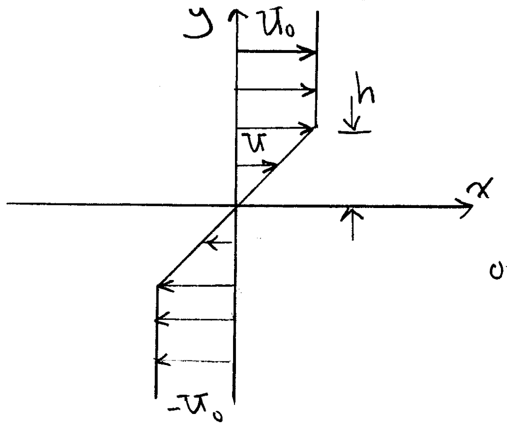
Turbulent Stability (importance of shear layers)

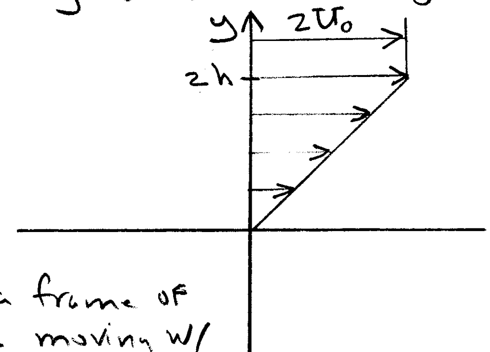
Jets, plumes, BL, wakes, etc. all involve shear layers and are generally unstable. Only in case of homogeneous turbulence, e.g. turbulence behind a grid, or turbulence in a box, is there no shear layer, and even in those shear layers were involved in their origins.

Inviscid shear layer instability (temporal) "Helmholtz"

consider the instability of the following (2D) shear layer flow:



or (in a frame of ref. moving w/ bottom stream)



Shear layer is 2h thick. Equations of motion are the Euler's eqns. (N-S eqns. w/o viscosity) or

$$\boxed{(U-c)(g''-\alpha^2 g) - g\,U'' = 0}$$

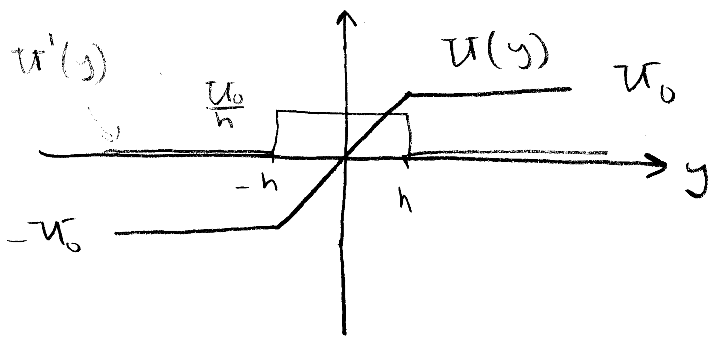
(governing equation for the Perturbations) "<u>Rayleigh Eqn.</u>"

Where $c = \beta / \alpha$ can be complex

(because α is real but β can be complex since we are considering temporal instability)

NOW consider the mean velocity:

look at U, U', U''



(turns on when ( ) becomes equal to 0 & stays on)

Unit step function (of y)

Can write $U'(y) = \dfrac{U_0}{h}\left[\underbrace{H(y+h)}_{\text{turns on at } -h} - \underbrace{H(y-h)}_{\text{turns off at } h}\right]$

(in terms of unit step function)

$$U''(y) = \frac{U_0}{h}\left[\delta(y+h) - \delta(y-h)\right]$$

Dirac-delta function (of y)
(spike at $\delta(0)$)

Substitute into eqn for $g(y)$:

$$(U-c)(g''-\alpha^2 g) - g\,\frac{U_0}{h}\left[\delta(y+h)-\delta(y-h)\right] = 0$$

The solution is simple except when δ is "on"

Everywhere except at $y = \pm h$, the eqn. becomes

$$(g''-\alpha^2 g) = 0$$

which has the general solution

$$g = K_1 e^{-\alpha y} + K_2 e^{\alpha y} \quad , \quad y \neq \pm h$$

Because $y = \pm h$ divides the flow into 3 regions, one has a solution of the form:

$$g(y) = \begin{cases} g_1(y) & y > h \\ g_2(y) & |y| < h \\ g_3(y) & y < -h \end{cases}$$

(Solution for Vertical Perturbation Velocity)

Since the solution must be bounded as $y \to \pm\infty$, the solution must be

$g_1 = A e^{-\alpha y}$ $\leftarrow$ ($K_2 \equiv 0$ since g would blow up as $y \to \infty$)

$g_2 = B e^{-\alpha y} + C e^{\alpha y}$

$g_3 = D e^{\alpha y}$ $\leftarrow$ ($K_1 \equiv 0$ " " " " " $y \to -\infty$)

A, B, C, D are constants.

B.C.: The 3 eqns. in g must be continuous at the interface of the domain of each of the functions g_1 g_2 g_3

The constants are determined from matching conditions at $y = \pm h$

4 are needed: At $y = h$ $g(h^+) = g(h^-)$

" $y = -h$ $g(-h^+) = g(-h^-)$

(i.e. normal (perturbation) velocity is continuous at $y = \pm h$)

The other 2 are obtained from the governing eqn.:

$$(U - c)(g'' - \alpha^2 g) - g \frac{U_0}{h}\left[\delta(y+h) - \delta(y-h)\right] = 0$$

integrating across the discontinuity at $y = \pm h$,
provides the proper jump in vorticity across $y = \pm h$ positions

Integrating we get:

at $y = h$: $$(U_0 - c)\left[g'(h^+) - g'(h^-)\right] + \underbrace{\int_{y=h^-}^{y=h^+} g \frac{U_0}{h} \delta(y-h)\,dy}_{} = 0$$

$\uparrow$ 0 since g is continuous

Note:
(properties of delta functions:

area under delta function $= 1$ i.e.

i.e. $$\int_{-\infty}^{\infty} \delta(x - x_0)\,dx = 1$$

and $$\int_{x_0 - c}^{x_0 + c} f(x)\,\delta(x - x_0)\,dx = f(x_0)$$)

for every $c > 0$

So eqn becomes:

(3) $$(U_0 - c)\left[g'(h^+) - g'(h^-)\right] + g(h)\frac{U_0}{h} = 0$$

and at $y = -h$:

(4) $$(U_0 + c)\left[g'(-h^+) - g'(-h^-)\right] + g(-h)\frac{U_0}{h} = 0$$

(4 eqns. for A, B, C, D.)

Substitute g's in the four matching conditions get.

1) $Ae^{-\alpha h} - Be^{-\alpha h} - Ge^{\alpha h} = 0$

2) $(U_0 - c)\left[-\alpha A e^{-\alpha h} - (-\alpha B e^{-\alpha h} + \alpha G e^{\alpha h})\right] + A \frac{U_0}{h} e^{-\alpha h} = 0$

3) $(U_0 + c)\left[(-\alpha B e^{\alpha h} + \alpha G e^{-\alpha h}) - \alpha D e^{-\alpha h}\right] + D \frac{U_0}{h} e^{-\alpha h} = 0$

4) $Be^{\alpha h} + Ge^{-\alpha h} - De^{-\alpha h} = 0$

(So need to find solution of these simultaneous linear equations)

For a non-trivial solution, determinant must $= 0$

Which gives the stability criterion (since β is contained in $c = \frac{\beta}{\alpha}$)

$$\begin{vmatrix} e^{-\alpha h} & -e^{-\alpha h} & -e^{\alpha h} & 0 \\ \left[\frac{U_0}{h} - (U_0 - c)\alpha\right] e^{-\alpha h} & (U_0 - c)\alpha e^{-\alpha h} & -(U_0 - c)\alpha e^{\alpha h} & 0 \\ 0 & -(U_0 + c)\alpha e^{\alpha h} & (U_0 + c)\alpha e^{-\alpha h} & \left[\frac{U_0}{h} - (U_0 + c)\alpha\right] e^{-\alpha h} \\ 0 & e^{\alpha h} & e^{-\alpha h} & -e^{-\alpha h} \end{vmatrix} = 0$$

(We want to simplify this matrix)

This matrix can be reduced to a tri-diagonal maxtrix by the following récipe (not unique): Multiply by $e^{\alpha h}$, add 1^{st} column to 2^{nd} column and keep the 1^{st}; add the last column to the 3^{rd} and keep last;

Multiply 1^{st} column by $e^{2\alpha h}$ and add to 3^{rd} and keep 1^{st}. Multiply last column by $e^{2\alpha h}$ and add to 2^{nd} and keep last column. Result:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{U_0}{h}-(U_0-c)\alpha & \frac{U_0}{h} & \left[\frac{U_0}{h}-2(U_0-c)\alpha\right]e^{2\alpha h} & 0 \\ 0 & \left[\frac{U_0}{h}-2(U_0+c)\alpha\right]e^{2\alpha h} & \frac{U_0}{h} & \frac{U_0}{h}-(U_0+c)\alpha \\ 0 & 0 & 0 & -1 \end{vmatrix} = 0$$

From which we obtain:

$$\left(\frac{U_0}{h}\right)^2 - \left[\frac{U_0}{h} - 2(U_0-c)\alpha\right]\left[\frac{U_0}{h} - 2(U_0+c)\alpha\right]e^{4\alpha h} = 0$$

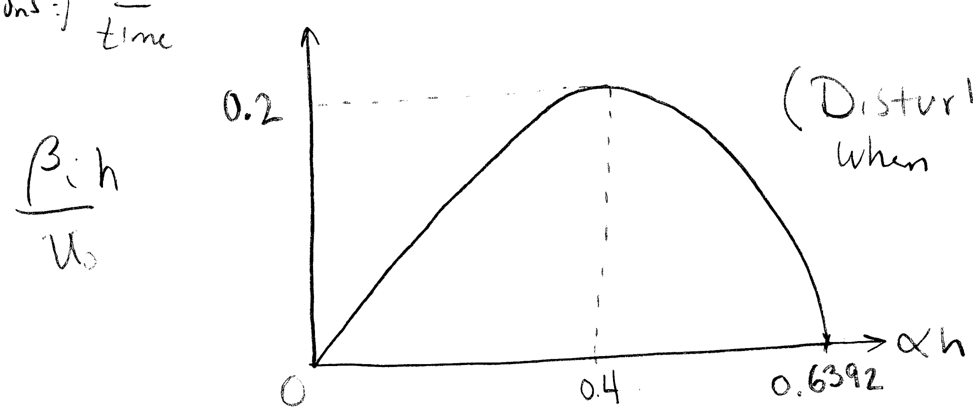
(We know that amplification factor is β_i, $\therefore$ need to find β)

recall $c = \frac{\beta}{\alpha}$, Solve for β: $(\beta = \beta_r + i\beta_i)$

find $$\beta_i = \frac{1}{2}\frac{U_0}{h}\sqrt{e^{-4\alpha h} - (1-2\alpha h)^2}$$

(Check of dimensions:) $\frac{1}{time}$

Wavenumber (spatial frequency) $\left[\frac{1}{length}\right]$



(Disturbance grows when $\beta_i > 0$)

The most unstable disturbance has a wavelength of

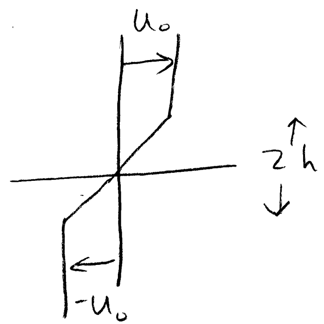
$$\lambda \equiv \frac{2\pi}{\alpha} = \frac{2\pi h}{0.4} \sim 8(2h)$$

($\uparrow$ distance between disturbances)

The shortest wavelength is

$$\lambda_{min} = \frac{2\pi}{0.639} h \approx 4.9(2h)$$

The Maximum amplification factor is $(\beta_i)_{max} = 0.2 \frac{u_0}{h}$



(for a given h.)
The stronger the shear layer $\Rightarrow$ larger β_i $\Rightarrow$ the quicker it becomes unstable.