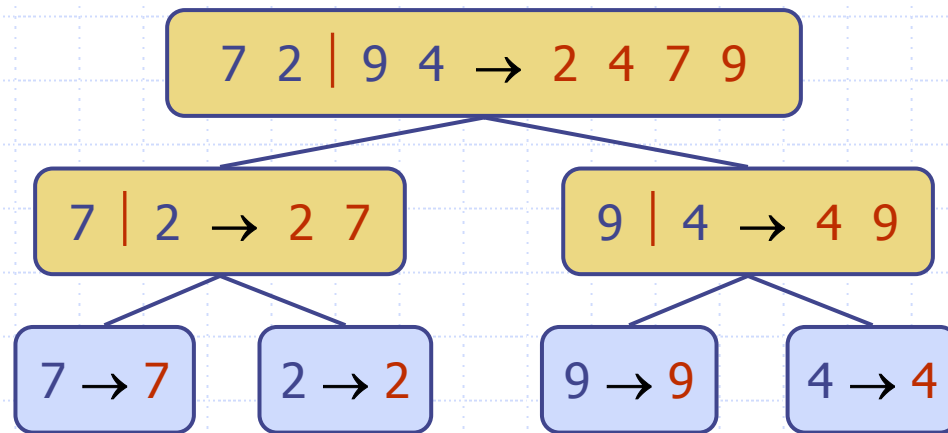


# Divide-and-Conquer

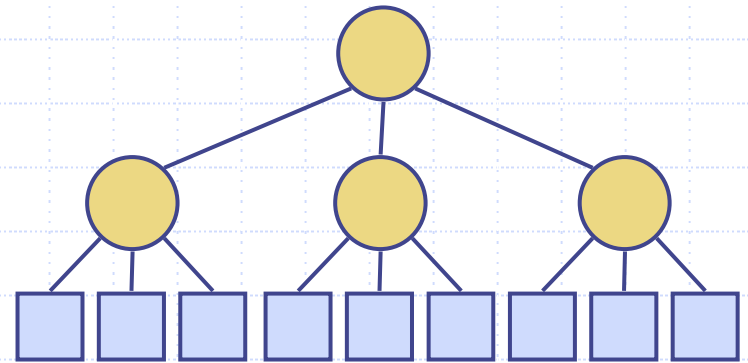


# Outline and Reading

- ◆ Divide-and-conquer paradigm (§10.1.1)
- ◆ Review Merge-sort (§10.1.3)
- ◆ Recurrence Equations (§10.1.3)
  - Iterative substitution
  - Recursion trees
  - Guess-and-test
  - The master method
- ◆ Integer Multiplication

# Divide-and-Conquer

- ◆ **Divide-and conquer** is a general algorithm design paradigm:
  - **Divide**: divide the input data  $S$  in two or more disjoint subsets  $S_1, S_2, \dots$
  - **Recur**: solve the subproblems recursively
  - **Conquer**: combine the solutions for  $S_1, S_2, \dots$ , into a solution for  $S$
- ◆ The base case for the recursion are subproblems of constant size
- ◆ Analysis can be done using **recurrence equations**



# Merge-Sort Review

- ◆ Merge-sort on an input sequence  $S$  with  $n$  elements consists of three steps:
  - **Divide**: partition  $S$  into two sequences  $S_1$  and  $S_2$  of about  $n/2$  elements each
  - **Recur**: recursively sort  $S_1$  and  $S_2$
  - **Conquer**: merge  $S_1$  and  $S_2$  into a unique sorted sequence

**Algorithm** *mergeSort*( $S, C$ )

**Input** sequence  $S$  with  $n$  elements, comparator  $C$

**Output** sequence  $S$  sorted according to  $C$

**if**  $S.size() > 1$

$(S_1, S_2) \leftarrow partition(S, n/2)$

*mergeSort*( $S_1, C$ )

*mergeSort*( $S_2, C$ )

$S \leftarrow merge(S_1, S_2)$

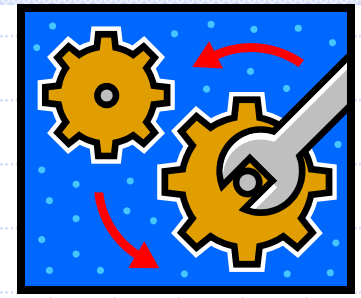
# Recurrence Equation Analysis



- ◆ The conquer step of merge-sort consists of merging two sorted sequences, each with  $n/2$  elements and implemented by means of a doubly linked list, takes at most  $bn$  steps, for some constant  $b$ .
- ◆ Likewise, the basis case ( $n < 2$ ) will take at  $b$  most steps.
- ◆ Therefore, if we let  $T(n)$  denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- ◆ We can therefore analyze the running time of merge-sort by finding a **closed form solution** to the above equation.
  - That is, a solution that has  $T(n)$  only on the left-hand side.



# Iterative Substitution

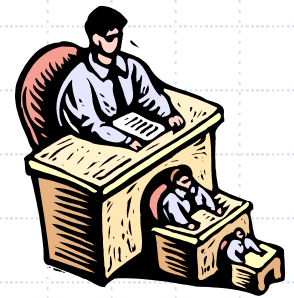
- ◆ In the iterative substitution, or “plug-and-chug,” technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern:

$$\begin{aligned}T(n) &= 2T(n/2) + bn \\&= 2(2T(n/2^2)) + b(n/2) + bn \\&= 2^2 T(n/2^2) + 2bn \\&= 2^3 T(n/2^3) + 3bn \\&= 2^4 T(n/2^4) + 4bn \\&= \dots \\&= 2^i T(n/2^i) + ibn\end{aligned}$$

- ◆ Note that base,  $T(n)=b$ , case occurs when  $2^i=n$ . That is,  $i = \log n$ .

- ◆ So, 
$$T(n) = bn + bn \log n$$

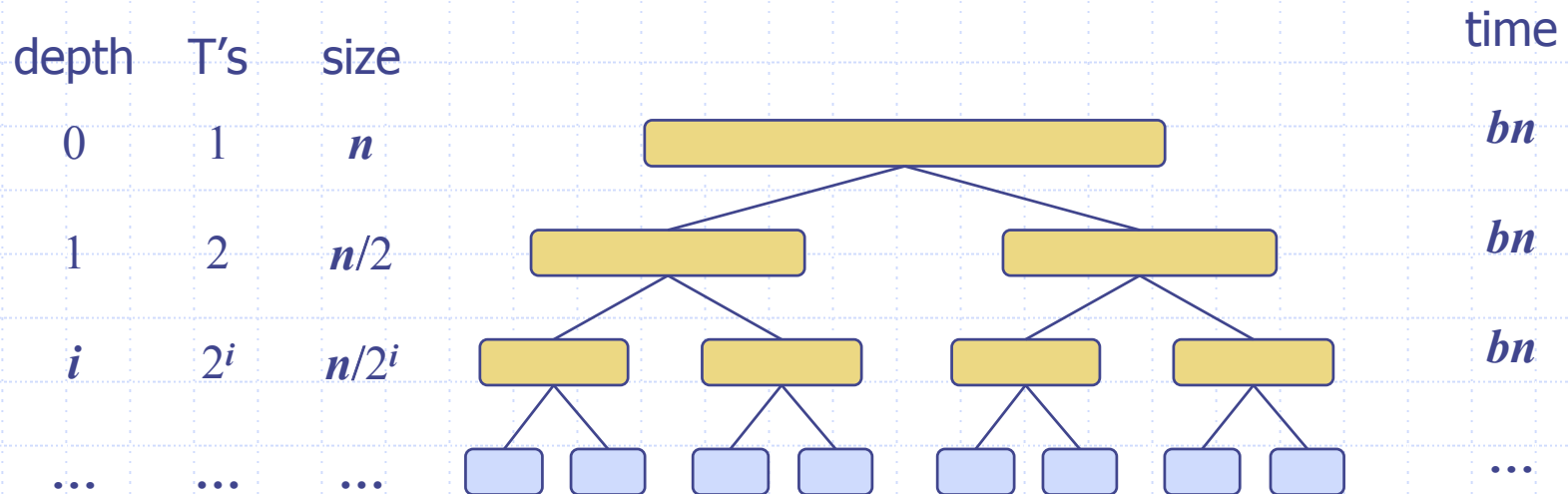
- ◆ Thus,  $T(n)$  is  $O(n \log n)$ .



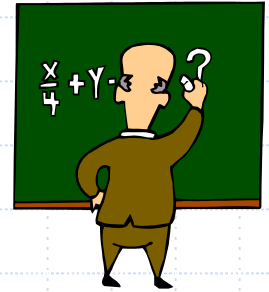
# The Recursion Tree

- ◆ Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$



Total time =  $bn + bn \log n$   
(last level plus all previous levels)



# Guess-and-Test Method

- ◆ In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

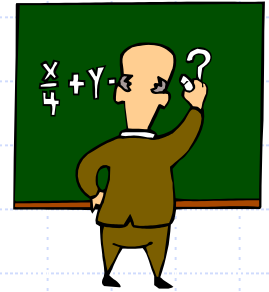
- ◆ Guess:  $T(n) < cn \log n$ .

$$\begin{aligned} T(n) &= 2T(n/2) + bn \log n \\ &= 2(c(n/2) \log(n/2)) + bn \log n \\ &= cn(\log n - \log 2) + bn \log n \\ &= cn \log n - cn + bn \log n \end{aligned}$$

- ◆ Wrong: we cannot make this last line be less than  $cn \log n$



# Guess-and-Test Method, Part 2



- ◆ Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

- ◆ Guess #2:  $T(n) < cn \log^2 n$ .

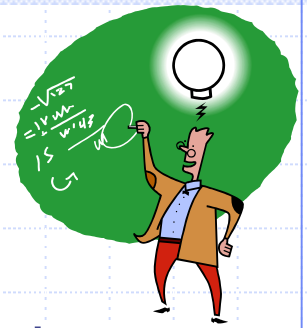
$$\begin{aligned} T(n) &= 2T(n/2) + bn \log n \\ &= 2(c(n/2) \log^2(n/2)) + bn \log n \\ &= cn(\log n - \log 2)^2 + bn \log n \\ &= cn \log^2 n - 2cn \log n + cn + bn \log n \\ &\leq cn \log^2 n \end{aligned}$$

- if  $c > b$ .

- ◆ So,  $T(n)$  is  $O(n \log^2 n)$ .

- ◆ In general, to use this method, you need to have a good guess and you need to be good at induction proofs.

# Master Method



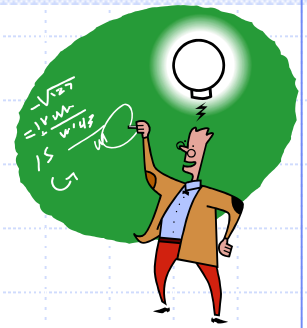
- ◆ Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

- ◆ The Master Theorem:

1. if  $f(n)$  is  $O(n^{\log_b a - \varepsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \varepsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

# Master Method, Example 1



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

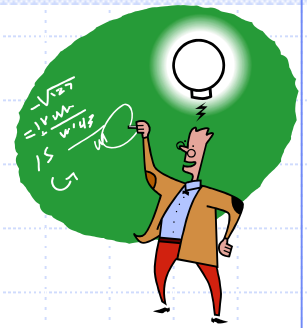
1. if  $f(n)$  is  $O(n^{\log_b a - \varepsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \varepsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

◆ Example:

$$T(n) = 4T(n/2) + n$$

Solution:  $\log_b a = 2$ , so case 1 says  $T(n)$  is  $O(n^2)$ .

# Master Method, Example 2



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

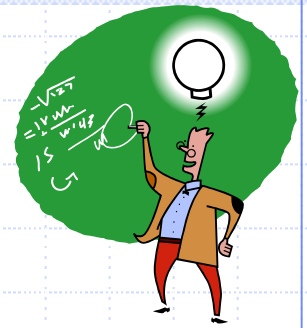
1. if  $f(n)$  is  $O(n^{\log_b a - \varepsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
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3. if  $f(n)$  is  $\Omega(n^{\log_b a + \varepsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

◆ Example:

$$T(n) = 2T(n/2) + n \log n$$

Solution:  $\log_b a = 1$ , so case 2 says  $T(n)$  is  $O(n \log^2 n)$ .

# Master Method, Example 3



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

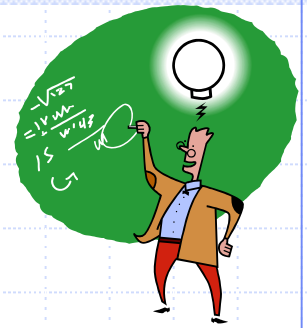
1. if  $f(n)$  is  $O(n^{\log_b a - \epsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \epsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

◆ Example:

$$T(n) = T(n/3) + n \log n$$

Solution:  $\log_b a = 0$ , so case 3 says  $T(n)$  is  $O(n \log n)$ .

# Master Method, Example 4



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

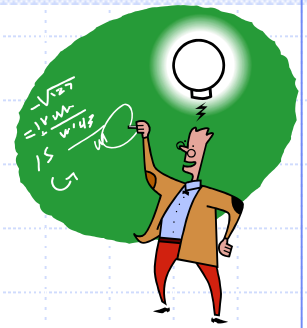
1. if  $f(n)$  is  $O(n^{\log_b a - \varepsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \varepsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

◆ Example:

$$T(n) = 8T(n/2) + n^2$$

Solution:  $\log_b a = 3$ , so case 1 says  $T(n)$  is  $O(n^3)$ .

# Master Method, Example 5



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

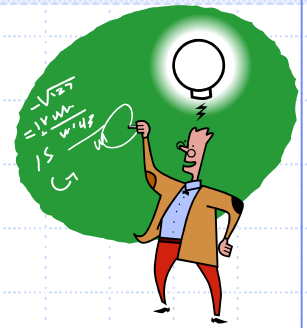
1. if  $f(n)$  is  $O(n^{\log_b a - \epsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \epsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

◆ Example:

$$T(n) = 9T(n/3) + n^3$$

Solution:  $\log_b a = 2$ , so case 3 says  $T(n)$  is  $O(n^3)$ .

# Master Method, Example 6



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

1. if  $f(n)$  is  $O(n^{\log_b a - \varepsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \varepsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

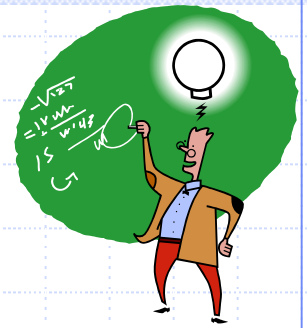
◆ Example:

$$T(n) = T(n/2) + 1 \quad (\text{binary search})$$

Solution:  $\log_b a = 0$ , so case 2 says  $T(n)$  is  $O(\log n)$ .



# Master Method, Example 7



◆ The form: 
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

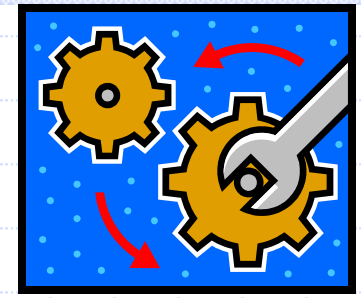
1. if  $f(n)$  is  $O(n^{\log_b a - \epsilon})$ , then  $T(n)$  is  $\Theta(n^{\log_b a})$
2. if  $f(n)$  is  $\Theta(n^{\log_b a} \log^k n)$ , then  $T(n)$  is  $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if  $f(n)$  is  $\Omega(n^{\log_b a + \epsilon})$ , then  $T(n)$  is  $\Theta(f(n))$ ,  
provided  $af(n/b) \leq \delta f(n)$  for some  $\delta < 1$ .

◆ Example:

$$T(n) = 2T(n/2) + \log n \quad (\text{heap construction})$$

Solution:  $\log_b a = 1$ , so case 1 says  $T(n)$  is  $O(n)$ .

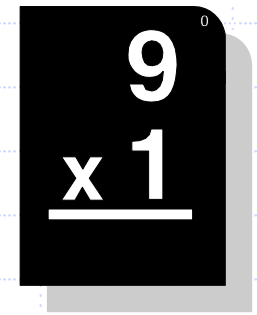
# Iterative “Proof” of the Master Theorem



- ◆ Using iterative substitution, let us see if we can find a pattern:

$$\begin{aligned}T(n) &= aT(n/b) + f(n) \\&= a(aT(n/b^2)) + f(n/b) + bn \\&= a^2T(n/b^2) + af(n/b) + f(n) \\&= a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \\&= \dots \\&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)\end{aligned}$$

- ◆ We then distinguish the three cases as
  - The first term is dominant
  - Each part of the summation is equally dominant
  - The summation is a geometric series



# Integer Multiplication

## ◆ Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

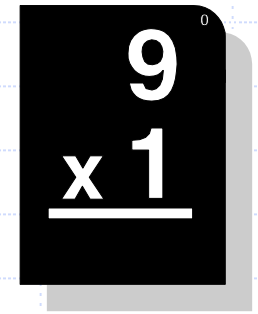
$$J = J_h 2^{n/2} + J_l$$

- We can then define  $I * J$  by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

- So,  $T(n) = 4T(n/2) + n$ , which implies  $T(n)$  is  $O(n^2)$ .
- But that is no better than the algorithm we learned in grade school.

# An Improved Integer Multiplication Algorithm



## ◆ Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- Observe that there is a different way to multiply parts:

$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

- So,  $T(n) = 3T(n/2) + n$ , which implies  $T(n)$  is  $O(n^{\log_2 3})$ , by the Master Theorem.
- Thus,  $T(n)$  is  $O(n^{1.585})$ .