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Studying the numerous papers that have appeared in the recent past that address "vortex breakdown," it may be difficult for a reader to avoid getting rather confused. It appears that various authors or even schools have conflicting views on the correct interpretation of the physics of vortex breakdown. Following the investigation by Keller *et al.* [Z. Angew. Math. Phys. **36**, 854 (1985)], in this paper, axisymmetric forms of vortex breakdown, as originally defined by Benjamin [J. Fluid Mech. **14**, 593 (1962)] are addressed. It is argued that at least some of the previous investigations have been concerned with different aspects of the same phenomena and may, in fact, not disagree. One of the most fundamental questions in this context concerns the properties of the distributions of total head and circulation on the downstream side of vortex breakdown transitions. Some previous investigators have suggested that the downstream flow would exhibit properties that are similar to those of a wake. For this reason the phenomenon of vortex breakdown is investigated for a class of distributions of total head and circulation in the domain of flow reversal that is substantially more general than in previous investigations. Finally, a variety of problems are discussed that are crucial for *a more complete theory of vortex breakdown*, but have not yet been solved. It is shown that for the typically small flow speeds in a domain of flow reversal produced by a vortex breakdown wave, the departures of both vortex core size and swirl number, with respect to the case of uniform total pressure in the zone of flow reversal, as discussed by Keller *et al.* [Z. Angew. Math. Phys. **36**, 854 (1985)], remain surprisingly small. As a consequence, the possible appearance of large departures from a Kirchhoff-type wake must be due to viscous diffusion at low and due to shear-layer instabilities at high Reynolds numbers. © *1995 American Institute of Physics.*

I. INTRODUCTION

Unfortunately the term "vortex breakdown" has been used for a variety of phenomena that have been observed on vortex flows. This may be one of the main reasons for the confusion regarding the interpretation of vortex breakdown. It should be pointed out that the present investigation is restricted to shock-like transitions between vortex flow states, essentially following Benjamin's[1] original definition. Benjamin has defined vortex breakdown as a transition between conjugate flow states, implying that both mass and momentum are conserved across such a transition. Furthermore, Benjamin's definition implies that vortex breakdown is a wave. Hence, in a frame of reference that moves with this wave the upstream flow cannot be subcritical. As the amplitude of this wave is not small, in general, the upstream flow is generally supercritical. Moreover, the definition implies that upstream and downstream flow states must be different. Phenomena in a vortex flow that are similar to a shock wave, a hydraulic jump, or a gravity current would clearly satisfy Benjamin's definition. On the other hand, a phenomenon that is similar to a solitary wave would not satisfy the definition, because the state of flow does not change across the wave. Furthermore, phenomena of instability, as discussed by Ludwieg,[2] Lessen and Sing,[3] and others, would not satisfy Benjamin's definition either, because they do not represent wave-like phenomena. It is, of course quite clear that other wave-like phenomena, similar to those discussed by Leibovich,[4] and instabilities, as discussed by Lessen and Singh,[3] appear in a variety of forms in vortex flows and may even play an important role in the general context of vortex breakdown, as defined by Benjamin.[1] However, they do not represent shock-like transitions between flow states. It is also rather generally accepted that, at least within the framework of perfect fluid theory, one or more forms of axisymmetric transitions between conjugate flow states do exist in vortex flows. At high Reynolds numbers, the flows downstream of vortex breakdown waves usually exhibit strong thin shear layers. The most unstable waves appearing on such shear layers are approximately orthogonal to the local flow direction. For this reason, Kelvin–Helmholtz-type instabilities typically lead to a nonaxisymmetric rollup of shear layers, and to the formation of one or more helical vortices. In certain cases instabilities may become so strong that the underlying axisymmetric character of the flow is almost completely lost. In this sense, instabilities generally play an essential role in the general context of transitions between vortex flow states.

It was probably Hall[5] who first suggested that vortex breakdown might be compared to boundary layer separation. The idea is that, under the influence of an adverse pressure gradient, an internal domain of flow reversal would appear that is similar to a wake. It should be pointed out that this idea is not at all in conflict with the concept of inertia waves. It merely focuses attention on the internal structure of vortex breakdown waves, whereas the concept of transitions between flow states highlights the properties of the flows upstream and downstream of the transitions without consideration of the internal structure of the vortex breakdown waves. Based on the analogy between thin vortex cores and boundary layers, numerous authors have tried to study vortex breakdown with the help of parabolized equations. While such an approach might be useful to study flows upstream of vortex breakdown waves, it is not expected to be useful to investigate the internal structure of vortex breakdown. A boundary layer theory loses its validity when separation occurs.

Although Benjamin's[1] theory already contained most ideas for a successful theory of axisymmetric vortex breakdown, at least two key elements were still missing. By implicitly restricting the discussion to analytic functions for the total head and the circulation, the possibility of discussing physically realistic zones of flow reversal was essentially ruled out. For the same reason the possibility of loss–free transitions was ruled out. Second, following the ideas of second-order theories for small-amplitude waves, Benjamin[1] applied a loss–free theory to transitions that lead to subcritical flows and, therefore, produce dissipation, without, however, having a criterion for vortex breakdown waves to be weak. Referring to the analogy between gravity currents and vortex breakdown, Keller *et al.*[6] have shown that vortex breakdown may lead to subcritical or supercritical flow. In the latter case, the transition is loss–free and may be followed by a second transition to subcritical flow that is similar to an internal hydraulic jump in a two-phase channel flow. A criterion for vortex breakdown waves to be weak has been derived by Keller.[7]

Recently, much has been learned about the properties of vortex breakdown waves at low Reynolds numbers from direct numerical solutions of the Navier–Stokes equations. The reader is referred to the investigations by Lopez,[8] Brown and Lopez,[9] and Neitzel,[10] for example. Numerical solutions have extended the information that has been available from LDA experiments. Unfortunately, numerical work is limited to low Reynolds numbers. The use of turbulence closure models to simulate vortex breakdown flows at high Reynolds numbers would probably not be very promising. The unsteadiness produced by instabilities combined with the highly nonisotropic entrainment would make any kind of turbulence modeling exceedingly difficult.

The main point of the present investigation is to show that, in order to include domains of flow reversal in an axisymmetric theory of vortex breakdown, the definition of the distributions of total head and circulation must be suitably extended beyond the upstream interval of streamfunction values. The simplest possibility, that has led to results of practical interest, is to assume that the total head is uniform and the circulation vanishes in the domain of flow reversal, as was done by Keller *et al.*[6] To simply choose an analytical extension of the distributions of total head and circulation beyond the upstream interval of definition, does not seem to lead to physically meaningful results.

II. THE GENERAL PROBLEM OF VORTEX BREAKDOWN

A. Loss–free axisymmetric flows

For loss–free axisymmetric flows, it is convenient to write the governing equation in the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial x^2}=\frac{dH}{d\psi}-\frac{C}{r^2}\frac{dC}{d\psi}, \tag{1}$$

where ψ denotes the streamfunction that is defined by

$$u=\frac{1}{r}\frac{\partial\psi}{\partial r}, \quad v=-\frac{1}{r}\frac{\partial\psi}{\partial x}, \tag{2}$$

and x and r are the axial and radial coordinates, respectively, and u and v refer to the corresponding velocity components. Here C is defined by

$$C=rw, \tag{3}$$

where w is the azimuthal velocity component, and H denotes the total head,

$$H=\frac{p}{\rho}+\frac{1}{2}[u^2+v^2+w^2], \tag{4}$$

p refers to the static pressure and ρ to the density. Furthermore, the integrals of motion, H and C, depend on the values of the streamfunction ψ only. The flow state upstream of a vortex breakdown transition can now be determined after defining H and C within a suitable interval of values of the streamfunction,

$$H=H(\psi), \quad C=C(\psi), \quad \text{if } 0\leqslant\psi<\psi_0. \tag{5}$$

Without restriction of generality, the value $\psi=0$ has been chosen for the streamline on the axis. The upper limit of the interval would typically correspond to a tube wall or to the stream tube at infinity.

Prior to proceeding further several fundamentally important points should be carefully considered. Within the framework of perfect fluid theory H and C do not need to be analytic functions. However, a condition that must be satisfied for a well-posed problem is that $H(\psi)$ should be continuous, which implies that Bernoulli's theorem is valid along streamlines and that the static pressure does not jump across (free-) stream surfaces. Furthermore, $H(\psi)$ should be positive for all possible values of ψ. Flow reversal first occurs at the stream tube $\psi=\psi_{\min}$, where $H(\psi)$ reaches its minimum $H_{\min}=H(\psi_{\min})$. It is, of course, possible that $\psi_{\min}$ is larger than zero. In this case the vortex breakdown bubble exhibits a toroidal shape, and no stagnation point appears on the axis. However, for most vortex flows of practical interest, H reaches its minimum at $\psi=\psi_{\min}=0$, and, as a consequence, a stagnation point appears at the upstream end of a vortex breakdown bubble.

Insisting on an analytic function $H(\psi)$ does, in general, rule out the possibility of unbounded domains of flow reversal. This is most easily seen in the case of a plane channel flow. In this case, the governing equation is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{dH}{d\psi}, \tag{6}$$

and the streamfunction is defined by

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}, \tag{7}$$

where x and y are Cartesian coordinates and u and v the corresponding velocity components. Suppose now that we

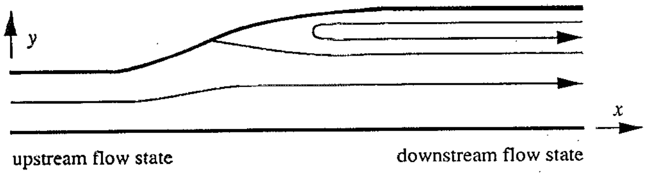

FIG. 1. Domain of flow reversal appearing after a diffuser section in a plane duct flow.

would like to discuss an unbounded domain of flow reversal that appears as a result of flow separation, as illustrated in Fig. 1.

For simplification, we assume that there are upstream and downstream flow states where the flow may be regarded as locally x independent. At such locations, Eq. (6) becomes x independent and can easily be integrated. Multiplying the equation by $\partial\psi/\partial y$ leads, after integration, to the following form of Bernoulli's theorem:

$$u=\frac{\partial\psi}{\partial y}=\sqrt{2[H-H_0]}, \tag{8}$$

where H_0 is the lowest value of H of the upstream flow. In other words, flow reversal first occurs on the streamline $\psi=\psi_0$ for which $H_0=H(\psi_0)$. To fix ideas, we consider a specific example. Suppose that the velocity profile of the upstream flow is given by

$$u_1(y)=U_L-(U_L-U_U)\cdot\frac{y}{h_1}, \tag{9}$$

where

$$0<U_U<U_L, \tag{10}$$

and h_1 refers to the upstream width of the channel (see Fig. 1). Hence, U_U and U_L denote the velocities at the upper and lower walls of the channel, respectively. Combining (8) and (9), we obtain

$$H=H_0+\frac{1}{2}U_L^2-[U_L-U_U]\frac{\psi}{h_1}, \tag{11}$$

where

$$0<\psi<\psi_0=\frac{h_1}{2}[U_L+U_U]. \tag{12}$$

Suppose now that the width of the channel grows rapidly in the flow direction and that the downstream width of the channel is substantially larger than the upstream width, as shown in Fig. 1, such that flow separation would be expected to occur at the upper wall. In this case, the governing equation (6) for the downstream flow state becomes

$$\frac{\partial^2\psi}{\partial y^2}=\frac{dH}{d\psi}=-\frac{U_L-U_U}{h_1}, \tag{13}$$

and its solution can be written in the form

$$\psi=\psi_2=-\frac{U_L-U_U}{2h_1}y^2+\left(\frac{h_1}{2h_2}[U_L+U_U]+\frac{h_2}{2h_1}[U_L-U_U]\right)y, \tag{14}$$

$$u_2(y)=\frac{d\psi_2}{dy}=-[U_L-U_U]\frac{y}{h_1}+\left(\frac{h_1}{2h_2}[U_L+U_U]+\frac{h_2}{2h_1}[U_L-U_U]\right), \tag{15}$$

where $y=h_2$ and $y=0$ refer to the upper and lower channel walls, respectively. The largest value of ψ ($=\psi_{\max}$, say) is reached at

$$y=y_{\max}=\frac{h_1}{[U_L-U_U]}\left(\frac{h_1}{2h_2}[U_L+U_U]+\frac{h_2}{2h_1}[U_L-U_U]\right), \tag{16}$$

where flow reversal begins and

$$\psi_{\max}=\psi_0+\frac{1}{8}\cdot\frac{h_1}{U_L-U_U}\left(\frac{h_1}{h_2}[U_L+U_U]-\frac{h_2}{h_1}[U_L-U_U]\right)^2. \tag{17}$$

Hence, $\psi_{\max}>\psi_0$ and, because

$$H(\psi_{\max})=H_0+\frac{1}{2}U_L^2-\frac{1}{8}\left(\frac{h_1}{h_2}[U_L+U_U]+\frac{h_2}{h_1}[U_L-U_U]\right), \tag{18}$$

$$H(\psi_{\max})<0,\quad \text{if } \frac{h_2}{h_1}>\frac{\sqrt{2H_0+U_L^2}+\sqrt{2H_0+U_U^2}}{U_L-U_U}. \tag{19}$$

It is now apparent that, according to (17), ψ exceeds the domain of definition (12) of $H(\psi)$. Furthermore, for sufficiently large values of h_2/h_1 the condition $H(\psi)>0$ is violated. As a conclusion, we may note that an analytical extension of the integrals of motion beyond the regime of definition does not only correspond to an ambiguous choice with respect to domains of flow reversal, but generally leads to results that are physically meaningless.

However, the downstream domain of flow reversal could easily be included by suitably extending the properties of H beyond the interval (12), without insisting on an analytical function. The same is, of course, true for H and C in the case of vortex breakdown flows.

Returning now to vortex flows again, it should be noted that, as far as the flow upstream of the vortex breakdown wave is concerned, H and C are only defined in a certain interval $0\leq\psi<\psi_0$. As was pointed out by Batchelor,[11] a domain of flow reversal consists of streamlines for which H and C are not defined by the upstream flow state. The corresponding values of ψ lie either outside the domain of definition, $0\leq\psi<\psi_0$, or belong to a different branch of a multivalued function $H(\psi)$. There is, of course, *an infinite number of* *choices* to extend the definition (5) of $H(\psi)$ and $C(\psi)$, in order to include domains of flow reversal. Promising choices would have to depend on certain downstream properties of the flow field. The situation is indeed very similar, as in the case of a wake. Considering the fact that the definition (5) can be extended in an infinite number of ways to discuss domains of flow reversal, restricting consideration to an analytical extension for $H(\psi)$ and $C(\psi)$ beyond the interval $0 \leqslant \psi < \psi_0$ corresponds to a choice that does neither account for the physics of the flow nor for mathematical requirements that would make it possible to include unbounded domains of flow reversal. In a recent paper, Leibovich and Kribus[12] have raised questions with respect to remarks by Keller *et al.*,[6] who pointed out that the investigations of Benjamin[1] and Fraenkel[13] failed to explain the basic physics of axisymmetric vortex breakdown, because the implicit assumption of analytic integrals of motion $H(\psi)$ and $C(\psi)$ made their investigations incomplete. With all respect for the pioneering work of Benjamin[1] and Fraenkel,[13] this remark should be reemphasized. Even for investigations that are limited to solitary wave-like phenomena, as those discussed by Leibovich and Kribus,[12] there does not seem to be a justification to use an analytic extension of $H(\psi)$ and $C(\psi)$, in order to discuss domains of local flow reversal. To arrive at a theory of vortex breakdown that includes domains of flow reversal, we should again look at the corresponding problem of the wake in a nonswirling flow. The simplest idea is to follow Kirchhoff and assume that the total pressure is uniform in a vortex breakdown bubble. With this assumption we simply ignore the appearance of a secondary flow in a vortex breakdown bubble. Although assuming a Kirchhoff-type wake clearly oversimplifies the general problem, it may lead to correct and accurate solutions for two-phase flows, as was shown by Keller *et al.*[6] for the case of air-filled vortex breakdown bubbles in water flows at high Reynolds numbers. Furthermore, it appears that this assumption may represent a useful approximation for many single-phase flows in the near field of the transition. Finally, it should be pointed out that the basic properties of vortex breakdown depend on a balance of flow forces. As is known from numerous technical approximations (e.g., the Borda jump diffuser), ignoring the dynamic pressure contributions from the secondary flow in a wake usually does not lead to a significant modification of the momentum balance.

To put the assumption of uniform total pressure within a vortex breakdown bubble to the test, we can easily extend the theory by Keller *et al.*[6] to a different distribution of total head within the vortex breakdown bubble. The situation considered is illustrated in Fig. 2. We investigate the properties of a vortex flow in a tube that undergoes vortex breakdown. Here it should be pointed out that the topology of a secondary flow in a vortex breakdown bubble strongly depends on the properties of $H(\psi)$ and $C(\psi)$ for $\psi<0$. In some cases it may look very different from that shown in Fig. 2, even in the absence of a second transition (see the next section), leading to subcritical flow.

It is assumed that the upstream flow can be represented as a Rankine vortex, defined by

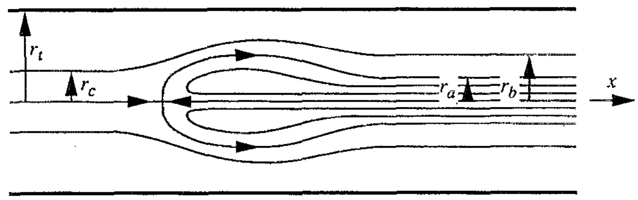

FIG. 2. Schematic drawing of a force– and loss–free transition of a vortex flow in a tube.

$$u_1(r)=U_1, \quad w_1(r)=\begin{cases}\Omega r, & \text{if } 0 \leqslant r \leqslant r_c,\\ \Omega r_c^2/r, & \text{if } r_c<r<r_t,\end{cases} \tag{20}$$

where the axial velocity U_1 and the angular velocity Ω are assumed to be constants, the subscript "1" refers to quantities that define the upstream flow state, and r_c and r_t refer to the radius of the vortex core and the tube radius. According to (20) we obtain the following integrals of motion:

$$\begin{aligned} C&=\begin{cases}k\psi, & \text{if } 0 \leqslant \psi \leqslant U_1 r_c^2/2,\\ kU_1 r_c^2/2, & \text{if } U_1 r_c^2/2<\psi<U_1 r_t^2/2,\end{cases}\\ H&=\begin{cases}U_1^2/2+U_1 k^2 \psi/2, & \text{if } 0 \leqslant \psi \leqslant U_1 r_c^2/2,\\ U_1^2/2+U_1^2 k^2 r_c^2/4, & \text{if } U_1 r_c^2/2<\psi<U_1 r_t^2/2,\end{cases}\end{aligned} \tag{21}$$

where

$$k=\frac{2\Omega}{U_1}. \tag{22}$$

To investigate the effect of a domain of flow reversal, we extend (21) by

$$C=0, \quad H=\frac{1}{2}U_1^2+\epsilon \frac{U_1}{2}k^2\psi, \quad \text{if } \psi<0, \tag{23}$$

where

$$0 \leqslant \epsilon \ll 1. \tag{24}$$

In principle, we could also include a departure from $C=0$ for $\psi<0$. However, within the framework of inviscid fluid flow theory $C=0$ if $\psi=0$. For this reason we would have to include domains with both positive and negative axial vorticity components. In general, there does not seem to be a justification for including axial vorticity components with both signs. Hence, we restrict the consideration to $C=0$ for $\psi<0$. The solutions presented by Keller *et al.*[6] correspond to the special case of a Kirchhoff-type wake, i.e., $\epsilon=0$. As was done in Ref. 6, the streamfunction ψ_2 of the downstream flow state can be expressed as the sum of the streamfunction ψ_1 of the upstream flow state and a departure $r\varphi$,

$$\psi=\psi_2=\psi_1+r\varphi, \tag{25}$$

where the subscript "2" refers to quantities to be taken at the second flow state. From (1), (21), and (23), we obtain the governing equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial x^2}$$

$$=\begin{cases}\epsilon U_1k^2/2, & \text{if } \psi<0,\\ U_1k^2/2-k^2\psi/r^2, & \text{if } 0\leqslant\psi\leqslant U_1r_c^2/2,\\ 0, & \text{if } U_1r_c^2/2<\psi<U_1r_t^2/2,\end{cases} \tag{26}$$

with a suitably extended domain of definition with respect to admissible values of the streamfunction. The x-independent solution of (26), that defines the downstream flow state, can be expressed as

$$\psi_2(r)=\begin{cases}-\frac{\epsilon}{16}U_1k^2r^2\cdot[r_a^2-r^2], & \text{if } 0\leqslant r\leqslant r_a,\\ \frac{U_1r^2}{2}+r\cdot[AJ_1(kr)+BY_1(kr)], & \text{if } r_a<r\leqslant r_b,\\ \frac{U_1r^2}{2}-\frac{U_1}{2}\cdot\frac{r_b^2-r_c^2}{r_t^2-r_b^2}[r_t^2-r^2], & \text{if } r_b<r<r_t,\end{cases} \tag{27}$$

where r_a is the radius of the stream tube defined by $\psi_2(r_a)=\psi_1(0)=0$ (see Fig. 2), r_b is the vortex core radius defined by $\psi_2(r_b)=\psi_1(r_c)$, and J_1 and Y_1 denote Bessel functions of the first and second kind. The coefficients A and B are determined by matching conditions for values of ψ and static pressure at $r=r_a\pm 0$ and $r=r_b\pm 0$. For details the reader is referred to Ref. 6. In contrast to Ref. 6, where $p_2(r_a)=p_S$, the static pressure at $r=r_a$ is now determined by

$$p_2(r_a)+\frac{\rho}{2}\cdot\left(\frac{U_1}{2}+\varphi'(r_a)\right)^2=p_S, \tag{28}$$

where

$$p_S=p_1(0)+\frac{\rho}{2}U_1^2 \tag{29}$$

refers to the stagnation pressure, and use has been made of $\psi_2(r_a)=\psi_1(0)=0$, i.e.,

$$U_1r_a^2/2+r_a\varphi(r_a)=0. \tag{30}$$

Making use of the fact that the circulation, $2\pi C$, remains constant along stream tubes, the azimuthal velocity component, which corresponds to the second flow state, can be expressed as

$$w_2(r)=\begin{cases}0, & \text{if } 0\leqslant r\leqslant r_a,\\ \Omega r+k\varphi(r), & \text{if } r_a<r\leqslant r_b,\\ \Omega r_c^2/r, & \text{if } r_b<r<r_t,\end{cases} \tag{31}$$

and $w_2(r)$ is continuous in the interval $0\leqslant r<r_t$. The distribution of static pressure, $p_2(r)$, is obtained with the help of the radial component of the momentum equation,

$$\frac{dp_2}{dr}=\rho\frac{w_2^2}{r}. \tag{32}$$

After this point, the solution procedure closely follows that given in Ref. 6 for the special case $\epsilon=0$. For this reason the discussion will focus on departures with respect to results given in Ref. 6. With the help of a lengthy but straightforward calculation, (32) can be integrated. We obtain the following distribution of static pressure:

$$\frac{p_2(r)-p_S}{\rho}=\begin{cases}-\frac{1}{8}\left(\frac{\epsilon\Omega^2r_a^2}{U_1}\right)^2, & \text{if } 0\leqslant r\leqslant r_a,\\ \frac{1}{2}\Omega^2r^2-2\Omega\chi-\frac{k^2}{2}\cdot[\chi^2+\varphi^2]-2\frac{\Omega^2}{k^2}, & \text{if } r_a<r\leqslant r_b,\\ -\frac{1}{2}\Omega^2\frac{r_c^4}{r^2}-\frac{1}{2}U_1^2\left(\frac{r_t^2-r_c^2}{r_t^2-r_b^2}\right)^2+\Omega^2r_c^2, & \text{if } r_b<r<r_t,\end{cases} \tag{33}$$

where

$$\varphi=AJ_1(kr)+BY_1(kr),\quad \chi=AJ_0(kr)+BY_0(kr), \tag{34}$$

and use has been made of the matching conditions,

$$\varphi(r_a)=-\frac{1}{2}U_1r_a,\quad \chi(r_a)=-\frac{U_1}{k}+\frac{\epsilon}{8}U_1kr_a^2,$$
$$\varphi(r_b)=-\frac{U_1}{2r_b}(r_b^2-r_c^2),\quad \chi(r_b)=\frac{U_1}{k}\cdot\frac{r_b^2-r_c^2}{r_t^2-r_b^2}. \tag{35}$$

We may note that the only differences with respect to Ref. 6 are due to the static pressure contribution $[p_2(r)-p_S]/\rho=-[\epsilon\Omega^2r_a^2/U_1]^2/8$ in the interval $0\leqslant r\leqslant r_a$ [see Eq. (33)] and the appearance of an additional term in the second matching condition (35). Another lengthy calculation leads to the correspondingly extended expression for the downstream flow force,

$$S_2=2\pi\int_0^{r_t}[p_2+\rho u_2^2]r\,dr$$
$$=\frac{1}{2}\rho U_1^2\pi r_t^2\left[1+\frac{1}{2}k^2r_c^2-\frac{1}{2}k^2r_c^2\frac{r_b^2}{r_t^2}\right.$$
$$+\frac{1}{8}k^2\frac{r_b^4-r_a^4}{r_t^2}-\frac{1}{4}k^2r_c^2\frac{r_c^2}{r_t^2}\ln\left(\frac{r_t^2}{r_b^2}\right)$$
$$\left.-\frac{1}{3r_t^2}\left(\frac{\epsilon\Omega^2r_a^3}{2U_1}\right)^2\right]. \tag{36}$$

The modification of the downstream flow force with respect to the expression given in Ref. 6 is due to the last term on the right-hand side of (36). As one would expect, accounting for a secondary flow within the vortex-breakdown bubble leads to a reduction of the downstream flow force. However, it can already be anticipated that the effects of this reduction on the vortex-breakdown swirl number, kr_a, and on the dimensionless radii r_a/r_t and r_b/r_t will be small as long as ϵ remains much smaller than 1. It may be worth pointing out that the situation is again very similar to that in the case of a wake in a nonswirling flow. Considering the change of static pressure across a Borda jump diffuser, for example, the static pressure contribution due to the secondary flow downstream of the backward facing step is generally ignored when upstream and downstream flow forces are calculated, because it does not lead to a substantial contribution to the momentum balance. The upstream flow force is, of course, the same as that given in Ref. 6. Making use of the dimensionless quantities,

$$R=r_c/r_t, \quad \xi=kr_c, \quad \eta=kr_b, \quad \zeta=kr_a, \tag{37}$$

the momentum balance, $S_2-S_1=0$, can be expressed as

$$[1+\epsilon^2\zeta^2/24]\zeta^4=[\eta^2-\xi^2][\eta^2-3\xi^2]+2\xi^4\ \ln(\eta^2/\xi^2). \tag{38}$$

For $\epsilon=0$, this expression agrees with the momentum balance derived in Ref. 6. The matching conditions at $r=r_b\pm 0$ for values of ψ and static pressure remain unchanged with respect to Ref. 6. They can be expressed as

$$\propto J_1(\eta)+\beta Y_1(\eta)=-\frac{1}{\pi}\cdot\frac{\eta^2-\xi^2}{\eta\zeta} \tag{39}$$

and

$$\alpha J_0(\eta)+\beta Y_0(\eta)=\frac{2}{\pi\zeta}\cdot\frac{\eta^2-\xi^2}{\xi^2/R^2-\eta^2}, \tag{40}$$

where

$$\alpha=Y_1(\zeta)-\tfrac{1}{2}\zeta Y_0(\zeta), \quad \beta=\tfrac{1}{2}\zeta J_0(\zeta)-J_1(\zeta). \tag{41}$$

With (38)–(41) we now have a complete set of equations to determine the swirl number ξ and the dimensionless radii η and ζ for given values of ϵ and R. The solution procedure is described in Ref. 6.

To illustrate cases with a substantial departure from the results presented in Ref. 6, we need to choose unrealistically large values for ϵ. Figure 3 shows the dimensionless upstream vortex core radius r_c/r_t versus the swirl number kr_c for $\epsilon=0$ (solid curve) and $\epsilon=0.2$ (dashed curve). Figure 4 shows the dimensionless bubble radii r_a/r_t and r_b/r_t versus the dimensionless radius r_c/r_t of the upstream vortex core for $\epsilon=0$ (solid curves) and $\epsilon=0.2$ (dashed curves). As expected, the departures turn out to be very small.

The only substantial difference between the cases $\epsilon=0$ and $\epsilon\neq 0$ appears in the limit of a free vortex, i.e., for $R\rightarrow 0$. In this limit, the equations (39)–(41) can be combined to

$$[\eta^2-\xi^2]J_0(\eta)+\zeta^2 J_2(\zeta)=0 \tag{42}$$

and

$$[\eta^2-\xi^2]Y_0(\eta)+\zeta^2 Y_2(\zeta)=0, \tag{43}$$

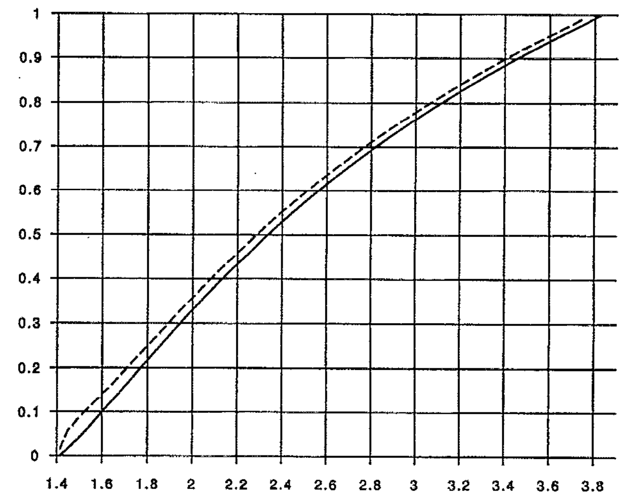

FIG. 3. Ratio of upstream vortex-core radius and tube radius, r_c/r_t, versus swirl number, kr_c, for $\epsilon=0$ (solid curve) and $\epsilon=0.2$ (dashed curve).

where use has been made of

$$J_1(\eta)Y_0(\eta)-J_0(\eta)Y_1(\eta)=\frac{2}{\pi\eta}. \tag{44}$$

The equations (38), (42), and (43) represent a complete set to determine the unknowns ξ, η, and ζ as functions of ϵ. The term in (38) that contains ϵ^2 as a factor is effectively the contribution of the secondary flow within the vortex breakdown bubble to the downstream flow force. This term cancels the degeneracy of the solution for $\epsilon=0$ in the limit $R\rightarrow 0$. In other words, for $\epsilon=0$, this type of (force– and loss–free) vortex breakdown does not exist in the limit of a free vortex, but for $\epsilon\neq 0$ the additional flow force contribution to (38) leads to a nontrivial solution for a force– and

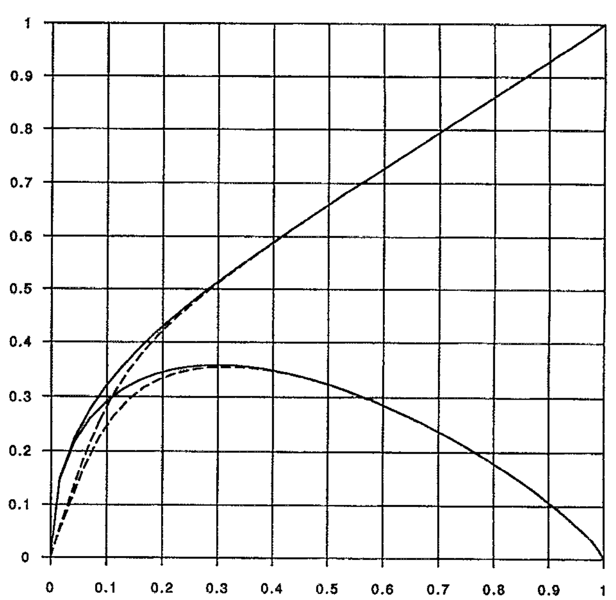

FIG. 4. Dimensionless bubble radii r_a/r_t (lower curves) and r_b/r_t (upper curves) versus the dimensionless upstream vortex-core radius r_c/r_t for $\epsilon=0$ (solid curves) and $\epsilon=0.2$ (dashed curves).

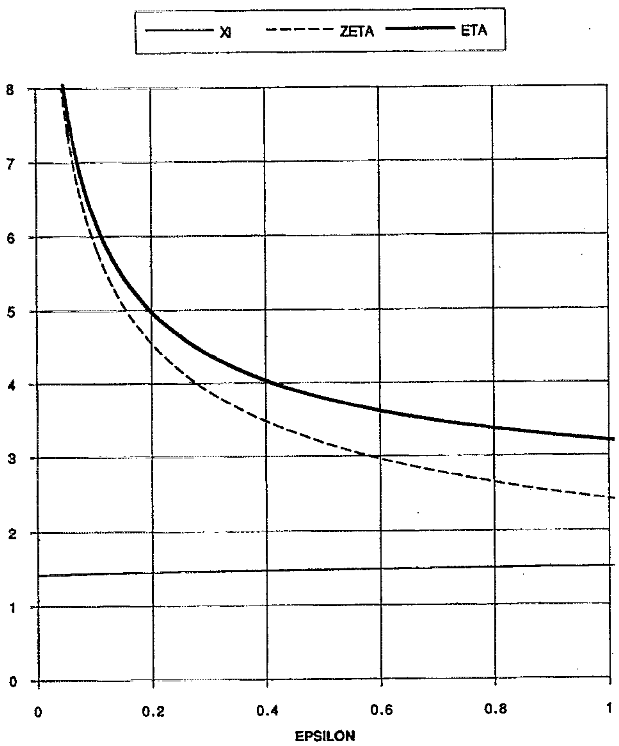

FIG. 5. Here $\xi(=kr_c)$, $\zeta(=kr_a)$, and $\eta(=kr_b)$ as functions of ϵ.

loss–free transition. It is interesting to note that the solution for ξ, η, and ζ only depends on the additional contribution to the flow force difference, and not on any details of the extended definition (23) of the integrals of motion. Figure 5 shows ξ, η, and ζ as functions of ϵ, according to Eqs. (38), (42), and (43).

As a conclusion of the investigation in this section, we may note that the theory presented by Keller *et al.*[6] should lead to results of good accuracy and reasonable practical value, even for single-phase flows, provided, of course, the flow speeds in the domain of flow reversal are sufficiently small. The investigation does not suggest any attempt to go beyond the simple idea of a Kirchhoff-type wake by looking for improved definitions of H and C. The flow in the domain of flow reversal is exceedingly complicated, and there appears to be little hope that going much beyond the assumption of uniform values H and C within this domain would lead to a better understanding of such flows. However, lacking promising alternatives, it is important to clearly recognize the limitations of simplifying assumptions. As the entrainment may be regarded as isotropic in an ordinary wake, there is at least some justification for the use of Reynolds-averaged equations to compute flows at high Reynolds numbers. In the case of a vortex breakdown bubble the entrainment is strongly nonisotropic and numerical calculations at high Reynolds numbers become very difficult.

B. Forms of vortex breakdown involving dissipation

It has been shown by Keller[14] at all axisymmetric vortex flows can be classed into two categories, A-type and B-type vortices. An A-type vortex has the characteristic property that it approaches its critical flow state when its cross-sectional area is increased, and departs from the critical flow state when its cross-sectional area is decreased. A B-type vortex has the opposite property. All vortex flows with analytic integrals of motion, $H(\psi)$ and $C(\psi)$, are A-type vortices. For a vortex to be B type, it is a necessary condition that $H(\psi)$ and $C(\psi)$ are not both analytic. From this fact it is again obvious that any analysis that is restricted to analytic integrals of motion, as that by Benjamin[1] or Leibovich and Kribus[12] cannot explain vortex breakdown transitions that are both force– and loss–free. The most general form of loss–free vortex breakdown, as discussed in the previous section, is a transition from a supercritical A-type to a supercritical B-type vortex.

There exists a second kind of vortex breakdown, which exhibits a close analogy to internal hydraulic jumps in two-phase channel flows, and produces a transition from a supercritical B-type vortex to a subcritical A- or B-type vortex. The type of vortex that appears after this transition depends on the downstream boundary condition. An example of a flow observation that shows the appearance of this second kind of vortex breakdown was given by Keller.[7] Depending on their relative propagation speeds, the two kinds of vortex breakdown may appear as separate waves or combine to a single dissipative transition from a supercritical A-type vortex to subcritical A- or B-type vortex. The complete structure of properties of vortex breakdown waves exhibits a close analogy with gravity currents in two-phase channel flows. The first kind of vortex breakdown corresponds to Benjamin's wave in a two-phase channel flow, and the second kind of vortex breakdown to an internal hydraulic jump. Depending on the relative propagation speeds of a Benjamin wave and an internal hydraulic jump, they may also appear as separate waves or combine to a single gravity current that leads to subcritical flow.

To discuss combined vortex-breakdown waves or the second kind of vortex breakdown we could, in principle, follow Benjamin's theory[1] after suitably extending the definitions of $H(\psi)$ and $C(\psi)$ to negative values of the streamfunction ψ, as long as the vortex breakdown waves remain weak. A criterion for vortex-breakdown waves to be weak has been derived by Keller.[7] However, restricting the consideration to weak waves would exclude a broad spectrum of technically interesting situations. For this reason the rather simple theoretical treatment presented by Keller,[15] which involves conservation of mass and flow force together with a postulate of self-similar flows, would appear to be less restrictive than Benjamin's theory.[1] However, in both cases (empirical) information about the radial redistribution of momentum on the downstream side of the vortex breakdown wave is required. In the case of Benjamin's theory,[1] this information is required for a suitable extension of the definitions of $H(\psi)$ and $C(\psi)$, and in the case of the treatment proposed by Keller[15] it is required to determine the shape parameters of the velocity profiles. It is again worth noting that almost exactly the same difficulty is encountered in a discussion of internal hydraulic jumps. To obtain highly accurate predictions of the subcritical flows on the downstream side of a combined wave or after a vortex breakdown wave of the second kind, a great deal of additional empirical knowledge would have to be generated first, in order to specify the shape parameters of the downstream velocity profiles.

It may seem disappointing that there appears to be no elegant way of predicting the subcritical flow after a combined wave. However, from a technical point of view combined waves are often not very dramatic events. Their influence on static pressure distributions along surfaces is typically rather small. On the other hand, the first kind of vortex-breakdown wave, as discussed in the previous section, produces very strong changes of static pressure on surfaces, as is well known in the cases of vortex breakdown appearing in diffusers or over delta wings.

III. OPEN QUESTIONS RELATED TO VORTEX BREAKDOWN

A fundamental question that has not been answered yet concerns the possible existence of genuine nonaxisymmetric forms of vortex breakdown. In other words, it is possible that a large variety of transitions to different types of standing helical waves on vortex flows might also exists. What is commonly called the "spiral form" of vortex breakdown does, in general, not seem to be a different phenomenon. The appearance of one or more helical vortex cores instead of a largely axisymmetric bubble can easily be explained on the basis of rapidly growing instabilities on the downstream side of certain vortex-breakdown transitions. In the absence of a rotating forcing mechanism, inertia wave theory does not seem to admit rotating helical inertia waves, whereas secondary helical vortices produced by instabilities exhibit an inherent time dependence. The only experimental evidence, known to the author of this paper, that might be regarded as an indication for the existence of nonaxisymmetric forms of vortex breakdown, is the appearance of sudden jumps leading to helical hollow-core vortices that was observed by Keller and Escudier.[16] It is important to note that in this case the helical pattern is "frozen," which is a clear indication for the wave-like nature of the phenomenon. Furthermore, the swirl number for the appearance of strong helical waves is typically much larger than that for the axisymmetric forms of vortex breakdown.

A second group of problems that requires further investigation concerns vortex rollup and entrainment on the downstream side of vortex breakdown waves. Although the basic forms of instability that lead to the formation of helical vortices are fairly well understood there is little knowledge about the properties of fully developed flows. From investigations of tornado-like vortices, it is apparent that the entrainment of a nearly critical swirling jet is by at least one order of magnitude smaller than that of a nonswirling jet. However, little is known about the precise dependence of radial momentum exchange and entrainment on the swirl number, as the swirl number of a vortex flow is increased from zero to subcritical values. Particular phenomena, like the Taylor–Proudman column and tornado-like vortex cores that grow extremely slowly, are indicators of the spectacular changes of entrainment with changing swirl number.

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