the the curl of the linear velocity associated with rigid-body rotation is just the angular velocity, thereby establishing a connection between curl and rotation.

In order to take this connection further, let us recall that the circulation Γ of a vector field $\boldsymbol{F}$ around a simple closed path C is given by the line integral

$$\Gamma = \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds,$$

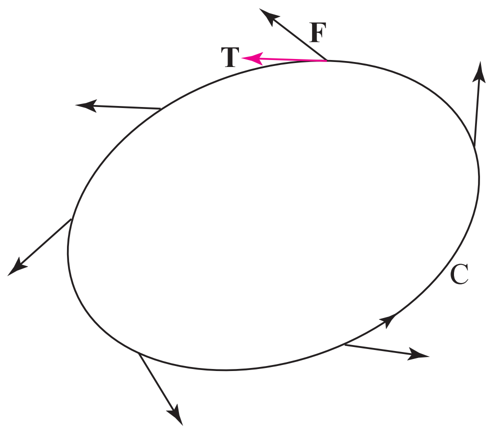

Figure 6: Circulation.

where $\boldsymbol{T}$ is the unit tangent vector on the path. The more closely aligned in direction $\boldsymbol{F}$ and $\boldsymbol{T}$ are along the path, the more of a rotational character the field $\boldsymbol{F}$ has, and the stronger is the circulation; see Figure 6. We now show that circulation of $\boldsymbol{F}$ is related to the curl of $\boldsymbol{F}$. Consider the example of a planar field,

$$\boldsymbol{F} = < F_1, F_2, 0 > .$$

Then

$$\nabla \times \boldsymbol{F} = \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle . \tag{3.2}$$

Consider the rectangular box $ABCD$ in the xy-plane, Figure 7, with corner A at (x, y) and with

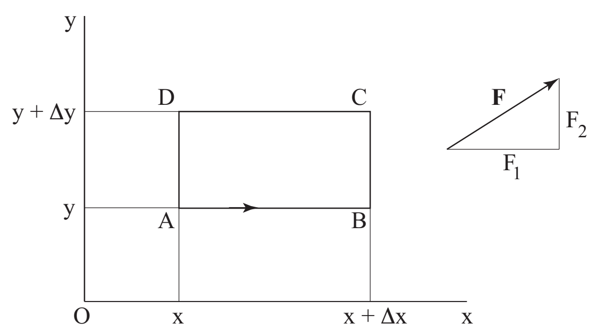

Figure 7: Meaning of curl.

vanishingly small dimensions Δx and Δy. Let $\boldsymbol{F} = < F_1, F_2, 0 >$ be a planar vector field, continuously differentiable. Then the circulation Γ of $\boldsymbol{F}$ around the box, traversed counterclockwise, is given by

$$\begin{aligned} \Gamma &= F_1(x,y)\Delta x + F_2(x+\Delta x, y)\Delta y - F_1(x, y+\Delta y)\Delta x - F_2(x,y)\Delta y \\ &= \Delta x \Delta y \left(\frac{F_2(x+\Delta x, y) - F_2(x,y)}{\Delta x} - \frac{F_1(x, y+\Delta y) - F_1(x,y)}{\Delta y} \right) . \end{aligned}$$

(Here we have assumed, to a good approximation, that the contribution to circulation from an edge of the box equals the length of the edge times the tangential component of the vector field evaluated at a convenient corner of the edge.) We divide by $\Delta x \Delta y$ and take the limit $\Delta x, \Delta y \to 0$. The result is

$$\lim_{\Delta x \Delta y \to 0} \frac{\Gamma}{\Delta x \Delta y} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

From (3.2) we note that the RHS above is the sole nonzero component of the curl of $\boldsymbol{F}$, or $\boldsymbol{k} \cdot (\nabla \times \boldsymbol{F})$. Thus we see that the curl is the circulation per unit area of the vector field, indicating yet again that curl is a measure of the rotational nature of the field.

4 Vector identities

The following identities are useful in applications, and can be checked in most cases by manipulating both the left-hand and the right-hand sides. The index notation makes the task simple, as we show in the example below.

In the following identities, a and b are scalar constants, f and g are scalar fields and $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields.

$$\nabla(af + bg) = a\nabla f + b\nabla g.$$

This identity shows that the del operator is a linear operator.

$$\begin{aligned} \nabla(fg) &= f\nabla g + g\nabla f, \\ \nabla(f/g) &= \frac{g\nabla f - f\nabla g}{g^2}. \end{aligned}$$

These identities reflect the fact that the del operator obeys the usual product and quotient rules of differentiation.

$$\begin{aligned} \nabla \times (\nabla f) &= \boldsymbol{0}, \\ \nabla \cdot (\nabla \times \boldsymbol{F}) &= 0. \end{aligned}$$

These identities show that the gradient of a scalar field is irrotational (or curl free) and that the curl of a vector field is incompressible (or divergence free).

$$\begin{aligned} \nabla(\boldsymbol{F} \cdot \boldsymbol{G}) &= \boldsymbol{F} \times (\nabla \times \boldsymbol{G}) - (\nabla \times \boldsymbol{F}) \times \boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} + (\boldsymbol{F} \cdot \nabla)\boldsymbol{G}, \\ \nabla \cdot (f\boldsymbol{F}) &= f\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla f, \\ \nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) &= \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G}), \\ \nabla \times (\boldsymbol{F} \times \boldsymbol{G}) &= \boldsymbol{F}(\nabla \cdot \boldsymbol{G}) - (\nabla \cdot \boldsymbol{F})\boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G}. \end{aligned}$$

We now define a new scalar differential operator, known as the Laplacian. It is the divergence of a gradient and is denoted by ∇^2. Thus,

$$\begin{aligned} \nabla^2 f &= \nabla \cdot (\nabla f) \\ &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\left((\boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

The Laplacian can also be applied to a vector field, *i.e.*,

$$\nabla^2 \boldsymbol{F} = \frac{\partial^2 \boldsymbol{F}}{\partial x^2} + \frac{\partial^2 \boldsymbol{F}}{\partial y^2} + \frac{\partial^2 \boldsymbol{F}}{\partial z^2}.$$

The Laplacian appears in the following identities.

$$\begin{aligned} \nabla \cdot (f\nabla g - g\nabla f) &= f\nabla^2 g - g\nabla^2 f, \\ \nabla \times (\nabla \times \boldsymbol{F}) &= \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}. \end{aligned}$$

Use of index notation. In the index notation, gradient, divergence and curl can be written in an economical fashion. For this purpose we shall use (x_1, x_2, x_3) as the coordinate system so that position vector $\boldsymbol{r}$ is

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = x_i\boldsymbol{e}_i,$$

where the $\boldsymbol{e}_i$ are the unit vectors in the coordinate directions and we have invoked the summation convention. If $f(x_1, x_2, x_3)$ is a scalar field then its gradient is written as

$$\nabla f : \frac{\partial f}{\partial x_i}.$$

Since i is a free index, the RHS above indicates a vector. Even more economically, we denote differentiation by a comma, so that $\partial f / \partial x_i$ is simply written as $f_{,i}$. Thus,

$$\nabla f : f_{,i}.$$

Let

$$\boldsymbol{F} = F_i \boldsymbol{e}_i$$

be a vector field. Then its divergence is written as

$$\nabla \cdot \boldsymbol{F} = F_{i,i}.$$

Recalling that the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as

$$\{\boldsymbol{a} \times \boldsymbol{b}\}_i = \epsilon_{ijk} a_j b_k,$$

we write the curl of $\boldsymbol{F}$ as

$$\{\nabla \times \boldsymbol{F}\}_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} F_k,$$

or as

$$\{\nabla \times \boldsymbol{F}\}_i = \epsilon_{ijk} F_{k,j}.$$

Example 4.1. Let us prove the identity

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}).$$

We start with the cross-product terms on the RHS.

$$\begin{aligned}
\{\boldsymbol{u} \times (\nabla \times \boldsymbol{v})\}_i + \{\boldsymbol{v} \times (\nabla \times \boldsymbol{u})\}_i &= \epsilon_{ijk} u_j \{\nabla \times \boldsymbol{v}\}_k + \epsilon_{ijk} v_j \{\nabla \times \boldsymbol{u}\}_k \\
&= \epsilon_{ijk} u_j \epsilon_{kmn} v_{n,m} + \epsilon_{ijk} v_j \epsilon_{kmn} u_{n,m} \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) u_j v_{n,m} + (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) v_j u_{n,m} \\
&= u_j v_{j,i} + v_j u_{j,i} - u_j v_{i,j} - v_j u_{i,j} \\
&= (u_j v_j), i - (\boldsymbol{u} \cdot \nabla) v_i - (\boldsymbol{v} \cdot \nabla) u_i \\
&= (\boldsymbol{u} \cdot \boldsymbol{v}), i - (\boldsymbol{u} \cdot \nabla) v_i - (\boldsymbol{v} \cdot \nabla) u_i.
\end{aligned}$$

Hence

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}).$$

We note that the operator $\boldsymbol{u} \cdot \nabla$ is defined as

$$\boldsymbol{u} \cdot \nabla = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \equiv u_i \frac{\partial}{\partial x_i}.$$