S19 Advanced Calculus MATH 4600: Practice problems for Test 1

1. The figure below shows a rectangular box. Its sides have lengths 3, 4 and 2.

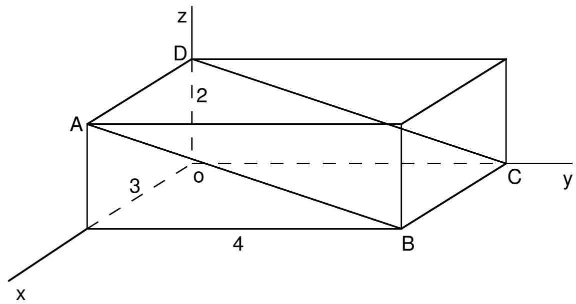

Figure 1: Problem 1.

   (a) Find a vector equation of the line connecting A and C.

   (b) Find an equation for the plane $ABCD$.

   (c) Suppose that the box is sliced into two parts by cutting along the plane $ABCD$. The lower half of the box is then laid on a horizontal table, with the face $ABCD$ flat on the table and the point O above it. What is the height of O above the table?

2. (a) Find the vector equation of the line through the point $P(1,1,2)$ that intersects the line $\boldsymbol{r} =< 3t+1, 5-7t, t+2 >$ orthogonally.

   (b) Find the equation of the plane, in the form $(\boldsymbol{r}-\boldsymbol{r}_0)\cdot\boldsymbol{n} = 0$, that is the perpendicular bisector of the line segment joining the points $(1,2,3)$ and $(3,-2,1)$.

   (c) Let $\boldsymbol{r}$ be the position vector of the point P. Describe the set of points P that satisfy the equation

$$\frac{\boldsymbol{r}\cdot\boldsymbol{e}_3}{\|\boldsymbol{r}\|} = \frac{1}{\sqrt{2}}.$$

   Here $\boldsymbol{e}_3$ is the unit vector along the z-axis.

3. Consider a sphere of radius b through which a cylindrical hole of radius $a < b$ has been drilled half way through the sphere, such that the axis of the hole coincides with a diameter of the sphere. Sketch a cross-section of the hollowed-out sphere by a plane that passes through the axis of the cylindrical hole. Describe, by means of appropriate inequalities, the hollowed-out sphere in cylindrical and spherical coordinates.

4. Suppose that the elevation of points on a mountain is given by

$$z = \frac{x^2}{3} + x^2y^2 + \frac{y^4}{2},$$

   where x and y are, respectively, the horizontal distances east and north of the origin on a topographic map.

   (a) If a climber were to start at the location 3 miles east and 1 mile south of the origin, and head southwest, is he going uphill or downhill? At what rate? At what angle to the horizontal?

   (b) In what direction(s) should he head if he wants to stay at the same elevation?

   (c) In what direction(s) will the elevation rise at a rate 1/4 of the rate of steepest ascent?

   (d) Suppose the climber puts on a personal jetpack and flies off the mountain along a straight path perpendicular to the mountain. Find a vector equation for the path.

5. At time $t = 0$, a particle is ejected from the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$ in a direction normal to the surface at a speed of 10 units per second. At what time does the particle cross the sphere $x^2 + y^2 + z^2 = 103$?

6. Consider the scalar field
$$f(x, y) = \frac{1}{8}x^2 + \frac{1}{4}y^2.$$
   (a) Find the vector field $\boldsymbol{F}$ for which $f(x, y)$ is the potential.
   (b) Compute, analytically, the flow lines of $\boldsymbol{F}$. (You will need to integrate elementary differential equations of form $dx/dt = ax$.)

7. Check if
$$\boldsymbol{F} =< 2xye^z + x\cos x, x^2e^z + 2e^y \ln z, x^2ye^z + \sin x + 2e^y/z >$$
is a gradient field. If so, then find its potential.

8. Let f be a scalar field and $\underline{F}$ and $\underline{G}$ vector fields. Indicate which of the following quantities are defined, and which ones are not. In each case, give a brief (one-sentence) explanation.
   (a) $\nabla \times (f\underline{F})$
   (b) $\nabla \cdot (f\underline{F})$
   (c) $\nabla f\underline{F}$
   (d) $\nabla(\underline{F} \cdot \underline{G})$
   (e) $\nabla \cdot (\underline{F} \cdot \underline{G})$
   (f) $\nabla \times (\underline{F} \cdot \underline{G})$
   (g) $(\nabla \cdot \underline{F})\ \underline{G}$
   (h) $\nabla f \cdot \underline{G}$
   (i) $(\nabla \times \underline{F}) \cdot \underline{G}$
   (j) $(\underline{F} \cdot \nabla)\underline{G}$
   (k) $\underline{F} \cdot (\nabla\underline{G})$
   (l) $(\underline{F} \cdot \nabla)f$

9. Suppose that a bird flies along the path $x = 2\cos t$, $y = 2\sin t$, $z = 3t$, where t is time, in an environment where the pressure varies from point-to-point as
$$P(x, y, z) = \frac{10x^2z}{y + 4}.$$
   (a) Use the chain rule to determine the rate of change of pressure with respect to time experienced by the bird at $t = \pi/4$.
   (b) Use the notion of the linear approximation to compute the approximate pressure at $t = \pi/4+0.01$.

10. Let
$$\boldsymbol{f}(\boldsymbol{x}) =< f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) >,$$
where $\boldsymbol{x} =< x_1, x_2, x_3 >$ and
$$f_1 = x_1^3x_2^2x_3,\ f_2 = e^{x_1}\sin x_2 \cos x_3.$$
Also, let
$$\boldsymbol{g}(\boldsymbol{z}) =< g_1(\boldsymbol{z}), g_2(\boldsymbol{z}), g_3(\boldsymbol{z}) >,$$
where $\boldsymbol{z} =< z_1, z_2 >$ and
$$g_1 = z_1^2z_2^2,\ g_2 = z_1 + z_2,\ g_3 = z_1/z_2.$$
   (a) Find $D\boldsymbol{f}(\boldsymbol{x})$, the derivative (or Jacobian matrix) of $\boldsymbol{f}$.
   (b) Find $D\boldsymbol{g}(\boldsymbol{z})$, the derivative (or Jacobian matrix) of $\boldsymbol{g}$.
   (c) Find $D(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{z})$.