Nonlinear Oscillation: Duffing's Equation

I. Introduction

When first dealing with oscillation and it's relation to differential equations, it is natural to start with an undamped and unforced mass-spring system, which can be represented by the following equation:

$$my'' + ky = 0$$

Where the second-order term is the mathematical representation of the gravitational force on a mass and the second term is the spring's force as a function of displacement from the equilibrium location. The solution to the ODE is a function that represents the mass's displacement. Assuming that either the initial displacement or the initial velocity is non-zero (or both), this equation predicts oscillation for forever.

Now, if we introduce a damping force, which changes based on the environment the system is in, we get the following equation:

$$my'' + \gamma y' + ky = 0$$

If we have a non-zero damping constant, γ, then our system will *eventually* reach a state of 0 energy such that the displacement and velocity both reach 0. This can be done through 3 different damping cases. An underdamped case is found when the roots of the ODE are complex. An overdamped case occurs when there are two distinct, real roots to the ODE. A critically damped case happens when we have a single repeated root. A critically damped case tells us that the situation provide will return the mass to it's equilibrium position faster than any other case.

Note: For the sake of simplicity, we will choose to keep the external forcing function at 0 for the remainder of this exploration report.

II. Adding Nonlinearity

The previous situations assume an ideal spring that has no distortion. However, real world springs are more complex. We can model these springs more accurately by adding a nonlinear term which comes about from the next odd term in the Taylor Series representation of the spring force. Again, let's remove drag from our equation for now. This is Duffing's Equation.

$$my'' + k_1 y + k_3 y^3 = 0$$

$$E = \frac{m(y')^2}{2} + \frac{k_1}{2} y^2 + \frac{k_3}{4} y^4$$

We can reach a matrix representation of this system:

$$my'' + k_1 y + k_3 y^3 = 0 \rightarrow \begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} v/m \\ -k_1 m - k_3 y^3 \end{bmatrix}$$

The third, nonlinear term in our force equation can give rise to three different kinds of springs. Assuming k_1 remains positive, the following value restrictions describe the three springs.

Soft Spring: k_3 < 0:

This creates a competitive nature between the two terms of the spring force. They will always be in opposite directions, and at larger displacements, the nonlinear term will dominate. This means that at larger displacements, the spring will no longer be able to oppose the movement of the mass.

Mass-Spring: k_3 = 0:

This is simply the linear representation of a mass-spring system. Notice that the spring force increases linearly with displacement from the equilibrium point.

Hard Spring: k_3 > 0:

In this case, both terms of the spring force collaborate, thus creating a stiffer spring. This tells us that the spring force will increase on a cubed trajectory, rather than linearly as with a normal spring system.

But what happens if k_1 is negative? Let's make $k_1 =- 2$ and $k_3 = 4$ such that our work equation becomes $v(y) =- y^2 + y^4$. On the y-v plane, we get the following.

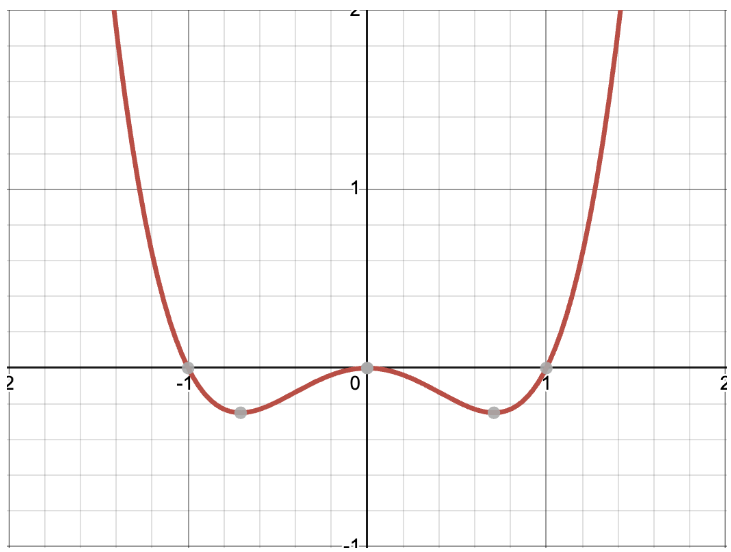

Figure 1: Graph of our work as a function of displacement

This tells us that when we have a large enough displacement, the work done will be positive, but before that happens the work will be in the opposite direction. In other words, our spring would need to be displaced a certain amount before the spring force pulls the mass back to the equilibrium position.

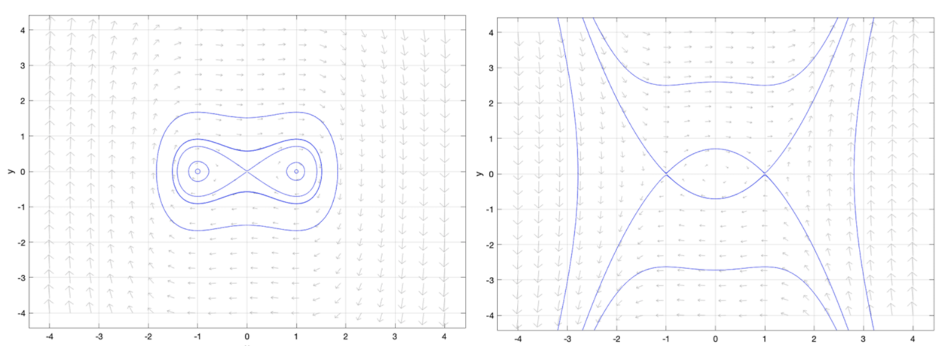

Figure 2: The phase planes of different scenarios. The left is when $k_1 < 0$ and $k_3 > 0$. The right is vice versa. Both have the same equilibrium points but with different stability.

III. Potential Exceptions to Duffing Equation

Duffings equation, while a powerful tool in describing nonlinear oscillations, isn't a one-size-fits-all solution for every physical system. Real-world complexities, like higher-order effects, unique material properties, or different boundary conditions, can challenge its accuracy. In highly chaotic or multidimensional systems, modifications or alternative models might be needed to truly capture the nuances of how these systems behave. Even in realms like quantum mechanics or relativistic physics, classical equations like Duffing's may not tell the full story, requiring more specialized approaches to understand the phenomena at play. Here we will challenge Duffing's equation to possible exceptions.

III a. Higher Order Effects

Duffing's equation assumes certain simplifications. In some cases, higher-order effects, such as damping, external forcing, or additional nonlinear terms, might need to be included for a more accurate representation of a specific system.

III b. Complex Systems

Real-world systems can be extremely complex, and Duffing's equation might not fully capture all the intricacies of such systems. For instance, in highly chaotic or turbulent systems, more sophisticated models or computational approaches might be necessary.

III c. Material Properties

In systems involving materials with non-standard characteristics or behavior, Duffing's equation might require modifications to reflect these unique properties accurately.

III d. Boundary Conditions

Duffing's equation assumes certain boundary conditions or constraints. Systems with different boundary conditions might require adjustments or a different modeling approach.

III e. Nonlinear Terms

Altering the form or introducing different types of nonlinear terms in the equation might lead to varied behaviors that aren't captured by the standard Duffing equation.

III f. Multidimensional Systems

Duffing's equation is typically used for one-dimensional systems. Extending it to multidimensional systems can be complex and might require different mathematical formalisms.

III g. Quantum and Relativistic Systems

In the realm of quantum mechanics or relativistic physics, classical equations like Duffing's might not directly apply, and specialized equations are used to describe phenomena in these domains.

While Duffing's equation provides a valuable framework for understanding nonlinear oscillations, its applicability might be limited in certain complex or specialized scenarios, requiring modifications or alternative models to better represent the underlying dynamics.

IV. Testing Potential Exceptions