MATH 4600: ADVANCED CALCULUS
Fall 2018

TEST I SOLUTIONS

NAME (Please print) ______________________________

NOTES

1. Please make sure that your answer book has 8 pages. The worksheets at the end are extra pages should you need them.
2. Attempt all four problems; these are equally weighted.
3. **Read the questions carefully before answering**.
4. If you would like full credit, then **justify your answers with appropriate, but brief, reasoning**.
5. Books, notes, crib sheets and calculators are not to be used.
6. Put your mobile devices away.
7. Best wishes.

1	
2	
3	
4	
TOTAL	

1. (a) Consider the path

$$\boldsymbol{r}(t) = \langle t\cos\pi t,\ t\sin\pi t,\ t\rangle\,.$$

Where will the tangent line to the path at $t = 5$ intersect the xy-plane?

(b) Find the shortest distance between the line L_1 determined by the points $A(-1,-1,1)$ and $B(0,0,0)$ and the line L_2 determined by the points $C(0,-2,0)$ and $D(2,0,5)$.

(a) The tangent vector at t is

$$\boldsymbol{r}'(t) =< \cos\pi t - \pi t\sin\pi t,\ \sin\pi t + \pi t\cos\pi t, 1 > .$$

At $t = 5$,

$$\boldsymbol{r}'(5) =< -1,\ -5\pi, 1 > .$$

A point on the line at $t = 5$ is

$$\boldsymbol{r}(5) =< -5, 0, 5 > .$$

Equation of the tangent line is

$$\boldsymbol{r}(s) = \boldsymbol{r}(5) + s\boldsymbol{r}'(5) =< -5, 0, 5 > +s < -1,\ -5\pi, 1 >=< -5 - s,\ -5\pi s,\ 5 + s > .$$

Intersection with the xy-plane corresponds to $z = 5 + s = 0$, or $s = -5$. Then the point is

$$\boxed{\boldsymbol{r}(-5) =< 0, 25\pi, 0 > .}$$

(b) A vector along L_1 is $\boldsymbol{v}_1 =< 0,0,0 > - < -1,-1,1 >=< 1,1,-1 >$.
A vector along L_2 is $\boldsymbol{v}_2 =< 0,-2,0 > - < 2,0,5 >=< -2,-2,-5 >$.
A vector perpendicular to both L_1 and L_2 is

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ -2 & -2 & -5 \end{bmatrix} =< -7, 7, 0 > .$$

The corresponding unit vector is

$$\boldsymbol{n} = \frac{1}{\sqrt{2}} < -1, 1, 0 > .$$

The shortest distance between the lines is

$$|\vec{AC}\cdot\boldsymbol{n}| = |< 1,-1,-1 > \cdot \frac{1}{\sqrt{2}} < -1,1,0 > | = \sqrt{2}.$$

2. (a) Use the chain rule to find $D(\boldsymbol{f} \circ \boldsymbol{g})(-2,1)$ for

$$\begin{aligned} \boldsymbol{f}(u,v,w) &= < v^2+uw, u^2+vw, u^2-v^2 >, \\ \boldsymbol{g}(x,y) &= < xy, x^2-y^2, 3x+5y > . \end{aligned}$$

(b) A boat is sailing Northeast at 20 km/hour. Assume that the temperature drops at a rate of $0.2°$ C per km in the northerly direction and $0.3°$ C per km in the easterly direction. What is the time rate of change of temperature as observed on the boat?

(a) According to the chain rule,

$$D(\boldsymbol{f} \circ \boldsymbol{g}) = D\boldsymbol{f}\ D\boldsymbol{g}.$$

Now,

$$D\boldsymbol{f} = \begin{bmatrix} f_{1_u} & f_{1_v} & f_{1_w} \\ f_{2_u} & f_{2_v} & f_{2_w} \\ f_{3_u} & f_{3_v} & f_{3_w} \end{bmatrix} = \begin{bmatrix} w & 2v & u \\ 2u & w & v \\ 2u & -2v & 0 \end{bmatrix},$$

$$D\boldsymbol{g} = \begin{bmatrix} g_{1_x} & g_{1_y} \\ g_{2_x} & g_{2_y} \\ g_{3_x} & g_{3_y} \end{bmatrix} = \begin{bmatrix} y & x \\ 2x & -2y \\ 3 & 5 \end{bmatrix}.$$

At $x=-2,\ y=1$,

$$u = xy = -2,\ v = x^2-y^2 = 3,\ w = 3x+5y = -1.$$

Then

$$D(\boldsymbol{f} \circ \boldsymbol{g}) = \begin{bmatrix} -1 & 6 & -2 \\ -4 & -1 & 3 \\ -4 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -4 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -31 & -20 \\ 9 & 25 \\ 20 & 20 \end{bmatrix}.$$

(b) Let the temperature be $T(x,y)$. Then, with x pointing eastwards and y pointing northwards,

$$T_x = -0.3, \quad T_y = -0.2.$$

Therefore

$$\nabla T = < T_x, T_y > = -< 0.3, 0.2 > .$$

The unit vector in the northeasterly direction is

$$\boldsymbol{u} = \frac{1}{\sqrt{2}} < 1,1 > .$$

The directional derivative in the northeasterly direction is

$$D_u T = \nabla T \cdot \boldsymbol{u} = -< 0.3, 0.2 > \cdot \frac{1}{\sqrt{2}} < 1,1 > = -\frac{0.5}{\sqrt{2}} \text{ degrees C per km.}$$

Time rate of change is the directional derivative times the velocity, *i.e.*,

$$\frac{dT}{dt} = -\frac{0.5}{\sqrt{2}} \times 20 = \boxed{-5\sqrt{2} \text{ degrees C per hour.}}$$

Alternatively, one can use the chain rule in the form

$$\frac{dT}{dt} = < T_x, T_y > \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle.$$

Note that $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$ is the velocity, of magnitude 20 and direction along the unit vector $(1/\sqrt{2}) < 1,1 >$. Therefore,

$$\begin{aligned} \frac{dT}{dt} &= < -0.3, -0.2 > \cdot 20(1/\sqrt{2}) < 1,1 > \\ &= \boxed{-5\sqrt{2} \text{ degrees C per hour.}} \end{aligned}$$

3. (a) Determine if

$$\boldsymbol{F} = \left\langle 2xe^z + ze^x,\ 3ze^y + 3yze^y,\ 3ye^y + x^2e^z + e^x \right\rangle$$

is a gradient field, and if so, find its potential.

(b) Show that $\nabla \cdot (\nabla f \times \nabla g) = 0$, where f and g are general, twice differentiable scalar fields.

(a) Let $\boldsymbol{F} = \nabla f =< f_x, f_y, f_z >$. Then

$$f_x = 2xe^z + ze^x \text{ so that } f = x^2e^z + ze^x + g(y,z).$$

$$f_y = g_y = 3ze^y + 3yze^y \text{ so that } g = 3ze^y + 3z(ye^y - e^y) + h(z) = 3yze^y + h(z).$$

Then

$$f = x^2e^z + ze^x + 3yze^y + h(z).$$

$$f_z = x^2e^z + e^x + 3ye^y + h'(z) = 3ye^y + x^2e^z + e^x.$$

Therefore $h'(z) = 0$ or $h =$ constant, say C. Then

$$f = x^2e^z + ze^x + 3yze^y + C.$$

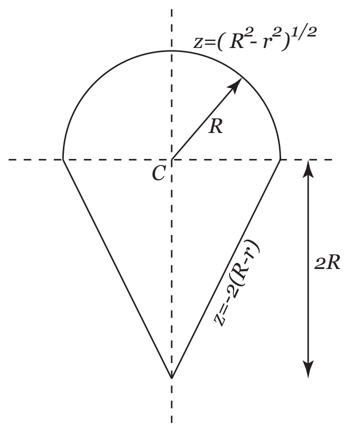
Successful computation of potential f means that $\boldsymbol{F}$ is a gradient field.

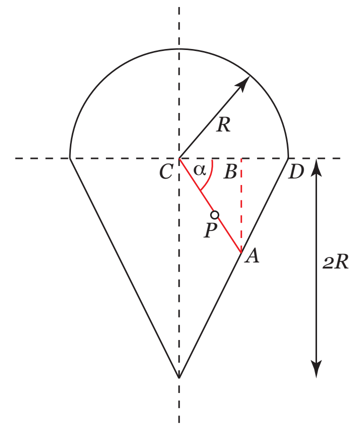
(b)

$$\nabla f \times \nabla g = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{bmatrix} =< f_yg_z - f_zg_y,\ f_zg_x - f_xg_z,\ f_xg_y - f_yg_x > .$$

$$\begin{aligned} \nabla \cdot (\nabla f \times \nabla g) &= \frac{\partial}{\partial x}(f_yg_z - f_zg_y) + \frac{\partial}{\partial y}(f_zg_x - f_xg_z) + \frac{\partial}{\partial z}(f_xg_y - f_yg_x) \\ &= f_{yx}g_z + f_yg_{zx} - f_{zx}g_y - f_zg_{yx} + f_{zy}g_x + f_zg_{xy} - f_{xy}g_z - f_xg_{zy} \\ &\quad + f_{xz}g_y + f_xg_{yz} - f_{yz}g_x - f_yg_{xz} \\ &= 0. \end{aligned}$$

4. The plane shape shown in the figure below is rotated about its vertical axis of symmetry to generate a solid in the shape of an ice cream cone. Describe the solid by means of appropriate inequalities in (i) cylindrical coordinates, (ii) spherical coordinates and (iii) cartesian coordinates. Use the point C as the origin.





(a) Cylindrical:

$$0 \le \theta < 2\pi, \quad 0 \le r \le R, \quad -2(R-r) \le z \le \sqrt{R^2 - r^2}.$$

(b) Cartesian:

$$-R \le x \le R, \quad -\sqrt{R^2 - x^2} \le y \le \sqrt{R^2 - x^2}, \quad -2[R - \sqrt{x^2 + y^2}] \le z \le \sqrt{R^2 - x^2 - y^2}.$$

(c) Spherical: Note that in the figure on the right,

$$CB = CA\cos\alpha, \quad BD = \frac{1}{2}AB = \frac{1}{2}CA\sin\alpha.$$

Therefore,

$$CB + BD = R = CA\left(\cos\alpha + \frac{1}{2}\sin\alpha\right),$$

and hence

$$CA = \frac{R}{\cos\alpha + \frac{1}{2}\sin\alpha}.$$

For any point P on the line CA, $\rho = CP$ and $\phi = \pi/2 + \alpha$. Also, $0 \le \rho \le CA$. Therefore we can write the following inequalities for the conical region.

$$0 \le \theta < 2\pi, \quad \pi/2 \le \phi \le \pi, \quad 0 \le \rho \le \frac{R}{\sin\phi - \frac{1}{2}\cos\phi}.$$

For the hemispherical region,

$$0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi/2, \quad 0 \le \rho \le R.$$