Usage of `genulens`

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GitHub: https://github.com/nkoshimoto/genulens
Paper for the Galactic model: Koshimoto, Baba, and Bennett (2021), ApJ, 917, 78
Paper for the code: Koshimoto and Ranc (2021), Zenodo.4784948

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The basic function of `genulens` is to simulate a large number of artificial microlensing events toward a specified (l, b) direction by the Monte Carlo method based on the Galactic model developed by Koshimoto, Baba, and Bennett (2021) [8] (hereafter K21), and to output the parameters. The output parameter space can be confined with observational constraints. For an individual event analysis, for example, this enables you to calculate a prior probability distribution of the lens-source relative proper motion $\boldsymbol{\mu}_{\rm rel}$ or of the microlens parallax $\boldsymbol{\pi}_{\rm E}$ to help the light curve modeling, or a posterior probability distribution of the lens mass and distance. The Galactic model is optimized for the bulge direction, so I do not recommend using it outside the box with $|l| \lesssim 10^\circ$ and $|b| \lesssim 7^\circ$.

To use `genulens`, you need to specify input parameter values or options in the arguments. A simple example is

```
(Example 1)
$./genulens Nsimu 1e+5 l 2.32 b -2.38 vEarthlb 1.10 -6.71 VERBOSITY 2 > tmp.dat
```

This simulates 10^5 (= `Nsimu`) artificial microlensing events toward the sky direction of $(l, b) = (2.32^\circ, -2.38^\circ)$. It uses the E+E$_{\rm X}$ model in K21 as the Galactic model by default. See Section 3-III for how to use other K21's models.

The output format can be specified by the parameter `VERBOSITY`, and Fig. 1 shows part of output lines of example 1. With "`VERBOSITY 2`", the output format is (from left to right):

- wt_j – Weight for jth line. **You need to weight each line by wt$_j$** when you combine them to make a probability distribution. wt_j takes mostly 1 and sometimes > 1 unless you use the "`SMALLGAMMA 1`" option.
- $t_{\rm E}$ – The Einstein radius crossing time in days.
- $\theta_{\rm E}$ – The angular Einstein radius in mas.
- $\pi_{\rm E,N}$ – Equatorial north component of the microlens parallax vector.
- $\pi_{\rm E,E}$ – Equatorial east component of the microlens parallax vector.
- $D_{\rm S}$ – Distance to the source system in pc.
- $\mu_{\rm S,l,hel}$ – Galactic longitudinal component of the source proper motion in the heliocentric frame in mas/yr.
- $\mu_{\rm S,b,hel}$ – Galactic latitudinal component of the source proper motion in the heliocentric frame in mas/yr.
- $i_{\rm S}$ – Flag to indicate which component the source system belongs to. The components are:
  - 0: Thin disk (0.00 - 0.15 Gyr)
  - 1: Thin disk (0.15 - 1.00 Gyr)
  - 2: Thin disk (1.00 - 2.00 Gyr)
  - 3: Thin disk (2.00 - 3.00 Gyr)
  - 4: Thin disk (3.00 - 5.00 Gyr)
  - 5: Thin disk (5.00 - 7.00 Gyr)
  - 6: Thin disk (7.00 - 10.00 Gyr)
  - 7: Thick disk (12.0 Gyr)
  - 8: Bulge (9 ± 1 Gyr)
  - 9: Nuclear stellar disk (7 ± 1 Gyr)
- $i_{\rm L}$ – Flag to indicate which component the lens system belongs to.
- $f_{\rm REM}$ – Flag to indicate evolutional stage of the lens.

```
#----- Output of Monte Carlo simulation w/ VERBOSITY= 2 and seed= 12304357 --------
#          wtj           tE       thetaE           piEN           piEE   D_S         muSl          muSb iS iL fREM
1.789982e+00 1.026014e+01 2.893354e-01 -2.670026e-02  4.126398e-02  8159 -6.23573e+00  4.13179e+00  8  8 0
1.000000e+00 1.449606e+01 2.533772e-01  2.800766e-01  2.697542e-01 13858 -6.68344e+00 -1.68848e-01  6  3 0
1.000000e+00 6.392391e+01 2.665412e-01 -2.151892e-02 -4.385056e-02  7157 -2.54691e+00 -1.72749e+00  8  8 0
1.000000e+00 4.701406e+00 1.194560e-01  4.380464e-02  3.561815e-02  8184 -1.55973e+01  2.67340e+00  8  8 0
1.000000e+00 1.739493e+01 2.549459e-01  9.644269e-02 -4.377408e-03  8763 -6.70109e+00 -1.46376e+00  8  8 0
1.000000e+00 5.888771e+00 1.094510e-01  6.294772e-02 -2.766520e-02  7904 -8.48463e+00 -1.29959e+00  8  8 0
```

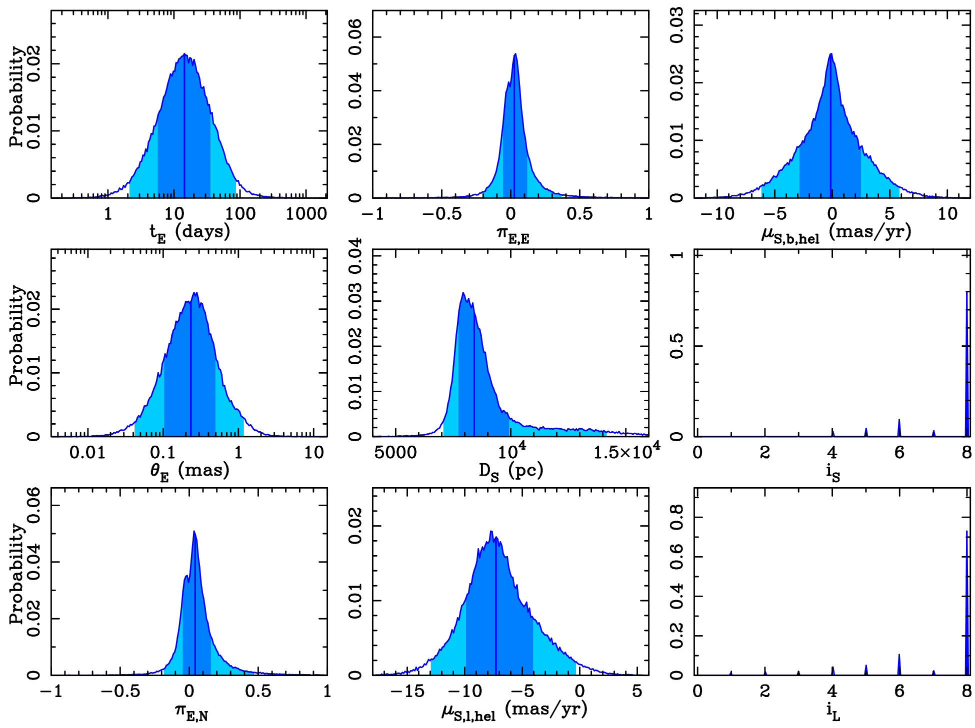

Figure 1: Part of the output lines of example 1 (top) and the probability distributions for each output parameter (bottom).

0: Main sequence

1: White dwarf

2: Neutron star

3: Black hole

Note that $f_{\rm REM}$ can be > 0 only when the "`REMNANT 1`" or "`onlyWD 1`" option is added to the argument, and can be > 1 only when the "`REMNANT 1`" option is added.

In example 1, the two values following `vEarthlb` are the galactic longitudinal and latitudinal components of the Earth velocity projected onto the sky plane in km/sec. `vEarthlb` is important to calculate the direction of $\boldsymbol{\pi}_{\rm E}$, so you do not have to specify it when you are not interested in the microlens parallax, or only interested in its magnitude. Technically, `vEarthlb` affects $t_{\rm E}$ value because it is used to convert $t_{\rm E,hel}(= \theta_{\rm E}/\mu_{\rm rel,hel})$ into $t_{\rm E}(= \theta_{\rm E}/\mu_{\rm rel,geo})$. However, in most cases, the difference in the resulted probability distributions of parameters other than the direction of the microlens parallax vector is very small between when you specify `vEarthlb` and when you do not specify it.

The output parameters with "`VERBOSITY 2`" are sufficient to calculate other (perhaps more interesting) microlensing parameters such as the lens mass and distance. Appendix A provides the formulae for the conversion. Alternatively, you can just use "`VERBOSITY 3`" option to have the lens mass and distance output if you are not struggling with low remaining storage on your computer (see Fig. 2 for the output parameters by `VERBOSITY 3`).

1 Posterior Calculation

For an individual event analysis, $t_{\rm E}$ is commonly measured, and $\theta_{\rm E}$ is mostly measured for a caustic-crossing event. A frequent situation is that you want to calculate the posterior probability distributions for the lens mass and distance by incorporating these constraints from observed parameters into the Galactic prior.

Let me use OGLE-2008-BLG-355 (Koshimoto et al. 2014 [9]) to show how to calculate a posterior distribution with `genulens`. You have measured $t_{\rm E} = 34.0 \pm 2.2$ days and $\theta_{\rm E} = 0.28 \pm 0.03$ mas for this event occurred at $(l, b) = (-0.08°, -3.45°)$.

```
#----- Output of Monte Carlo simulation w/ VERBOSITY= 3 and seed= 12304357 --------
#           wtj           M_L   D_L   D_S            t_E      theta_E          pi_E          pi_EN          pi_EE       mu_rel        mu_Sl         mu_Sb     I_L iS iL fREM
1.090589e+00 6.073099e-01  9798 11270 3.337356e+01 2.567712e-01 5.191626e-02  1.268154e-02  5.034359e-02 2.810178e+00 -8.36728e+00  1.97348e+00  99.000  7  8 0
1.000000e+00 8.924341e-01  7467  8010 3.675385e+01 2.568369e-01 3.533856e-02 -2.433739e-02 -2.562236e-02 2.552377e+00 -4.00786e+00 -2.02992e-01  99.000  8  8 0
1.000000e+00 5.172451e-01  7701  9233 3.254831e+01 3.012192e-01 7.150788e-02  4.492107e-02  5.563698e-02 3.380217e+00 -7.02563e+00 -3.33934e-01  99.000  4  8 0
1.000000e+00 4.967772e-01  8126  9339 3.581116e+01 2.542376e-01 6.284139e-02 -1.548968e-02 -6.090247e-02 2.593054e+00 -4.58248e+00 -6.41778e-01  99.000  8  6 0
1.000000e+00 5.501188e-01  6733  7838 3.407724e+01 3.062447e-01 6.835647e-02 -4.835272e-02 -4.831792e-02 3.282422e+00 -3.57722e+00 -4.30568e+00  99.000  8  8 0
1.000000e+00 7.642087e-01  7927  8963 3.233027e+01 3.013273e-01 4.841657e-02 -3.061996e-02 -3.750443e-02 3.404233e+00 -6.64013e+00  1.77290e+00  99.000  8  8 0
```

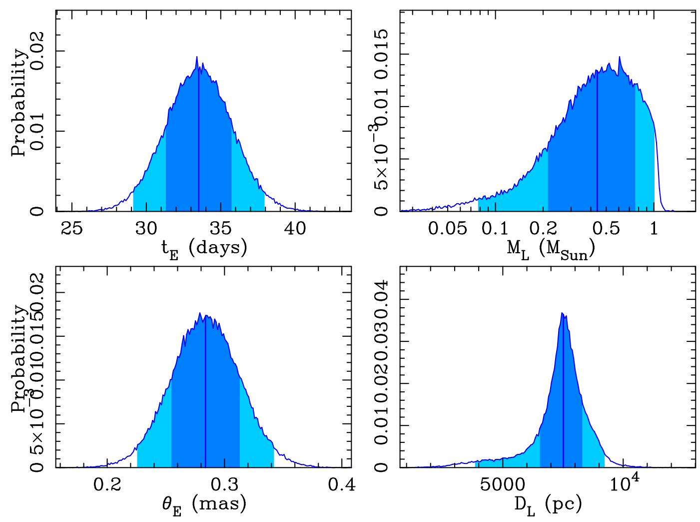

Figure 2: Part of the output lines of example 2 (top) and the probability distributions for some parameters (bottom). Note that the lens apparent magnitude in I-band, $I_{\rm L}$, only takes 99 unless the extinction value for red clump centroid $A_{I,{\rm rc}}$ is specified by `AIrc`.

1.1 Simple way: Independent Gaussian assumption

You can calculate the posterior probability distribution by

```
(Example 2)
$./genulens NlikeMIN 1e+5 l -0.08 b -3.45 tE 34.0 2.2 thetaE 0.28 0.03 VERBOSITY 3 > tmp.dat
```

when the lens is assumed to be a normal star (not a white dwarf)[1]. Fig. 2 shows part of the output lines (with `VERBOSITY` = 3) and the distribution for some parameters. The "`VERBOSITY 3`" option is more redundant than "`VERBOSITY 2`", and it outputs $M_{\rm L}$, $D_{\rm L}$, $\pi_{\rm E}$, $\mu_{\rm rel,geo}$, and $I_{\rm L}$, in addition to the output parameters with `VERBOSITY 2`. In example 2, I do not specify `vEarthlb` assuming that we are not interested in the direction of $\pi_{\rm E}$ vector now.

The "`tE` [`val`] [`err`]" option multiplies a Gaussian probability of

$$\mathcal{L}_p = \exp\left(-0.5\left(\frac{\texttt{val} - p}{\texttt{err}}\right)^2\right) \tag{1}$$

by the Galactic prior by default, where $p = t_{\rm E}$ in this case. When "`UNIFORM 1`" is added to the arguments, it multiplies a uniform probability of

$$\mathcal{L}_p = \begin{cases} 1 & \text{when } |\texttt{val} - p| \leq \texttt{err} \\ 0 & \text{when } |\texttt{val} - p| > \texttt{err} \end{cases} \tag{2}$$

by the Galactic prior. The same is true for the "`thetaE` [`val`] [`err`]" option but with $p = \theta_{\rm E}$.

In practice, when you add the `tE` and `thetaE` constraints, `genulens` generates many microlensing events following the Galactic model, and outputs its parameters only if the condition "$r < \mathcal{L}_{t_{\rm E}}\mathcal{L}_{\theta_{\rm E}}$" is true, where r is a random number generated from the uniform distribution between 0 and 1.

Example 2 outputs $\sim 10^5$ (= `NlikeMIN`) lines that passed the condition "$r < \mathcal{L}_{t_{\rm E}}\mathcal{L}_{\theta_{\rm E}}$", which means the number of simulated events are much larger than `NlikeMIN`. This is the difference between `NlikeMIN` and `Nsimu` used in example 1. If you use "`Nsimu 1e+5`" instead of "`NlikeMIN 1e+5`" in example 2, it would simulate 10^5 microlensing events, and would output parameters for small fraction of them that passed the condition.

The "`NlikeMIN`" option is useful, but you need to be careful when using it because you cannot easily tell the amount of time it will take unlike the "`Nsimu`" option. **I recommend that you always check the computational time with a small `NlikeMIN` value before you try a calculation with a large `NlikeMIN` value,** because the computational time with a certain `NlikeMIN` value totally depends on the constraints you used for `tE`, `thetaE`, or other parameters (see Section 2 for some notes on computation time). In the case of example 2, it takes $\sim$ 80 secs with my laptop to simulate $\sim 1.4 \times 10^7$ events and accept $\sim 10^5$ events out of them.

[1] The "`onlyWD 1`" option can be used to consider a white dwarf lens possibility. Maybe this is preferable now that a planet around a white dwarf is discovered in the microlensing sensitive region (Blackman et al. 2021 [2]).

1.2 More rigorous way: Incorporation of parameter distribution from light curve modeling

In example 2, I incorporated the measured $t_{\rm E}$ and $\theta_{\rm E}$ values assuming they are independent Gaussian distributions. This assumption mostly works for events with good measurements for the parameters that you want to incorporate with the Galactic prior (i.e., $t_{\rm E}$ and $\theta_{\rm E}$ for OGLE-2008-BLG-355). The assumption becomes not good if there is a non-negligible correlation among the parameters, or the parameter distribution is heavily skewed. So, ideally, we want to convolve the Galactic prior with a parameter distribution from light curve modeling.

Currently, `genulens` only supports incorporation of the observed constraints in either a Gaussian probability form (Eq. 1) or a uniform probability form (Eq. 2), and cannot directly incorporate the parameter distribution from your light curve analysis like MCMC.

An alternative way, in the case of OGLE-2008-BLG-355, is to conduct a similar calculation to example 2, but with the "`UNIFORM 1`" option and larger error bars for $t_{\rm E}$ and $\theta_{\rm E}$, e.g.,

```
(Example 3)
$./genulens NlikeMIN 1e+6 l -0.08 b -3.45 tE 34.0 9.0 thetaE 0.28 0.12 VERBOSITY 2
 UNIFORM 1 > tmp.dat
```

The error bars should be taken so that it covers major parameter space. Note that "`NlikeMIN 1e+6`" is probably too small to have a good resolution.

Separately, you should be able to calculate a joint probability distribution of $t_{\rm E}$ and $\theta_{\rm E}$, $P_{\rm LC}(t_{\rm E}, \theta_{\rm E})$, from your light curve analysis. By multiplying $P_{\rm LC}(t_{\rm E}, \theta_{\rm E})$ by ${\rm wt}_j$ in each line of the output of example 3, you can calculate a posterior probability distribution for $t_{\rm E}$ and $\theta_{\rm E}$ (and other output parameters) including the correlation between $t_{\rm E}$ and $\theta_{\rm E}$, and non-Gaussian shape of them.

This is just my idea to incorporate it, and maybe there is a better way. Please let me know if you find a better idea.

2 Notes on computation time

Computation time could be an issue with the calculation using `genulens`. As introduced in Section 3, you can put constraints on $\boldsymbol{\pi}_{\rm E}$, $\boldsymbol{\mu}_{\rm S}$, $I_{\rm L}$, similarly to $t_{\rm E}$ and $\theta_{\rm E}$. However, the computation time to obtain sufficient statistics for the posterior distribution will very easily increase when more parameters are used for constraints and/or when very small error bars are specified.

Some people may not like this, but I often specify a larger error value to the extent that it does not affect the result. In particular, since $t_{\rm E}$ has a relatively low sensitivity to the lens property, even when $t_{\rm E}$ is measured with an error of 1%, I often increase the error to about 5% to reduce calculation time. In my experience, this has never affected the resulted posterior distribution.

If you find that your desired calculation would take too long as a result of a short trial run, you can divide the calculation into multiple runs with different random seed values using the `seed` option. For example, you can run

```
$./genulens [options for calculation] seed 1 > tmp1.dat
$./genulens [options for calculation] seed 2 > tmp2.dat
.
.
$./genulens [options for calculation] seed 50 > tmp50.dat
```

in parallel if you have sufficient cores in your machine(s), and combine the output files to make a posterior probability distribution.

Importance sampling could be another way to reduce the computation time, but I haven't implemented it yet. I would consider about it if there is a demand.

3 Input Parameters

Here I explain keywords for input parameters you might find useful. The default values are used when you do not specify the argument.

I. Parameters to put observational constraints

1. Event coordinate

   Event coordinate must be given in the galactic coordinate system.

   — `l [double]` (deg., default: 1.0) :
   Galactic longitude.

   — `b [double]` (deg., default: -3.9) :
   Galactic latitude.

ex.) `$./genulens l 3.0 b -3.0 VERBOSITY 3 > tmp.dat`

2. For conversion between the heliocentric and the geocentric frames

   For calculation of $\pi_{\rm E,N}$ and $\pi_{\rm E,E}$ (and technically $\mu_{\rm rel,geo}$ and $t_{\rm E} = \theta_{\rm E}/\mu_{\rm rel,geo}$), the Earth velocity projected onto the sky plane toward (l, b) at the reference time for the geocentric coordinate (e.g., event peak time) must be given in either the galactic or equatorial coordinate system.

   — `vEarthlb [double] [double]` (km/s, default: 11.9392, -17.9020) :
   Galactic longitudinal and latitudinal components of the projected Earth velocity.

   ex.) `$./genulens l 2.32 b -2.38 vEarthlb 1.10 -6.71 VERBOSITY 3 > tmp.dat`

   — `vEarthEN [double] [double]` (km/s, default: 0, 0) :
   Equatorial east and north components of the projected Earth velocity.

   ex.) `$./genulens l 2.32 b -2.38 vEarthEN 6.37 -2.37 VERBOSITY 3 > tmp.dat`

3. For source distance distribution

   3.1. *When extinction ($A_{I,\rm rc}$) is given*

   The number of source stars in the magnitude range $I_{\rm S,min}$ to $I_{\rm S,max}$ and the distance range $D_{\rm S}$ to $D_{\rm S} + dD_{\rm S}$ is

$$dN_{\rm S} \propto n(D_{\rm S}) D_{\rm S}^2 \, dD_{\rm S} \int_{I_{\rm S,min}}^{I_{\rm S,max}} L_{\mathcal{M}_I}(I_{\rm S} - A_I(D_{\rm S}) - 5\log[D_{\rm S}/10\,{\rm pc}]) dI_{\rm S}, \tag{3}$$

   where $L_{\mathcal{M}_I}(\mathcal{M}_I)$ is the luminosity function for the I-band absolute magnitude.
   If mean extinction for the red clump (RC) toward the sky direction, $A_{I,\rm rc}$, is given, `genulens` calculates Eq. (3) for the $D_{\rm S}$ distribution using the PARSEC isochrone model [3, 4, 17] for $L_{\mathcal{M}_I}(\mathcal{M}_I)$ and

$$A_I(D_{\rm S}) = A_{I,\rm RC} \frac{1 - \exp[-D_{\rm S}/(h_{\rm dust}/\sin|b|)]}{1 - \exp[-D_{\rm RC}/(h_{\rm dust}/\sin|b|)]}, \tag{4}$$

   where $h_{\rm dust}$ is the dust scale height and $D_{\rm RC}$ is the mean RC distance. Related parameters are given by

   — `AIrc [double]` (default: 0) :
   Mean RC extinction in the target field in I-band, $A_{I,\rm rc}$. It can be taken from the OGLE-III extinction map [12].

   — `Isrange [double] [double]` (default: 14.0, 21.0) :
   $I_{\rm S,min}$ and $I_{\rm S,max}$ in Eq. (3). The default range corresponds to the OGLE-IV survey sensitivity [10]. The range can be taken from the $I_{\rm S}$ measurement. In such a case, make sure to take the $I_{\rm S,min}$ and $I_{\rm S,max}$ range conservatively to account for all the uncertainties of (a) the source flux measurement from the light curve modeling, (b) calibration to the standard (Cousins I) magnitude system, (c) the extinction (not only the $A_{I,\rm rc}$ measurement, but also the $A_I(D_{\rm S})$ model), and (d) the isochrone model, because none of them is considered in other places in the calculation.

   — `hdust [double]` (pc, default: 164.0) :
   Dust scale height, $h_{\rm dust}$. Default is 164 pc taken from Nataf et al. (2013) [12]. K21 used the default value in their fit, so changing this is not highly recommended without a strong reason.

   — `DMrc [double]` (default: $14.3955 - 0.0239\,l + 0.0122\,|b| + 0.128$) :
   Mean distance modulus for the RC, $\rm DM_{rc}$. The mean distance $D_{\rm RC}$ in Eq. (4) is given by $D_{\rm RC} = 10^{1+0.2{\rm DM_{rc}}}$ pc. $\rm DM_{rc}$ can be taken from the μ value in the OGLE-III extinction map [12]. The default formula is from Eqs. (2) - (3) of Nataf et al. (2016) [11].

   ex.) In the MB14472 analysis (Ranc et al. 2021 [13]), we have $I_S = 15.8 \pm 0.2$ mag and $A_{I,\rm rc} = 1.30^{+0.26}_{-0.23}$ mag. The combined error in quadrature is ~ 0.32. So, I conservatively took the range for `Isrange` with 15.8 ± 0.4 mag and run the following to have the Galactic prior for $\mu_{\rm rel}$ (because the finite source effect was weak);

```
$./genulens NlikeMIN 1e+6 l 2.32 b -2.38 vEarthlb 1.10 -6.71 tE 14.6 0.8
 Isrange 15.4 16.2 AIrc 1.3 DMrc 14.479 VERBOSITY 3 > tmp.dat
```

   You can add the constraints from the $(V-I)_{\rm S}$ measurement in addition to the $I_{\rm S}$ measurement.

   — `EVIrc [double]` (default: 0) :
   Mean RC reddening in the target field, $E(V-I)_{\rm rc}$, which can be taken from the OGLE-III extinction map [12].

— `VIsrange [double] [double]` (default: 0.0, 0.0) :
$(V-I)_\mathrm{S}$ range considered. This can be determined based on the $(V-I)_\mathrm{S}$ measurement from the light curve modeling. Similarly to the `Isrange` case, the $(V-I)_\mathrm{S}$ range should be taken conservatively.

ex.) In the OB181185 analysis (Kondo et al. 2021 [7]), we have $I_S = 20.082 \pm 0.012$ mag, $(V-I)_S = 2.34 \pm 0.03$ mag, $A_{I,\mathrm{rc}} = 1.96 \pm 0.05$ mag, and $E(V-I) = 1.66 \pm 0.07$ mag. I conservatively took 20.1 ± 0.3 mag for `Isrange` and 2.34 ± 0.10 mag for `VIsrange` and run the following (50 times in parallel with different seed values) to have the Galactic prior to be combined with the MCMC result

```
$./genulens NlikeMIN 1e+6 l 2.47 b -2.00 vEarthlb -24.4 13.1 AIrc 1.96
 Isrange 19.8 20.4 EVIrc 1.66 VIsrange 2.24 2.44 DMrc 14.47 tE 16.0 0.5
 thetaE 0.207 0.05 UNIFORM 1 SMALLGAMMA 1 VERBOSITY 2 seed 1 > tmp.dat
```

3.2. *When extinction ($A_{I,\mathrm{rc}}$) is NOT given*

Eq. (3) is often simplified by

$$dN_\mathrm{S} \propto n(D_\mathrm{S}) D_\mathrm{S}^{\gamma'}\, dD_\mathrm{S}, \tag{5}$$

assuming the integration of $L_{\mathcal{M}_I}$ in Eq. (3) is $\propto D_\mathrm{S}^{\gamma'-2}$ (or $\propto 10^{0.2(2-\gamma')\mathcal{M}_I}$) [6].
When $A_{I,\mathrm{rc}}$ is not given, `genulens` uses Eq. (5) for the D_S distribution and the slope γ' can be specified by

— `gammaDs [double]` (default: 0.5) :
γ' in Eq. (5). A reasonable range is $-1 \le \gamma' \le 1$ [6]. For example, Bennett et al. (2014) [1] uses the default value of $\gamma' = 0.50$, Sumi et al. (2011) [16] uses $\gamma' = 0$, and Zhu et al. (2017) [19] uses $\gamma' = -0.85$. Zhu et al. (2017) [19] demonstrated that one's choice of γ' value has little impact on distance estimates for the lens and source[2].

ex.)
```
$./genulens l 3.0 b -3.0 gammaDs 0.0 VERBOSITY 3 > tmp.dat
```

In my experience, the D_S distribution with $\gamma' = 0.5$ is reasonably similar to the one using an $A_{I,\mathrm{rc}}$ value and $I_{\mathrm{S,min}} = 14$ and $I_{\mathrm{S,max}} = 21$. This should depends on the $A_{I,\mathrm{rc}}$ value, though. So I would use the $A_{I,\mathrm{rc}}$ option whenever it is available.

4. For likelihood calculation

When observed values for a parameter set $\boldsymbol{p}$ is specified, `genulens` outputs only jth event that satisfies $\prod_i \mathcal{L}_{p_i} < r_j$ out of the simulated events, where $\mathcal{L}_{p_i}$ is given by Eq. (1) or (2) and r_j is a random number generated from the uniform distribution between 0 and 1.

— `tE [double] [double]` (days, default: 54.5, 99999999999.0) :
Observed Einstein radius crossing time, t_E, value and its error. Used for $\mathcal{L}_{t_\mathrm{E}}$ calculation.

— `thetaE [double] [double]` (mas, default: 0, 0) :
Observed angular Einstein radius, θ_E, value and its error. Used for $\mathcal{L}_{\theta_\mathrm{E}}$ calculation.

ex.)
```
$./genulens NlikeMIN 1e+4 l -0.08 b -3.45 tE 34.0 2.2 thetaE 0.28 0.03
 VERBOSITY 3 > tmp.dat
```

— `piE [double] [double]` (default: 0, 0) :
Observed microlens parallax, π_E, value and its error. Used for $\mathcal{L}_{\pi_\mathrm{E}}$ calculation.

— `piEN [double] [double]` (default: 0, 0) :
Observed north component of microlens parallax vector, $\pi_{\mathrm{E,N}}$, value and its error. Used for $\mathcal{L}_{\pi_{\mathrm{E,N}}}$ calculation.

— `piEE [double] [double]` (default: 0, 0) :
Observed east component of microlens parallax vector, $\pi_{\mathrm{E,E}}$, value and its error. Used for $\mathcal{L}_{\pi_{\mathrm{E,E}}}$ calculation.

ex.)
```
$./genulens NlikeMIN 1000 l -0.2 b -1.7 vEarthlb -1.19 8.71 tE 80.9 2.0
 piEN 0.11 0.02 piEE 0.14 0.01 VERBOSITY 3 > tmp.dat
```

— `musl [double] [double]` (mas/yr, default: 0, 0) :
Observed galactic longitudinal component of source proper motion in the heliocentric coordinate, $\mu_{\mathrm{S,l}}$, value and its error. Used for $\mathcal{L}_{\mu_{\mathrm{S,l}}}$ calculation.

— `musb [double] [double]` (mas/yr, default: 0, 0) :
Observed galactic latitudinal component of source proper motion in the heliocentric coordinate, $\mu_{\mathrm{S,b}}$, value and its error. Used for $\mathcal{L}_{\mu_{\mathrm{S,b}}}$ calculation.

[2] Note that Zhu et al. (2017) use a different parameter γ, which is related to γ' in this document as $\gamma = 2 - \gamma'$. I have denoted γ' to avoid confusion with their γ.

— `musN [double] [double]` (mas/yr, default: 0, 0) :
Observed equatorial north component of source proper motion in the heliocentric coordinate, $\mu_{\rm S,N}$, value and its error. Used for $\mathcal{L}_{\mu_{\rm S,N}}$ calculation.

— `musE [double] [double]` (mas/yr, default: 0, 0) :
Observed equatorial east component of source proper motion in the heliocentric coordinate, $\mu_{\rm S,E}$, value and its error. Used for $\mathcal{L}_{\mu_{\rm S,E}}$ calculation.

ex.)
```
$./genulens NlikeMIN 1000 l -1.2 b -2.3 vEarthlb 15.0 -24.1 tE 11.4 0.2
 thetaE 0.237 0.024 musE -3.7 0.5 musN -10.4 0.4 VERBOSITY 2 > tmp.dat
```

— `musRCG [int]` (default: 0) :
If 1, a given $\boldsymbol{\mu}_{\rm S}$ vector is considered to be relative to the mean RC proper motion, where RC stars are assumed to have mean velocity of 0 at the mean distance $D_{\rm RC}$.

ex.) Shvartzvald et al. (2019) [14] measured the source proper motion $(\mu_{\rm S,N}, \mu_{\rm S,E}) = (-5.10, -3.15) \pm (0.46, 0.44)$ mas/yr relative to the red clump giants. A probability distribution of, e.g., $(\pi_{\rm E,N}, \pi_{\rm E,E})$, based on the Galactic model can be calculated by
```
$./genulens NlikeMIN 1e+5 l 0.69 b 2.01 vEarthlb -24.9 14.8 tE 14.9 1.1
 thetaE 0.14 0.02 musE -3.15 0.44 musN -5.10 0.46 musRCG 1 VERBOSITY 2 > tmp.dat
```

— `[pname]det [int]` (default: 2 for `[pname]` = `IL`, 0 for others) :
Parameter to consider an upper/lower limit constraint for a parameter p. p can be $t_{\rm E}$, $\theta_{\rm E}$, $\pi_{\rm E}$, or $I_{\rm L}$ for `[pname]` = `tE`, `thetaE`, `piE`, or `IL`, respectively.
`[pname]det` = 1 is the upper limit option, and the likelihood is given by

$$\mathcal{L}_p = \begin{cases} \exp\left(-0.5\left(\frac{\texttt{val}-p}{\texttt{err}}\right)^2\right) & \text{when } p > \texttt{val} \\ 1 & \text{when } p \leq \texttt{val}\ , \end{cases} \tag{6}$$

where `val` and `err` are the values specified by a "`[pname] [val] [err]`" argument.
Similarly, `[pname]det` = 2 is the lower limit option (and default for $p = I_{\rm L}$), and the likelihood is given by

$$\mathcal{L}_p = \begin{cases} \exp\left(-0.5\left(\frac{\texttt{val}-p}{\texttt{err}}\right)^2\right) & \text{when } p < \texttt{val} \\ 1 & \text{when } p \geq \texttt{val}. \end{cases} \tag{7}$$

`[pname]det` = 0 is the detection option (and default for $p = t_{\rm E}, \theta_{\rm E}$, and $\pi_{\rm E}$), and the likelihood is given by Eq. (1) or Eq. (2).

ex.) Dong et al. (2009) [5] measured the 3σ lower limit of $\theta_{\rm E} > 0.60$ mas for OB05071.
```
$./genulens NlikeMIN 5e+4 l -4.42 b -3.79 tE 68.1 1.2 thetaE 0.600 0.001
 thetaEdet 2 VERBOSITY 2 > tmp.dat
```
where the error 0.001 is an arbitrarily chosen small value so that $\theta_{\rm E} > 0.6$ mas is rarely accepted. (So this is not a 3σ limit but a hard limit at 0.6 mas.)

— `IL [double] [double]` (mag, default: 14.0, 0.01 with `ILdet` = 2) :
Observed lens magnitude in I-band. Used for $\mathcal{L}_{I_{\rm L}}$ calculation. You have to input an $A_{I,\rm rc}$ value to use this option. By default, a lower limit of $I_{\rm L} > 14.0$ is applied when an $A_{I,\rm rc}$ is given assuming a star brighter than 14.0 mag in I-band should be saturated and undetectable. Note that the lens magnitude $I_{\rm L}$ is calculated using an empirical mass-luminosity relation for a main-sequence star. So this does not work when the lens is a giant star.

ex.) Wyrzykowski et al. (2016) [18] discovered a black-hole candidate OGLE3-ULENS-PAR-02 and put a lower limit on the lens I-band magnitude of $I_L > 17.7$ mag based on their measurement of the blend magnitude. A lens mass posterior can be calculated by
```
$./genulens NlikeMIN 1000 l 1.45 b -2.13 vEarthlb -13.2 19.7 tE 288.8 20.0
 piEN -0.033 0.004 piEE -0.073 0.004 AIrc 1.42 DMrc 14.5 IL 17.7 0.01 ILdet 2
 REMNANT 1 SMALLGAMMA 1 VERBOSITY 3 > tmp.dat
```

— `UNIFORM [int]` (default: 0) :
If 1, the likelihood for a parameter p, $\mathcal{L}_p$, is given by Eq. (2). If 0, $\mathcal{L}_p$ is given by Eq. (1). See section 1.2 for when this might be useful.

ex.) In the OB181185 analysis (Kondo et al. 2021 [7]), which was also introduced above in the description of `VIsrange`, we wanted to convolve the Galactic prior with the probability distribution resulted from the MCMC on the light curve. Following the idea described in section 1.2, we took the error range for $t_{\rm E}$ and $\theta_{\rm E}$ largely enough (5σ) and run the following (50 times in parallel with different seed values)
```
$./genulens NlikeMIN 1e+6 l 2.47 b -2.00 vEarthlb -24.4 13.1 AIrc 1.96
 Isrange 19.8 20.4 EVIrc 1.66 VIsrange 2.24 2.44 DMrc 14.47 tE 16.0 0.5
 thetaE 0.207 0.050 UNIFORM 1 SMALLGAMMA 1 VERBOSITY 2 seed 1 > tmp.dat
```

II. Parameters to control the simulation

There are three categories of events handled in the Monte Carlo simulation by `genulens`: "generated events", "simulated events", and "accepted events"[3]. In each trial of the Monte Carlo simulation, `genulens` at first randomly generates a microlensing event (i.e., one source-lens pair) based on the mass function, density distribution, and velocity distribution. The event rate Γ is calculated for the "generated event" to decide whether or not to consider this as a "simulated event" (see the description of `SMALLGAMMA` below for details of this decision). If it does, the simulated event's parameters are next compared with the input observational constraints for a parameter set $\boldsymbol{p}$. If it meets the likelihood condition $\prod_i \mathcal{L}_{p_i} < r$ (r is a random number between 0 and 1), it is finally considered as an "accepted event", and its parameters are printed if `VERBOSITY` is not 0. Note that if you use no observational constraint, the number of simulated events equals to the number of accepted events.

1. Simulation length

   You can use `Nsimu` or `NlikeMIN` to control the simulation length.

   — `Nsimu [long]` (default: 100000) :
   Number of simulated events you wish to have. The simulation ends when the number of simulated events reaches `Nsimu` unless the number of accepted events is less than `NlikeMIN` at that time.

   ex.) `$./genulens Nsimu 1e+5 tE 34.0 2.2 thetaE 0.28 0.03 VERBOSITY 3 > tmp.dat`

   — `NlikeMIN [long]` (default: 0) :
   Minimum number of accepted events. The simulation ends as soon as the number of accepted events gets > `NlikeMIN` after the number of simulated events reaches `Nsimu`. Again, I recommend that you always check the computation time with a small `NlikeMIN` value before you try a large `NlikeMIN` value.

   ex.) `$./genulens NlikeMIN 1e+5 tE 34.0 2.2 thetaE 0.28 0.03 VERBOSITY 3 > tmp.dat`

2. For events with small lensing probability

   `SMALLGAMMA` controls the condition for a "generated event" to be considered as a "simulated event".

   — `SMALLGAMMA [int]` (default: 0) :
   If 0 (default), the event rate $\Gamma = C D_{\rm L}^2\, \theta_{\rm E}\, \mu_{\rm rel}$ is calculated for each generated event, where C is an arbitrarily determined constant[4] to limit $\Gamma \lesssim 1$. When a generated event satisfies $r < \Gamma$, it is considered as a simulated event, where r is a random number between 0 and 1.

   If 1, all generated events are considered as simulated events. This will include events with extremely small lensing probabilities. So this option is useful to calculate Galactic priors for events with parameters that seem unlikely, or to restrict parameter space that is only loosely constrained by observation alone.

   The output parameter wt_j differs depending on the `SMALLGAMMA` value, and it is given by

   $$\mathrm{wt}_j = \begin{cases} \max(1, \Gamma) & \text{when } \texttt{SMALLGAMMA} = 0 \\ \Gamma & \text{when } \texttt{SMALLGAMMA} = 1, \end{cases} \tag{8}$$

   where $\max(a, b)$ takes a larger value among a and b.

   Note that the `SMALLGAMMA` = 1 option requires a larger `NlikeMIN` or `Nsimu` value to have the same resolution of the probability distribution as the `SMALLGAMMA` = 0 option.

   ex.) Again in the OB181185 analysis (Kondo et al. 2021 [7]), which was also introduced above in the descriptions of `VIsrange` and `UNIFORM`, we wanted to quantitatively calculate the Galactic prior probability distribution of $(\pi_{\rm E,N}, \pi_{\rm E,E})$ over a wide area of the parameter space, including very low probability parts, to compare it with what preferred from the ground-based and *Spitzer* light curves. So I used the `SMALLGAMMA` option and run the following (50 times in parallel with different seed values)

   ```
   $./genulens NlikeMIN 1e+6 l 2.47 b -2.00 vEarthlb -24.4 13.1 AIrc 1.96
    Isrange 19.8 20.4 EVIrc 1.66 VIsrange 2.24 2.44 DMrc 14.47 tE 16.0 0.5
    thetaE 0.207 0.05 UNIFORM 1 SMALLGAMMA 1 VERBOSITY 2 seed 1 > tmp.dat
   ```

   See Fig. 5 in Kondo et al. (2021) [7] for a result how this worked.

3. For parallel/independent simulations

   Please make sure to use different `seed` values for parallel or independent runs.

[3] This terminology is introduced just to describe the options in this part, and might not consistent through this document.
[4] $C = 8 \times 10^{-9}$ is currently used with $D_{\rm L}$ in pc, $\theta_{\rm E}$ in mas, and $\mu_{\rm rel}$ in mas/yr.

— `seed [long]` (default: 12304357) :
Random seed value (must be positive integers).

ex.)
```
$./genulens VERBOSITY 3 seed 4 > tmp4.dat &
$./genulens VERBOSITY 3 seed 5 > tmp5.dat &
```

4. For remnant possibility

One of the output parameters, $f_{\rm REM}$, is the flag to specify the type of lens. By default, it only takes $f_{\rm REM} = 0$, which indicates a normal star. You can allow $f_{\rm REM}$ to be > 0 and consider a possibility of remnant lens by the `REMNANT` or `onlyWD` option.

— `REMNANT [int]` (default: 0) :
If 1, the lens is allowed to be not only a normal star ($f_{\rm REM} = 0$), but also a remnant, i.e., a white dwarf ($f_{\rm REM} = 1$), a neutron star ($f_{\rm REM} = 2$), or a black hole ($f_{\rm REM} = 3$). These are simulated by combining the initial mass function and the initial-final mass relationships. See section 2.1 in the K21 paper and references therein for the initial-final mass relationships used.

ex.) `REMNANT` option can be used for an analysis of a black-hole lens candidate. Here is an example for a black-hole candidate OGLE3-ULENS-PAR-02 (Wyrzykowski et al. 2016 [18]), which is also introduced for the `IL` option,
```
$./genulens NlikeMIN 1000 l 1.45 b -2.13 vEarthlb -13.2 19.7 tE 288.8 20.0
 piEN -0.033 0.004 piEE -0.073 0.004 AIrc 1.42 DMrc 14.5 IL 17.7 0.01 ILdet 2
 REMNANT 1 SMALLGAMMA 1 VERBOSITY 3 > tmp.dat
```

— `onlyWD [int]` (default: 0) :
If 1, the lens is allowed to be not only a normal star ($f_{\rm REM} = 0$), but also a white dwarf ($f_{\rm REM} = 1$). This is an option prepared for a planetary event analysis because in the current knowledge, a neutron star or black hole host is unlikely whereas a normal star or white dwarf host is likely. Now that a planet around a white dwarf is discovered in the microlensing sensitive region (Blackman et al. 2021 [2]), you might want to use this option for an analysis of a planetary microlensing event unless the lens is confirmed to be luminous.

ex.) If you want to consider a white dwarf host possibility, example 2 in section 1.1 would be
```
$./genulens NlikeMIN 1e+5 l -0.08 b -3.45 tE 34.0 2.2 thetaE 0.28 0.03
 onlyWD 1 VERBOSITY 3 > tmp.dat
```
and a sharp peak of the white dwarf population would appear around $\sim 0.52\ M_\odot$ in the lens mass probability distribution. Note that if you do like this, you will implicitly assume that the planet-hosting probability for white dwarfs is the same for the normal stars'.

5. For output parameters

You can change the output parameters for each event by `VERBOSITY`.

— `VERBOSITY [int]` (default: 0) :
`VERBOSITY` = 0 does not output parameters of each event, and only show the header and footer part. Among non-zero `VERBOSITY`, `VERBOSITY` = 2 and `VERBOSITY` = 3 are currently active and practical. See Figs. 1 and 2 for what parameters are outputted with `VERBOSITY` = 2 and 3, respectively. Other values are for development purposes only and are not recommended for use, even if they are implemented in the program. (`VERBOSITY` = 1 is NOT active due to a historical reason...)

Again, you can calculate all the parameters outputted with `VERBOSITY` = 3 from the parameters outputted with `VERBOSITY` = 2, except for $I_{\rm L}$. See Appendix A for the conversion.

ex.)
```
$./genulens VERBOSITY 2 > tmp2.dat &
$./genulens VERBOSITY 3 > tmp3.dat &
```

6. For optical depth and event rate

This is not very related to the main function of `genulens`, but you can calculate the optical depth and event rate toward the input direction by `CALCTAU`.

— `CALCTAU [int]` (default: 0) :
With `CALCTAU` = 1, `genulens` calculates the optical depth τ toward the input (l, b) direction for $I_{\rm S,min} < I_S < I_{\rm S,max}$. Here, $I_{\rm S,min}$ and $I_{\rm S,max}$ are the ones given by the `Isrange` option, and you need to input the extinction value by the `AIrc` option as well. Then, `genulens` takes the mean of $t_{\rm E}$ for the "generated events" in the Monte Carlo simulation and calculate the

mean $t_{\rm E}$ value, $\langle t_{\rm E}\rangle$. After the simulation, the event rate is calculated by $2\tau/(\pi \langle t_{\rm E}\rangle)$. The calculated values are shown in a line starting from "# avetE= " in the footer part. Make sure not to use a too small `Nsimu` value to guarantee a certain degree of accuracy of the event rate calculation (or $\langle t_{\rm E}\rangle$).

ex.) Optical depth and event rate toward the center coordinate at the BLG511 field of the OGLE-IV survey are calculated by

```
$./genulens l 3.28 b -2.52 Isrange 14.0 21.0 AIrc 1.77 CALCTAU 1 |grep avetE=
```

This returns a line starting from

```
# avetE= 23.270 days, medtE= 15.477 days, tau= 1.424360e-06 ,
```

and you see the optical depth is $\sim 1.42\times10^{-6}$ and the event rate is $\sim 1.42\times10^{-5}\,{\rm star}^{-1}{\rm year}^{-1}$ or $\sim 138\,{\rm deg.}^{-2}{\rm year}^{-1}$. These values are moderately agree with the ones for the BLG511 field by Mróz et al. (2019) [10], $(1.43 \pm 0.15) \times 10^{-6}$, $(1.35 \pm 0.10) \times 10^{-5}\,{\rm star}^{-1}{\rm year}^{-1}$, and $113.9 \pm 8.1\,{\rm deg.}^{-2}{\rm year}^{-1}$, respectively. Note that the model calculation is for the one input direction, and ignored the bumpy extinction distribution in the bulge field.

III. Parameters for Galactic model

Although there are many parameters implemented in `genulens` that enable you to modify the Galactic model, I do not recommend changing them as they can easily skew the K21 fitting results. However, if you are interested in planetary mass object population or systematic errors due to model selection, the parameters described here may be changeable without ruining the K21's results.

1. Model choice from the four K21 models

   K21 presented four models with different functional forms for the bulge profile. The default model is the E+E$_{\rm X}$ model, that showed the lowest $\tilde{\chi}^2_{\rm sum}$ value in the paper.

   If you want to try another model, e.g., to see systematic error due to the model selection, you can use one of the following options to specify the model.

   — `E_fg0 [int]` (default: 0) :
   If 1, use the E model in K21.

   — `G_fg0 [int]` (default: 0) :
   If 1, use the G model in K21.

   — `EXE_fg0 [int]` (default: 0) :
   If 1, use the E+E$_{\rm X}$ model in K21.

   — `GXG_fg0 [int]` (default: 0) :
   If 1, use the G+G$_{\rm X}$ model in K21.

   ex.) `$./genulens l 3.0 b -3.0 G_fg0 1 VERBOSITY 3 > tmp.dat`

2. Planetary mass region of IMF

   `genulens` uses an initial mass function (IMF) of a broken power-law given by

$$\frac{dN}{dM} \propto \begin{cases} M^{\alpha_1} & \text{when } M_1 < M < 120\,M_\odot \\ M^{\alpha_2} & \text{when } 0.08\,M_\odot \le M < M_1 \\ M^{\alpha_3} & \text{when } M_3 \le M < 0.08\,M_\odot \\ M^{\alpha_4} & \text{when } M_l \le M < M_3\ . \end{cases} \tag{9}$$

   Any of the break masses or slopes can be changed, but it is not recommended to change them because the agreement with the data fitted by K21 would be messed up. However, the following three parameters for planetary mass region could be changed without corrupting the fitting result because their influence on the fit is small;

   — `M3 [double]` ($M_\odot$, default: 0.01) :
   Break mass between α_3 and α_4. Must be smaller than 0.08 $M_\odot$, and $< 0.01 M_\odot$ is recommended not to corrupt the fit result in K21.

   — `Ml [double]` ($M_\odot$, default: 0.001) :
   Minimum cutoff mass of the mass function.

   — `alpha4 [double]` (default: α_3) :
   Slope for $M_l \le M < M_3$. Default is α_3, the slope for brown dwarf mass range. ($\alpha_3 = -0.176$ for the E+E$_{\rm X}$ model.)

   So, when you want to apply, e.g., a steep slope $\alpha_4 = -2.0$ between $10^{-6} M_\odot$ and $0.001 M_\odot$ to consider a posterior distribution for an event with a short timescale of $t_{\rm E} = 0.50 \pm 0.05$ days toward $(l, b) = (3.0, -3.0)$, run ex.) `$./genulens l 3.0 b -3.0 NlikeMIN 1e+4 VERBOSITY 3 Ml 1e-06 M3 0.001 alpha4 -2.0 tE 0.50 0.05 > tmp.dat`

IV. Parameters for Galactic Central region

It turned out that there is a demand on the Galactic Central (GC) region after the K21 analysis. The following `NSD` and `CenSgrA` options were added after the K21 analysis to apply `genulens` (and `genstars`, a star version of `genulens`) to the GC region. **These are applied by default** and affect when $|b| \lesssim 0.5°$. Note that the K21 analysis was not motivated to apply the model to the GC region and any fit to real data in the GC region is not conducted. Particularly, the nuclear star cluster (NSC), which dominates the central $\sim 0.1° \times 0.1°$ region, is not yet included. Nevertheless, I have confirmed that star number prediction in the GC region works reasonably well there by comparing with the GALACTICNUCLEUS data.

— `NSD [int]` (default: if ($|l| < 5° \wedge |b| < 2°$) 1; else 0) :
If 1, add a nuclear stellar disk component based on Sormani et al. (2022) [15] to the K21 model.

— `CenSgrA [int]` (default: 1) :
If 1, locate the Galactic Center at the Sgr A* position, $(l, b, D) = (-0.056°, -0.046°, 8160\ \mathrm{pc})$.
If 0, locate the Galactic Center at $(l, b, D) = (0°, 0°, 8160\ \mathrm{pc})$.

A Formulae for Conversion

The following summarizes formulae to convert the output parameters with `VERBOSITY` = 2 into other common parameters used in microlensing studies. PA is the position angle between the equatorial east direction and the galactic longitudinal direction on the sky plane toward (l, b), which is given in the header part of a `genulens` output. Note that most parameters are outputted if you use `VERBOSITY` = 3.

$$\pi_{\rm E} = \sqrt{\pi_{\rm E,N}^2 + \pi_{\rm E,E}^2} \tag{10}$$

$$M_{\rm L} = \frac{\theta_{\rm E}}{\kappa \pi_{\rm E}} \quad (\kappa = 8.144\ \mathrm{mas}\ M_\odot^{-1}) \tag{11}$$

$$D_{\rm L} = \frac{\rm AU}{\pi_{\rm E}\theta_{\rm E} + {\rm AU}/D_{\rm S}} \tag{12}$$

$$\boldsymbol{\mu}_{\rm rel,geo} = \frac{\theta_{\rm E}}{t_{\rm E}} \frac{\boldsymbol{\pi}_{\rm E}}{\pi_{\rm E}} \tag{13}$$

$$\text{or} \quad \begin{pmatrix} \mu_{\rm rel,geo,N} \\ \mu_{\rm rel,geo,E} \end{pmatrix} = \frac{\theta_{\rm E}}{t_{\rm E}\,\pi_{\rm E}} \begin{pmatrix} \pi_{\rm E,N} \\ \pi_{\rm E,E} \end{pmatrix} \tag{14}$$

$$\text{or} \quad \begin{pmatrix} \mu_{\rm rel,geo,b} \\ \mu_{\rm rel,geo,l} \end{pmatrix} = \begin{pmatrix} \cos {\rm PA} & -\sin {\rm PA} \\ \sin {\rm PA} & \cos {\rm PA} \end{pmatrix} \begin{pmatrix} \mu_{\rm rel,geo,N} \\ \mu_{\rm rel,geo,E} \end{pmatrix} \tag{15}$$

$$\boldsymbol{\mu}_{\rm rel,hel} = \boldsymbol{\mu}_{\rm rel,geo} + \frac{\pi_{\rm E}\,\theta_{\rm E}}{\rm AU} \boldsymbol{v}_{\oplus,\perp} \tag{16}$$

$$\boldsymbol{\mu}_{\rm L,hel} = \boldsymbol{\mu}_{\rm rel,hel} + \boldsymbol{\mu}_{\rm S,hel} \tag{17}$$

$$\boldsymbol{v}_{\rm L,\perp} = D_{\rm L}\,\boldsymbol{\mu}_{\rm rel,hel} + \boldsymbol{v}_{\odot,\perp} \tag{18}$$

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