observer moving at c (to the right)

(for a right-running wave)

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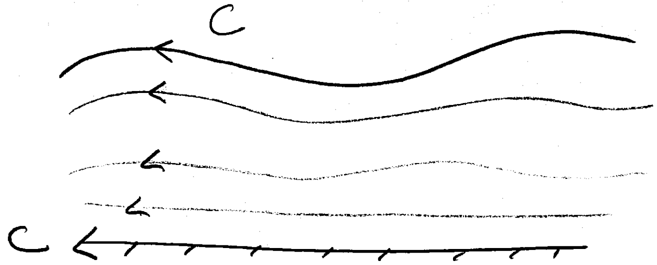

(What is observed is a "Stationary wave" as opposed to the "Standing" waves of problem 3)

Says that the phase speed, c, is the relevant speed for Re.

(c is of order 10's of cm/s to many m/s) thus $Re \gg 1$

(So it makes sense to neglect viscosity, at least for the time being)

Eqns: Continuity $\quad \text{div}\, \vec{u} = 0 \quad$ (for ρ = const.)

momentum

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}}_{\text{Unsteady \& Convective acceleration}} = \underbrace{-\frac{1}{\rho}\nabla P}_{\text{pressure gradient}} \underbrace{- \nabla(gy)}_{\text{gravity}} + \cancel{\nu \nabla^2 \vec{u}}$$

(We are neglect viscosity)

Eulers Eqn.

(or we can write the acceleration terms in terms of vorticity:)

$$\vec{u}_t + \nabla \frac{|\vec{u}|^2}{2} + (\text{curl}\, \vec{u}) \times \vec{u} = \quad ''$$

↖ subscript implies differentiation

(at this point, it is generally assumed that)

Irrotational Flow for (linear) Surface Waves

(it is usually argued that we've started with fluid at rest, w/o vorticity and the process adds no torque, hence no vorticity added)

$$\Rightarrow \boxed{\text{Vorticity}\overset{(\vec{\omega})}{} \equiv \text{curl}\, \vec{u}} = 0$$

(potential function, not to be confused with angle)

and since ρ = const, $\Rightarrow \quad \vec{u} = \nabla \phi \quad$ i.e. there must be a velocity potential i.e. $u = \frac{\partial \phi}{\partial x}$ denote ϕ_x ; $v = \frac{\partial \phi}{\partial y}$ denot ϕ_y

$\&\ \omega_z = 0 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Continuity eqn. gives $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\right)$ $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\boxed{\nabla^2 \phi = 0}$

$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$

irrot + incomp = potent

(although we are going to look at vorticity laws in detail a bit later, it is worthwhile to stop at this point & more rigorously justify the use of potential flow for small amplitude oscillatory flow)

Linear wave $\Rightarrow$ max η (denote A) $<< \lambda$ (wavelength) (Landau & Lifshitz)

Consider order of magnitude of various terms in Euler's eqn.

$$\frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u} \cdot \nabla)\vec{u}}_{u\frac{\partial u}{\partial x} \text{ in 1-D}} = -\frac{1}{\rho}\nabla P$$

Can show that $u = O\left(\frac{A}{\tau}\right)$ τ is period of oscillation

$$\frac{\partial u}{\partial t} = O\left(\frac{A}{\tau^2}\right)$$

$$\frac{\partial u}{\partial x} = O\left(\frac{A}{\tau \lambda}\right)$$

$$\frac{A}{\tau^2} >> \frac{A}{\tau}\frac{A}{\tau \lambda}$$ if $A << \lambda$ (true for a linear surface wave)

$\Rightarrow$ $\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\nabla P$, take curl of both sides :

$\frac{\partial}{\partial t}(\text{curl } \vec{u}) = 0$, hence curl $\vec{u}$ = constant

and since in oscillatory motion the mean velocity is zero $\Rightarrow$ Curl $\vec{u} = 0$ $\therefore$ Can treat $\wedge$ (this or any other small disturbance) as potential flow

For now, we will retain the $\nabla\frac{|\vec{u}|^2}{2}$ (nonlinear) part of the convective acceleration term

(back to the wave problem formulation)

(using the velocity potential,) Momentum eqn. becomes:

$$\nabla \phi_t + \nabla\left(\frac{|\nabla \phi|^2}{2}\right) + \nabla\left(\frac{P}{\rho}\right) + \nabla(gy) = 0$$

Integrate (& multiply thru by ρ)

$$\boxed{P - P_0 = -\rho\,\phi_t - \rho\,\frac{|\nabla\phi|^2}{2} - \rho g y}$$

Unsteady Bernoulli eqn. for incompressible, irrotational flow

(if we evaluate this when flow is undisturbed $\Rightarrow$ RHS $= 0 \Rightarrow P = P_0$ @ $y=0$)

$\therefore$ P_0 = atmospheric pressure

BC at bottom $y = -h$: $\phi_y = 0$

at FS $y = \eta(x,t)$

Bernoulli's eqn. evaluated at FS gives:

$$\boxed{\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g\eta = 0} \quad \text{at } y = \eta$$

(this provides us with the)
Dynamic Free Surface Condition. (nonlinear)

(since we want to solve Laplace's eqn. we need BC all around the fluid, our problem is that FS boundary is unknown and to get it we need another BC at FS; so we need the kinematic BC, in other words this BC has introduced a new unknown, η and by itself won't help us solve $\nabla^2\phi = 0$)

(to get the Kin. BC, we can argue that fluid velocity, in vertical direction, is equal to $\frac{\partial \eta}{\partial t}$, but to be more exact:)

$$V = \frac{D\eta}{Dt}$$

(Vertical comp. of velocity)

implicity says that fluid element at surface remains there.

or $\phi_y = \frac{D\eta}{Dt}$ $\Rightarrow$ $\boxed{\phi_y = \eta_t + \phi_x \eta_x}$ at $y=\eta$

<u>Kinematic B.C.</u>

i.e. Vertical Component of Fluid Velocity at Surface

equals Surface elevation acceleration (i.e. its rate of change)

If amplitudes are sufficiently small, (we'll discuss H.O.T. later)

1) expand in Taylor Series about $y=0$
2) neglect all quadratic terms, i.e. Linearize

(Dynamic FS Condition)

$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0$ becomes $\boxed{\phi_t + g\eta = 0}$ (linearized dyn. FS Cond.)

(Kinematic FS condition)

$\phi_y = \eta_t + \phi_x \eta_x$ $\boxed{\phi_y = \eta_t}$ (linearized Kin. FS Con.)

eliminate η: $\boxed{\phi_{tt} + g\phi_y = 0}$ at $y = 0$

Now we can use this combined BC (only has ϕ) to solve the Laplace eqn.:

(For an) <u>Infinite Wave Train</u>: $\phi_{xx} + \phi_{yy} = 0$ $-h < y < 0$

bc: $\phi_y = 0$ at $y = -h$

$\phi_{tt} + g\phi_y = 0$ at $y = 0$
not $y = \eta$!

Separation of Variables Shows (that ϕ must be of form:) $\phi = e^{\pm ikx} \cdot e^{ky}$ with coef. a function of t

Consider waves traveling in positive x-direction. Try:

$$\phi = A e^{i(kx-\omega t)} \cosh k(y+h)$$

or

$$= A e^{\overbrace{ik(x-ct)}} \underbrace{\cosh k(y+h)}$$

} Real Part understood

usual form of a right-running wave — to satisfy bottom condition

Where k = wave number = $\frac{2\pi}{\lambda}$ ← wave length

ω = angular (circular) frequency

$c = \frac{\omega}{k}$ Phase speed

A = related to max. wave height

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Substitute this solution in the combined BC: $\phi_{tt} + g\phi_y = 0$ at $y=0$

Find: $\boxed{\omega^2 = gk \tanh kh}$

"dispersion relation"

recall: $\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$, $\sinh\theta = (e^{\theta}-e^{-\theta})/2$, $\cosh\theta = (e^{\theta}+e^{-\theta})/2$ T.P.

and: $c^2 = \frac{g}{k}\tanh kh$

Says: Surface waves are dispersive, i.e. their propagation speed depends on their wave length, in this case propagation speed increases with wave length $\approx \left(\frac{1}{k}\right)^{1/2}$

(This is why surface waves are intrinsically more difficult to deal with than sound waves. When you make a sound composed of many different frequencies, some complex tone, for, say, 1 second duration, someone a long distance away would hear the same tone, albeit attenuated, and for the same exact 1 second duration. But in water, any disturbance, unless it is composed of one pure sinusoidal component, will disperse, with longest waves leaving behind shorter ones

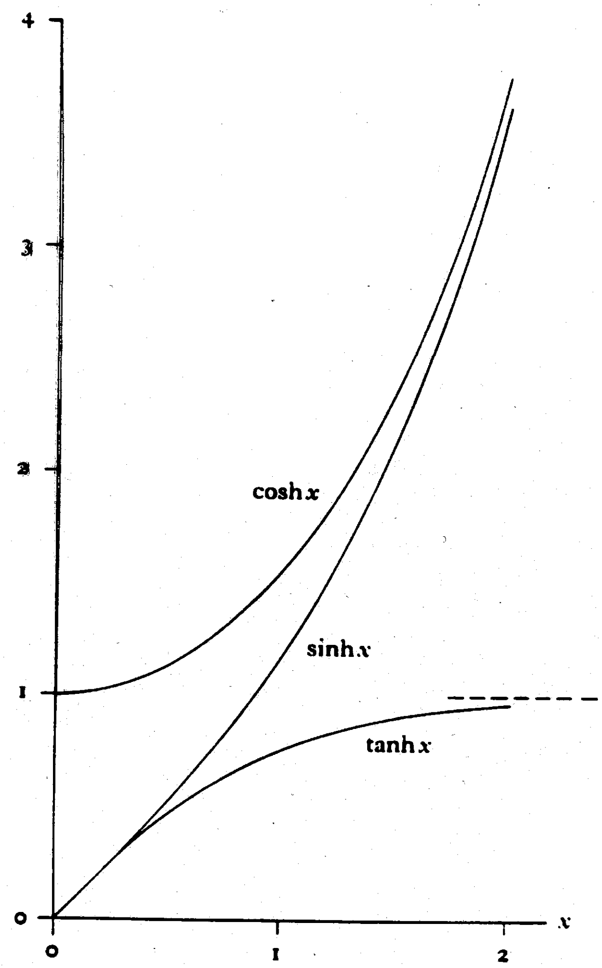

Figure 51. Full lines: graphs of the hyperbolic functions cosh x, sinh x and tanh x. Broken line: asymptotic value, 1, taken by tanh x for large x.

Corresponding η: $\quad \eta = i\,\dfrac{Ak}{\omega}\, e^{i(kx-\omega t)} \sinh(kh)$
(using $\phi_t + g\eta = 0$)

Real part understood

Note that this is a linear theory: small amplitude

(means:)

$$\underbrace{\frac{Ak}{\omega}\left(=\frac{A}{c}\right)}_{\text{(max wave height)}} \ll \underbrace{\frac{1}{k}}_{\propto \text{ wavelength}}$$

To obtain fluid velocity:

$$u = \phi_x = ikA\, e^{ik(x-ct)} \cosh k(y+h)$$

$$v = \phi_y = kA\, e^{ik(x-ct)} \sinh k(y+h)$$

Particle Paths: $\quad \dfrac{dx}{dt} \equiv u \qquad \dfrac{dy}{dt} \equiv v$

Linearize about a point near the surface (x_0, y_0), write in real form, then integrate, get:

$$x - x_0 = -\frac{A}{c} \sin k(x_0 - ct) \cosh k(y_0 + h)$$

$$y - y_0 = -\frac{A}{c} \cos k(x_0 - ct) \sinh k(y_0 + h)$$

Combining $(\sin^2\theta + \cos^2\theta = 1)$

$$\left[\frac{x - x_0}{\frac{A}{c}\cosh k(y_0+h)}\right]^2 + \left[\frac{y - y_0}{\frac{A}{c}\sinh k(y_0+h)}\right]^2 = 1$$

So particle paths are elliptical. T.P.

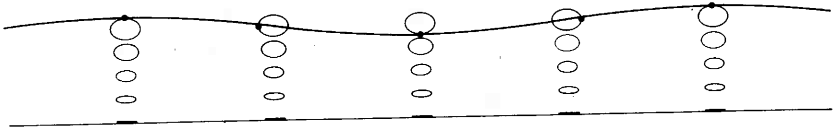

Figure 55. Paths of fluid particles in a sinusoidal wave of length λ travelling from left to right on water of depth $h = 0.16\lambda$. As in figure 50, the maximum surface elevation is 0.02λ. A particle's instantaneous position on its elliptical path is here shown only for those in the top row, but the motion of every particle in the same vertical line is (once more) in phase.

(Limiting Cases:)

(<u>Deep water</u> (wavelength << depth) $kh >> 1$

(we had $\omega^2 = gk \tanh kh$)

($\tanh kh \Rightarrow 1$ T.P.)

$\Rightarrow \boxed{\omega^2 = gk}$ $\boxed{c = \sqrt{\frac{g}{k}}}$

or $c = 1.25\sqrt{\lambda}$ (SI units & earth g)

e.g. (double the wavelength and wave speed goes up by $\sqrt{2}$)

($1 < \lambda < 100m \Rightarrow 1.25 < c < 12.5$ m/s)

using $\phi = A' e^{i(kx-\omega t)} e^{ky}$

noting that for large θ, $\cosh\theta \to \frac{e^\theta}{2}$ and replacing $\frac{1}{2}Ae^{kh}$ by A'

$\eta = \frac{ik}{\omega} A' e^{i(kx-\omega t)}$

$u = ikA' e^{i(kx-\omega t)} e^{ky}$ and $v = kA' e^{i(kx-\omega t)} e^{ky}$ Note constant speed

Also Note the exponential decrease in magnitude of velocity as depth increases:

→ Ex: at a depth equal to λ ($y = -\lambda$), Velocity decreases by a factor of $e^{-2\pi}$ or $\frac{1}{535}$

(Now,) Particle Paths become circular (cosh θ and sinh θ become nearly the same for large θ)

($(x - x_0)^2 + (y - y_0)^2 = \left(\frac{A}{c} e^{ky_0}\right)^2$ T.P.

(Physically, this simply means that as the depth becomes large compared to the wavelength, the effect of the bottom goes away & the particle paths go from elliptical to circular)

(by looking at the limit of Shallow Water, we'll get an understanding of why waves disperse)

<u>Shallow water</u> (wavelength >> h) $kh << 1$

$\tanh kh \to kh$, thus, $\omega^2 = gk(kh) = ghk^2$

($c = \sqrt{gh}$ note: indep. of k!

∴ Shallow water waves are non-dispersive. i.e. waves of every wavelength travel at the same speed, determined by depth. T.P. (the entire range)

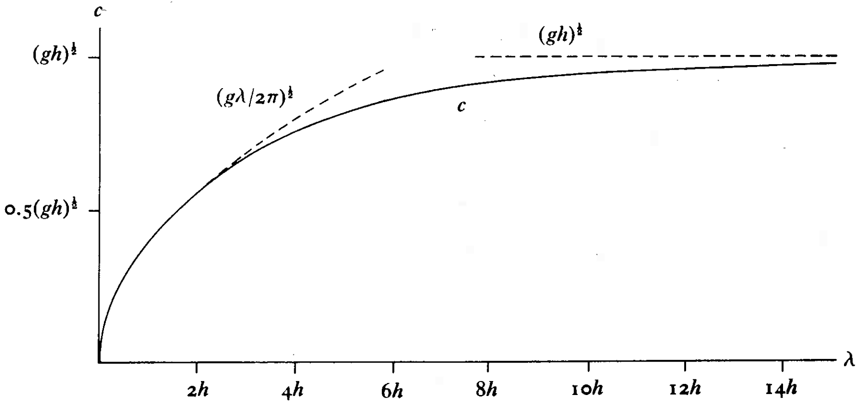

Figure 52. The wave speed c given by linear theory for waves of varying length λ on water of uniform depth h. Note the transition between the deep-water value $(g\lambda/2\pi)^{\frac{1}{2}}$ and the long-wave value $(gh)^{\frac{1}{2}}$.