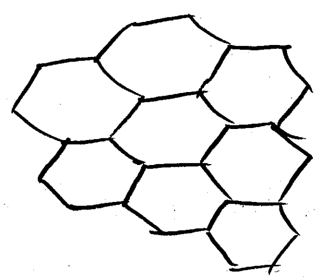
b) Thermal Instability

Bénard Convection ("BAY-NAHR")

(Verbal introduction) Instability is expected if a fluid is heated from below (due to buoyancy effects.) If the temperature difference across a horizontal layer of fluid is large enough, destablizing buoyancy effect is greater than stablizing effect of viscosity. A balance between these effects can lead to a steady perturbed state, with perodic vertical polygonal cells.

Experiments of Bénard showed hexagonal cells. Velocity is upward at all centers, downward at cell boundaries.

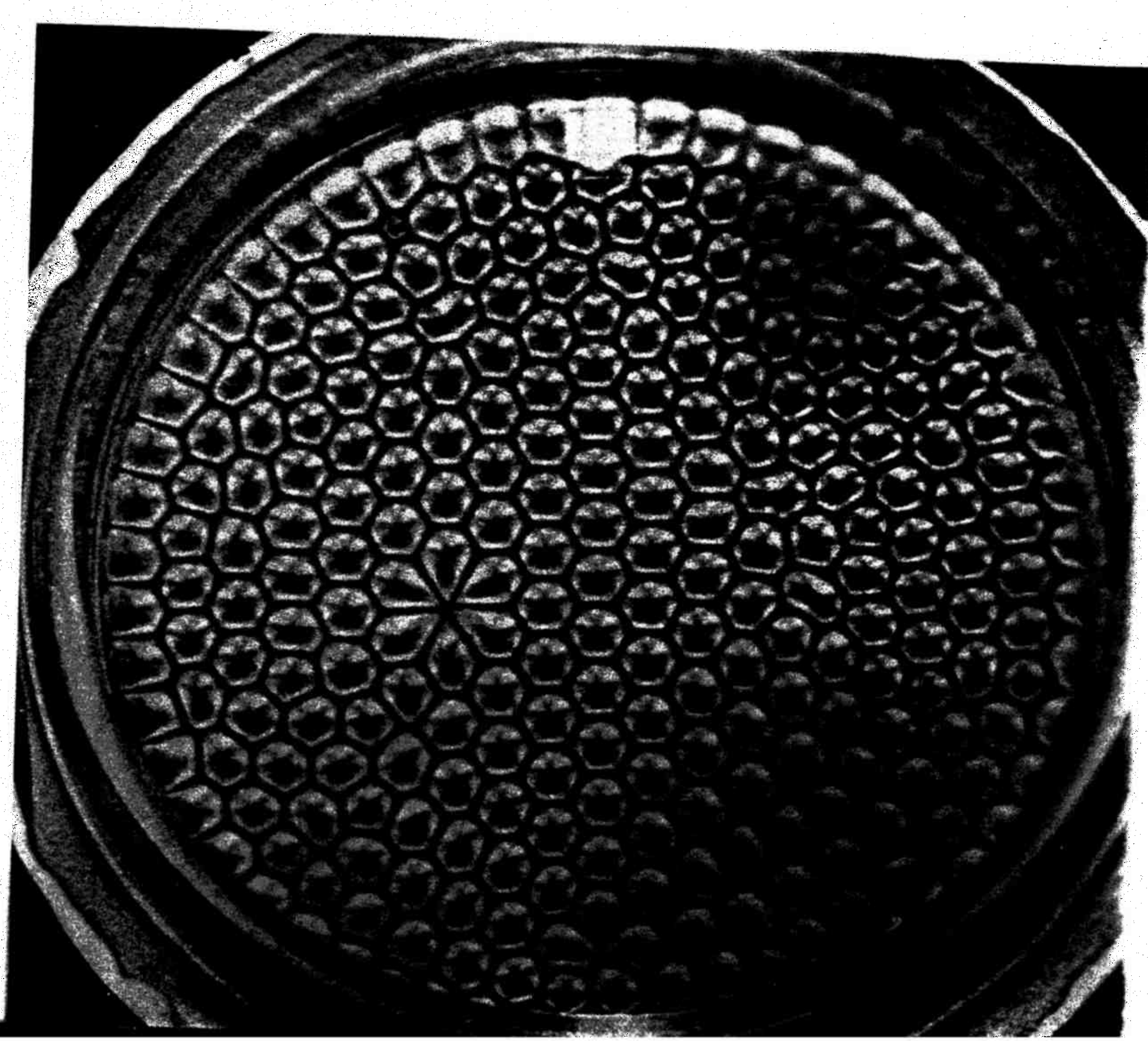


T.P. or (demo) either FS or solid wall

(Give Refs before starting the analysis:)

Refs ⇒ Chandrasekhar Hydrodynamic & Hydromagnetic Stability

(also) Drazin & Reid (Hydrodynamic Stability) from Van Dyke, An Album of Fluid Motion



142. Imperfections in a hexagonal Bénard convection pattern. The hexagonal pattern of cells typical of convective instability driven primarily by surface tension is seen to accommodate itself to a circular boundary. Aluminum powder shows the flow in a thin layer of silicone oil of kinematic viscosity 0.5 cm^2/s on a uniformly heated copper plate. A tiny dent in the plate causes the imperfection at the left, forming diamond-shaped cells. This shows how sensitive the pattern is to small irregularities. *Koschmieder 1974*

(Drazin)

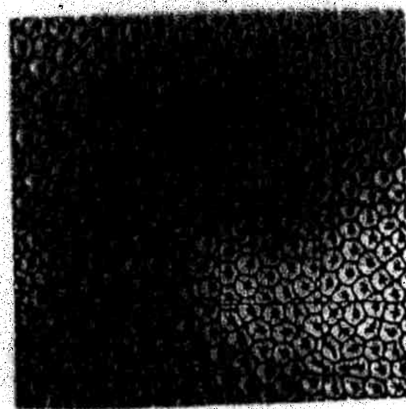


Figure 6.1 Plan view of the surface of a layer of spermaceti wax heated from below. After Bénard (1900).

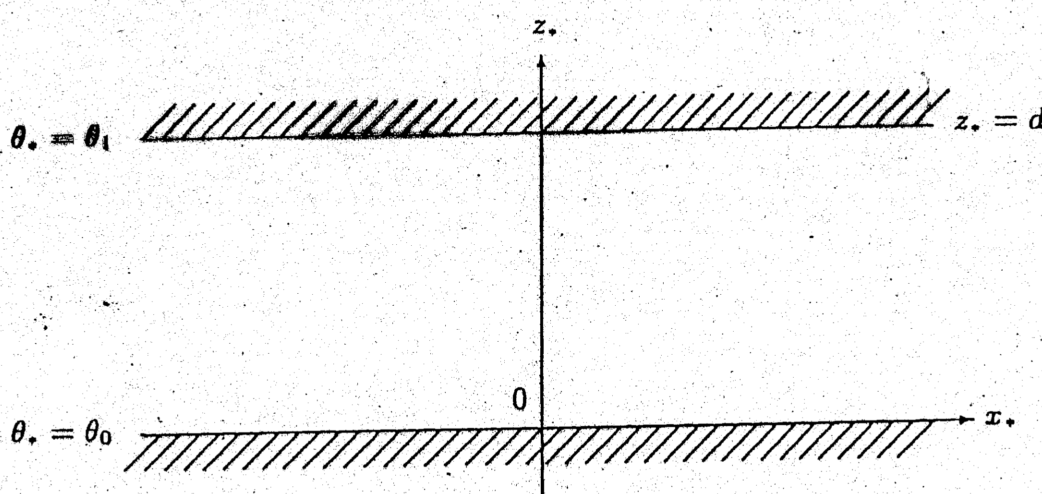


Figure 6.2 The configuration of Rayleigh–Bénard convection.

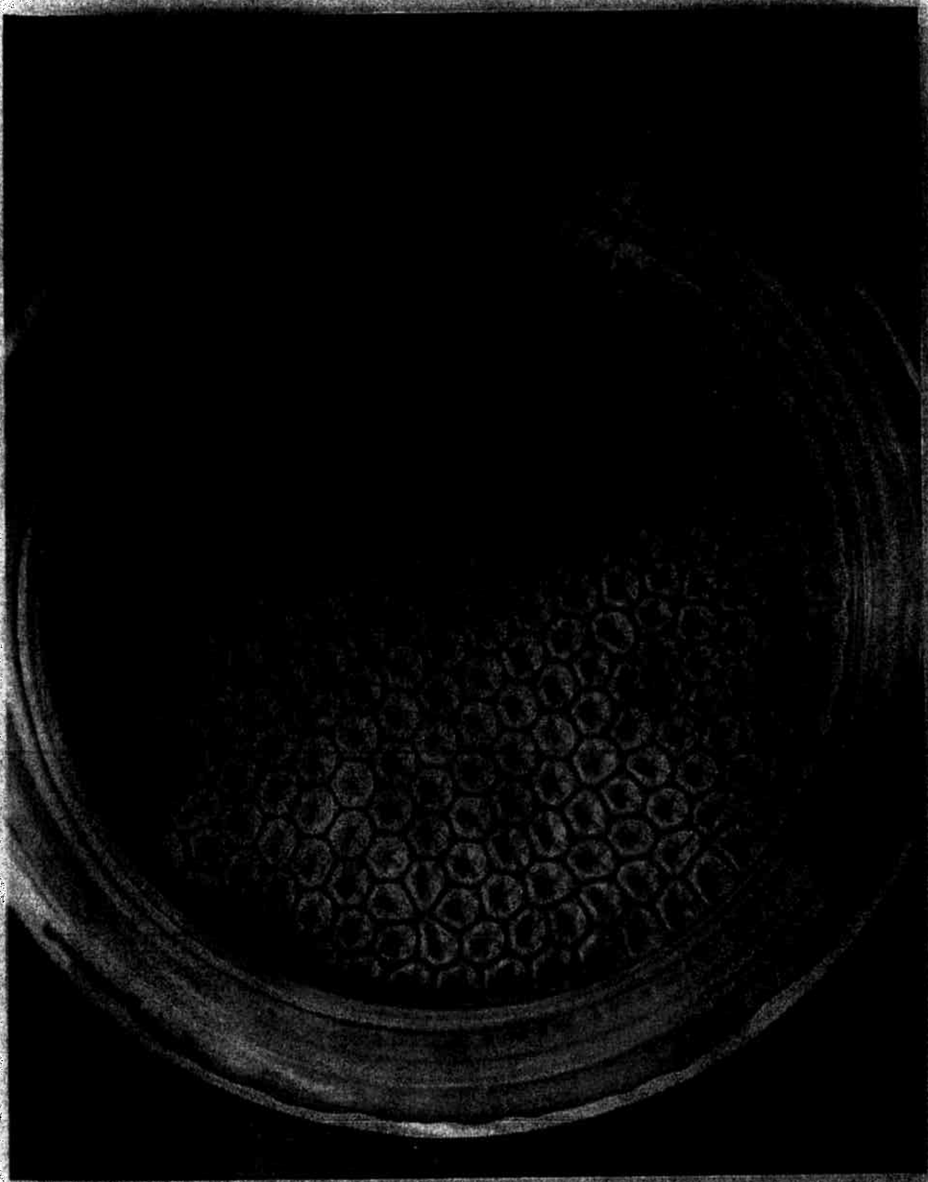


Fig. 2.1. Bénard cells under an air surface. (From Koschmieder & Pallas 1974.)

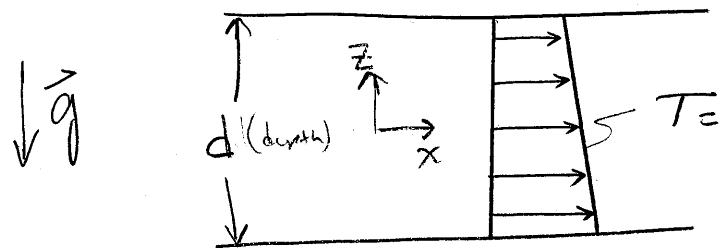
Drazin & Reid
Hydrodynamic Stability

can also be seen in a short film sequence of Mollo-Christensen (FL 1968a).

Stimulated by Bénard's experiments, Rayleigh (1916a) formulated the theory of convective instability of a layer of fluid between horizontal planes. He chose equations of motion and boundary conditions to model the experiments, and derived the linear equations for normal modes. He then showed that instability would occur only when the adverse temperature gradient was so large that the dimensionless parameter $g\alpha\beta d^4/\kappa\nu$ exceeded a certain critical value. Here g is the acceleration due to gravity, α the coefficient of

Analysis (Consider) A thin horizontal layer with linear temperature variation

$$T = T_0 - \beta z \qquad -\frac{d}{2} < z < \frac{d}{2}$$



temp at Center

$T = T_0 - \beta z$ (w/o mean initial flow, only a temp. gradient)

$\beta > 0$ "adverse temperature gradient"

(The fundamental reason for this type of instability is the unstable buoyancy effect, so we need to consider the variation of density)

For small changes (in T):

density in the center

$$\rho = \rho_0 \left[1 - \alpha (T - T_0)\right] \qquad \alpha > 0$$

α is the coefficient of thermal expansion is a small number compared to one

and since $\dfrac{dp}{dz} = -g\rho$

$$= -g\rho_0 \left[1 - \alpha(T - T_0)\right]$$

(do the usual procedure for linear analysis)

Introduce small disturbances $\vec{u}$, p', T'

(where $\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$)

Linearized disturbance eqns. are:

Continuity $\quad \operatorname{div} \vec{u} = 0$

All other terms in the full continuity eqn. include factor α, hence are small. (So conservation mass is essentially incompressible continuity eqn.)

(although the flow itself is not exactly incompressible, since following a fluid element you would see change in density resulting from heat conduction and subsequent thermal expansion)

Momentum $\vec{u}_t + \frac{1}{\rho_0}\nabla p' = \underbrace{\alpha T' g\vec{k}}_{\text{body force due to buoyancy}} + \nu\nabla^2\vec{u}$

$\frac{\partial\vec{u}}{\partial t}$

$\nu = \frac{\mu}{\rho}$, taken to be constant (although neither is exactly constant since T varies)

(convective acceleration term is of second order compared to remaining terms)

(we know that if we neglect viscosity, the unstable stratification would lead to disturbances that continue to grow, so to understand what happens in nature, the stabilizing influence of viscosity needs to be considered)

Energy $T'_t - \beta w = K\nabla^2 T'$

(Note on the energy eqn)
Starting from general consv. of energy eqn:

$$\rho\frac{De}{Dt} = \operatorname{div}(k\nabla T) - p\,\cancel{\operatorname{div}\vec{u}} + \Phi$$

internal energy (e)

$\operatorname{div}\vec{u} \to 0$ (as shown)

k: thermal conductivity; take to be constant

Φ: dissipation function $= 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$ when $\operatorname{div}\vec{u} = 0$, and since we have small disturbances, this term is second order

and since $de = c_v\,dT$

$$\rho c_v\frac{DT}{Dt} = k\nabla^2 T$$

and since $c_v \approx c_p$ for any liquid, we can write this as:

$$\frac{DT}{Dt} = K\nabla^2 T$$

where K is the thermal diffusivity

$K \equiv \frac{k}{\rho c_p}$ (units of $[K] = \frac{m^2}{s}$)

(notation in Drazin & Reid)

or $\frac{\partial T}{\partial t} + (\vec{u}\cdot\nabla)T = K\nabla^2 T$

$\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right)$

which gives the above disturbance eqn.

(end of note)

Introduce dimensionless variables:

$$\frac{x}{d},\ \frac{y}{d},\ \frac{z}{d},\ \frac{Kt}{d^2},\ \frac{\vec{u}}{K/d},\ \frac{p'}{\rho_0 (K/d)^2},\ \frac{T'}{\beta d}$$

To keep notation simple, in the following, let:

$x, y, z, t, \vec{u}, p, T$ denote these dimensionless quantities.

- $$\left.\begin{array}{l} \operatorname{div} \vec{u} = 0 \\ \vec{u}_t + \nabla P = R_a \, P_r \, T \, \vec{k} + P_r \nabla^2 \vec{u} \\ T_t - W = \nabla^2 T \end{array}\right\}$$ (non dim., linearized disturb. eqns)

where $P_r \equiv \frac{\mu c_p}{k} = \frac{\nu}{K}$ Prandtl Number $\left(\frac{\text{Viscous diffusion}}{\text{thermal} \quad "}\right)$
(dimensionless No. that is a property of the fluid)

$R_a \equiv \frac{g \alpha \beta d^4}{K \nu}$ Rayleigh " $\left(\frac{\text{buoyancy force}}{\text{Viscous Force}}\right)$
(dimension number that depends on both fluid properties & flow)

Rayleigh showed that if $R_a > R_{a\,critical}$, get instability.

- (Question is how do we find $R_{a\,critical}$)

By eliminating other variables, can obtain a single eqn. for W or T.
(Similar to what we did in internal wave analysis, when we chose between P_1 & V_1)
$\quad$ & $\quad$ " $\quad$ " $\quad$ " $\quad$ " Rayleigh $\quad$ " $\quad$ " $\quad$ " u & r
Then we separate variables, find that T (& W) has following form:

$$T = e^{\sigma t} f(x,y)\, \Theta(z)$$ (σ is dimless frequency, not surface tension)

and where f satisfies $f_{xx} + f_{yy} + \lambda^2 f = 0$

For neutral stability ($\sigma = 0$), $\Theta(z)$ is found to satisfy

$$\left(\frac{d^2}{dz^2} - \lambda^2\right)^3 \Theta + \lambda^2 R_a \Theta = 0$$ (6th order eqn.)

- With suitable boundary conditions

(upper & lower boundaries may be rigid or free, i.e. either no slip or no stress)

This gives an eigenvalue problem for λ and R_a

A Particular value of λ gives the minimum Ra for stability, the "critical" Rayleigh number

- Linear theory does not give info. about cell shape.

If we assume a cell shape, then we can solve for f
(in Pde: $f_{xx} + f_{yy} + \lambda^2 f = 0$)

e.g. for hexagonal cells

$$f = \text{Const.} \left\{ \cos \frac{\lambda}{2}(\sqrt{3}x + y) + \cos \frac{\lambda}{2}(\sqrt{3}x - y) + \cos \lambda y \right\}$$

Observe : (1) Satisfies the Pde

(2) is Periodic in x and y

(3) Symmetrical for a $60°$ rotation about $x=0$ $y=0$

- Additional Stability Refs.

Rosenhead (ed.) <u>Laminar Boundary Layers</u> article by Stewart

Lin (a classic in linear theory) <u>Hydrodynamic Stability</u>

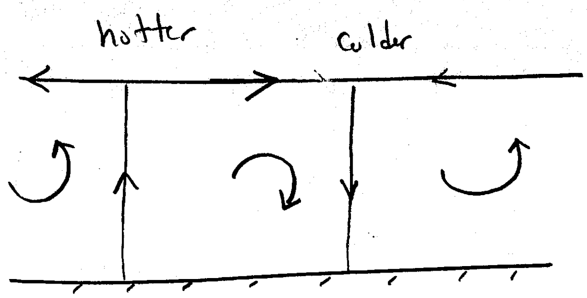
Gollub & Swinney (1981) <u>Hydrodynamic Instabilities & Transition to Turbulence</u>

(incidently, an important physical factor missing from all these theories:)

$\Rightarrow$ Benard's experiments had rigid lower & free upper surface, and now appears that the dependence of surface tension on temperature was an important instability mechanism in these experiments.

Gradient of temperature gives gradient of surface tension, hence

- surface shear stress & subsequently surface motion. In general, surface tension decreases with increasing surface temperature, shear stress tends to drag fluid from hotter to colder areas



Viscous forces then give motion as shown
(i.e. still unstable at $Ra = 0$)
i.e. in absence of gravity

This is a form of "Marangoni" convection (in particular, this is called thermo-capillary motion; a.k.a. thermo-Marangoni)

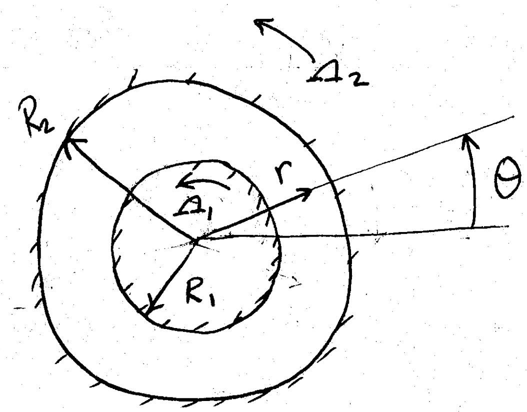
Buoyancy forces are also present if $Ra > 0$

→ Other patterns have been document, some due to imperfection (T.P.)

C) <u>Taylor-Couette Instability</u>: Flow between rotating cylinders

"Centrifugal Instability" (cf ~~gravitational~~, capillary & thermal)

Consider viscous flow between concentric circular cylinders



N-S eqn. for incomp. flow (dimensional)

$$\operatorname{div} \vec{u} = 0 \qquad \text{Continuity}$$

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{u} \qquad \text{momentum}$$

Write N-S eqns in cylindrical coordinates (r, θ, z). For basic state (unperturbed)

$$\frac{\partial}{\partial t} = 0 \qquad \frac{\partial}{\partial z} = 0 \qquad \frac{\partial}{\partial \theta} = 0 \ ; \quad u_1 = u_r = 0, \quad u_2 = u_\theta = V, \quad u_3 = u_z = 0$$

Velocity in r-direction (not to be mistaken as a partial derivative)

(see Schlichting P. 66)

Continuity: $$\frac{1}{r}\frac{\partial}{\partial r}(r \cancel{u_r}^{\,0}) + \frac{1}{r}\frac{\cancel{\partial u_\theta}^{\,0}}{\partial \theta} + \frac{\cancel{\partial u_z}^{\,0}}{\partial z} = 0$$

∴ every term in continuity is zero
(this would've told us that $u_r = 0$ if we didn't know already)