MATH 4600: ADVANCED CALCULUS
Fall 2018

TEST II Solutions

NAME (Please print) ____________________________

NOTES

1. Please make sure that your answer book has 8 pages. The worksheets at the end are extra pages should you need them.
2. Attempt all four problems.
3. **Read the questions carefully before answering**.
4. If you would like full credit, then **justify your answers with appropriate, but brief, reasoning**.
5. Books, notes, crib sheets and calculators are not to be used.
6. Put your mobile devices away.
7. Best wishes.

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TOTAL	

1. (30 points) Consider the solid body $\mathcal{B}$ in the first octant bounded by the three coordinate planes and the surfaces $x^2 + y^2 = 4$ and $x^2 + y^2 = z$.

   (a) (8 points) Draw a neat sketch of the body. In a coordinate system of your choice describe the body by means of inequalities satisfied by each of the three coordinates.

   (b) (8 points) Assuming that the density of the body is y, write down, and evaluate, an appropriate integral for the moment of inertia of the body about the z-axis.

   (c) (6 points) Parametrize S, the portion of the boundary of $\mathcal{B}$ that corresponds to the surface $x^2 + y^2 = z$.

   (d) (8 points) Write down, and evaluate, an appropriate integral for the area of S.

(a) We elect to use cylindrical coordinates. The body $\mathcal{B}$ is described by

$$0 \leq r \leq 2, \quad 0 \leq \theta < \pi/2, \quad 0 \leq z \leq r^2.$$

(b) The moment of inertia is

$$\begin{aligned} I &= \iiint_{\mathcal{B}} (x^2 + y^2) y \, dV \\ &= \int_0^{\pi/2} \int_0^2 \int_0^{r^2} r^3 \sin\theta \, dz \, r dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^2 r^6 \sin\theta \, dr \, d\theta \\ &= \int_0^{\pi/2} \frac{2^7}{7} \sin\theta \, d\theta \\ &= \frac{2^7}{7}. \end{aligned}$$

(c) Parametrization for S is

$$\boldsymbol{r} =< r\cos\theta, r\sin\theta, r^2 >, \quad 0 \leq r \leq 2, \; 0 \leq \theta \leq \pi/2.$$

For the scale factor we need

$$\begin{aligned} \boldsymbol{r}_r \times \boldsymbol{r}_\theta &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} \\ &= < -2r^2\cos\theta, -2r^2\sin\theta, r > . \end{aligned}$$

Then the scale factor is

$$||\boldsymbol{r}_r \times \boldsymbol{r}_\theta|| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}.$$

(d) The surface area is given by the integral

$$\begin{aligned} A &= \iint_S dS \\ &= \int_0^{\pi/2} \int_0^2 ||\boldsymbol{r}_r \times \boldsymbol{r}_\theta|| \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^2 r\sqrt{4r^2 + 1} \, dr \, d\theta \\ &= \frac{1}{8} \int_0^{\pi/2} \int_0^2 8r\sqrt{4r^2 + 1} \, dr \, d\theta \\ &= \frac{1}{8} \int_0^{\pi/2} \frac{2}{3} (4r^2 + 1)^{3/2} \Big|_0^2 \, d\theta \\ &= \frac{\pi}{6} [(17)^{3/2} - 1]. \end{aligned}$$

2. (25 points) Let C be the curve of intersection of the surface $z = x^2 + y^2$ and the plane $z = 3 + 2y$, traversed counterclockwise around the z-axis when viewed from the top.

   (a) (10 points) Show that C can also be viewed as the intersection of the cylinder $x^2+(y-1)^2 = 4$ and the plane $z = 3 + 2y$. Use polar coordinates centered at $x = 0$, $y = 1$ to find a parametrization of C.

   (b) (15 points) Find the circulation of the vector field $\boldsymbol{F} =< y - 1, z - 5, x >$ around C.

(a) Elimination of z from the equations of the two surfaces leads to

$$x^2 + y^2 = 3 + 2y,$$

which can be written as

$$x^2 + (y-1)^2 = 4,$$

and identified as the equation of a right-circular cylinder. Thus the curve C can also be viewed as the intersection of this cylinder and the plane $z = 3 + 2y$. Therefore a parametrization is

$$\boldsymbol{r} =< 2\cos\theta, 1 + 2\sin\theta, 5 + 4\sin\theta >, \quad 0 \le \theta < 2\pi.$$

This parametrization has the required counterclockwise orientation.

(b) The circulation can be written as

$$\begin{aligned}
\Gamma &= \int_C [(y-1)\ dx + (z-5)\ dy + x\ dz] \\
&= \int_0^{2\pi} \left((y-1)\frac{dx}{d\theta} + (z-5)\frac{dy}{d\theta} + x\frac{dz}{d\theta} \right) d\theta \\
&= \int_0^{2\pi} \left[(2\sin\theta)(-2\sin\theta) + (4\sin\theta)(2\cos\theta) + 8\cos^2\theta \right] d\theta \\
&= \int_0^{2\pi} [-4\sin^2\theta + 8\sin\theta\cos\theta + 8\cos^2\theta] d\theta \\
&= 4\int_0^{2\pi} \left[\frac{\cos 2\theta - 1}{2} + \sin 2\theta + \cos 2\theta + 1 \right] d\theta \\
&= 4\pi.
\end{aligned}$$

3. (25 points) Consider the solid body $\mathcal{B}$ whose upper surface S_1 is the cone $z = R - \sqrt{x^2+y^2}$ and the lower surface S_2 is the hemisphere $z = -\sqrt{R^2 - x^2 - y^2}$.

   (a) (6 points) Provide parametric descriptions for S_1 and S_2.

   (b) (8 points) Find unit normals on S_1 and S_2.

   (c) (11 points) Find the outward flux of the vector field $\boldsymbol{F} =< x, y, z >$ across the entire surface of $\mathcal{B}$.

   (a) Both the cone and the hemisphere have the projection $x^2 + y^2 = R^2$ in the xy-plane. Therefore the two surfaces can be parametrized as follows.
   In cartesian coordinates,

$$\begin{aligned} S_1: \quad & \boldsymbol{r} =< x, y, R - \sqrt{x^2+y^2} >, \;\; x^2 + y^2 \le R^2, \\ S_2: \quad & \boldsymbol{r} =< x, y, -\sqrt{R^2 - x^2 - y^2} >, \;\; x^2 + y^2 \le R^2. \end{aligned}$$

   In polar coordinates,

$$\begin{aligned} S_1: \quad & \boldsymbol{r} =< r\cos\theta, r\sin\theta, R - r >, \quad 0 \le r \le R, \; 0 \le \theta < 2\pi, \\ S_2: \quad & \boldsymbol{r} =< r\cos\theta, r\sin\theta, -\sqrt{R^2 - r^2} >, \quad 0 \le r \le R, \; 0 \le \theta < 2\pi. \end{aligned}$$

   (b) Consider first the surface S_1. For it the normal is given by

$$\begin{aligned} \boldsymbol{r}_r \times \boldsymbol{r}_\theta &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} \\ &= < r\cos\theta, r\sin\theta, r > . \end{aligned}$$

   We have $||\boldsymbol{r}_r \times \boldsymbol{r}_\theta|| = \sqrt{2}r$ so that the unit normal is

$$\boldsymbol{n} = \frac{1}{\sqrt{2}} < \cos\theta, \sin\theta, 1 > .$$

   The $\boldsymbol{k}$-component is positive indicating that the normal points upwards and hence outwards from the cone.
   For the hemisphere S_2, the normal is just the radial vector so that the unit normal

$$\boldsymbol{n} = \frac{1}{R} < x, y, z > .$$

   As $z < 0$ the normal points downwards, and hence outwards.

   (c) In each case the flux is given by

$$\int\!\!\int_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS,$$

   where $\boldsymbol{n}$ is the *outward* unit normal to S.
   On S_1,

$$\boldsymbol{F} =< x, y, z >=< r\cos\theta, r\sin\theta, R - r > .$$

   Therefore,

$$\boldsymbol{F} \cdot \boldsymbol{n} = \frac{1}{\sqrt{2}}(r\cos^2\theta + r\sin^2\theta + R - r) = \frac{R}{\sqrt{2}}.$$

   Thus the contribution of S_1 to the flux is

$$\begin{aligned} I_1 &= \int\!\!\int_{S_1} \boldsymbol{F} \cdot \boldsymbol{n} \, dS \\ &= \int_0^{2\pi} \int_0^R (\boldsymbol{F} \cdot \boldsymbol{n}) \, ||\boldsymbol{r}_r \times \boldsymbol{r}_\theta|| \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{2}} \sqrt{2} r \, dr \, d\theta \\ &= \pi R^3. \end{aligned}$$

On S_2,

$$\boldsymbol{F} \cdot \boldsymbol{n} = \frac{1}{R}(x^2 + y^2 + z^2) = \frac{1}{R}R^2 = R.$$

Therefore the flux is

$$\begin{aligned} I_2 &= \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\, dS \\ &= R\iint_{S_2} dS \\ &= R(2\pi R^2) = 2\pi R^3, \end{aligned}$$

since the area of the hemisphere is $2\pi R^2$. Then the total flux is $I_1 + I_2 = 3\pi R^3$.

4. (20 points) Use the method of Lagrange multipliers to find the minimum and maximum values of $f(x, y, z) = x + yz$ on the sphere $x^2 + y^2 + z^2 \leq 1$.

First, consider the critical points within the sphere. The necessary conditions for criticality are

$$f_x = f_y = f_z = 0.$$

Since $f_x = 1$, there is a contradiction, so that there are no critical points within the sphere. On the boundary of the sphere we have a constrained problem, the constraint being

$$g = x^2 + y^2 + z^2 - 1 = 0. \tag{1}$$

Then the conditions for criticality are

$$\begin{aligned} f_x - \lambda g_x = 1 - \lambda 2x &= 0, && (2)\\ f_y - \lambda g_y = z - \lambda 2y &= 0, && (3)\\ f_z - \lambda g_z = y - \lambda 2z &= 0. && (4) \end{aligned}$$

Elimination of z from (3) and (4) leads to

$$y(1 - 4\lambda^2) = 0 \tag{5}$$

so that either $y = 0$ or $\lambda = \pm 1/2$. If $y = 0$ then from (3), $z = 0$. Substitution into (1) shows that $x = \pm 1$, yielding candidate points $P_1(1, 0, 0)$ and $P_2(-1, 0, 0)$. For P_1, (1) finds that $\lambda = 1/2$ and for P_2, likewise, $\lambda = -1/2$. Thus all possibilities have been exhausted.

At P_1, $f = 1$ and at P_2, $f = -1$. Thus f has 1 as the maximum and -1 as the minimum.