(Next topic:)

Weakly Nonlinear Waves (in deep water)

(due to Rayleigh)

Assume a steady motion in coordinate system moving at speed c (phase speed):

$$\phi = -c(x-ct) + a\,c\,\sin k(x-ct)\,e^{ky}$$

(for a right-running wave)

$\uparrow$ uniform flow to left (not y dependent)

$\uparrow$ amplitude

(this is the full potential, not just the perturbation potential)

Since $\frac{\partial\phi}{\partial x} = u \equiv \frac{\partial\psi}{\partial y}$ and $\frac{\partial\phi}{\partial y} = v \equiv -\frac{\partial\psi}{\partial x}$

(also give the 2-d cylindrical from $v_r \equiv \frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $v_\theta \equiv -\frac{\partial\psi}{\partial r}$ and axisymmetric 3-d case: $v_r \equiv -\frac{1}{r}\frac{\partial\psi}{\partial z}$ $v_z \equiv \frac{1}{r}\frac{\partial\psi}{\partial r}$)

$$\Rightarrow \quad \psi = -cy + a\,c\,\cos k(x-ct)\,e^{ky}$$

Take $\psi = 0$ at free surface (i.e. fs is a streamline)

(we've been denoting the position of the free surface) $y = \eta$

Solve for η:

$$0 = -c\eta + a\,c\,\cos k(x-ct)\,e^{k\eta} \quad ----(1)$$

(for linear case, $k\eta \to 0$ and find $\eta = a\cos k(x-ct)$),

(but, for a weakly nonlinear analysis,)

Still assume small amplitude, i.e. $ka \ll 1$, $(\propto \frac{a}{\lambda})$

but keep higher order terms.

Try: (expanding the normalized surface elevation in powers of the small quantity in this problem:)

$$\frac{\eta}{a} = \eta_1 + ka\,\eta_2 + (ka)^2\,\eta_3 + \text{H.O.T.}$$

(η_n are dimensionless functions)

Substitute this in Eqn (1) ($\psi = 0$ @ $y = \eta$)

$$\eta_1 + ka\,\eta_2 + (ka)^2\eta_3 + \cdots = \cos k(x-ct)\, e^{ka\eta_1 + (ka)^2\eta_2 + \cdots}$$

(expand in Taylor series since ka is still small)

$$= \cos k(x-ct)\,\left(1 + ka\,\eta_1 + (ka)^2\eta_2 + \cdots\right)$$

Equate terms of order 1, $O(ka)$, $O(ka)^2$ ...

$$\eta_1 = \cos k(x-ct)$$ (this is what we got from linear theory)

$$\eta_2 = \eta_1 \cos k(x-ct) = \cos^2 k(x-ct) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2k(x-ct)$$

$$\eta_3 = \eta_2 \cos k(x-ct) = \cdots$$

$\vdots$

Thus, $\eta - \frac{1}{2}ka^2 = a\cos k(x-ct) + \frac{1}{2}(ka)\, a\cos 2k(x-ct) + \cdots$

This is equivalent to the Kinematic Surface Condition (which impl[ies] that fluid element at surface remains there)

Also.
Choose origin so that mean value of η is zero:

(Reality check:) $\eta = a\cos k(x-ct) + \frac{1}{2}(ka)\, a\cos 2k(x-ct) + \cdots$

Is this eqn. consistent with Dynamic FS Condition?
(i.e. is the form of η we chose appropriate)

(Since at surface:) $P = P_0$ (atmospheric pressure)
(substituting back into the unsteady Bernoulli eqn:)

$$\text{Constant} = g\eta + \phi_t + \tfrac{1}{2}\left(\phi_x^2 + \phi_y^2\right) \quad \text{at } y = \eta$$ (with new origin)

(Since our origin was moved) (Substitute in the η:)

$$\text{New Constant} = g\left(a\cos k(x-ct) + \cdots\right)$$

$$- k a c^2 \cos k(x-ct)\, e^{k(a \cos k(x-ct) + \cdots)}$$

$$+ \frac{1}{2} k^2 a^2 e^{2ka \cos k(x-ct) + \cdots} + \text{HOT}$$

$$\text{Newer Constant} = \left\{ ga - kac^2 + \frac{k^2a^2c^2}{2} 2ka \right\} \cos k(x-ct) + $$

(Since LHS is constant and can not depend on x or t)

Eqn is satisfied only if $\{ \quad \} = 0$, i.e.

$$0 = ga - kac^2 + \frac{k^2a^2c^2}{2} 2ka$$

or

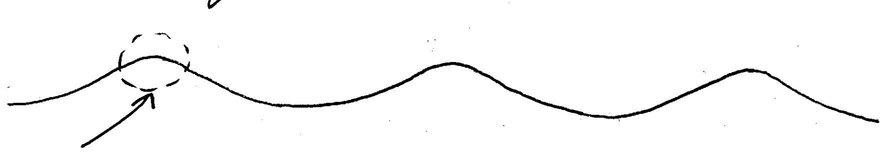
$$c^2 = \frac{g\cancel{a}}{k\cancel{a}(1-k^2a^2)} \Rightarrow \boxed{c^2 = \frac{g}{k}(1 + k^2a^2 + \cdots)} \quad \text{since } ka \ll 1$$

No longer just $c^2 = \frac{g}{k}$, but also a small correction.

So, the dispersion relation now depends on amplitude.

Using $c = \frac{\omega}{k}$ $\Rightarrow$ $\boxed{\omega = \sqrt{gk}\left(1 + \frac{1}{2}k^2a^2 + \cdots\right)}$

The correction to η makes crests sharper & stretches out the troughs.



By a local solution can show angle, at now a sharp crest $\Rightarrow 120°$ as $a \to a_{max}$ (see Lamb or Yih)

By taking additional terms, find $k \cdot a_{max} \approx 0.44$ ($a_{max} \approx 7\%$ of λ)

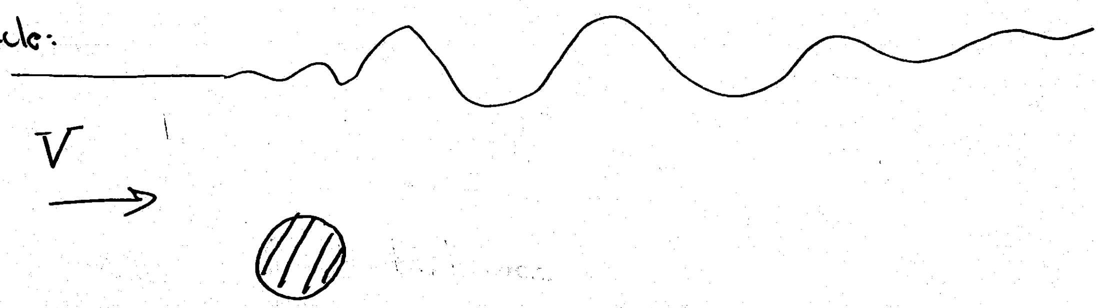
$$\Rightarrow C_{max} \approx 1.1 \sqrt{\frac{g}{k}}$$

(i.e. up to a 10% increase in phase speed due to non linearity. However instabilities & wave breaking (white caps) complicate things significantly)

(Final topic in surface waves:)

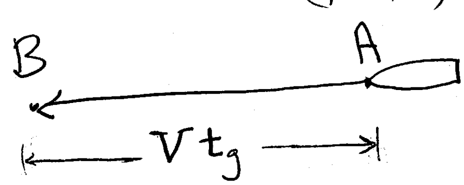
<u>Ship waves</u> ("Kelvin Wave Pattern") (Ref. Lighthill)

When a disturbance moves through water, the wave field is identical to that of a uniform flow past a stationary obstacle:

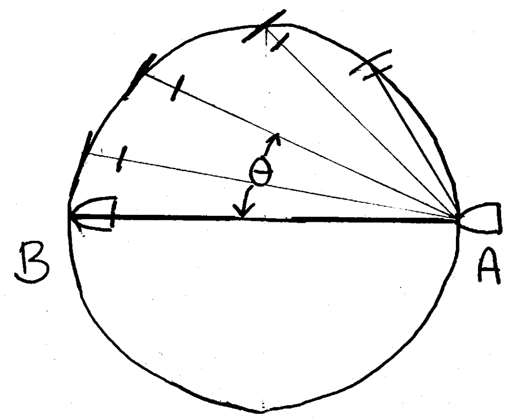


Thus stationary waves are possible when phase speed C equals V.

Consider a surface vessel moving at a constant velocity $\vec{V}$.



t_g is time since generating a given set of waves at some previous position

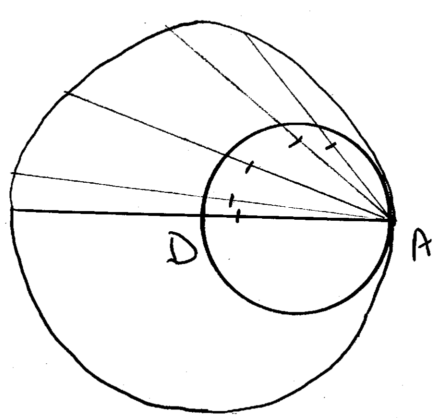


Circle AB would contain the waves traveling with the ship; different wave lengths going at different speeds, the longest wavelength λ_{max} corresponding to $V = c = \sqrt{\frac{g}{k}}$ traveling in the direction of the ship, all others moving at

$c = V \cos\theta$, i.e. shortest waves moving laterally.

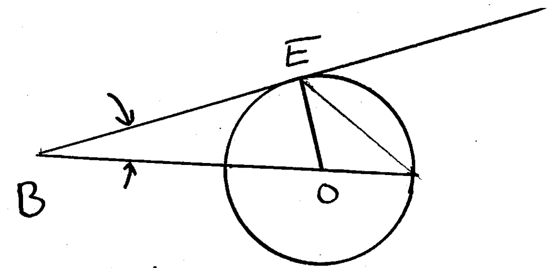
But we know that the waves associated with a ship are of gravity type $\left(c = \sqrt{\frac{g}{k}}\right)$ and a packet of gravity waves moves at group velocity $c_g = \frac{1}{2} c$

Thus, ship waves can only go $\frac{1}{2}$ as far as depicted above.



Therefore, waves traveling with the ship that were generated when ship was at A will be contained in Circle AD

So, when the Ship gets to B, its waves must be contained inside a triangle:

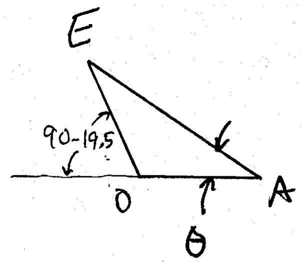


So the "Kelvin Ship-Wave" wedge has a half angle of:

$$\sin^{-1}\left(\frac{1}{3}\right) \doteq 19.5^\circ$$

The waves traveling at the center of the wedge have a phase speed $c = V = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$

$\Rightarrow \lambda_{max} = \frac{2\pi V^2}{g}$. Waves at the boundary of wedge (E) are traveling at angle θ



$$\theta = \frac{1}{2}(90 - 19.5^\circ) = 35.3^\circ$$

Thus, their phase speed $c = V\cos\theta = 0.816\,V$.

and their $\lambda = \frac{2}{3}\lambda_{max}$

Overall Wave Pattern: T.P. Fig 70 Lighthill (Note two sets of waves.)
T.P. Fig 71

Recent Observations. (optical vs. SAR images) T.P. Visual T.P. SAR

Munk, Scully-Power & Zachariasen Proc. Royal Soc. London A 412 231-254 (1987) Narrow V & Wedges.

Fig 70 Lighthill

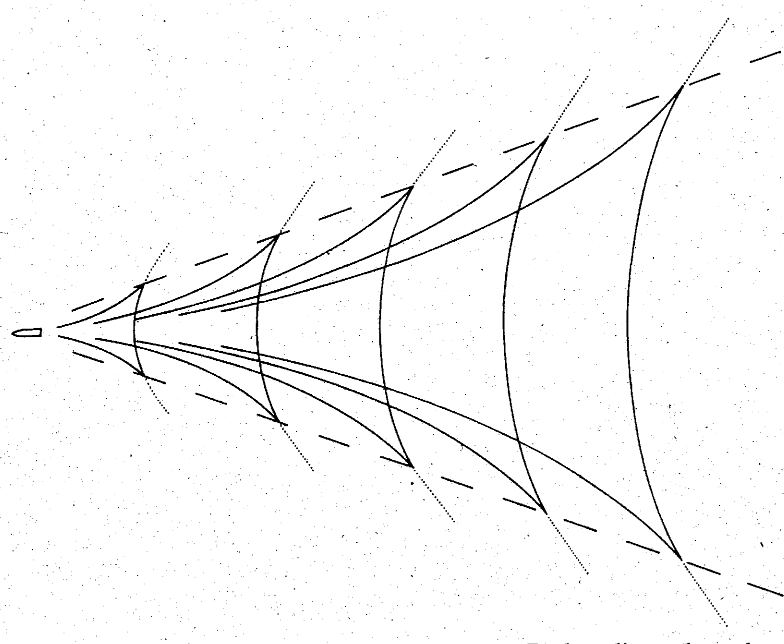


Figure 70. Plain lines: Kelvin ship-wave pattern. Broken lines: boundary of Kelvin wedge. Dotted lines: extension of waves beyond the Kelvin wedge indicated by the theory of sections 4.11 and 4.12.

Fig. 71 Lighthill (p. 278)

Figure 71. An observed ship-wave pattern. [*Courtesy of Aerofilms Ltd.*]