Divergence Theorem (Gauss' Theorem)

Let S be a closed surface, with outward unit normal $\mathbf{n}$, bounding the volume V. Then if $\mathbf{F}(\mathbf{x},t)$ is a *vector* field, the divergence theorem is

$$\int_S \mathbf{n} \cdot \mathbf{F} \, dA = \int_V \nabla \cdot \mathbf{F} \, dV \tag{B.29}$$

where dA is an increment of area on S and dV is an increment of volume in V. This can also be written

$$\int_S n_i F_i \, dA = \int_V F_{i,i} \, dV \tag{B.30}$$

If $F(\mathbf{x},t)$ is a *scalar* field, the divergence theorem is

$$\int_S \mathbf{n} F \, dA = \int_V \nabla F \, dV \tag{B.31}$$

This can be derived directly from (B.29). The corresponding indicial form is

$$\int_S \mathbf{e}_i n_i F \, dA = \int_V \mathbf{e}_i F_{,i} \, dV \tag{B.32}$$

with the component statements

$$\int_S n_i F \, dA = \int_V F_{,i} \, dV \tag{B.33}$$

If $\mathsf{F}(\mathbf{x},t)$ is a *tensor* field, the divergence theorem in dyadic notation is [compare (B.29)]

$$\int_S \mathbf{n} \cdot \mathsf{F} \, dA = \int_V \nabla \cdot \mathsf{F} \, dV \tag{B.34}$$

This can be put in indicial notation from the correspondences $\mathbf{n} \cdot \mathbf{F} \sim n_i F_{ik}$ and $\nabla \cdot \mathbf{F} \sim F_{ik,i}$, yielding [compare (B.30)]

$$\int_S n_i F_{ik}\, dA = \int_V F_{ik,i}\, dV \tag{B.35}$$

Stokes' Theorem

Let S be a surface bounded by the simple closed curve C. Then if $\mathbf{F}(\mathbf{x},t)$ is a continuous vector field, Stokes' theorem is

$$\oint_C \mathbf{F} \cdot \mathbf{t}\, dl = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \tag{B.36}$$

where $\mathbf{n}$ is a unit vector normal to S, $\mathbf{t}$ is a unit vector tangential to C, and dl is an increment of length along C.

Determinant Expansion for the Curl

In Cartesian coordinates, if $\mathbf{F}(\mathbf{x},t)$ is a vector field

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{B.37}$$

Vector Identities[1]

$$\nabla(uv) = u\, \nabla v + v\, \nabla u \tag{B.38}$$

$$\nabla \cdot (\phi \mathbf{v}) = \phi \nabla \cdot \mathbf{v} + \nabla \phi \cdot \mathbf{v} \tag{B.39}$$

$$\nabla \times (\phi \mathbf{v}) = \phi \nabla \times \mathbf{v} + \nabla \phi \times \mathbf{v} \tag{B.40}$$

$$\nabla \times (\nabla \phi) = 0 \tag{B.41}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0 \tag{B.42}$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{u} \tag{B.43}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\mathbf{v} \tag{B.44}$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \tag{B.45}$$

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} \tag{B.46}$$

$$(\mathbf{v}\cdot\nabla)\mathbf{r} = \mathbf{v} \tag{B.47}$$

$$\nabla\cdot\mathbf{r} = 3 \tag{B.48}$$

$$\nabla\times\mathbf{r} = 0 \tag{B.49}$$

$$\nabla\cdot(r^{-3}\mathbf{r}) = 0 \tag{B.50}$$

$$d\mathbf{f} = (d\mathbf{r}\cdot\nabla)\mathbf{f} + \frac{\partial\mathbf{f}}{\partial t}\,dt \tag{B.51}$$

$$d\phi = d\mathbf{r}\cdot\nabla\phi + \frac{\partial\phi}{\partial t}\,dt \tag{B.52}$$