# ADVANCED CALCULUS
# FALL 2018

## LESSON 6: Cartesian Tensors

This lesson is a brief introduction to cartesian tensors. Here are some references for this material.

- Cartesian Tensors, Harold Jeffreys, Dover.
- Cartesian Tensors: An Introduction, George Temple, Dover.
- Mathematics Applied to Continuum mechanics, Lee Segel, SIAM (Chapters 1 and 2).
- Tensor Analysis, Theory and Applications, I. S. Sokolnikoff, John Wiley.
- Most books on analytical mechanics or continuum mechanics contain a discussion of tensors.

# 1 Introduction

An elementary introduction to vectors generally begins by defining the term *scalar*. One is reminded that a physical quantity such as mass, area, volume, speed and temperature can be fully expressed in terms of its magnitude once the unit of measurement is specified. Such a quantity is termed a scalar, perhaps because an instrument measuring it typically has a pointer moving over a scale. Other quantities such as force, velocity, acceleration and magnetic field require, in addition to magnitude, a direction for complete specification. Mathematically a scalar is represented by a real number while a vector can be represented by a directed line segment with its length specifying the magnitude and the arrow pointing the direction. Computations involving vectors, though capable of being defined geometrically (dot and cross products, for example), can be carried out more conveniently if the geometric representation is replaced by an equivalent algebraic representation. In $\mathcal{R}^3$ for example, once a coordinate frame, or basis, consisting of three linearly independent vectors is chosen, any vector can be expressed as a linear combination of the basis vectors, and the three coefficients in the linear combination (also referred to as coordinates or components) form an ordered triple that provides an algebraic representation of the vector. We speak of a Cartesian representation when the basis is orthonormal. Vector operations can now be expressed as algebraic operations on the components.

In a different basis the same vector would be represented by a different triple. We emphasize that *the vector as a physical entity does not depend upon the frame of reference but the ordered triple that allows us to represent or reconstruct it does.* Once the two bases are specified, there is a well-defined rule that yields the components of the vector in the new basis in terms of those in the old basis.

While scalars and vectors suffice to represent many physical quantities, there are others such as stress, strain and moment of inertia that are more complex and hence require more elaborate mathematical representations. In particular, while a vector possesses three components in $\mathcal{R}^3$ these more complicated quantities, as we shall see, possess more. Let us take the example of stress $\boldsymbol{T}$ which, being force per unit area, may be thought of as $\boldsymbol{T} = \boldsymbol{F}/\boldsymbol{A}$. The force $\boldsymbol{F}$ is a vector of course, but so is the area $\boldsymbol{A}$ as it has both size and orientation, the latter corresponding to the direction normal to the area. As division of vectors is not defined, it would make sense to interpret the above relation as $\boldsymbol{F} = \boldsymbol{T}\boldsymbol{A}$. Thus the stress $\boldsymbol{T}$ is a physical quantity that determines, for a given area $\boldsymbol{A}$, the force $\boldsymbol{F}$ acting on it. We can consider $\boldsymbol{T}$ as having *two* directions associated with it, those of $\boldsymbol{F}$ and $\boldsymbol{A}$, and therefore requiring *nine* components to represent it in a given basis. We call $\boldsymbol{T}$ a *tensor of rank two* or a *tensor of second order* and expect that its components will transform according to a well-defined rule, just as those of a vector do, under a change of basis. While a vector can be easily visualized as a physical or geometric entity that exists without reference to a basis, a tensor is less amenable to such a visual grasp even though it is also an object that exists without reference to a basis. It is this more abstract nature of a tensor that puts additional emphasis on its representation in a basis, and on the rule of transformation of this representation under a change of basis. In fact we can, and shall, define a tensor as *an object whose representation in a basis transforms in a certain way when the basis is changed.* Thus it is the rule of transformation that becomes the essence of the notion of the tensor.

We shall see that a scalar can be viewed as a rank-zero tensor and a vector as a rank-one tensor, and that tensors of rank higher than two are also relevant. The frame-invariant property of tensors makes them an ideal tool for the mathematical representation and study of the laws of physics which themselves are frame-invariant.

## 1.1 Stress tensor

Before launching into the algebra and calculus of tensors, it is worth describing in some detail how a tensor can arise in practice. We elaborate on the example of the stress tensor referred to above.

Consider the motion and/or deformation of a body of solid or fluid. There are two kinds of forces that act on matter in bulk. Forces of the first kind are known as body forces, which are long-range forces like gravity or electromagnetic force that penetrate into the interior of the body and act on each element of the body. The body force on an element is proportional to the size of the element. If $\Delta V$ is a small volume element of the body surrounding a point whose position vector is $\boldsymbol{x}$ then the body force on the element at time $t$ can be expressed as

$$\boldsymbol{F}(\boldsymbol{x}, t)\rho(\boldsymbol{x}, t)\Delta V,$$

where $\rho$ is the density. Here $\boldsymbol{F}$ is the body force per unit mass; $\boldsymbol{F} = \boldsymbol{g}$ in the case of gravity.

Forces of the second kind are surface forces. These are of molecular origin and are exerted on the common boundary between two interacting elements in direct contact. The surface force at a point within the body is specified by considering a small planar element of area $\Delta A$ surrounding the point, and identifying the force exerted by matter on one side of the element by the matter on the other side. The total force across the element is proportional to $\Delta A$ and for an element located at position $\boldsymbol{x}$ the force at time $t$ is given by

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\Delta A.$$

The force per unit area, $\boldsymbol{t}$, is called stress, and in addition to $\boldsymbol{x}$ and $t$ it also depends upon the orientation

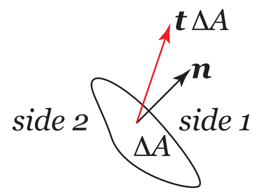


Figure 1: Force $\boldsymbol{t}\,\Delta A$ on a surface normal to $\boldsymbol{n}$.

of the surface element as determined by the unit normal vector $\boldsymbol{n}$. The usual sign convention about the direction of $\boldsymbol{n}$ is that $\boldsymbol{t}$ is the stress exerted by the matter on the side of the surface element to which $\boldsymbol{n}$ points (side 1 ), on the matter on the side which $\boldsymbol{n}$ points away from (side 2 in Figure 1). Thus the normal component of $\boldsymbol{t}$ having the same sense as $\boldsymbol{n}$ represents a tension. Of course the force exerted across the surface element by side 2 on side 1 is equal and opposite, *i.e.*,

$$\boldsymbol{t}(\boldsymbol{x}, t, -\boldsymbol{n})\Delta A = -\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\Delta A.$$

Since the surface element $\Delta A$ at the position $\boldsymbol{x}$ can have an infinity of possible orientations, it would appear that specifying the dependence of $\boldsymbol{t}$ upon $\boldsymbol{n}$ is a daunting task. We now show that such is not the case. Consider the forces acting instantaneously on a material element of volume $\Delta V$ in the shape of a tetrahedron embedded within the body of material and shown in Figure 2. The three orthogonal faces of

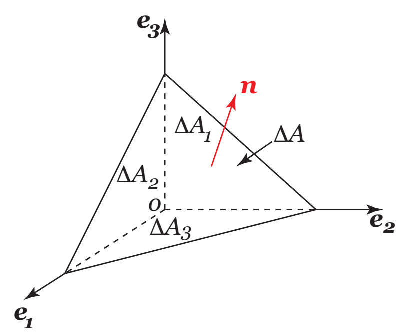


Figure 2: Tetrahedral volume element.

the tetrahedron have areas $\Delta A_1$, $\Delta A_2$ and $\Delta A_3$, while the slant face has area $\Delta A$. The three orthogonal unit vectors emerging from the vertex $O$ of the tetrahedron are $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. The outward unit normal to the face with area $\Delta A_j$ is $-\boldsymbol{e}_j$, and the outward unit normal to the slant face is $\boldsymbol{n}$. Surface forces act on each face of the tetrahedron and their sum is

$$\boldsymbol{t}(\boldsymbol{n})\Delta A + \boldsymbol{t}(-\boldsymbol{e}_1)\Delta A_1 + \boldsymbol{t}(-\boldsymbol{e}_2)\Delta A_2 + \boldsymbol{t}(-\boldsymbol{e}_3)\Delta A_3.$$

The dependence of $\boldsymbol{t}$ on $\boldsymbol{x}$ and $t$ is not displayed here, as $t$ is the same for each term in the above expression while $\boldsymbol{x}$ is approximately the same, for the small tetrahedron. Since $\Delta A_j = \boldsymbol{e}_j \cdot \boldsymbol{n}\ \Delta A$ and $\boldsymbol{t}(-\boldsymbol{e}_j) = -\boldsymbol{t}(\boldsymbol{e}_j)$, the sum of the surface forces can be written as

$$[\boldsymbol{t}(\boldsymbol{n}) - \{(\boldsymbol{e}_1 \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_1) + (\boldsymbol{e}_2 \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_2) + (\boldsymbol{e}_3 \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_3)\}]\Delta A. \tag{1.1}$$

Motion of the tetrahedron takes place according to Newton's law,

$$\text{mass} \times \text{acceleration} = \text{resultant of body forces} + \text{resultant of surface forces}.$$

Let $\Delta L$ characterize a typical linear dimension of the tetrahedral element. Then its volume $\Delta V$, and hence its mass, is of the order of $(\Delta L)^3$ while the area of each face, proportional to $\Delta A$, is of the order of $(\Delta L)^2$. Let $\Delta L \to 0$ without a change in the shape of the tetrahedron. Then the LHS of the above equation, as well as the body force term on the RHS, approach zero as $(\Delta L)^3$ while the surface force term approaches zero as $(\Delta L)^2$. Then satisfaction of the above equation in the limit necessarily requires the vanishing of the coefficient of $\Delta A$ in (1.1). Thus *the stresses are locally in equilibrium* and we can write

$$\boldsymbol{t}(\boldsymbol{n}) = (\boldsymbol{e}_1 \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_1) + (\boldsymbol{e}_2 \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_2) + (\boldsymbol{e}_3 \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_3). \tag{1.2}$$

The above equation states that *the stress on the slant face is determined entirely by the stresses on the three orthogonal faces and the orientation of the slant face.* Once the stresses on the three orthogonal faces are known, *any* slant face (*i.e.*, an infinity of possible orientations) can be accommodated by assigning the corresponding $\boldsymbol{n}$. Thus the stress system is not as complicated as feared earlier.

The discussion thus far has made no reference to a coordinate system; the vectors appearing in the expression (1.2) are all frame-independent entities. Let us now take the orthogonal triad $\boldsymbol{e}_j$, $j = 1, 2, 3$, as the coordinate system of choice. Let $\boldsymbol{e}_j \cdot \boldsymbol{n} = n_j$ and denote by $T_{ji}$ the $i$-th component of the vector $\boldsymbol{t}(\boldsymbol{e}_j)$, *i.e.*, the $i$-th component of the stress exerted on a planar face normal to the $j$-direction. Then the $i$-th component of $\boldsymbol{t}(n)$ can be written as

$$t_i(\boldsymbol{n}) = n_1 T_{1i} + n_2 T_{2i} + n_3 T_{3i} = \sum_{j=1}^{3} n_j T_{ji}. \tag{1.3}$$

We reiterate that the vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ do not depend upon the coordinate system employed, but their components $t_i$ and $n_j$ do. Therefore the quantity whose components in the system of choice are given by $T_{ji}$ must also be intrinsically frame-independent. In view of our discussion in the Introduction above it is proper to identify this quantity with nine components as a tensor of second order. This is the stress tensor; *it defines completely the state of stress at a point and allows the stress on a planar surface through the point with an orientation $\boldsymbol{n}$ to be computed by the equation above.*

Since $T_{ji}$ is the $i$-th component of the vector $\boldsymbol{t}(\boldsymbol{e}_j)$, we can write

$$\boldsymbol{t}(\boldsymbol{e}_j) = \sum_{i=1}^{3} T_{ji}\boldsymbol{e}_i. \tag{1.4}$$

Before proceeding further we pause to establish some useful notation and rules of manipulation.

# 2 Tensors: transformation and definition

In section 1.1 we saw that the stress tensor is a set of nine numbers $T_{ji}$ which are associated in a certain way with the coordinate system of choice. This association is symbolized by the two subscript indices; specifically, $T_{ji}$ is the $i$-th component of the stress on the coordinate plane orthogonal to the basis vector $\boldsymbol{e}_j$. We also observed that the stress tensor is a frame-independent object. An obvious question arises: given a set of nine numbers represented by a symbol with two subscript indices, say $S_{mn}$, how do we decide that these numbers are the components of a frame-independent object and that the set $S_{mn}$ can therefore rightly be called a tensor? We need to establish the constraint that the set of numbers must obey to qualify. The constraint, imposed by frame-independence, is that the set $S_{mn}$ must transform in a certain way under a change of coordinates. Such a transformation rule can then act as the basis for the definition of a tensor.

## 2.1 The transformation rule

Before discussing the transformation rule for tensors, it is instructive to see how the components of a vector transform under a change of basis.

### 2.1.1 Transformation rule for vectors

Let $x_i$ be the components of vector $\boldsymbol{x}$ in the unprimed basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $x'_j$ its components in the primed basis $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$. Then,

$$\boldsymbol{x} = x_i\boldsymbol{e}_i = x'_j\boldsymbol{e}'_j.$$

Now,

$$\boldsymbol{e}'_j = (\boldsymbol{e}_i \cdot \boldsymbol{e}'_j)\boldsymbol{e}_i = \alpha_{ij}\boldsymbol{e}_i, \tag{2.1}$$

where $\alpha_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}'_j$ is the cosine of the angle between the basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}'_j$. (Note that the indices $i, j$ in $\alpha_{ij}$ are chosen so that the first index $i$ goes with the unprimed vector $\boldsymbol{e}_i$ and the second index $j$ with the primed vector $\boldsymbol{e}_j$. We shall adhere to this convention throughout.) Then

$$x'_j\boldsymbol{e}'_j = \alpha_{ij}x'_j\boldsymbol{e}_i = x_i\boldsymbol{e}_i.$$

The second equality above can be rewritten as

$$(x_i - \alpha_{ij}x'_j)\boldsymbol{e}_i = \boldsymbol{0}.$$

As the basis vectors $\boldsymbol{e}_i$ are linearly independent, the above equation implies that

$$x_i = \alpha_{ij}x'_j. \tag{2.2}$$

Analogous arguments prove the converse,

$$x'_j = \alpha_{ij}x_i. \tag{2.3}$$

The last two equations comprise the *transformation rule for vectors.* (We repeat that in these relations *the second subscript goes with the prime.* This phrase will be a good aid to memory.) We call this set of nine quantities the *transformation array.*

**Exercise.** Show that the matrix forms of the above equations are, respectively,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix},$$

and

$$\begin{bmatrix} x'_1 & x'_2 & x'_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix},$$

or, more economically,

$$\boldsymbol{x} = \boldsymbol{L}\boldsymbol{x}' \quad \text{and} \quad \boldsymbol{x}'^T = \boldsymbol{x}^T\boldsymbol{L}, \tag{2.4}$$

where

$$\boldsymbol{L} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \tag{2.5}$$

is the transformation array written in a matrix form.

**Remark.** An important property of the transformation array is worth noting. By changing the dummy index from $i$ to $k$ equation (2.3) can also be written as $x'_j = \alpha_{kj}x_k$. Then it can be combined with equation (2.2) to yield

$$x_i = \alpha_{ij}\alpha_{kj}x_k.$$

However, we also have the relation

$$x_i = \delta_{ik}x_k.$$

Elimination of $x_i$ from the above pair of equations leads to

$$(\delta_{ik} - \alpha_{ij}\alpha_{kj})x_k = 0.$$

Since the vector components $x_k$ are arbitrary, the above relation can hold only if

$$\alpha_{ij}\alpha_{kj} = \delta_{ik}. \tag{2.6}$$

Similarly it can be shown that

$$\alpha_{ji}a_{jk} = \delta_{ik}. \tag{2.7}$$

In matrix form, equation (2.6) can be written as

$$\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{I}, \quad \text{or} \quad \boldsymbol{L}^{-1} = \boldsymbol{L}^T, \tag{2.8}$$

where $\boldsymbol{L}$ is defined above in equation (2.5). (The above result also follows from equations (2.4).) Matrices satisfying (2.8) are known as *orthogonal matrices*. Hence the transformation between the primed and the unprimed bases is known as an orthogonal transformation. An immediate consequence of (2.8) is the following constraint on the determinant of $\boldsymbol{L}$,

$$(\det \boldsymbol{L})^2 = 1 \quad \text{so that} \quad \det \boldsymbol{L} = \pm 1.$$

**Exercise.** What is the geometric interpretation of the signs in the above equation?

### 2.1.2 Transformation rule for tensors

We start by recalling that in a system with basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, the stress $\boldsymbol{t}(\boldsymbol{n})$ on an area element with unit normal $\boldsymbol{n}$ is given by equation (1.2), reproduced below in the summation notation.

$$\boldsymbol{t}(\boldsymbol{n}) = (\boldsymbol{e}_j \cdot \boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_j).$$

Here, $\boldsymbol{t}(\boldsymbol{e}_j)$ is the stress on the plane normal to the basis vector $\boldsymbol{e}_j$. For ease of writing we change the notation slightly, writing $\boldsymbol{t}_j$ for $\boldsymbol{t}(\boldsymbol{e}_j)$. Then the above equation becomes

$$\boldsymbol{t}(\boldsymbol{n}) = (\boldsymbol{e}_j \cdot \boldsymbol{n})\boldsymbol{t}_j. \tag{2.9}$$

With this provision equation (1.4) can be rewritten as

$$\boldsymbol{t}_j = T_{ji}\boldsymbol{e}_i, \tag{2.10}$$

where $T_{ji}$ are the components of the stress tensor in the basis being employed.

Let us now suppose that the coordinate system is changed via a rigid rotation so that the new basis vector triad is $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$, that the stress vector on the plane orthogonal to $\boldsymbol{e}'_m$ is $\boldsymbol{t}'_m$, and that the components of the stress tensor in the new system are $T'_{mk}$. On replacing $\boldsymbol{n}$ by $\boldsymbol{e}'_m$ in (2.9) we get

$$\boldsymbol{t}(\boldsymbol{e}'_m) = \boldsymbol{t}'_m = (\boldsymbol{e}_j \cdot \boldsymbol{e}'_m)\boldsymbol{t}_j.$$

Use of (2.10) in the above equation yields

$$\boldsymbol{t}'_m = (\boldsymbol{e}_j \cdot \boldsymbol{e}'_m)T_{ji}\boldsymbol{e}_i.$$

The analog of equation (2.10) in the primed system is

$$\boldsymbol{t}'_m = T'_{mk}\boldsymbol{e}'_k.$$

Elimination of $\boldsymbol{t}'_m$ from the last two equations leads to

$$T'_{mk}\boldsymbol{e}'_k = (\boldsymbol{e}_j \cdot \boldsymbol{e}'_m)T_{ji}\boldsymbol{e}_i. \tag{2.11}$$

With

$$\boldsymbol{e}_i = (\boldsymbol{e}_i \cdot \boldsymbol{e}'_k)\boldsymbol{e}'_k,$$

(2.11) becomes

$$\begin{aligned} T'_{mk}\boldsymbol{e}'_k &= (\boldsymbol{e}_j \cdot \boldsymbol{e}'_m)T_{ji}(\boldsymbol{e}_i \cdot \boldsymbol{e}'_k)\boldsymbol{e}'_k \\ &= \alpha_{jm}\alpha_{ik}T_{ji}\boldsymbol{e}'_k. \end{aligned} \tag{2.12}$$

where $\alpha_{jm}$ and $\alpha_{ik}$ are the elements of the transformation array introduced above. As the vectors $\boldsymbol{e}'_k$ are linearly independent the above equation reduces to [1]

$$T'_{mk} = \alpha_{jm}\alpha_{ik}T_{ji}. \tag{2.13}$$

By similar arguments the converse of the above result can be derived.

$$T_{ji} = \alpha_{jm}\alpha_{ik}T'_{mk}. \tag{2.14}$$

These two relations together constitute the transformation rule that relates the components of the stress tensor between the two coordinate systems. Note that again, the second subscript goes with the prime. Note also that each of the above relations is in fact a set of nine equations, another example of the economy of the index notation.

**Exercise.** Show that the matrix forms of equations (2.13) and (2.14) are, respectively,

$$\boldsymbol{T}' = \boldsymbol{L}^T\boldsymbol{T}\boldsymbol{L} \quad \text{and} \quad \boldsymbol{T} = \boldsymbol{L}\boldsymbol{T}'\boldsymbol{L}^T.$$

### 2.1.3 Definition of a tensor

We are now ready to define a cartesian tensor of rank 2.

**Def.** Let $\boldsymbol{S}$ be an entity with 9 components $S_{ij}$ in a given basis. Then $\boldsymbol{S}$ is a tensor of rank 2 if under a change of basis the components of $\boldsymbol{S}$ change according to the tensor transformation rule, namely,

$$S'_{mk} = \alpha_{jm}\alpha_{ik}S_{ji}.$$

We can go on to define higher-order tensors in a similar way. Thus $S_{jip}$, a set of 27 numbers, is a third-order cartesian tensor if it obeys the transformation rule

$$S'_{mkq} = \alpha_{jm}\alpha_{ik}\alpha_{pq}S_{jip}.$$

Generalization to an $n$-th order tensor consisting of $3^n$ components is straightforward.

**Remarks.**

- We emphasize that a tensor is not characterized simply by the number of components and their association with a certain number of indices. The essential features are that (i) the indices must refer to a single basis and that (ii) the components must obey the transformation rule. Thus the transformation array $\alpha_{ij}$ does not constitute a tensor as its two indices refer to two different bases.
- Let $\boldsymbol{T}$ be an entity with 9 components $T_{ij}$ in a given basis. According to the definition above, $\boldsymbol{T}$ is a tensor if under a change of basis the components of $\boldsymbol{T}$ change according to the tensor transformation rule. The quotient rule described below provides a short cut in establishing that $\boldsymbol{T}$ is a tensor.

[1] Equation (2.12) can be written as

$$[T'_{mk} - \alpha_{jm}\alpha_{ik}T_{ji}]\boldsymbol{e}'_k = \boldsymbol{0}.$$

For each $m$ the LHS of the equation is a linear combination of the linearly independent vectors $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$. As the linear combination adds up to the zero vector, the coefficients $T'_{mk} - \alpha_{jm}\alpha_{ik}T_{ji}$ in the combination must vanish.

**Quotient rule.** Let $\boldsymbol{a}$ be an arbitrary vector and let $b_i = T_{ij}a_j$. If $\boldsymbol{b}$ transforms as a vector, then $\boldsymbol{T}$ is a second-rank tensor.

To prove this, consider transformation to a primed basis so that

$$b'_i = T'_{ij}a'_j.$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ transform as vectors, we have

$$\alpha_{mi}b_m = T'_{ij}\alpha_{mj}a_m,$$

or, upon replacing $b_m$ by $T_{mj}a_j$,

$$\begin{aligned} \alpha_{mi}T_{mj}a_j &= T'_{ij}\alpha_{mj}a_m \\ &= T'_{im}\alpha_{jm}a_j \end{aligned}$$

where we have interchanged the dummy indices $j$ and $m$ on the RHS. Then,

$$(\alpha_{mi}T_{mj} - \alpha_{jm}T'_{im})a_j = 0.$$

Since the components of $\boldsymbol{a}$ are arbitrary, the above equation reduces to

$$\alpha_{mi}T_{mj} = \alpha_{jm}T'_{im}.$$

Upon multiplying both sides by $\alpha_{jk}$ we get

$$\alpha_{mi}\alpha_{jk}T_{mj} = \alpha_{jm}\alpha_{jk}T'_{im}.$$

Since $\alpha_{jm}\alpha_{jk} = \delta_{mk}$, the above reduces to

$$T'_{ik} = \alpha_{mi}\alpha_{jk}T_{mj},$$

which proves that $\boldsymbol{T}$ transforms as a tensor.

**Remark.** The quotient rule $b_i = T_{ij}a_j$ allows a tensor to be interpreted as the ratio, or quotient, of two vectors.

## 2.2 Examples of tensors

(i) A scalar is a zero-order tensor.

(ii) Any vector is a first-order tensor. Consider, for example, the gradient of a scalar-valued function $f(x_1, x_2, x_3)$. Under a change of coordinates from the unprimed to the primed system the components of $\nabla f$ transform as follows.

$$\frac{\partial f}{\partial x'_j} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial x'_j} = \frac{\partial f}{\partial x_i}\frac{\partial(\alpha_{ij}x'_j)}{\partial x'_j} = \alpha_{ij}\frac{\partial f}{\partial x_i}, \tag{2.15}$$

where we have made use of equation (2.2). Since $\nabla f$ obeys the transformation rule for vectors, it is a first-order tensor.

(iii) Some examples of tensors of second order are given below.

(a) Consider the transformation of the set of nine quantities defined by the product $x_jx_i$ where the $x_i$ are components of a vector $\boldsymbol{x}$. Application of the vector transformation rules derived above leads immediately to the relations

$$x_jx_i = \alpha_{jm}x'_m\alpha_{ik}x'_k = \alpha_{jm}\alpha_{ik}x'_mx'_k, \quad \text{and conversely,} \quad x'_mx'_k = \alpha_{jm}\alpha_{ik}x_jx_i. \tag{2.16}$$

A comparison of the above equations with (2.13) and (2.14) shows that the components of the stress tensor transform in exactly the same way as do the products $x_jx_i$.

(b) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors with components $x_i$ and $y_j$. Then the 9 components of the product $x_i y_j$ form a second-order tensor. The argument is the same as that leading to the derivation of (2.16). This tensor is known as the *tensor product* of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. The tensor product of two vectors is also called a *dyad.*

(c) The Kronecker delta is a second-order tensor. To show that such is the case, one must prove that

$$\delta'_{mn} = \alpha_{im}\alpha_{jn}\delta_{ij}.$$

The RHS immediately reduces to $\alpha_{im}\alpha_{in}$ which, in view of (2.7), reduces further to $\delta_{mn}$. Therefore,

$$\delta'_{mn} = \delta_{mn}$$

*i.e.*, the set of numbers $\delta_{mn}$ is transformed into itself by the tensor transformation rule. Hence the Kronecker delta is a second-order tensor.

(d) **Moment of inertia tensor**

Consider a point mass $m$ rotating with angular velocity $\boldsymbol{\Omega}$. With the origin on the axis of rotation, let $\boldsymbol{x}$ be the position of the mass in a coordinate system attached to the body. Then the velocity of the mass relative to a stationary frame outside the body is $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{x}$ and the angular momentum $\boldsymbol{h} = \boldsymbol{x} \times m\boldsymbol{v} = m\boldsymbol{x} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$. The corresponding expression for the angular momentum of a rigid body $\mathcal{B}$ is

$$\boldsymbol{H} = \iiint_{\mathcal{B}} \boldsymbol{x} \times (\boldsymbol{\Omega} \times \boldsymbol{x})\rho(\boldsymbol{x})\, dV,$$

where $\rho(\boldsymbol{x})$ is the density of the body. The index notation allows $\boldsymbol{x} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$ to be manipulated as follows.

$$\begin{aligned}
\big(\boldsymbol{x} \times (\boldsymbol{\Omega} \times \boldsymbol{x})\big)_i &= \epsilon_{ijk} x_j (\boldsymbol{\Omega} \times \boldsymbol{x})_k \\
&= \epsilon_{ijk} x_j (\epsilon_{kmn}\Omega_m x_n) \\
&= \epsilon_{ijk}\epsilon_{kmn} x_j \Omega_m x_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\Omega_m x_j x_n \\
&= \Omega_i x_j x_j - \Omega_j x_j x_i.
\end{aligned}$$

Continuing with the index notation,

$$\begin{aligned}
H_i &= \iiint_{\mathcal{B}} \rho(\Omega_i x_k x_k - \Omega_j x_j x_i)\, dV \\
&= \iiint_{\mathcal{B}} \rho(\delta_{ij} x_k x_k - x_j x_i)\Omega_j\, dV \\
&= \left[\iiint_{\mathcal{B}} \rho(\delta_{ij} x_k x_k - x_j x_i)\, dV\right]\Omega_j \\
&= I_{ij}\Omega_j.
\end{aligned}$$

Here,

$$I_{ij} = \iiint_{\mathcal{B}} \rho(\delta_{ij} x_k x_k - x_j x_i)\, dV \tag{2.17}$$

is the *moment-of-inertia tensor* of the rigid body.