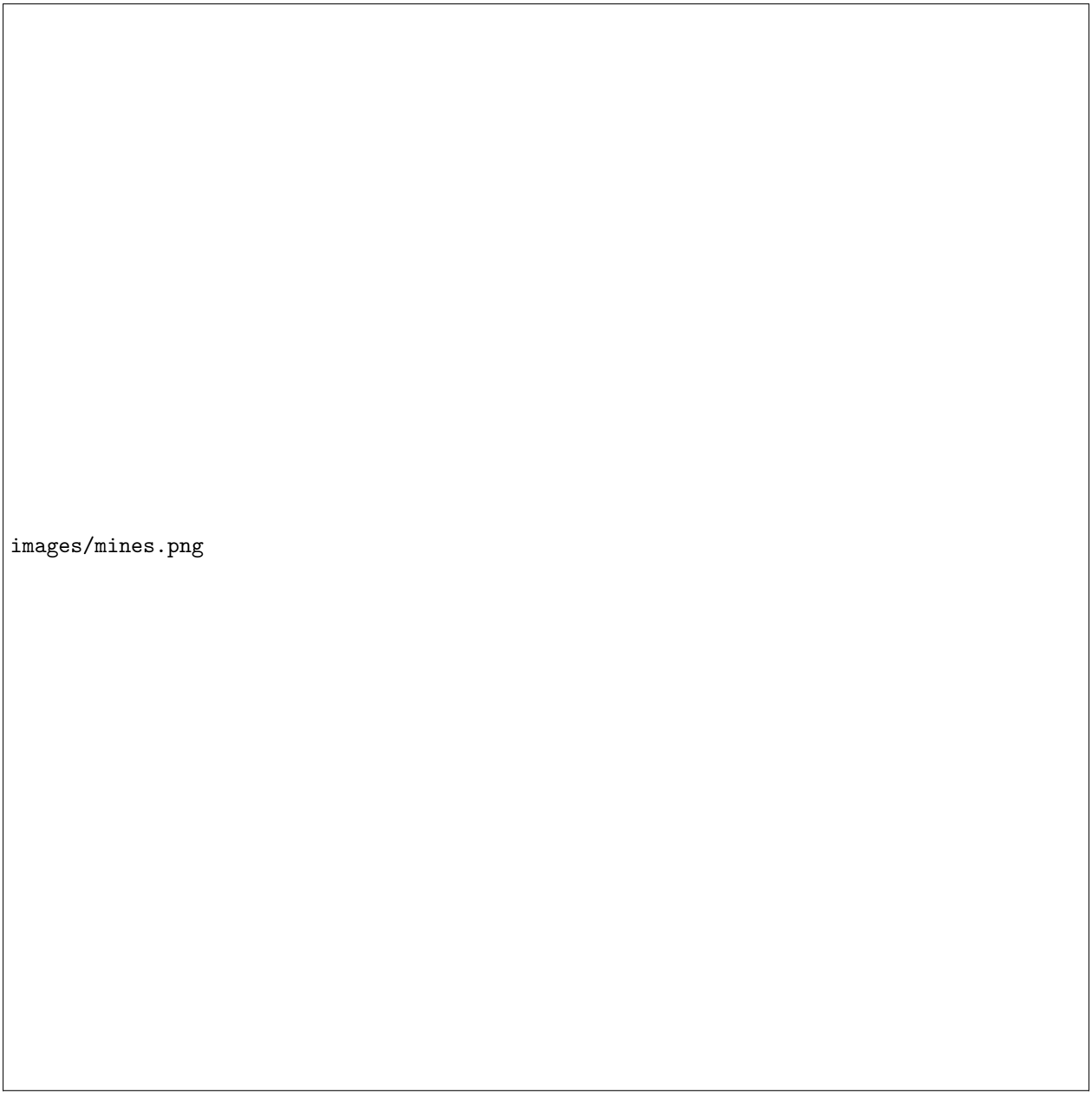
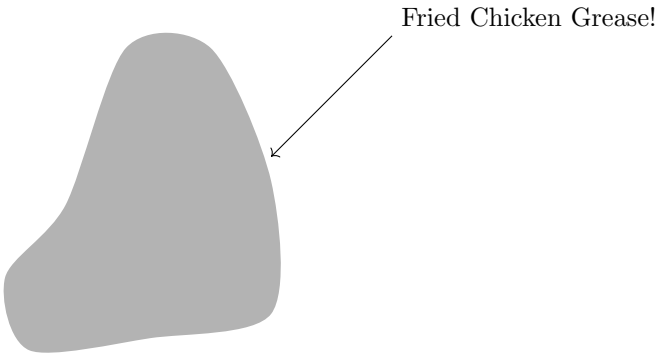


DEPARTMENT OF PHYSICS

PHGN 361 - INTERMEDIATE ELECTROMAGNETISM

Lecture Notes by

Dr. Pat Kohl



Lecture 1: Introduction and Framing

Electromagnetism is a field theory. As such, it requires us to specify two things:

(1) How matter produces fields

(2) How fields affect matter

Note that for E&M, "matter" means charges.

Point (1) is handled by the Maxwell equations. These are our source equations (*i.e.* equations that describe how sources make fields).

Maxwell's equations and their interpretation.

Equation	Interpretation
$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o}$	$\boldsymbol{E}$-fields with divergence come from charges (electric monopoles).
$\nabla \cdot \boldsymbol{B} = 0$	$\boldsymbol{B}$-fields with divergence don't exist (no magnetic monopoles).
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\boldsymbol{E}$-fields with curl come from time-varying $\boldsymbol{B}$-fields — and those *only*.
$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \epsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$ $= \mu_o \left(\boldsymbol{J} + \boldsymbol{J}_D\right)$	$\boldsymbol{B}$-fields with curl come from currents (moving electric monopoles) and from time-varying $\boldsymbol{E}$-fields. Sometimes we refer to $\epsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$ as displacement current and bundle the J's together.

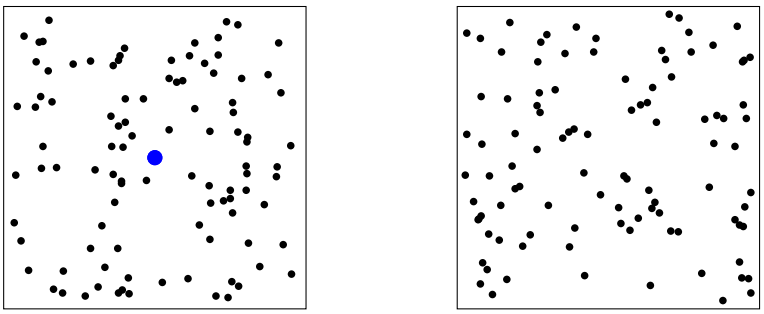
Point (2) comes from the Lorentz force law: $\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$.

Charges feel forces from E-fields *always* and from B-fields when the charges are in motion. This means that observers in different inertial reference frames might disagree on whether a particular charge is experiencing a magnetic force at all, but that's okay.
Eventually we discover that fields themselves are different in different reference frames. This isn't a throw-away result since fields are very real things, carrying energy and momentum.

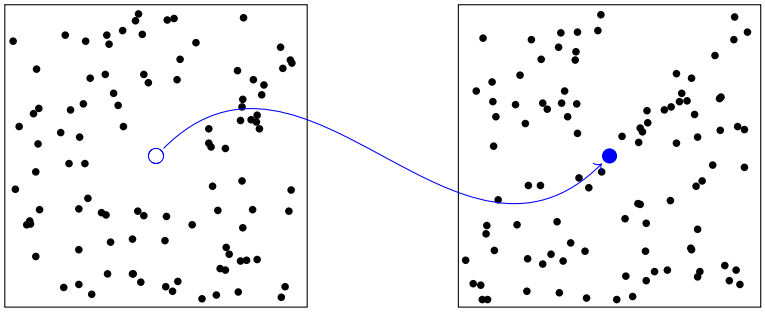
There's another equation that often gets listed as fundamental — the so-called continuity equation:

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}.$$

This is a statement of local conservation of charge: not only is charge conserved globally, in that the total amount never changes, but it's also conserved locally, meaning that to get from here to there, it has to move through the intervening space.



A blue dot in the left box.



A blue dot in the right box.

An illustration of continuity. In order for the blue dot to have move from the left box to the right box, it must necessarily have crossed through the intervening space.

Conservation of charge is certainly fundamental, but it isn't a postulate - we can derive it from other laws. Start with the Ampere-Maxwell equation:

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \epsilon_o \frac{\partial \boldsymbol{E}}{\partial t}.$$

Take the divergence of both sides:

$$\begin{aligned}
\nabla \cdot (\nabla \times \boldsymbol{B}) &= \nabla \cdot \left(\mu_o \boldsymbol{J} + \mu_o \epsilon_o \frac{\partial \boldsymbol{E}}{\partial t} \right) \\
0 &= \mu_o \left[\nabla \cdot \boldsymbol{J} + \epsilon_o \nabla \cdot \left(\frac{\partial \boldsymbol{E}}{\partial t} \right) \right] \\
0 &= \nabla \cdot \boldsymbol{J} + \frac{d}{dt} \underbrace{(\epsilon_o \nabla \cdot \boldsymbol{E})}_{= \rho} \\
\implies & \boxed{\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}}
\end{aligned}$$

For charge to be entering or leaving a region, there must be a divergence in the current. Note that this gives another way to tell that Ampére's law ($\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$) is incomplete without Maxwell's correction.

Interlude: What are divergences and curls?

These Maxwell equations are in differential form, which is arguably the cleanest form, describing what's happening at a single point in space, as compared to the the integral forms that sample an entire region. We move back and forth between these forms using the divergence theorem and Stokes' theorem, which are higher dimensional generalizations of the fundamental theorem of calculus.

Group activity: convert Gauss's law from differential form to integral form and back using the divergence theorem.

Correct integral expressions aren't always super transparent. (Faraday clicker)
For instance, $\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\dfrac{d}{dt}\int \boldsymbol{B} \cdot d\boldsymbol{A}$ with no qualifiers is simply false and will lead you to conclude that curly $\boldsymbol{E}$-fields exist when they don't. Consider the following.
$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int \dfrac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$ is true always. And we can get the time-derivative out using the 3D Leibniz rule.

$$\text{1D Leibniz Rule: } \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t) + \int_a^b \frac{\partial}{\partial t}f(x,t)\, dx$$

$$\text{3D Leibniz Rule: } \frac{d}{dt}\int\limits_{A(t)} \boldsymbol{F}(\boldsymbol{x},t)\cdot d\boldsymbol{A} = -\oint (\boldsymbol{v}\times\boldsymbol{F})\cdot d\boldsymbol{\ell} + \int \left(\frac{\partial \boldsymbol{F}}{\partial t} + (\nabla\cdot\boldsymbol{F})\,\boldsymbol{v}\right)\cdot d\boldsymbol{A}$$

Letting $\boldsymbol{B}$ be the vector field gives

$$\frac{d}{dt}\int\limits_{A(t)} \boldsymbol{B}(\boldsymbol{x},t)\cdot d\boldsymbol{A} = \int \left(\frac{\partial \boldsymbol{B}}{\partial t} + \cancelto{0}{(\nabla\cdot\boldsymbol{B})}\,\boldsymbol{v}\right)\cdot d\boldsymbol{A} - \oint (\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{\ell}$$

$$\implies \frac{d}{dt}\int \boldsymbol{B}(\boldsymbol{x},t)\cdot d\boldsymbol{A} + \oint (\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{\ell} = \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{A}$$

Again, we know that $\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int \dfrac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$ is true always. So sub into that we get

$$\oint \boldsymbol{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int \boldsymbol{B}\cdot d\boldsymbol{A} - \oint (\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{\ell}$$

$$\implies \underbrace{\oint (\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{\ell}}_{\text{emf}} = \underbrace{-\frac{d}{dt}\int \boldsymbol{B}\cdot d\boldsymbol{A}}_{-\frac{d\Phi}{dt}}$$

Which is the correct integral form of Faraday's law, and explicitly shows both the $\boldsymbol{E}$-field and the motional contributions to the emf.
Altogether, Maxwell's equations

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_o} \qquad \nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla\times\boldsymbol{B} = \mu_o\boldsymbol{J} + \mu_o\epsilon_o\frac{\partial \boldsymbol{E}}{\partial t}$$

and the Lorentz force law

$$\boldsymbol{F} = q\,(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$$

give us all of classical E&M. The rest is sorting out the details.

Lecture 2: Coulomb's and Gauss's Laws

The general set of equations governing electromagnetism is as follows:

$$\text{Maxwell's equations} \begin{cases} \nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \epsilon_o \dfrac{\partial \boldsymbol{E}}{\partial t} \end{cases} \qquad \underbrace{\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)}_{\text{Lorentz force law}}$$

But we don't generally try to tackle that all at once. We'll get started by studying what happens when all the charges in the neighborhood are fixed in place — what we call electrostatics. Then, all the time-varying terms go away, as do the currents, and in fact all sources of $\boldsymbol{B}$-fields, leaving us with just

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o} \\ \nabla \times \boldsymbol{E} = \boldsymbol{0} \end{cases} \qquad \boldsymbol{F} = q\boldsymbol{E}$$

And as an added bonus, you can even derive $\nabla \times \boldsymbol{E} = \boldsymbol{0}$ for Gauss's law, leaving us with only two unique equations. We might look at that derivation later on if we have time.
Now, you may not be accustomed to seeing Gauss's law in differential form, so let's convert it to integral form. Integrating over some volume:

$$\int (\nabla \cdot \boldsymbol{E})\, d^3x' = \int \frac{\rho}{\epsilon_o}\, d^3x'$$

Note that there are several conventions for writing a differential volume element. Using dV is not preferred, since we sometimes use it for a differential voltage. You'll see d^3x, dv, and even $d\tau$.

On the right, integrating a volume charge density over a volume gives a charge:

$$\int \frac{\rho}{\epsilon_o}\, d^3x' \quad = \quad \frac{1}{\epsilon_o} \underbrace{\int \rho\, d^3x'}_{\text{charge}} \quad = \quad \frac{Q_{\text{encl}}}{\epsilon_o}.$$

On the left, we can apply the divergence theorem:

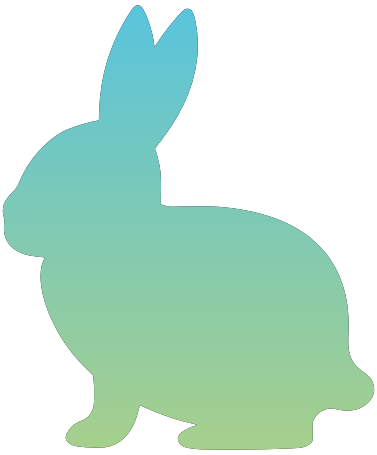
$$\int (\nabla \cdot \boldsymbol{E})\; d^3x' = \oint \boldsymbol{E} \cdot d\boldsymbol{A}.$$

Putting the two together, we obtain the familiar intro version of Gauss's law:

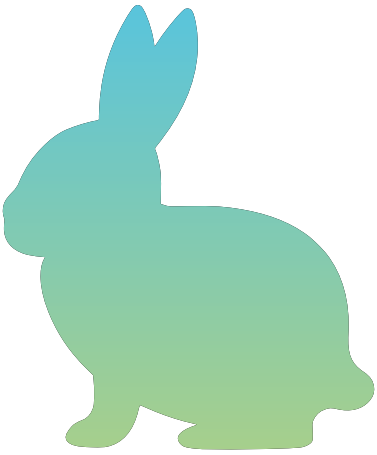
$$\boxed{\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{\text{enc}}}{\epsilon_o}}$$

You can take the integral form and derive it from the differential form. The steps are almost identical but in reverse, except for one tricky part that I won't spoil for you.

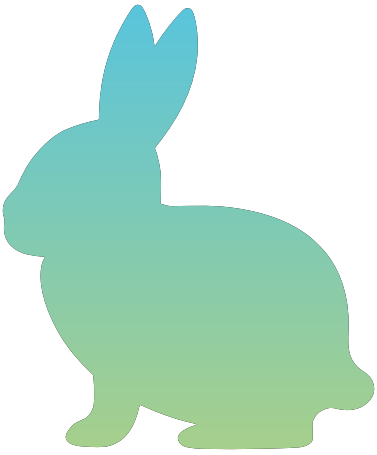
These forms are equivalent insofar as you can derive one from the other, but say slightly different things. The differential form is a statement about single points in space — if there's some non-zero ρ at a point, there's also a diverging $\boldsymbol{E}$-field there. The integral form is a statement about whole regions — the flux through some shape is proportional to the amount of charge it encloses.
I promised that you can get all of electrostatics from $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o}$ and $\boldsymbol{F} = q\boldsymbol{E}$, and you can. That includes Coulomb's law, as long as we include one more not-terribly-controversial postulate. Start with a positive point charge of magnitude q at the origin:



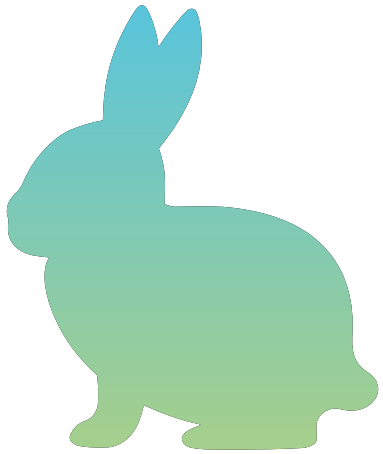
If we posit that space is rotationally invariant (changing the orientation of a system doesn't change the physics), we can immediately conclude that a point charge (being spherically symmetric) must produce a strictly radial field. To see this, try a little proof by contradiction.



Suppose the charge makes some non-radial field at some point. Then suppose suppose we take the whole system and rotate it 180° about the indicated axis.



The actual charge is unchanged, but now it makes the opposite field! This is kind of nonsense, and our claim that there could be a non-radial field must have been false.



So we have a radial field. Let's draw a spherical Gaussian surface of radius r around the point charge, and write down Gauss's law.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{\text{enc}}}{\epsilon_o}$$

We're integrating $\boldsymbol{E} \cdot d\boldsymbol{A}$ all around the Gaussian sphere. The differential area vector $d\boldsymbol{A}$ corresponds to a little patch of that surface, and points radially outward. The electric field $\boldsymbol{E}$ also points radially outward everywhere, so at any point on the Gaussian surface $\boldsymbol{E} \cdot d\boldsymbol{A} = E\ dA$. (And $Q_{\text{enc}} = q$, naturally.)

$$\implies \oint E\ dA = \frac{q}{\epsilon_o}$$

Now, this system is rotationally invariant (it has spherical symmetry), so the magnitude of $\boldsymbol{E}$ has to be the same everywhere on the domain of integration, and we can pull it out.

$$\implies E \oint dA = \frac{q}{\epsilon_o}$$

$$\implies E\left(4\pi r^2\right) = \frac{q}{\epsilon_o}$$

$\oint dA$ is just the area of the Gaussian surface, or $4\pi r^2$ in this case.

$$\implies E = \frac{q}{4\pi\epsilon_o\ r^2}$$

And we know the the electric field is radial, so

$$\boxed{\boldsymbol{E}_{\text{point}} = \frac{q}{4\pi\epsilon_o\ r^2}\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_o}\frac{\boldsymbol{r}}{r^3}}$$

Which is indeed Coulomb's law

Coulomb's & Gauss's Laws

Coulomb's Law is derived empirically and reads

$$\boldsymbol{F} = \frac{kq_1q_2\ \hat{\mathbf{r}}}{r^2} = \frac{kq_1q_2\ \boldsymbol{r}}{r^3} \qquad \text{or} \qquad \boldsymbol{F} = \frac{kq_1q_2\ (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3}$$

We define the electric field via

$$\left.\begin{array}{c} \boldsymbol{E} = \dfrac{\boldsymbol{F}}{q} \implies \boldsymbol{E} = \dfrac{kq\ \hat{\mathbf{r}}}{r^2} \\ \Updownarrow \\ \boldsymbol{E} = \dfrac{1}{4\pi\epsilon_o}\dfrac{q\ (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3} \end{array}\right\}$$ Often still called Coulomb's law. Note that $k = \dfrac{1}{4\pi\epsilon_o}$

Superposition applies. That is, the net electric field at a point is the sum of the contributions from the (discrete) constituent charges:

$$\boldsymbol{E}_{\text{net}} = \sum_{i=1}^{N} \frac{kq_i\ \hat{\mathbf{r}}}{r^2} \qquad \text{or} \qquad \boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \sum_{i=1}^{N} \frac{q_i\ (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3}$$

This can be generalized to continuous distributions by considering a differential amount of charge dq:

$$d\boldsymbol{E} = \frac{k\ dq\ \boldsymbol{r}}{r^3} \qquad \text{or} \qquad d\boldsymbol{E} = \frac{1}{4\pi\epsilon_o} \frac{dq\ (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3}$$

Warning on Notation

We frequently encounter problems involving two points:

- Source points, $\boldsymbol{x'}$, which are denoted with primed coordinates. These represent the locations of whatever is producing the field (*i.e.* the location of charges).
- Field points, $\boldsymbol{x}$, which are denoted with unprimed coordinates. These represent the location of interest (*i.e.* the point at which the electric field is being calculated).

Sometimes, a short-hand notation is adopted in which these vectors are combined into one *separation vector* $\boldsymbol{r}$:

$$\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x'}.$$

Note that Griffiths denotes the separation vector with a script $\boldsymbol{\imath}$.

Its magnitude is

$$r = |\boldsymbol{x} - \boldsymbol{x'}|\,,$$

and a unit vector in the direction from $\boldsymbol{x'}$ to $\boldsymbol{x}$ is

$$\hat{\mathbf{r}} = \frac{\boldsymbol{r}}{r} = \frac{\boldsymbol{x} - \boldsymbol{x'}}{|\boldsymbol{x} - \boldsymbol{x'}|}.$$

For consistency, we will usually keep source and field vectors separate.

Suppose the source of the electric field is instead distributed continuously over some region. Then the sum becomes an integral:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \int \frac{dq\ (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3}.$$

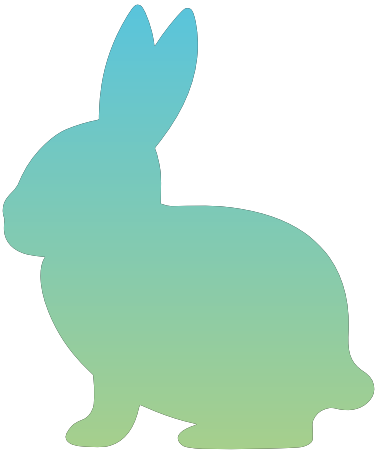
What we use for dq amounts to what dimension the charge is distributed over.

Continuous charge distribution	Differential charge	Electric field expression
Line charge, λ	$dq = \lambda\ dx'$ $= \lambda\ dl'$	$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \int \frac{\lambda(\boldsymbol{x'})\ dx' \quad (\boldsymbol{x} - \boldsymbol{x'})}{\lvert\boldsymbol{x} - \boldsymbol{x'}\rvert^3}$
Surface charge, σ	$dq = \sigma\ d^2x'$ $= \sigma\ dA'$	$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \int \frac{\sigma(\boldsymbol{x'})\ d^2x' \quad (\boldsymbol{x} - \boldsymbol{x'})}{\lvert\boldsymbol{x} - \boldsymbol{x'}\rvert^3}$
Volume charge, ρ	$dq = \rho\ d^3x'$ $= \rho\ d\tau'$	$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \int \frac{\rho(\boldsymbol{x'})\ d^3x' \quad (\boldsymbol{x} - \boldsymbol{x'})}{\lvert\boldsymbol{x} - \boldsymbol{x'}\rvert^3}$

Coulomb's law + superposition + some math generates all of electrostatics.

(Clicker question)

Let's do a nice Coulomb's law derivation. We have seen via Gauss's law that spherically symmetric charge distributions behave identically to point charges viewed from the outside. Moreover, symmetric shells have zero electric field inside. We'll show this via integration.
First, we choose to believe that the spherical symmetry gives us a strictly radial field (more on this later) and as a fun trick, we look at the field at some distance along the z-axis.



Start with the general expression for the electric field made by a surface charge:

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o}\int \frac{dq\ \ (\boldsymbol{x}-\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|^3} = \frac{1}{4\pi\epsilon_o}\int \frac{\sigma\ dA'\ \ (\boldsymbol{x}-\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|^3}$$

$$dq = \sigma\ dA' = \sigma\ R^2 \sin\theta'\ d\theta'\ d\phi'$$

Consider the source and field position vectors:

$$\begin{cases} \boldsymbol{x} = z\ \hat{\mathbf{k}} \\ \boldsymbol{x'} = R\ \hat{\mathbf{r}} = R\sin\theta'\cos\phi'\ \hat{\mathbf{i}} + R\sin\theta'\sin\phi'\ \hat{\mathbf{j}} + R\cos\theta'\ \hat{\mathbf{k}} \end{cases}$$

Forming their difference gives

$$\boldsymbol{x}-\boldsymbol{x'} = (-R\sin\theta'\cos\phi')\,\hat{\mathbf{i}} + (-R\sin\theta'\sin\phi')\,\hat{\mathbf{j}} + (z - R\cos\theta')\,\hat{\mathbf{k}}$$

The magnitude of the separation vector is

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{x'}| &= \sqrt{(-R\sin\theta'\cos\phi')^2 + (-R\sin\theta'\sin\phi')^2 + (z-R\cos\theta')^2} \\ &= \sqrt{R^2\sin^2\theta'\left(\cos^2\phi' + \sin^2\phi'\right) + (z-R\cos\theta)^2} \\ &= \sqrt{R^2\sin^2\theta' + z^2 - 2Rz\cos\theta' + R^2\cos^2\theta'} \\ &= \sqrt{R^2 + z^2 - 2Rz\cos\theta'} \end{aligned}$$

Putting it all together,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= \frac{1}{4\pi\epsilon_o}\int \frac{\sigma dA'\ \ (\boldsymbol{x}-\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|^3} \\ &= \frac{1}{4\pi\epsilon_o}\left(\cancelto{0}{\int \frac{\sigma\ dA'\ \ (-R\sin\theta'\cos\phi')\,\hat{\mathbf{i}}}{|\boldsymbol{x}-\boldsymbol{x'}|^3}} + \cancelto{0}{\int \frac{\sigma\ dA'\ \ (-R\sin\theta'\sin\phi')\,\hat{\mathbf{j}}}{|\boldsymbol{x}-\boldsymbol{x'}|^3}} + \int \frac{\sigma\ dA'\ \ (z-R\cos\theta')\,\hat{\mathbf{k}}}{|\boldsymbol{x}-\boldsymbol{x'}|^3}\right) \\ &= \frac{1}{4\pi\epsilon_o}\int_0^{2\pi}\int_0^{\pi} \frac{\sigma\ R^2\sin\theta'\ d\theta'\ d\phi'\ \ \ (z-R\cos\theta')}{(R^2+z^2-2Rz\cos\theta')^{3/2}}\ \hat{\mathbf{k}}. \end{aligned}$$

The terms corresponding to the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components of the separation vector $\boldsymbol{x}-\boldsymbol{x'}$ vanish (note we are integrating $\cos\phi'$ and $\sin\phi'$, respectively, from 0 to 2π — physically this corresponds to the symmetry about the z-axis). This leaves us with

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{4\pi\epsilon_o}\int_0^{2\pi}\int_0^{\pi} \frac{\sigma\ R^2\sin\theta'\ d\theta'\ d\phi'\ \ \ (z-R\cos\theta')}{(R^2+z^2-2Rz\cos\theta')^{3/2}}\ \hat{\mathbf{k}} \\ &= \frac{\sigma\ R^2}{2\epsilon_o}\int_0^{\pi} \frac{\sin\theta'\ d\theta'\ \ (z-R\cos\theta')}{(R^2+z^2-2Rz\cos\theta')^{3/2}} \end{aligned}$$

Substituting $u = \cos\theta'$ and $du = -\sin\theta'\ d\theta'$,

$$\boldsymbol{E} = \frac{\sigma R^2}{2\epsilon_o} \int_{-1}^{1} \frac{-du\ \ (z - Ru)}{\left(R^2 + z^2 - 2Rzu\right)^{3/2}}\ \hat{\mathbf{k}}$$

Using integration tables or software,

$$\begin{aligned}
\boldsymbol{E} &= \frac{\sigma R^2}{2\epsilon_o} \left(\frac{zu - R}{z^2\sqrt{R^2 + z^2 - 2Rzu}} \right) \Bigg|_{-1}^{1} \hat{\mathbf{k}} \\
&= \frac{\sigma R^2}{2\epsilon_o\ z^2} \left[\left(\frac{z - R}{\sqrt{R^2 + z^2 - 2Rz}} \right) + \cancel{\left(\frac{z + R}{\sqrt{R^2 + z^2 + 2Rz}} \right)^{1}} \right] \hat{\mathbf{k}} \\
&= \frac{\sigma R^2}{2\epsilon_o\ z^2} \left[\frac{z - R}{\sqrt{(z - R)^2}} + 1 \right] \hat{\mathbf{k}} \\
&= \frac{\sigma R^2}{2\epsilon_o\ z^2} \left[\frac{z - r}{|z - R|} + 1 \right] \hat{\mathbf{k}}.
\end{aligned}$$

If $z < R$ (inside the sphere), then $\dfrac{z - r}{|z - R|} = -1$ and we get

$$\boldsymbol{E}_{\text{inside}} = \frac{\sigma R^2}{2\epsilon_o\ z^2} \left[-1 + 1\right]\ \hat{\mathbf{k}} = \mathbf{0}.$$

If $z > R$ (outside the sphere), then $\dfrac{z - r}{|z - R|} = 1$ and we get

$$\boldsymbol{E}_{\text{outside}} = \frac{\sigma R^2}{2\epsilon_o\ z^2} \left[1 + 1\right]\ \hat{\mathbf{k}} = \frac{\sigma\ R^2}{\epsilon_o\ z^2}\ \hat{\mathbf{k}}.$$

Since the charge is uniformly distributed, $\sigma = \dfrac{Q}{4\pi R^2}$. With this substitution, the electric field outside becomes

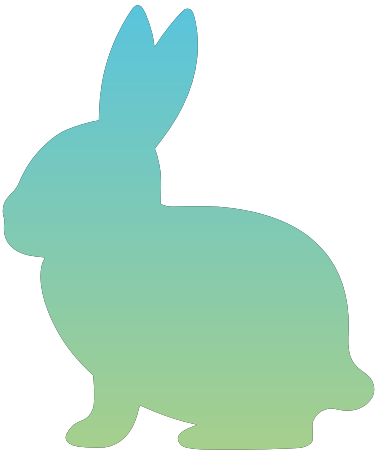
$$\boldsymbol{E}_{\text{outside}} = \frac{Q}{4\pi\ R^2} \frac{R^2}{\epsilon_o\ z^2}\ \hat{\mathbf{k}} = \frac{Q}{4\pi\epsilon_o\ z^2}\ \hat{\mathbf{k}}.$$

We've gotten this result before with Gauss's law, and as it turns out Coulomb's law implies Gauss's law, and vice-versa (almost). Deriving Gauss's law from Coulomb's law is hard to do formally, so let's skip the derivation for now. Instead, recall the definition of flux:

$$\Phi \equiv \int \boldsymbol{E} \cdot d\boldsymbol{A}.$$

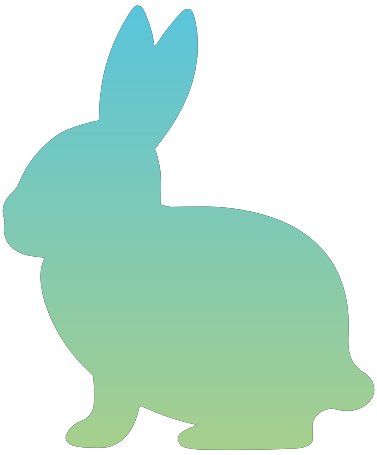
Gauss's law is a statement about the flux through a closed surface, and is <u>always</u> true. If we have good symmetry (maybe planar, cylindrical, or spherical in nature), we will <u>also</u> be able to solve for $\boldsymbol{E}$. But we make many remarkably subtle arguments along the way.

Let's consider an infinite homogeneous sheet of charge, in Cartesian coordinates, with charge density σ.



Suppose there exists a component of $\boldsymbol{E}$ in the $\hat{\mathbf{i}}$-direction. Now, the sheet has a variety of symmetries, including symmetry under reflection and rotation. Twisting the sheet about the z-axis doesn't change it and thus can't change the field. So assuming E_x is non-zero leads to a contradiction. Similarly, there can't be a component of the field in the $\hat{\mathbf{j}}$-direction. To wit: $E_x = E_y = 0$.

The $\hat{\mathbf{k}}$-component of the electric field, E_z, respects all available symmetries and so can exist. Assuming a positively charged sheet, the electric field points away from the sheet. I therefore draw a Gaussian surface like so.



$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{\text{enc}}}{\epsilon_o}$$

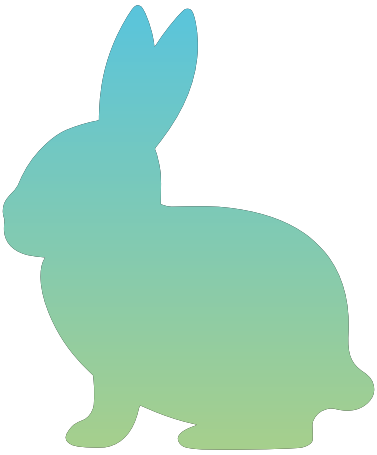
We will work the left-hand side of Gauss's law first.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \underset{\substack{\text{top +}\\ \text{bottom}}}{\int} \boldsymbol{E} \cdot d\boldsymbol{A} = \underset{\substack{\text{top +}\\ \text{bottom}}}{\int} E \; dA = E \underset{\substack{\text{top +}\\ \text{bottom}}}{\int} dA = E\,(2A_{\text{top}})$$

Every single step came with a reason. Know them.

Now, $Q_{\text{enc}} = \sigma A_{\text{top}}$, so

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{\text{enc}}}{\epsilon_o} \quad \Longrightarrow \quad E\,(2A_{\text{top}}) = \frac{\sigma A_{\text{top}}}{\epsilon_o} \quad \Longrightarrow \quad \boxed{E_{\text{sheet}} = \frac{\sigma}{2\epsilon_o}}$$

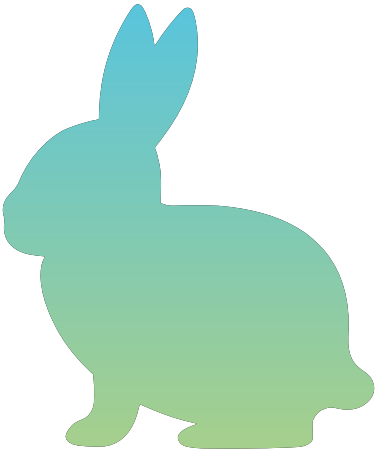


Notice the discontinuity in the perpendicular component of the $\boldsymbol{E}$-field (denoted $E^{\perp}$):

$$\boxed{E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_o}}$$

This is always true. We can always zoom in until the surface is locally flat and draw a Gaussian box.

This is a boundary condition. It will come back, and it will bring a friend.



Consider the component of the $\boldsymbol{E}$-field parallel to the surface, $E^{\parallel}$. Assume $\nabla \times \boldsymbol{E} = \boldsymbol{0}$.

Then, applying Stokes' theorem to the loop shown,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = \int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A} = 0$$

As the loop gets small, we must have

$$\boxed{E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}}}$$

That is, $E^{\parallel}$ is always continous across surfaces.

Now, how is it that $\nabla \times \boldsymbol{E} = \boldsymbol{0}$? This is easy to show with Faraday's law, which reads

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{And nothing has time dependence.}$$

A harder way to show that $\boldsymbol{E}$ is curl-free is straight from Coulomb's law:

$$\boldsymbol{E} = \int \frac{k\ \rho(\boldsymbol{x'})\ d^3x'\quad (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}.$$

Knowing that $\nabla \frac{1}{r} = \frac{\boldsymbol{r}}{r^3}$, we can say

$$\nabla \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \frac{\boldsymbol{x} - \boldsymbol{x'}}{|\boldsymbol{x} - \boldsymbol{x'}|^3}.$$

So Coulomb's law can be re-written:

$$\boldsymbol{E} = \int \frac{k\ \rho(\boldsymbol{x'})\ d^3x'\quad (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} = \int k\ \rho(\boldsymbol{x'})\ \left(\nabla \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|}\right)\ d^3x'$$

Note that the gradient operates on the real variable $\boldsymbol{x}$, not on the dummy variable $\boldsymbol{x'}$. However, the integration is with respect to source variable $\boldsymbol{x'}$. As such, we can pull the gradient out of the integral:

$$\boldsymbol{E} = \nabla \underbrace{\int \frac{k\ \rho(\boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|}}_{\text{scalar}}$$

Taking the curl of both sides,

$$\nabla \times \boldsymbol{E} = \nabla \times \nabla(\text{scalar}) = \boldsymbol{0},$$

since the curl of a gradient is always zero.

Lecture 3: Voltage and Energy (and Delta Functions)

At this point, we have

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o} \qquad \text{and} \qquad \nabla \times \boldsymbol{E} = \boldsymbol{0},$$

plus two boundary conditions that follow from those:

$$\begin{cases} E_1^{\perp} - E_2^{\perp} = \dfrac{\sigma}{\epsilon_o} \\ E_1^{\parallel} - E_2^{\parallel} = 0. \end{cases}$$

We can also derive Coulomb's law from Gauss's law:

$$\boldsymbol{E}(\boldsymbol{x}) = \int \frac{k\ \rho(\boldsymbol{x'})\ d^3x' \quad (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3}.$$

Strictly speaking, this is more than enough to do electrostatics. Knowing charges gives us the fields, which lead to forces via $\boldsymbol{F} = q\boldsymbol{E}$.

But, as you may recall from intro physics, sometimes we prefer to cast things in terms of voltage and energy instead of field and force.

Given some curl-free field ($\nabla \times \boldsymbol{E} = \boldsymbol{0}$ everywhere), we can define some scalar function V such that

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla V(\boldsymbol{x}),$$

which goes by a few different names, including the *voltage*, the *electric potential*, and just the *potential*.

Notably, the potential V carries the same information as the field $\boldsymbol{E}$, but the potential is a bit more pleasant to deal with for being a scalar, and it also hooks into energy pretty directly. We've seen before that

$$\Delta U = q\Delta V.$$

And we've seen that we can construct V according to either

$$\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{\ell}$$

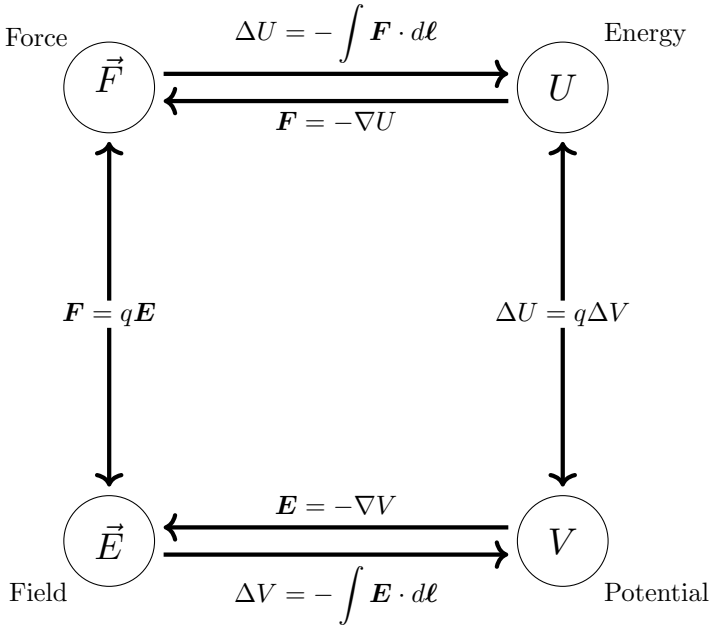
A statement about the difference in voltage between two points

or

$$V(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \int \frac{\rho(\boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|}$$

A generalization of the potential from a point charge, $V = \dfrac{1}{4\pi\epsilon_o}\dfrac{dq}{r}$

We can arrange the relationships between all these in a handy little square:



Everything thus far is from intro physics, so let's start adding some new stuff. We have both

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o} \qquad \text{and} \qquad \boldsymbol{E} = -\nabla V.$$

Substituting the latter into the former yields

$$\nabla \cdot \underbrace{(-\nabla V)}_{\boldsymbol{E}} = \frac{\rho}{\epsilon_o} \quad \Longrightarrow \quad \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_o}}$$

which is Poisson's equation — the partial differential equation that yields V in electrostatics given a known source ρ.

We will be solving Poisson's equation a lot in the near future, and to solve PDEs, we need boundary conditions. What might be the boundary conditions on V be?

Well, we know

$$\Delta V = V_{\text{above}} - V_{\text{below}} = -\int \boldsymbol{E} \cdot d\boldsymbol{\ell},$$

and real $\boldsymbol{E}$-fields are always finite. Thus, for sufficiently small $d\boldsymbol{\ell}$, $\Delta V \to 0$. In other words, as the path length shrinks to zero, so too does the integral:

$$V_{\text{above}} = V_{\text{below}} \qquad \boxed{\text{So } V \text{ is always continuous.}}$$

That being said, we know there can be discontinuities in $\boldsymbol{E}$. And $\boldsymbol{E}$-fields come about by taking derivatives of V (recall that $\boldsymbol{E} = -\nabla V$). So consider some boundary, and let n denote the direction normal to the boundary (pointing from "below" to "above"). We have the boundary condition on the perpendicular component of the electric field:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_o}$$

Using $E^{\perp} = -\dfrac{\partial V}{\partial n}$, gives us

$$\boxed{\frac{\partial V_{\text{below}}}{\partial n} - \frac{\partial V_{\text{above}}}{\partial n} = \frac{\sigma}{\epsilon_o}}$$

So while V is always continuous, derivatives of V aren't necessarily. At least, not the derivative perpendicular to a boundary. Since $E^{\parallel}$ is continuous, so must be the derivative of V in any direction parallel to the surface.

(Clicker question)

Energy of a Charge Distribution

What is the work it takes to move a charge q from $\boldsymbol{x} = \boldsymbol{a}$ to $\boldsymbol{x} = \boldsymbol{b}$? We know work is given by $\int \boldsymbol{F} \cdot d\boldsymbol{\ell}$. The electric field exerts a force on the charge on the charge according to $\boldsymbol{F} = q\boldsymbol{E}$. Thus, the minimum force needed to overcome this field is $\boldsymbol{F} = -q\boldsymbol{E}$. It follows that the work done in moving a charge is

$$\text{Work } = \int \boldsymbol{F} \cdot d\boldsymbol{\ell} = -q \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = q\left[V(\boldsymbol{b}) - V(\boldsymbol{a})\right].$$

Assuming we set the potential to be zero at infinity, the work required to bring in a charge from infinity to some location $\boldsymbol{x}$ is

$$\text{Work } = q\left[V(\boldsymbol{x}) - \cancelto{0}{V(\infty)}\right] = qV(\boldsymbol{x}).$$

Imagine bringing a charge q_1 from infinity to the origin in empty space. This takes no work, since there is no field to fight against. However, once at the origin, the charge q_1 sets up a potential according to

$$V = \frac{1}{4\pi\epsilon_o}\frac{q_1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \frac{1}{4\pi\epsilon_o}\frac{q_1}{r}.$$

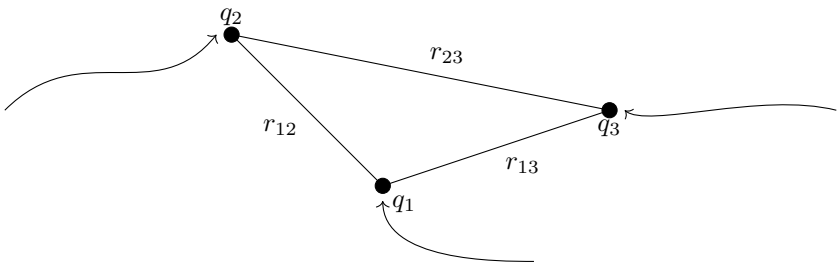
The work done in bringing in another charge q_2 from infinity to some position r is

$$W_2 = q_2 \underbrace{\left(\frac{1}{4\pi\epsilon_o}\frac{q_1}{r}\right)}_{V} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r}$$

This is the energy of interaction U between two charges separated a distance r.

Now we fix q_1 and q_2 in place, and let r_{12} represent the distance between them. The work done in bringing in a third charge is

$$W_3 = q_3 \underbrace{\left(\frac{1}{4\pi\epsilon_o}\right)\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)}_{\text{potential from } q_1 \text{ and } q_2}.$$



The total work in assembling these three charges is

$$W = \frac{1}{4\pi\epsilon_o}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

The work done in assembling a collection of discrete charges is also the energy we'd get if we dismantled the system. That is, it represents the energy stored in the configuration. We can generalize to n number of particles:

$$U = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_o}\right)\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}},$$

where the factor of 1/2 arises because we're double counting each pair. We can re-write this expression by pulling out a factor of q_i from the second sum:

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_o}\frac{q_j}{r_{ij}}\right)}_{\substack{\text{potential at } r_i \\ \text{(position of } q_i\text{)} \\ \text{due to all other} \\ \text{charges}}} = \frac{1}{2}\sum_{i=1}^{n} q_i V$$

We can generalize this expression to a continuous charge distribution:

$$\boxed{U = \frac{1}{2}\int \rho(\boldsymbol{x'})V(\boldsymbol{x'})\ d^3x'} \tag{1}$$

Alternate derivation of (1)

Start with the double sum that gives the energy of a collection of discrete charges:

$$U = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_o}\right)\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}}$$

Now, instead of a collection of discrete charges, consider some continuous charge distribution, with differential charge elements dq_1 and dq_2 (located on the same charge distribution). Then the sum becomes an integral and we have

$$U = \frac{1}{2}\iint \frac{k\ dq_1\ dq_2}{r_{12}} = \frac{1}{2}\int \frac{k\ \rho(\boldsymbol{x}_1)\ \rho(\boldsymbol{x}_2)\ d^3x_1\ d^3x_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}.$$

Again, $\rho(\boldsymbol{x}_1)$ and $\rho(\boldsymbol{x}_2)$ describe the same charge distribution — we're breaking the charge distribution into little dq's and check each against all the others.

As an alternative, note that $\displaystyle\int \frac{k\ \rho(\boldsymbol{x}_1)\ d^3x_1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$ is how we'd write the voltage due to ρ at $\boldsymbol{x}_2$ (the location of the second charge in a particular pair). Thus, we can re-write the energy of the charge distribution as

$$U = \frac{1}{2}\int \rho(\boldsymbol{x'})V(\boldsymbol{x'})\ d^3x'$$

The expression for the energy of the system explicitly reference charge. This is not shocking — we're used to potential energy being a thing associated with pairs of charges. But here's where it gets interesting. We're going to re-write the energy in terms of the field, thereby eliminating ρ and V in favor of $\boldsymbol{E}$.

From Gauss's law, we know that $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o}$. Thus,

$$\rho = \epsilon_o\left(\nabla \cdot \boldsymbol{E}\right).$$

Substituting into (1) gives us

$$U = \frac{\epsilon_o}{2}\int \left(\nabla \cdot \boldsymbol{E}\right)\ V\ d^3x.$$

For any scalar-valued function f and vector-valued function $\boldsymbol{A}$, the following holds:

$$\nabla \cdot (f\boldsymbol{A}) = f\left(\nabla \cdot \boldsymbol{A}\right) + \boldsymbol{A} \cdot \left(\nabla f\right).$$

Taking $f = V$ and $\boldsymbol{A} = \boldsymbol{E}$, we have

$$\nabla \cdot (V\boldsymbol{E}) = V(\nabla \cdot \boldsymbol{E}) + \boldsymbol{E} \cdot (\nabla V) \implies \underbrace{(\nabla \cdot \boldsymbol{E})\ V}_{\substack{\text{matches} \\ \text{integrand} \\ \text{above}}} = \nabla \cdot (V\ \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla V)$$

Substituting, we get

$$\begin{aligned} U &= -\frac{\epsilon_o}{2} \int \boldsymbol{E} \cdot \underbrace{(\nabla V)}_{-\boldsymbol{E}}\, d^3x + \frac{\epsilon_o}{2} \int \nabla \cdot (V\ \boldsymbol{E})\ d^3x \\ &= \frac{\epsilon_o}{2} \int \underbrace{(\boldsymbol{E} \cdot \boldsymbol{E})}_{E^2}\ d^3x + \frac{\epsilon_o}{2} \int \nabla \cdot (V\ \boldsymbol{E})\ d^3x \end{aligned}$$

Applying the divergence theorem to the second term yields

$$U = \frac{\epsilon_o}{2} \int E^2\ d^3x + \frac{\epsilon_o}{2} \oint (V\ \boldsymbol{E}) \cdot d\boldsymbol{A}$$

But what volume are we integrating over? Clearly, looking at the expression for energy in (1), we must at least integrate over the volume that encloses our charge distribution. But what's to stop us from integrating over all space? After all, $\rho = 0$ outside the charge distribution, so the extra space contributes nothing to the integral. Then the surface integral in the second term vanishes, since it examines V and $\boldsymbol{E}$ at the edge of all space (which, for any real, finite source, is zero). Thus,

$$\boxed{U = \frac{\epsilon_o}{2} \int E^2\ d^3x}$$

This is entirely in terms of fields, not charge. So we can look at the fields themselves as being very real things with real energy.

Just for fun, let's take a look at the energy associated with the field made by a point charge at the origin. The field is

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o} \frac{q\ \hat{\mathbf{r}}}{r^2}$$

Substituting this into the energy expression and integrating over all space yields

$$\begin{aligned} U &= \frac{\epsilon_o}{2} \int \boldsymbol{E} \cdot \boldsymbol{E}\ d^3x \\ &= \frac{\epsilon_o}{2} \left(\frac{q}{4\pi\epsilon_o}\right)^2 \int \left(\frac{\hat{\mathbf{r}}}{r^2} \cdot \frac{\hat{\mathbf{r}}}{r^2}\right)\ d^3x. \end{aligned}$$

Note that $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$. Taking the differential volume element to be $r^2\ \sin\theta\ dr\ d\theta\ d\phi$, we get

$$\begin{aligned} U &= \frac{\epsilon_o}{2} \left(\frac{q}{4\pi\epsilon_o}\right)^2 \int \frac{1}{r^4} r^2\ \sin\theta\ dr\ d\theta\ d\phi \\ &= \frac{\epsilon_o}{2} \left(\frac{q}{4\pi\epsilon_o}\right)^2 \underbrace{\left(\int_0^{2\pi} d\phi\right)}_{2\pi} \underbrace{\left(\int_0^{\pi} \sin\theta\ d\theta\right)}_{2} \left(\int_0^{\infty} \frac{dr}{r^2}\right) \\ &= \frac{q^2}{8\pi\epsilon_o} \int_0^{\infty} \frac{dr}{r^2} \\ &= \frac{q^2}{8\pi\epsilon_o} \left(-\frac{1}{r}\right)\Big|_0^{\infty} \end{aligned}$$

which kind of diverges. That's bad.

What we just worked shows that $\boldsymbol{E}$-fields from point charges should contain infinite energy, thereby implying that point charges shouldn't be possible. But every experiment ever done indicates that an electron is a zero-radius true point. Fixing this apparent contradiction is one of the great achievements of quantum electrodynamics.

Delta Functions and Point Sources

We've already seen indications that point charges behave a bit wonky, even though they do seem to exist experimentally. I'm afraid this is going to get a bit worse before it gets better.

Let's take a look at Gauss's law again. If we take the divergence of an $\boldsymbol{E}$-field, we should recover the charge density that produced that field:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o}.$$

What happens if we take the divergence of the field made by a point charge at the origin? Applying the divergence operator in spherical coordinates gives us

$$\nabla \cdot \underbrace{\left(\frac{q}{4\pi\epsilon_o} \frac{\hat{\mathbf{r}}}{r^2} \right)}_{\boldsymbol{E}_{\text{point}}} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\cancel{r^2} \frac{q}{4\pi\epsilon_o \cancel{r^2}} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{q}{4\pi\epsilon_o} \right) = 0 \; ?$$

Well, according to that, the divergence of something like $\dfrac{\hat{\mathbf{r}}}{r^2}$ is zero everywhere. Thus, so must ρ be zero everywhere. But that can't be right. So what's the catch?

The catch is that the operator $\dfrac{1}{r^2} \dfrac{\partial}{\partial r} \left(r^2 \right)$ isn't super well-defined at the origin. All we can conclude from the above is that $\nabla \cdot \boldsymbol{E}$ is zero everywhere <u>but</u> the origin. To deal with the origin, let's take a look at delta functions first.

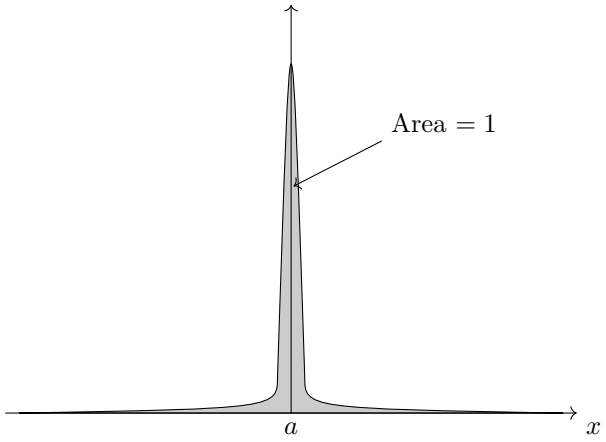
A delta function is used to represent a finite amount of stuff compressed into an essentially zero-dimensional domain. In one dimension, a delta function $\delta(x)$ is <u>defined</u> as the thingy that satisfies these two properties: $\delta(x)$ is a delta function if:

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \text{undefined}, & \text{if } x = 0 \end{cases}$$

and

$$\int \delta(x-a)\; f(x)\; dx = \begin{cases} f(a), & \text{if } a \text{ is in the domain of integration} \\ 0, & \text{otherwise} \end{cases}$$

Basically, it is a very sharply peaked function that, when present in an integral, plucks out the value of another function at one point:



So how do we represent ρ for a point charge of size q in 3D? How about a delta function:

$$\rho(\boldsymbol{x}) = q\ \delta^3(\boldsymbol{x}) \qquad \begin{array}{l}(\text{Or } q\ \delta^3(\boldsymbol{x}-\boldsymbol{x'}) \text{ if the point charge}\\ \text{is at } \boldsymbol{x'} \text{ instead of the origin})\end{array}$$

With that in mind, since $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o}$, it should be the case that

$$\nabla \cdot \underbrace{\left(\frac{q}{4\pi\epsilon_o}\frac{\hat{\mathbf{r}}}{r^2}\right)}_{\boldsymbol{E}_{\text{point}}} = \frac{1}{\epsilon_o}\underbrace{q\ \delta^3(\boldsymbol{x})}_{\rho}$$

And therefore

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{x}) \qquad \text{Is this true?}$$

Well, it's true if

(1) $\nabla \cdot \left(\dfrac{\hat{\mathbf{r}}}{r^2}\right) = 0$ everywhere but the origin. And we've established that.

(2) $\displaystyle\int \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}\right)\ d^3x = 4\pi$ for a domain of integration that includes the origin.

Let's integrate $\nabla \cdot \left(\dfrac{\hat{\mathbf{r}}}{r^2}\right)$ over a sphere of radius R. Just doing that as a volume integral is tricky, since the integrand is undefined at the origin, so let's dodge the bad part by using the divergence theorem:

$$\begin{aligned}\int \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}\right)\ d^3x &= \oint \frac{\hat{\mathbf{r}}}{r^2} \cdot d\boldsymbol{A}\\ &= \oint \frac{\hat{\mathbf{r}}}{r^2} \cdot \left(r^2\ \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{r}}\right)\\ &= \oint \sin\theta\ d\theta\ d\phi\\ &= \underbrace{\left(\int_0^{\pi} \sin\theta\ d\theta\right)\left(\int_0^{2\pi} d\phi\right)}_{=\ 4\pi}\end{aligned}$$

So it checks out. $\nabla \cdot \left(\dfrac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\ \delta^3(r)$, and then

$$\nabla \cdot \left(\frac{q}{4\pi\epsilon_o}\frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{1}{\epsilon_o} q\ \delta^3(r) = \frac{\rho}{\epsilon_o}$$

and Gauss's law holds.

Lecture 4: Multipole Expansions

We should be getting used to the idea that we can expand complex functions in terms of simpler functions. In calculus, we learned about Taylor expansions. For $f(x)$ near a:

$$f(x) = f(a) + f'(a)\ (x-a) + \frac{f''(a)\ (x-a)^2}{2!} + \frac{f'''(x)\ (x-a)^3}{3!} + \cdots$$

which will converge more or less quickly depending of f and a. Common Taylor series include the trig functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

You've also seen Fourier series expansions — re-expressions of complicated functions in terms of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

where

$$\begin{cases} a_0 = \displaystyle\int_{-\pi}^{\pi} f(x)\ dx \\ a_n = \displaystyle\int_{-\pi}^{\pi} f(x)\ \cos(nx)\ dx \\ a_n = \displaystyle\int_{-\pi}^{\pi} f(x)\ \sin(nx)\ dx \end{cases}$$

And there are many, many other ways to express a function in some basis. You're probably learning some general approaches in quantum right now, involving bras & kets.

In electrostatics, the basic potential function for a point at the origin goes like $\frac{1}{r}$:

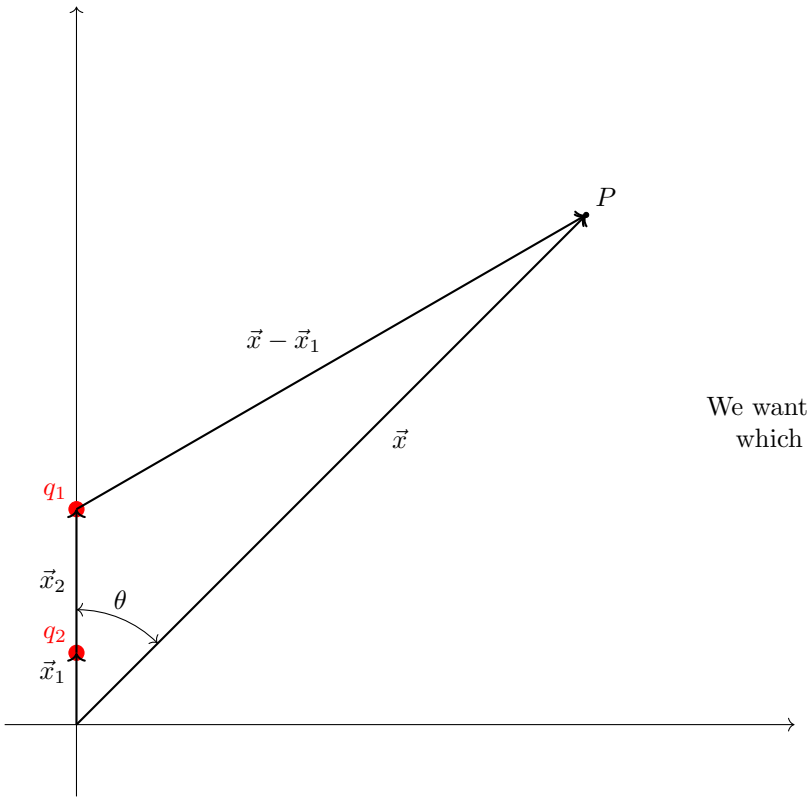
$$V_{\text{point}}(r) = \frac{k\ q}{r}.$$

More complicated systems (including points off the origin make more complicated potentials, but if we're decently far away we can still expand $V(\boldsymbol{x})$ as a reciprocal power series:

$$V(\boldsymbol{x}) = \frac{\text{thing}_1}{r} + \frac{\text{thing}_2}{r^2} + \frac{\text{thing}_3}{r^3} + \cdots$$

We call this the multipole expansion, for reasons that will become apparent. Note that conventionally, we set this up so that r is the radial coordinate, not $|\boldsymbol{x} - \boldsymbol{x'}|$.

Let's start with a simple example: two point charges q_1 and q_2 at locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively.



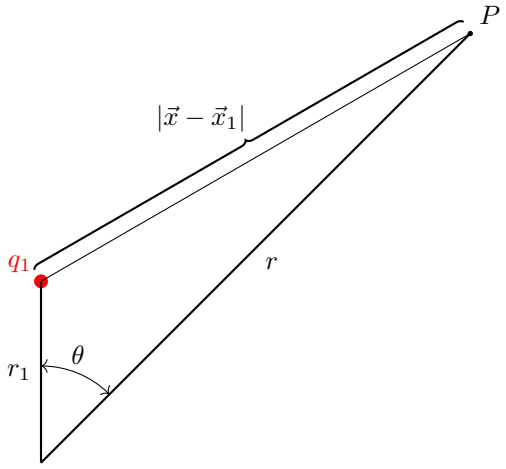
We want to find the voltage at point P, which is at some arbitrary angle θ.

I'll let the $\theta = 0$ axis lay along the line that includes the charges. Additionally, let $|\boldsymbol{x}| = r$, $|\boldsymbol{x}_1| = r_1$, and $|\boldsymbol{x}_2| = r_2$. We're interested in the régime where $r \gg r_1 + r_2$ (point P is far from the charges).

The exact expression for the voltage at P is

$$V(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o}\frac{q_1}{|\boldsymbol{x} - \boldsymbol{x}_1|} + \frac{1}{4\pi\epsilon_o}\frac{q_2}{|\boldsymbol{x} - \boldsymbol{x}_2|} \tag{1}$$

Consider the triangle formed above:



Using the law of cosines, we have $|\boldsymbol{x} - \boldsymbol{x}_1|^2 = r^2 + r_1^2 - 2rr_1\ \cos\theta$

Expanding the denominator of the first term of (1) gives us

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_1|} = \frac{1}{\sqrt{r^2 - 2rr_1\cos\theta + r_1^2}}$$

Pulling out a $1/r$,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} = \frac{1}{\sqrt{r^2\left(1+\frac{r_1^2-2rr_1\ \cos\theta}{r^2}\right)}} = \frac{1}{r\ \sqrt{1+\frac{r_1^2-2rr_1\ \cos\theta}{r^2}}} = \frac{1}{r}\underbrace{\left(1+\frac{r_1^2-2rr_1\ \cos\theta}{r^2}\right)^{-1/2}}_{(1+\text{small})^n}$$

We can use a binomial expansion on the term in parentheses. The general formula is

$$(1+x)^{-1/2} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots,$$

or by taking $n = 1/2$,

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots$$

which converges absolutely for $|x| < 1$. Here, we take $x = \dfrac{r_1^2 - 2rr_1\ \cos\theta}{r^2}$. Since $r \gg r_1$, the convergence condition is satisfied. Thus, we have

$$\begin{aligned}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} &\simeq \frac{1}{r}\left[1 + \frac{1}{2}\frac{2rr_1\ \cos\theta - r_1^2}{r^2} + \frac{3}{8}\left(\frac{4r^2r_1^2\ \cos^2\theta - 4rr_1^3\ \cos\theta + r_1^4}{r^4}\right)\right] \\ &= \frac{1}{r} + \frac{r_1\ \cos\theta}{r^2} - \frac{r_1^2}{2r^3} + \frac{3}{2}\frac{r_1^2\ \cos^2\theta}{r^3} - \frac{3}{2}\frac{r_1^3\ \cos\theta}{r^4} + \frac{3}{8}\frac{r_1^4}{r^5}.\end{aligned}$$

Again, we're interested the potential far away from the point charges (so $r \gg r_1$), meaning we can safely drop all terms higher order than $1/r^3$. Moreover, since we're far away, the angle θ for the different sources are about the same. As such, we can follow an identical process for the second term in (1), thereby giving us

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|} = \frac{1}{r} + \frac{r_2\ \cos\theta}{r^2} - \frac{r_2^2}{2r^3} + \frac{3}{2}\frac{r_2^2\ \cos^2\theta}{r^3}$$

Superposition gives us the voltage at P:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_o}\left[\underbrace{\frac{q_1+q_2}{r}}_{\text{monopole}} + \underbrace{\frac{(q_1r_1+q_2r_2)\ \cos\theta}{r^2}}_{\text{dipole}} + \underbrace{\frac{q_1r_1^2+q_2r_2^2}{2r^3}\left(3\cos^2\theta-1\right)}_{\text{quadrupole}}\right]$$

Note the presence of three terms (to wit: the monopole, dipole, and quadrupole terms). Physically, we interpret them as follows:

A **monopole** (a point charge) makes a voltage that goes like $1/r$ (and a field like $1/r^2$). A system of charges has a term in its voltage that goes like kq_{tot}/r, where q_{tot} is the total charge of the system.

A standard **dipole** is two charges of the same magnitude q and opposite sign, separated by some distance d. The net charge of a true dipole is zero (no monopole moment), so far away it has no $1/r$ potential. It does, however, have some leftover $1/r^2$ potential. The equal & opposite charges screen away some, but not all, of V.
The dipole moment of this pair is defined as

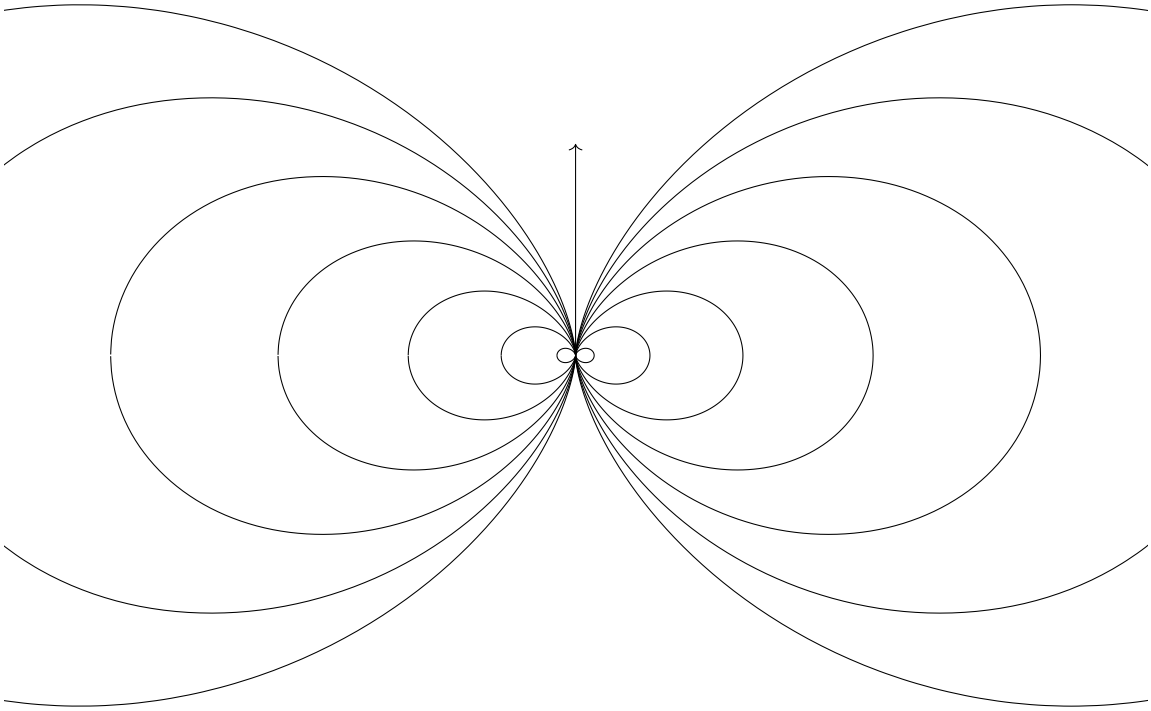
$$\boldsymbol{p} \equiv q\boldsymbol{d}$$

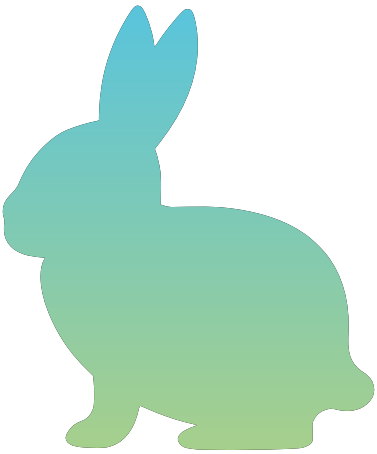
and its potential far away looks like

$$V(\boldsymbol{x}) = \frac{\boldsymbol{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_o\ r^2}$$

And a general definition for the dipole moment of any charge is $\boldsymbol{p} = q\boldsymbol{x'}$, where $\boldsymbol{x'}$ is the location of the charge.

So something that looks like $\frac{qr\cos\theta}{r^2}$ is exactly a dipole potential.





Note that most molecules are polar to some degree. For instance, the water molecule has the dipole moment $\boldsymbol{p}$ shown. You can find tables of dipole moments easily enough.

A quadrupole is two dipoles back to back in such a way that their dipole moments cancel, as do their voltages that go like $1/r^2$, leaving $1/r^3$ as a remainder.
We've seen quadrupoles before in field session (mass spectrometry unit).
Deriving an expression for the potential of a quadrupole takes a bit more work but is essentially what we did before. For now, take my word for it that the third term in (1) is a quadrupole-like term.
So now we can see what a multipole expansion is, physically. We're expanding a potential function is a basis,

where the elements of the basis include the kinds of fields made by a monopole, a dipole, a quadrupole, and so on.

What we did with the two charge system above is generalizable. For any localized charge distribution, if we're far from the source,

$$V(\boldsymbol{x}) \simeq \frac{1}{4\pi\epsilon_o}\left[\frac{Q_{\text{net}}}{r} + \frac{\hat{\mathbf{r}}\cdot\boldsymbol{p}}{r^2} + \right]$$

Lecture 5: Conductors, Capacitors, and the Method of Images

Coulomb's law gives us a recipe for finding the electric field anywhere in space given a static distribution of charge. This includes charges on insulating material.

But what if we add conductors? Charges can shift around and so now we don't know $\rho(\boldsymbol{x})$ everywhere ahead of time. What now?

Now, we learn how to solve Poisson's equation, $\nabla^2 V = -\dfrac{\rho}{\epsilon_o}$. A lot.

First, some properties of conductors:

(1) (Some) charges are free to move.

(2) Charges exist in vast numbers, even if the net charge is zero.

(3) The electric field $\boldsymbol{E}$ is zero in a conductor in electrostatic equilibrium. Achieving electrostatic equilibrium takes very little time, about 10×10^{-19} s in copper (which is effectively zero)

(3a) Voltages in conductors are constant, since $\Delta V = \displaystyle\int \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$

(4) Inside a conductor, $\rho(\boldsymbol{x}) = 0$. This follows from Gauss's law.

(4a) Net charges therefore live on the surface.

(5) At the surface of a conductor, $E^{\parallel} = 0$ and $E^{\perp} = \frac{\sigma}{\epsilon_o}$. This is a special (and most relevant) case of our old boundary condition.

Note that this is absolute and very powerful: if we know σ at the surface of a conductor we know $\boldsymbol{E}$ and vice-versa.

This is all stuff we can infer without much in the way of mathematical hardware. But to move on we need more.

Poisson's equation: $\nabla^2 V = -\frac{\rho}{\epsilon_o}$

Laplace's equation: $\nabla^2 V = 0$

And these have unique solutions, as long as we know all the boundary conditions — just like we got used to in diffy q.

We can use this information on the uniqueness of solutions to prove something I stated as fact in Phys 200.

(6) Given any chunk of conductor with any kind of cavity (hollow) in electrostatic equilibrium, $\boldsymbol{E} = \boldsymbol{0}$ in the cavity.

Proof. Since there is no charge inside the cavity, we must satisfy $\nabla^2 V = 0$. The edge of the cavity are at some constant potential, V_o (boundary condition).

Guess a solution: $V = V_o$ throughout the cavity. This satisfies $\nabla^2 V = 0$ and the boundary condition, so it's a solution to Laplace's equation. But Laplace's equation only admits unique solutions. So $V = V_o$ in the cavity (and hence $\boldsymbol{E} = \boldsymbol{0}$ is the solution.

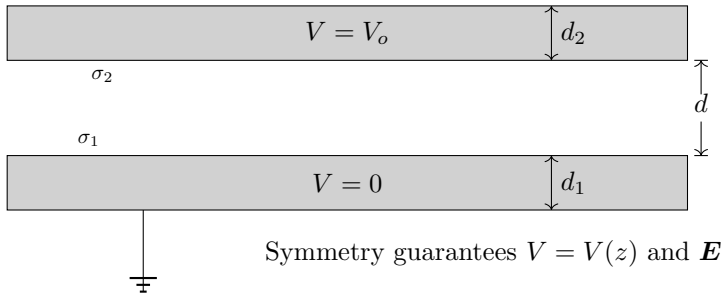
■

(7) Extending this further, we can discover that even if the conductor has charge, a hollow will have no charge on the interior surface.

So what makes a conductor a conductor? (Periodic table slide) The theory of conduction (band theory) is quantum mechanical in nature and is coming up soon (Griffiths 5.3). Keep an eye out!

Now let's get back to the general problem of solving Laplace's equation with conductors.

We will start with two "large" $\left(\sqrt{\text{Area}} \gg d\right)$ square metal plates, the prototypical capacitor. Ground one plate ($V = 0$) and hold the other at some V_o (these are our boundary conditions). Let's even let the plates have thicknesses, d_1 and d_2.



Symmetry guarantees $V = V(z)$ and $\boldsymbol{E} = E(z)\ \hat{\mathbf{k}}$

In between the plates Laplace's equation ($\nabla^2 V = 0$) applies (since there can't be any charge there), so let's guess and check. Taking $V =$ constant won't satisfy the boundary conditions, but a linear potential ($V = c_1 z + c_2$, where c_1 and c_2 are constants) will. So let's guess

$$V(z) = \frac{V_o}{d} z$$

This fits the boundary conditions and satisfies Laplace's equation, so it must be the unique solution

$$\implies \boldsymbol{E} = -\frac{V_o}{d}\ \hat{\mathbf{k}}$$

That was easy. Now, can we infer the surface charge densities σ_1 and σ_2 on the interior sheets? Well, for conductors, we have the boundary condition

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_o}\hat{\mathbf{n}} \qquad \text{at an edge.}$$

Recall that the unit normal vector always points outward from a surface. So for the bottom plate (surface 1), the unit normal vector is $\hat{\mathbf{n}}_1 = \hat{\mathbf{k}}$ and

$$\boldsymbol{E}_1 = \frac{\sigma_1}{\epsilon_o}\hat{\mathbf{k}} = -\frac{V_o}{d}\hat{\mathbf{k}} \quad \implies \quad \sigma_1 = -\frac{V_o \epsilon_o}{d}.$$

Once the unit normal vector is defined, it is signed. So since the unit normal vector points outward from the top plate (surface 2), we take $\hat{\mathbf{n}}_2 = -\hat{\mathbf{k}}$ and

$$\boldsymbol{E}_2 = -\frac{\sigma_2}{\epsilon_o}\hat{\mathbf{k}} = -\frac{V_o}{d}\hat{\mathbf{k}} \quad \implies \quad \sigma_2 = \frac{V_o \epsilon_o}{d}.$$

In the regions above and below

Lecture 6: Spherical Symmetry

Lecture 7: Cylindrical Symmetry

Lecture 8: Separation of Variables, Hyperbolic Functions, and Series Solutions

Suppose we have a non-trivial differential equation like, oh I don't know, Laplace's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

We've learned methods for highly symmetric cases and gimmicks for a few others, but no general techniques. We're going to use separation of variables and series solutions to learn how to solve Laplace's equation for any configuration.

Assume a solution of the form of products:

$$V(x, y, z) = X(x)Y(y)Z(z).$$

This looks like a severe constraint, but we get lucky and solutions of this form are such that linear combinations of multiple solutions can be used to build up anything else. That is, the solutions will form a complete set (span the space of functions). Let's take our ansatz ($V(x, y, z) = X(x)Y(y)Z(z)$) and substitute it into Laplace's equation:

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0.$$

Dividing both sides by $V(x, y, z) = X(x)Y(y)Z(z)$ gives

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\substack{\text{depends}\\ \text{only on } x}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\substack{\text{depends}\\ \text{only on } y}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\substack{\text{depends}\\ \text{only on } z}} = 0.$$

Notice how each term is independent (that is, this equation assumes the form $f(x) + g(y) + h(z) = 0$). Then the only way for this equation to hold is for each term individually to be a constant. (Fiddling with x, say, while holding y and z constant must leave the sum of the three terms untouched at zero — this implies $f(x)$ is constant). So we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \kappa_1^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = \kappa_2^2, \qquad \text{and} \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = \kappa_3^2 \qquad \begin{array}{l}\text{with the constraint}\\ \kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 0.\end{array}$$

We just turned a single PDE in three variables into three ODEs. Notice that the ODEs are the same ones that describe a simple harmonic oscillator. The general solutions to these ODEs is

$$\begin{cases} X(x) = C_1 e^{\kappa_1 x} + D_1 e^{-\kappa_1 x} \\ Y(y) = C_2 e^{\kappa_2 y} + D_2 e^{-\kappa_2 y} \\ Z(z) = C_3 e^{\kappa_3 z} + D_3 e^{-\kappa_3 z} \end{cases}$$

where the separation constants κ_1, κ_2, and κ_3 are complex numbers.

> **An interlude: Trigonometric and Hyperbolic Functions.**
>
> Trigonometric and hyperbolic functions have similar structure and behavior. Both are composed of

exponential functions. Hyberbolic functions use reals, trigonometric use complex:

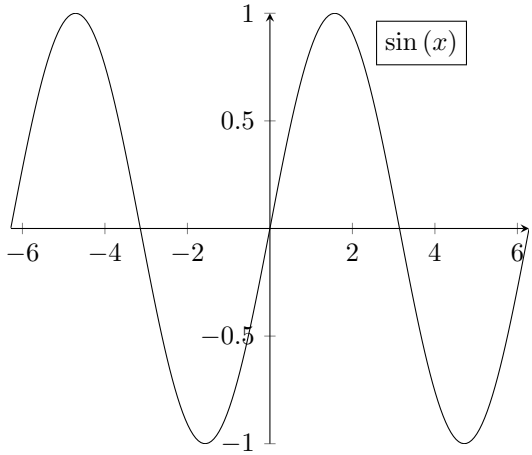
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \sinh(x) = \frac{e^{x} - e^{-x}}{2}$$
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \qquad \cosh(x) = \frac{e^{x} + e^{-x}}{2}.$$

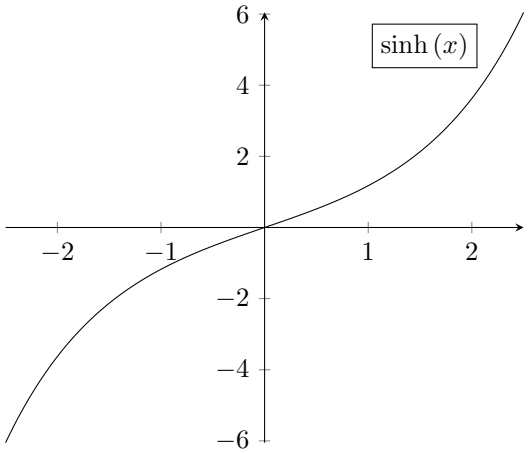
Why bother with these functions at all instead of using $\left\{e^{x}, e^{-x}, e^{ix}, e^{-ix}\right\}$ as our basic set?

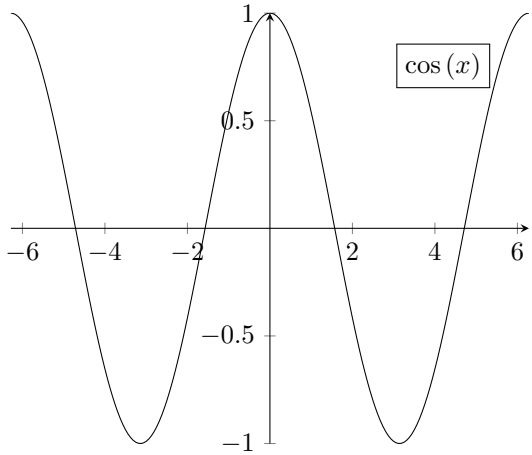
Trigonometric ($\sin(x)$ and $\cos(x)$) and hyperbolic ($\sinh(x)$ and $\cosh(x)$) functions are constructed to be even or odd. You can think of them as the even and odd "components" of the exponentials:

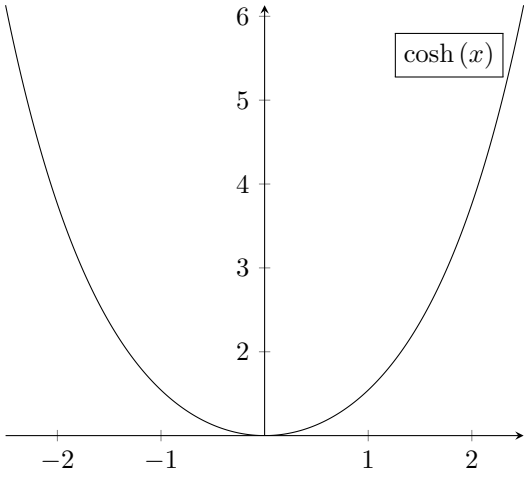
$$e^{ix} = \cos x + i \sin x$$
$$e^{x} = \cosh x + \sinh x$$

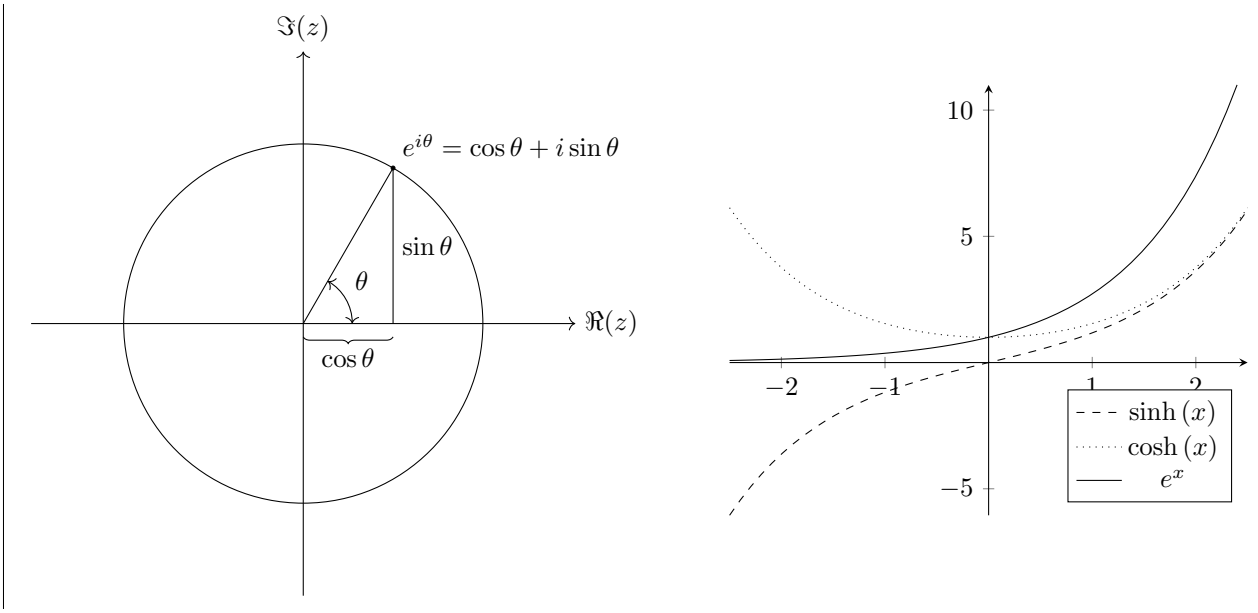
Here are some graphs:











In short, trigonometric and hyperbolic functions are basically rewrites of exponential functions and have some pleasant properties. So we use them to build up our solutions to Laplace's equations. So the general solution to $\dfrac{d^2X}{dx^2} = \kappa^2 X$ is

$$X(x) = Ae^{\kappa x} + Be^{-\kappa x} \qquad \Longleftrightarrow \qquad X(x) = \begin{cases} C_1 \cosh(\kappa x) + C_2 \sinh(\kappa x), & \text{if } \kappa \text{ is real} \\ C_1 \cos(\kappa x) + C_2 \sin(\kappa x), & \text{if } \kappa \text{ is imaginary} \end{cases}$$

Now, Laplace's equation is linear, so any combination of solutions is also a solution. So our general approach will be to use a series solution. Perhaps something like

$$V(x,y) = \sum_{n=0}^{\infty} V_n \sin\left(\frac{n\pi x}{L}\right) \cosh(y).$$

Remember from Fourier analysis: sines and cosines are orthogonal functions. That means for any symmetric interval of length $2L$, the following hold:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 2L, & m = n = 0 \\ L, & m = n \neq 0 \\ 0, & m \neq n \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0,$$

where n, m are integers. Orthogonality allows us to find the Fourier series of a known function, say f, over some interval (here, on $-L < x < L$):

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Let's multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from $-L$ to L. Doing so gives

$$\int_{-L}^{L} f(x)\ \sin\left(\frac{m\pi x}{L}\right)\ dx = \int_{-L}^{L}\left[\frac{a_o}{2}\sin\left(\frac{m\pi x}{L}\right)\right]\ dx\ +\ \int_{-L}^{L}\left[\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\right]\ dx$$
$$+\ \int_{-L}^{L}\left[\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\right]\ dx$$

Interchanging the sum and the integral,

$$\int_{-L}^{L} f(x)\ \sin\left(\frac{m\pi x}{L}\right)\ dx = \frac{a_o}{2}\underbrace{\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\ dx}_{0}\ +\ \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\ dx}_{0}$$
$$+\ \sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\ dx.$$

The first two terms vanish for all integer m and n. The last term vanishes if $m \neq n$. So we have

$$\int_{-L}^{L} f(x)\ \sin\left(\frac{m\pi x}{L}\right)\ dx = \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\ dx}_{=L,\ \text{if } m=n}$$

Solving for b_n,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx, \qquad n = 1, 2, 3, \ldots$$

Following a similar procedure for a_n gives us the complete Fourier coefficients:

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)\ dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx.$$

But what if we don't know your $f(x)$? Like if it's $V(x)$? That's where boundary conditions come in.

Lecture 9: Separation of Variables in Spherical Coordinates

The Laplacian is extra fun is spherical coordinates. Laplace's equation reads

$$\nabla^2 V \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}\right] = 0.$$

We'll make life <u>much</u> easier by considering ϕ-independent problems (such problems are said to have *azimuthal symmetry*). Later on I'll comment on what happens when you add ϕ back in. If V is independent of ϕ, then $\dfrac{\partial V}{\partial\phi} = 0$. So Laplace's equation becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

We assume a product solution $V(r,\theta) = R(r)\Theta(\theta)$ and put it through Laplace's equation:

$$\Theta(\theta)\frac{d}{dr}\left(r^2\ \frac{dR}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)R(r) = 0.$$

Diving through by $V(r,\theta) = R(r)\Theta(\theta)$,

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{depends only on } r} + \underbrace{\frac{1}{\Theta\ \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\text{depends only on } \theta} = 0.$$

Since the first term depends only on r and the second term only on θ, it must be the case that each term individually is a constant (let's call it κ):

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{=\kappa,\ \text{say}} + \underbrace{\frac{1}{\Theta\ \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{=-\kappa,\ \text{say}} = 0.$$

Let's work with the angular equation first.

$$\frac{1}{\Theta\ \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\kappa \qquad \Longrightarrow \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\ \frac{d\Theta}{d\theta}\right) + \kappa\Theta = 0.$$

To solve this ODE, we will use the substitution

$$u = \cos\theta \qquad \Longrightarrow \qquad du = -\sin\theta\ d\theta$$

and recast $\Theta(\theta)$ as $P(u)$. (That is, we will solve the ODE for $P(\cos\theta)$, rather than $\Theta(\theta)$). As per the chain rule,

$$\frac{d\Theta}{d\theta} = \frac{dP}{du}\frac{du}{d\theta} = -\sin\theta\ \frac{dP}{du} = -\sqrt{1-u^2}\ \frac{dP}{du}$$

Note that since $\sin^2\theta + \cos^2\theta = 1$, we have that $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-u^2}$

Lecture 10: Separation of Variables in Cylindrical Coordinates

Laplace's equation in cylindrical coordinates reads

$$\nabla^2 V \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

As usual, we'll cut ourselves a break and only consider "long" objects such that there's no z-dependence in V and we can cut down to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

Also as usual, we hope for solutions of the form $V(r, \phi) = R(r)\Phi(\phi)$. Putting this ansatz into Laplace's equation gives

Lecture 11: Separation of Variables - A 3D Cartesian Problem

Lecture 12: Introduction to Polarization/Dielectrics

The term 'dielectric' is a fancy word for 'insulator'. Insulators are characterized by the general immobility of the electrons, both the electrons that are already there and any added charge density. But problems involving dielectrics are not quite as dull as they seem.
Clicker Question on Balloons
Atoms and molecules can become polar two different ways:

(1) Intrinsic polarization: Molecules that are naturally polar. If, for example, you work out the occupied orbitals for a water molecule, you find the electron cloud is a little denser near the oxygen.

So the molecule has a dipole moment $\boldsymbol{p}$ (1.85 Debye) 1 Debye is equivalent to a proton & electron separated by 0.21 Angstroms.

Incidentally, the intrinsic polarity of water molecules makes it a good solvent and raises the boiling point. Polarity (or lack thereof) is also the foundation of how soap works.

(2) Induced polarization: Applied $\boldsymbol{E}$-fields can cause the electron cloud and nucleus to separate slightly, giving any dielectric material a dipole moment.

The proportionality is defined by

$$\boldsymbol{p} = \alpha \boldsymbol{E}, \qquad \text{where } \alpha \text{ is the atomic polarizability.}$$

This is a microscopic equation, describing how fields interact with individual atoms & molecules. Noble elements (those with filled outer electron shells) tend to not be very polarizable. Alkali metals (the ones with loosely bound valence electrons) tend to polarize easily.
To figure out α quantum mechanically, we solve the time-independent Schrödinger equation with the atomic potential and the potential from a constant electric field E_o (say, in the z-direction).

$$\hat{H}\psi = E\psi \quad \Longrightarrow \quad -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$
$$\Longrightarrow \quad -\frac{\hbar^2}{2m}\nabla^2\psi - \left[\frac{(Z\cdot e_f)}{4\pi\epsilon_o\, r} + Z\cdot e_f\,(E_o\; z)\right]\psi = E\psi$$

where Z is the atomic number and e_f is the fundamental charge. With ψ in hand you can calculate anything via its expectation value:

$$\boldsymbol{p} = \langle\psi\,|\,-(Ze_f)\,\boldsymbol{x}\,|\,\psi\rangle$$

And you get $\boldsymbol{p} = (\text{stuff})\,\boldsymbol{E}$, where the stuff is the proportionality constant α. This is again for a single atom or molecule

Now working towards a macroscopic picture, let's consider an aggregate of intrinsically polar molecules. Left alone, they'd organize randomly (with some tendency to stick together).
Applying an $\boldsymbol{E}$-field will align these molecules somewhat.
More thermal energy means more randomization. We can calculate the average dipole moment $\boldsymbol{p}$ at a particular temperature using statistical mechanics. Recall that the average value of some arbitrary function f over the interval is

$$\langle f\rangle = \frac{\displaystyle\int_a^b f(x)\;dx}{\displaystyle\int_a^b dx}$$

Thermal averages

Lecture 13: The Displacement Field

Lecture 14: Clausius-Mossitti, Boundary Conditions, and Examples

Lecture 14 (Supplement): Dealing with that $\ell = 0$ Term

Lecture 15: Electric Current

Lecture 16: Current and Resistance

Lecture 17: Current Dynamics (Getting Progressively Less Static)

Lecture 18: Magnetism

Lecture 19: Sources of Magnetic Fields

Lecture 20: Ampere's Law

Lecture 21: The Vector Potential

Lecture 22: The Magnetic Dipole

A long, long time ago we learned how to do multipole expansions for V:

$$V(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\boldsymbol{p}}{r^2} + \frac{\hat{\mathbf{r}}\cdot\overleftrightarrow{Q}\cdot\hat{\mathbf{r}}}{r^3}\right],$$

where Q is the monopole moment, $\boldsymbol{p}$ is the dipole moment, and $\overleftrightarrow{Q}$ is the quadrupole moment vector.
We can do a similar expansion of $\boldsymbol{A}$. We can reasonably expect one major difference though. There are no magnetic monopoles, so the leading term in the expansion should be the dipole term.
We have

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi}\int \frac{J(\boldsymbol{x}')\ d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}.$$

And as with the V expansion, we start from

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\boldsymbol{x}'}{r^2} + \mathcal{O}\left(\frac{r'^2}{r^3}\right),$$

with

$$r = |\boldsymbol{x}| \qquad \text{and} \qquad r' = |\boldsymbol{x}'| \ll r.$$

We're going to keep this expansion short and clean, so we keep only the first two terms. That gives us

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi}\left[\frac{1}{r}\int \boldsymbol{J}(\boldsymbol{x}')\ d^3x' + \frac{1}{r^2}\int \boldsymbol{J}(\boldsymbol{x}')\,(\hat{\mathbf{r}}\cdot\boldsymbol{x}')\ d^3x'\right].$$

In magnetostatics, $\nabla\cdot\boldsymbol{J} = -\frac{\partial\rho}{\partial t} \implies \nabla\cdot\boldsymbol{J} = 0$. That is, current doesn't out of nowhere or disappear; $\boldsymbol{J}$ has no divergence. Only complete circuits of current exist. Thus, it should be completely palatable that

$$\int \boldsymbol{J}(\boldsymbol{x}')\ d^3x' = 0 \qquad \text{(try writing in terms of divergence).}$$

We'll prove it too. We note that

$$\nabla\cdot(x_i\boldsymbol{J}) = \boldsymbol{J}\cdot(\nabla x_i) + (\nabla\cdot\boldsymbol{J})\,x_i,$$

and

$$\nabla\cdot\boldsymbol{J} = 0, \qquad \nabla x_i = \hat{\mathbf{e}}_i.$$

So $J_i = \nabla\cdot(x_i\boldsymbol{J})$ and we have

$$\begin{aligned}\int J_i(\boldsymbol{x}')\ d^3x' &= \int \nabla\cdot(x_i\boldsymbol{J})\ d^3x' \\ &= \oint (x_i\boldsymbol{J})\cdot d\boldsymbol{a} \qquad \text{(divergence theorem).}\end{aligned}$$

And since the integral is all over space, as long as $\boldsymbol{J}$ falls off faster than $1/r$, the integrand is zero at the "area" bounding infinity. This implies

$$\int \boldsymbol{J}(\boldsymbol{x}')\ d^3x' = 0.$$

So the monopole term in the multipole expansion is zero, leaving us with must be the dipole term:

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi r^2}\int \boldsymbol{J}(\boldsymbol{x}')\,(\hat{\mathbf{r}}\cdot\boldsymbol{x}')\ d^3x'$$

As with the voltage expansion, we need to decide how to write the dipole moment.

There, we wrote $\frac{1}{4\pi\epsilon_o}\frac{1}{r^2}\,\hat{\mathbf{r}}\cdot\boldsymbol{p}$. Here, we will write $\frac{\mu_o}{4\pi}\frac{1}{r^2}\,\boldsymbol{m}\times\hat{\mathbf{r}}$. These are very analogous. We just have to figure out what $\boldsymbol{m}$ is from that $\boldsymbol{J}$ integral. The derivation is lengthy and is in the book. The result is

$$\boldsymbol{m}=\frac{1}{2}\int \boldsymbol{x}'\times\boldsymbol{J}(\boldsymbol{x}')\;d^3x'.$$

At which point,

$$A(\boldsymbol{x})=\frac{\mu_o}{4\pi r^2}\boldsymbol{m}\times\hat{\mathbf{r}}.$$

The B-field is

$$\boxed{\boldsymbol{B}=\nabla\times\boldsymbol{A}=\frac{\mu_o}{4\pi}\frac{\left[3\hat{\mathbf{r}}\,(\boldsymbol{m}\cdot\hat{\mathbf{r}})-\boldsymbol{m}\right]}{r^3}.}$$

Optional Derivation

We have that

$$\boldsymbol{B}(\boldsymbol{x})=\frac{\mu_o}{4\pi}\nabla\times\left(\boldsymbol{m}\times\frac{\hat{\mathbf{r}}}{r^2}\right)$$

Using an identity,

$$\boldsymbol{B}(\boldsymbol{x})=\frac{\mu_o}{4\pi}\left[\underbrace{m\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right)}_{\text{(1)}}{}^{0}-\underbrace{\frac{\hat{\mathbf{r}}}{r^2}\left(\nabla\cdot\boldsymbol{m}\right)}_{\text{(2)}}{}^{0}+\underbrace{\left(\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\right)\boldsymbol{m}}_{\text{(3)}}{}^{0}-\underbrace{(\boldsymbol{m}\cdot\nabla)\frac{\hat{\mathbf{r}}}{r^2}}_{\text{(4)}}\right]$$

(1) is zero everywhere but at $r=0$ (the location of the dipole), since $\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}=4\pi\delta^3(r)$. And since the field is going to be divergent at $r=0$ anyway, this term contributes nothing useful.

(2) and (3) involve derivatives of $\boldsymbol{m}$, which is a constant vector.

Only (4) matters. What does $(\boldsymbol{m}\cdot\nabla)$ mean? It's easy to express in Cartesian:

$$(\boldsymbol{m}\cdot\nabla)\,\boldsymbol{f}=\left(m_x\frac{\partial}{\partial x}\right)f_x\,\hat{\mathbf{i}}+\left(m_y\frac{\partial}{\partial y}\right)f_y\,\hat{\mathbf{j}}+\left(m_z\frac{\partial}{\partial z}\right)f_z\,\hat{\mathbf{k}}.$$

And so,

$$\begin{aligned}\left[(\boldsymbol{m}\cdot\nabla)\frac{\hat{\mathbf{r}}}{r^2}\right]_x&=m_x\frac{\partial}{\partial x}\frac{x}{r^3}\\&=m_x\frac{\partial}{\partial x}\left[x\left(x^2+y^2+z^2\right)^{-3/2}\right]\\&=m_x\left[\frac{1}{r^3}+x\left(-\frac{3}{2}\right)r^{-5}\,(2x)\right]\\&=\frac{m_x}{r^3}-\frac{3x^2m_x}{r^5}\\&\;\;\vdots\\&\text{Using } m_xx=(\boldsymbol{m}\cdot\boldsymbol{r})_x \text{ and } x=r_x,\\&\;\;\vdots\\&=\frac{m_x}{r^3}-\frac{3\,(\boldsymbol{m}\cdot\hat{\mathbf{r}})_x\,r_x}{r^5}\end{aligned}$$

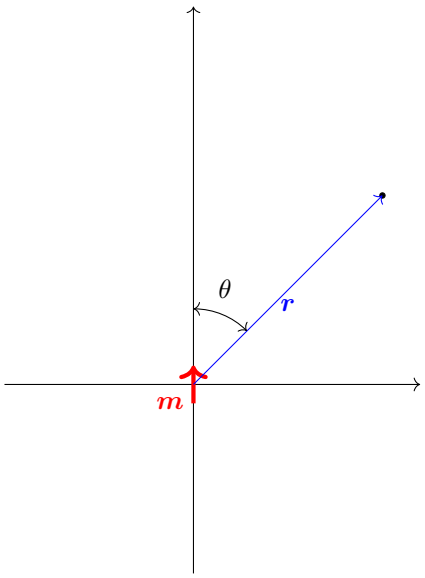
Thus,

$$(\boldsymbol{m}\cdot\nabla)\,\frac{\hat{\mathbf{r}}}{r^2}=\frac{\boldsymbol{m}}{r^3}-\frac{3\,(\boldsymbol{m}\cdot\boldsymbol{r})\,\boldsymbol{r}}{r^5}=\frac{\boldsymbol{m}}{r^3}-\frac{3\hat{\mathbf{r}}\,(\boldsymbol{m}\cdot\hat{\mathbf{r}})}{r^3}.$$

With the overall minus and the constants,

$$\boldsymbol{B}(\boldsymbol{x})=\frac{\mu_o}{4\pi}\frac{[3\hat{\mathbf{r}}\,(\boldsymbol{m}\cdot\hat{\mathbf{r}})-\boldsymbol{m}]}{r^3}.$$

We can clean up $\boldsymbol{A}$ and $\boldsymbol{B}$ by letting $\boldsymbol{m}$ point in the $\hat{\mathbf{k}}$ direction (or line up the z-axis with $\boldsymbol{m}$).



Then, in spherical,

$$\boldsymbol{A}(\boldsymbol{x})=\frac{\mu_o}{4\pi r^2}m\hat{\mathbf{k}}\times\hat{\mathbf{r}},\qquad\text{and}\qquad\hat{\mathbf{k}}\times\hat{\mathbf{r}}=|\hat{\mathbf{k}}||\hat{\mathbf{r}}|\sin\theta\;\hat{\boldsymbol{\phi}}=\sin\theta\;\hat{\boldsymbol{\phi}}$$

$$\boxed{\boldsymbol{A}(\boldsymbol{x})=\frac{\mu_o}{4\pi r^2}\frac{\sin\theta}{}\;\hat{\boldsymbol{\phi}}}$$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x})&=\frac{\mu_o}{4\pi}\frac{[3\hat{\mathbf{r}}\,(\boldsymbol{m}\cdot\hat{\mathbf{r}})-\boldsymbol{m}]}{r^3}\\&=\frac{\mu_o}{4\pi r^3}\left[3\hat{\mathbf{r}}\left(m\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}\right)-m\hat{\mathbf{k}}\right]\qquad\text{and }\hat{\mathbf{k}}=\cos\theta\;\hat{\mathbf{r}}-\sin\theta\;\hat{\boldsymbol{\theta}}\\&=\frac{\mu_o}{4\pi r^3}\left[3m\cos\theta\;\hat{\mathbf{r}}-m\cos\theta\;\hat{\mathbf{r}}+m\sin\theta\;\hat{\boldsymbol{\theta}}\right]\end{aligned}$$

$$\boxed{\boldsymbol{B}(\boldsymbol{x})=\frac{\mu_o}{4\pi r^3}\left[2\hat{\mathbf{r}}\cos\theta+\hat{\boldsymbol{\theta}}\sin\theta\right]}$$

What does this look like? I'd have no idea without the help of a computer, except for the fact that the form matches that of an electric dipole:

$$\boldsymbol{E}(\boldsymbol{x})=\frac{\rho_o}{4\pi\epsilon_o\;r^3}\left(2\cos\theta\;\hat{\mathbf{r}}+\sin\theta\;\hat{\boldsymbol{\theta}}\right),\qquad\text{(3.101, pg. 78)}$$

which looks like

Or rather like the field lines of a bar magnet, everyone's first dipole.

The parallels between electric and magnetic dipoles are legion.

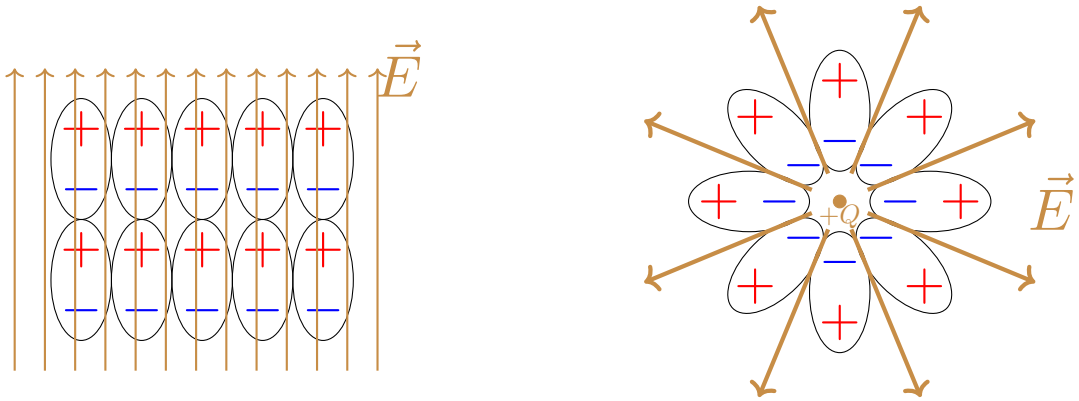
A comparison of electric and magnetic dipoles

	Electric	Magnetic
Dipole moment, general	$\boldsymbol{p} = \int \boldsymbol{x'}\ \rho(\boldsymbol{x'})\ d^3x'$	$\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{x'} \times \boldsymbol{J}(\boldsymbol{x'})\ d^3x'$
Dipole moment, basic	$\boldsymbol{p} = q\boldsymbol{d}$ (two charges $\pm q$ separation d)	$\boldsymbol{m} = I\boldsymbol{A}$ (current loop of area A)
Potential	$V(\boldsymbol{x}) = \frac{\boldsymbol{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_o\ r^2}$	$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o\ \boldsymbol{m}\times\hat{\mathbf{r}}}{4\pi\ r^2}$
Field of	$\boldsymbol{E}(\boldsymbol{x}) = \frac{3\hat{\mathbf{r}}\,(\boldsymbol{p}\cdot\hat{\mathbf{r}}) - \boldsymbol{p}}{4\pi\epsilon_o\ r^3}$	$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_o}{4\pi\ r^3}\left[3\hat{\mathbf{r}}\,(\boldsymbol{m}\cdot\hat{\mathbf{r}}) - \boldsymbol{m}\right]$
Torque on	$\boldsymbol{N} = \boldsymbol{p}\times\boldsymbol{E}$	$\boldsymbol{N} = \boldsymbol{m}\times\boldsymbol{B}$
Energy of	$U = -\boldsymbol{p}\cdot\boldsymbol{E}$	$U = -\boldsymbol{m}\cdot\boldsymbol{B}$

Pretty much the same stuff across the board. What's really different about the E and B cases is that for B, the dipole appears to be the most basic building block, whereas for E, the monopole is.

Lecture 23: Magnetization and Bound Currents

In chapter 6 we discussed electric fields in matter, and how matter becomes polarized, which is to say lots of induced dipoles show up at the atomic level.



We also get local accumulations of charge that exist entirely as a result of the polarizing field: bound charge (volume and surface).

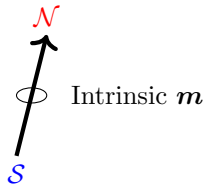
In simple situations, $\boldsymbol{p} \propto \boldsymbol{E}$ ($\boldsymbol{p} = \epsilon_o\ \chi_e \boldsymbol{E}$). As we discussed, many situations are not so simple.

Very much the same thing happens in magnetism.

Atomically, there are two places we find magnetic dipoles:

1. Intrinsic magnetic moments of elementary particles

   An electron (like all fermions) has an intrinsic magnetic moment related to its intrinsic angular momentum, both of which are referred to as spin.

   I can give you no good reason as to why it exists. It simply does. The classical model of a spinning ball of charge falls apart under even the most casual prodding (see homework).

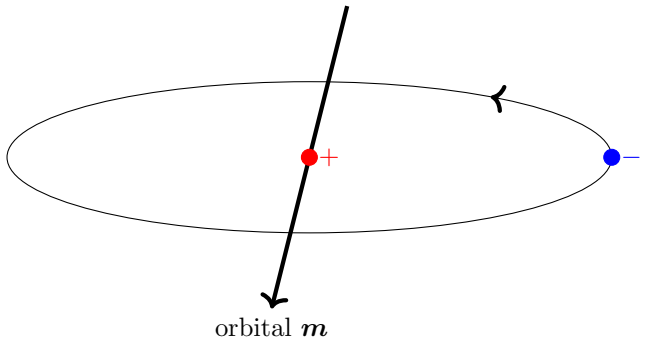
   The spin magnetic moment of an electron is

   $$|\boldsymbol{m}_e| = \frac{e\ \hbar}{2m} = \mu_B,$$

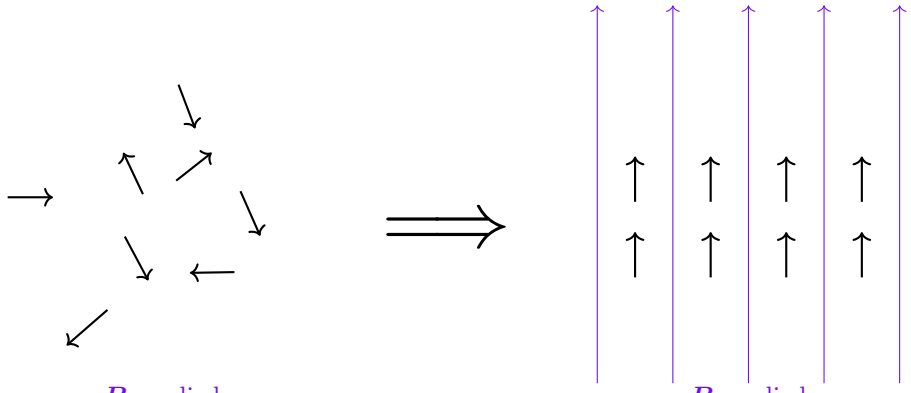
   where μ_B is a unit of magnetic moment called the Bohr magnetron.

2. Orbital magnetic moments of atoms

   There exist magnetic moments due to the interactions between electrons and nuclei. Classically we can think of this as because of the orbiting electron constituting a current, but this model also falls apart quickly



However these atomic magnetic moments arise, they're usually randomly oriented and the overall average dipole moment per volume $\boldsymbol{M}$ (like magnetization) is zero.
But an applied magnetic field can align these dipoles, either with or against $\boldsymbol{B}$, and give us nonzero $\boldsymbol{M}$.



Generally,

$$\boxed{\boldsymbol{m} \propto \boldsymbol{B}}$$

We'll discuss the mechanisms by which this happens (paramagnetism, diamagnetism, ferromagnetism) later. For now we'll just accept that it can happen.

Recall that the vector potential due to a dipole of moment $\boldsymbol{m}$ is

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}.$$

So if we have a function $\boldsymbol{M}(\boldsymbol{x})$ describing the volume density of the magnetic dipoles we can say

$$d\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \frac{\boldsymbol{M} \times \boldsymbol{r}}{r^3} d^3x, \qquad (\text{with } d\boldsymbol{m} = \boldsymbol{M} d^3x)$$

This implies

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \int \frac{\boldsymbol{m}(\boldsymbol{x'}) \times (\boldsymbol{x} - \boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|}$$

This is general, but can be written in a more useful form using a derivation extremely similar to the one we did with $V(\boldsymbol{x})$ in chapter 6.

We get

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \int \frac{\nabla' \times \boldsymbol{M}(\boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|} + \frac{\mu_o}{4\pi} \int \frac{\boldsymbol{M}(\boldsymbol{x'}) \times \hat{\mathbf{n}}\ dA'}{|\boldsymbol{x} - \boldsymbol{x'}|}.$$

Last chapter we derived

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|} \qquad \text{and} \qquad \boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \int \frac{\boldsymbol{K}(\boldsymbol{x'})\ dA'}{|\boldsymbol{x} - \boldsymbol{x'}|}.$$

Comparing, we see these match up if $\boldsymbol{J}(\boldsymbol{x}) = \nabla \times \boldsymbol{M}(\boldsymbol{x'})$ and $\boldsymbol{K}(\boldsymbol{x'}) = \boldsymbol{M}(\boldsymbol{x'}) \times \hat{\mathbf{n}}$.

Since these "currents" only exist because of magnetization $\boldsymbol{M}$, we call them volume and surface bound currents.

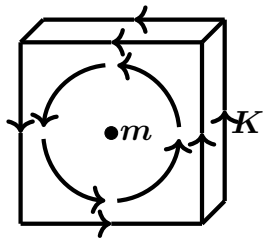
Putting it all together, the $\boldsymbol{A}$ resulting from the $\boldsymbol{M}$ is

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_o}{4\pi} \int_{\mathcal{V}} \frac{\boldsymbol{J}_b(\boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|} + \frac{\mu_o}{4\pi} \int_{\mathcal{A}} \frac{\boldsymbol{K}_b(\boldsymbol{x'})\ dA'}{|\boldsymbol{x} - \boldsymbol{x'}|},$$

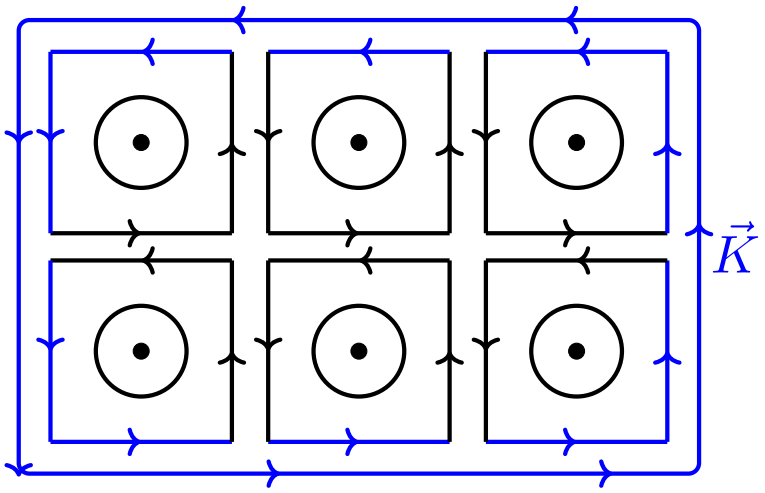
where the volume integral is over the space filled by $\boldsymbol{M}$, and the area integral is over the area bounding that space.

Visualizing bound currents

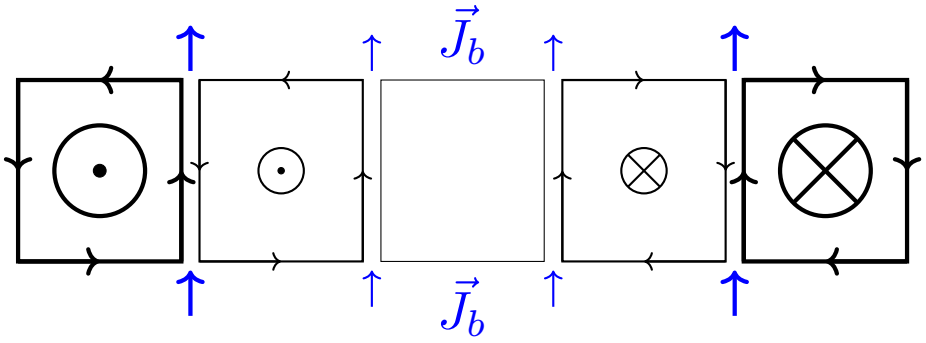
Classically, every magnetic dipole can be related back to a current loop, which we'll visualize as occupying a square unit cell.



Many cells together, assuming uniform $\boldsymbol{M}$, will result in a lot of cancellation and some leftover current on the perimeter (the bound surface current).



If we have a nonuniform $\boldsymbol{M}$ — probably (but not necessarily) coming from a nonuniform $\boldsymbol{B}$ — we may end up with bound currents in the interior also.



Lecture 24: B, H, and Boundary Conditions

Let's briefly review how bound charge let to the idea of the displacement field $\boldsymbol{D}$.

The source equation $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_o$ refers to <u>all</u> charges, both free and bound:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_f + \rho_b}{\epsilon_o}$$

This is inconvenient, since we often can only specify the free charge. Since $\rho_b = -\nabla \cdot \boldsymbol{P}$, this invites some rearrangement:

$$\epsilon_o \left(\nabla \cdot \boldsymbol{E}\right) = \rho_f + \rho_b = \rho_f - \nabla \cdot \boldsymbol{P} \implies \nabla \cdot \left(\epsilon_o \boldsymbol{E} + \boldsymbol{P}\right) = \rho_f.$$

So, if we can define a composite field $\boldsymbol{D} \equiv \epsilon \boldsymbol{E} + \boldsymbol{P}$, we can recover a source equation in the style of Gauss's law, but written only in terms of free charge:

$$\nabla \cdot \boldsymbol{D} = \rho_f.$$

We're going to do something analogous for magnetism. Ampere's law says $\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$. That current density $\boldsymbol{J}$ is <u>all</u> $\boldsymbol{J}$, free and bound:

$$\nabla \times \boldsymbol{B} = \mu_o \left(\boldsymbol{J}_f + \boldsymbol{J}_b\right).$$

Since $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$, we can do some reshuffling:

$$\nabla \times \frac{\boldsymbol{B}}{\mu_o} = \boldsymbol{J}_f + \nabla \times \boldsymbol{M}$$
$$\implies \nabla \times \left(\frac{\boldsymbol{B}}{\mu_o} - \boldsymbol{M}\right) = \boldsymbol{J}_f,$$

Define $\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_o} - \boldsymbol{M}$ and we get

$$\boxed{\nabla \times \boldsymbol{H} = \boldsymbol{J}_f} \qquad \text{Ampere's Law in Matter}$$

Interlude: Naming $\boldsymbol{H}$ & $\boldsymbol{B}$ (or, "grumpy old people having a fight about stupid bullshit")

So $\boldsymbol{E}$ is fundamental in that it is responsible for forces quite directly ($\boldsymbol{F} = q\boldsymbol{E}$) and that it is the field that gets produced by all charges, not just a special subset of charges ($\nabla \cdot \boldsymbol{E} = \rho_{\text{total}}/\epsilon_o$). Thus, $\boldsymbol{E}$ is always the electric field, and the composite object $\boldsymbol{D}$ is given a special name, the displacement field.

Similarly, $\boldsymbol{B}$ is the field in the force law ($\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$) and is the field produced by all forms of current. So obviously everyone calls $\boldsymbol{B}$ the magnetic field and $\boldsymbol{H}$ something else, right?

Ha ha. No. For some reason that I cannot find (and I've looked), some people call $\boldsymbol{B}$ the magnetic induction and $\boldsymbol{H}$ the magnetic field. And other people do other stuff.

Names I've heard for B and H

$\boldsymbol{B}$	$\boldsymbol{H}$
Magnetic field	Magnetic field
Magnetic induction	Auxilary magnetic field
Magnetic flux density	Magnetic field strength

And people will fight to death over this sort of thing! We're going to keep it clean. $\boldsymbol{B}$ is the magnetic field. Period. And $\boldsymbol{H}$ we'll just call 'H'.

Anyway, back to Ampere's law in matter:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f \qquad \text{or} \qquad \oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_{f,\,\text{enc}}.$$

Let's continue to parallel what was done for polarized dielectrics. There, $\boldsymbol{P} = \epsilon_o \chi_e \boldsymbol{E}$ for linear, isotropic materials and $\boldsymbol{D} = \epsilon \boldsymbol{E}$, with $\epsilon = \epsilon\,(1 + \chi_e)$. Here, once again for linear isotropic materials, we define

$$\begin{aligned} \boldsymbol{M} &= \chi_m \boldsymbol{H} \qquad (\chi_m \text{ is the magnetic susceptibility} \\ \boldsymbol{B} &= \mu \boldsymbol{H} \qquad \text{with } \mu = \mu_o\,(1 + \chi_m);\ \mu \text{ is named the 'permeability'} \end{aligned}$$

χ_m is very small for almost all materials save ferromagnetic ones ($\chi_m \sim 10^{-3}$ — 10^{-8}). Thus for most materials magnetization is much less relevant than polarization.

Also, we can derive an important constraint on $\boldsymbol{J}_b$ now: Since $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$, if $\boldsymbol{M} = \chi_m \boldsymbol{H}$, then

$$\boldsymbol{J}_b = \nabla \times \chi_m \boldsymbol{H} \qquad \text{and} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J}_f.$$

So

$$\boxed{\boldsymbol{J}_b = \chi_m \boldsymbol{J}_f}$$

Therefore, if our material is linear, uniform, and isotropic, $\boldsymbol{J}_b$ can exist only if $\boldsymbol{J}_f$ exists.

Boundary Conditions on $\boldsymbol{B}$ and $\boldsymbol{H}$

We get these in the same way we got the conditions on $\boldsymbol{E}$ and $\boldsymbol{D}$.

$\nabla \cdot \boldsymbol{B} = 0$ is equivalent to $\oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0$. So if we draw a small box around a surface and look at the flux through that box, only the flux through the top and the bottom matter if we make the box thin enough. And so,

$$\oint \boldsymbol{B} \cdot d\boldsymbol{A} = B_{\perp,\text{above}} A - B_{\perp,\text{below}} A = 0 \implies \boxed{B_{\perp,\text{above}} = B_{\perp,\text{below}}}$$

So the normal component of $\boldsymbol{B}$ is continuous across surfaces.

$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$ is equivalent to $\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_o I_{\text{enc}}$.

Draw an Amperian loop enclosing a surface and make it very thin. Crank out what we've done before. Then

$$\begin{aligned} &B_{\parallel,\text{above}} L - B_{\parallel,\text{below}} L = \mu_o K L \\ &B_{\parallel,\text{above}} \cancel{L} - B_{\parallel,\text{below}} \cancel{L} = \mu_o K \cancel{L} \\ &\implies B_{\parallel,\text{above}} - B_{\parallel,\text{below}} = \mu_o K. \end{aligned}$$

But we actually have to be a bit more careful this time, because only surface currents that pass through the loop count, which is to say only surface currents that are perpendicular to the loop face and thus to the parallel B are being considered. Orientation matters. We respect this by writing

$$\boxed{\boldsymbol{B}_{\parallel,1} - \boldsymbol{B}_{\parallel,2} = \mu_o \boldsymbol{K} \times \hat{\mathbf{n}}}$$

where $\boldsymbol{n}$ is the normal vector to the surface. We're coming at this slightly back-ackwards. Instead of trying to write the component of $\boldsymbol{K}$ that goes through the loop, we're looking at the component of $\boldsymbol{K}$ that when crossed with $\hat{\mathbf{n}}$ is parallel to the B's under consideration. That is the component that matters.

For $\boldsymbol{H}$ it's much the same:

$$\nabla \cdot \boldsymbol{H} = \nabla \cdot \left(\frac{\boldsymbol{B}}{\mu_o} - \boldsymbol{M} \right) = 0$$

as long as $\nabla \cdot \boldsymbol{M} = 0$. This is usually true (it may be possible to rig an exception with ferromagnetic materials).

$$\text{So usually} \qquad \boldsymbol{H} \cdot \boldsymbol{H} = 0 \implies \boxed{H_{\perp,\text{above}} - H_{\perp,\text{below}} = 0}$$

$$\text{And } \nabla \times \boldsymbol{H} = \boldsymbol{J}_f \implies \boxed{H_{\parallel,1} - H_{\parallel,2} = \boldsymbol{K}_f \times \hat{\mathbf{n}}}$$

Full boundary conditions (in statics) summarized

For E	$E_{\perp,1} - E_{\perp,2} = \sigma/\epsilon_o$, $E_\parallel$ is continuous
For B	$B_\perp$ is continous, $B_{\parallel,1} - B_{\parallel,2} = \mu_o \boldsymbol{K} \times \hat{\mathbf{n}}$
For D	$D_{\perp,1} - D_{\perp,2} = \sigma_f$, $D_\parallel$ is continous if $\nabla \times \boldsymbol{P} = 0$
For H	$H_\perp$ is continous if $\nabla \cdot \boldsymbol{M} = 0$. $H_{\parallel,1} - H_{\parallel,2} = \boldsymbol{K}_f \times \hat{\mathbf{n}}$

You could memorize all of these and it wouldn't be a waste of time, but they all come from nearly identical derivations (one of two, anyway)

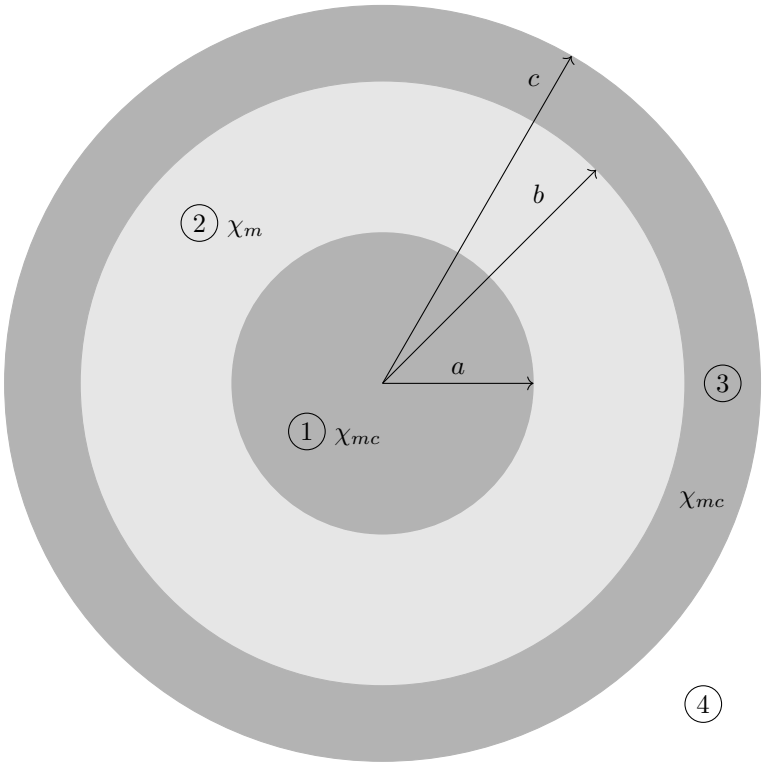
$\nabla \cdot$ (field) equations lead to conditions on field$_\perp$
$\nabla \times$ (field) equations lead to conditions on field$_\parallel$

This is a mathematical and totally general result.

Lecture 25: Examples, Including the Magnetic Scalar Potential

One problem we've solved in the past is the electric field in a coaxial cable with a dielectric spacer. Let's solve the analogous problem in magnetics.

Take a coaxial cable. The interior wire has radius a. The return sheath has inner and outer radii b and c, respectively. The gap is filled with an insulator with magnetic susceptibility χ_m. The conductor has susceptibility χ_{mc}.



If there's a current I, we have free current densities

$$\begin{cases} J_{f,\text{ in}} = \dfrac{I}{\pi a^2}(+\hat{\mathbf{k}}) & \text{for } r < a \\ J_{f,\text{ out}} = \dfrac{I}{\pi\left(c^2 - b^2\right)}(-\hat{\mathbf{k}}) & \text{for } b < r < c \end{cases}$$

Note that the $\boldsymbol{J}$ vectors are in opposite directions.

We can't use $\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$, because $\boldsymbol{J}$ includes bound currents, so we instead work from $\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$ or $\oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_{f,\text{ enc}}$.

To find $\boldsymbol{H}$ we draw an Amperian loop in one of the four regions:

(1) Inside the interior wire:

$$\oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_f \implies H\left(2\pi r\right) = J_{f,\text{ in}}\left(\pi r^2\right) = \frac{I\, r^2}{a^2}$$

$$\implies \boxed{\boldsymbol{H} = \frac{I\, r}{2\pi a^2}\hat{\boldsymbol{\phi}}, \qquad \text{for } r < a.}$$

(2) Inside the insulator:

Note that $I_{\text{enc}} = I$, so

$$\boxed{\boldsymbol{H} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}}, \qquad \text{for } a < r < b.}$$

(3) Inside the return sheath:

We enclose some current going in both directions, so

$$I_{\text{enc}} = I - J_{f,\text{ out}}\left(\pi r^2 - \pi b^2\right) = I - \frac{I\left(r^2 - b^2\right)}{c^2 - b^2} = I\left[\frac{c^2 - b^2}{c^2 - b^2} - \frac{r^2 - b^2}{c^2 - b^2}\right]$$

$$\implies \boxed{\boldsymbol{H} = \frac{I\left(c^2 - r^2\right)}{2\pi r\left(c^2 - b^2\right)}\hat{\boldsymbol{\phi}}, \qquad \text{for } b < r < c.}$$

(4) Outside the coaxial cable:

We now enclose all the current in both directions so $I_{\text{enc}} = 0$.

$$\implies \boxed{\boldsymbol{H} = \boldsymbol{0}, \qquad \text{for } r > c.}$$

Note that throughout this problem we've been taking advantage of the cylindrical symmetry of the situation and the knowledge that $\boldsymbol{H}$ has to point in the $\hat{\boldsymbol{\phi}}$ direction.
So let's find everything else. We know $\boldsymbol{B} = \mu\boldsymbol{H}$, so

$$\boldsymbol{B} = \begin{cases} \dfrac{\mu_o\left(1 + \chi_{mc}\right) Ir}{2\pi a^2}\hat{\boldsymbol{\phi}} & (1) \\ \dfrac{\mu_o\left(1 + \chi_m\right) I}{2\pi r}\hat{\boldsymbol{\phi}} & (2) \\ \dfrac{\mu_o\left(1 + \chi_{mc}\right) I\left(c^2 - r^2\right)}{2\pi r\left(c^2 - b^2\right)}\hat{\boldsymbol{\phi}} & (3) \\ 0 & (4) \end{cases}$$

And the magnetization $\boldsymbol{M}$ is given by $\boldsymbol{M} = \chi\boldsymbol{H}$, so

$$\boldsymbol{M} = \begin{cases} \dfrac{\chi_{mc} Ir}{2\pi a^2}\hat{\boldsymbol{\phi}} & (1) \\ \dfrac{\chi_m I}{2\pi r}\hat{\boldsymbol{\phi}} & (2) \\ \dfrac{\chi_{mc} I\left(c^2 - r^2\right)}{2\pi r\left(c^2 - b^2\right)}\hat{\boldsymbol{\phi}} & (3) \\ 0 & (4) \end{cases}$$

We can get the bound current densities $\boldsymbol{J}_b$ and $\boldsymbol{K}_b$ from the magnetization $\boldsymbol{M}$.
What should be the net bound current?

The magnetic scalar potential

You may recall that $\nabla \times \boldsymbol{E} = \boldsymbol{0}$ implies $\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$, which in turn implies that $\int \boldsymbol{E} \cdot d\boldsymbol{\ell}$ is path-independent and we can use it to build a scalar potential function V such that $\boldsymbol{E} = -\nabla V$.

This doesn't work so well for the magnetic field since $\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$ and there's usually non-zero $\boldsymbol{J}$ if we're bothering to talk about $\boldsymbol{B}$. But there's a loophole.
We have that $\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$ and sometimes $\boldsymbol{J}_f$ is zero when $\boldsymbol{J}_b$ isn't. So $\nabla \times \boldsymbol{B} \neq \boldsymbol{0}$ but $\nabla \times \boldsymbol{H} = \boldsymbol{0}$ and we have a legitimate situation where we can use a scalar potential for $\boldsymbol{H}$:

$$\boldsymbol{H} = -\nabla \phi_m \qquad \text{This works anytime } \boldsymbol{J}_f \text{ is zero.}$$

But why do we bother? What does this buy us? Well, $\nabla \cdot \boldsymbol{H} = \nabla \cdot (\mu \boldsymbol{B})$. But Maxwell's equations tell us magnetic fields don't have divergence ($\nabla \cdot \boldsymbol{B} = 0$). So if μ is spatially uniform,

$$\nabla \cdot \boldsymbol{H} = 0.$$

Substituting the scalar potential for $\boldsymbol{H}$, we get

$$\nabla^2 \phi_m = 0 \qquad \begin{array}{c}\text{This is a good thing. We know all kinds of} \\ \text{dirty tricks for solving this equation.}\end{array}$$

As Pollack says, we can "bring to bear" the "considerable mathematical armamentarium" we have for such problems. In other words, we just made magnetostaics look like electrostatics (if $\boldsymbol{J}_f = 0$).
This really works, too. Check out an example:
A sphere of radius a made of a material with susceptibility chi_m is placed in a uniform magnetic field $\boldsymbol{B} = B_o \hat{\mathbf{k}}$.
There is no $\boldsymbol{J}_f$, so we can solve $\nabla^2 \phi_m = 0$ and go nuts. Which the book does. But you know what's better? We already solved this problem.
Back in chapter 6 we did a polarizable sphere in a uniform $\boldsymbol{E}$-field. This is structurally identical. And what does a physicist do when they see a problem they've already solved? The write down the answer. From page 210:

$$\begin{cases} V_{\text{inside}} = -C_1 r \cos\theta \\ \\ V_{\text{outside}} = -E_o r \cos\theta + \dfrac{C_2 a^3}{r^2} \cos\theta \end{cases} \qquad \text{with} \quad C_1 = \left(\frac{3}{\kappa + 2}\right) E_o \quad \text{and} \quad C_2 = \left(\frac{\kappa + 1}{\kappa + 2}\right) E_o$$

Now we just fix it up a little. First off, κ is the dielectric "constant," $\kappa = \dfrac{\epsilon}{\epsilon_o}$. So we replace it with $\kappa_m = \frac{\mu}{\mu_o}$.
And in electrostatics, $\boldsymbol{E} = -\boldsymbol{\nabla V}$, but in magnetostatics, $\boldsymbol{H} = -\nabla \phi_m$. As such, we replace E_o with H_o, not B_o. Using $H_o = \dfrac{B_o}{\mu_o}$, we get

$$\phi_{m,\text{ in}} = \underbrace{\frac{-3B_o}{(\kappa_m + 2)\,\mu_o}}_{C_1} r\ \cos\theta \qquad \text{and} \qquad \phi_{m,\text{ out}} = \frac{-B_o r\ \cos\theta}{\mu_o} + \underbrace{\left(\frac{\kappa_m - 1}{\kappa_m + 2}\right)\frac{B_o}{\mu_o}}_{C_2} \frac{a^3\ \cos\theta}{r^2}.$$

Just like that. Now, there are some major physical differences between electrostatics and magnetostatics. In the former, $\kappa \geqslant 1$ (always), but in the latter κ_m can be less than 1.
Also, in electostatics, polarizing an object with $\kappa > 1$ leads to a charge separation that opposes the applied field. But in magnetostatics, magnetizing an object with $\kappa_m > 1$ enhances the applied field.
Let's see how this plays out in the above example for the interior electric and magnetic fields.
For the electric field, we have

$$V_{\text{inside}} = \frac{-3E_o}{\kappa + 2} r\ \cos\theta.$$

Taking the negative gradient gives

$$\begin{aligned} \boldsymbol{E}_{\text{inside}} &= -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} \\ &= \frac{3E_o}{\kappa + 2}\cos\theta\ \hat{\mathbf{r}} - \frac{3E_o}{\kappa + 2}\sin\theta\ \hat{\boldsymbol{\theta}} \end{aligned}$$

Using $\hat{\mathbf{k}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}$ gives

$$\boxed{\boldsymbol{E}_{\text{inside}} = \frac{3E_o}{\kappa + 2}\hat{\mathbf{k}}}$$

For the magnetic field, we know that

$$\boldsymbol{H}_{\text{inside}} = \frac{3H_o}{\kappa_m + 2}\hat{\mathbf{k}}.$$

Using $\boldsymbol{B} = \mu\boldsymbol{H}$, we get

$$\underbrace{\frac{\boldsymbol{B}_{\text{inside}}}{\mu}}_{\boldsymbol{H}_{\text{inside}}} = \frac{3\overbrace{(B_o/\mu_o)}^{H_o}}{\kappa_m + 2}\ \hat{\mathbf{k}}$$

Since $\kappa_m = \dfrac{\mu}{\mu_o}$,

$$\boxed{\boldsymbol{B}_{\text{inside}} = \frac{3\kappa_m B_o}{\kappa_m + 2}\ \hat{\mathbf{k}}}$$

Sketching these out, we see.
Strongly diamagnetic materials significantly exclude applied $\boldsymbol{B}$-fields. Superconductors have $\kappa_m = 0$ and repel $\boldsymbol{B}$ completely, much as conductors have $\kappa \to \infty$ so that $\boldsymbol{E}_{\text{inside}} = \mathbf{0}$.
Demo: Pyrolytic graphite, a form of graphite that consists of sheets of graphene (planar arrangemetns of carbon atoms).
It is more diamagnetic at room temperature than any other known material.
A semi-classical explanation of this is that the carbon rings allow for strong induced currents when a $\boldsymbol{B}$-field is brought near. By Lenz's law, those currents will be in such a way as to create $\boldsymbol{B}$-fields that oppose the change in $\boldsymbol{B}$ that induced the currents, thereby leading to the possibility of levitation.

Magnetic Shielding

I probably won't have to work very hard to convince you that sometimes you'll want to keep external magnetic fields away from certain objects or devices. Let's look at the principle of magnetic shielding.
Consider a spherical shell of material with permeability μ and inner and outer radii R_i and R_o, respectively. We apply a magnetic field of the form $\boldsymbol{B} = B_o\hat{\mathbf{k}}$. There are no free currents, so we can define $\boldsymbol{H} = -\nabla\phi_m$. The external $\boldsymbol{H}$ is $\dfrac{\boldsymbol{B}}{\mu_o}$, so

$$\phi_{m,\text{ external}} = -\frac{B_o\ z}{\mu_o} = -\frac{B_o}{\mu_o}r\ \cos\theta.$$

We've solved problems like this before. Basically, we looked at the solution to $\nabla^2 V = 0$ in spherical and pulled out the terms proportional to $\cos\theta$. Those went like $r\ \cos\theta$ and $\dfrac{\cos\theta}{r^2}$.
For $r < R_i$, we exclude $\frac{\cos\theta}{r^2}$, since that blows up at $r = 0$:

$$\phi_{m,\text{ in}}(r,\theta) = \alpha r\ \cos\theta.$$

For $R_i < r < R_o$, we keep both terms:

$$\phi_{m,\text{ mid}}(r,\theta) = \gamma r\ \cos\theta + \frac{\delta\ \cos\theta}{r^2}.$$

For $r > R_o$, we know the $r\ \cos\theta$ term must match the applied potential at large r, so

$$\phi_{m,\text{ out}}(r,\theta) = -\frac{B_o}{\mu_o}r\ \cos\theta + \beta\frac{\cos\theta}{r^2}$$

In summary:

$$\phi_m = \begin{cases} \alpha r\ \cos\theta & r < R_i \\ \gamma r\ \cos\theta + \dfrac{\delta\ \cos\theta}{r^2} & R_i < r < R_o \\ -\dfrac{B_o}{\mu_o} r\ \cos\theta + \beta\dfrac{\cos\theta}{r^2} & r > R_o \end{cases}$$

Four constants to find: α, β, γ, and δ. We need to invoke four boundary conditions. There are many possible choices; as always there's a bit of an art to choosing easy ones. I, for one, like to use the continuity of the potential whenever possible. The book seems to agree.

$$\phi_{m,\ \text{in}}(R_i) = \phi_{m,\ \text{mid}}(R_i) \implies \alpha R_i = \gamma R_i + \frac{\delta}{R_i^2}$$

$$\phi_{m,\ \text{mid}}(R_o) = \phi_{m,\ \text{out}}(R_o) \implies \gamma R_o + \frac{\delta}{R_o^2} = -\frac{B_o}{\mu_o} + \frac{B_o}{R_o^2}$$

And we can get two more equations from the continuity of $B_\perp$:

$$\mu_o \frac{\partial \phi_{m,\ \text{in}}(R_i)}{\partial r} = \mu \frac{\partial \phi_{m,\ \text{mid}}(R_i)}{\partial r} \qquad \text{and} \qquad \mu_o \frac{\partial \phi_{m,\ \text{mid}}(R_o)}{\partial r} = \mu \frac{\partial \phi_{m,\ \text{out}}(R_o)}{\partial r}$$

We end up with four linear equations in four unknowns. Matrix mojo ensues. The complete list of coefficients is in the book. We'll just look at α since its pertinent to B_{inside}:
For large κ_m, the denominator is proportional to
The more permeable the material, the weaker the field in the shielded region. There exist materials engineered to have very high κ_m, such as the alloy μ-meta (Fe, Ni, Cu, Cr), with κ_m up to 10^5.
Note that this has applications not just in keeping $\boldsymbol{B}$-fields out of a place, but also in getting them from one place to another (see transformer design, for example).

Lecture 26: Diamagnetism, Paramagnetism, and Ferromagnetism

Lecture 27: Motional EMF and Faraday's Law I

"Mr. Faraday, of what use are your discoveries?"
"Madam, of what use is a baby?"

Michael Faraday

We're now moving into full-blown electrodynamics, where there will almost always be some time-dependence somewhere. We need to start with a little vocabulary:

Electromotive Force (emf)

A measure of how much work a system is doing on a unit charge. Has units of volts (V), (energy/charge).
Physically, very much like voltage, though "voltage" or "electric potential" tends to be used in situations where a scalar potential can be defined.

Since work is $\int \boldsymbol{F} \cdot d\boldsymbol{\ell}$ and the Lorentz force law is $\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{E})$, emf will be of the form $\int \frac{\boldsymbol{F}}{q} \cdot d\boldsymbol{\ell} = \int (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}$. Generally, we will focus on either electric or magnetic forces. We've seen Faraday's law written two different ways in Phys 200:

$$\oint (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int \boldsymbol{B} \cdot d\boldsymbol{A} \qquad \text{and} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

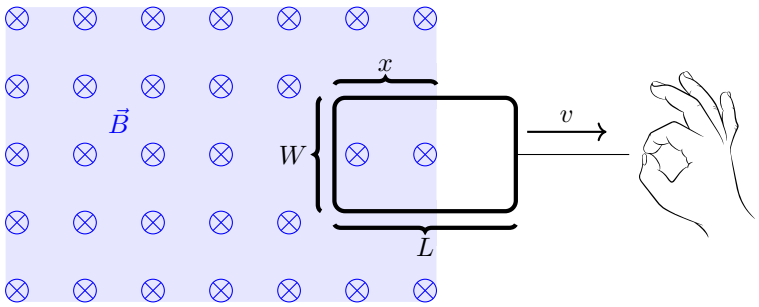
And we're going to add a thrid:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Let's work through a few different situations and see what kind of insight we can obtain.

EMF from a static $\boldsymbol{B}$-field

Let's suppose we have a $\boldsymbol{B}$-field that occupies some region of space. It's uniform, with magnitude B, into the page. Everywhere else B is zero. (This field is actually unphysical — do you see why?)
We also have a wire loop with dimensions W and L partially in the field. We're pulling it out with some speed v.



The charge carriers in the wires (let's call them positive for convenience) feel an emf due to the Lorenz force:

$$\mathcal{E} = \oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}$$

Now, the right-hand-rule shows us that only charges in the leftmost wire segment are acted on (only there is $\boldsymbol{v} \times \boldsymbol{B}$ parallel to $d\boldsymbol{\ell}$.

This implies

$$\mathcal{E} = vBW.$$

Let's look at the change in magnetic flux, $\dfrac{d\Phi}{dt}$, also. Note that $\Phi = \displaystyle\int \boldsymbol{B} \cdot d\boldsymbol{A}$, or in this case, $\Phi = BWx$. Thus

$$\frac{d\Phi}{dt} = \frac{d}{dt}(BWx) = BW \underbrace{\frac{dx}{dt}}_{v}$$

So we find that

$$\boxed{\mathcal{E} = \oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}}$$

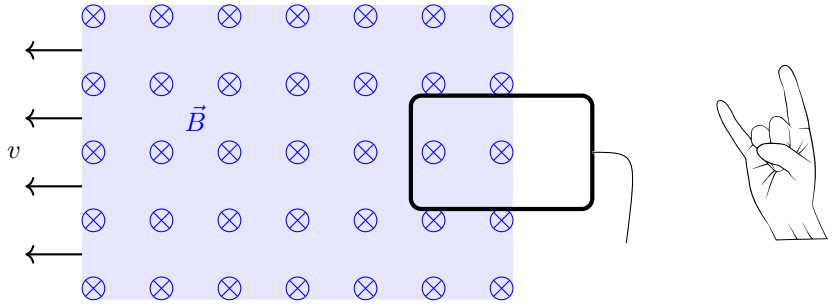
(What's that sign about?)
You'll notice there is no $\boldsymbol{E}$ anywhere in the problem.
This turns out to be a general result for any moving loop in a $\boldsymbol{B}$-field. You can prove it using $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. To emphasize for future use:

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{and} \qquad \boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$
$$\implies \mathcal{E} = -\frac{d\Phi}{dt}$$

With no electric field, and being derivable from existing principles, this is not Faraday's Law (not when the emf comes from a strictly magnetic force). But it is quite closely related. And you'll get the same answers for straight wire, no loop.

Electromagnetic Induction and Faraday's (actual) Law

Now, what of the reciprocal situation where the wire loop is stationary and the region occupied by the field is moving, thereby providing a time-varying $\boldsymbol{B}$-field?



We cannot derive what happens from what we currently know. We need a brand new, empirically obtained law. Enter Faraday's law in differential form:

$$\boxed{\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}}$$

This law tells us that E-fields have another source besides charges: time-varying magnetic fields

This law will let us figure out what happens in any situation where the magnetic field is changing, thus inducing a curly emf such that

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} \neq 0$$ And so we come full circle.

We have to proceed carefully, though. Many people (and books) screw up what follows as we try to get Faraday's law into integral form. First, integrate both sides over some open surface:

$$\int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

Apply Stokes Theorem on the left

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

Right-hand side is a spatial integral only, so we can pull out $\frac{\partial}{\partial t}$ and make it a total derivative

$$\boxed{\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int \boldsymbol{B} \cdot d\boldsymbol{A}}$$

That seemed simple enough. And it's true that $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ and $\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\dfrac{d}{dt}\int \boldsymbol{B} \cdot d\boldsymbol{A}$ are equivalent if and only if $\boldsymbol{B}$ is the only thing with a time dependence.
What happens if we go back to the first situation, where the magnetic field isn't changing but the loop is moving? Start with:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Integrate both sides and apply Stokes' Theorem on the left

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

Now we don't pull out $\frac{\partial}{\partial t}$. Why not?

So what we have here is that $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ and the above are equivalent with no restrictions whatsoever. But we can't necessarily get that $\dfrac{\partial}{\partial t}$ out if the area is changing, because the bounds of the integral are time dependent!

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int_{s(t)} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

(2) Note that this does give a consistent result in the case of the moving loop, since $\frac{\partial \boldsymbol{B}}{\partial t} = 0$ and $\boldsymbol{E} = 0$ there.

But can we do better? What can we do with that $\displaystyle\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$ term? Let's look at

$$-\frac{d}{dt}\int_{s(t)} \boldsymbol{B} \cdot d\boldsymbol{A}$$

and see if we can relate it. You take the derivative of functions whose bounds have variables using the Leibniz rules. You've seen one in one-dimension:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t) + \int_a^b \frac{\partial}{\partial t} f(x,t)\, dx.$$

It's got a vaguely product/chain rule flavor to it, where to apply the derivative completely, we need to hit the b-endpoint, the a-endpoint and the integrand. This generalizes to to vector fields instead of scalar functions, but not cleanly. Letting $\boldsymbol{B}$ be our vector field, the three-dimensional Leibniz rule gives:

$$\frac{d}{dt}\int \boldsymbol{B}(\boldsymbol{x},t) \cdot d\boldsymbol{A} = \int \left(\frac{\partial \boldsymbol{B}}{\partial t} + (\nabla \cdot \boldsymbol{B})\, \boldsymbol{v}\right) \cdot d\boldsymbol{A} - \oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell},$$

where $\boldsymbol{v}$ comes from the time derivative of the spatial coordinates defining the are (so, for example, the velocity of the wire loop). Recall that $\nabla \cdot \boldsymbol{B} = 0$, so we can find that

$$-\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A} = -\oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} - \frac{d}{dt}\int \boldsymbol{B}(\boldsymbol{x},t) \cdot d\boldsymbol{A}$$

Substituting this into (2) yields

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} - \frac{d}{dt} \int \boldsymbol{B} \cdot d\boldsymbol{A}$$

$$\implies \boxed{\oint (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int \boldsymbol{B} \cdot d\boldsymbol{A}}$$

Which is true generally. It's basically a version of $\mathcal{E} = -\frac{d\Phi}{dt}$ that shows explicitly how $\boldsymbol{E}$ and $\boldsymbol{B}$ can both contribute to the emf.

Five forms of Faraday's law

$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{E}}{\partial t}$	Faraday's law proper, in differential form. Relates time-varying $\boldsymbol{B}$-fields and curly $\boldsymbol{E}$-fields at single points in space.
$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$	Integral form of the above. Always true. Relates $\boldsymbol{E}$ and $\boldsymbol{B}$ over entire regions.
$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int \boldsymbol{B} \cdot d\boldsymbol{A}$	True only if the loop in question is entirely static. If misapplied generally, forces you to conclude that there's a curly $\boldsymbol{E}$-field present anytime there's a time-varying flux for any reason. This can lead to some serious substantive errors - it's note mere pedantry.
$\oint (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int \boldsymbol{B} \cdot d\boldsymbol{A}$	Corrected form of the above. Adding a motional emf term fixes it.
$\mathcal{E} = -\frac{d\Phi}{dt}$	A field-agnostic version of "Faraday's law." Says that time-varying magnetic fluxes are associated with voltages. Doesn't actually say time-varying $\boldsymbol{B}$-fields have anything to do with curly $\boldsymbol{E}$-fields. So is it really Faraday's law? Seems like "Faraday's law" has evolved into kind of an umbrella term. And that's okay, if you understand the bits and pieces.

Lecture 28: Denouement