d.e.'s $\quad \text{div}\, \vec{u} = 0 \quad$ (continuity)

(N-S eqns. written in Vector form) $\quad \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \frac{1}{R_e}\nabla^2\vec{u} \quad$ (momentum)

Introduce small disturbances $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}$

$$u = u_0 + \tilde{u}, \quad v = \tilde{v}, \quad w = \tilde{w}, \quad p = p_0 + \tilde{p}$$

Linearized disturbance equations become:

(continuity gives) $\quad \tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0$

(x-mom. gives) $\quad \tilde{u}_t + u_0\tilde{u}_x + \tilde{v}\,u_0' = -\tilde{p}_x + \frac{1}{R_e}\nabla^2\tilde{u}$

(y- " ") $\quad \tilde{v}_t + u_0\tilde{v}_x = -\tilde{p}_y + \frac{1}{R_e}\nabla^2\tilde{v}$

(z) $\quad \tilde{w}_t + u_0\tilde{w}_x = -\tilde{p}_z + \frac{1}{R_e}\nabla^2\tilde{w}$

$$\left.\begin{array}{l}\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0 \\ \tilde{u}_t + u_0\tilde{u}_x + \tilde{v}\,u_0' = -\tilde{p}_x + \frac{1}{R_e}\nabla^2\tilde{u} \\ \tilde{v}_t + u_0\tilde{v}_x = -\tilde{p}_y + \frac{1}{R_e}\nabla^2\tilde{v} \\ \tilde{w}_t + u_0\tilde{w}_x = -\tilde{p}_z + \frac{1}{R_e}\nabla^2\tilde{w}\end{array}\right\} \quad (\text{I})$$

where $u_0' \equiv \frac{du_0}{dy}$

Choose exponential form for disturbances

$$\left.\begin{array}{l}\tilde{u} = u_1(y)\,E \\ \tilde{v} = v_1(y)\,E \\ \tilde{w} = w_1(y)\,E \\ \tilde{p} = p_1(y)\,E\end{array}\right\} \quad (\text{II})$$

Where $E = e^{i[\alpha(x-ct)+\beta z]}$

α and β Wavenumber in x and z directions and are real

but c is complex, i.e. $c = c_r + i c_i$

For instability, $c_i > 0 \Rightarrow$ instabilities grow in time.

(With this formulation we are deriving the three-dimensional Orr-Sommerfeld equation which describes the stability of a general shear flow and has applications beyond the Poiseuille flow)

Substitute into the linearized equations of motion (Ⅱ into Ⅰ)

① $$i\alpha u_1 + v_1' + i\beta w_1 = 0$$

② $$-i\alpha c u_1 + i\alpha u_0 u_1 + v_1 u_0' = -i\alpha p_1 + \frac{1}{Re}\left(\frac{d^2}{dy^2}-\alpha^2-\beta^2\right)u_1$$

③ $$-i\alpha c v_1 + i\alpha u_0 v_1 = -p_1' + \frac{1}{Re}\left(\frac{d^2}{dy^2}-\alpha^2-\beta^2\right)v_1$$

④ $$-i\alpha c w_1 + i\alpha u_0 w_1 = -i\beta p_1 + \frac{1}{Re}\left(\frac{d^2}{dy^2}-\alpha^2-\beta^2\right)w_1$$

4 equations for 4 unknowns (u_1, v_1, w_1, p_1).

Combine these equations to get a single equation containing only v_1, you accomplish this by adding α② + β④, eliminating $\alpha u_1 + \beta w_1$ using ①. Differentiate that result to find p_1' and substitute in ③.

Find:

$$\boxed{(u_0-c)\left(\frac{d^2}{dy^2}-\alpha^2-\beta^2\right)v_1 - u_0'' v_1 = \frac{-i}{\alpha Re}\left(\frac{d^2}{dy^2}-\alpha^2-\beta^2\right)^2 v_1}$$

"Orr-Sommerfeld equation" (which governs the vertical component of disturbance velocity in a shearing flow, when flow can be 3D but base state shearing)

(you can learn a very profound fact about the difference between stability of 2-dimensional and 3-dimensional disturbances)

Can relate this eqn. to an equivalent 2D disturbance eqn. by replacing $\underset{\left(\frac{2\pi u}{\nu}\right)}{\alpha Re}$ with $\underset{\left(\frac{2\pi u}{\nu}\right)}{(\alpha^2+\beta^2)^{1/2}} Re_{(2D)}$

i.e.
$$Re_{2D} \equiv \underbrace{\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}_{\text{like a direction cosine}} \underset{\overset{\uparrow}{3D}}{Re}$$
observe that this will always be <1

$\Rightarrow Re_{2D} < Re$.

Leads to Squire's theorem:

Equivalent 2D problem has a lower Reynolds number, thus, a 3D instability corresponds to a 2D instability at a lower Reynolds number. i.e. 2D disturbances become unstable at a lower Reynolds number "Squire's Theorem"

Stated another way: 3D disturbances in a wave traveling in a different direction from the basic flow, only the component of basic flow velocity in the direction of wave affects the disturbance.

(this explains fundamentally why we always see 2D disturbances growing first, then we see 3D disturbances.) e.g. T-S waves in BL, Kelvin-Helmholtz instability, i.e. roll-up of a vortex sheet

(So, because we are interested in a linear theory) we only consider 2D disturbances.

(We can either avoid writing the equation for $\tilde{w}$ or let the wavelength of disturbances in z-direction go to ∞, hence) $\beta = 0$

Orr-Sommerfeld eqn. for 2D disturbances becomes:

$$(U_0 - c)\left(\frac{d^2}{dy^2} - \alpha^2\right) V_1 - U_0'' V_1 = -\frac{i}{\alpha R_e}\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 V_1$$

B.C. (for 2D Channel): $V_1 = 0$ at $y = \pm 1$

and

$\frac{dV_1}{dy} = 0$ at $y = \pm 1$

↖ (get from continuity)

recall V_1 is defined by: $V = \tilde{V} = V_1(y)\,E$, $E = e^{i\alpha(x-ct)}$

In terms of Perturbation stream function: $\psi = \phi(y)\,E$

$\tilde{u} = \frac{\partial \psi}{\partial y} = \phi'(y)\,E$ $\qquad$ $\tilde{V} = -\frac{\partial \psi}{\partial x} = -i\alpha\,\phi(y)\,E$

$\Downarrow$

$u_1 = \phi'$ $\qquad$ $V_1 = -i\alpha\phi$

Thus,
Orr-Sommerfeld eqn. has identical form for ϕ as for V_1

$$(U_0 - c)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi - U_0''\phi = -\frac{i}{\alpha R_e}\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 \phi$$

B.C. $\phi = \frac{d\phi}{dy} = 0$ at $y = \pm 1$

We will consider the limiting case of inviscid flow ($R_e \to \infty$)

(OS eqn becomes)

$$(U_0 - c)(\phi'' - \alpha^2\phi) - U_0''\phi = 0$$ "Rayleigh Eqn."

(from this we can show why a velocity profile with an inflection point is unstable)

(Show this by:)

multiply by $-\dfrac{\phi^*}{U_0-c}$, where ϕ^* is the complex conjugate of ϕ ($\phi = \phi_r + i\,\phi_i$

$\downarrow$ 3-28-11 $\Rightarrow \phi^* = \phi_r - i\,\phi_i$)

and integrate (w.r.t. y).

$$-\int_{-1}^{1} \phi^* \phi'' \, dy + \alpha^2 \int_{-1}^{1} \phi^* \phi \, dy + \int_{-1}^{1} \frac{U_0'' \phi^* \phi}{(U_0-c)} \, dy = 0$$

Note: $\phi^* \phi = |\phi|^2 \; (>0)$

(in general if $Z = x + iy$ and its complex conjugate $Z^* = x - iy$

$$\Rightarrow Z^* Z = x^2 + y^2 \equiv |Z|^2$$

(also $\dfrac{1}{U_0 - c} \cdot \dfrac{U_0 - c^*}{U_0 - c^*} = \dfrac{U_0 - c^*}{|U_0 - c|^2}$ (since U_0 is real)

Integrate the first integral by parts:

$$-\underset{0}{\cancel{\phi^* \phi'}}\Big|_{-1}^{1} + \int_{-1}^{1} \phi'^* \phi' \, dy$$

Thus eqn. becomes

$$\int_{-1}^{1} \left(|\phi'|^2 + \alpha^2 |\phi|^2 \right) dy + \int_{-1}^{1} \frac{U_0'' (U_0 - c^*) \, |\phi|^2}{|U_0 - c|^2} \, dy = 0$$

(

Write $c = c_r + i\,c_i$, $c^* = c_r - i\,c_i$

and consider the imaginary part of the equation:

$$c_i \int_{-1}^{1} \frac{U_0'' \, |\phi|^2}{|U_0 - c|^2} \, dy = 0$$

(Says that if $c_i \neq 0$, intergrand must change sign in $-1 < y < 1$ and since $|\phi|^2$ and $|U_0 - c|^2$ are always positive, U_0'' must be zero somewhere in $-1 < y < 1$)

Recall disturbances are of the form: e.g. $\tilde{V} = V_1(y) \, e^{i[\alpha(x-ct) + \beta z]}$

and since α which is the wave number (in x-direction) $= \frac{2\pi}{\lambda}$ is real, thus $c_i = 0$ is neutrally stable $\left(e^{i\alpha(-i c_i)t} \to e^{\alpha c_i t}\right)$ and $c_i < 0 \Rightarrow$ disturbances dampen and since there is no viscosity (which is the only dampening mechanism) is not a possibility.

Thus, "a necessary condition for instability ($c_i \neq 0$) when $Re \to \infty$ ($\nu \to 0$) is that $U_0(y)$ must have an inflection point" (Rayleigh)

Notice however, $U_0(y)$ for Poiseuille flow ($U_0 = 1 - y^2 \Rightarrow U'' \neq 0$) has no inflection point. Hence in the inviscid limit Poiseuille flow is stable. Since instability is observed experimentally, must conclude that in this case viscosity is destablizing. "Viscous Instability" (to see how this is possible, briefly,)

If you integrate disturbance kinetic energy $\frac{|\vec{v}|^2}{2}$ across the channel find that phase changes in $\tilde{u}$ and $\tilde{v}$, Caused by Viscosity, can be such that the Reynolds Stress term $(\tilde{u}\tilde{v})$ leads to instability.

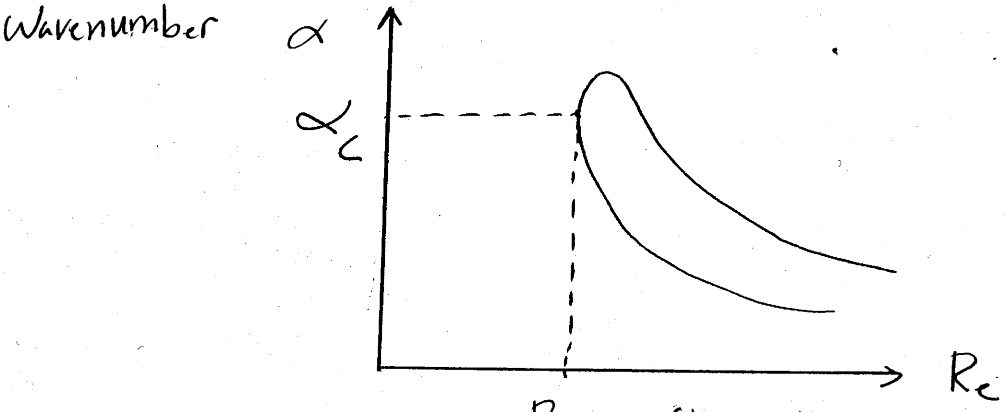
—— (Finally)

To obtain a stability boundary, set $c_i = 0$ in O-S eqn. The Problem becomes eigenvalue type of the form:

$$F(\alpha, c_r, Re) = 0$$

Solution was obtained numerically

(Orszag 1971, JFM 50, 689-703)

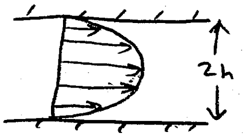


(note, you get a similar picture for the instability of a laminar BL, Blasius profile doesn't have an inflection point in it either) (T.P.)

Re_{crit} (below this Re, infinitesimal disturbances of any wavelength are dampened by viscosity)

$$\boxed{\begin{aligned} Re_c &= 5772 \\ \alpha_c &= 1.0205 \\ c_r &= 0.264 \end{aligned}}$$

$\left(Re = \frac{h\, U_{0max}}{\nu} \right)$



(recall) $\alpha = \frac{2\pi}{\lambda}$ $\quad \lambda_c = \frac{2\pi}{1.0205} = 6.157$ (in dimensional form, $\lambda_c = 6.157\,h$)

$(2\text{-D} \Rightarrow \beta = 0)$

(recall) $\tilde{u} = u_1(y)\, e^{i\alpha(x-ct)}$

$\tilde{v} = V_1(y) \ldots$

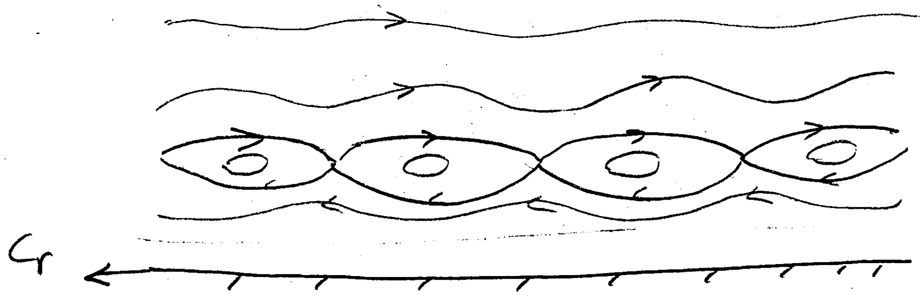
$\tilde{p} = \ldots$

this c_r is speed that the most unstable wavelength travels at.

The "critical layer", where $U_0 = c_r$, which in this case is close to the wall:

it occurs: $c_r = 0.264 = U_0 = 1 - y^2 \Rightarrow y = \pm 0.858$

If you travel at c_r, you would see:



Kelvin "Cat's-eye" pattern (coordinate system moving at c_r)

(Video on stability)

III) Vorticity dynamics

(first, a review)

(we will try and cover selected portions of material covered in the first 16 chapters of Lugt, Introduction to Vortex Theory)

a) Vortex Laws

i) Definitions: Vorticity $\boxed{\vec{\omega} \equiv \text{curl } \vec{u}}$ or $\vec{\omega} = \nabla \times \vec{u}$

(note: some books define as $\frac{1}{2}$ curl)

In cartesian coordinates: (x, y, z)

$$\vec{u} = u\vec{i} + v\vec{j} + w\vec{k} \quad \text{and} \quad \text{curl } \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\vec{\omega} = \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{\substack{\omega_x \text{ (or } \omega_1) \\ x\text{-component of vorticity}}} \vec{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{\substack{\omega_y \ (\omega_2) \\ y\text{-comp}}} \vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\substack{\omega_z \ (\omega_3) \\ z\text{-comp.}}} \vec{k}$$

(since vorticity is related to fluid motion, it's worthwhile to think about fluid) Kinematics

(unlike in solid mechanics, where deformations are usually quite small, in fluids deformations are large. Thus in fluid mechanics we deal with deformation rate)

e.g. We've seen $\frac{\partial u_i}{\partial x_i}$ is linear strain, and the general cartesian tensor

$$\frac{\partial u_i}{\partial x_j} = \overbrace{\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{"deformation" or "rate of strain tensor"}}}^{\text{symmetric part}} + \overbrace{\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{"rate of rotation tensor"}}}^{\text{anti-symmetric part}}$$

Rate-of-strain tensor:

(in 3-D Cartesian coordinates)

$$\varepsilon_{ij} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

(notice that it is symmetric) There are <u>6</u> numbers altogether (not 9)

Rate of rotation tensor: $\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$

This is described by 3 numbers:

$$(2\Omega_{ij}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Note using convention

$$\begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix}$$

where $\vec{\omega} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$ is the Vorticity Vector

(or $\omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$)

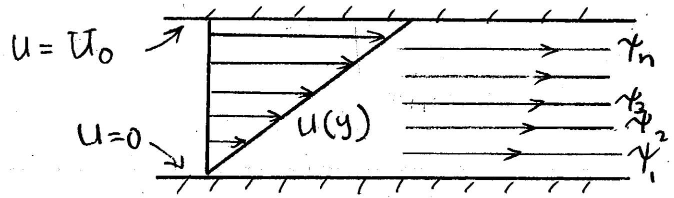
Thus the Vorticity <u>Vector</u> is associated with the antisymmetric tensor Ω_{ij} (rate of rotation tensor).

This is one way of interpreting vorticity $\vec{\omega}$ i.e. twice the angular velocity of fluid element.

<u>Examples:</u> Consider a Plane Couette flow (Laminar) $u = (\frac{y}{h})U_0$, $V = W = 0$

2D

$\left(W = \frac{\partial()}{\partial z} \equiv 0\right)$



$$\varepsilon_{ij} = \begin{pmatrix} 0 & \frac{U_0}{2h} & 0 \\ \frac{U_0}{2h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Omega_{ij} = \begin{pmatrix} 0 & \frac{U_0}{2h} & 0 \\ -\frac{U_0}{2h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Says that there is only one component of Vorticity $\left(\text{cf. } 2\Omega_{ij} = \begin{matrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{matrix}\right)$ $\underset{\omega_3 \nearrow}{\omega_z} = -\dfrac{U_0}{h}$

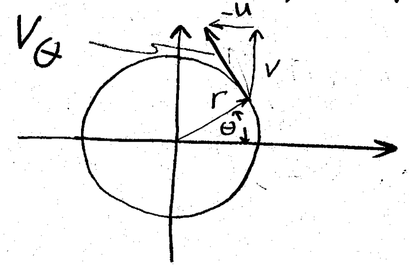
Note (in this flow) ① Vorticity is not a function of y (same everywhere)

② negative, i.e. Clockwise □ later → ▱

Also note: finite Vorticity does not necessarily imply circular or even curved streamlines.

(Another example which illustrates this fact)

Example: Consider fluid flow, in the absence of deformation, around a point "solid body rotation"



Flow field is defined in cylindrical coordinates as:

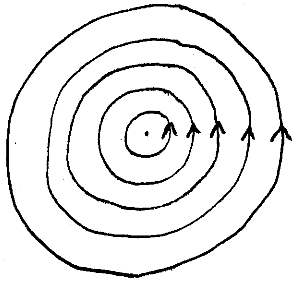
$$\left\{\begin{array}{l} V_r = 0 \\ V_\theta = \Omega r \end{array}\right\}$$

and in terms of Cartesian velocity components:

$$-u = V_\theta \sin\theta = \Omega r \sin\theta = \Omega y$$

Streamlines:

$$v = V_\theta \cos\theta = \Omega r \cos\theta = \Omega x$$



And Vorticity: $\omega_x = \omega_y = 0$ $\left(W = \dfrac{\partial()}{\partial z} = 0\right)$ since 2D

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Omega - (-\Omega) = 2\Omega$$

Note that it is the same everywhere.

(to avoid having to convert the velocity when given in cylinder into Cartesian coord. and then only being able to find Vorticity in Cartesian coord.)

Vorticity in Cylindrical Coordinates: (Yih)

$$\nabla \times \vec{u} = \begin{vmatrix} \frac{1}{r}e_r & e_\theta & \frac{1}{r}e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix}$$

$$\vec{\omega} = \nabla \times \vec{u}$$

$$= \underbrace{\frac{1}{r}\left(\frac{\partial V_z}{\partial \theta} - \frac{\partial (r V_\theta)}{\partial z}\right)}_{\omega_r} \underset{\substack{\uparrow \\ \text{Unit Vector} \\ \text{in r-direction}}}{\vec{e}_r} + \underbrace{\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)}_{\omega_\theta} \vec{e}_\theta + \underbrace{\left(\frac{1}{r}\left[\frac{\partial (r V_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta}\right]\right)}_{\omega_z} \underset{\substack{\uparrow \\ \text{Sa} \\ \text{as} \\ \vec{k}}}{\vec{e}_z}$$

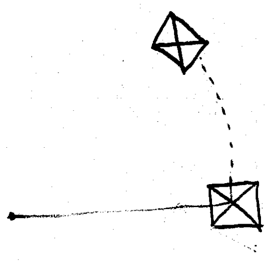
In this flow $0 = \underbrace{\frac{\partial ()}{\partial \theta}}_{\text{(axisymmetric)}} = \frac{\partial ()}{\partial z} = V_r = V_z \Rightarrow \omega_r = \omega_\theta = 0$

(and)

For any axisymmetric flow:

$$\omega_z = \frac{1}{r}\left(\frac{\partial (r V_\theta)}{\partial r} - \cancelto{0}{\frac{\partial V_r}{\partial \theta}}\right) = \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}$$

For solid-body rotation $(V_\theta = \Omega r)$: $\omega_z = 2\Omega$ (Same result, of course, as with ω_z using Cartesian form of equations)



Rotation of fluid element, defined as the average rotation of the two orthogonal line segments, is equal to Ω and its vorticity is twice Ω.

(Final)

Example Consider a Point Vortex

"Potential" "

"Free" "

"Line" " (due to the two-dimensionality of this concept.)

$\Gamma \equiv \oint \vec{u} \cdot d\vec{\ell}$

Flow field is defined by: $\begin{cases} V_\theta = \frac{\Gamma}{2\pi r} \\ V_r = 0 \end{cases}$ Where Γ is the circulation ($\frac{\Gamma}{2\pi}$ denoted as K, is often referred to as "Vortex strength")

for $r > 0$

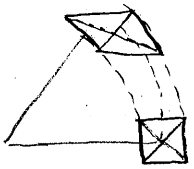
Since it is also two-dimensional, the r and θ-Components of Vorticity are identically Zero. $(\omega_r = \omega_\theta = 0)$

$$\omega_z = \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} \quad \text{(Since it's axisymmetric)}$$

$$= \frac{\Gamma}{2\pi r^2} + \left(\frac{\Gamma}{2\pi}(-r^{-2})\right) = 0 \qquad r > 0$$

Means that for $r > 0$, there is no Vorticity (flow is irrotational, and that's why we can use Potential flow theory to describe its motion) Note that the Velocity is singular at $r=0$, where the Vorticity must be infinite

(Physical explaination of why no Vorticity:)



there is deformation but not rotation.

(instantaneous idea, can not extend to large times)

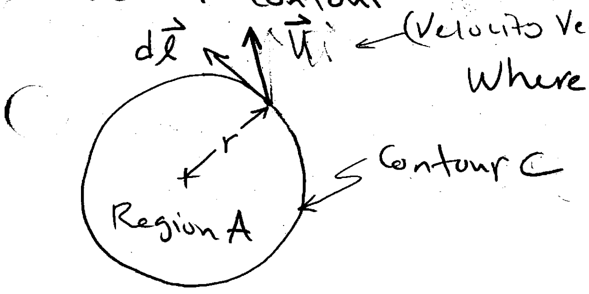
(We can define a Vortex then following Lugt:)

"A Vortex is the rotating motion of a multitude of material Particles around a common center"

(a Vortex appears to be a more intuitive concept then vorticity, but ironically it is in fact more troublesome to define and treat rigorously than Vorticity)

Kinematics of rotation

Consider the angular velocity around a circular contour



Where $d\vec{l} = \vec{e}_t \, |d\vec{l}|$

↑ unit vector tangent to C

Angular Velocity at any Point on the Contour is: $\zeta = \dfrac{\vec{u}(r,\theta)\cdot\vec{e}_t}{r}$

Average angular Velocity around the contour is:

(ccw contour, i.e. region always to left of path) - - - →

$$\text{Avg. } \zeta = \frac{\oint_C \frac{\vec{u}}{r}\cdot d\vec{l}}{\oint_C \vec{e}_t \cdot d\vec{l}} = \frac{1}{2\pi r^2}\oint \vec{u}\cdot d\vec{l}\,.$$

Using Stokes theorem:

$$\oint_C \vec{u}\cdot d\vec{l} = \iint_S (\text{curl}\,\vec{u})\cdot\vec{n}\; ds$$

(right-hand rule understood)

$$\underbrace{\Gamma \equiv \qquad\qquad}_{\text{circulation around contour c}} = \underbrace{\iint_S \vec{\omega}\cdot\vec{n}\; ds}_{\text{Flux of Vorticity through enclosed Surface S}}$$

(using Stokes theorem, we find:) Tip!

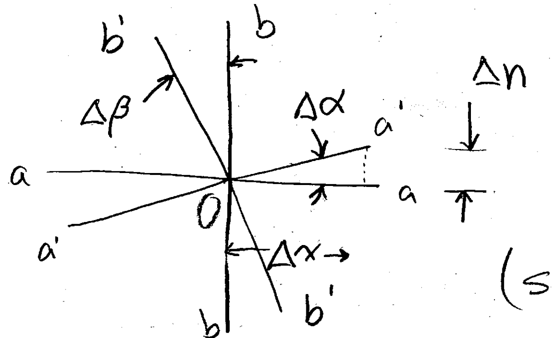
in the limit $r \to 0$

$$\text{Avg } \zeta = \frac{1}{2\pi r^2}(\vec{\omega}\cdot\vec{n})\pi r^2 = \frac{1}{2}(\vec{\omega})_n \leftarrow \text{normal component}$$

Thus the normal component of vorticity represents rotation, $\zeta_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

(again, you get the result that Vorticity is 2 times the avg. rate of rotation)

Alternatively, we can define rate of rotation as the average rotation of 2 initially ⊥, infinitesimal line elements in the fluid, i.e.



$$\zeta_{aa} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t}$$

$$(\text{similarly,})\ \zeta_{bb} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t}$$

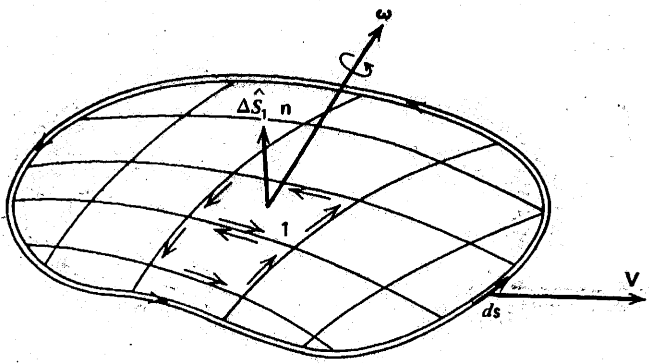


Fig. 15. Stokes' theorem.

Define fluid element rotation as: $\zeta \equiv \frac{1}{2}(\zeta_{aa} + \zeta_{bb})$

($\frac{\Delta \alpha}{\Delta t} = \frac{\Delta n / \Delta x}{\Delta t}$

If the vertical velocity at Point 0 is v_0, the velocity at Point a is:

$$v_a = v_0 + \frac{\partial v}{\partial x} \Delta x$$

Length $\overline{aa'} = \Delta n = \frac{\partial v}{\partial x} \Delta x \, \Delta t$

Substituting we get: $\zeta_{aa} = \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overbrace{\frac{\partial v}{\partial x} \Delta x \, \Delta t}^{\Delta n} / \Delta x}{\Delta t} = \frac{\partial v}{\partial x}$

Similarly find $\zeta_{bb} = -\frac{\partial u}{\partial y}$, thus the fluid rotation is:

$$\zeta_z = \frac{1}{2}(\zeta_{aa} + \zeta_{bb}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

((the nice thing about the ⊥ fluid elements is that it Provides an insight into what a rotational fluid is, i.e. the average rate of rotation of 2 initially ⊥ line elements, it's Physical & distinct from angular momentum)

Qualitatively:

* If both elements rotate by ≈ equal amounts: you have a Vortex flow

* If only one element rotates (shear flow) in which case the flow is rotational but there are no vortices (e.g. Couette flow)

(of course you might have irrotational flow around a circle, as we saw in the case of the Potential Vortex, where $V_\theta \propto \frac{1}{r}$) and neither element rotates

T.P. (K&C Fig 11&12)

(we can now try to define a vortex in more rigorous terms)

For inviscid flows (ideal fluid), Saffman (Vortex Dynamics, 1992) ((Like Lugt's book, this is also on reserve) defines a vortex:

"A Vortex is a finite, simply connected, but deformable region of Vorticity surrounded by an irrotational fluid flow"

(if viscosity is not negligible, then diffusion won't allow Vorticity to remain compact)

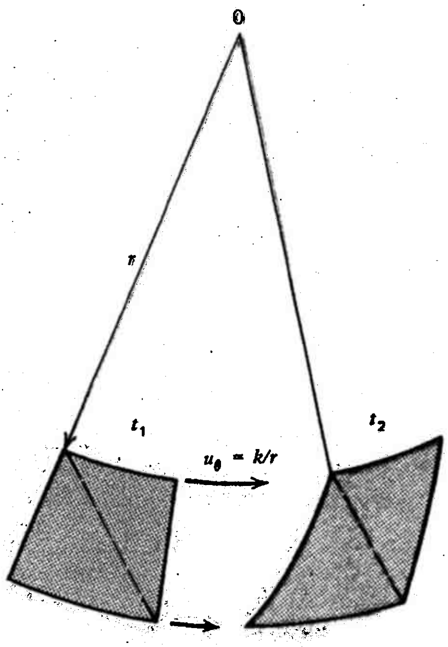


Fig. 11. Fluid element in a vortex flow.
(Kuethe & Chow)

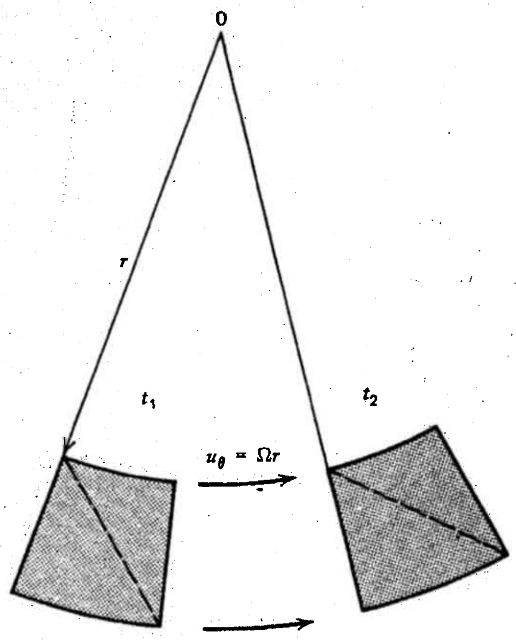
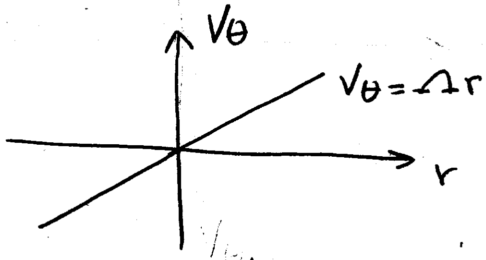


Fig. 12. Fluid element in a flow in solid-body rotation.
(Kuethe & Chow)

According to this definition the couette flow is not a vortex, (thank God!) (also) Solid-body rotation is a vortex only if the region of solid-body rotation is bounded by a region of irrotational flow.

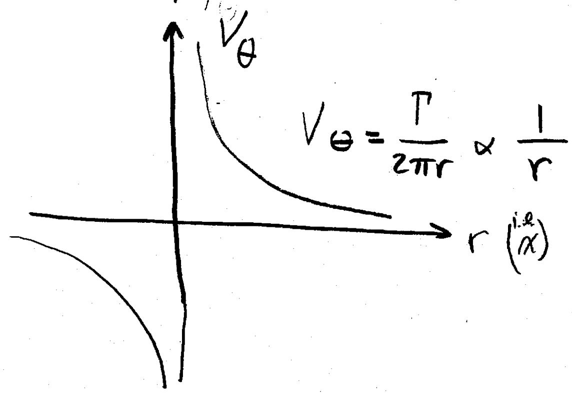
i.e.



Solid-body rotation flow field

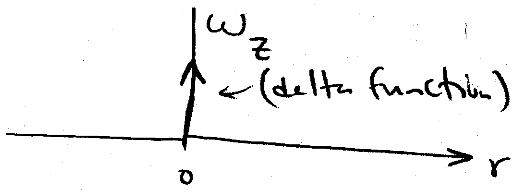
(this is not a vortex, since it is everywhere rotational)

Point Vortex

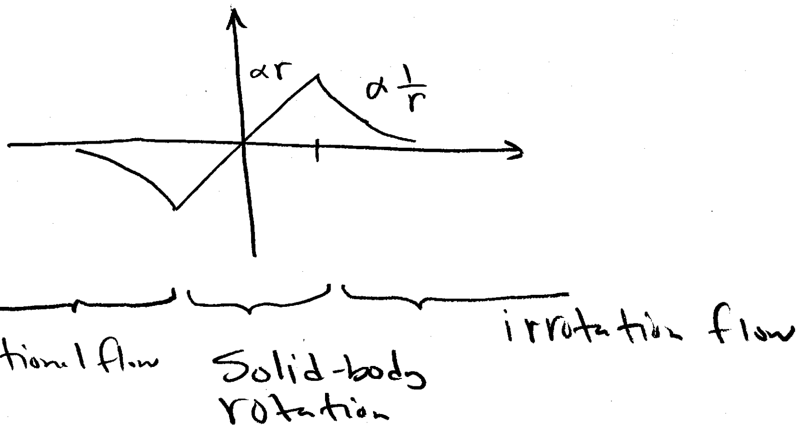


$$V_\theta = \frac{\Gamma}{2\pi r} \propto \frac{1}{r}$$

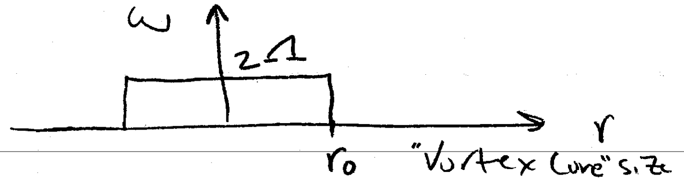
(this qualifies as a vortex)



The Solid-body rotation surrounded by a potential flow would be a vortex:



ω = constant (= 2Ω)



"RanKine Vortex"

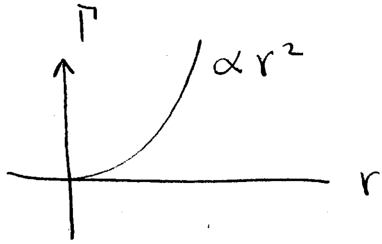
Note the corresponding circulation for each:

$$\Gamma = \oint \vec{u} \cdot d\vec{l}$$

(according to Stokes Thm

$$\Gamma = \underbrace{\iint_S \vec{\omega} \cdot \vec{n}\, ds}_{\text{area integral of vorticity}} = \underbrace{2\Omega}_{\omega} \underbrace{(\pi r^2)}_{\text{area}}$$

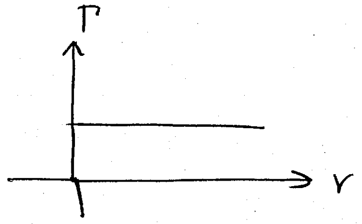
* Solid-body rotation $\Gamma = \oint (\Omega r)\, dl = (\Omega r)(2\pi r) = 2\pi \Omega r^2$



(bigger the contour, larger the circulation)

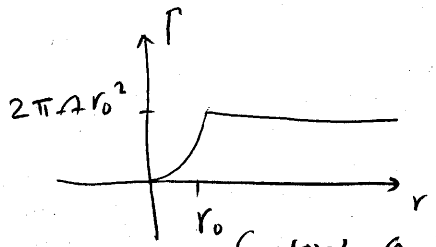
* Potential vortex

$$\oint \frac{\Gamma}{2\pi r}\, d\vec{l} = \Gamma$$



Not a function of r, i.e. any finite contour around the origin will give the same circulation
(And according to Stokes theorem, area integral results in the same circulation for any finite area, no matter how large, consistent with the fact that the area under delta function is constant)

* Rankine Vortex



$2\pi \Omega r^2$ $r < r_0$ r^2 increase

$2\pi \Omega r_0^2$ $r \geq r_0$ Constant (outside of vortex core, it's constant)

(Now, a more in-depth look at Vorticity)

(the) Alternate definition of Vorticity ($\vec{\omega} = \text{curl}\, \vec{u}$) is based on circulation

$$\vec{n} \cdot \vec{\omega} = \lim_{A \to 0} \frac{1}{A} \oint_C \vec{u} \cdot d\vec{r}$$

where A is a plane area, $\vec{n}$ is the normal to A, C is the curve enclosing A. This is a limiting case of Stokes Theorem for $A \to 0$

(Says:) $$\omega_z \equiv \frac{\Gamma}{A} \quad \text{as } A \to 0$$

Note This definition of Vorticity is useful, for example, when you have discrete data: