Lecture 3: Voltage, Energy, and Delta Functions

At this point, we have

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o} \qquad \text{and} \qquad \nabla \times \boldsymbol{E} = \boldsymbol{0},$$

plus two boundary conditions that follow from those:

$$\begin{cases} E_1^{\perp} - E_2^{\perp} = \dfrac{\sigma}{\epsilon_o} \\ E_1^{\|} - E_2^{\|} = 0. \end{cases}$$

We can also derive Coulomb's law from Gauss's law:

$$\boldsymbol{E}(\boldsymbol{x}) = \int \frac{k\ \rho(\boldsymbol{x'})\ d^3x' \quad (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3}.$$

Strictly speaking, this is more than enough to do electrostatics. Knowing charges gives us the fields, which lead to forces via $\boldsymbol{F} = q\boldsymbol{E}$.

But, as you may recall from intro physics, sometimes we prefer to cast things in terms of voltage and energy instead of field and force.

Given some curl-free field ($\nabla \times \boldsymbol{E} = \boldsymbol{0}$ everywhere), we can define some scalar function V such that

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla V(\boldsymbol{x}),$$

which goes by a few different names, including the *voltage*, the *electric potential*, and just the *potential*.

Notably, the potential V carries the same information as the field $\boldsymbol{E}$, but the potential is a bit more pleasant to deal with for being a scalar, and it also hooks into energy pretty directly. We've seen before that

$$\Delta U = q\Delta V.$$

And we've seen that we can construct V according to either

$$\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{\ell}$$ A statement about the difference in voltage between two points

or

$$V(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_o} \int \frac{\rho(\boldsymbol{x'})\ d^3x'}{|\boldsymbol{x} - \boldsymbol{x'}|}$$ A generalization of the potential from a point charge, $V = \dfrac{kq}{r}$.

We can arrange the relationships between all these in a handy little square:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\text{Force} \quad \vec{F} \arrow[r, shift left, "\Delta U = -\int \boldsymbol{F} \cdot d\boldsymbol{\ell}"] \arrow[d, leftrightarrow, "\boldsymbol{F} = q\boldsymbol{E}"'] & U \quad \text{Energy} \arrow[l, shift left, "\boldsymbol{F} = -\nabla U"] \arrow[d, leftrightarrow, "\Delta U = q\Delta V"] \\
\text{Field} \quad \vec{E} \arrow[r, shift right, "\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{\ell}"'] & V \quad \text{Potential} \arrow[l, shift right, "\boldsymbol{E} = -\nabla V"']
\end{tikzcd}
$$

Everything thus far is from intro physics, so let's start adding some new stuff. We have both

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o} \qquad \text{and} \qquad \boldsymbol{E} = -\nabla V.$$

Substituting the latter into the former yields

$$\nabla \cdot \underbrace{(-\nabla V)}_{\boldsymbol{E}} = \frac{\rho}{\epsilon_o} \qquad \Longrightarrow \qquad \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_o}}$$

which is Poisson's equation — the partial differential equation that yields V in electrostatics given a known source ρ.

We will be solving Poisson's equation a lot in the near future, and to solve PDEs, we need boundary conditions. What might be the boundary conditions on V be?

Well, we know

$$\Delta V = V_{\text{above}} - V_{\text{below}} = -\int \boldsymbol{E} \cdot d\boldsymbol{\ell}$$

and real $\boldsymbol{E}$-fields are always finite. Thus, for sufficiently small $d\boldsymbol{\ell}$, $\Delta V \to 0$. In other words, as the path length shrinks to zero, so too does the integral:

$$V_{\text{above}} = V_{\text{below}} \qquad \boxed{\text{So } V \text{ is always continuous.}}$$

That being said, we know there can be discontinuities in $\boldsymbol{E}$. And $\boldsymbol{E}$-fields come about by taking derivatives of V (recall that $\boldsymbol{E} = -\nabla V$). So consider some boundary, and let n denote the direction normal to the boundary (pointing from "below" to "above"). We have the boundary condition on the perpendicular component of the electric field:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_o}$$

Using $E^{\perp} = -\dfrac{\partial V}{\partial n}$, gives us

$$\boxed{\frac{\partial V_{\text{below}}}{\partial n} - \frac{\partial V_{\text{above}}}{\partial n} = \frac{\sigma}{\epsilon_o}}$$

So while V is always continuous, *derivatives* of V aren't necessarily. At least, not the derivative perpendicular to a boundary. Since $E^{\parallel}$ is continuous, so must be the derivative of V in any direction parallel to the surface.

Interlude

Clicker question

Energy of a Charge Distribution

What is the work it takes to move a charge q from $\boldsymbol{x} = \boldsymbol{a}$ to $\boldsymbol{x} = \boldsymbol{b}$? We know work is given by $\int \boldsymbol{F} \cdot d\boldsymbol{\ell}$. The electric field exerts a force on the charge on the charge according to $\boldsymbol{F} = q\boldsymbol{E}$. Thus, the minimum force needed to overcome this field is $\boldsymbol{F} = -q\boldsymbol{E}$. It follows that the work done in moving a charge is

$$\text{Work } = \int \boldsymbol{F} \cdot d\boldsymbol{\ell} = -q \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = q\left[V(\boldsymbol{b}) - V(\boldsymbol{a})\right].$$

Assuming we set the potential to be zero at infinity, the work required to bring in a charge from infinity to some location $\boldsymbol{x}$ is

$$\text{Work } = q\left[V(\boldsymbol{x}) - \cancelto{0}{V(\infty)}\right] = qV(\boldsymbol{x}).$$

Imagine bringing a charge q_1 from infinity to the origin in empty space. This takes no work, since there is no field to fight against. However, once at the origin, the charge q_1 sets up a potential according to

$$V = \frac{1}{4\pi\epsilon_o}\frac{q_1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \frac{1}{4\pi\epsilon_o}\frac{q_1}{r}.$$

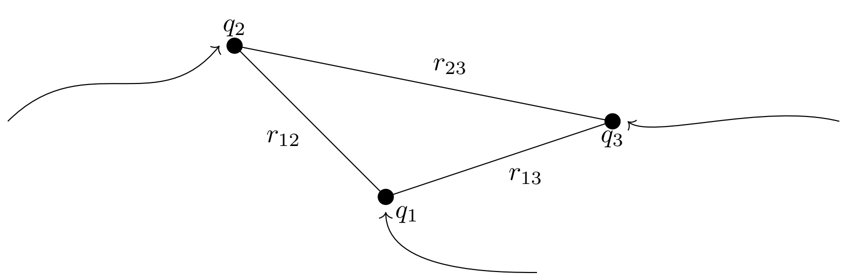
The work done in bringing in another charge q_2 from infinity to some position r is

$$W_2 = q_2 \underbrace{\left(\frac{1}{4\pi\epsilon_o}\frac{q_1}{r}\right)}_{V} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r}$$

This is the energy of interaction U between two charges separated a distance r.

Now we fix q_1 and q_2 in place, and let r_{12} represent the distance between them. The work done in bringing in a third charge is

$$W_3 = q_3 \underbrace{\left(\frac{1}{4\pi\epsilon_o}\right)\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)}_{\text{potential from } q_1 \text{ and } q_2.}$$



The total work in assembling these three charges is

$$W = \frac{1}{4\pi\epsilon_o}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

The work done in assembling a collection of discrete charges is also the energy we'd get if we dismantled the system. That is, it represents the energy stored in the configuration. We can generalize to n number of particles:

$$U = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_o}\right)\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}},$$

where the factor of 1/2 arises because we're double counting each pair. Now, instead of a collection of discrete charges, consider some continuous charge distribution, with differential charge elements dq_1 and dq_2 (located on the same charge distribution). Then the sum becomes an integral and we have

$$U = \frac{1}{2}\iint \frac{k\ dq_1\ dq_2}{r_{12}} = \frac{1}{2}\int \frac{k\ \rho(\boldsymbol{x}_1)\ \rho(\boldsymbol{x}_2)\ d^3x_1\ d^3x_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} \tag{1}$$

Again, $\rho(\boldsymbol{x}_1)$ and $\rho(\boldsymbol{x}_2)$ describe the <u>same</u> charge distribution — we're breaking the charge distribution into little dq's and check each against all the others.

As an alternative, note that $\displaystyle\int \frac{k\ \rho(\boldsymbol{x}_1)\ d^3x_1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$ is how we'd write the voltage due to ρ at $\boldsymbol{x}_2$ (the location of the second charge in a particular pair). Thus, we can re-write the energy of the charge distribution as

$$\boxed{U = \frac{1}{2}\int \rho(\boldsymbol{x'})\ V(\boldsymbol{x'})\ d^3x'} \tag{2}$$

Alternate derivation of (2)

Start with the double sum that gives the energy of a collection of discrete charges:

$$U = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_o}\right)\sum_{i=1}^{n}\sum_{j\neq 1}^{n}\frac{q_i\ q_j}{r_{ij}}.$$

We can re-write this expression by pulling out a factor of q_i from the second sum:

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_o}\frac{q_j}{r_{ij}}\right)}_{\substack{\text{potential at}\\ r_i \text{ (position}\\ \text{of } q_i\text{) due to}\\ \text{all other}\\ \text{charges}}} = \frac{1}{2}\sum_{i=1}^{n} q_i V$$

Now generalizing to a continuous charge distribution:

$$\boxed{U = \frac{1}{2}\int \rho(\boldsymbol{x'})\ V(\boldsymbol{x'})\ d^3x'}$$

The expression for the energy of the system explicitly reference charge. This is not shocking — we're used to potential energy being a thing associated with pairs of charges. But here's where it gets interesting. We're going to re-write the energy in terms of the field, thereby eliminating ρ and V in favor of $\boldsymbol{E}$.

From Gauss's law, we know that $\nabla\cdot\boldsymbol{E} = \dfrac{\rho}{\epsilon_o} \implies \rho = \epsilon_o\,(\nabla\cdot\boldsymbol{E})$.. Substituting that into (2) gives us

$$U = \frac{\epsilon_o}{2}\int (\nabla\cdot\boldsymbol{E})\ V\ d^3x.$$

For any scalar-valued function f and vector-valued function $\boldsymbol{A}$, the following holds:

$$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla f).$$

Taking $f = V$ and $\boldsymbol{A} = \boldsymbol{E}$, we have

$$\nabla \cdot (V\boldsymbol{E}) = V(\nabla \cdot \boldsymbol{E}) + \boldsymbol{E} \cdot (\nabla V) \implies \underbrace{(\nabla \cdot \boldsymbol{E})\ V}_{\substack{\text{matches}\\ \text{integrand}\\ \text{above}}} = \nabla \cdot (V\ \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla V)$$

Substituting, we get

$$\begin{aligned} U &= -\frac{\epsilon_o}{2} \int \boldsymbol{E} \cdot \underbrace{(\nabla V)}_{-\boldsymbol{E}} d^3x + \frac{\epsilon_o}{2} \int \nabla \cdot (V\ \boldsymbol{E})\ d^3x \\ &= \frac{\epsilon_o}{2} \int \underbrace{(\boldsymbol{E} \cdot \boldsymbol{E})}_{E^2}\ d^3x + \frac{\epsilon_o}{2} \int \nabla \cdot (V\ \boldsymbol{E})\ d^3x \end{aligned}$$

Applying the divergence theorem to the second term yields

$$U = \frac{\epsilon_o}{2} \int E^2\ d^3x + \frac{\epsilon_o}{2} \oint (V\ \boldsymbol{E}) \cdot d\boldsymbol{A}$$

But what volume are we integrating over? Clearly, looking at the expression for energy in (2), we must at least integrate over the volume that encloses our charge distribution. But what's to stop us from integrating over all space? After all, $\rho = 0$ outside the charge distribution, so the extra space contributes nothing to the integral. Then the surface integral in the second term vanishes, since it examines V and $\boldsymbol{E}$ at the edge of all space (which, for any real, finite source, is zero). Thus,

$$\boxed{U = \frac{\epsilon_o}{2} \int E^2\ d^3x}$$

This is entirely in terms of fields, not charge. So we can look at the fields themselves as being very real things with real energy.

Just for fun, let's take a look at the energy associated with the field made by a point charge at the origin. The field is

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o} \frac{q\ \hat{\boldsymbol{r}}}{r^2}$$

Substituting this into the energy expression and integrating over all space yields

$$\begin{aligned} U &= \frac{\epsilon_o}{2} \int \boldsymbol{E} \cdot \boldsymbol{E}\ d^3x \\ &= \frac{\epsilon_o}{2} \left(\frac{q}{4\pi\epsilon_o}\right)^2 \int \left(\frac{\hat{\boldsymbol{r}}}{r^2} \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\right)\ d^3x. \end{aligned}$$

Note that $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} = 1$. Taking the differential volume element to be $r^2\ \sin\theta\ dr\ d\theta\ d\phi$, we get

$$\begin{aligned} U &= \frac{\epsilon_o}{2} \left(\frac{q}{4\pi\epsilon_o}\right)^2 \int \frac{1}{r^4} r^2\ \sin\theta\ dr\ d\theta\ d\phi \\ &= \frac{\epsilon_o}{2} \left(\frac{q}{4\pi\epsilon_o}\right)^2 \underbrace{\left(\int_0^{2\pi} d\phi\right)}_{2\pi} \underbrace{\left(\int_0^{\pi} \sin\theta\ d\theta\right)}_{2} \left(\int_0^{\infty} \frac{dr}{r^2}\right) \\ &= \frac{q^2}{8\pi\epsilon_o} \int_0^{\infty} \frac{dr}{r^2} \\ &= \frac{q^2}{8\pi\epsilon_o} \left(-\frac{1}{r}\right)\Bigg|_0^{\infty} \end{aligned}$$

which kind of diverges. That's bad.

What we just worked shows that $\boldsymbol{E}$-fields from point charges should contain infinite energy, thereby implying that point charges shouldn't be possible. But every experiment ever done indicates that an electron is a zero-radius true point. Fixing this apparent contradiction is one of the great achievements of quantum electrodynamics.

Delta Functions and Point Sources

We've already seen indications that point charges behave a bit wonky, even though they do seem to exist experimentally. I'm afraid this is going to get a bit worse before it gets better.

Let's take a look at Gauss's law again. If we take the divergence of an $\boldsymbol{E}$-field, we should recover the charge density that produced that field:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o}.$$

What happens if we take the divergence of the field made by a point charge at the origin? Applying the divergence operator in spherical coordinates gives us

$$\nabla \cdot \underbrace{\left(\frac{q}{4\pi\epsilon_o}\frac{\hat{\boldsymbol{r}}}{r^2}\right)}_{\boldsymbol{E}_{\text{point}}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{q}{4\pi\epsilon_o\,\cancel{r^2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_o}\right) = 0\ ?$$

Well, according to that, the divergence of something like $\dfrac{\hat{\boldsymbol{r}}}{r^2}$ is zero everywhere. Thus, so must ρ be zero everywhere. But that can't be right. So what's the catch?

The catch is that the operator $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\right)$ isn't super well-defined at the origin. All we can conclude from the above is that $\nabla \cdot \boldsymbol{E}$ is zero everywhere <u>but</u> the origin. To deal with the origin, let's take a look at delta functions first.

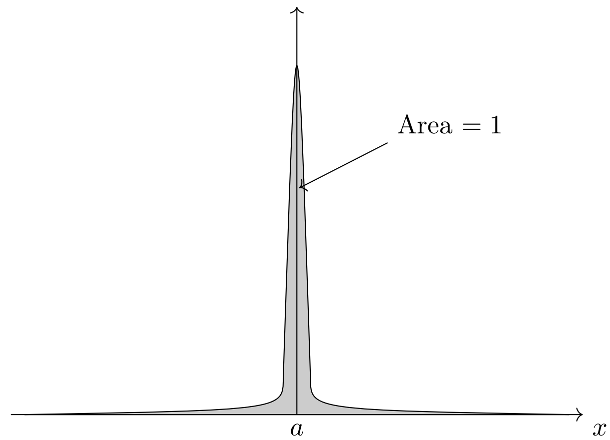
A delta function is used to represent a finite amount of stuff compressed into an essentially zero-dimensional domain. In one dimension, a delta function $\delta(x)$ is <u>defined</u> as the thingy that satisfies these two properties: $\delta(x)$ is a delta function if:

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \text{undefined}, & \text{if } x = 0 \end{cases}$$

and

$$\int \delta(x-a)\ f(x)\ dx = \begin{cases} f(a), & \text{if } a \text{ is in the domain of integration} \\ 0, & \text{otherwise} \end{cases}$$

Basically, it is a very sharply peaked function that, when present in an integral, plucks out the value of another function at one point:



So how do we represent ρ for a point charge of size q in 3D? How about a delta function:

$$\rho(\boldsymbol{x}) = q\ \delta^3(\boldsymbol{x}) \qquad \text{(Or } q\ \delta^3(\boldsymbol{x}-\boldsymbol{x'}) \text{ if the point charge is at } \boldsymbol{x'} \text{ instead of the origin)}$$

With that in mind, since $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_o}$, it should be the case that

$$\nabla \cdot \underbrace{\left(\frac{q}{4\pi\epsilon_o}\frac{\hat{\boldsymbol{r}}}{r^2}\right)}_{\boldsymbol{E}_{\text{point}}} = \frac{1}{\epsilon_o}\underbrace{q\ \delta^3(\boldsymbol{x})}_{\rho}$$

And therefore

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{x}) \qquad \text{Is this true?}$$

Well, it's true if

(1) $\nabla \cdot \left(\dfrac{\hat{\boldsymbol{r}}}{r^2}\right) = 0$ everywhere but the origin. And we've established that.

(2) $\displaystyle\int \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\right)\ d^3x = 4\pi$ for a domain of integration that includes the origin.

Let's integrate $\nabla \cdot \left(\dfrac{\hat{\boldsymbol{r}}}{r^2}\right)$ over a sphere of radius R. Just doing that as a volume integral is tricky, since the integrand is undefined at the origin, so let's dodge the bad part by using the divergence theorem:

$$\begin{aligned}
\int \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\right)\ d^3x &= \oint \frac{\hat{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{A} \\
&= \oint \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \left(r^2\ \sin\theta\ d\theta\ d\phi\ \hat{\boldsymbol{r}}\right) \\
&= \oint \sin\theta\ d\theta\ d\phi \\
&= \underbrace{\left(\int_0^{\pi} \sin\theta\ d\theta\right)\left(\int_0^{2\pi} d\phi\right)}_{=\ 4\pi}
\end{aligned}$$

So it checks out. $\nabla \cdot \left(\dfrac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\ \delta^3(r)$, and then

$$\nabla \cdot \left(\frac{q}{4\pi\epsilon_o}\frac{\hat{\boldsymbol{r}}}{r^2}\right) = \frac{1}{\epsilon_o} q\ \delta^3(r) = \frac{\rho}{\epsilon_o}$$

and Gauss's law holds.