Solution Set # 3

1) The ith crest in a wave packet would maintain its position if its (phase) speed is equal to the speed of the packet (group velocity).

The phase speed of the waves within each packet is that corresponding to its wavenumber, which is average of k_1 and $(1+\epsilon)k_1$, or $(1+\frac{\epsilon}{2})k_1$, which can be approximated as k_1.

Thus, the phase speed, $C = \frac{\omega}{k} = \frac{\sqrt{gk + \frac{\sigma}{\rho}k^3}}{k} = \sqrt{\frac{g}{k_1} + \frac{\sigma}{\rho}k_1}$

The group velocity, $C_g = \frac{d\omega}{dk} = \frac{1}{2}\left(gk_1 + \frac{\sigma}{\rho}k_1^3\right)^{-1/2}\left(g + \frac{3\sigma}{\rho}k_1^2\right)$

Equating these (and denoting k_1 as k), $C_g = C$ or:

$$\frac{1}{2}\left(gk + \frac{\sigma}{\rho}k^3\right)^{-1/2}\left(g + \frac{3\sigma}{\rho}k^3\right) = \sqrt{\frac{g}{k} + \frac{\sigma}{\rho}k}$$

Symplifying, get $k = \sqrt{\frac{g\rho}{\sigma}}$, which is also the wavenumber for k_m, i.e. waves with minimum phase speed.

For water ($\rho = 1\ \text{gram/cm}^3$ and $\sigma = 72$ dynes/cm),

$$k_m = \sqrt{\frac{(981)(1)}{72}} \doteq 3.7\ \text{cm}^{-1} \quad \text{or} \quad \lambda = \frac{2\pi}{k} = 1.7\ \text{cm}$$

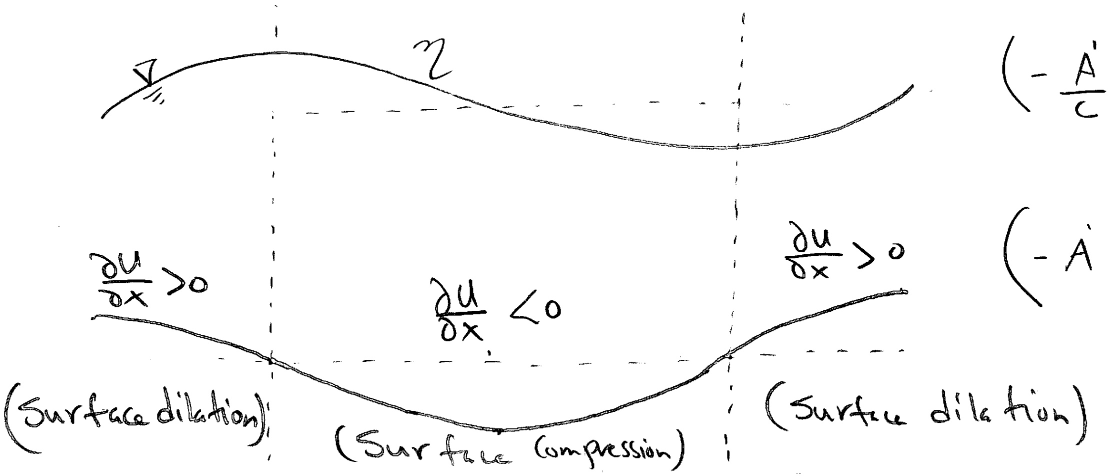
This excercise shows that the wave energy and therefore the wave packet moves at the same speed as the wave crests only when the wavelength corresponds to that of the minimum (phase) wave speed. This is consistent with the fact that those waves are equally affected by gravitional forces (with the corresponding group velocity of $\frac{1}{2}C$) and by capillary forces (with the corresponding group velocity of $\frac{3}{2}C$).

2) The surface divergence can be described by $\frac{\partial u_s}{\partial s}$ where u_s is the surface velocity along the natural (surface) coordinate s. For small amplitude (linear) waves, the surface divergence can be estimated by: $\frac{\partial u_s}{\partial s} \approx \frac{\partial u}{\partial x}$. This strain rate can be computed from the given potential:

$$\phi = A' \cos(kx-\omega t) e^{ky}$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial \phi_x}{\partial x} = \phi_{xx} \Rightarrow \frac{\partial u}{\partial x} = -A' k^2 \cos(kx-\omega t)$ at $y=0$

Therefore, for the wavelength shown:



$\left(-\frac{A'}{c} \sin(kx-\omega t)\right)$

$\left(-A' k^2 \cos(kx-\omega t)\right)$

For standing waves there will be an additional sinusoidal variation with t.

3) writing the ideal gas law ($PV = mRT$) for each bubble:

(because the bubble has 2 interfaces, one inside and one outside)

bubble ①: $\left(P_0 + \frac{4\sigma}{a_1}\right)\left(\frac{4}{3}\pi a_1^3\right) = m_1 RT$

" ② $\left(P_0 + \frac{4\sigma}{a_2}\right)\left(\frac{4}{3}\pi a_1^3\right) = m_2 RT$

Coalesced bubble $\left(P_0 + \frac{4\sigma}{r}\right)\left(\frac{4}{3}\pi r^3\right) = (m_1 + m_2) RT$

(where P_0 is the pressure outside the bubbles)

Combine the three equations, get:

$$\left(P_0 + \frac{4\sigma}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(P_0 + \frac{4\sigma}{a_1}\right)\left(\frac{4}{3}\pi a_1^3\right) + \left(P_0 + \frac{4\sigma}{a_2}\right)\left(\frac{4}{3}\pi a_2^3\right)$$

Simplify to get:

$$P_0 r^3 + 4\sigma r^2 = P_0 (a_1^3 + a_2^3) + 4\sigma (a_1^2 + a_2^2)$$

Q.E.D.

4 a) $\Delta P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$; here, $R_1 = R$ and $R_2 = \frac{H/2}{\cos\theta_E}$ (since meniscus shape must be circular arc, in the absence of gravity)

$\Rightarrow \underline{\Delta P = \sigma\left(\frac{1}{R} - \frac{\cos\theta_E}{H/2}\right)}$ b) If $H \ll R$, $\underline{\Delta P = -\frac{2\sigma \cos\theta_E}{H}}$

c) force becomes repulsive (i.e. pressure inside capillary bridge becomes larger than atmospheric pressure)

When $\cos\theta_E$ flips sign $\Rightarrow$ $\underline{\theta_E > \pi/2}$ marks change from attraction to repulsion

d) $\Delta P = 0.072\left(\frac{1}{0.01} - \frac{1}{2.5\times10^{-6}}\right) = \underline{-28,800 \text{ Pa}}$ ($\approx 1/3$ atm) ; $F = \Delta P . A = \Delta P \pi R^2 = \underline{9 \text{ N}}$

(Note that this force is enough to hold the weight of ~1 liter of water, a million times the weight of the liquid bridge itself)