Lecture 6: Conductors and Spherical Symmetry

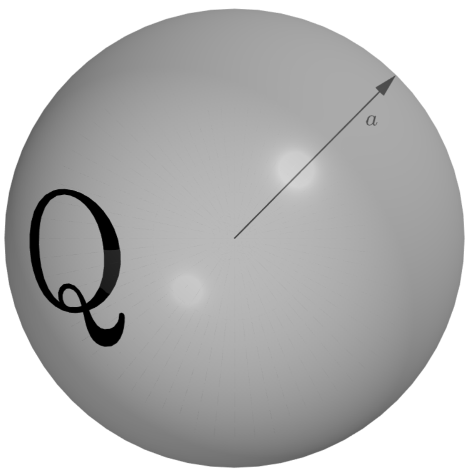
March 12, 2021

More math! Expect this class to pretty much be math phys for a while (at least until we get to chapter 6).

Interlude

Clicker question

Today, we'll solve Laplace's equation a few times in situations with spherical symmetry. We'll start with the easiest possible example: A charged spherical conductor, radius a and total charge Q.



We've done this via Gauss's law and $\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{\ell}$ in intro physics. We'll get E, V, and σ in short order.

$$E(r) = \frac{k\,Q}{r^2} \text{ outside,} \quad E = 0 \text{ inside}$$
$$V(r) = \frac{k\,Q}{r} \text{ outside,} \quad V = \frac{k\,Q}{a} \text{ inside}$$
$$\sigma = \frac{Q}{4\pi a^2}$$

Now let's do the same thing with Laplace's equation. We need to shart from conditions on V. The boundary conditions are

$$\begin{cases} V(a) = V_o \\ \\ V(\infty) = 0 \end{cases}$$

We have spherical symmetry, so $\nabla^2 V = 0$ becomes $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial V}{\partial r}\right) = 0$.

The mixed r's, $\frac{1}{r}$'s, and $\frac{\partial}{\partial r}$'s suggest a polynomial solution or maybe an exponential. Let's guess polynomial first, so our ansatz is $V(r) = r^p$. Substituting this into Laplace's equation yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\,(r^p)}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(p r^{p+1}\right) = p\,(p+1)\,r^{p+2} = 0$$

If this is to hold for all r (and it should), then it must be the case that $p = 0$ or $p = -1$. Then our solution becomes

$$V(r) = \frac{A}{r} + B$$

We created a solution with two free parameters for a problem with two BCs. We should all stop and feel happy for a second.

Now let's apply the boundary conditions. The requirement that the voltage be zero at infinty gives

$$V(\infty) = 0 \qquad \Longrightarrow \qquad B = 0.$$

The voltage at the radius of the sphere is fixed at V_o. This gives

$$V(a) = \frac{A}{a} = V_o \qquad \Longrightarrow \qquad A = V_o a.$$

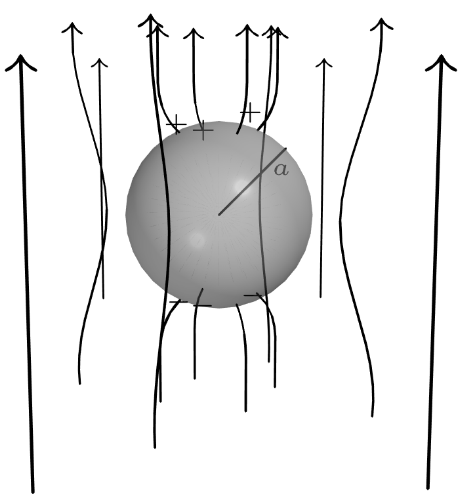
So we have

$$V(r) = \begin{cases} V_o, & r < a, \\ \\ \frac{V_o a}{r}, & r \geqslant a \end{cases} \qquad \text{and} \qquad \boldsymbol{E}(r) = \begin{cases} 0, & r < a, \\ \\ \frac{V_o a}{r^2}, & r \geqslant a \end{cases}.$$

(Where we obtained $\boldsymbol{E}$ via the relationship $\boldsymbol{E} = -\nabla^2 V$). Finally, we can get the charge density σ from the boundary conditions.

This is the process of solving Laplace's equation, from beginning to end.

Now let's spice it up: Take a conducting sphere at some fixed voltage (let's call it $V = 0$, so that it's grounded), and stick it in a uniform electric field. There'll be a charge separation on the sphere, and the solution will now depend on r and θ. The electric field will probably also be altered in response to the redistributed charges.



Let's hit this with Laplace's equation. And for that we need boundary conditions. At large r (*i.e.* far away from the conducting sphere) the electric field points straight up:

$$\boldsymbol{E} = E_o \hat{\boldsymbol{k}} \qquad \Longrightarrow \qquad V(r,\theta) = -E_o \underbrace{r\cos\theta}_{z}$$

We need building blocks to construct our solution. What we used before, $V(r) = \frac{A}{r} + B$, is inadequate, because there's no θ dependence.

There existst an infinite collection of solutions to Laplace's equation. There are known as the *spherical harmonics*. The book spots us a couple fo the simplest ones involving r and $\cos\theta$, allowing us to construct a trial V:

$$V(r,\theta) = \frac{A}{r} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

We have four undetermined constants and two boundary conditions. We need to invoke some physics knowlege to make two of those constants vanish.

First, V is arbitrary up to a constant, so $B = 0$. We suspect A is also zero, though we can't prove it solidly without techniques from chapter 5. (Recall that A/r is the potential of a uniformly charged sphere, so it's probably not necessary to build up our solution.) The cool thing is that we can just *guess* $A = 0$, and if we get it to work, uniqueness say's we're right!

So let's work with $V(r,\theta) = \dfrac{C\ \cos\theta}{r^2} + D\ r\cos\theta$. Now invoke the boundary condition from earlier: at large r, $V(r,\theta) = -E_o r\cos\theta$. Using our ansatz, this gives us

$$V(r,\theta) = \cancelto{0}{\frac{C\ \cos\theta}{r^2}} + Dr\cos\theta \overset{\text{set}}{=} -E_o r\cos\theta \qquad \Longrightarrow \qquad D = -V_o$$

That means, we so far have $V(r,\theta) = \dfrac{C\ \cos\theta}{r^2} - E_o r\cos\theta$. Now we also know that on the surface of the sphere (that is, at $r = a$) the voltage is fixed at zero. This gives us

$$V(a,\theta) = \frac{C\ \cos\theta}{a^2} - E_o a\cos\theta = 0 \qquad \Longrightarrow \qquad C = E_o a^3$$

And so we get

$$V(r,\theta) = -E_o r\cos\theta + \frac{E_o a^3\cos\theta}{r^2}$$

We can get the field components from $-\nabla V$:

$$\begin{cases} E_r(r,\theta) = E_o\left(1 + \frac{2a^3}{r^3}\right)\cos\theta \\ E_\theta(r,\theta) = -E_o\left(1 - \frac{a^3}{r^3}\right)\sin\theta. \end{cases}$$

It would be a fun exercise to make sure these satisfy the generic boundary conditions for $E^{\parallel}$ and $E^{\perp}$. A fun exercise *for you*

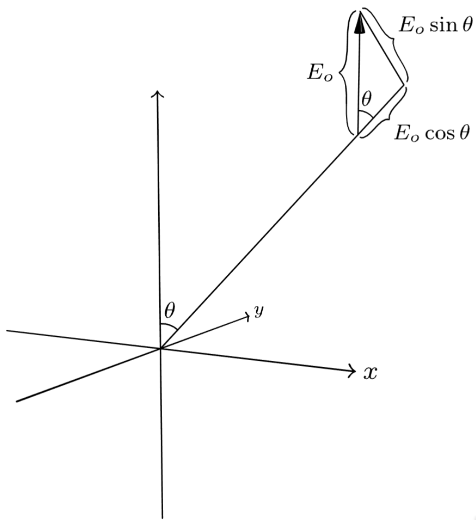
As $r \to \infty$, we get

$$\begin{cases} E_r = E_o\cos\theta \\ E_\theta = -E_o\sin\theta \end{cases}$$

Solid so far. And if we go to find σ, we get

$$\sigma = 3\epsilon_o E_o\cos\theta,$$

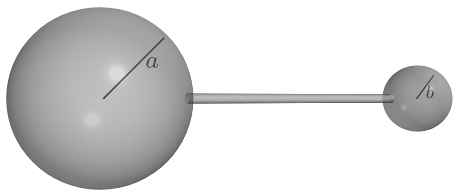
which, when inegrated over the whole sphere, gives zero.



A partial generalization to arbitrary objects.

When things aren't spheres, math gets hard (or easy, if you're good at numerical analysis). But here's a general principle that's pretty handy.

Consider two connected, charged conducting spheres of radii a and b.



If these are reasonably well separated,

$$V_a(a) = \frac{Q_a}{4\pi\epsilon_o a} = \frac{Q_b}{4\pi\epsilon_o b} = V_b(b),$$

and note that $\sigma_a = \frac{Q_a}{4\pi a^2}$. Then

$$\frac{Q_a}{4\pi a^2 b} = \frac{Q_b}{4\pi b^2 a} \implies \frac{\sigma_a}{b} = \frac{\sigma_b}{a} \qquad \implies \frac{\sigma_a}{\sigma_b} = \frac{b}{a}.$$

And electric field is proportional to surface charge density, so places with small radii of curvature (*i.e.* pointy bits) exhibit strong E-fields. This has a variety of practical applications and interesting manifestations.