Lecture 5: Conductors, Capacitors, and the Method of Images

Coulomb's law gives us a recipe for finding the electric field anywhere in space given a static distribution of charge. This includes charges on insulating material.

But what if we add conductors? Charges can shift around and so now we don't know $\rho(\boldsymbol{x'})$ everywhere ahead of time. What now?

Now, we learn how to solve Poisson's equation, $-\nabla^2 V = \dfrac{\rho}{\epsilon_o}$. A lot. Basically all of chapters 4 and 5.

First, some *properties of conductors*:

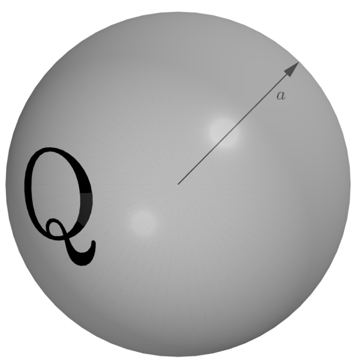
(1) (Some) charges are free to move.

(2) Charges exist in vast numbers, even if there's no net charge (that is, $Q_{\text{net}} = 0$).

(3) There is zero electric field ($\boldsymbol{E} = \boldsymbol{0}$) in a conductor in electrostatic equilibrium. This takes very little time, $\sim 10^{-19}$ s in copper, about zero for all but the very fastest processes like x-ray radiation.

   (3a) Voltages in conductors are constant. That is, $\displaystyle\int \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$ for any path in a conductor.

(4) The charge density is zero ($\rho(\boldsymbol{x}) = 0$) inside a conductor. Follows from Gauss's law:



$\boldsymbol{E} = \boldsymbol{0}$ in a conductor, so $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o} = 0$.

   (4a) Net charges therefore reside on the surface of conductors.

Interlude

Clicker question

(5) At the surface of a conductor, $E^{\parallel} = 0$ and $E^{\perp} = \frac{\sigma}{\epsilon_o}$. This is a special (and the most relevant) case of our old boundary condition.

Note this is absolute and very powerful: If we know σ at the surface of a conductor, we automatically know $\boldsymbol{E}$, and vice versa.

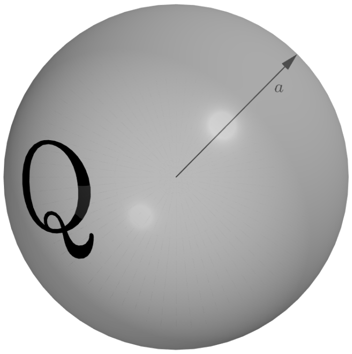
This is all stuff we can infer without much in the way of mathematical hardware. But to move on we need more.

$$\left.\begin{array}{ll} \text{Poisson's equation:} & \nabla^2 V = -\frac{\rho}{\epsilon_o} \\ & \\ \text{Laplace's equation:} & \nabla^2 V = 0 \end{array}\right\}$$

And these have *unique* solutions, as long as we know all the boundary conditions — just like we got used to in Differential Equations.

We can use this to prove something I stated as fact in intro physics.

(6) Given any chunk of conductor with any kind of hollow in electrostatic equilibrium, $\boldsymbol{E} = \boldsymbol{0}$ in the cavity.



Proof. We know that $\nabla^2 V = 0$ inside the cavity. The edges of the cavity are at some constant potential, say V_o (a boundary condition). Guess a solution: $V = V_o$ throughout the cavity.

This certainly satisfies $\nabla^2 V = 0$ and the boundary condition, so it is a solution to Laplace's equation. But Laplace's equation only admits *unique* solutions. So it must be the case that $V = V_o$ in the cavity (and hence $\boldsymbol{E} = \boldsymbol{0}$) is *the* solution. ∎

(7) Extending this further, we can discover that even if the conductor has charge, a hollow will have now charge on the interior surface.

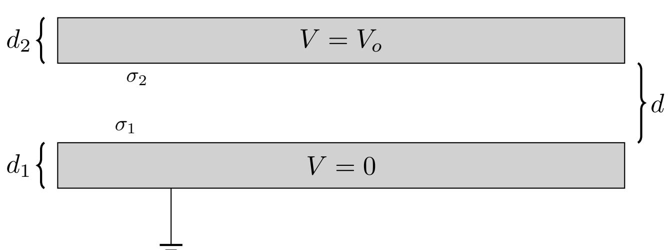
So what makes a conductor a conductor?

1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
19 K	19 Ca	21 Sc	22 Ti	23 Va	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	57-71	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	89-103	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
			57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
			89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr

Theory of conduction (band theory) is strictly quantum mechanical and is coming up soon. Keep an eye out!

Now let's get to the general problem of solving Laplace's equation with conductors.

We'll start with two large [1] square metal plates, the prototypical capacitor



Let's even let the plates have thicknesses, d_1 and d_2.

The symmetry guarantees

$$V = V(z) \quad \text{and} \quad \boldsymbol{E} = E(z)\,\hat{\boldsymbol{k}}.$$

[1] For our purposes, take "large" to mean $\sqrt{\text{Area}} \gg d$.

In between the plates $\nabla^2 V = 0$ applies, so let's guess and check. A constant voltage won't satisfy the boundary conditions, but a linear voltage will, so let's guess

$$V(z) = \frac{V_o}{d} z \quad \left.\vphantom{\frac{V_o}{d}}\right\} \text{This fits the boundary conditions and satisfies } \nabla^2 V = 0 \text{ so it must be the unique solution.}$$

$$\implies \boldsymbol{E} = -\frac{V_o}{d}\hat{\boldsymbol{k}}$$

That was easy. Now, can we infer σ_1 and σ_2, the charges on the interior sheets? Well, for conductors, we have the boundary condition

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_o}\,\hat{\boldsymbol{n}} \quad \text{at an edge, so}$$

$$\boldsymbol{E}_1 = \frac{\sigma_1}{\epsilon_o}\hat{\boldsymbol{k}} = -\frac{V_o}{d}\hat{\boldsymbol{k}} \quad \implies \quad \sigma_1 = -\frac{V_o \epsilon_o}{d}$$

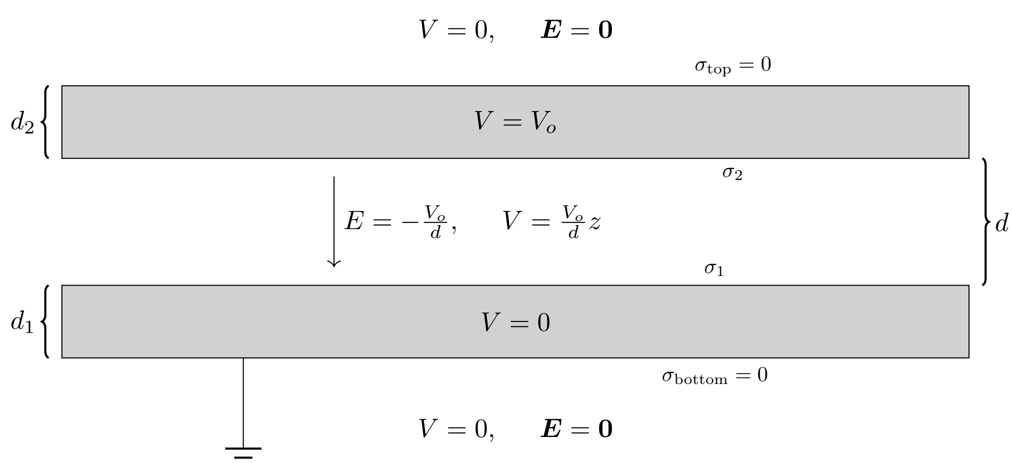
Here's the tricky part: the unit normal $\hat{\boldsymbol{n}}$ once defined, is signed, so for the surface 2 we have that $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{k}}$:

$$\boldsymbol{E}_2 = -\frac{\sigma_2}{\epsilon_o}\hat{\boldsymbol{k}} = -\frac{V_o}{d}\hat{\boldsymbol{k}} \quad \implies \quad \sigma_2 = \frac{V_o \epsilon_o}{d}$$

In the regions above and below the plates, $\nabla^2 V = 0$ admits $V = V_o$ and $V = 0$, respectively, yielding

$$\begin{cases} \boldsymbol{E}_{\text{above}} = \boldsymbol{E}_{\text{below}} = \boldsymbol{0} \\ \sigma_{\text{top}} = \sigma_{\text{bottom}} = 0. \end{cases}$$

Here's an illustration of what we have:



If we define $C = \dfrac{Q}{V}$, we get $C = \dfrac{\epsilon_o A}{d}$ for this parallel plate capacitor. Remember, a capacitor is two distinct chunks of metal. You can use one to store charge (and, by extention, energy).

Calculating capacitance *does* happen sometimes, usually in the context of stray capacitance. There are two primary approaches

(1) Start with Q/V. Calculate V in terms of Q or σ (or vice versa), usually using $\Delta V = \displaystyle\int \boldsymbol{E} \cdot d\boldsymbol{\ell}$ or $\nabla^2 V = \dfrac{\rho}{\epsilon_o}$ (which often becomes $\nabla^2 V = 0$). Divide Q by V. Done.

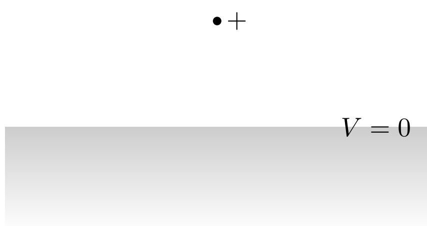
(2) Recall equations for energy stored in a capacitor:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2.$$

Find U via $U = \frac{1}{2}\epsilon_o \int E^2 \; d^3x$. Then set the U's equal and solve for C.

Supposedly there exist cases where method (2) is easier than method (1). I don't know any off the top of my head. It's quite commom that they're of similar difficulty, though.

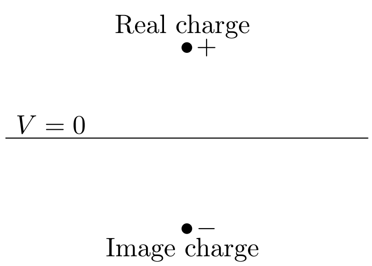
Method of Images



If we add a point charge to a system with a half-inifinite block of metal, we break our symmetry (some of it, anyway) and induce a non-uniform charge distribution in the metal. This sounds bad, but we will again abuse uniqueness to get an easy answer.

At the plane, the potential is constant. Let's call it 0.

We want to solve $-\nabla^2 V = \frac{\rho}{\epsilon_o}$ for $z > 0$ subject to that boundary condition. (We're dealing with a *conducting* block of metal, so $V = 0$ for $z \leqslant 0$.) A different problem that satisfies the same boundary condition is *two* point charges:



We still have $V = 0$ along that conductor.

For $z > 0$, this would give us

$$V(x, y, z) = \frac{q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{x^2 + y^2 + (z - z_o)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + z_o)^2}}\right].$$

This satisfies Poisson's equation for $z > z_o$, because $\frac{1}{\sqrt{x^2+y^2+(z-z_o)^2}}$ is basically $\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|}$, and $\nabla^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|} = 4\pi\delta(\boldsymbol{x})$ (if there is a point charge at $\boldsymbol{x}$.)

Taking $-\nabla$, we end up with

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_o}\left[\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (z - z_o)\,\hat{\boldsymbol{k}}}{\left(x^2 + y^2 + (z - z_o)^2\right)^{3/2}} - \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + (z - z_o)\,\hat{\boldsymbol{k}}}{\left(x^2 + y^2 + (z - z_o)^2\right)^{3/2}}\right].$$

I still can't believe that ackchually werks.

Interlude

Slide of image charge

One last matter: finding σ. We have a boundary condition for that:

$$E_z = \frac{\sigma}{\epsilon_o}, \qquad \text{since the normal direction is } \hat{\boldsymbol{k}}.$$

The problem has rotational symmetry, so rewriting $x^2 + y^2 = r^2$, we get

$$\begin{aligned}\sigma(r) = \epsilon_o E(r,0) &= \epsilon_o \cdot \frac{q}{4\pi\epsilon_o}\left[\frac{-z_o}{\left(r^2+z_o^2\right)^{3/2}} - \frac{z_o}{\left(r^2+z_o^2\right)^{3/2}}\right] \\ &= -\frac{q z_o}{2\pi\left(r^2+z_o^2\right)^{3/2}}\end{aligned}$$

And the total charge is $-q$, as expected. The book shows a few other neat applications of images in situations with planes.

Major take-home point: Look at how versatile and powerful boundary conditions can be.