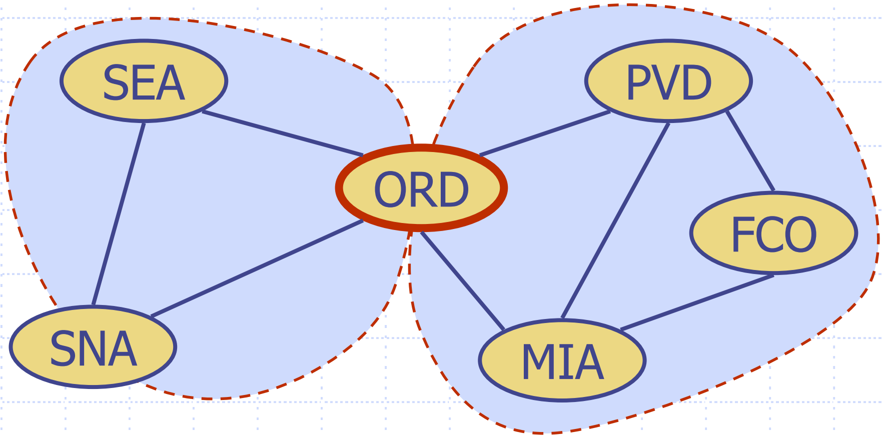
Biconnectivity

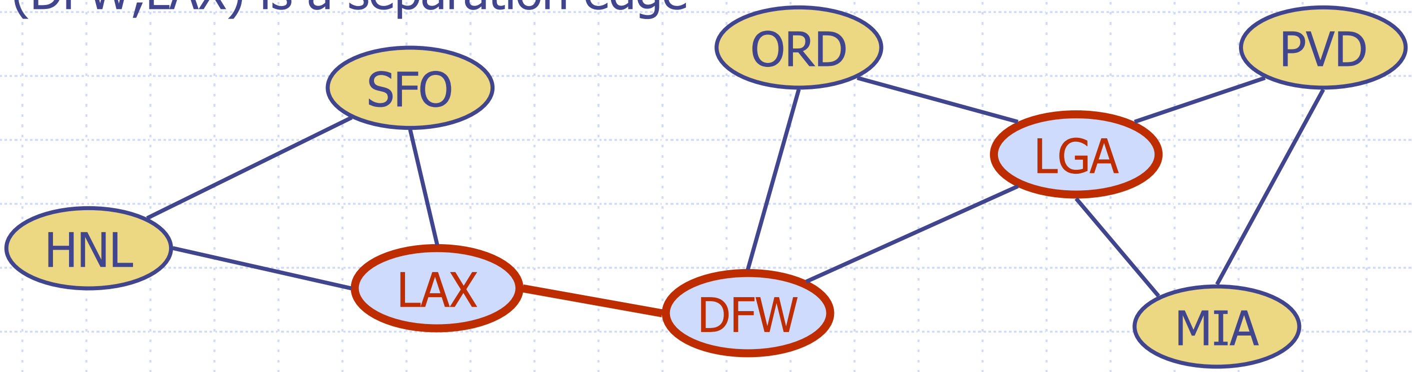


Outline and Reading

- Definitions
  - Separation vertices and edges
  - Biconnected graph
  - Biconnected components
  - Equivalence classes
  - Linked edges and link components
- Algorithms
  - Auxiliary graph
  - Proxy graph

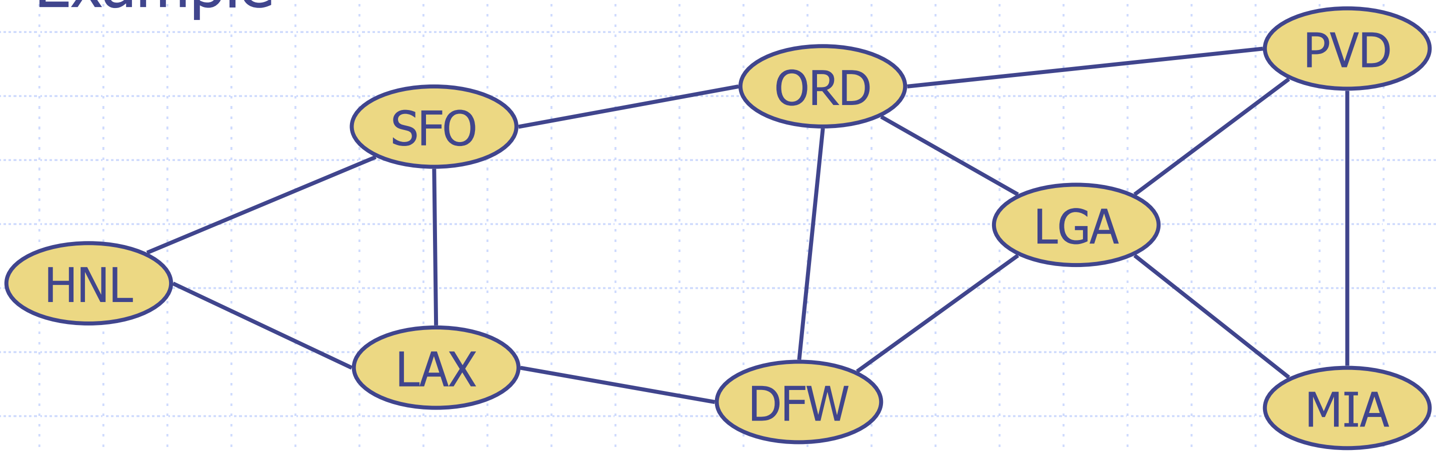
Separation Edges and Vertices

- Definitions
  - Let G be a connected graph
  - A separation edge of G is an edge whose removal disconnects G
  - A separation vertex of G is a vertex whose removal disconnects G
- Applications
  - Separation edges and vertices represent single points of failure in a network and are critical to the operation of the network
- Example
  - DFW, LGA and LAX are separation vertices
  - (DFW,LAX) is a separation edge



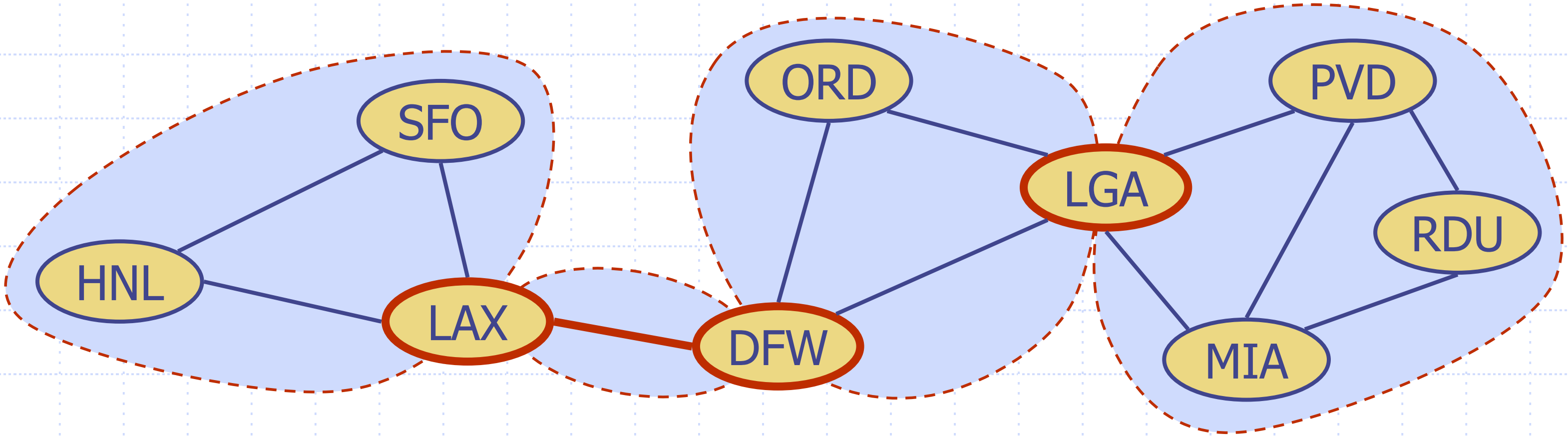
Biconnected Graph

- Equivalent definitions of a biconnected graph G
  - Graph G has no separation edges and no separation vertices
  - For any two vertices u and v of G, there are two disjoint simple paths between u and v (i.e., two simple paths between u and v that share no other vertices or edges)
  - For any two vertices u and v of G, there is a simple cycle containing u and v
- Example



Biconnected Components

- Biconnected component of a graph *G*
  - A maximal biconnected subgraph of *G*, or
  - A subgraph consisting of a separation edge of *G* and its end vertices
- Interaction of biconnected components
  - An edge belongs to exactly one biconnected component
  - A nonseparation vertex belongs to exactly one biconnected component
  - A separation vertex belongs to two or more biconnected components
- Example of a graph with four biconnected components



Equivalence Classes

- Given a set S, a relation R on S is a set of ordered pairs of elements of S, i.e., R is a subset of $S \times S$
- An equivalence relation R on S satisfies the following properties

  Reflexive: $(x,x) \in R$

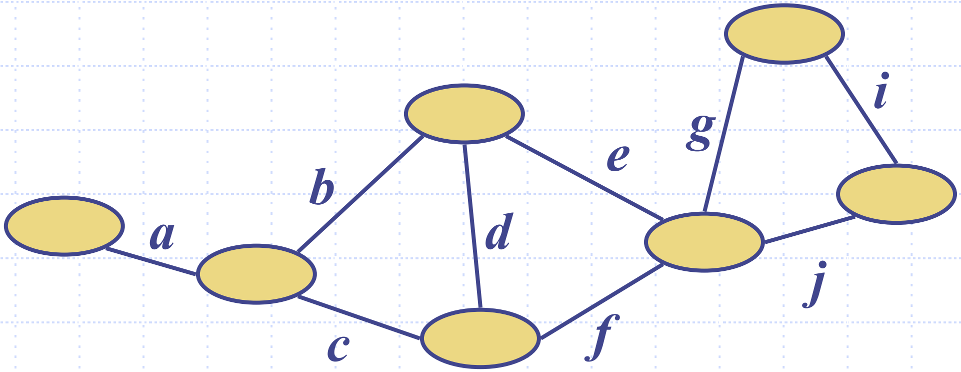
  Symmetric: $(x,y) \in R \Rightarrow (y,x) \in R$

  Transitive: $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$
- An equivalence relation R on S induces a partition of the elements of S into equivalence classes
- Example (connectivity relation among the vertices of a graph):
  - Let V be the set of vertices of a graph G
  - Define the relation

    $C = \{(v,w) \in V \times V$ such that G has a path from v to $w\}$
  - Relation C is an equivalence relation
  - The equivalence classes of relation C are the vertices in each connected component of graph G

Link Relation

- Edges e and f of connected graph G are linked if
  - $e = f$, or
  - G has a simple cycle containing e and f



Theorem:

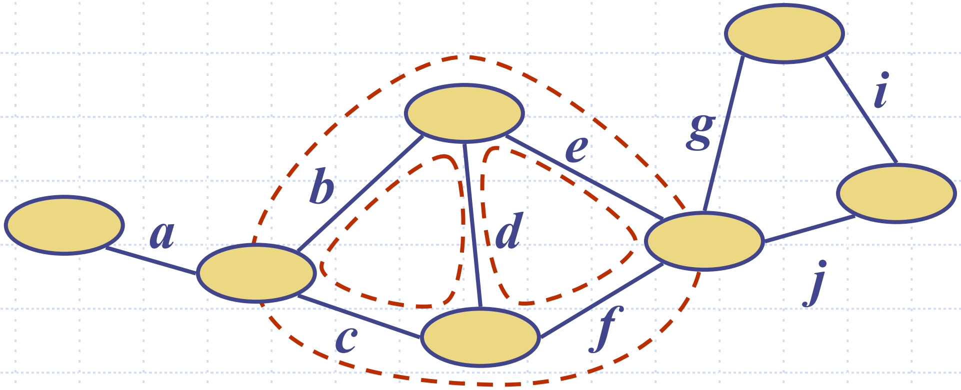
The link relation on the edges of a graph is an equivalence relation

Proof Sketch:

- The reflexive and symmetric properties follow from the definition
- For the transitive property, consider two simple cycles sharing an edge

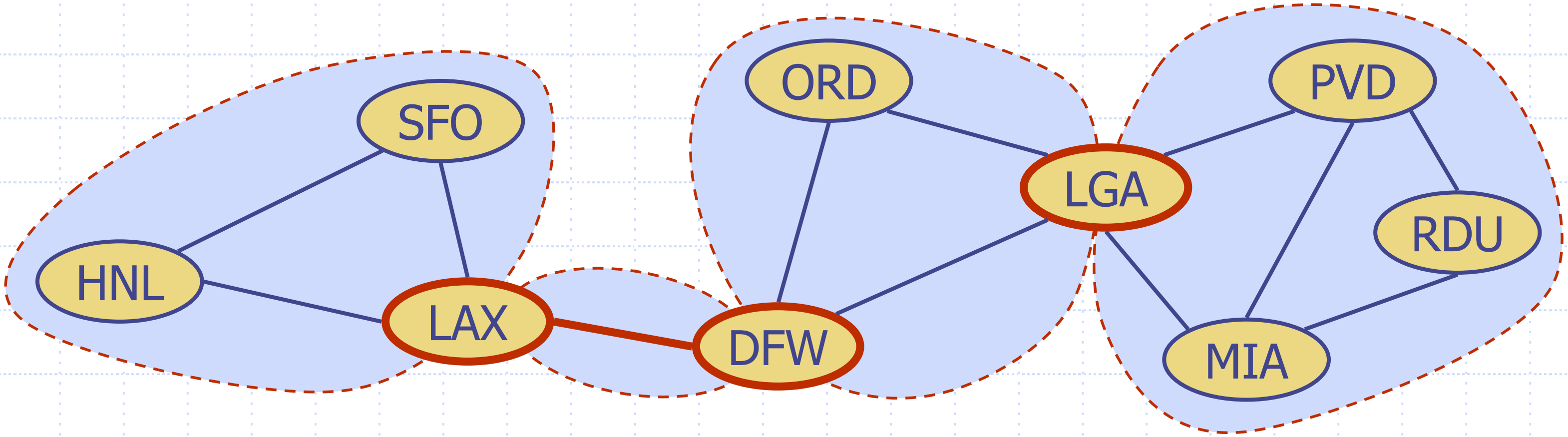
Equivalence classes of linked edges:

$\{a\}$ $\{b, c, d, e, f\}$ $\{g, i, j\}$



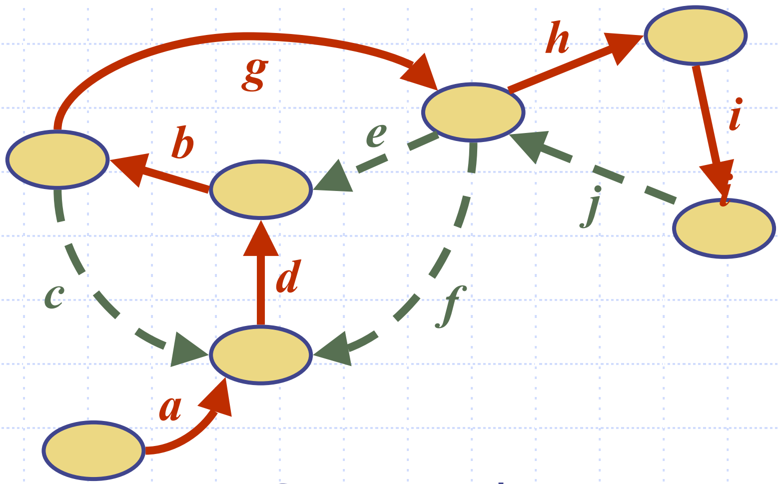
Link Components

- The link components of a connected graph G are the equivalence classes of edges with respect to the link relation
- A biconnected component of G is the subgraph of G induced by an equivalence class of linked edges
- A separation edge is a single-element equivalence class of linked edges
- A separation vertex has incident edges in at least two distinct equivalence classes of linked edge

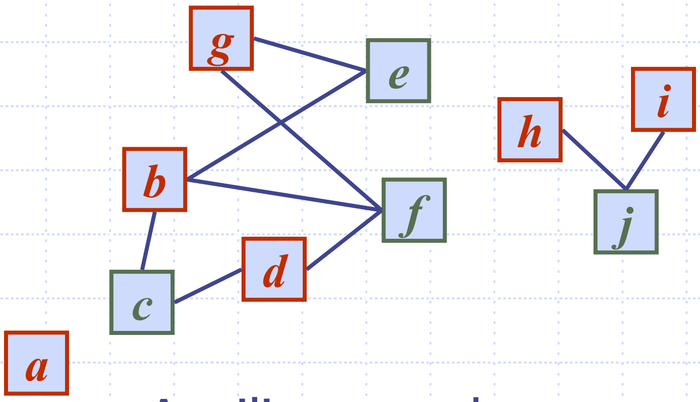


Auxiliary Graph

- Auxiliary graph B for a connected graph G
  - Associated with a DFS traversal of G
  - The vertices of B are the edges of G
  - For each back edge e of G, B has edges (e,f_1), (e,f_2) , …, (e,f_k), where $f_1, f_2, \ldots, f_k$ are the discovery edges of G that form a simple cycle with e
  - Its connected components correspond to the the link components of G



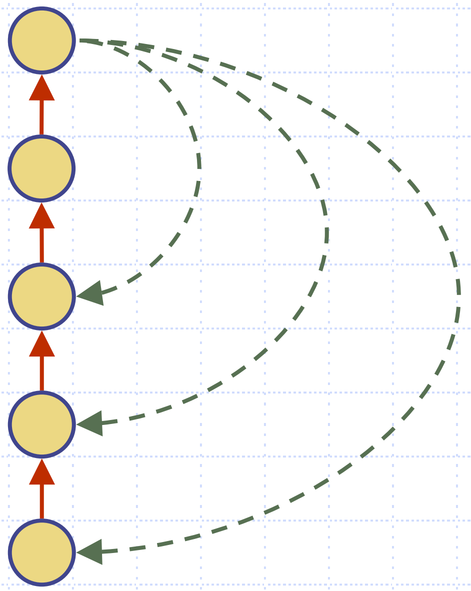
DFS on graph G



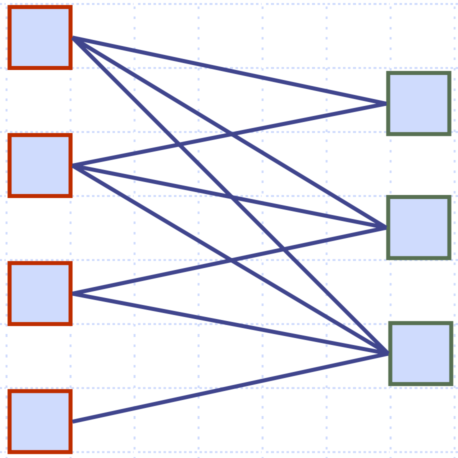
Auxiliary graph B

Auxiliary Graph (cont.)

- In the worst case, the number of edges of the auxiliary graph is proportional to nm



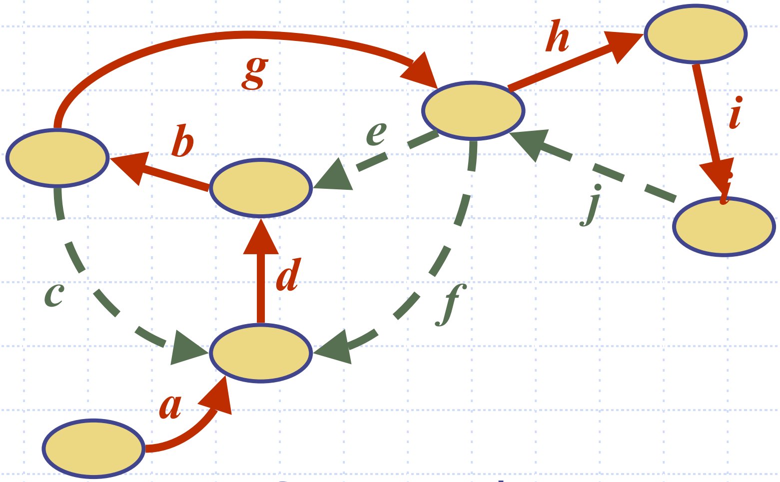
DFS on graph G



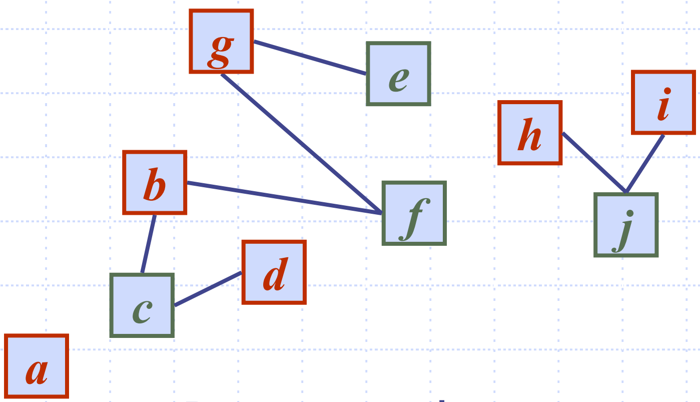
Auxiliary graph B

Proxy Graph

```
Algorithm proxyGraph(G)
  Input connected graph G
  Output proxy graph F for G
  F ← empty graph
  DFS(G, s) { s is any vertex of G}
  for all discovery edges e of G
    F.insertVertex(e)
    setLabel(e, UNLINKED)
  for all vertices v of G in DFS visit order
    for all back edges e = (u,v)
      F.insertVertex(e)
      repeat
        f ← discovery edge with dest. u
        F.insertEdge(e,f,∅)
        if f getLabel(f) = UNLINKED
          setLabel(f, LINKED)
          u ← origin of edge f
        else
          u ← v { ends the loop }
      until u = v
  return F
```



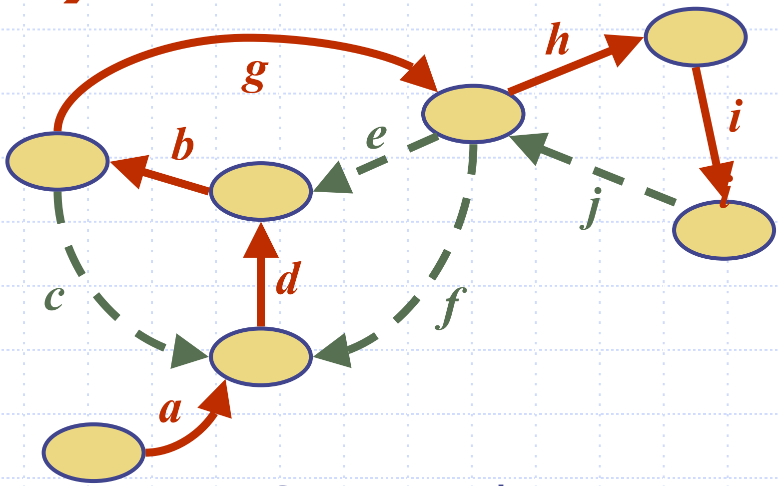
DFS on graph G



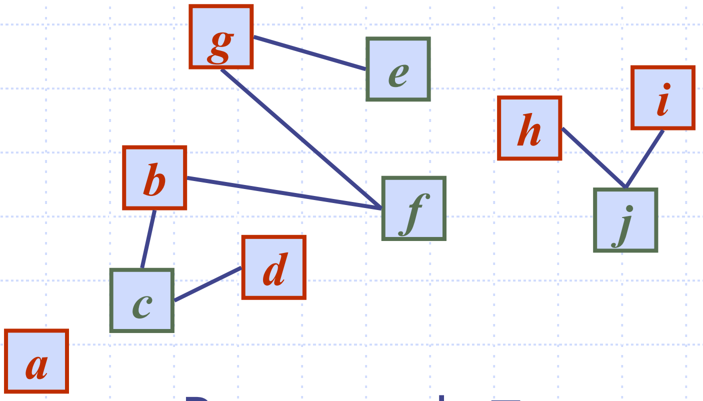
Proxy graph F

Proxy Graph (cont.)

- Proxy graph F for a connected graph G
  - Spanning forest of the auxiliary graph B
  - Has m vertices and $O(m)$ edges
  - Can be constructed in $O(n + m)$ time
  - Its connected components (trees) correspond to the the link components of G
- Given a graph G with n vertices and m edges, we can compute the following in $O(n + m)$ time
  - The biconnected components of G
  - The separation vertices of G
  - The separation edges of G



DFS on graph G



Proxy graph F