II) Flow Instabilities

(define) Stability: $\text{Velocity} = U e^{i(kx-\omega t)} = U e^{ik(x-ct)}$

& $e^{i\theta} = \cos\theta + i\sin\theta$ (Polar notation)

∴ Temporal Stability Problem:

$$\boxed{\begin{array}{l} k \text{ real} \\ \hline \omega \text{ complex } (= \omega_r + i\omega_i) \end{array}}$$

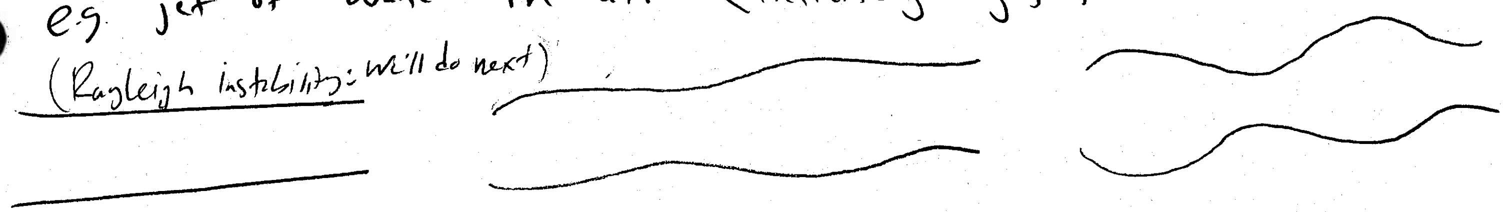
i.e. $$\text{Velocity} = \underbrace{\left\{ U e^{i(kx-\omega_r t)} \right\}}_{\substack{\text{original} \\ \text{wave}}} \underset{\substack{\uparrow \\ \text{growth} \\ \text{rate}: \ \omega_i}}{e^{-\omega_i t}}$$

∴ $\omega_i > 0$ instability in time

$\omega_i < 0$ dampening in time

e.g. jet of water in air (infinitely long jet, start it up & observe)

(Rayleigh instability: will do next)



time ⟶

Spatial stability Problem

$$\boxed{\begin{array}{l} k \text{ complex } (= k_r + ik_i) \\ \hline \omega \text{ real} \end{array}}$$

get $$\text{Velocity} = \left\{ \quad \right\} \underset{\substack{\uparrow \\ \text{growth rate}: \ -k_i}}{e^{-k_i x}}$$

∴ $k_i < 0$ instability (growth w/ distance)

$k_i > 0$ dampening (w/ distance)

$\therefore$ in temporal stability $c = \dfrac{\omega_r}{k}$ is phase speed

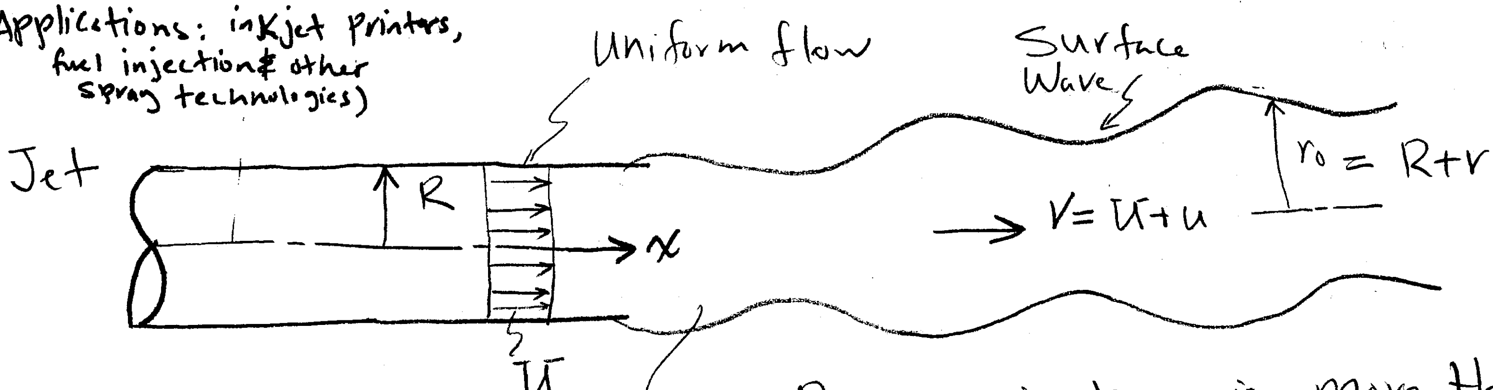
" spatial " $c = \dfrac{\omega}{k_r}$

(Consider the)

a) Stability of a liquid jet (in gas) to capilary forces: 1/9

"Rayleigh"

(Applications: inkjet printers, fuel injection & other spray technologies)



Pressure in here is more than P_{atm} because of Surface tension (but clearly, r and u can be positive or negative)

Assume Inviscid Flow

(this flow physically, appears to be unstable, since a perturbation which causes a narrowing of the jet → flow will accelerate → Pressure will decrease → surface tension will narrow the jet further.) (but this doesn't tell you if all wavelengths are equally unstable, or how fast instabilities grow)

(to do this formally, we must) Consider the possibility of a wave growing in time.

$V = U + u$ ← fluctuations (about the mean)

U ↑ mean (constant)

$r_0 = R + r$

assume Velocity at any point is same across the jet (for a given downstream location) at any instant

(linear analysis) $|r/R| << 1$, $|u/U| << 1$ } (small perturbations)

$r = r(x,t)$, $u = u(x,t)$ (i.e. not dep. on r)

Also assume that pressure is uniform across the jet at any instant so $P = P(x,t)$

The equations of motion (for incompressible & inviscid fluid)

(Cartesian coord:)

mass: $\dfrac{\partial r_0^2}{\partial t} + \dfrac{\partial r_0^2 V}{\partial x} = 0$ (obtained by considering a thin disc-shaped control volume and neglecting higher order terms)

note that this is non-linear ($r_0^2 V$)

x-momentum: $\dfrac{\partial V}{\partial t} + V\dfrac{\partial V}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}$ (ordinary approach to mom.)

(this is also non-linear)

(Can't do much with these eqns. since they are non-linear pde's)

We can linearize the equations if we consider small perturbations (critical step)

substitute $r_0 = R + r$ $V = U + u$ and linearize

Mass: $\dfrac{\partial (R^2 + 2rR + r^2)}{\partial t} + \dfrac{\partial (R^2 + 2rR + r^2)(U+u)}{\partial x} = 0$

$R^2 = \text{const}$; $2rR + r^2 = 2R^2\left[\dfrac{r}{R} + \dfrac{1}{2}\left(\dfrac{r}{R}\right)^2\right]$

neglect (compared to $\frac{r}{R}$)

Find: (linearized eqns. of motion)

$$\frac{\partial r}{\partial t} + U\frac{\partial r}{\partial x} + \frac{R}{2}\frac{\partial u}{\partial x} = 0 \qquad \text{(Continuity)}$$

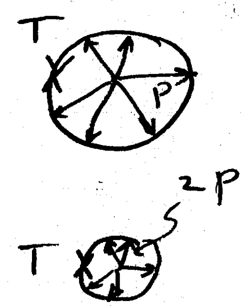
$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(x-momentum)}$$

(instability requires surface tension)

Surface tension becomes important when jet is small.

(because pressure resulting from surface tension is inversely proportional to curvature of the jet)

(think of it as a cable stretched around a cylinder; for a fixed cable tension, the pressure exerted on the cylinder will double if the cylinder radius was made in half)



If the jet was large, P would just equal P_{atm} (& the flow appears to be stable)

(We are interested in cases when diameter is small) and flow is possibly unstable

(Formally,)

The pressure in the jet above the atmosphere is: $P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

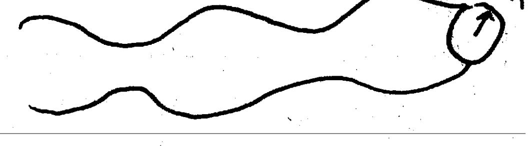
"Laplace's formula"

σ is the surface tension
(mN/m or dynes/cm)

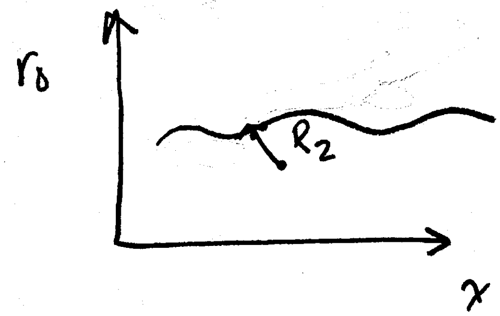
R_1 & R_2 are the principal radii of curvature.

R_1 = radius in the plane normal ($\perp$) to the x-axis

$$\Rightarrow R_1 = r_0$$



R_2 is radius of curvature in the plane which contains x-axis :



(From Calculus)

$$R_2 = \frac{-\left[1+\left(\frac{\partial r_0}{\partial x}\right)^2\right]^{3/2}}{\left(\frac{\partial^2 r_0}{\partial x^2}\right)} \quad ; \quad \text{linearize} \Rightarrow R_2 = \frac{-1}{\left(\frac{\partial^2 r_0}{\partial x^2}\right)}$$

Then the momentum eqn. becomes

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{\sigma}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{r_0} - \frac{\partial^2 r_0}{\partial x^2}\right)$$

or

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u = \frac{\sigma}{\rho}\left[\frac{1}{R^2}\frac{\partial r}{\partial x} + \frac{\partial^3 r}{\partial x^3}\right]$$

from Continuity:

$$\left(\frac{\partial}{\partial x}\right)u = -\frac{2}{R}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)r$$

elliminating u:

$$\underbrace{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 r}_{} = -\frac{\sigma R}{2\rho}\left(\frac{1}{R^2}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^4 r}{\partial x^4}\right)$$

$$\frac{\partial^2 r}{\partial t^2} + 2U\frac{\partial^2 r}{\partial x \partial t} + U^2\frac{\partial^2 r}{\partial x^2}$$

↓ $\therefore$ need 2 conditions in t
and 4 " in r

(Now we have a single Pde that describes r as a function of x & t)

introduce a disturbance of the form:

$$r = a e^{i(kx - \omega t)}$$

(5)

(We want to see what happens to it, does it grow or decay)

here, a is amplitude, k is real (since interested in temporal instability) is the wave number (and $\lambda = \frac{2\pi}{k}$ is the wave length)

ω is complex, ω_r is the angular freq. (as before) of oscillation, ω_i is the amplification factor (or growth rate) since:

$$r = a e^{\omega_i t} e^{i(kx - \omega_r t)}$$

(for $\omega_i < 0$ disturbances dampened
$\omega_i > 0$ " grow with time)

Also note $c_r = \frac{\omega_r}{k}$ is the propagation rate of wave in the x-dir. (phase speed)

Substitute the assumed form for r (into the pde for r & obtain)

$$(\omega - Uk)^2 = \frac{\sigma k^2}{2\rho R}(k^2 R^2 - 1)$$

(Since ω_i indicates the stability,) Solve for ω

(find for waves moving to the right:)

$$\omega = Uk + \sqrt{\frac{\sigma}{2\rho R^3}}\, kR \sqrt{k^2 R^2 - 1} = \omega_r + i\omega_i$$

Observe

1) for $kR \gg 1$ (then ω is always real $\omega_i = 0$)

disturbance is neutrally stable

(Since $k = \frac{2\pi}{\lambda}$, find:) $\lambda \leq 2\pi R$

Short waves are neutrally stable, neither grow nor decay.

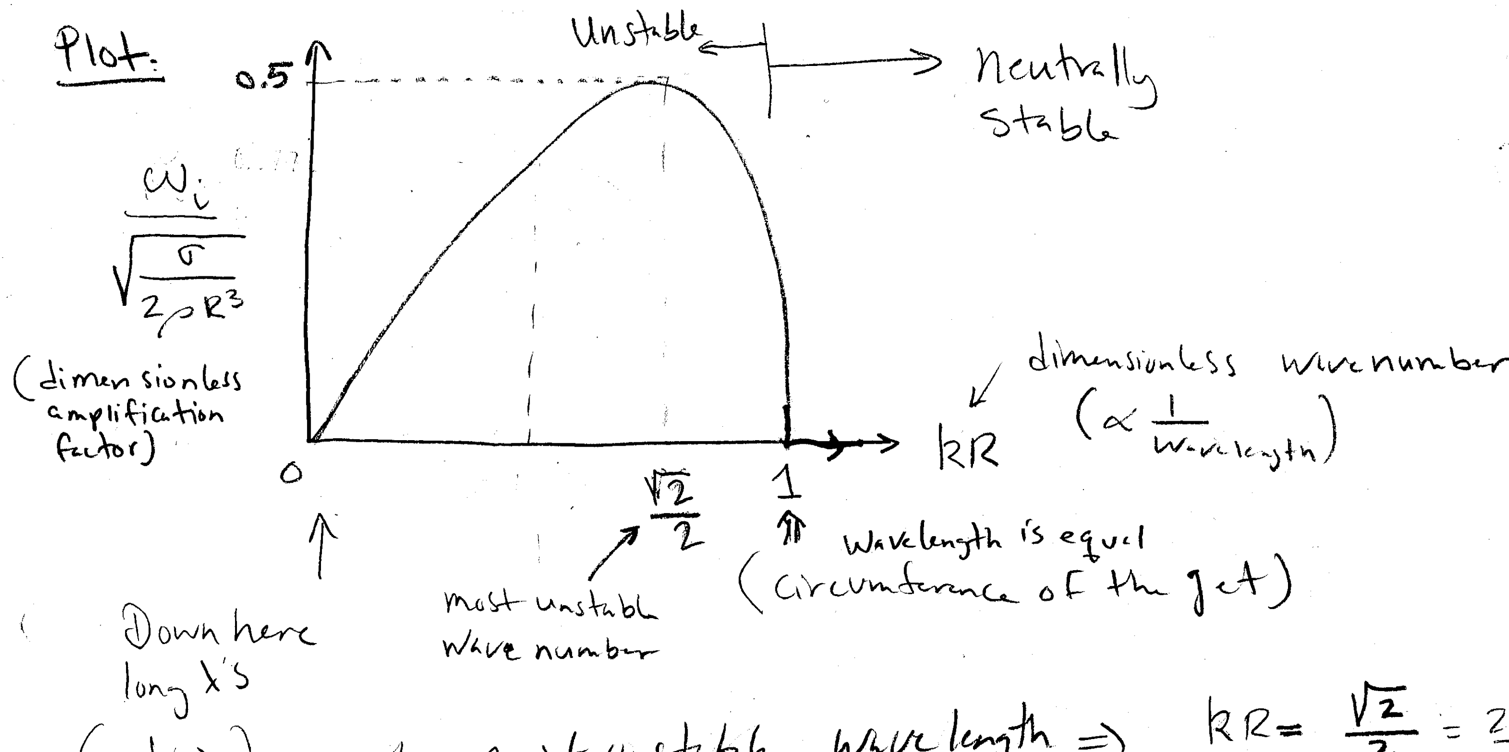
(these are capillary waves on a liquid jet : can be easily observed)

2) for $kR < 1$ $\quad \omega = Uk + i\sqrt{\frac{\sigma}{2\rho R^3}}\, kR\sqrt{1-k^2R^2}$

Find $c_r = \frac{\omega_r}{k} = U$ i.e. waves travel with the jet

$\omega_i = \sqrt{\frac{\sigma}{2\rho R^3}}\, kR\sqrt{1-k^2R^2} > 0 \quad \therefore$ disturbances grow

Plot:



The most unstable wavelength $\Rightarrow$ $kR = \frac{\sqrt{2}}{2} = \frac{2\pi R}{\lambda}$

$\lambda = 4.44$ Diameters

(Its amplification factor is : $\tilde{\omega}_i = 0.5 \sqrt{\frac{\sigma}{2\rho R^3}}$)

(Physical interpretation :) ⑦

(· Since there is no viscosity, waves can't dampen.
Very short waves (capillary waves) aren't affected by the jet to make it unstable ∴ they are neutrally stable.
long waves affect the pressure inside the jet & can grow.)