(Now we can physically explain wave dispersion:)

Longer waves (larger λ and smaller k) have an effect that penetrates deeper into the bulk, thus there is a stronger restoring effect due to gravity. In shallow water, the effect of all the (longer) waves reaches the bottom, thus the restoring effect is independant of k and only a function of h. Thus, deep water waves are dispersive and shallow water waves are not

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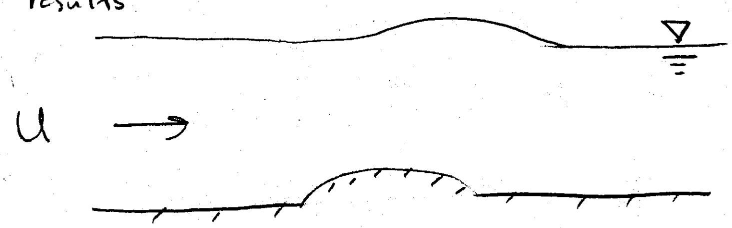
(one application of shallow water results) <u>critical flow</u> in hydraulics

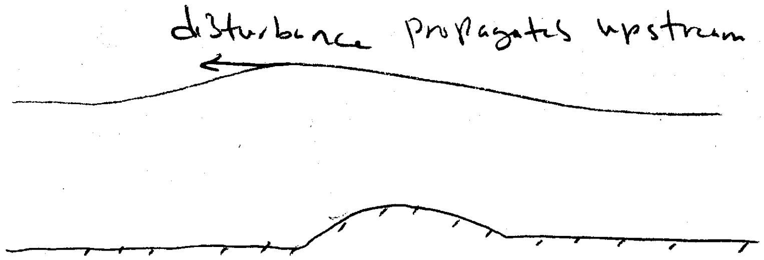


if $U > \sqrt{gh}$ "super critical"

results in a stationary wave

(analogous to sound not being able to propagate upstream in a supersonic field)

if $U < \sqrt{gh}$ "subcritical"



Hence Froude number, $F_r \equiv \dfrac{U}{\sqrt{gh}}$, if $F_r > 1$ supercritical

$F_r < 1$ sub critical

(there are other interpretations of F_r; we'll discuss later)

See Voit, Annual Review 19, 217-236

(Wave dispersion relations also help us understand the castrophic process of Tsunamis: Large scale disturbance in deep ocean produced by

Seismic activity. When $h = \mathcal{O}(\text{kilometers})$, $\eta_{max} = \mathcal{O}(1\,m)$ only! (Very linear wave)

$\Rightarrow$ if $\lambda = \mathcal{O}(h)$, then $c = \sqrt{\frac{g}{k}} \approx 40\ m/s$
$\Rightarrow$ " $\lambda > \mathcal{O}(h)$, " $c = \sqrt{gh} \approx 100\ m/s$ } in deep ocean

($M = 0.1 \to 0.3$!)

and as wave reaches the continental shelf and nears the beach, $h \to 0$, non linearities take over and λ decreases but general not as quickly as h, so $\frac{\lambda}{h}$ increases, $c \to \sqrt{gh}$ wave slows, refracts, and steepens ($\eta_{max} \to \mathcal{O}(10\text{-}100\ m)$!)

nonlinear & beyond the scope of present analysis

Surface Tension

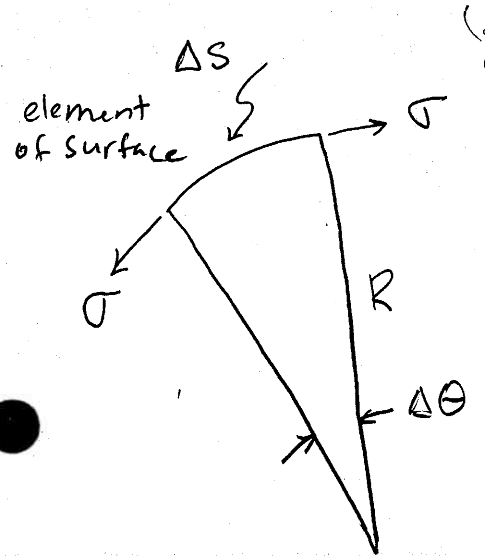
(see Adamson 3rd ed. pp 7-18)

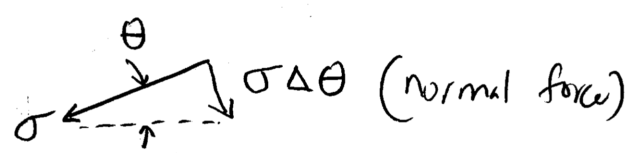
(in addition to gravity, surface tension acts as a restoring force for a perturbed surface and like gravity, it over shoots in its action, thus surface tension also produces waves) e.g. ink jet!

(σ is important not only in context of surface waves, but in determining flow on surface of liquids at all scales, but most significant small scales)

An extension of the surface, for example due to wave motion, increases the surface energy since attractive molecular force is present only below, hence requires work. The effect is the same as if a tangential force, σ, (per unit length) were present at surface. (hence the term surface tension)

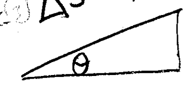
(some texts use γ)
(also interpreted as energy/area in terms of free energy)





Normal force / area (at a given time) is:

$\sigma \dfrac{\partial \theta}{\partial s}$ but: ① $\Delta s = \sqrt{1+\eta_x^2}\,\Delta x$



② $\Delta y = \dfrac{\partial \eta}{\partial x}\Delta x = \eta_x \Delta x$ (↑ same notation as before)

① Δx

(by chain rule:)

$$= \sigma \frac{\partial}{\partial s}\left(\tan^{-1}\eta_x\right) = \sigma \underbrace{\frac{\partial}{\partial x}\left(\tan^{-1}\eta_x\right)}_{\frac{1}{1+\eta_x^2}\eta_{xx}} \underbrace{\frac{\partial x}{\partial s}}_{\frac{1}{\sqrt{1+\eta_x^2}}}$$

$$= \sigma \frac{1}{1+\eta_x^2}\,\eta_{xx}\,\frac{1}{\sqrt{1+\eta_x^2}} = \frac{\sigma\,\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}$$

This term tells you how much the pressure changes across the interface due to surface tension

↓ 1-30-12

Dynamic free surface condition becomes:

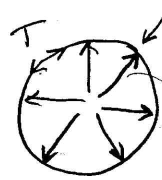
$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g\eta - \underbrace{\frac{\sigma}{\rho}\frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}}_{} = 0 \quad \text{at } y = \eta$$

(exact)

(Note its sign: ⇒ negative η_{xx} ⇒ pressure jump (+))

Linearizing: $\boxed{\phi_t + g\eta - \dfrac{\sigma}{\rho}\eta_{xx} = 0}$ at $y = 0$

Note, the same result is obtained from Laplace's Circumferential tension law: the tension on the wall of a tube is $r_0 \cdot P_e$ where P_e is excess pressure



T P_e (difference between inside pressure and outside pressure)

$T = r_0 \cdot P_e \Rightarrow P_e = \dfrac{T}{r_0}$. This law can be written for any liquid surface:

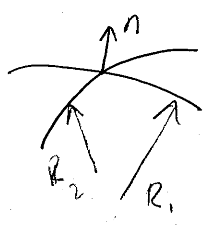
$$P - P_0 = \Delta P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

"Laplace Pressure" aka Young-Laplace eqn

P: Press. inside liquid; P_0: atmospheric pressure

R_1 and R_2 are the principal radii, and are obtained by passing a pair of orthogonal planes through the surface normal

(illustrate Young-Laplace eqn. Fig. II-3 Adamson) T.P.



(back to surface wave problem)

For a linear theory, $\Delta P = -\sigma\left(\eta_{xx} + \eta_{yy}\right)$, and for 2-D waves

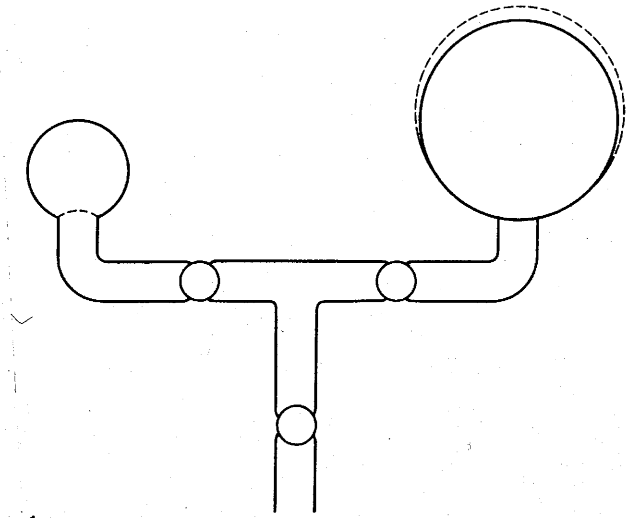


Fig. II-3. Illustration of the Young and Laplace equation.

Adamson Physical Chemistry of Surfaces (5th Edn.)

$\Delta P = -\sigma \, \eta_{xx}$

Combining the linearized dynamic free surface condition with Kinematic condition,

$$\phi_y - \eta_t = 0 \quad \text{at} \quad y = 0$$

(same as before)

dispersion relation is found to be, for deep water:

$$\boxed{\omega^2 = k\left(g + \frac{\sigma}{\rho}k^2\right)} \qquad \text{and} \qquad c^2 \equiv \frac{\omega^2}{k^2} = \frac{1}{k}\left(g + \frac{\sigma}{\rho}k^2\right)$$

Observe that surface tension effects are negligible if $k^2 \ll \frac{\rho g}{\sigma}$,

So when wave number is small, i.e. wave length is large, get <u>gravity waves</u>.

Gravitational effects are negligible if $k^2 \gg \frac{\rho g}{\sigma}$ $\Rightarrow$ <u>Capillary waves</u>.

(at intermediate k) When both effects are important,
get "capillary-gravity waves" (c-g) sometimes called "ripples"

Note: c has a minimum at $k = \boxed{k_m = \sqrt{\frac{\rho g}{\sigma}}}$ with $\boxed{c_m = \sqrt{\frac{2g}{k_m}}}$.

min. phase speed

For water (at room temp.) $\sigma = 0.072$ Newtons/m or 72 dynes/cm in cgs units

$\rho = 1000$ kg/m^3 (1 in cgs units)

$\Rightarrow \underbrace{\lambda_m = \frac{2\pi}{k_m} = 1.7 \text{ cm}}$ and $\underbrace{c_m = 23 \text{ cm/s}}$ T.P.

(<u>Sidenote:</u> aside from comparing k^2 with $\frac{\rho g}{\sigma}$ to see balance between surface tension and gravity, can define "Bond number" by considering the lengthscale at which the two effects match: $\frac{1}{L^2} \equiv \frac{\rho g}{\sigma}$ $\Rightarrow L = \sqrt{\frac{\sigma}{\rho g}} \approx 0.27$ cm

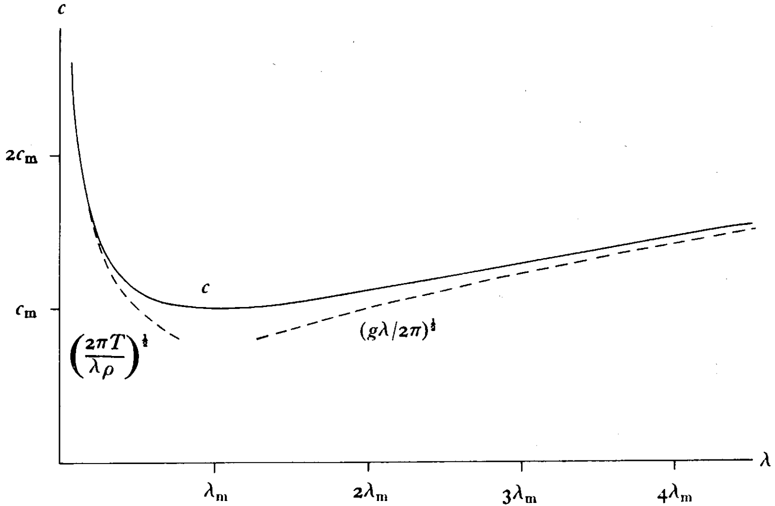


Figure 56. The wave speed c for ripples on deep water. Note the transition betwee[n] the capillary-wave value $(2\pi T/\lambda\rho)^{\frac{1}{2}}$ and the gravity-wave value $(g\lambda/2\pi)^{\frac{1}{2}}$. Thi[s] occurs around $\lambda = \lambda_m$, the wavelength for minimum wave velocity given b[y] equation (55).

(or 2.7 mm) for water at earth g. 5/3

<u>Group Velocity</u> (first, a physical view of this concept)

((til now, we had in mind a single wavenumber. Now consider some range of k; for simplicity, say there are a pair of wavenumbers, close to one another)

Consider a wave composed of two frequency components ω and ω',

Where $\omega \doteq \omega'$ and $k \doteq k'$

$$\eta = a \sin(kx - \omega t) + a \sin(k'x - \omega' t)$$

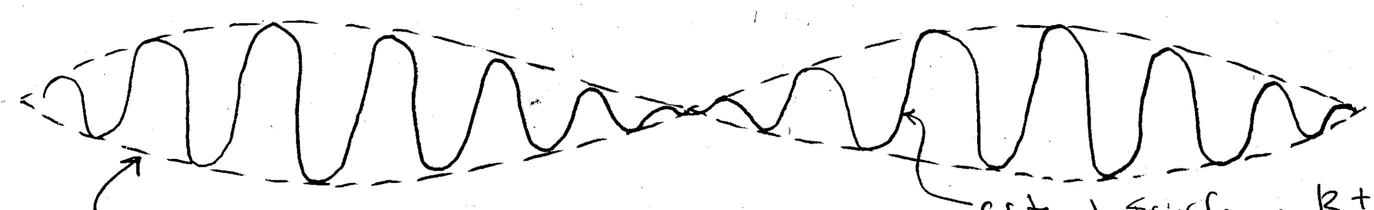
since $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$, find:

$$\eta = 2a \sin\left(\frac{k+k'}{2} x - \frac{\omega+\omega'}{2} t\right) \cos\left(\frac{k'-k}{2} x - \frac{\omega'-\omega}{2} t\right)$$

$$\left(\approx 2a \sin(kx - \omega t) \cos \frac{k'-k}{2}\Big(x - \underbrace{\frac{\omega'-\omega}{k'-k}}_{\text{"group velocity" } c_g} t\Big)\right)$$

↓ 2-3-11

(because when you add two waves of similar wavelength & frequency, get heterodyning, as in FM radio operation)



big envelope: small wave number $\frac{k'-k}{2}$, travels at c_g

actual surface: $\frac{k+k'}{2}$ (travels at $\frac{\omega+\omega'}{k+k'} \approx \frac{\omega}{k} = c$)

We can generalize this mathematically: define group velocity c_g:

(

$$c_g \equiv \frac{d\omega(k)}{dk}$$ (in this case, $\frac{d\omega}{dk} = \frac{\omega'-\omega}{k'-k}$ is the phase speed of the wave packet)

6/3

For deep water gravity waves (recall): $\omega = \sqrt{gk}$, $c = \sqrt{\frac{g}{k}}$ $\left(\text{since } c = \frac{\omega}{k} = \frac{\lambda\omega}{2\pi}\right) = \lambda \cdot f$

(thus,) group velocity $c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c$

(on the other hand,)
When surface tension effects dominate:

recall: $\left(\omega^2 = k\left(g + \frac{\sigma}{\rho}k^2\right)\right)$ $\omega^2 = \frac{\sigma}{\rho}k^3 \Rightarrow \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{\sigma k}{\rho}} = \frac{3}{2}c$

Says that group velocity is slower than the phase speed (by a factor of $\frac{1}{2}$) for gravity waves, and group velocity is larger than phase velocity (by a factor of $\frac{3}{2}$) for capillary waves.

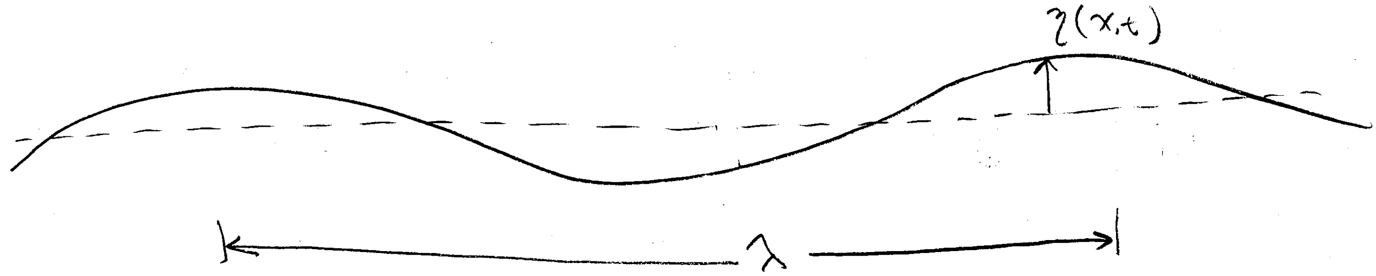
Physical significance: gravity waves run behind the body, e.g. wake of ships (Kelvin wave  pattern, discuss later)

In case of capillary waves, waves run in front of body, e.g. finger in front of smoothly running tap, waves will propagate upstream

ex:  $c = -V$

another example: fishing line moving in water

Wave Energy & Relation to Group Velocity



Consider deep-water gravity waves (no surface tension)

$$\eta = -a \sin k(x-ct)$$

$$\phi = ac \cos k(x-ct)\, e^{ky}$$

Average Kinetic energy in 1 wavelength $\left(\lambda = \frac{2\pi}{k}\right)$

(no need to put η, since we have linear theory)

$$KE = \frac{1}{2}\rho \int_0^\lambda \int_{-\infty}^0 (\phi_x^2 + \phi_y^2)\, dy\, dx$$

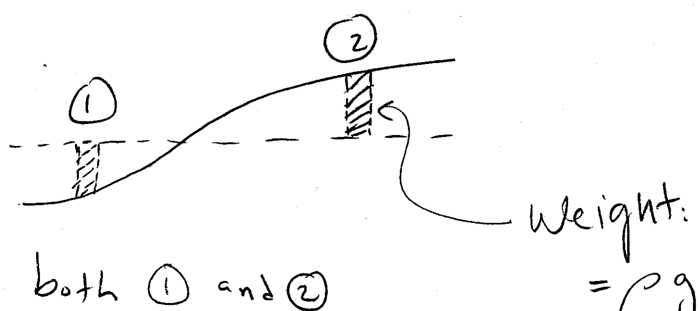
$$= \frac{1}{2}\rho \int_0^\lambda \left(\frac{a^2 c^2 k}{2}\right) dx$$

$$\Rightarrow \boxed{KE = \frac{1}{4}\rho a^2 c^2 k \lambda}$$

Potential energy:

$$PE = \rho g \int_0^\lambda \int_0^\eta y\, dy\, dx$$

$$= \rho g \int_0^\lambda \frac{\eta^2}{2}\, dx \quad \text{(to see physically why it's } \eta^2 \text{ term:)}$$



both ① and ② contribute positively to potential energy (positive displacement adds fluid with positive potential energy, negative displacement removes fluid with negative potential energy)

$$= \frac{1}{4}\rho g a^2 \lambda,$$

and since $c^2 = \frac{g}{k}$ for deep water waves,

$$\Rightarrow PE = \frac{1}{4}\rho a^2 c^2 k \lambda \quad , \quad \therefore \boxed{PE = KE}$$

Hence, we have equipartition of energy for deep water waves

The total wave energy in one wave length is $= \frac{1}{2}\rho g a^2 \lambda$ 8/3

and the total energy per unit length is $= \frac{1}{2}\rho g a^2$

$\underline{\text{Work done}}$ in 1 Period, on fluid to the right of $x = \text{const.}$ is

$$W = \int_0^{\frac{2\pi}{\omega}} \int_{-\infty}^{0 \leftarrow (\eta \approx 0)} P \phi_x \, dy \, dt$$

, but from unsteady Bernoulli's eqn:

$$P - P_0 = -\rho \phi_t - \rho \frac{|\nabla\phi|^2}{2} - \rho g y$$

(set the arbitary atmospheric pressure to zero and neglect the nonlinear terms)

$$P = -\rho\phi_t - \rho g y$$

and we had $\phi = ac \cos k(x-ct)\, e^{ky}$

$$W = \frac{1}{4}\rho a^2 k c^3 \frac{2\pi}{\omega} = \frac{1}{4}\rho a^2 k c^2 \lambda \quad \left(\text{since } \lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega}\frac{\omega}{k} = \frac{2\pi}{\omega}c\right)$$

(thus)
Average work done per unit time equals average energy transported

$$\underset{\left(\frac{1}{4}\rho a^2 k c^3\right)}{\text{Average work done}} \qquad \underbrace{\left(\text{average energy/unit area}\right)}_{=\frac{1}{2}\rho a^2 c^2 k} \times \underbrace{(\text{speed})}_{=c_g}$$

"group velocity"

i.e. $\frac{1}{4}\rho a^2 k c^3 = \frac{1}{2}\rho a^2 c^2 k \, c_g$

$\therefore \boxed{c_g = \frac{1}{2}c}$

Thus, group velocity (c_g) is also the speed at which wave energy propagates, which for gravity waves is $\frac{c}{2}$

Same result as before, when we computed group velocity from its definition $\frac{d\omega}{dk}$