

# Illustrating Good Practice in L<sup>A</sup>T<sub>E</sub>X

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## 1 Introduction

This document is not a guide on how to use L<sup>A</sup>T<sub>E</sub>X, since plenty of those already exist. Instead, what follows is an incoherent collection of examples that demonstrate good practice. For more of my awesome L<sup>A</sup>T<sub>E</sub>X resources, visit <https://github.com/nkosten4/LaTeX-templates>.

## 2 Eigenvalues of a Hermitian Operator

It is not hard to show that the eigenvalues of a Hermitian operator are real. Let  $|a\rangle$  be an eigenstate of the Hermitian operator  $A$  corresponding to eigenvalue  $a$ , then

$$A|a\rangle = a|a\rangle. \quad (2.1)$$

Then we have

$$\langle a|A|a\rangle = \langle a|a|a\rangle = a\langle a|a\rangle. \quad (2.2)$$

Taking the Hermitian conjugate of both sides of (2.1) gives

$$\langle a|A^\dagger = a^*\langle a|. \quad (2.3)$$

Then we also have

$$\langle a|A^\dagger|a\rangle = a^*\langle a|a\rangle \quad (2.4)$$

Since  $A$  is Hermitian, we have  $\langle a|A^\dagger|a\rangle = \langle a|A|a\rangle$ . Then comparing (2.2) and (2.4) gives

$$(a - a^*)\langle a|a\rangle = 0.$$

This, of course, implies that  $a = a^*$ , since  $\langle a|a\rangle \neq 0$ . Then it must be the case that  $a$  is real.

## 3 Unit Basis Vectors

Let  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ , and  $\hat{\mathbf{k}}$  be unit vectors parallel to the  $x$ ,  $y$ , and  $z$  axes, respectively. An arbitrary vector  $\mathbf{a}$  can be expanded in terms of these *basis vectors*:

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}. \quad (3.1)$$

One can label a point  $P$  by its Cartesian coordinates  $(x, y, z)$ , but sometimes it is more convenient to use *spherical* coordinates  $(r, \theta, \phi)$ ;  $r$  is the distance from the origin (the magnitude of the position vector  $\mathbf{r}$ ),  $\theta$  (the angle down from the  $z$ -axis) is called the *polar angle*, and  $\phi$  (the angle around from the  $x$ -axis) is the *azimuthal angle*. The unit vectors  $\hat{\mathbf{r}}$ ,  $\hat{\boldsymbol{\theta}}$ ,  $\hat{\boldsymbol{\phi}}$  constitute an orthogonal basis set. In terms of the Cartesian unit vectors,

$$\left. \begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}. \end{aligned} \right\} \quad (3.2)$$

Finally, one can label a point  $P$  by its cylindrical coordinates  $(\rho, \phi, z)$ ;  $\phi$  has the same meaning as in spherical coordinates, and  $z$  is the same as Cartesian, but  $\rho$  is the distance to  $P$  from the  $z$ -axis. The unit vectors are

$$\left. \begin{aligned} \hat{\rho} &= \cos \phi \, \hat{\mathbf{i}} + \sin \phi \, \hat{\mathbf{j}}, \\ \hat{\phi} &= -\sin \phi \, \hat{\mathbf{i}} + \cos \phi \, \hat{\mathbf{j}}, \\ \hat{z} &= \hat{\mathbf{k}}. \end{aligned} \right\} \quad (3.3)$$

## 4 Classical Electromagnetism

Here are the Maxwell equations:

**Table 4.1:** Maxwell’s equations and their interpretation.

| Equation   | Interpretation   |
|--|--|
| $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_o}$  | $\mathbf{E}$ -fields with divergence come from charges (electric monopoles).   |
| $\nabla \cdot \mathbf{B} = 0$  | $\mathbf{B}$ -fields with divergence don’t exist (no magnetic monopoles).  |
| $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$   | $\mathbf{E}$ -fields with curl come from time-varying $\mathbf{B}$ -fields — and those <i>only</i> .   |
| $\begin{aligned} \nabla \times \mathbf{B} &= \mu_o \mathbf{J} + \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \\ &= \mu_o (\mathbf{J} + \mathbf{J}_D) \end{aligned}$ | $\mathbf{B}$ -fields with curl come from currents (moving electric monopoles) and from time-varying $\mathbf{E}$ -fields. Sometimes we refer to $\epsilon_o \frac{\partial \mathbf{E}}{\partial t}$ as displacement current and bundle the $\mathbf{J}$ ’s together. |

In regions of space where there is no charge or current, Maxwell’s equations read

$$\left. \begin{aligned} \nabla \cdot \mathbf{E} &= 0, & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{B} &= \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \right\}$$

The Lorentz force law reads

$$\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Finally, most experts agree that

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

## 5 Matrix Multiplication

Consider some  $A \in \mathbb{R}^{p \times q}$  and  $B \in \mathbb{R}^{q \times p}$ . We can represent these matrices as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & a_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qp} \end{pmatrix}$$

respectively. Then the product  $AB$  can be written

$$AB = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1q}b_{q1} & a_{11}b_{12} + \cdots + a_{1q}b_{q2} & \cdots & a_{11}b_{1p} + \cdots + a_{1q}b_{qp} \\ a_{21}b_{11} + \cdots + a_{2q}b_{q1} & a_{21}b_{12} + \cdots + a_{2q}b_{q2} & \cdots & a_{21}b_{1p} + \cdots + a_{2q}b_{qp} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}b_{11} + \cdots + a_{pq}b_{q1} & a_{p1}b_{12} + \cdots + a_{pq}b_{q2} & \cdots & a_{p1}b_{1p} + \cdots + a_{pq}b_{qp} \end{pmatrix}.$$

## 6 Sinusoidal Waves

A sinusoidal wave assumes the form

$$g(z, t) = A \cos [k(z - vt) + \delta].$$

Figure (6.1) shows this function at  $t = 0$ . The amplitude of the wave is  $A$ . The argument of the cosine is called the phase, and  $\delta$  is the phase constant. At  $z = vt - \delta/k$ , the phase is zero; this is known as the “central maximum.” Observe that  $\delta/k$  is the distance by which the central maximum is “delayed.” Finally,  $k$  is the wave number; it is related to the wavelength  $\lambda$  by the equation

$$\lambda = \frac{2\pi}{k},$$

for when  $z$  advances by  $2\pi/k$ , the cosine executes one complete cycle.



**Figure 6.1:** A sinusoidal wave train. With the passage of time, the entire wave train proceeds to the right.

At any fixed point  $z$ , the string vibrates up and down, undergoing one full cycle in a period

$$T = \frac{2\pi}{kv}.$$

## 7 Electric Fields on Pointy Things

The pointy bits on a conducting surface should produce greater electric fields than the flat bits. Let’s show that numerically. Laplace’s equation, in two dimensions, reads

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \quad (7.1)$$

We can solve Laplace’s equation numerically using a finite difference method for a rectangular region. That is, we partition a rectangle into a lattice of grid points. If the spacing in the  $x$ -direction is  $\Delta x$ , the second derivative  $\frac{\partial^2 V}{\partial x^2}$  is

$$\frac{\partial^2 V}{\partial x^2} = \frac{V(x + \Delta x, y) - 2V(x, y) + V(x - \Delta x, y)}{\Delta x^2}. \quad (7.2)$$

Similarly, if the spacing in the  $y$ -direction is  $\Delta y$ , the second derivative  $\frac{\partial^2 V}{\partial y^2}$  is

$$\frac{\partial^2 V}{\partial y^2} = \frac{V(x, y + \Delta y) - 2V(x, y) + V(x, y - \Delta y)}{\Delta y^2}. \quad (7.3)$$

For convenience, let's say  $\Delta x = \Delta y = h$ . The Laplacian operator in two-dimensions becomes

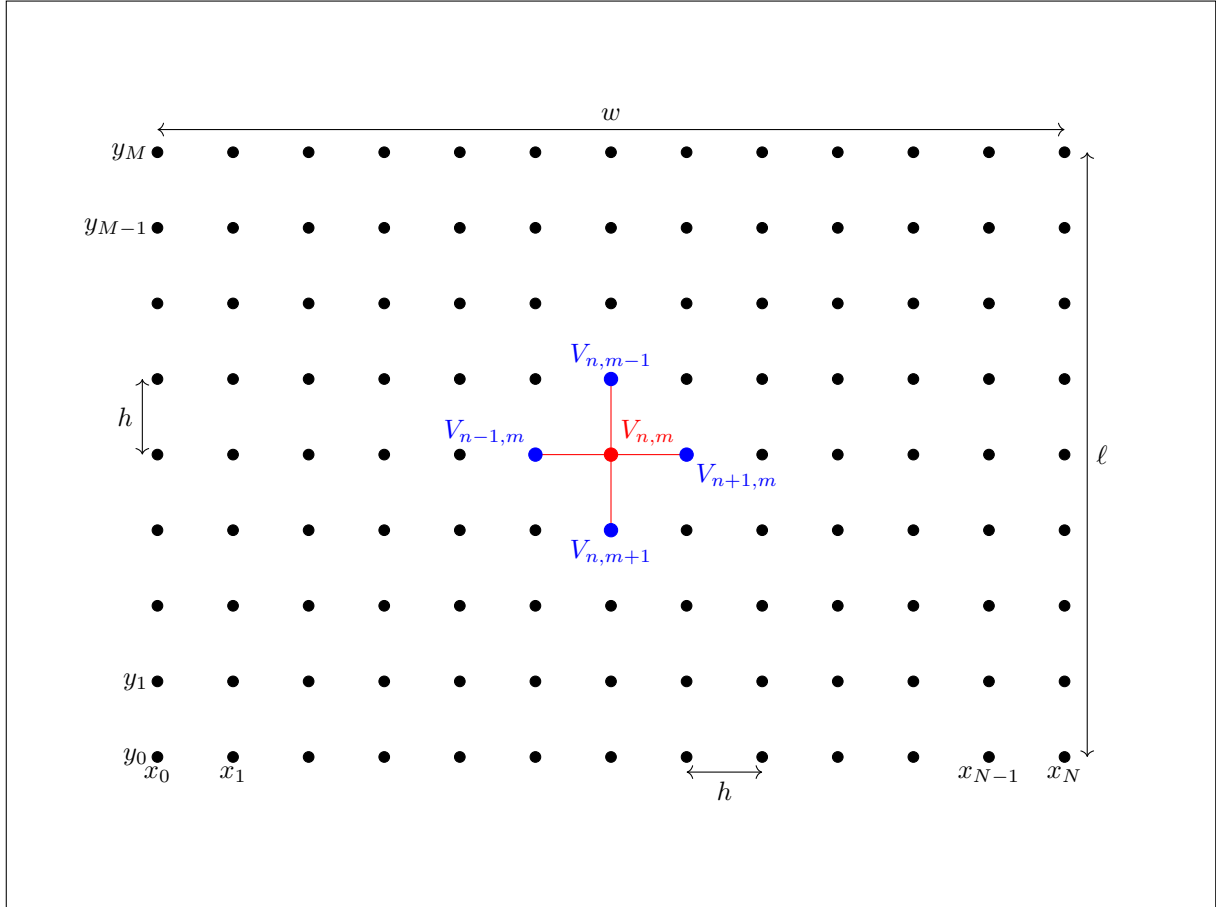
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V(x + h, y) + V(x - h, y) + V(x, y + h) + V(x, y - h) - 4V(x, y)}{h^2}. \quad (7.4)$$

Then Laplace's equation becomes

$$V(x + h, y) + V(x - h, y) + V(x, y + h) + V(x, y - h) - 4V(x, y) = 0 \quad (7.5)$$

$$\Rightarrow \boxed{V(x, y) = \frac{V(x + h, y) + V(x - h, y) + V(x, y + h) + V(x, y - h)}{4}}. \quad (7.6)$$

That is, the potential  $V(x, y)$  is the average of its four neighboring points on the lattice. Let's make this concrete by drawing a diagram.



**Figure 7.1:** A two-dimensional lattice of points.

As shown in Fig. (7.1), we partition the interval  $0 < x < w$  into  $N + 1$  evenly spaced points. That is,

$$x_n = nh, \quad \text{for } n \in \{0, 1, \dots, N\}. \quad (7.7)$$

Likewise, we divide the interval  $0 < y < \ell$  into  $M + 1$  evenly spaced points. That is,

$$y_m = mh, \quad \text{for } m \in \{0, 1, \dots, M\}. \quad (7.8)$$

What follows is the implementation of this numerical method.

```

1 """
2 Created on Mon Feb 03 13:12:20: 2020
3 Author: Nicholas D. Kostin
4 Description: Numerical solution to Laplace's equation for a rectangle
5 """
6
7 import numpy as np, matplotlib.pyplot as plt
8
9 plt.rcParams['mathtext.fontset'] = 'stix'
10 plt.rcParams['font.family'] = 'STIXGeneral'
11
12 # Physical constants
13 eps0 = 8.85e-12 # permittivity of free space
14
15 # Dimensions of region
16 region_width = 120 # Horizontal length of region
17 region_length = 80 # Vertical length of region
18
19 # Dimensions of conducting rectangle
20 rectangle_width = 60 # Horizontal length of conducting rectangle
21 rectangle_length = 40 # Vertical length of conducting rectangle
22
23 # Location of bottom-left corner of conducting rectangle
24 rect_corner = 20
25
26 # Set potential at boundaries of region to be zero
27 boundary_potential = 0
28
29 # Dirichlet Boundary conditions on conducting rectangle
30 # All sides at the same potential
31 V_box = 9
32
33 # Set color interpolation and color map
34 color_interpolation = 20
35 color_map = plt.cm.coolwarm # dope-ass colors
36
37 # Create lattice of points
38 X, Y = np.meshgrid(np.arange(0, region_width), np.arange(0, region_length))
39
40 # Set array size and the interior value with some guess
41 V = np.ones((region_length, region_width))
42 V.fill(0)
43
44 # Set potential at boundaries of region
45 V[1, :] = boundary_potential # potential at bottom
46 V[region_length - 1, :] = boundary_potential # potential at top
47 V[:, 1] = boundary_potential # potential at left
48 V[:, region_width - 1] = boundary_potential # potential at right
49
50
51 # Set potential at boundaries of conducting rectangle
52 V[rect_corner, rect_corner:rect_corner + rectangle_width] = V_box # potential at bottom
53 V[rect_corner + rectangle_length, rect_corner:rect_corner + rectangle_width] = V_box #
54   potential at top
55 V[rect_corner:rect_corner + rectangle_length, rect_corner] = V_box # potential at left
56 V[rect_corner:rect_corner + rectangle_length, rect_corner + rectangle_width] = V_box #
57   potential at right
58
59 # Implement 1000 iterations (and hope that's enough for convergence)
60 for iteration in range(0,1000):
61     for i in range(1, region_length - 1, 1):
62         for j in range(1, region_width - 1, 1):
63             # Conducting surfaces are equipotentials
64             if rect_corner <= i <= rect_corner + rectangle_length and rect_corner <= j <=
65               rect_corner + rectangle_width:
66                 V[i, j] = V[i, j]
67             # Implement finite difference method

```

```

65         else:
66             V[i, j] = (1/4) * (V[i - 1][j] + V[i + 1][j] + V[i][j - 1] + V[i][j + 1])
67
68 # Create array for electric field
69 E = np.zeros((region_length, region_width, 2))
70
71 # Compute electric field
72 for i in range(region_length):
73     for j in range(region_width):
74         # Electric field is zero inside a conductor
75         if rect_corner <= i <= rect_corner + rectangle_length and rect_corner <= j <=
            rect_corner + rectangle_width:
76             E[i, j] = [0, 0]
77         # Electric field is derivative of potential otherwise
78         elif i != 0 and i != region_length - 1 and j != 0 and j != region_width - 1:
79             E[i, j] = [(V[i + 1, j] - V[i - 1, j]) / (2), (V[i, j + 1] - V[i, j - 1]) / (2)]
80
81 # Create array for magnitude of the electric field
82 E_magnitude = np.zeros((region_length, region_width))
83
84 # Compute the magnitude of the electric field
85 for i in range(region_length):
86     for j in range(region_width):
87         E_magnitude[i, j] = (E[i, j, 0]**2 + E[i, j, 1]**2)**(1/2)
88
89 # Plot a contour map of the potential
90 plt.figure(1)
91 plt.title("Contour of Potential")
92 plt.contourf(X, Y, V, color_interpolation, cmap=color_map)
93 plt.colorbar() # Set colorbar
94 plt.show() # Display the color map
95
96 # Plot a contour map of the electric field magnitude
97 plt.figure(2)
98 plt.title("Contour of Magnitude of Electric Field")
99 plt.contourf(X, Y, E_magnitude, color_interpolation, cmap=color_map)
100 plt.colorbar() # Set colorbar
101 plt.show() # Display the color map

```

## 8 Infinite Square Well Potential

Suppose we have a quantum particle with mass  $m$  in an infinite square well potential:

$$V(x) = \begin{cases} 0, & 0 \leq x \leq \infty \\ \infty & \text{otherwise} \end{cases}.$$

The expectation value of the position  $\hat{x}$  of the particle measured in the energy eigenstate  $|\psi_n\rangle$  that is represented by  $\psi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ , we use

$$\langle \psi_n | \hat{x} | \psi_n \rangle = \int_0^\pi \psi_n^*(x) x \psi_n(x) dx = \frac{2}{\pi} \int_0^\pi x \sin^2(nx) dx = \frac{\pi}{2}. \quad (8.1)$$

The expectation value of the momentum  $\hat{p}$  in the eigenstate  $|\psi_n\rangle$  is given by

$$\langle \psi_n | \hat{p} | \psi_n \rangle = \int_0^\pi \psi_n^*(x) \left( -i\hbar \frac{d}{dx} \right) \psi_n(x) dx = 0. \quad (8.2)$$

## 9 Commuting Hermitian Matrices

Let  $A, B \in \mathbb{C}^{p \times p}$  be Hermitian matrices. Prove that  $AB$  is Hermitian if and only if  $A$  and  $B$  commute (*i.e.* they don't work from home).

*Proof.* A matrix  $\mathcal{A} \in \mathbb{C}^{p \times p}$  is Hermitian if it satisfies

$$\mathcal{A}^\dagger = \mathcal{A}.$$

That is, a matrix  $\mathcal{A} \in \mathbb{C}^{p \times p}$  is Hermitian if it is its own conjugate transpose. Moreover, for  $\mathcal{A} \in \mathbb{C}^{p \times q}$  and  $\mathcal{B} \in \mathbb{C}^{q \times r}$ , we have the identity

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger \mathcal{A}^\dagger.$$

We will start by proving the forward direction. That is, we will show that if  $A$  and  $B$  commute, then  $AB$  must be Hermitian. Using the identity above to take the conjugate transpose of  $AB$  we get

$$(AB)^\dagger = B^\dagger A^\dagger.$$

But since we assumed  $A$  and  $B$  to both be Hermitian, we have that  $B^\dagger = B$  and  $A^\dagger = A$ . Explicitly,

$$(AB)^\dagger = B^\dagger A^\dagger = BA.$$

But since we assumed  $A$  and  $B$  commute, then  $BA = AB$ . And we have

$$(AB)^\dagger = B^\dagger A^\dagger = BA = AB$$

Thereby  $AB$  satisfies the definition of a Hermitian matrix. Now we will show the reverse direction: that if  $AB$  is Hermitian then  $A$  and  $B$  must necessarily commute. If  $AB$  is Hermitian, then

$$(AB)^\dagger = AB$$

But we already showed that we can rewrite the left-hand side of the equation above as  $(AB)^\dagger = B^\dagger A^\dagger = AB$ , since we assumed  $A$  and  $B$  to both be Hermitian. Then we have

$$BA = AB.$$

And clearly,  $A$  and  $B$  must necessarily commute. ■

## 10 Blind Text

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper.

## 11 Some Results from Real Analysis

Here we have a *theorem*.

### **Bolzano-Weierstrass Theorem**

Every bounded sequence has a convergent subsequence.

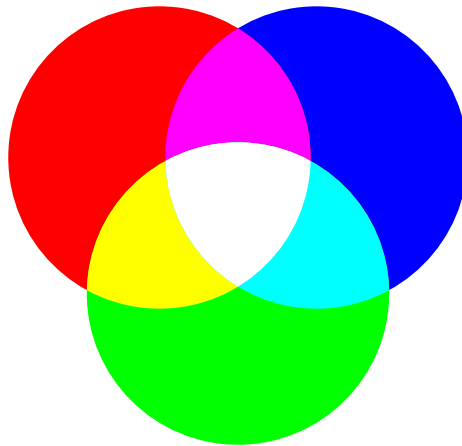
What follows is a *lemma*.

### **Lemma**

Cauchy sequences are bounded.

## 12 Computer Colors

HTML color codes are hexadecimal triplets representing the colors red, green and blue.



## 13 Awesome Circled Numbers and Highlighting

Here are some cool circled numbers. The first argument is whatever is to be circled, and the **second argument** is the background color. The **third argument** is the outline color, and the **fourth argument** is color of the first argument.



Isn't this cool?

These circled numbers work in equations too!

$$\int_{\textcircled{a}}^{\textcircled{b}} f(x)dx + \int_{\textcircled{b}}^{\textcircled{a}} f(x)dx = \textcircled{0}$$

Don't forget, however, that too much color is cringe, so be exercise caution!



## 14 Consider a Graph

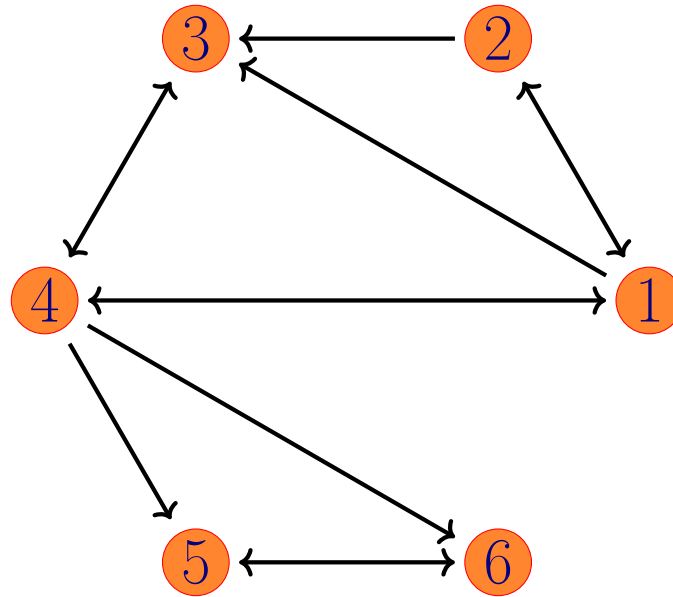
Consider a graph with vertices

$$V = \{1, 2, 3, 4, 5, 6\}$$

and edges

$$E = \{1 \rightarrow 2, 1 \rightarrow 3, 1 \rightarrow 4, 2 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 4, 4 \rightarrow 1, 4 \rightarrow 3, 4 \rightarrow 5, 4 \rightarrow 6, 5 \rightarrow 6, 6 \rightarrow 5\}$$

The directed graph for this vertex set  $V$  and edge set  $E$  is



## 15 A Particle in 3D

A particle in 3D has mass  $m$  and is in the potential  $V(x, y, z) = \frac{1}{2}m\omega^2 (x^2 + y^2 + z^2)$ .

### 15.1 Energies of the Lowest Four Eigenstates

Suppose we need to find the energies of the four lowest eigenstates of the Hamiltonian and their corresponding degeneracies. We can start by expanding the potential as

$$V(x, y, z) = \underbrace{\frac{1}{2}m\omega^2 x^2}_{V_x(x)} + \underbrace{\frac{1}{2}m\omega^2 y^2}_{V_y(y)} + \underbrace{\frac{1}{2}m\omega^2 z^2}_{V_z(z)}.$$

And the total eigenenergy is

$$E_{n_x n_y n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \hbar\omega \left( n_x + \frac{1}{2} \right) + \hbar\omega \left( n_y + \frac{1}{2} \right) + \hbar\omega \left( n_z + \frac{1}{2} \right) \quad (15.1)$$

where  $n_x, n_y, n_z = 0, 1, 2, \dots$  are independent from each other. The ground state should have  $n_x = n_y = n_z = 0$ , so its energy is given by

$$E_{000} = 3\hbar\omega \left( \frac{1}{2} \right) = \frac{3}{2}\hbar\omega.$$

The first excited state has energy

$$\underbrace{E_{100} = E_{010} = E_{001}}_{\text{degeneracy 3}} = \frac{\hbar}{\omega} \left( \frac{3}{2} + \frac{1}{2} + \frac{1}{2} \right) = \frac{5}{2} \hbar \omega.$$

The second excited state has energy

$$\underbrace{E_{200} = E_{020} = E_{002} = E_{110} = E_{101} = E_{011}}_{\text{degeneracy 6}} = \frac{\hbar}{\omega} \left( \frac{5}{2} + \frac{1}{2} + \frac{1}{2} \right) = \frac{7}{2} \hbar \omega.$$

And finally, the third excited state has energy

$$\underbrace{\left\{ \begin{array}{l} E_{300} = E_{030} = E_{003} = E_{111} = E_{102} \\ E_{120} = E_{201} = E_{021} = E_{210} = E_{012} \end{array} \right\}}_{\text{degeneracy 10}} = \frac{\hbar}{\omega} \left( \frac{7}{2} + \frac{1}{2} + \frac{1}{2} \right) = \frac{9}{2} \hbar \omega.$$

These results are summarized in the table below.

**Table 15.1:** Energies and corresponding degeneracies of the 3D QHO

| State                | Energy                     | Degeneracy |
|----------------------|----------------------------|------------|
| Ground state         | $\frac{3}{2} \hbar \omega$ | 1          |
| First excited state  | $\frac{5}{2} \hbar \omega$ | 3          |
| Second excited state | $\frac{7}{2} \hbar \omega$ | 6          |
| Third excited state  | $\frac{9}{2} \hbar \omega$ | 10         |

## 15.2 Degeneracy of an Arbitrary Excited State

Now suppose we need to find the degeneracy  $g_n$  of the  $n^{\text{th}}$  excited state as a function of  $n$ . For our three-dimensional harmonic oscillator, we can define

$$n = n_x + n_y + n_z.$$

It follows from Eq. (15.1) that the energy is strictly a function of  $n$ . That is, any combination of  $n_x$ ,  $n_y$ , and  $n_z$ , as long as they produce the same  $n$ , will yield the same energy. Such states are said to be *degenerate*. Our goal is to find how many possible combinations of  $n_x$ ,  $n_y$ , and  $n_z$  exist for a given  $n$ .

To explicitly determine the degeneracy  $g_n$  for a given state, we fix the value of  $n$ . Because of the restriction imposed by setting  $n = n_x + n_y + n_z$ , we really have two free parameters (say,  $n_x$  and  $n_y$ ). We let  $n_x$  assume any allowable integer value:

$$n_x \in \{0, 1, \dots, n\}.$$

Then  $n_y$  can assume any integer value given the sum  $n_x + n_y$  does not exceed  $n$ :

$$n_y \in \{0, 1, \dots, n - n_x\}.$$

In particular, for any value of  $n_x$ , there are  $n - n_x + 1$  unique choices for  $n_y$  (note the size of the set of permissible  $n_y$  values). Finally,  $n_z$  is automatically fixed:

$$n_z = n - n_x - n_y.$$

For each permissible value of  $n_x$  (from 0 to  $n$ ), there are  $n - n_x + 1$  choices for  $n_y$ . Summing the number of these choices for all allowable values of  $n_x$  yields

$$\begin{aligned} \sum_{n_x=0}^n (n - n_x + 1) &= (n+1) + n + (n-2) + \cdots + \cancel{(n-n)} \rightarrow 0 \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

The series above is well known. Thus, we find that the degeneracy is

$$g_n = \frac{(n+1)(n+2)}{2}.$$