Vortices in Potential Flow

We've already looked at the potential vortex. To go further and learn more about vortex dynamics, we want to see how one or more isolated vortices behave. Before we can do that it's worth briefly reviewing:

Complex-function theory applied to two-dimensional flows (Lugt p. 123)

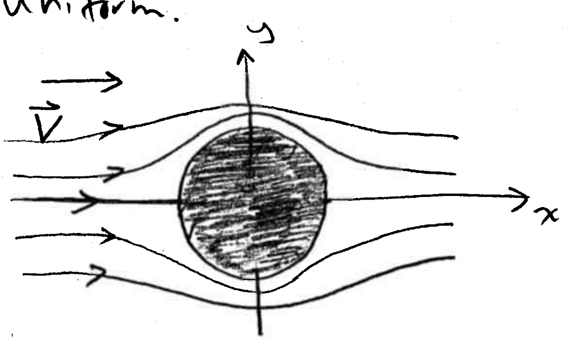
(Churchill & Brown, Complex Variables & Applic.)

$\mathrm{curl}\,\vec{u}\ (=\vec{\omega}) = 0$

and $\vec{u} = \nabla\phi$ which along with continuity gives $\nabla^2\phi = 0$ (the Laplace eqn.)

Such a potential is called a "harmonic" function. We use the fact that conformal mapping allows the solutions of harmonic functions to be transformed, e.g. from complicated geometries to simple ones.

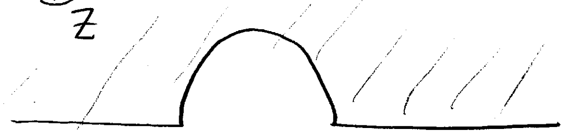
Example: We can compute the flow around a cylinder ($x^2+y^2=1$) in the z-plane ($z = x+iy$) by conformal transformation from the F-plane ($F = p+iq$) in which the flow is uniform.



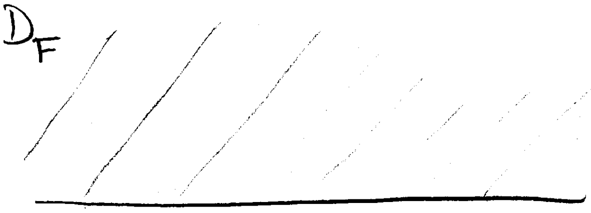
(Because of symmetry about x-axis, it suffices to compute the flow in the upper half)

Domain

D_Z (Physical)



D_F



The transformation $F = z + \frac{1}{z}$ maps D_Z onto D_F and uniform flow in D_F gives us the desired flow in D_Z

Introduce the "Complex potential" W

$$\boxed{W(z) = \phi + i\psi}$$

, where $z = x + iy$ is a complex number and its conjugate is $\bar{z} = x - iy$

and "Complex Velocity"

$$\boxed{\frac{dW}{dz} = u - iv}$$

and Velocity Vector becomes $\vec{V} = u + iv = \overline{\frac{dW}{dz}}$

where

ϕ is the Velocity potential

ψ " " Stream function

Recall the Cauchy-Riemann equations which apply to analytic function $f(z)$:

$$f(z) = p(x,y) + i\,q(x,y)$$

are: $p_x = q_y$, $p_y = -q_x$ must be satisfied for f' to exist

thus,

$$\boxed{\phi_x = \psi_y} \qquad \boxed{\phi_y = -\psi_x}$$

u - Component of Velocity $\qquad$ v - Component of Velocity

Real and imaginary parts of any analytic function are harmonic, i.e. satisfy the Laplace eqn.

$$\Rightarrow \quad \nabla^2\phi = 0 \quad \text{and} \quad \nabla^2\psi = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\omega_z = 0$$

Incompressible Flow $\qquad$ Irrotational Flow

Some Complex functions of importance in fluid flow:

1) Source $v_r = \frac{m}{r}$ where m is the source strength

$(v_\theta = 0)$ (a negative real m results in a sink)

This source can be written in terms of complex potential, as:

$$W = m \log z$$

2) Potential Vortex

$$v_\theta = \frac{\Gamma}{2\pi r} = \frac{K}{r}$$ K is vortex stregth

$$W = -iK \log z \quad (= \phi + i\psi)$$

(and some others)

(Before we can study the motion of vortex systems, which posses certain integral invariants, it is worth considering the general invariant principles that apply to all 2-dim. flows)

Invariants for 2D motion Refs. Lugt ; Batchelor (1967) An Intro. to Fluid Dynamic

Recall that we proved for inviscid flow,

$$\frac{D\Gamma}{Dt} = 0 \quad \text{(Kelvin's circulation theorem)}$$

this leads to an integral invariant quantity (in 2D):

$$\iint \omega \, ds \quad \text{(circulation)}$$ i.e. circulation is a constant in the flow

(So, for a general treatment) Consider an infinite region with no interior boundaries, and $\vec{\omega} \neq 0$ only in some finite Part of the region.

(We find that our notions from Conservation Laws can't be used to come up with integral invariants.)

e.g. both momentum and Kinetic energy are unbounded, even for some simple flows such as the Point Vortex, $V_\theta \propto \frac{1}{r}$

So we need quantities related to momentum & K.E., but not themselves

Recall the Velocity induced by a continuous distribution of Vorticity:

$$\vec{u}(\vec{r}) = \frac{1}{4\pi} \iiint_V \frac{\vec{\omega}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, dv(\vec{r}')$$

(Recall this is for Point $\vec{r}$ whereas Vorticity is at $\vec{r}'$)

in 2D; $\vec{\omega} = \omega \vec{k}$ when $\omega = \omega_z = \omega(x,y)$.

Integrate over z:

$$\vec{u} = \frac{1}{2\pi} \vec{k} \times \iint \frac{\omega(\vec{r}') \, (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|} \, ds$$

For $\vec{r} \to \infty$ i.e. large distances compared to region w/ vorticity

$$\boxed{\vec{u} \sim \frac{1}{2\pi} \frac{\vec{k} \times \vec{r}}{r^2} \iint \omega \, ds}$$

(asymptotically approaches)

(also,) from

stream function $\psi = -\frac{1}{2\pi}\iint \omega(\vec{r}\,') \ln|\vec{r}-\vec{r}\,'| \, ds(\vec{r}\,')$ 5

for $\vec{r} \to \infty$ $\psi \sim -\frac{1}{2\pi} \ln(r) \left(\iint \omega \, ds\right)$

(back to invariants)

We can list the invariant quantities (in 2D flow):

① $\iint \omega \, ds(\vec{r}\,')$ (Circulation)

② $\iint \vec{r} \times \omega \vec{k} \, ds$ (related to fluid impulse $\frac{1}{2} \iiint (\vec{r} \times \vec{\omega}) \, dV$ necessary to impart by body on fluid to cause given flow

See Batchelor for all the details

(others are:)

③ $\iint \vec{r} \times (\vec{r} \times \vec{\omega}) \, ds$ (related to angular impulse)

④ $\iint ds \iint ds(\vec{r}) \, \omega(\vec{r}) \, \omega(\vec{r}\,') \ln|\vec{r}-\vec{r}\,'|$ (related to energy)

outline of proofs of ① & ②

① Follows from (Kelvin's Circulation theorem) $\frac{D\Gamma}{Dt} = 0$

② First consider x-component:

$$\frac{d}{dt}\iint y \, \omega \, ds = \iint y \frac{\partial \omega}{\partial t} = \iint y \left(-u\frac{\partial \omega}{\partial x} - v\frac{\partial \omega}{\partial y}\right) ds$$

since $\frac{D\vec{\omega}}{Dt} = 0$ (in 2D inviscid)

Integrate over large rectangle, find:

$$\frac{d}{dt}\iint y\omega \, ds = \iint v \, \omega \, ds = \iint v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) ds$$

Find that this is zero. $\left(\frac{\partial \omega}{\partial x} \to 0, \frac{\partial \omega}{\partial y} = 0\right)$

thus $\iint y \omega \, ds$ is an invariant.

Now, similarly for the y-component, find $\iint x \omega \, ds$ is invariant.

Thus, find that the "Center of Vorticity" (X, Y) is fixed. (analogous to center of mass, cg)

(where,)

$$X \equiv \frac{\iint x \omega \, ds}{\iint \omega \, ds}, \quad Y \equiv \frac{\iint y \omega \, ds}{\iint \omega \, ds}$$

similarly, from invariant #3

Can define a distance D:

$$D^2 \equiv \frac{1}{\iint \omega \, ds} \iint \left\{ (x-X)^2 + (y-Y)^2 \right\} \omega \, ds$$

Where D measures how dispersed the vorticity is about (X, Y) and is a constant (another invariant)

(Proofs of 3 and 4 take us too far away from our goals and can be found in Batchelor)

Vortex Motions:

(begin with) A) Motion of two vortices:

(Keep in mind that)

A line vortex is a vortex tube contracted to a curve with its strength $\Gamma = \iint \omega \, ds$ held constant.

A vortex tube, and thus also a line vortex, moves with the fluid and its strength remains constant ($\nu = 0$).

Specialize the integral relations above (which are for any distribution of vorticity) by representing line vortices in terms of delta functions.

recall (by definition) Dirac delta function: $\delta(x - x_0) \to \infty$ at $x = x_0$

$\delta(x - x_0) = 0$ at $x \neq x_0$

and $$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$$

$$\int_{x_0 - c}^{x_0 + c} f(x)\, \delta(x - x_0)\, dx = f(x_0) \quad \text{for every } c > 0$$

(back to motion of two vortices)

For 2D motion, let (x_1, y_1) and (x_2, y_2) be the time-dependent coordinates of two line vortices with strength Γ_1 and Γ_2

We know that Γ_1 and Γ_2 are constant (our first invariant) and also X, Y, D are constant (follow from the second and third invariants)

For convenience, take $X = Y = 0$ unless $\Gamma_1 + \Gamma_2 = 0$

Integral invariant (#2) give (by definition of center of vorticity, X, Y)

- $\Gamma_1 x_1 + \Gamma_2 x_2 \equiv (\Gamma_1 + \Gamma_2) X = 0$

  $\Gamma_1 y_1 + \Gamma_2 y_2 \equiv (\Gamma_1 + \Gamma_2) Y = 0$

and again for $X = Y = 0$, invariant #3 (by definition of distance D) give

$$\Gamma_1 (x_1^2 + y_1^2) + \Gamma_2 (x_2^2 + y_2^2) = (\Gamma_1 + \Gamma_2) D^2$$

substitute for (x_2, y_2) above, get:

- $\Gamma_1 (x_1^2 + y_1^2) + \Gamma_2 \left\{ \left(-\frac{\Gamma_1}{\Gamma_2} x_1\right)^2 + \left(-\frac{\Gamma_1}{\Gamma_2} y_1\right)^2 \right\} = (\Gamma_1 + \Gamma_2) D^2$

gives $\boxed{x_1^2 + y_1^2 = \underbrace{\frac{\Gamma_2}{\Gamma_1} D^2}_{\text{constant}}}$

and similarly, find

$$\boxed{x_2^2 + y_2^2 = \overbrace{\frac{\Gamma_1}{\Gamma_2} D^2}}$$

i.e. * two vortices move on circular paths.

- * distance between vortices is constant:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2^2) = \left(1 + \frac{\Gamma_1}{\Gamma_2}\right)^2 (x_1^2 + y_1^2)$$ (from $\Gamma_1 x_1 + \Gamma_2 x_2 = 0$, $\Gamma_1 y_1 + \Gamma_2 y_2 = 0$)

so $d^2 = \left(\frac{\Gamma_1}{\Gamma_2} + 1\right)^2 \frac{\Gamma_2}{\Gamma_1} D^2$ (from above) (everything on RHS is invariant of flow, Γ_1, Γ_2, D. Thus,) d must be constant

<u>Velocities:</u> Knowing that each Vortex moves only as a resut of velocity induced at its location by the other vortex:

(general induced Velocity law for 2-D flow, for Velocity at $\vec{r}$ and Vorticity at $\vec{r}\,'$)

$$\vec{u}(\vec{r}) = \frac{1}{2\pi}\,\vec{k} \times \iint \frac{\omega(\vec{r}\,')\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,ds$$

which, as we saw, becomes for "large" $\vec{r}$

$$\vec{u}(\vec{r}) = \frac{1}{2\pi}\,\frac{\vec{k}\times\vec{r}}{r^2}\underbrace{\iint \omega(\vec{r}\,')\,ds}$$

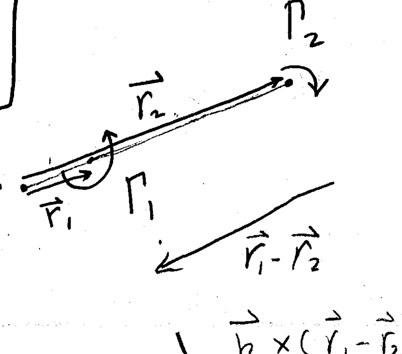
Vector that locates first Vortex, i.e. Vector from origin (center of Vorticity X, Y) to (x_1, y_1).

$$\vec{u}(\vec{r}_1) = \frac{1}{2\pi}\,\frac{\vec{k}\times(\vec{r}_1-\vec{r}_2)}{d^2}\,\Gamma_2$$

or

$$\boxed{\frac{d\vec{r}_1}{dt} = \frac{\Gamma_2}{2\pi d^2}\,\vec{k}\times(\vec{r}_1-\vec{r}_2)}$$

say, origin



As expected, induced Velocity is Purely tangential.

Similarly, $$\frac{d\vec{r}_2}{dt} = \frac{\Gamma_1}{2\pi d^2}\,\vec{k} \times (\vec{r}_2 - \vec{r}_1)$$

- Angular Velocity:

$$\frac{1}{r} u_{tangential}$$

$$\frac{1}{r_1}\left|\frac{d\vec{r}_1}{dt}\right| = \frac{1}{r_1}\frac{\Gamma_2}{2\pi d}$$

$$= \frac{1}{\sqrt{x_1^2+y_1^2}}\frac{\Gamma_2}{2\pi d}$$

(from above)

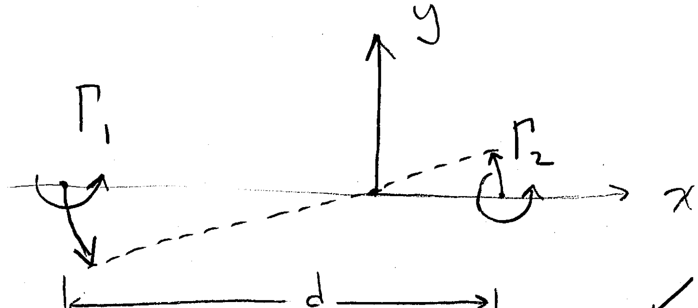
$$= \left(\frac{\Gamma_1}{\Gamma_2}+1\right)\frac{1}{d}\frac{\Gamma_2}{2\pi d}$$

$$\boxed{= \frac{(\Gamma_1+\Gamma_2)}{2\pi d^2}}$$

$$\text{(and)} \quad = \frac{1}{r_2}\left|\frac{d\vec{r}_2}{dt}\right|$$

So, the physical picture for two vortices that you get from the linear and angular velocities computed above is as follows:

1) If $\Gamma_2 > \Gamma_1 > 0$



recall delta function's properties

Center of Vorticity (& thus center of rotation of system)

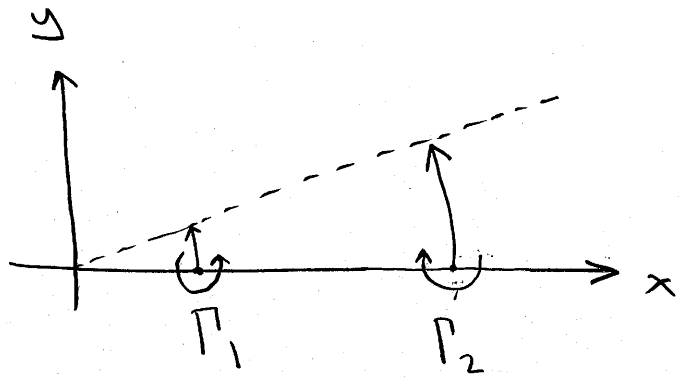
$$X = \frac{\iint x\,\omega\,ds}{\iint \omega\,ds} = \frac{x_1\Gamma_1 + x_2\Gamma_2}{\Gamma_1+\Gamma_2} \equiv 0$$

(since we've fixed the coordinate on Center of Vorticity) thus $x_1 = -x_2 \dfrac{\Gamma_2}{\Gamma_1}$ as shown above

(i.e. relative to origin, Vortex-1 is on the negative x-axis and its distance to origin is $\dfrac{\Gamma_2}{\Gamma_1}$ times as far when compared to that of vortex 1)

Note angular velocity is positive (i.e. CCW) since $\dfrac{\Gamma_1+\Gamma_2}{2\pi d^2} > 0$

2) If $\Gamma_1 > -\Gamma_2 > 0$

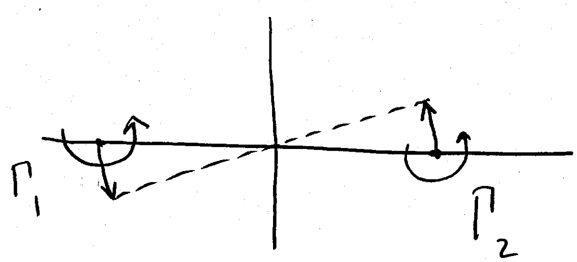


since $x_1 = -x_2 \dfrac{\Gamma_2}{\Gamma_1}$

and now that $\dfrac{\Gamma_2}{\Gamma_1}$ is negative, x_1 is positive (unlike case-1)

Here, angular velocity is also positive, since $\Gamma_1+\Gamma_2$ is still >0

3) If $\Gamma_1 = \Gamma_2$



(In this case, D has an obvious meaning)

since (by definition) $\Gamma_1(x_1^2+y_1^2) + \Gamma_2(x_2^2+y_2^2) = (\Gamma_1+\Gamma_2)D^2$

(dividing through by Γ)

$$\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = 2D^2$$

$$D = \frac{d}{2}$$ (Which makes sense, because D measures the dispersion of vorticity about (X, Y))

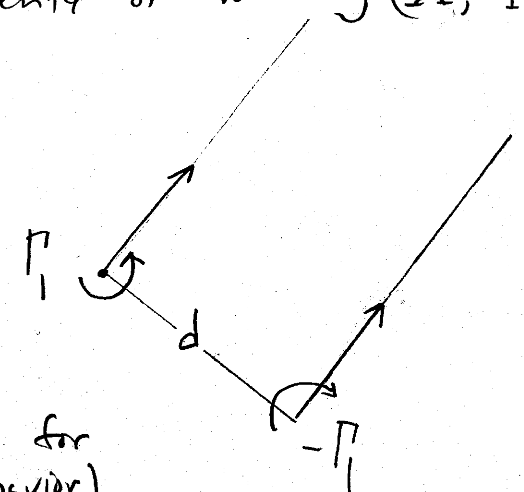
final case.

4) If $\Gamma_1 = -\Gamma_2$ "Vortex Pair"

- From the eqn. relating $\underline{d}$ and $\underline{D}$: $d^2 = \left(\frac{\Gamma_1}{\Gamma_2}+1\right)^2 \frac{\Gamma_2}{\Gamma_1} D^2$

$$D^2 = d^2 \frac{\Gamma_1}{\Gamma_2}\left(\frac{\Gamma_1}{\Gamma_2}+1\right)^{-2} = -\frac{d^2}{(-1+1)^2} \quad \text{is unbounded}$$

and Center of vorticity (X, Y) is indeterminant $\left(\frac{0}{0}\right)$



- (the reason for this behavior)

each Vortex induces a $\frac{\Gamma_1}{2\pi d}$ speed upon the other, thus the Vortices travel along straight paths with speed $\frac{\Gamma_1}{2\pi d}$

(the Vortex pair geometry, i.e. Case 4 when $\Gamma_1 = -\Gamma_2$, has many Practical applications including the trailing Vortex Pair generated by any lift Producing wing)

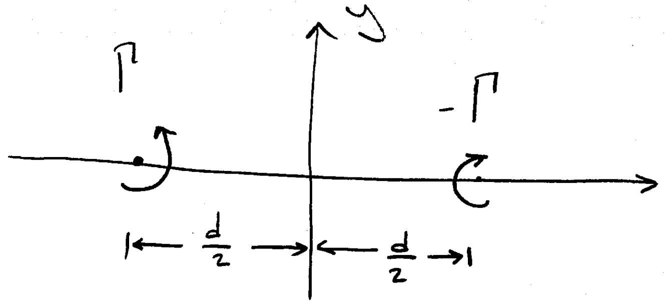
(the invariant quantities are not useful in this Particular Case, so we

- analyze this & more complicated Vortex flows using:)

Potential Flow Analysis:

Choose coordinate System as follows:

$Z = x + iy$



Complex Velocity is: $\frac{dW}{dz} = u - iv$.

Since $W = -i\frac{\Gamma}{2\pi}\log z$ for a vortex at origin, it follows that:

$\frac{dW}{dz} = -i\frac{\Gamma}{2\pi}\frac{1}{z - z_0}$ where z_0 is the location of vortex.

Thus, complex velocity for this vortex pair system is:

$$u - iv = \frac{dW}{dz} = \underbrace{-i\frac{\Gamma}{2\pi}\frac{1}{z + \frac{d}{2}}}_{\text{due to left vortex}} \underbrace{- \frac{i(-\Gamma)}{2\pi}\frac{1}{z - \frac{d}{2}}}_{\text{right vortex}}$$

(if vortex was on $+y$ axis, z_0 would've been $i\frac{d}{2}$ instead of $-\frac{d}{2}$)

$$\boxed{(u - iv) = \frac{i\Gamma d}{2\pi\left\{z^2 - \left(\frac{d}{2}\right)^2\right\}}}$$

(this provides the velocity everywhere except at $z = \pm\frac{d}{2}$)

and vortices move upward at speed $\frac{\Gamma}{2\pi d}$.

* If this velocity is subtracted, coordinate system moves with the vortices and the flow is steady state.

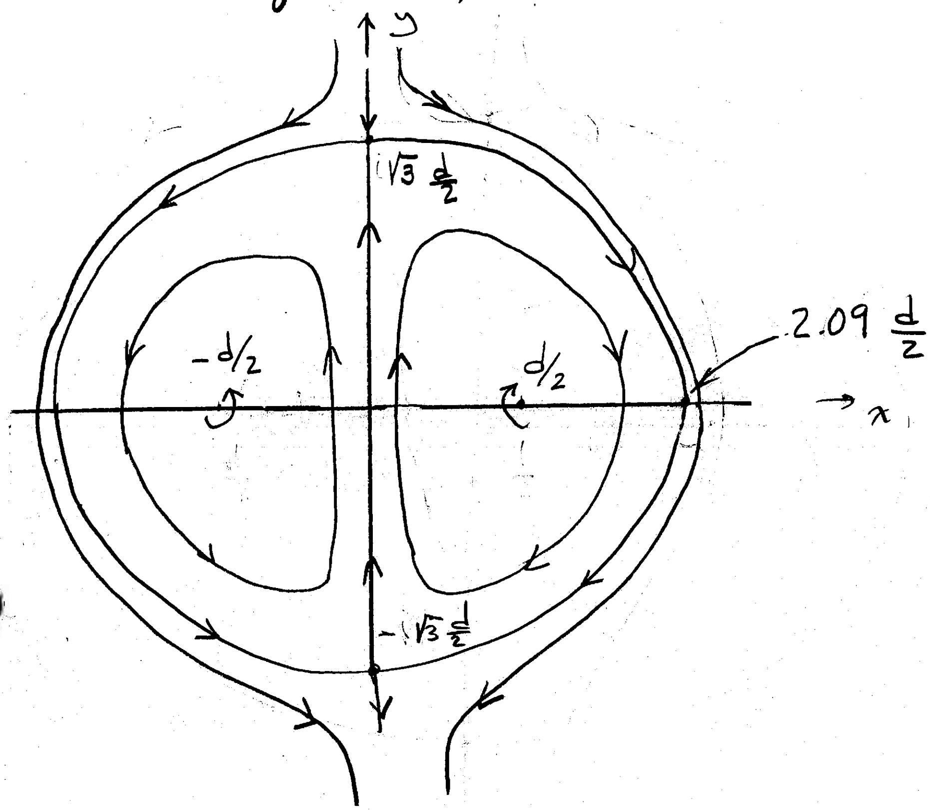
Then:

$$u - iv = \frac{i\Gamma}{2\pi d} + \frac{i\Gamma d}{2\pi\left\{z^2 - \left(\frac{d}{2}\right)^2\right\}}$$

- Velocity in the negative y direction

Find Stagnation points in flow at $z = \pm i\sqrt{3}\frac{d}{2}$ $(= x+iy)$

i.e., at $(0, \sqrt{3}\frac{d}{2})$ and $(0, -\sqrt{3}\frac{d}{2})$



T.P.

Hence there is a closed streamline, and fluid inside this closed streamline is carried by vortices ("Kelvin oval")
(turbulent boundary layers: signal associated with hair-pin vortices)
We will see this also for vortex rings (although vortex rings are more complicated)
(in addition to Chap. 7 of Lugt.)
Ref. for vortex pairs: Lamb, Hydrodynamics, article 155

For motion of 3 or more vortices, see H. Aref, Annual Review Vol. 15 (1983).

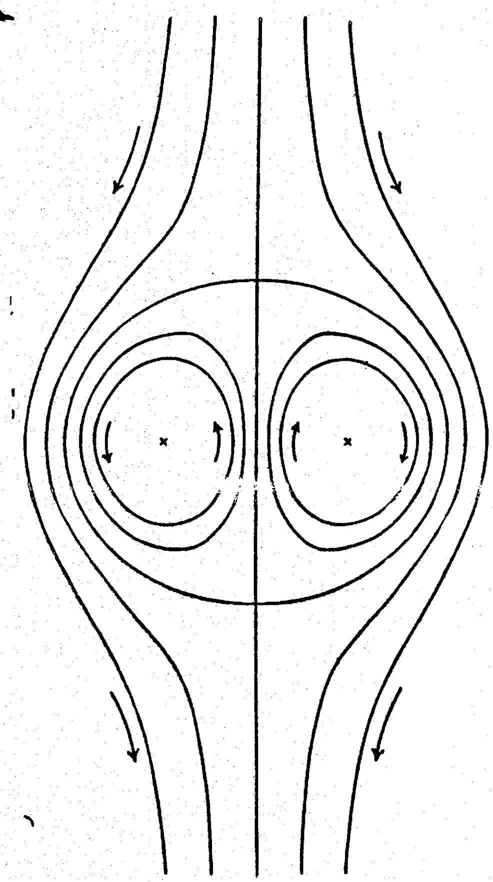
- Vortex / boundary interactions:

For inviscid flow, motion of each vortex next to a wall is simulated exactly by replacing the wall with an "image" vortex. T.P. T.P.

pp. 140, 141 Lugt

frame of reference
fixed to vortex

Kelvin oval Area:
$\doteq 11.4\, a^2$ (a is half spacing)



F. of R. Stationary

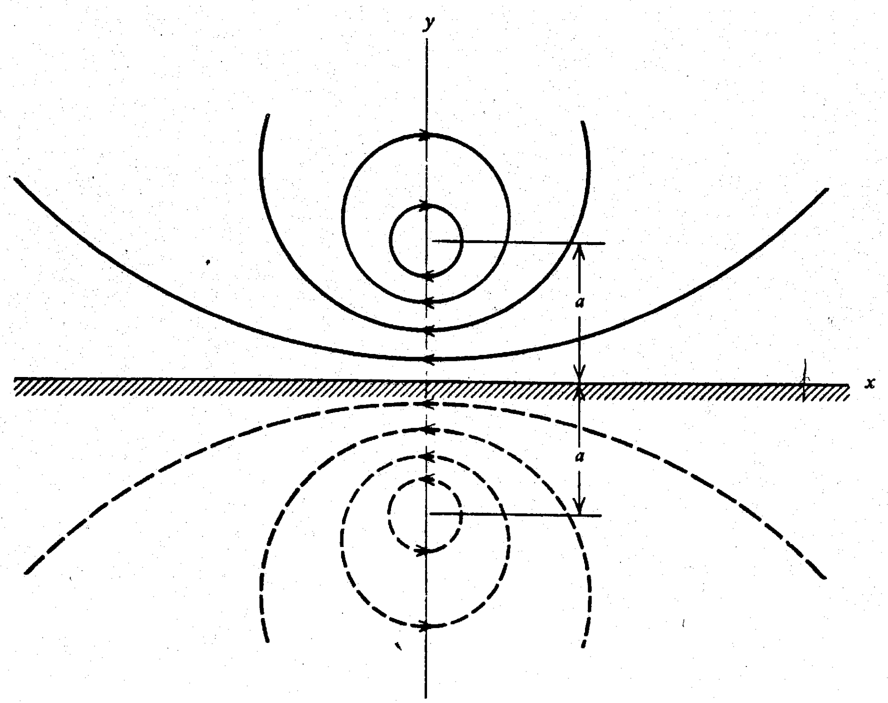


Fig. 18. A vortex near a plane wall.

Frame of reference stationary

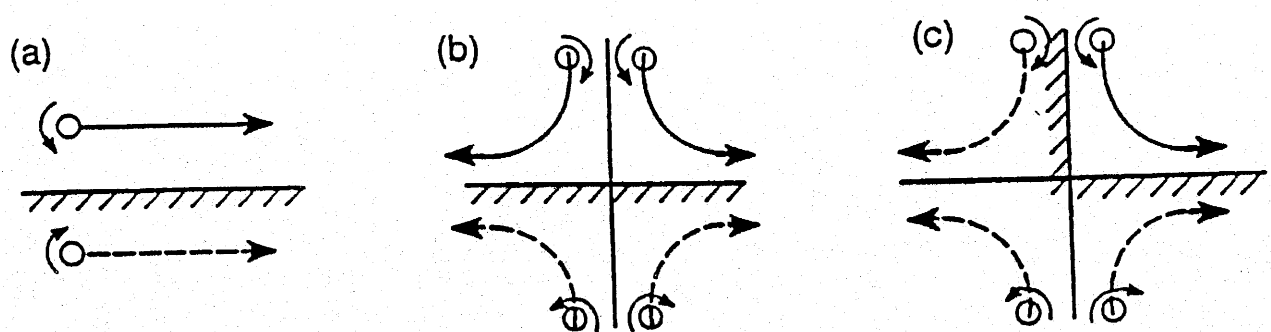
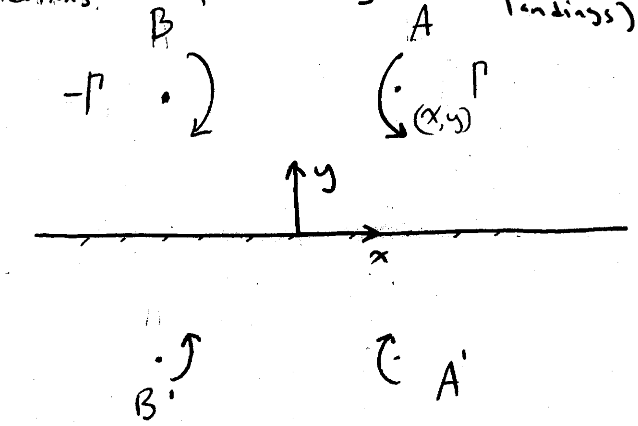


Fig. 7.3: Symmetry lines on which the normal velocity component vanishes can be interpreted as boundary lines or solid walls in potential flows. (a) Infinitely long straight wall generated by an image vortex; (b) two vortices approaching a straight wall; and (c) movement of a vortex in a rectangular corner.

(Lugt)

e.g. normal Collision of a Vortex Pair With a Wall:

Ex: Wake of an aircraft near ground (neglecting vortex inclination angle and the axial flow)
(implications in airport Safety vis-a-vis time between landings)



(as in the HW, we can find the motion of each vortex by computing the vector sum of the velocity induced at that location by each of the other three vortices)

The motion of vortex A is simply described by:

$$\dot{x} = \frac{\Gamma}{4\pi}\frac{x^2}{y r^2} \qquad \dot{y} = -\frac{\Gamma}{4\pi}\frac{y^2}{x r^2}, \quad \text{where } r^2 = x^2+y^2$$

($\dot{x}$ is only due to A' and B' and $\dot{y}$ is only due to B and B'

Limiting Cases: discuss Large x and large y

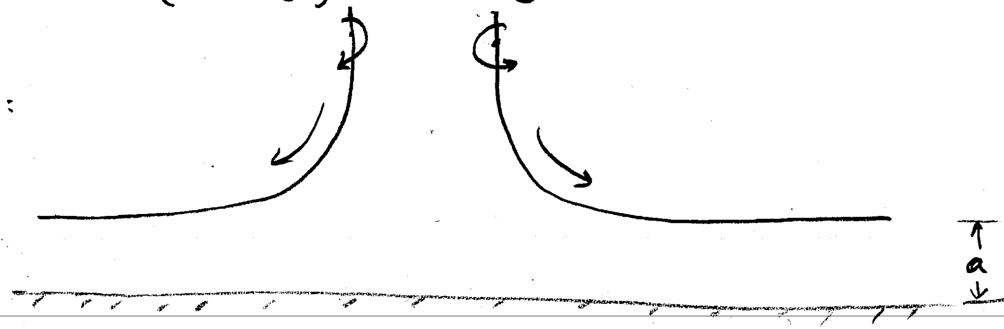
Dividing these by each other, get the differential eqn. for Path

$$\frac{dx}{x^3} + \frac{dy}{y^3} = 0$$

Which has the solution:

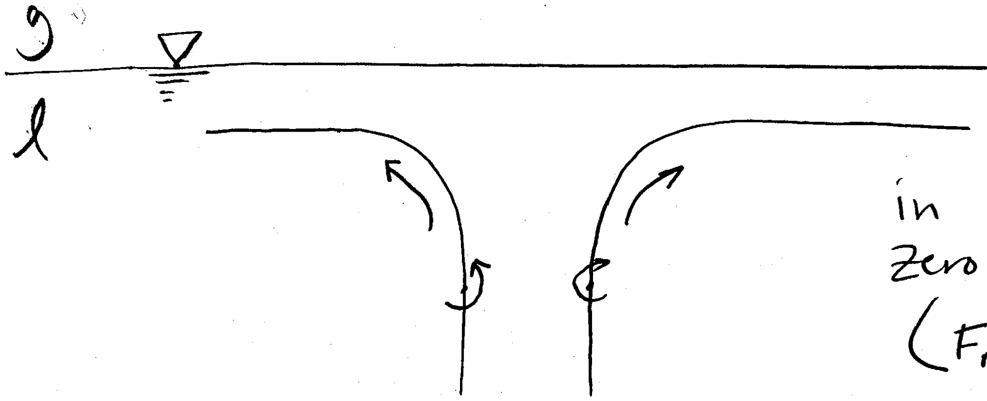
$$a^2(x^2+y^2) = 4x^2y^2 \,,$$ where a is an arbitrary constant

Plot:



Q: Is this physically realized?

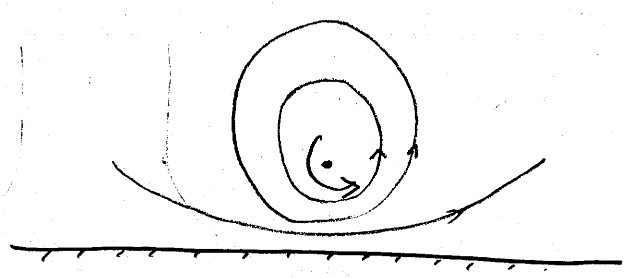
Yes, if boundary is allowed to move tangentially, as in a (clean) free surface



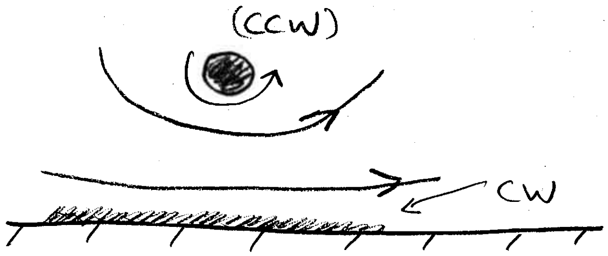
in the limit of Zero Froude Number ($F_r = \frac{\Gamma}{\sqrt{g d^3}}$ ← inertial effects / ← gravitational ")

(but in the original problem, viscosity can affect the process)

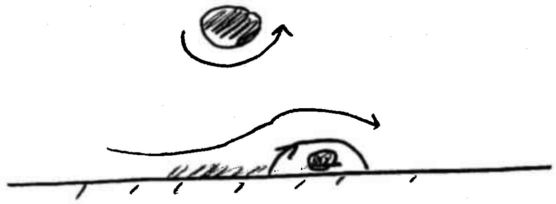
See Lugt 15.1

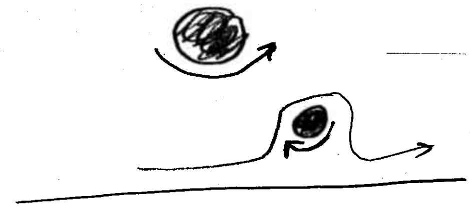


Velocity induced by the vortex near the wall must diminish due to no-slip condition. A boundary layer is formed at the wall, with vorticity of opposite-sign

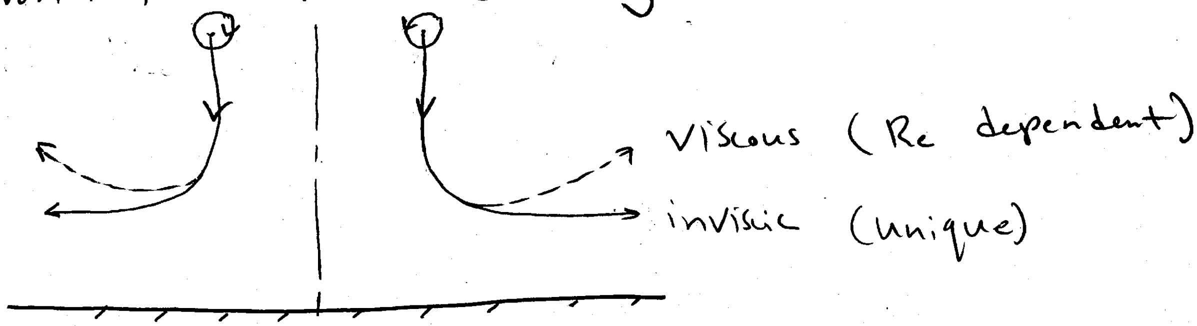


The adverse pressure gradient encountered in the boundary layer downstream of the primary vortex causes the roll-up of the boundary-layer vorticity into a "Secondary Vortex." The BL separation (roll-up) process is followed by a "BL eruption"





thus a Vortex Pair in real fluid colliding with a solid wall rebounds from it.



(we'll look at other vortex-boundary interactions next lecture, after we studied vortex rings, it turns out rings also rebound - recall "Shapiro's vorticity movie)

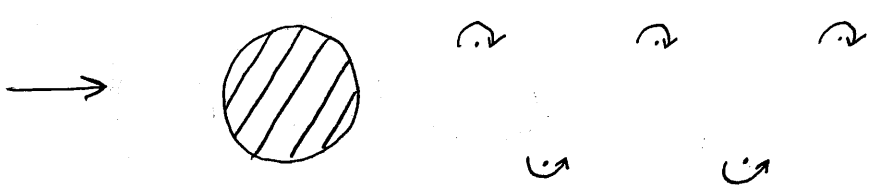
(another vortex system that is of fundamental interest both in vortex dynamics and in flow stability and transition to turbulence is :)

Kármán Vortex Street

(Von Karman gave a series of lectures at Cornell in 1954 and this material is in a small book "Aerodynamics")

(also, see p. 137 of Saffman)

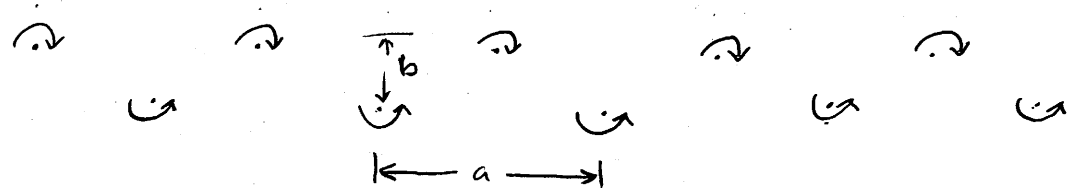
(Probably the most widely studied flow geometry, namely) uniform flow past a cylinder



This is the antisymmetric vortex array behind a cylinder for a certain range of Re (say, $50 \lesssim Re \lesssim 10^5$)

(we can readily analyze this flow)

If we assume that the array extends to ∞ in both directions, taking coordinate origin at a vortex in bottom row find complex potential:



$$W(z) = \phi + i\psi = -i\frac{\Gamma}{2\pi}\ln\left(\sin\frac{\pi z}{a}\right) + i\frac{\Gamma}{2\pi}\ln\left(\sin\frac{\pi}{a}\left[z-\frac{a}{2}-ib\right]\right)$$

(on p.146 Lagt shows how you get the $\ln(\sin)$ from $\Sigma \ln z - na$)

notice that it has the proper singularity at each vortex.

Complex velocity is:

$$\frac{dW}{dz} = W'(z) = u - iv$$

$$= -i\frac{\Gamma}{2a}\coth\left(\frac{\pi z}{a}\right) + i\frac{\Gamma}{2a}\coth\frac{\pi}{a}\left(z-\frac{a}{2}-ib\right)$$

Speed at which the array moves equals $W'(z)$ due to one vortex row evaluated at a vortex in the other row (analogeous to the simple case of a vortex pair)

e.g. take $z = \frac{a}{2} + ib$ in the first term above

get $u = -i\frac{\Gamma}{2a}\coth\frac{\pi}{a}\left(\frac{a}{2}+ib\right) = \boxed{-\frac{\Gamma}{2a}\tanh\frac{\pi b}{a}}$

i.e. vortices move to left at this speed. (tanh)

Now, we must determine the ratio $\frac{a}{b}$.

- This is done using a stability analysis: by perturbing vortex locations slightly, find that displacements grow with time, unless

$$\cosh \frac{\pi b}{a} = \sqrt{2}$$

This gives neutral stability, i.e.

$$\boxed{\frac{b}{a} = 0.281}$$

(Kármán; see Lamb, article 156)

(However, recent nonlinear stability analysis have shown this to be unstable.)

- i.e. Retaining higher order terms shows that this configuration is in fact unstable to finite disturbances. This leaves unsettled the question of why the vortex street is observed experimentally.

(even recent calculations w/ finite core have shown the same unstable behavior)

(final topic in vortex dynamics:)

Vortex Rings

(to understand these vortex structures which are among the most common, but not always easily observable fluid flows in nature and technology.)

- T.P.

(also see Batchelor; also food coloring in water)