



COLORADO SCHOOL OF MINES
EARTH • ENERGY • ENVIRONMENT

DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS

MATH 400 — SPECTRAL ANALYSIS OF LINEAR OPERATORS

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HOMEWORK 1

A Vector in a Hilbert space

The state of a quantum system can be described by a vector in a Hilbert space. Once a basis is chosen for the Hilbert space, the quantum state can be represented by a column vector with complex number entries. Using a column vector to describe a quantum state has the drawback of being basis-dependent. Suppose the basis of a Hilbert space is formed by $\{\chi_1, \chi_2, \dots, \chi_N\}$. We now have two ways to represent a quantum state.

$$|\psi\rangle = \sum_{i=1}^N a_i |\chi_i\rangle \quad \Longleftrightarrow \quad \psi = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \quad (1.1.1)$$

The Inner Product

For two real column vectors \mathbf{u} and \mathbf{v} , the inner product between them is defined as $\mathbf{u}^T \mathbf{v} = \sum_i u_i v_i$. This is the same as the dot product between the two corresponding vectors in the Euclidean space $\vec{u} \cdot \vec{v}$. The dot product is invariant under a change of basis. For two vectors in the Hilbert space $|\psi\rangle$ and $|\phi\rangle$, we denote their inner product as $\langle\psi|\phi\rangle$. If a specific basis of the Hilbert space is chosen, we can represent $|\psi\rangle$ and $|\phi\rangle$ using two column vectors:

$$\psi = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \quad (1.2.1)$$

Correspondingly, the bra $\langle\psi|$ would be represented by the row vector

$$\psi^\dagger = (a_1^* \cdots a_n^*). \quad (1.2.2)$$

The Dirac notation $\langle\psi|\phi\rangle$ can then be regarded as the matrix product of ψ^\dagger and ϕ :

$$\langle\psi|\phi\rangle = \psi^\dagger \phi = \sum_{i=1}^n a_i^* b_i. \quad (1.2.3)$$

Note that the inner product is also invariant under the change of basis.

Orthonormal Basis

The basis vectors of a Hilbert space are typically made of states that are normalized and orthogonal to each other. For example, if $\{\chi_1, \chi_2, \dots, \chi_N\}$ form a basis, then for any $i, j \in \{1, 2, \dots, N\}$, we have

$$\langle\chi_i|\chi_j\rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \quad (1.3.1)$$

Such a basis is said to be *orthonormal*.

Time Evolution

For a **time-independent** Hamiltonian \hat{H} , the state of the system at time t is given by the solution to the Schrödinger equation $i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle. \quad (1.4.1)$$

HOMEWORK 2

Cauchy Sequences

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Lemma

Cauchy sequences are bounded.

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$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad (2.1.1)$$

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Convergent Subsequences

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$$\oint \mathbf{B} \cdot d\mathbf{a} = 0 \quad (2.2.1)$$

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