A Vector in a Hilbert space

The state of a quantum system can be described by a vector in a Hilbert space. Once a basis is chosen for the Hilbert space, the quantum state can be represented by a column vector with complex number entries. Using a column vector to describe a quantum state has the drawbrack of being basis-dependent. Suppose the basis of a Hilbert space is formed by $\{\chi_1, \chi_2, \cdots, \chi_N\}$. We now have two ways to represent a quantum state.

$$|\psi\rangle = \sum_{i=1}^{N} a_i |\chi_i\rangle \qquad \Longleftrightarrow \qquad \boldsymbol{\psi} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \tag{0.1.1}$$

The Inner Product

For two real column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the inner product between them is defined as $\boldsymbol{u}^T\boldsymbol{v} = \sum_i u_i v_i$. This is the same as the dot product between the two corresponding vectors in the Euclidean space $\vec{u} \cdot \vec{v}$. The dot product is invariant under a change of basis. For two vectors in the Hilbert space $|\psi\rangle$ and $|\phi\rangle$, we denote their inner product as $\langle\psi|\phi\rangle$. If a specific basis of the Hilbert space is chosen, we can represent $|\psi\rangle$ and $|\phi\rangle$ using two column vectors:

$$\boldsymbol{\psi} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\phi} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{0.2.1}$$

Correspondingly, the bra $\langle\psi|$ would be represented by the row vector

$$\boldsymbol{\psi}^\dagger = \begin{pmatrix} a_1^* \cdots a_n^* \end{pmatrix}. \tag{0.2.2}$$

The Dirac notation $\langle\psi|\phi\rangle$ can then be regarded as the matrix product of $\boldsymbol{\psi}^\dagger$ and $\boldsymbol{\phi}$:

$$\langle\psi|\phi\rangle = \boldsymbol{\psi}^\dagger\boldsymbol{\phi} = \sum_{i=1}^{n} a_i^* b_i. \tag{0.2.3}$$

Note that the inner product is also invariant under the change of basis.

Orthonormal Basis

The basis vectors of a Hilbert space are typically made of states that are normalized and orthogonal to each other. For example, if $\{\chi_1, \chi_2, \cdots, \chi_N\}$ form a basis, then for any $i, j \in \{1, 2, \cdots, N\}$, we have

$$\langle\chi_i|\chi_j\rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{0.3.1}$$

Such as basis is said to be *orthonormal*.

Time Evolution

For a time-independent Hamiltonian $\hat{H}$, the state of the system at time t is given by the solution to the Schrödinger equation $i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle\,. \tag{0.4.1}$$