Incompressible Flow MANE-6560
(Hydrodynamic Theory)

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Syllabus (Lugt & Lighthill: both good reading, especially Lugt - gives fresh Perspective on fluid mechanics)

Why study advanced fluid mechanics?
See nice write-up in <u>Trends in Fluid Mechanics</u> about applications

"Vortices are the muscles & sinews of fluid motion" Kuchemann. (T.P.) transpirencies
→ instablity (very important)

(To be able to do original reseach in fluid mechanics, i.e. basic reseach, need at least a reasonable understanding of several areas in fluid mech.:
(typical course work:)

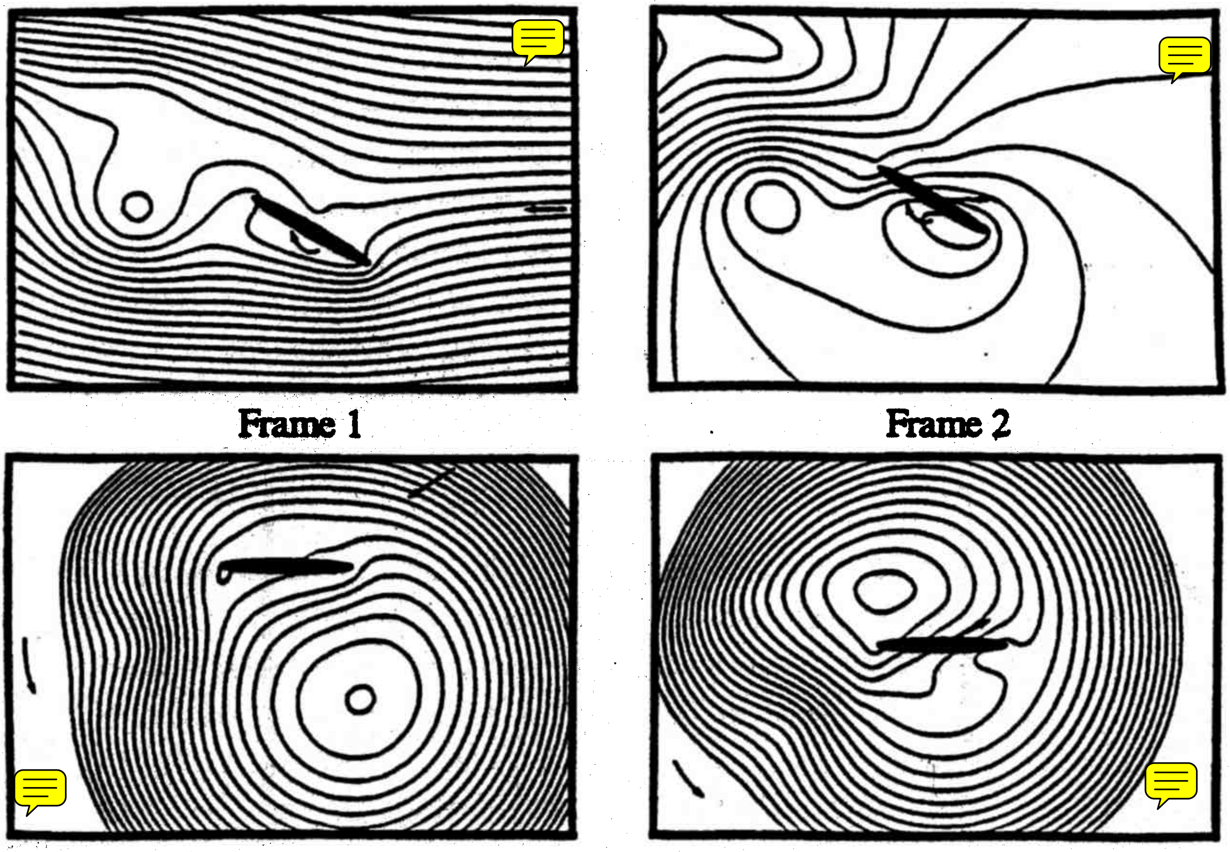
- Fluid mechanics (I)
- Viscous flow & boundary layer theory ⇓
- Turbulence
- Compressible flow
- Kinetic theory
- Continuum theory of fluids ←--- (hydrodynamics, including interfacial hydrodynamics; Present course but with emphasis on the physics of incompressible flow, including modern developments)
- Combustion
- Aerodynamics
- ...

↓ Specialty courses

(these are all interconnected & help in clarifying the "big Picture" no mater what areas of fluid mechanics you work in, be it Convective htx, viscous flow, turbulence, combustion, microfluidics, etc.)

<u>Incompressible flow</u>: Density changes following the fluid element are negligible.
(flow in the atmosphere & oceans is, for the most Part, incompressible)
(Strictly speaking, even in a slowly flowing water stream there are density changes, but these are negligible & not dynamically important to the flow

(Even for a gas, density (ρ) changes very little if Mach number of flow is small.
To see why) Consider the isentropic flow equations for Stagnation density, ρ_0



Frame 1

Frame 2

Frame 3

Frame 4

Fig. 1.10: Computer-generated streamline patterns for the flow past a rotating elliptic cylinder (wing) in four different reference frames (Lugt and Ohring 1977).

Lugt (1996)

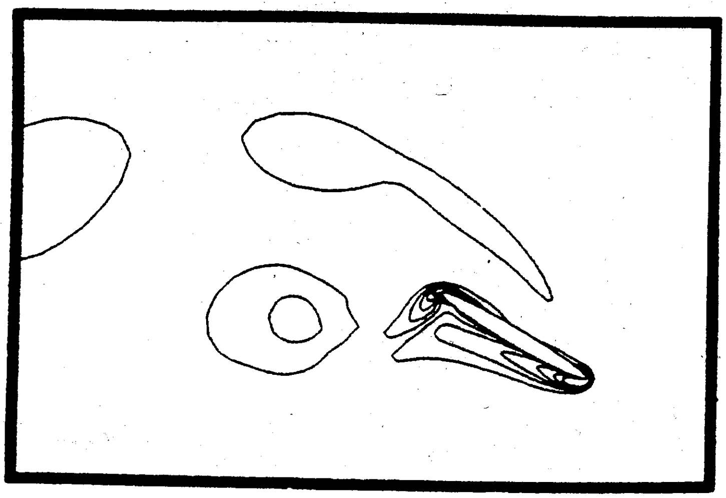


Fig. 1.11: Equivorticity lines for the flow situation displayed in Fig. 1.10 (Lugt and Ohring 1977).

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}$$ where $M = \frac{V}{a}$ Mach number

V: Characteristic Velocity Scale

a: Speed of sound

(eqn. applies essentially if inviscid & as long as we are not doing work to our ideal gas, e.g. in a compr. or turbine it can fail, but not always so) mangle

$\gamma = \frac{c_p}{c_v}$ Ratio of Constant Pressure to constant Volume specific heat (≈ 1.4 @room temp for diatomic gas, like air

As long as flow speed is < 100 m/s (224 mph) with corresponding $M < 0.3$ for air,

$$\frac{\Delta \rho}{\rho} < 5\%$$ & compressibility is not very important

And for liquids, speed of sound is even greater, e.g. $V = 100$ m/s in water corresponds to only $M = 0.07$ (compressibility unimportant) (not to say that this eqn applies in water!)

Note: Incompressible flow does not necessarily mean uniform density, i.e. it can include heterogeneous flow:

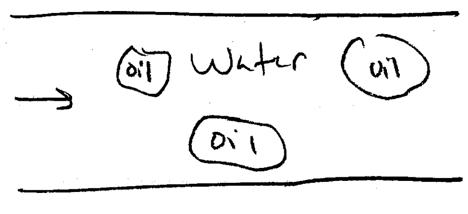
Consider Continuity eqn.: $$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{V}) = 0$$ for any fluid, Compressible or otherwise

$$\Rightarrow \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \operatorname{grad} \rho}_{\frac{D\rho}{Dt}} + \rho \operatorname{div} \vec{V} = 0$$

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \vec{V} = 0$$

Incompressible flow implies $\frac{D\rho}{Dt} = 0$ i.e. Substantial (aka "Eulerian" or "material") derivative of density is zero, and not always $\frac{\partial \rho}{\partial t} = 0$, since in an incompressible flow of an inhomogeneous flow, $\frac{\partial \rho}{\partial t} \neq 0$.

Example: two-phase flow of oil & water



density at a fixed location ≠ const

$$\frac{\partial \rho}{\partial t} \neq 0$$

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For most of our work we'll be concerned with incompressible flow of homogeneous fluid, ρ=constant ($\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} = 0$), thus continuity reduces to:

$$\text{div } \vec{V} = 0$$

(in vector notation, and in index or tensorial form:)

$$\frac{\partial u_i}{\partial x_i} = 0$$

(Summation over repeated indices is implied)

e.g., in Cartesian coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

I) Surface Waves

(See general refs. in syllibus)

Lighthill Waves in Fluids (Chap3

Yih Fluid Mechanics

Lamb Hydrodynamics

Whitham Linear & nonlinear Wave

(Note that sound waves and their nonlinear counterpart, shock waves, are studied in the class compressible flow theory. When fluid is incompressible, $a \to \infty$ and sound waves cease to be a useful concept. Here we consider surface waves, since their most important application, namely water waves, is a major topic of incompressible flow, in part because > 70% of the surface of our planet is covered with water and that the transport of mass, momentum & energy between the ocean and the atmosphere greatly depends on water waves) & aerosols/sprays
(e.g. breaking waves which cause "white caps" in ocean greatly enhance transport of CO_2)

(Water waves are in some ways more difficult than sound waves, even for the case of linear water waves, due to the

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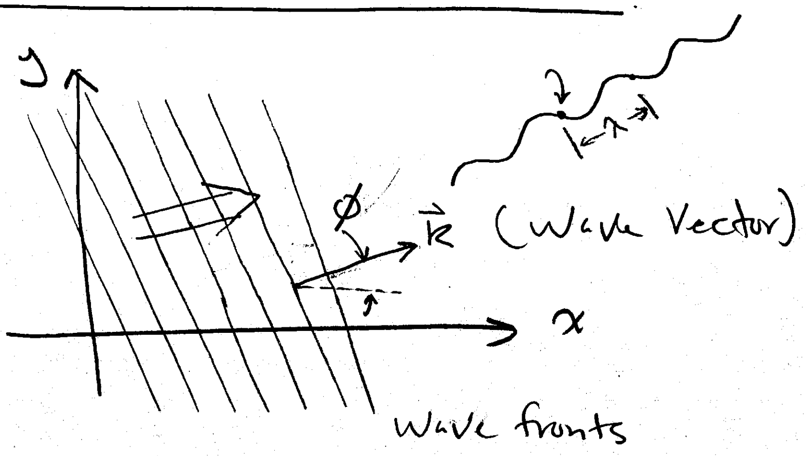
(...dispersive nature.: Similarly, nonlinear water waves are more difficult to analyze than nonlinear waves in the interior of a fluid, such as shock waves, again due to dispersio

(first, some) <u>General concepts about waves:</u>

(in 2-D)



Wave vector: * Points in the direction of wave propagation

* its magnitude, $|\vec{k}|$ = "Wavenumber"

Spatial frequency (# of radians) / Unit length (at a fixed tim

its relation to wavelength, λ: $k = \frac{2\pi}{\lambda}$

(a wave can be represented by:)

$$\xi(\underset{\substack{\uparrow\\ \text{location}\\ \text{in 3-D}\\ \text{space}}}{\vec{x}},\underset{\substack{\uparrow\\ \text{time}}}{t}) = \underbrace{A_{k,\omega}}_{\substack{\text{amplitude:}\\ \text{function of}\\ k,\omega}} \underbrace{e^{i(\vec{k}\cdot\vec{x}-\omega t)}}_{\text{Phase}}$$

(from Complex Variables,) Recall:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\leftarrow$ (a real quantity)

(Can think of this as just a definition of $e^{i\theta}$)

here,

$$\theta(\vec{x},t) = \vec{k}\cdot\vec{x} - \omega t \equiv \text{Phase function}$$

and $\vec{k}\cdot\vec{x} = kx\cos\phi + ky\sin\phi$

ω is temporal frequency (# of radians/unit time) at a fixed locatio

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ϕ is angle of propagation

To ride on a fixed position of wave, e.g. crest,

$$\theta(\vec{x},t) = \text{const.}$$

$$\Rightarrow d\theta = \vec{K}\cdot d\vec{x} - \omega dt = 0$$

in one-dim.:

$$= K\,dx - \omega dt = 0$$

hence speed of propagation is: $\frac{dx}{dt} = \left(\frac{\omega}{k}\right) \equiv c$ "Phase speed"

Speed you have to move in the direction of the wave vector in order to remain at a particular part of the wave.

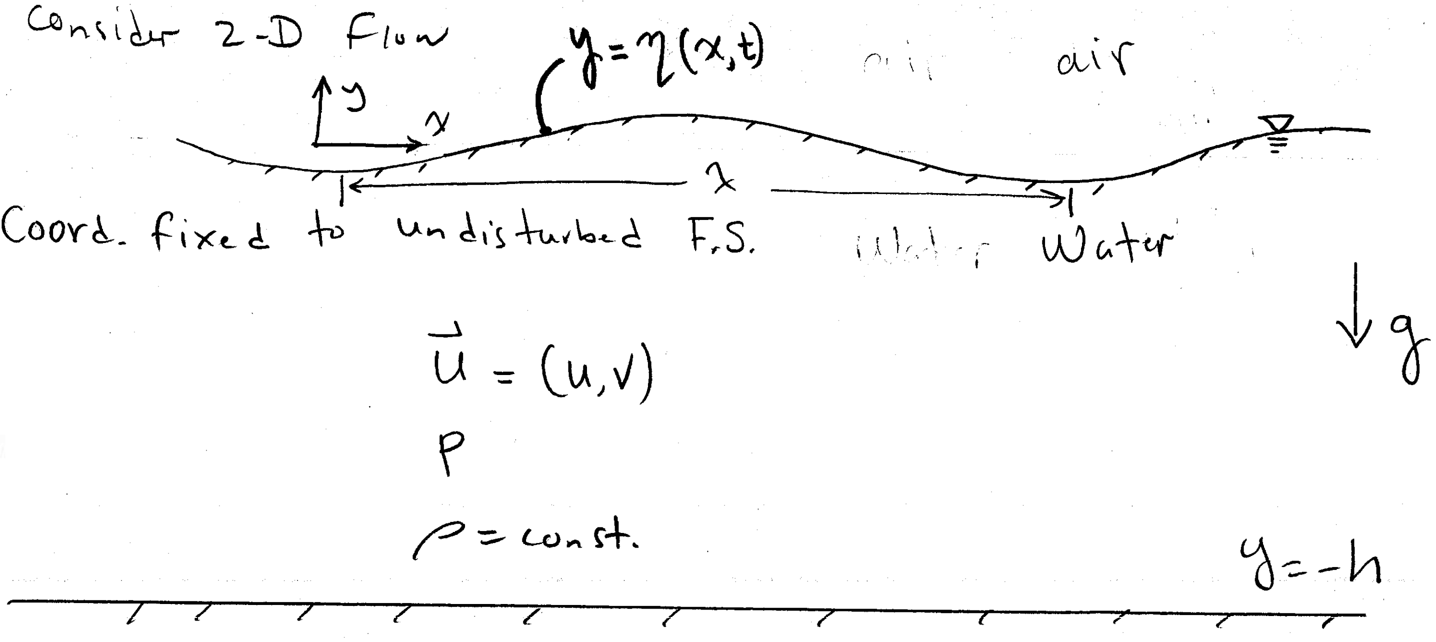
(end of wave review)

(We begin our study of surface waves by considering:)

Linear dispersive waves (we'll consider nonlinearity later)

(Small amplitude waves on the surface of any liquid under the influence of gravity)

Consider 2-D Flow



$\vec{u} = (u,v)$

p

$\rho = \text{const.}$

first Q:
(Why a traveling wave?)

$P_1 = P_{atm}$ $\qquad P_2 = P_{atm} + \rho g \eta$ (in absence of surface tension, considered later, or motion, considered next)

fluid at 2 will try to move to the right (say)

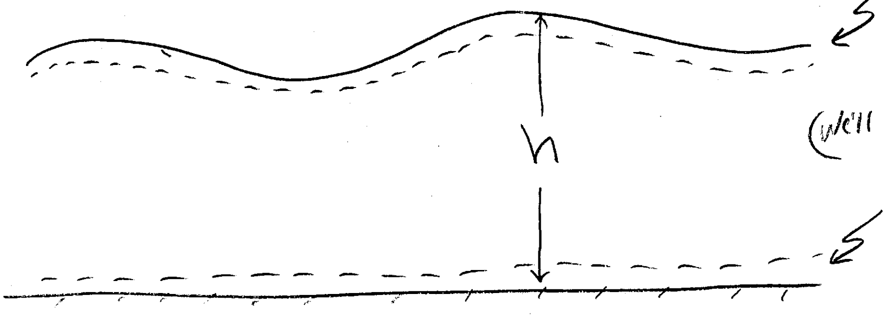
(but more fundamentally, why a wave in the first place?)

(obviously, the f.s. of a liquid in equilibrium in a gravitational field is a plane. If under some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. And like a pendulum, when perturbed, gravity does not merely return the surface back to equilibrium, but overshoots that point and creates a wave motio

(for this analysis, at least for the time being) ↓ 1-23-11

⇒ Assume $\mu = 0$. (To be exact,) When Reynolds number, $Re \left(= \frac{Uh}{\nu}\right)$, where $\nu = \frac{\mu}{\rho}$ is large enough, the viscous effect is concentrated at 2 boundary regions:

recall dim. less NS: $\frac{\partial \vec{u}^*}{\partial t} + (u^* \cdot \nabla) u^* = -\nabla p^* + \frac{1}{Re} \nabla^2 u^*$ $\boxed{T.P.}$



this BL exists since the shear stresses have to become zero, if clea
(We'll discuss this limitation more later)

this BL is due to oscillatory motio next to a no-slip surface and does not grow with x.
(Similar to Stokes' 2nd Problem)

(must have)

$$Re = \frac{Uh}{\nu} \gg 1$$

What to use for U?

(turns out that)
Particle velocity will give a deceptively low Re.
Instead, take a different point of view: one that makes the flow steady state;

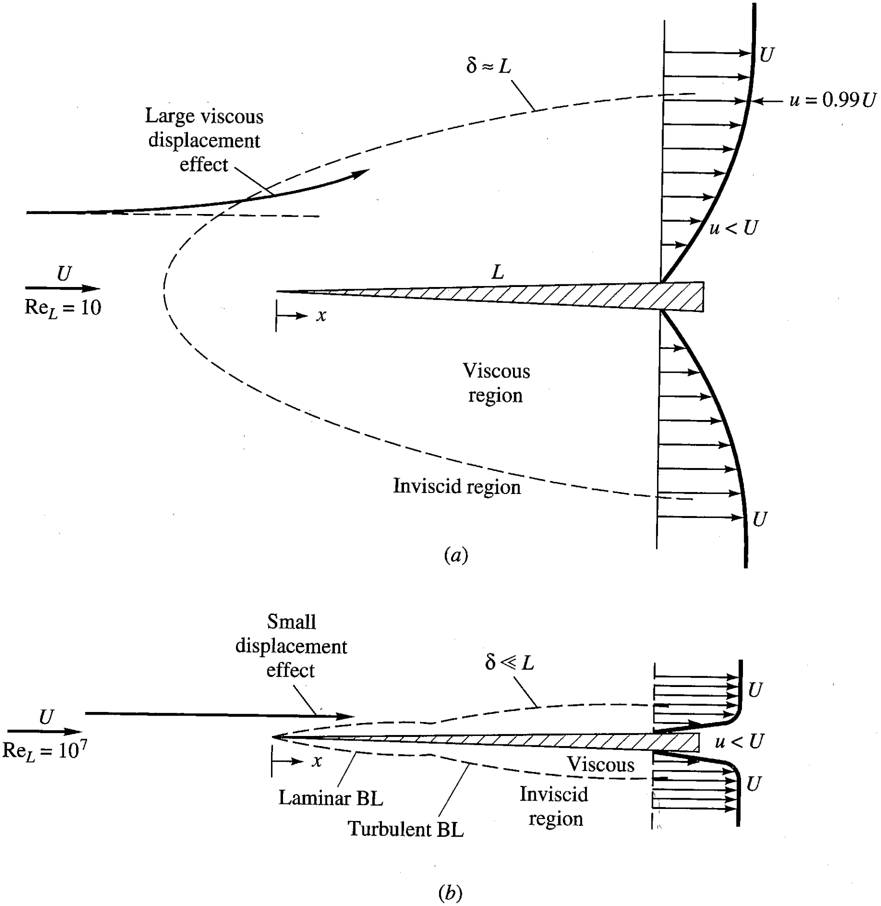


Fig. 7.1 Comparison of flow past a sharp flat plate at low and high Reynolds numbers: (*a*) laminar, low-Re flow; (*b*) high-Re flow.