Scalars, vectors and tensors

A scalar — a property with no preferred direction

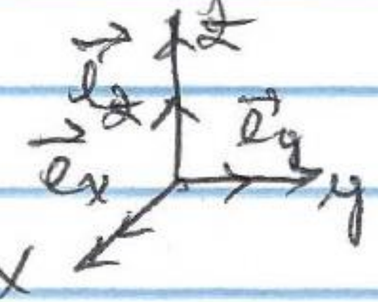
ρ, P, T, μ, h, s

$c = a+b \qquad c = ab \qquad c = \frac{a}{b}$

A vector is a property with a preferred direction

$\vec{V}$ velocity vector $\qquad V = |\vec{V}|, \quad \vec{e}_{\vec{V}}, \quad \vec{V} = V\vec{e}_{\vec{V}}$

In 3D $\qquad \vec{V} = V_x\vec{e}_x + V_y\vec{e}_y + V_z\vec{e}_z$



$$\vec{V} = \sum_{i=x,y,z} V_i\vec{e}_i = V_i\vec{e}_i$$

$$= (V_x, V_y, V_z) = (V_i)$$

$$\vec{c} = k\vec{a} = (k a_i)\vec{e}_i \quad ; \quad c_i = k a_i$$

$$\vec{c} = \vec{a} \pm \vec{b} = (a_i \pm b_i)\vec{e}_i \quad , \quad c_i = a_i \pm b_i$$

$$c = \vec{a}\cdot\vec{b} = \sum_{i=x,y,z} a_i b_i = a_i b_i$$

$$\vec{a}\cdot\vec{b} = 0 \iff \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \perp \vec{b}$$

$$\vec{c} = \vec{a}\times\vec{b} = \det\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \vec{0} \iff \vec{a} = 0 \text{ or } \vec{b} = 0 \text{ or } \vec{a} \parallel \vec{b}$$

$$\vec{c} \perp \vec{a} \text{ and } \vec{c} \perp \vec{b}$$

A <u>tensor</u> of 2nd order is a property determined by two directions

$$\underline{\underline{T}} = \vec{a}\vec{b} \qquad \vec{b}\vec{a} \neq \vec{a}\vec{b}$$

$$\underline{\underline{T}} = T_{xx}\vec{e}_x\vec{e}_x + T_{xy}\vec{e}_x\vec{e}_y + T_{xz}\vec{e}_x\vec{e}_z$$
$$T_{yx}\vec{e}_y\vec{e}_x + T_{yy}\vec{e}_y\vec{e}_y + T_{yz}\vec{e}_y\vec{e}_z$$
$$T_{zx}\vec{e}_z\vec{e}_x + T_{zy}\vec{e}_z\vec{e}_y + T_{zz}\vec{e}_z\vec{e}_z = \sum_{\substack{i=x,y,z\\ j=x,y,z}} T_{ij}\vec{e}_i\vec{e}_j = T_{ij}\vec{e}_i\vec{e}_j$$

$$= \left(T_{ij}\right)$$

$$\underline{\underline{T}}_3 = \underline{\underline{T}}_1 \pm \underline{\underline{T}}_2 \quad , \quad T_{3ij} = T_{1ij} \pm T_{2ij}$$

$$\underline{\underline{T}}_1 = k\underline{\underline{T}} = (k T_{ij})$$

product from right

$$\underline{\underline{T}} \cdot \vec{c} = (\underline{\underline{\vec{a}\vec{b}}}) \cdot \vec{c} = \vec{a}(\vec{b}\cdot\vec{c})$$

$$(\underline{\underline{\vec{e}_x\vec{e}_y}}) \cdot \vec{e}_y = \vec{e}_x(\vec{e}_y\cdot\vec{e}_y) = \vec{e}_x 1$$

$$(\underline{\underline{\vec{e}_x\vec{e}_y}}) \cdot \vec{e}_x = 0\,\vec{e}_x$$

product from left

$$\vec{c} \cdot \underline{\underline{T}} = \vec{c} \cdot (\vec{a}\vec{b}) = (\vec{c}\cdot\vec{a})\,\vec{b}$$

$$\vec{e}_x \cdot (\underline{\underline{\vec{e}_x\vec{e}_y}}) = (\vec{e}_x\cdot\vec{e}_x)\,\vec{e}_y = 1\,\vec{e}_y$$

$$c = \underline{\underline{T}}_1 : \underline{\underline{T}}_2 = \underline{\underline{(\vec{a}_1 \vec{b}_1)}} : (\vec{a}_2 \vec{b}_2) = (\vec{a}_1 \cdot \vec{b}_2)(\vec{b}_1 \cdot \vec{a}_2)$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z$$

$$\vec{\nabla} c = \frac{\partial c}{\partial x}\vec{e}_x + \frac{\partial c}{\partial y}\vec{e}_y + \frac{\partial c}{\partial z}\vec{e}_z \quad \text{gradient of a scalar } c$$

$$\vec{\nabla} \cdot \vec{a} = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right) \cdot \left(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z\right)$$

$$= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad \text{a divergence of } \vec{a}$$

$$= \sum_{i=x,y,z} \frac{\partial a_i}{\partial i} = \frac{\partial a_i}{\partial x_i}$$

$$\vec{\nabla} \times \vec{a} = \det \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$ the rotor of $\vec{a}$

when $\vec{V}$ is velocity vector: $\vec{\nabla}\cdot\vec{V}$ and $\vec{\nabla}\times\vec{V}=\vec{\omega}$ vorticity

$(\vec{\nabla}\vec{V})$ gradient of velocity vector = a tensor of 2^{nd} order

$$= \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right)\left(V_x\vec{e}_x + V_y\vec{e}_y + V_z\vec{e}_z\right)$$

$$= \frac{\partial V_x}{\partial x}\vec{e}_x\vec{e}_x + \frac{\partial V_y}{\partial x}\vec{e}_x\vec{e}_y + \frac{\partial V_z}{\partial x}\vec{e}_x\vec{e}_z$$

$$+ \frac{\partial V_x}{\partial y}\vec{e}_y\vec{e}_x + \frac{\partial V_y}{\partial y}\vec{e}_y\vec{e}_y + \frac{\partial V_z}{\partial y}\vec{e}_y\vec{e}_z$$

$$+ \frac{\partial V_x}{\partial z}\vec{e}_z\vec{e}_x + \frac{\partial V_y}{\partial z}\vec{e}_z\vec{e}_y + \frac{\partial V_z}{\partial z}\vec{e}_z\vec{e}_z = \left(\frac{\partial V_i}{\partial j}\right)_{\substack{i=x,y,z\\ j=x,y,z}}$$

$$\vec{V}\cdot\left(\underline{\underline{\vec{\nabla}\vec{V}}}\right)=\left(V_x\vec{e}_x+V_y\vec{e}_y+V_z\vec{e}_z\right)\cdot\left(\underline{\underline{\vec{\nabla}\vec{V}}}\right)$$

$$=\vec{e}_x\left(V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}\right)$$

$$+\vec{e}_y\left(V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}\right)$$

$$+\vec{e}_z\left(V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}\right)$$

$$=\sum_{i=x,y,z}\left(V_x\frac{\partial V_i}{\partial x}+V_y\frac{\partial V_i}{\partial y}+V_z\frac{\partial V_i}{\partial z}\right)\vec{e}_i$$ = the convective terms of $\vec{V}$

$\vec{V}\cdot\vec{\nabla}p =$ the convective term of a scalar p

$$= V_x \frac{\partial p}{\partial x} + V_y \frac{\partial p}{\partial y} + V_z \frac{\partial p}{\partial z} = \sum_{i=x,y,z} V_i \frac{\partial p}{\partial i}$$

trace of $\underline{\underline{(\vec{\nabla}\vec{V})}} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \vec{\nabla}\cdot\vec{V}$