ADVANCED CALCULUS
SPRING 2019

LESSON 3: Multiple Integrals

This lesson considers the following topics.

1. Double and triple integrals.
2. Change of variables in integral evaluation.

Supplements Chapter 5 and 6 of the Marsden-Tromba text.

1 Introduction

Integration is a *summation* procedure. It answers the question: how much of something is there in total? In single-variable calculus the integral

$$\int_a^b f(x)\, dx$$

may compute the area under a curve, or the mass of a rod, or the distance travelled for a given velocity distribution. In multiple integration this definition is extended to answer such questions as: what is the area of a region with a boundary of arbitrary shape, what is the moment of inertia of a solid of arbitrary shape about a given axis of rotation, how much energy due to surface tension is stored in a soap film, what is the lift on the wing of an airplane given the pressure distribution on it, or how much heat is escaping through the walls of a building? In mathematical terms, we shall consider double and triple integrals, and in the next chapter, surface integrals. *This lesson will be covered with dispatch as much of it is a review of topics from the Multivariable Calculus and Linear Algebra course.*

2 The double integral

We begin with the double integral of $f(x, y)$ over a region D, written symbolically as

$$\iint_D f(x, y)\, dA.$$

Like the ordinary integral, the double integral is also defined as the limit of a Riemann sum.

Def. Let $f(x, y)$ be a continuous function defined on a region D bounded by a simple, closed, piecewise smooth curve. Consider a partition $\mathcal{P}_n$ that divides D into n subregions D_i, $i = 1, 2, \cdots, n$, each bounded by a simple, closed, piecewise smooth curve. Let ΔA_i be the area of D_i and (x_i^*, y_i^*) an arbitrary point in D_i. Consider the sum

$$S_n = \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta A_i.$$

Let $n \to \infty$ such that each $\Delta A_i \to 0$. If $I = \lim_{n\to\infty} S_n$ exists and is independent of the partition $\mathcal{P}_n$ and of the choice of (x_i^*, y_i^*), then f is said it be *integrable* over D and I is said to be the double integral or the area integral of f over D. We write

$$I = \iint_D f(x, y)\, dA.$$

Let $g(x, y)$ be another continuous function defined on D. Then the following properties of the integral can be deduced from the definition.

1. Linearity:
$$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA.$$

2. Homogeneity:
$$\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA.$$

3. Monoticity: If $f(x, y) \geq g(x.y)$, then,
$$\iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA.$$

4. Additivity: If $D = D_1 \cup D_2$, then,

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

5. For $f(x,y) = 1$ the integral yields the area A of D. Thus,

$$\iint_D dA = A.$$

6. Mean-value theorem:

$$\iint_D f(x,y)\,dA = f(x_0,y_0)A,$$

where A, as above, is the area of D and (x_0,y_0) is some point within D.

Remarks.

- The above definition of the double integral assumes that $f(x,y)$ is continuous on the bounded domain D. The requirement of continuity can be relaxed to include functions that are discontinuous across curves in D that are a finite union of graphs of continuous functions of the form $y = \phi(x)$ or $x = \psi(y)$.
- For $f(x,y) \geq 0$, the double integral can be thought to represent the volume of a cylindrical object whose base is the region D in the xy-plane and whose top is the surface $z = f(x,y)$.

2.1 Evaluation of the double integral: rectangular regions

Let us recall that in single-variable calculus, the integral

$$\int_a^b f(x)\,dx$$

is seldom computed from its definition as the limit of a Riemann sum. Instead, we invoke the Fundamental Theorem of Calculus which tells us that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$. There is no direct counterpart of the Fundamental Theorem for double integrals. However, for certain shapes of region D the double integral can be evaluated as an *iterated integral,* a process that treats the double integral as a succession of two single integrals. The simplest of such regions is the rectangle $a \leq x \leq b$, $c \leq y \leq d$ (also expressed as $[a,b] \times [c,d]$). With $dA = dx\,dy$ we write I either as

$$\iint_D f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

or as

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

In the first case integration is carried out first with respect to x and then with respect to y and the process is reversed in the second case. The iterated integral may be evaluated in any order, in principle, although one order may require less effort than the other, or, evaluation may be possible in one order but not in the other.

Example 2.1. Consider

$$I = \iint_D x^3 e^{x^2 y}\, dA, \qquad D = [1,4] \times [0,2].$$

Here, integration first with respect to y and then with respect to x is the only feasible choice. Accordingly we write

$$\begin{aligned} I &= \int_1^4 \int_0^2 x^3 e^{x^2 y}\, dy\, dx \\ &= \int_1^4 x^3 \left[\frac{e^{x^2 y}}{x^2} \right]_{y=0}^{y=2} dx \\ &= \int_1^4 \left(x e^{2x^2} - x \right) dx = \left[\frac{1}{4} e^{2x^2} - \frac{x^2}{2} \right]_1^4 = \frac{1}{4}(e^{32} - e^2) - \frac{15}{2}. \end{aligned}$$

2.2 Evaluation of the double integral: vertically and horizontally simple regions

A region D that is vertically simple (also referred to as a region of type I) is described as

$$D : \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b.$$

Similarly a region D that is horizontally simple (also referred to as a region of type II) is described as

$$D : \psi_1(x) \le x \le \psi_2(y), \quad c \le y \le d.$$

Here $\phi_1(x)$, $\phi_2(x)$, $\psi_1(y)$ and $\psi_2(y)$ are continuous functions. A vertically simple region (Figure 1) is

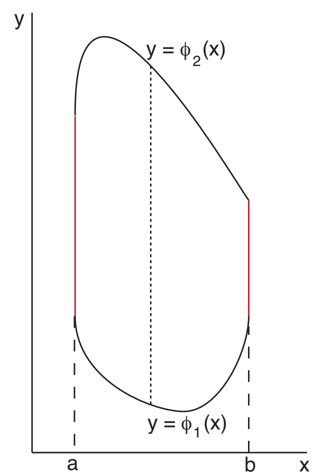

Figure 1: Vertically simple region.

bounded by vertical boundaries, and on such a region the integral is evaluated by integrating first in the y-direction and then in the x-direction. Thus,

$$\iint_D f(x,y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy\, dx.$$

A horizontally simple region (Figure 2) is bounded by horizontal boundaries and on it, integration proceeds first in the x-direction and then in the y-direction. Thus,

$$\iint_D f(x,y)\, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy.$$

Some regions can be of either type and on these, integration can be done in any order (although, again, one order may require lesser effort than the other).

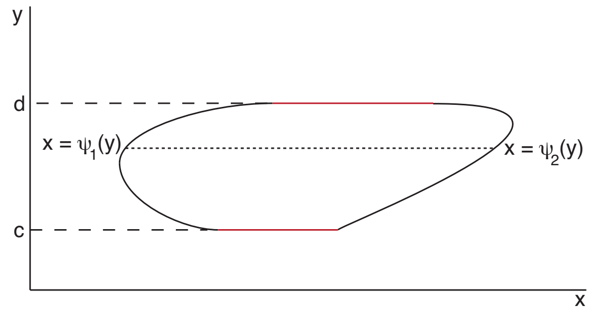

Figure 2: Horizontally simple region.

Example 2.2. Consider

$$I = \iint_D x(1+y)\, dA, \qquad D : \sin x \le y \le 3\sin x,\ 0 \le x \le \pi.$$

The region is vertically simple (draw it) and is treated as follows.

$$\begin{aligned} I &= \int_0^{\pi} \int_{y=\sin x}^{y=3\sin x} x(1+y)\, dy\, dx \\ &= \int_0^{\pi} x \left[y + \frac{y^2}{2} \right]_{y=\sin x}^{y=3\sin x} dx \\ &= \int_0^{\pi} x \left(2\sin x + 4\sin^2 x\right) dx \\ &= \int_0^{\pi} x\,(2\sin x + 2 - 2\cos 2x)\, dx. \end{aligned}$$

On using by-parts integration,

$$\begin{aligned} I &= \left[x^2 + x(-2\cos x - \sin 2x) - \left(-2\sin x + \frac{1}{2}\cos 2x \right) \right]_0^{\pi} \\ &= \pi^2 + 2\pi. \end{aligned}$$

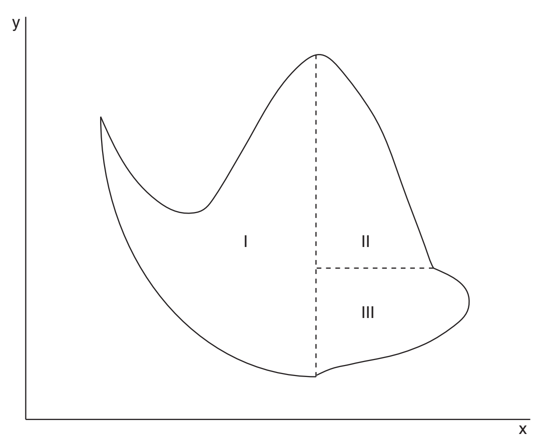

Figure 3: Mixed region.

Remark. Regions that are neither vertically nor horizontally simple are subdivided so that each subdivision is either horizontally or vertically simple. Such a region is shown in Figure 3, where subregions I and II are vertically simple while subregion III is horizontally simple.

3 The triple integral

Just as single integrals are defined for functions of one variable and double integrals for functions of two variables, triple integrals are defined for functions of three variables. The notion of a Riemann sum applies again. The triple integral of $f(x,y,z)$ over a region B is denoted by

$$\iiint_B f(x,y,z)\,dV.$$

For the integral to exist (over a reasonably smooth, bounded domain of integration), continuity of the integrand is a sufficient condition. From a practical point of view, and for appropriately shaped regions, triple integrals are evaluated by expressing them as iterated integrals. This is done most simply when B is a rectangular box, defined as

$$B = [a,b] \times [c,d] \times [r,s].$$

Example 3.1.

$$I = \iiint_D xyz^2\,dV, \qquad B = [0,1] \times [-1,2] \times [0,3].$$

Any of the six possible orders of integration can be used. We proceed as follows.

$$\begin{aligned}
I &= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dx\,dy\,dz \\
&= \int_0^1 \int_{-1}^2 xy \left[\frac{z^3}{3}\right]_0^3 dx\,dy \\
&= \int_0^1 \int_{-1}^2 9xy\,dx\,dy \\
&= \frac{9}{2} \int_{-1}^2 x\left[y^2\right]_{-1}^2 dx \\
&= \frac{27}{2} \int_{-1}^2 x\,dx \\
&= \frac{81}{4}.
\end{aligned}$$

3.1 Elementary solid regions: types I, II and III

Iterated integration can be applied to elementary regions that are shaped like a straight pipe, or cylinder, with caps at both ends. The axis of the pipe is parallel to one of the coordinate axes, giving rise to three possible regions. These regions are described by the relevant inequalities below.

$$\begin{aligned}
&\text{Type I:} \quad B = \{(x,y,z) \mid (x,y) \in D,\ \phi_1(x,y) \le z \le \phi_2(x,y)\}, \\
&\text{Type II:} \quad B = \{(x,y,z) \mid (y,z) \in D,\ \phi_1(y,z) \le x \le \phi_2(y,z)\}, \\
&\text{Type III:} \quad B = \{(x,y,z) \mid (z,x) \in D,\ \phi_1(z,x) \le y \le \phi_2(z,x)\}.
\end{aligned}$$

A type-I region is a cylinder with its axis parallel to the z-axis. (Draw a sketch.) D is its projection in the xy-plane (which we may view as the *base* of B) and $z = \phi_1(x,y)$ and $z = \phi_2(x,y)$ are its lower and upper bounding surfaces or caps. Starting at a point $(x,y,0)$ in the base, as one increases z, one enters the body B at $z = \phi_1(x, y)$ and exits the body at $z = \phi_2(x, y)$. For such a body, iterated integration is first performed in the z-direction, which reduces the integral to a double integral over D as follows. We write $dV = dz\, dA$ to get

$$\iiint_B f(x, y, z)\, dV = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\, dz \right] dA,$$

with similar reductions for solids of types II and III.

Example 3.2. Consider

$$\iiint_B z\, dV,$$

where B is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

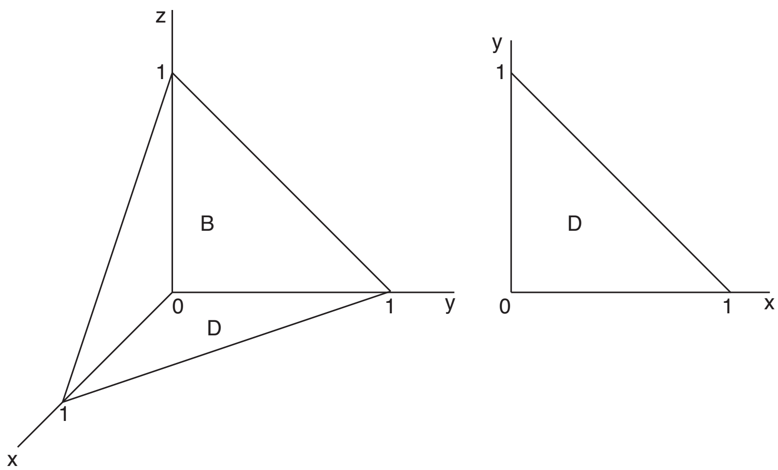

Figure 4: Tetrahedron B and its projection in the xy-plane, D.

Here the base of the body can be taken to be the triangular region D in the xy-plane, with $z = 0$ the lower cap and $z = 1 - x - y$ the upper cap, so that we have a body of type I. We can write the integral as

$$\begin{aligned} I &= \iint_D \int_{z=0}^{z=1-x-y} z\, dz\, dA \\ &= \iint_D \frac{1}{2}(1 - x - y)^2\, dA. \end{aligned}$$

The region D is both vertically and horizontally simple. Treating it as vertically simple we have

$$\begin{aligned} I &= \int_0^1 \int_0^{1-x} \frac{1}{2}(1 - x - y)^2\, dy\, dx \\ &= -\frac{1}{6} \int_0^1 \left[(1 - x - y)^3 \right]_0^{1-x} dx \\ &= \frac{1}{6} \int_0^1 (1 - x)^3\, dx \\ &= \frac{1}{24}. \end{aligned}$$

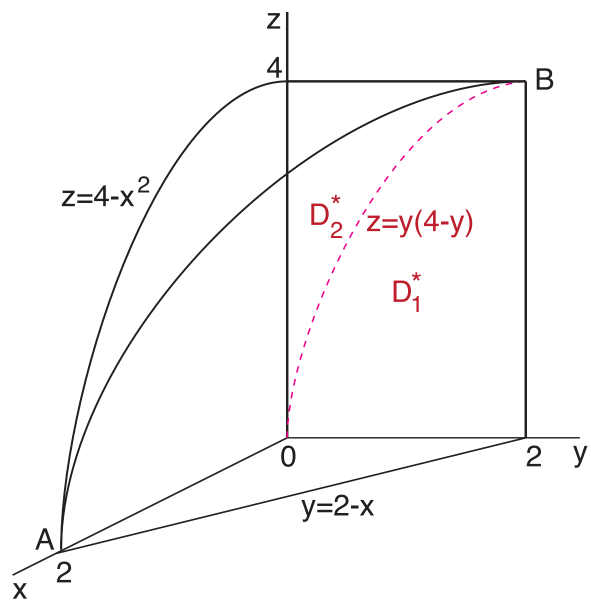

Figure 5: Solid of Example 3.3.

Example 3.3. Find the volume of the solid bounded by $z = 4 - x^2$, $x + y = 2$, and the coordinate planes.

The solid, drawn in Figure 5, can be viewed as a solid of type I, type II or type III, and we shall consider all three options.

First as a solid of type I whose projection, or base, in the xy-plane is the triangle $D : 0 \leq y \leq 2 - x$, $0 \leq x \leq 2$, and whose lower and upper caps are $z = 0$ and $z = 4 - x^2$ respectively. Then,

$$\begin{aligned} I &= \iint_D \int_{z=0}^{z=4-x^2} dz\, dA \\ &= \iint_D (4 - x^2)\, dA \\ &= \int_0^2 \int_0^{2-x} (4 - x^2)\, dydx \\ &= \int_0^2 (4 - x^2)(2 - x)\, dx \\ &= \frac{20}{3}. \end{aligned}$$

Now we consider the solid as one whose projection in the xz-plane is the region $\hat{D} : 0 \leq z \leq 4 - x^2$, $0 \leq x \leq 2$, and whose left and right boundaries are $y = 0$ and $y = 2 - x$ respectively. Then,

$$\begin{aligned} I &= \iint_{\hat{D}} \int_{y=0}^{y=2-x} dy\, dA \\ &= \iint_{\hat{D}} (2 - x)\, dA \\ &= \int_0^2 \int_0^{4-x^2} (2 - x)\, dzdx \\ &= \int_0^2 (4 - x^2)(2 - x)\, dx \\ &= \frac{20}{3}. \end{aligned}$$

Finally, consider the solid to have the rectangle $D^* : 0 \le y \le 2,\ 0 \le z \le 4$ as its base in the yz-plane. Then a portion of the base, labelled D_1^* in Figure 5, is capped by the plane $y = 2 - x$ and the remainder of the base, labelled D_2^*, is capped by the cylinder $z = 4 - x^2$. The boundary between D_1^* and D_2^* is the projection of the curve AB (the intersection of surfaces $y = 2 - x$ and $z = 4 - x^2$) onto the base. The equation of this projection is $z = y(4 - y)$, obtained by eliminating x between $y = 2 - x$ and $z = 4 - x^2$. We now write the integral as the sum

$$\begin{aligned} I &= \iint_{D_1^*} \int_0^{\sqrt{4-z}} dx dA + \iint_{D_2^*} \int_0^{2-y} dx dA \\ &= \int_0^2 \int_{y(4-y)}^4 \sqrt{4-z}\ dz dy + \int_0^2 \int_0^{y(4-y)} (2-y)\ dz dy \\ &= -\frac{2}{3} \int_0^2 (4-z)^{3/2} \Big|_{y(4-y)}^4\ dy + \int_0^2 (2-y)y(4-y)\ dy \\ &= \frac{2}{3} \int_0^2 (2-y)^3\ dy + \int_0^2 (y^3 - 6y^2 + 8y)\ dy \\ &= \frac{8}{3} + 4 = \frac{20}{3}. \end{aligned}$$

Example 3.4.

$$\iiint_B \sqrt{x^2 + z^2}\ dV$$

where B is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

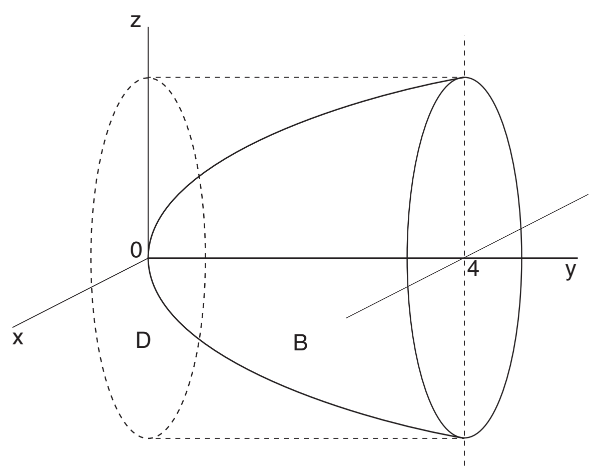

Figure 6: Solid B in Example 3.4 and its projection D in the xz-plane.

This is a solid of type III with the circular disk $D : x^2 + z^2 \le 4$ as its projection in the yz-plane; see Figure 6. We can write the integral as

$$\begin{aligned} I &= \iint_D \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\ dy\ dA \\ &= \iint_D \sqrt{x^2 + z^2}\ \Big[y\Big]_{x^2+z^2}^4\ dA \\ &= \iint_D \sqrt{x^2 + z^2}\ [4 - (x^2 + z^2)]\ dA. \end{aligned}$$

Evaluation of this integral requires a change of variables, which is the topic of the next section. We shall return to this example later. (See if the solid can be thought of as a type-I solid, with base in the xy-plane. If so, then describe it by means of inequalities and write down the corresponding iterated integral.)

4 Change of variables

We recall that evaluation of integrals in R^1 can often be facilitated by employing a suitable change of variables, or substitution. We now explore how this notion can be extended to multiple integrals.
Consider the integral

$$J = \int_1^4 e^{\sqrt{x}}\, dx.$$

We do not know what the antiderivative of $e^{\sqrt{x}}$ is, so we consider the change of variables

$$x = u^2$$

which maps the interval $I^* : 1 \le u \le 2$ on the u-line into the interval of integration $I : 1 \le x \le 4$ on the x-line. We differentiate the transformation to get

$$dx = 2u\, du.$$

This expression tells us that as u is incremented by a small amount to $u + du$, the corresponding increment $x(u + du) - x(u)$ in x is well-approximated by the differential dx, the linear approximation to the exact increment. Note that the increments du and dx are related by the *scale factor* $2u$.
Transforming to the u-variable the integral becomes

$$J = 2\int_1^2 e^u u\, du = 2\big[ue^u - e^u\big]_1^2 = 2e^2.$$

More generally, consider the integral

$$J = \int_a^b F(x)\, dx,$$

to be evaluated by the change of variables

$$x = f(u),$$

a transformation from the $u-$ line to the $x-$ line. If the interval $I : a \le x \le b$ is the image of the interval $I^* : c \le u \le d$ under this transformation, then the integral J transforms into

$$\int_c^d F(f(u))\, f'(u)\, du.$$

Note the role played by the derivative $f'(u)$ as the scale factor between the elementary lengths du and dx.

4.1 Change of variables in R^n

We start with $n = 2$. Consider the integral

$$I = \iint_D F(x, y)\, dA$$

and the change of variables, or mapping,

$$x = f_1(u, v),\ y = f_2(u, v), \quad \text{or in vector form,} \quad \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u}),$$

where

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Let $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u})$ map the region D^* in the uv-plane into the region D in the xy-plane. We assume that the mapping has the following properties.

- $\boldsymbol{f}$ is continuously differentiable.
- $\boldsymbol{f}$ is *one-to-one*, *i.e.*, $\boldsymbol{f}(\boldsymbol{u}_1) = \boldsymbol{f}(\boldsymbol{u}_2)$ only if $\boldsymbol{u}_1 = \boldsymbol{u}_2$. (No two points in D^* have the same image in D.)
- The map is *onto*, *i.e.*, every $\boldsymbol{x} \in D$ is the image if some $\boldsymbol{u}$ in D^*.

The above properties make the mapping $\boldsymbol{f}$ *invertible, i.e.*, there exists another mapping $\boldsymbol{g} : D \to D^*$ such that $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{u})) = \boldsymbol{u}$ for all $\boldsymbol{u} \in D^*$ and $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{x}$ for all $\boldsymbol{x} \in D$.

The linear map. The simplest mapping is the linear mapping,

$$x = f_1(u, v) = au + bv, \quad y = f_2(u, v) = cu + dv,$$

which can be written in vector or matrix form as

$$\boldsymbol{x} = \boldsymbol{T}\boldsymbol{u}, \quad \boldsymbol{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The mapping is invertible if the determinant

$$\det \boldsymbol{T} = ad - bc \neq 0.$$

Then the inverse matrix $\boldsymbol{T}^{-1}$ exists, and we can write

$$\boldsymbol{u} = \boldsymbol{T}^{-1}\boldsymbol{x}.$$

The mapping $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{u}$ takes a vector $\boldsymbol{u}$, multiplies it with the materix $\boldsymbol{T}$ and generates a new vector $\boldsymbol{x}$.

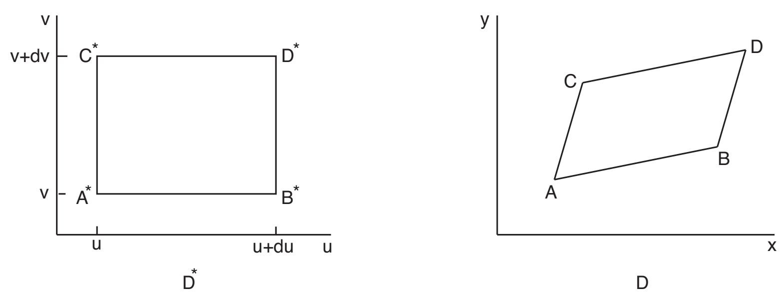

Figure 7: Mapping of a rectangle from D^* to D.

The operation can be viewed as a rotation and a stretch of $\boldsymbol{u}$ to transform it into $\boldsymbol{x}$. As such, parallelograms in D^* are transformed into parallelograms in D.

Let us construct the image in D of the rectangle $[u, u + du] \times [v, v + dv]$ in D^*; see Figure 7. The point $A^*(u, v)$ in D^* maps into the point $A(au + bv, cu + dv)$ in D. Similarly the point $B^*(u + du, v)$ in D^* maps into the point $B\big(a(u + du) + bv, c(u + du) + dv\big)$ in D. Therefore the vector $\overrightarrow{A^*B^*}$ in D^* maps into the vector $\overrightarrow{AB} =< a, c > du$ in D. Similarly the point $C^*(u, v + dv)$ in D^* maps into the point $C\big(au + b(v + dv), cu + d(v + dv)\big)$ in D so that the vector $\overrightarrow{A^*C^*}$ in D^* maps into the vector $\overrightarrow{AC} =< b, d > dv$. Now the area of the parallelogram in D is

$$\begin{aligned} dA_{xy} &= ||\overrightarrow{AB} \times \overrightarrow{AC}|| \\ &= || < a, c > du \times < b, d > dv|| \\ &= |ac - bd| \; dudv \\ &= |ac - bd| dA_{uv} \end{aligned}$$

where $dA_{uv} = dudv$ is the area of the rectangle in D^*. Thus we see that under the linear mapping, areas are scaled by the factor $|\det \boldsymbol{T}|$.

Nonlinear maps. We return to the general mapping $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u})$ and note that under such a map, straight lines are transformed into curves in general so that a rectangle will be transformed into a shape with curved boundaries. Consider, for example the nonlinear mapping between cartesian and polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Under this map the rectangle $[0, \pi/2] \times [0, 1]$ is transformed into a sector of the unit circle; see Figure 8.

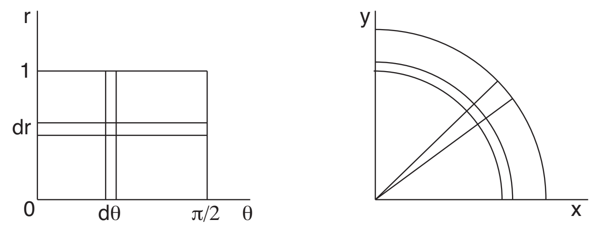

Figure 8: Transformation from polar to cartesian coordinates.

However, a small rectangle in the $r\theta$-plane is transformed into a small shape with nearly straight sides in the xy-plane.

Let us now compute the scale factor between elementary areas for the general nonlinear transformation $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u})$. Figure 9 shows the rectangle $[u, u+du] \times [v, v+dv]$ in D^* and its image in D. We note that the

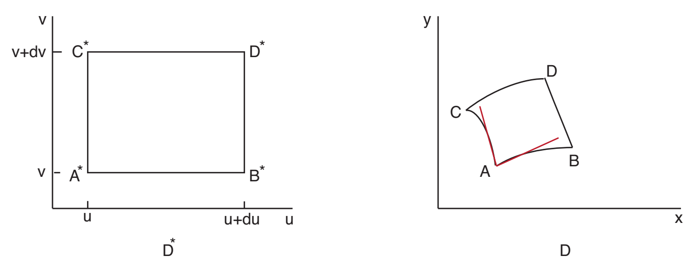

Figure 9: Transformation of a small rectangle under the general nonlinear map.

point $A^* : \boldsymbol{u} \in D^*$ maps into the point $A : \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u}) \in D$. A neighboring point $\boldsymbol{u}+d\boldsymbol{u} \in D^*$ will map into the point $\boldsymbol{f}(\boldsymbol{u}+d\boldsymbol{u}) \in D$ which, under the linear approximation, is well-approximated by $\boldsymbol{f}(\boldsymbol{u})+D\boldsymbol{f}(\boldsymbol{u})d\boldsymbol{u}$. Here, $D\boldsymbol{f}(\boldsymbol{u})$ is the derivative of the mapping, given by

$$D\boldsymbol{f}(\boldsymbol{u}) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

Thus the point B^* for which $d\boldsymbol{u} = \begin{bmatrix} du \\ 0 \end{bmatrix}$ maps approximately into the point $B : \boldsymbol{f}(\boldsymbol{u}) + D\boldsymbol{f}(\boldsymbol{u}) \begin{bmatrix} du \\ 0 \end{bmatrix}$.
Similarly the point C^* for which $d\boldsymbol{u} = \begin{bmatrix} 0 \\ dv \end{bmatrix}$ maps approximately into the point $B : \boldsymbol{f}(\boldsymbol{u})+D\boldsymbol{f}(\boldsymbol{u}) \begin{bmatrix} 0 \\ dv \end{bmatrix}$.
Therefore the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in D^* map respectively into the vectors $D\boldsymbol{f}(\boldsymbol{u}) \begin{bmatrix} du \\ 0 \end{bmatrix}$ and $D\boldsymbol{f}(\boldsymbol{u}) \begin{bmatrix} 0 \\ dv \end{bmatrix}$ in D. The rectangle in D^* has the area $A_{uv} = dudv$. The corresponding image in D, approximately shaped

like a parallelogram, has the area

$$\begin{aligned}
dA_{xy} &= ||\overrightarrow{AB} \times \overrightarrow{AC}|| \\
&= \left\| D\boldsymbol{f}(\boldsymbol{u}) \begin{bmatrix} du \\ 0 \end{bmatrix} \times D\boldsymbol{f}(\boldsymbol{u}) \begin{bmatrix} 0 \\ dv \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} du \\ 0 \end{bmatrix} \times \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} 0 \\ dv \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} x_u \\ y_u \end{bmatrix} du \times \begin{bmatrix} x_v \\ y_v \end{bmatrix} dv \right\| \\
&= |x_u y_v - y_u x_v| dudv \\
&= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}.
\end{aligned}$$

Here the symbol within the absolute value signs multiplying A_{uv} on the RHS is known as the Jacobian of the transformation, and equals the determinant of the derivative matrix, *i.e.*,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - y_u x_v.$$

Thus we find that under a nonlinear transformation, small areas scale by the factor $dA_{xy}/dA_{uv} = |J| = |\det D\boldsymbol{f}(\boldsymbol{u})|$. Therefore the double integral transforms as follows.

$$\iint_D F(x,y)\, dxdy = \iint_{D^*} F(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

Remarks.

- For the transformation between polar and cartesian coordinates, $x = r\cos\theta,\ y = r\sin\theta,$ we have

  $$|J| = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = |x_r y_\theta - x_\theta y_r| = |\cos\theta\ r\cos\theta + r\sin\theta\sin\theta| = r$$

  so that $dxdy = rdrd\theta$, a familiar result.

- Extension to $n > 2$ is straightforward. For example, consider the triple integral

  $$\iiint_D F(x,y,z)\, dV,$$

  to be transformed under the mapping $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u})$ where now,

  $$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

  Then,

  $$\iiint_D F(x,y,z)\, dxdydz = \iint_{D^*} F(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw,$$

  where now,

  $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}.$$

Transformation between cartesian and cylindrical coordinates and between cartesian and spherical coordinates are particularly useful. We now compute the corresponding Jacobians.

Cylindrical coordinates. The transformation is

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$$

The Jacobian is

$$\begin{aligned} J &= \det \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix} \\ &= \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= r. \end{aligned}$$

Spherical coordinates. The transformation is

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$$

The Jacobian is

$$\begin{aligned} J &= \det \begin{bmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{bmatrix} \\ &= \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} \\ &= -\rho^2\sin\phi. \end{aligned}$$

We now illustrate the above procedure by means of examples.

Example 3.4 revisited. Consider the integral

$$I = \iint_D \sqrt{x^2+z^2}[4-(x^2+z^2)]\ dA,$$

where D is the disk $x^2+y^2 \le 4$. We introduce polar coordinates $x = r\cos\theta,\ z = r\sin\theta$. Then the domain in the $r\theta$-plane is $D^* : 0 \le \theta \le 2\pi,\ 0 \le r \le 2$. With $dA = rdrd\theta$, the transformed integral is

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^2 r(4-r^2)r\ drd\theta \\ &= \int_0^{2\pi} \left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{64}{15}\ d\theta = \frac{128\pi}{15}. \end{aligned}$$

Example 4.1. Consider the integral

$$I = \iint_D \frac{dA}{\sqrt{1-x^2-y^2}}, \qquad D: x^2 + \left(y - \frac{1}{2}\right)^2 \le \frac{1}{4}.$$

Again we transform to polar coordinates, $x = r\cos\theta,\ y = r\sin\theta$. Then the boundary of D is the image of $D^* : 0 \le r \le \sin\theta,\ 0 \le \theta \le \pi$ in the $r\theta$-plane. Then the integral transforms into

$$\begin{aligned}
I &= \int_0^{\pi}\int_0^{\sin\theta} \frac{r\ dr d\theta}{\sqrt{1-r^2}} \\
&= -\int_0^{\pi} \sqrt{1-r^2}\Big|_0^{\sin\theta}\ d\theta \\
&= \int_0^{\pi} (1 - |\cos\theta|)\ d\theta \\
&= \int_0^{\pi/2} (1-\cos\theta)\ d\theta + \int_{\pi/2}^{\pi} (1+\cos\theta)\ d\theta \\
&= \pi - 2.
\end{aligned}$$

Example 4.2. Evaluate

$$I = \iiint_B y\ dV,$$

where B is the region in the first octant bounded by the plane $x+y+z=2$, the cylinder $x^2+z^2=1$ and the coordinate planes.

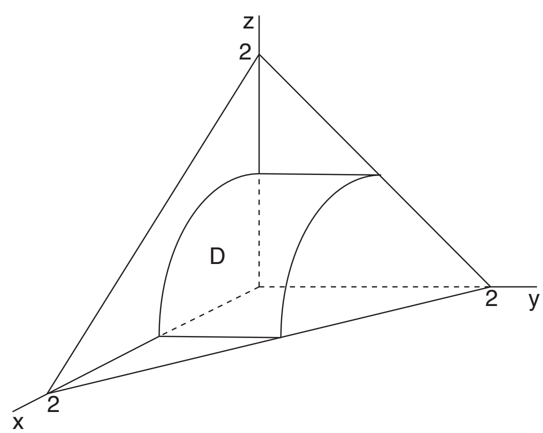

Figure 10: Solid of Example 4.2.

The solid is shown in Figure 10 as the cylinder $x^2+z^2=1$ capped on the left by the plane $y=0$ and on the right by the plane $y = 2-x-z$. Its projection D in the $y=0$ plane is the quarter circle $x^2+z^2 \le 1,\ x \ge 0,\ z \ge 0$. Thus we write

$$\begin{aligned}
I &= \iint_D \int_{y=0}^{y=2-x-z} y\ dy\ dA \\
&= \frac{1}{2}\iint_D (2-x-z)^2\ dA.
\end{aligned}$$

We now use polar coordinates $x = r\cos\theta,\ z = r\sin\theta$, where $dA = r\ drd\theta$ and $0 \le r \le 1,\ 0 \le \theta \le \pi/2$.

Therefore,

$$
\begin{aligned}
I &= \int_0^{\pi/2}\int_0^1 (2 - r\cos\theta - r\sin\theta)^2 r \, dr d\theta \\
&= \frac{1}{2}\int_0^{\pi/2}\int_0^1 [4r - 4r^2\cos\theta - 4r^2\sin\theta + r^3 + r^3\sin 2\theta] \, dr d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \left(2 - \frac{4}{3}\cos\theta - \frac{4}{3}\sin\theta + \frac{1}{4} + \frac{1}{4}\sin 2\theta\right) \, d\theta \\
&= \frac{1}{2}\left[\frac{9}{4}\theta - \frac{4}{3}\sin\theta + \frac{4}{3}\cos\theta - \frac{1}{8}\cos 2\theta\right]_0^{\pi/2} \\
&= \frac{9\pi}{16} - \frac{29}{24}.
\end{aligned}
$$

Example 4.3. Consider the integral

$$I = \iint_D (x^2 - y^2) \, dA,$$

where D is bounded by $y = x,\ y = 1 + x,\ xy = 2,\ xy = 4.$

We introduce the transformation

$$y - x = u,\ xy = v,$$

so that D^* becomes $0 \le u \le 1,\ 2 \le v \le 4$. In order to compute the scale factor we need the mapping in the form $x = x(u, v),\ y = y(u, v)$. However, here we are given the inverse map $u = u(x, y),\ v = v(x, y)$. Although the mapping can be inverted explicitly in this example, let us be more general and pretend that such an inversion is not possible. We now show that to compute the scale factor an explicit inversion is not needed. Let the forward mapping be $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u})$ and the inverse mapping $\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{x})$. Then,

$$\boldsymbol{x} = \boldsymbol{f}\big(\boldsymbol{g}(\boldsymbol{x})\big) = (\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}).$$

We now differentiate both sides with respect to $\boldsymbol{x}$ and use the fact that the derivative of a composite funtion

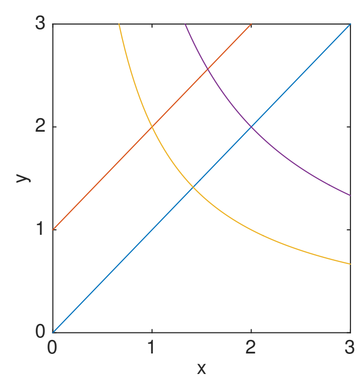

Figure 11: Region D in the xy-plane.

is the product of the derivatives. Also, the derivative of $\boldsymbol{x}$ with respect to itself is just the identity matrix $\boldsymbol{I}$, of size 2×2 in this case. Thus,

$$\boldsymbol{I} = D\boldsymbol{f}(\boldsymbol{u}) D\boldsymbol{g}(\boldsymbol{x}).$$

Taking determinants of both sides,

$$1 = \det D\boldsymbol{f}(\boldsymbol{u}) \det D\boldsymbol{g}(\boldsymbol{x})$$

Therefore the scale factor is given by

$$|J| = |\det D\boldsymbol{f}(\boldsymbol{u})| = \frac{1}{|\det D\boldsymbol{g}(\boldsymbol{x})|}.$$

In this example,

$$\begin{aligned}
|J| &= 1\Big/\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \\
&= 1\Big/\left|\det\begin{bmatrix} -1 & 1 \\ y & x \end{bmatrix}\right| \\
&= \frac{1}{|-(x+y)|} \\
&= \frac{1}{x+y}.
\end{aligned}$$

In the last step we have used the fact that D lies in the first quadrant; see Figure 11. Now the integral transforms into

$$\begin{aligned}
I &= \iint_{D^*} (x^2 - y^2)\frac{1}{x+y}\, dudv \\
&= \int_2^4 \int_0^1 (x-y)\, dudv \\
&= \int_2^4 \int_0^1 (-u)\, dudv \\
&= -\frac{1}{2}\int_2^4 dv = -1.
\end{aligned}$$

Example 4.4. Consider the integral

$$I = \iint_D \frac{(2x+y-3)^2}{(2y-x+6)^2}\, dA,$$

where D is the square with vertices $(0,0)$, $(1,-2)$, $(3,-1)$ and $(2,1)$. The four edges of the square have

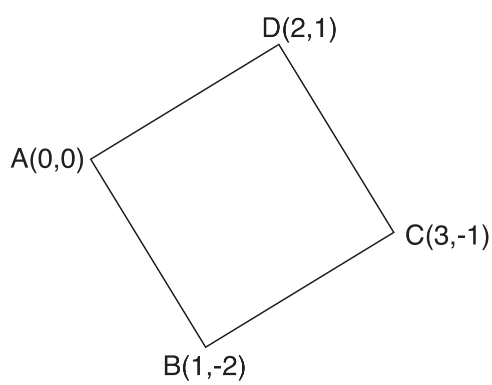

Figure 12: Region D in Example 4.4.

the following equations:

$$AB : y + 2x = 0, \quad BC : 2y - x = -5, \quad CD : y + 2x = 5, \quad DA : 2y - x = 0.$$

This suggest the following mapping.

$$2y - x = u, \quad 2x + y = v.$$

Then D is the image of $D^* : -5 \le u \le 0,\ 0 \le v \le 5$. The Jacobian of the mapping from (x, y) to (u, v) is

$$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - v_x u_y = (-1)(1) - (2)(2) = -5.$$

The scale factor is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \Big/ \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{5}.$$

Therefore,

$$\begin{aligned} I &= \int_0^5 \int_{-5}^0 \frac{(v-3)^2}{(u+6)^2} \frac{1}{5}\, dudv \\ &= \int_0^5 (v-3)^2 \left[\frac{-1}{u+6}\right]_{-5}^0 dv \\ &= \frac{5}{18}\left[(v-3)^3\right]_0^5 = \frac{175}{18}. \end{aligned}$$

Example 4.5. Evaluate the triple integral

$$I = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 \frac{e^z}{\sqrt{x^2+y^2}}\, dzdydx.$$

The domain of integration is the solid

$$B : \sqrt{x^2+y^2} \le z \le 3, \quad -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, \quad -3 \le x \le 3.$$

This is a cone with vertex at the origin, a flat top, and height 3, shown in Figure 13. It is convenient to

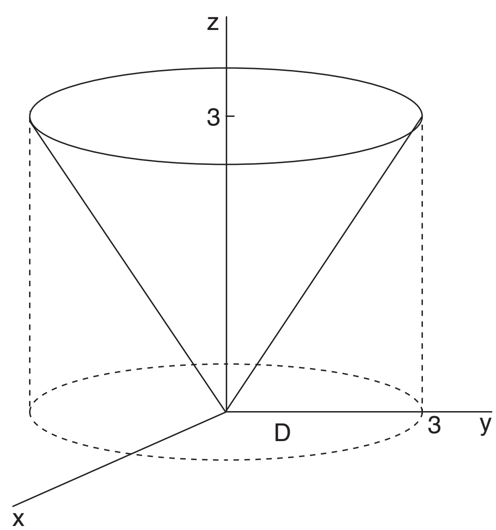

Figure 13: Solid of Example 4.5.

transform to cylindrical coordinates, for which the scale factor is r. Then the integral transforms into

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^3 \int_r^3 \frac{e^z}{r} r\, dzdrd\theta \\ &= \int_0^{2\pi} \int_0^3 [e^3 - e^r]\, drd\theta \\ &= \int_0^{2\pi} [re^3 - e^r]_0^3\, d\theta = 2\pi(2e^3+1). \end{aligned}$$

Example 4.6. Determine the centroid of the region bounded above by the sphere $x^2+y^2+z^2=18$ and below by the paraboloid $3z=x^2+y^2$.

Because of symmetry, the centroid will lie on the z-axis. The location of the centroid is given by

$$\bar{z}=\frac{I_2}{I_1}$$

where

$$I_1=\iiint_B \delta(x,y,z)\,dV, \quad I_2=\iiint_B \delta(x,y,z)z\,dV,$$

$\delta(x,y,z)$ being the density of the solid material. Here the density is a constant, and we take it to be 1

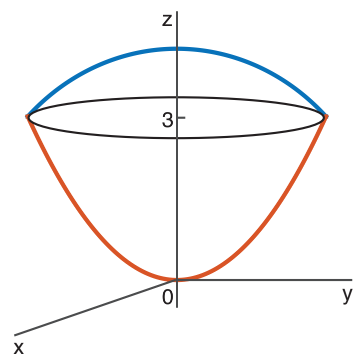

Figure 14: Solid of Example 4.6.

without loss of generality. Now, the sphere and the paraboloid intersect at $3z=18-z^2$, yielding $z=3$ and $z=-6$, the former being the applicable root. At $z=3$ the two surfaces intersect in the circle $D: x^2+y^2=9$, which is then the projection of the solid B in the xy-plane. Then we can write

$$\begin{aligned}
I_1 &= \iiint_B dV \\
&= \iint_D \int_{(x^2+y^2)/3}^{\sqrt{18-x^2-y^2}} dz\,dA \\
&= \iint_D \left(\sqrt{18-x^2-y^2}-\frac{x^2+y^2}{3}\right)\,dA.
\end{aligned}$$

We now switch to polar coordinates, $x=r\cos\theta$, $y=r\sin\theta$. Then D is the image of $D^*: 0\le\theta\le 2\pi$, $0\le r\le 3$. Therefore,

$$\begin{aligned}
I_1 &= \int_0^{2\pi}\int_0^3 \left((18-r^2)^{1/2}-\frac{r^2}{3}\right) r\,drd\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{3}(18-r^2)^{3/2}-\frac{r^4}{12}\right]_0^3 d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{3}\left\{(18)^{3/2}-9^{3/2}\right\}-\frac{81}{12}\right] d\theta \\
&= \left[9\{2^{3/2}-1\}-\frac{27}{4}\right]2\pi=\frac{9}{2}\left(8\sqrt{2}-7\right)\pi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= \iiint_B z\, dV \\
&= \iint_D \int_{(x^2+y^2)/3}^{\sqrt{18-x^2-y^2}} z\, dz\, dA \\
&= \frac{1}{2}\iint_D \left(18 - x^2 - y^2 - \frac{(x^2+y^2)^2}{9}\right)\, dA \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^3 \left(18 - r^2 - \frac{r^4}{9}\right) r\, drd\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left[9r^2 - \frac{r^4}{4} - \frac{r^6}{54}\right]_0^3\, d\theta \\
&= \frac{189}{4}\pi.
\end{aligned}$$

Therefore,

$$\bar{z} = \frac{I_2}{I_1} = \frac{21}{2(8\sqrt{2}-7)}.$$

Example 4.7. The moment of inertia of a continuous body with density $\delta(x,y,z)$ about an axis of rotation is given by

$$I = \iiint_B \delta(x,y,z)\{d(x,y,z)\}^2\, dV,$$

where $d(x,y,z)$ is the distance from the axis of rotation of the point (x,y,z) in the body. Consider a solid ball of radius a, with density distribution $\delta = x^2+y^2+z^2$. It moment of inertia about the z-axis is given by

$$I = \iiint_B \delta d^2\, dV = \iiint_B (x^2+y^2+z^2)(x^2+y^2)\, dV.$$

We use spherical coordinates in which the ball of radius a centered at the origin is described by $0 \le \rho \le a$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. With

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi,$$

and with

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi,$$

we have

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2\rho^2\sin^2\phi\rho^2\sin\phi\, d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{a^7}{7}\sin^3\phi\, d\phi d\theta \\
&= \frac{a^7}{7}\int_0^{2\pi}\frac{4}{3}\, d\theta = \frac{8\pi}{21}a^7.
\end{aligned}$$

Final remarks. As stated in the Introduction and illustrated by the last two examples, multiple integrals arise in a variety of contexts. The domain of integration is usually described in physical or geometric terms, which must be translated into suitable inequalities so that integration limits can be assigned to each integral in the iterative evaluation. A sketch is often helpful. If a change of variables is contemplated, then the motivation is either a simplification of the domain, or that of the integrand, or both. The precise mappings are problem-dependent, in general. Cylindrical or spherical geometries will often call for corresponding curvilinear coordinates.