Incompressible Flow
Exam-2 (open book, open notes)

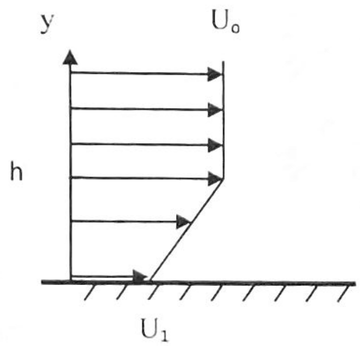
Due: **Noon**, Thursday April 9, 2020

Please sign honor pledge

"I have neither given nor received aid during this exam"

Name: Chris Nkunthorn Signature:

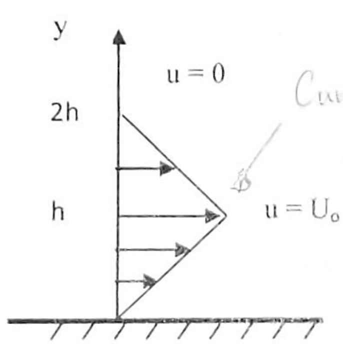
i) Examine the inviscid stability of this base state:



Very similar to example if

$u \to U_0 - U_1$

ii) Examine the inviscid stability of this base state:



Curvature flip

Inherently Unstable

Incompressible Flow Exam 2

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Setup

Rayleigh Equation

$$(u-c)\overbrace{(g^{''}-\alpha g)}^{\text{harmonic}}-u^{''}g=0 \tag{1}$$

Problem 1: No Curvature Change

Graph of the 1D velocity profile looks very similar to the example profile if the frame of reference is shifted moving with one of the velocity profiles. From this perspective, the movement of the upper layer is at speed $U_o - U_1$

Velocity Profiles

$$u(y)=\begin{cases}(U_0-U_1)\frac{y}{h} & 0\le y\le h\\ (U_0-U_1) & h<y\end{cases} \tag{2}$$

From this perspective, while in the shear layer the velocity increase from the base flow U_1 is proportionate to the relative position y toward the total height of the shear layer h, so that the expression $\frac{y}{h}$ is in the unit interval. Beyond h, the flow takes the full value.

The first derivative of the velocity profile with respect to elevation $u'(y)$, has nonzero values only in the shear layer as the both U_0 and U_1 are uniform. Ideally, move away from a piecewise function and write using the Heaviside function $H(y)$, so that it turns *on* at $y=0$ and then off after.

$$u^{'}(y)=\frac{(U_0-U_1)}{h}\left[\overbrace{H(y)}^{\text{on}}-\overbrace{H(y-h)}^{\text{off}}\right] \tag{3}$$

Taking the second derivative, the Heaviside function becomes the Dirac Delta $\delta(y)$.

$$u^{''}(y) = \frac{(U_0 - U_1)}{h}\left[\overbrace{\delta(y)}^{\text{on}} - \overbrace{\delta(y-h)}^{\text{off}}\right] \tag{4}$$

We are now ready to substitute into Rayleigh's Equation. The harmonic part has a generic solution in the form

$$g(y) = k_1 e^{-\alpha y} + k_2 e^{\alpha y} \tag{5}$$

which must be piece wise C_0 continuous. This is because this is the small wave perturbation going in either direction . IN the cases at the far field $\lim y \to \pm\infty$ the perturbations would blow up. This is the opportunity for simplification

$$g(y) = \begin{cases} g_1 & = Ae^{-\alpha y} + Be^{\alpha y} \quad & 0 \leq y \leq h \\ g_2 & = Ce^{-\alpha y} & h < y \end{cases} \tag{6}$$

Boundary Conditions

For 3 coefficients, 3 boundary conditions are required

- $y = 0; g_1(y = 0) = 0$... perturbations die at the wall. Either they reflect or are absorbed, but the result is the same: the wall serves as an anti node as it is not itself vibrating
- $y = h$;then we need piecewise continuity $g_1(y = h) = g_2(y = h)$
- $\left(\overbrace{(U_0 - U_1)}^{u} - c\right)[g'(h^+) - g'(h^-)] + g(h) \cdot \frac{U_0 - U_1}{h} = 0$ by integration of the Rayleigh equation

Matrix Form

Using the equations

$$\begin{cases} 0 = Ae^{-\alpha} + Be^{-\alpha} - Ce^{-\alpha} \\ 0 = Ae^{0} + Be^{0} + 0 \\ 0 = (U_0 - U - _1 + c)\left[A(-\alpha)e^{-\alpha h} + B\alpha e^{-\alpha h} - Ce^{-\alpha h}\right] + C \cdot \frac{U_0 - U_1}{h} \end{cases} \tag{7}$$

has a stability criterion is an eigenproblem of a system of simultaneous linear equations

$$\overbrace{\begin{vmatrix} 1 & 1 & 0 \\ 1 & e^{2\alpha h} & 1 \\ (U_0 - U_1 - c)(-\alpha) & (U_0 - U_1 - c)\alpha e^{2\alpha h} & (U_0 - U_1 - c)\alpha + \frac{U_0 - U - 1}{h} \end{vmatrix}}^{\text{identically } 0} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} \tag{8}$$

The determinant is found by method of principal minors along the first row. This produces

$$\boxed{0 = e^{2\alpha h}\left(\frac{U_0 - U_1}{h}\right) - \left[2(U_0 - U_1 - c)\alpha + \left(\frac{U_0 - U_1}{h}\right)\right]} \tag{9}$$

Problem 2: Curvature Change

The second derivative of the velocity profile produces a change and can be written as a single expression. A manner similar to part a produces

$$u^{''}(y) = (\delta(y + h) - 2\delta(y) + \delta(y - h)) \tag{10}$$

I can't get to a useful useful stability criterion for the 4 unknowns and 4 BC's. But I wouldn't expect this to be stable either as of the sign change about the inflection point.

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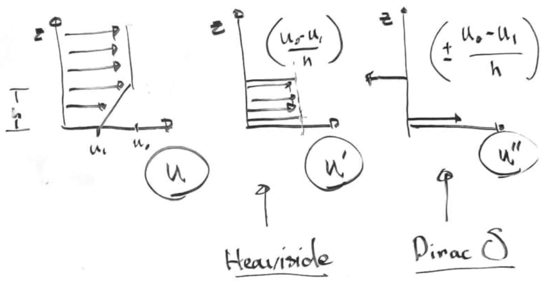
- Part a) No Curvature Change
  - Graphs of 1D Velocity look very similar to example profile, if



> frame of reference @ "y = 0"
> and moving at "$u = U_1$";
> conclude that simple mapping (?)
> of "$2h \rightarrow h$" & "$U_0 \rightarrow U_0 - U_1$"
> in derivation expected.

  - Apply Rayleigh Eqn ... $(u-c)\cdot(g'' - \alpha^2 g) - g \cdot u'' = 0$, where "$\frac{d}{dy}H(y) = \delta(y)$"

"H(y)" is the Heaviside Step Function and "δ(y)" is the Dirac Delta

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- Part a) No Curvature Change
  - Given "$y \in \mathbb{R}^+$", "$u(y)$" has piecewise form

$$u(y) = \begin{cases} u_1(y) = \frac{u_0 - u_1}{h} \cdot y & , y \in [0, h] \\ u_2(y) = u_0 - u_1 & , y \in (h, \infty) \end{cases} \rightarrow u''(y) = \begin{cases} \frac{u_0 - u_1}{h} & , y = 0 \\ \frac{u_1 - u_0}{h} & , y = h \, ; \\ 0 & , \text{else} \end{cases}$$

> $-(u_0 - u_1) = u_1 - u_0$
> but better written
> in terms of 'δ'

  - Substitution of "$u''(y)$" $\rightarrow$ Reynolds Eqn
    - $u''(y) = \frac{u_0 - u_1}{h} \left(\delta(y) - \delta(y - h) \right)$
    - $u'(y) = \frac{u_0 - u_1}{h} \left(H(y) - H(y - h) \right)$

$$\longmapsto (u - c) \cdot \underbrace{(g'' - \alpha^2 g)} - g \cdot \left[\frac{u_0 - u_1}{h} \cdot \underbrace{\left(\delta(y) - \delta(y - h) \right)} \right] = 0$$

Simple Harmonic solution, except @ 0 ; h"

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- Part a) No Curvature Change
  - Solution of " $g'' - \alpha g = 0$ " is simple harmonic ODE has a generic solution

    " $g(y) = K_1 e^{-\alpha y} + K_2 e^{\alpha y}$ ", but for interval (h, ∞), " $K_2 = 0$ " to stay bounded

  - $$g(y) = \begin{cases} g_1(y) = A e^{-\alpha y} + B e^{\alpha y}, & y \in [0, h) \\ g_2(y) = C e^{-\alpha y}, & y \in (h, \infty) \end{cases}$$; > (A, B, C) are all in $\mathbb{R}$; > 3 eqns for 3 unknowns

  - Boundary Conditions
    - $y = 0$; $g_1(y=0) = 0$
    - $y = h$; $\lim_{y \to h} \left(g_1(y) = g_2(y) \right)$
    - $((U_0 - U_1) + c)\left[g'(h^+) - g'(h^-)\right] + g(h) \cdot \frac{U_0 - U_1}{h} = 0$

$$\boxed{\begin{aligned} 0 &= A e^{-\alpha h} + B e^{\alpha h} - C e^{-\alpha h} \\ 0 &= A e^{0} + B e^{0} + 0 \\ 0 &= (U_0 - U_1 + c)\left[-\alpha A e^{\alpha h} + \alpha B e^{-\alpha h} - C e^{-\alpha h}\right] + C \frac{U_0 - U_1}{h} e^{-\alpha h} \end{aligned}}$$

> 3 Eqns for ABC

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$(3,3)$ term... $e^{-\alpha h}\left[(u_0-u_1-c)\alpha+\left(\frac{u_0-u_1}{h}\right)\right]$

· Part a) No Curvature Change

- Stability Criterion is eigen problem

$$\underset{\sim}{0}=\begin{vmatrix} e^{-\alpha h} & e^{\alpha h} & e^{-\alpha h} \\ 1 & 1 & 0 \\ (u_0-u_1-c)(-\alpha)e^{-\alpha h} & (u_0-u_1-c)(\alpha)e^{\alpha h} & (u_0-u_1-c)\alpha e^{-\alpha h}+\frac{u_0-u_1}{h}e^{-\alpha h} \end{vmatrix}\begin{Bmatrix} A \\ B \\ C \end{Bmatrix}$$

- Swap rows ① & ②, multiply ② & ③ by $e^{\alpha h}$

$$\underset{\sim}{0}=\begin{vmatrix} 1 & 1 & 0 \\ 1 & e^{2\alpha h} & 1 \\ (u_0-u_1-c)(-\alpha) & (u_0-u_1-c)\alpha e^{2\alpha h} & (u_0-u_1-c)\alpha+\frac{u_0-u_1}{h} \end{vmatrix}\begin{Bmatrix} A \\ B \\ C \end{Bmatrix}$$

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- Part a) No Curvature Change

  - Determinant of Square Matrix by Principal minors along row # 1

$$0 = (1)\begin{vmatrix} e^{2\alpha h} & 1 \\ (u_0-u_1-c)\alpha e^{2\alpha h} & (u_0-u_1-c)\alpha + \left(\frac{u_0-u_1}{h}\right) \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ (u_0-u_1-c)(-\alpha) & (u_0-u_1-c)\alpha + \left(\frac{u_0-u_1}{h}\right) \end{vmatrix} + 0$$

$$0 = \left(e^{2\alpha h}\right)\left[(u_0-u_1-c)\alpha + \left(\frac{u_0-u_1}{h}\right)\right] - (u_0-u_1-c)\alpha e^{2\alpha h} - \left[(u_0-u_1-c)\alpha + \left(\frac{u_0-u_1}{h}\right) - (u_0-u_1-c)(-\alpha)\right]$$

$$0 = \cancel{e^{2\alpha h}(u_0-u_1-c)\alpha} + e^{2\alpha h}\left(\frac{u_0-u_1}{h}\right) - \cancel{e^{2\alpha h}(u_0-u_1-c)\alpha} - \left[2(u_0-u_1-c)\alpha + \left(\frac{u_0-u_1}{h}\right)\right]$$

$$\boxed{0 = e^{2\alpha h}\left(\frac{u_0-u_1}{h}\right) - \left[2(u_0-u_1-c)\alpha + \left(\frac{u_0-u_1}{h}\right)\right]}$$

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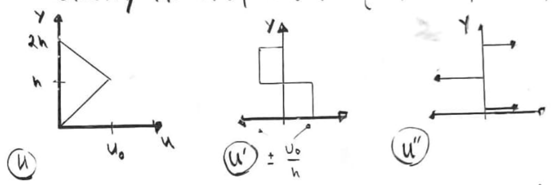
· Part b) Yes, Curvature Change

- Velocity 1D Profiles & Eqns, Shift Reference for Symmetry of $y=h$



$$u(y) = \begin{cases} u_1(y) = -\frac{U_0}{h} y, & y \in [0, h] \\ u_2(y) = +\frac{U_0}{h} y, & y \in [-h, 0) \\ 0, & \text{else} \end{cases}$$

- $u'(y) = \frac{U_0}{h} \left(H(y+H) - 2H(y) + H(y-h) \right)$
- $u''(y) = \frac{U_0}{h} \left(\delta(y+H) - 2\delta(y) + \delta(y-h) \right)$

> Though inviscid base flow is
> symmetric, peturbation, '$g(y)$'
> probably isn't.

- Rayleigh Eqn ... $(u-c)\cdot(g'' - \alpha g) - g\left[\frac{U_0}{h}\left(\delta(y+h) - 2\delta(y) + \delta(y-h)\right)\right] = 0$

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- Part b) Yes, Curvature Change
  - Perturbation ... 4 unknowns, 4 BC's
  - Can't write a useful stability criterion ... might just be my mistake ?
  - However, don't expect it to be stable either, given shape of u''
    - image of a very thin string being drawn (perpendicularly wrt lenghth) through quiescent field.
  - Rayleigh Eqn ... $(u-c)\cdot(g''-\alpha g) - g\left[\frac{U_0}{h}\left(\delta(y+h) - 2\delta(y) + \delta(y-h)\right)\right] = 0$