PHGN 200 — Electromagnetism and Optics

Block III: Magnetics

Block III Review Slides

A Quick Introduction

We learned in Block I that *stationary charges* produce *electric fields*.

Then in Block II we learned that *current* is nothing more than *moving charges*.

So what kind of fields do *moving charges* produce? Magnetic fields!

Biot-Savart Law

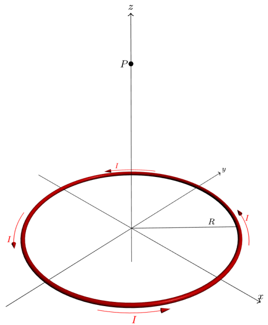
The Biot-Savart Law reads

$$d\vec{B} = \frac{\mu_o I \; d\vec{\ell} \times \vec{r}}{4\pi \; r^3}$$

This gives us a recipe for finding magnetic fields from current-carrying wires.

Worked Example — A Current-Carrying Loop

- Consider a loop of radius R that carries a current I. What is the magnetic field at some point P on the axis of the loop?



Solution. Let's start by building up all the pieces of the Biot-Savart law.

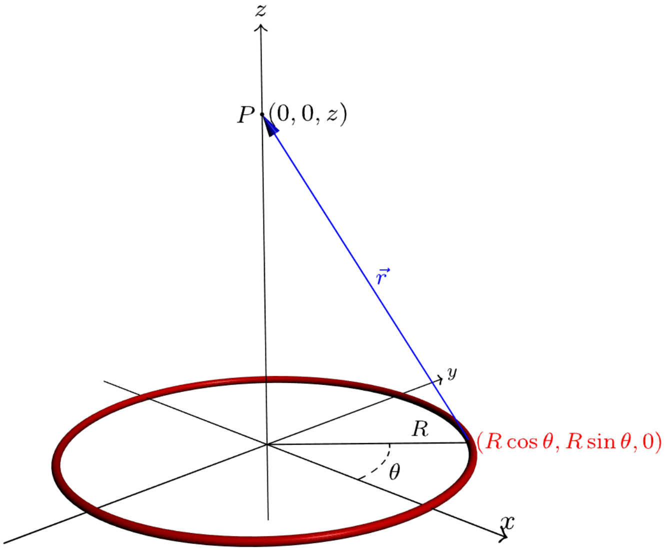
$$d\vec{B} = \frac{\mu_o I \ d\vec{\ell} \times \vec{r}}{4\pi \ r^3}$$

Worked Example — A Current-Carrying Loop

Let's start with with $\vec{r}$-vector. As always, the $\vec{r}$-vector points from the source (the current-carrying loop) to the observation point.

The observation point P is situated on the z-axis. So its coordinates are $(0, 0, z)$.

Worked Example — A Current-Carrying Loop



Worked Example — A Current-Carrying Loop

So the $\vec{r}$-vector is

$$\vec{r} = (0 - R\cos\theta)\,\hat{\boldsymbol{\imath}} + (0 - R\sin\theta)\,\hat{\boldsymbol{\jmath}} + (z - 0)\,\hat{\boldsymbol{k}}$$

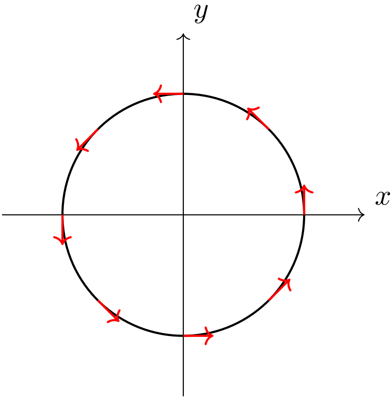
$$\boxed{\vec{r} = -R\cos\theta\ \hat{\boldsymbol{\imath}} - R\sin\theta\ \hat{\boldsymbol{\jmath}} + z\ \hat{\boldsymbol{k}}}$$

and its magnitude is

$$\begin{aligned} r &= \sqrt{(-R\cos\theta)^2 + (-R\sin\theta)^2 + (z)^2} \\ &= \sqrt{R^2\cos^2\theta + R^2\sin^2\theta + z^2} \\ &= \sqrt{R^2\cancel{\left(\cos^2\theta + \sin^2\theta\right)} + z^2} \\ &= \sqrt{R^2 + z^2}. \end{aligned}$$

Worked Example — A Current-Carrying Loop

Now we need $\vec{d\ell}$. This is the infinitesimal segment of the current-carrying wire.



Worked Example — A Current-Carrying Loop

We know that infinitesimal arc length is

$$\left|d\vec{\ell}\right| = d\ell = R\ d\theta$$

This holds for any circular path. However, if we want to get the components of the $d\vec{\ell}$ vector, we need to attenuate each term by the appropriate trig factor:

$$d\vec{\ell} = \left(\underbrace{\qquad\qquad}_{\text{trig factor}} R\ d\theta\right)\hat{\boldsymbol{\imath}} + \left(\underbrace{\qquad\qquad}_{\text{trig factor}} R\ d\theta\right)\hat{\boldsymbol{\jmath}}$$

Finding which term goes like $\sin\theta$ and which goes like $\cos\theta$ is by far easiest to do with limiting values.

Worked Example — A Current-Carrying Loop

$$d\vec{\ell} = \overbrace{\left(\underbrace{\qquad\qquad}_{\text{trig factor}} R\ d\theta\right)}^{dx} \hat{\boldsymbol{\imath}} + \overbrace{\left(\underbrace{\qquad\qquad}_{\text{trig factor}} R\ d\theta\right)}^{dy} \hat{\boldsymbol{\jmath}}$$

At $\theta = \frac{\pi}{2}$, the $d\vec{\ell}$ vector points straight to the left. So we want a trig factor that respects $dy = 0$ when $\theta = \frac{\pi}{2}$. Immediately, this tells us that $dy = R\cos\theta\ d\theta$, since $\cos\left(\frac{\pi}{2}\right) = 0$. Conversely, we find that $dx = -R\sin\theta\ d\theta$, since the $d\vec{\ell}$ vector points to the left. So we have

$$\boxed{d\vec{\ell} = -R\sin\theta\ d\theta\ \hat{\boldsymbol{\imath}} + R\cos\theta\ d\theta\ \hat{\boldsymbol{\jmath}} + 0\ \hat{\boldsymbol{k}}}$$

We can check this against the other limiting value, $\theta \to 0$. Indeed, we get

$$d\vec{\ell}\,\Big|_{\theta=0} = \cancelto{0}{-R\sin(0)\ d\theta\ \hat{\boldsymbol{\imath}}} + R\cos(0)\ d\theta\ \hat{\boldsymbol{\jmath}} = R\ d\theta\ \hat{\boldsymbol{\jmath}}$$

Worked Example — A Current-Carrying Loop

We can also start from first principles. We know, that $d\vec{\ell}$, in its most general form, is $d\vec{\ell} = dx\ \hat{\boldsymbol{\imath}} + dy\ \hat{\boldsymbol{\jmath}}$. By convention, the positive $d\vec{\ell}$ vector points such that a counter-clockwise path is traced out along a circle. Moreover, we know that along a circle (with θ taken from the positive x-axis),

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta.$$

Taking a derivative to get dx and dy, we have

$$dx = -R\sin\theta\ d\theta \qquad \text{and} \qquad dy = R\cos\theta\ d\theta.$$

Putting everything together,

$$d\vec{\ell} = dx\ \hat{\boldsymbol{\imath}} + dy\ \hat{\boldsymbol{\jmath}} = \boxed{-R\sin\theta\ d\theta\ \hat{\boldsymbol{\imath}} + R\cos\theta\ d\theta\ \hat{\boldsymbol{\jmath}}}$$

Worked Example — A Current-Carrying Loop

Now we need the cross-product $d\vec{\ell} \times \vec{r}$.

$$\begin{aligned} d\vec{\ell} \times \vec{r} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -R\sin\theta\ d\theta & R\cos\theta\ d\theta & 0 \\ -R\cos\theta & -R\sin\theta & z \end{vmatrix} \\ &= Rz\cos\theta\ d\theta\ \hat{\boldsymbol{\imath}} + Rz\sin\theta\ d\theta\ \hat{\boldsymbol{\jmath}} + \\ &\qquad + \underbrace{\left(R^2\sin^2\theta\ d\theta + R^2\cos^2\theta\ d\theta\right)}_{=\ R^2\ d\theta}\ \hat{\boldsymbol{k}} \end{aligned}$$

So we have

$$d\vec{\ell} \times \vec{r} = Rz\cos\theta\ d\theta\ \hat{\boldsymbol{\imath}} + Rz\sin\theta\ d\theta\ \hat{\boldsymbol{\jmath}} + R^2\ d\theta\ \hat{\boldsymbol{k}}$$

Worked Example — A Current-Carrying Loop

We don't care about the $\hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{\jmath}}$ components. Why?

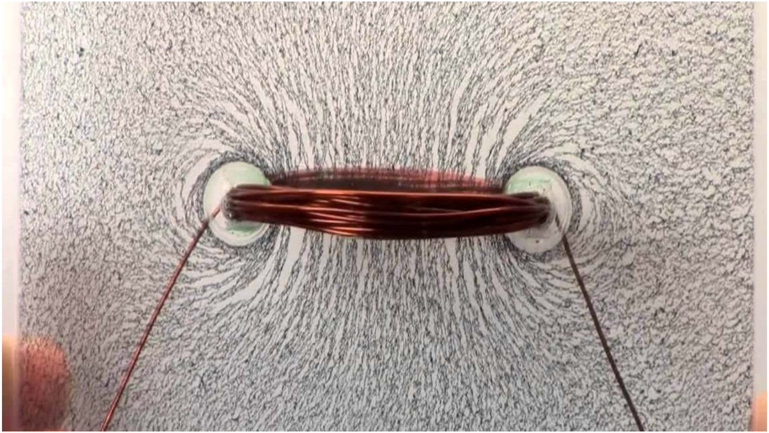
$$d\vec{\ell} \times \vec{r} = \underbrace{Rz\cos\theta\ d\theta}_{\left(d\vec{\ell}\times\vec{r}\right)_x}\ \hat{\boldsymbol{\imath}} + Rz\sin\theta\ d\theta\ \hat{\boldsymbol{\jmath}} + R^2\ d\theta\ \hat{\boldsymbol{k}}$$

Say we want to determine the $\hat{\boldsymbol{\imath}}$-component of the magnetic field. We would assemble the Biot-Savart law as usual:

$$B_x = \int dB_x = \int_0^{2\pi} \frac{\mu_o I\ \ (Rz\cos\theta\ d\theta)}{4\pi\left(\sqrt{R^2+z^2}\right)^3}\ \hat{\boldsymbol{\imath}}$$

We're integrating with respect to θ. And everything in that integral barring $\cos\theta$ can be pulled out of the integral. But we know that $\displaystyle\int_0^{2\pi} \cos\theta\ d\theta = 0$.

Worked Example — A Current-Carrying Loop



Worked Example — A Current-Carrying Loop

So now we know the magnetic field, on the axis of the loop, is strictly in the $\hat{\boldsymbol{k}}$ direction. Let's finish up by assembling the pieces of the Biot Savart law and integrating:

$$\vec{B} = \int d\vec{B} = \int_0^{2\pi} \frac{\mu_o\ I\ \left(R^2\ d\theta\right)}{4\pi\ \left(R^2 + z^2\right)^{3/2}}\ \hat{\boldsymbol{k}}$$

$$= \frac{\mu_o\ I\ R^2}{4\pi\ \left(R^2 + z^2\right)^{3/2}} \cancelto{2\pi}{\int_0^{2\pi} d\theta}\ \hat{\boldsymbol{k}}$$

And we end up with a familiar expression:

$$\boxed{\vec{B} = \frac{\mu_o I R^2}{2\left(R^2 + z^2\right)^{3/2}}\ \hat{\boldsymbol{k}}}\ \blacktriangleleft$$

Ampére's Law

Ampére's law reads

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_o I_{\text{thru}} + \underbrace{\cancel{\mu_o \epsilon_o \frac{d\Phi_E}{dt}}}_{\substack{\text{don't worry}\\ \text{about}\\ \text{displacement}\\ \text{current}}}$$

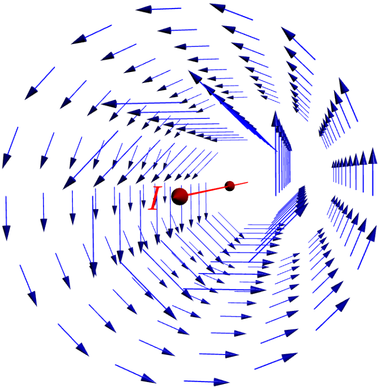
The left-hand side of Ampére's law is a *closed* line integral. Term of art: we integrate the magnetic field over an *Ampérian loop*.

The right-hand side of Ampére's law is the current enclosed by that Ampérian loop. The most general method to determine enclosed current is to integrate the current density:

$$I_{\text{thru}} = \int \vec{J} \cdot d\vec{A} = \int J \, dA$$

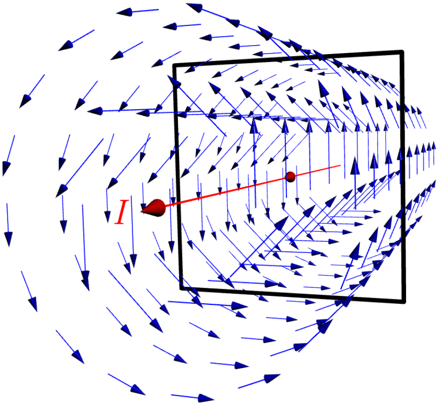
Remarks on Ampére's Law

Ampére's law is a *law of nature*. That is, it holds for *any* closed loop. Consider, for instance, the current-carrying wire below.



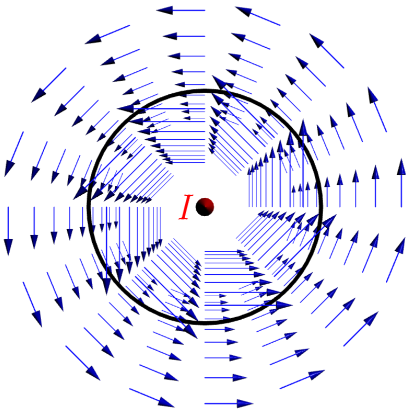
Remarks on Ampére's Law

The value of the integral $\oint \vec{B} \cdot d\vec{\ell}$ is the same whether the Ampérian loop looks like this:



Remarks on Ampére's Law

or like this:



Put another way, Ampére's law — the statement about the value of the integral $\oint \vec{B} \cdot d\vec{\ell}$ — is true always, but is only useful sometimes.

Using Ampére's law to Solve for Magnetic Fields

Under certain circumstances, the symmetry of a problem will enable us to solve for magnetic fields with Ampére's law. If the magnetic field is always parallel to the $d\vec{\ell}$ vector of the Ampérian loop, then

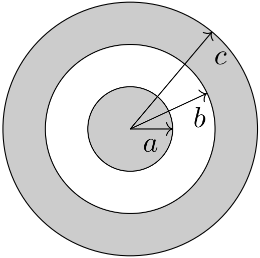
$$\oint \vec{B} \cdot d\vec{\ell} = \oint B \, d\ell,$$

since a dot product of parallel vectors is immaterial. Moreover, if the magnetic field is *uniform* on the entirety of the loop then

$$\oint B \, d\ell = B \underbrace{\left(\oint d\ell \right)}_{\substack{\text{length of} \\ \text{Ampérian} \\ \text{loop}}}$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

- A wire with radius a carries a uniform current I_o. An annular cylinder with inner radius b and outer radius c is concentric with the inner wire, and carries a current such that its current density is given by $J_{\text{annular cylinder}} = \frac{1}{r^2}$ (assume the units on J are A/m^2). Find the magnitude of the magnetic field at a distance r from the center axis, where



(a) $r < a$,

(b) $a \leqslant r \leqslant b$,

(c) $b \leqslant r \leqslant c$, and

(d) $r > c$.

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Solution. Ampére's law reads

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_o I_{\text{thru}}$$

We need the magnitude of the magnetic field at a distance r from the center axis, where (a) $r < a$, (b) $a \leqslant r \leqslant b$, (c) $b \leqslant r \leqslant c$, and (d) $r > c$. For each of these, we'll need to calculate the the current enclosed.

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (a). Here the current enclosed by the Ampérian loop is confined to the inner conductor, where the current is uniformly distributed. Therefore, it follows that the current density is $J = \frac{\text{total current}}{\text{total cross-sectional area}} = \frac{I}{\pi a^2}$. So the enclosed current is

$$\begin{aligned} I_{\text{thru}} = \int J \, dA &= \overbrace{\left(\frac{I_o}{\pi a^2}\right)}^{J} \int dA \\ &= \frac{I_o}{\pi a^2} \int_0^r 2\pi r' \, dr' \\ &= \frac{2\pi I_o}{\pi a^2} \left(\frac{r'^2}{2}\right) \Big|_0^r \\ &= \frac{I_o \, r^2}{a^2} \end{aligned}$$

(The cross sectional area is a circle. So $A = \pi r^2$. Taking a derivative gives $dA = 2\pi r \, dr$.)

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (b). Here the Ampérian loop is nested between the inner conductor and the outer conductor. There's no current in the region $a \leqslant r \leqslant b$, so the enclosed current is

$$\begin{aligned} I_{\text{thru}} = \int J\, dA &= \overbrace{\left(\frac{I_o}{\pi a^2}\right)}^{J} \int dA \\ &= \frac{I_o}{\pi a^2} \int_0^a 2\pi r'\, dr' \\ &= \frac{I_o\, \cancel{a^2}}{\cancel{a^2}} \end{aligned}$$

That's reassuring. It should indeed be the case that the current through the Ampérian loop is just the current carried by the inner wire.

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (c). Here the Ampérian loop is nested inside the annular cylinder (the outer conductor), such that $b \leqslant r \leqslant c$. We're still enclosing the current from the inner conductor, but now we also have some current from the outer conductor, whose current density is given by the function $J_{\text{ann cyl}} = \frac{1}{r^2}$. As such, the enclosed current is

$$\begin{aligned} I_{\text{thru}} &= I_o + \int_b^r \overbrace{\left(\frac{1}{r'^2}\right)}^{J_{\text{ann cyl}}} \overbrace{(2\pi r' \, dr')}^{dA} \\ &= I_o + 2\pi \int_b^r \frac{dr'}{r'} \\ &= I_o + 2\pi \ln\left(\frac{r}{b}\right) \end{aligned}$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (d). Here the Ampérian loop is outside the annular cylinder, such that $r > c$. We're still enclosing the current from the inner conductor, but now we also have all of the current from the outer conductor, whose current density is given by the function $J_{\text{ann cyl}} = \frac{1}{r^2}$. As such, the enclosed current is

$$
\begin{aligned}
I_{\text{thru}} &= I_o + \int_b^c \overbrace{\left(\frac{1}{r'^2}\right)}^{J_{\text{ann cyl}}} \overbrace{(2\pi r' \; dr')}^{dA} \\
&= I_o + 2\pi \int_b^c \frac{dr'}{r'} \\
&= I_o + 2\pi \ln\left(\frac{c}{b}\right)
\end{aligned}
$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

In all cases, the appropriate symmetry arguments are satisfied. The magnetic field is parallel to the $d\vec{\ell}$ vectors, and the magnetic field is always uniform along the Ampérian loop. So we have, for the left-hand side of Ampére's law,

$$\oint \vec{B} \cdot d\vec{\ell} = \oint B \; d\ell = B \underbrace{\left(\oint d\ell \right)}_{\substack{\text{length of} \\ \text{Ampérian} \\ \text{loop}}} = B\,(2\pi \; r)$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (a). Putting everyting together, the magnetic field inside the inner wire is

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_o I_{\text{thru}}$$

$$B\,(2\pi\ r) = \mu_o \left(\frac{I_o\ r^2}{a^2} \right)$$

$$\boxed{B = \frac{\mu_o\ I_o\ r}{2\pi a^2}}$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (b). The magnetic field in between the inner wire and the annular cylinder is

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_o I_{\text{thru}}$$

$$B\,(2\pi\ r) = \mu_o I_o$$

$$\boxed{B = \frac{\mu_o\ I_o}{2\pi r}}$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (c). For the magnetic field in the annular cylinder, we get

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_o I_{\text{thru}}$$

$$B\,(2\pi\ r) = \mu_o \left(I_o + 2\pi \ln \left(\frac{r}{b} \right) \right)$$

$$\boxed{B = \frac{\mu_o\ \left(I_o + 2\pi\ \ln\left(\frac{r}{b}\right)\right)}{2\pi\ r}}$$

Worked Example — Using Ampére's law to Solve for Magnetic Fields

Part (d). And finally, the magnetic field outside the annular cyliner is

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_o I_{\text{thru}}$$

$$B\,(2\pi\ r) = \mu_o \left(I_o + 2\pi \ln\left(\frac{c}{b}\right)\right)$$

$$\boxed{B = \frac{\mu_o\ \left(I_o + 2\pi\ \ln\left(\frac{c}{b}\right)\right)}{2\pi\ r}} \blacktriangleleft$$

Magnetic Flux

Magnetic flux is given by

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

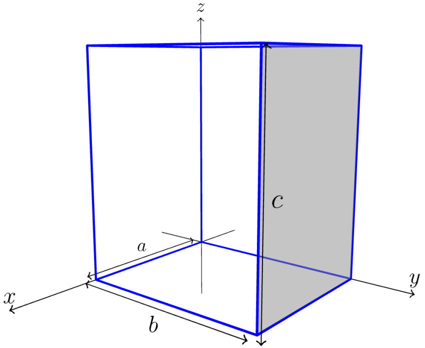
If the surface is completely closed, then

$$\Phi_{B,\text{ net}} = \oint \vec{B} \cdot d\vec{A} = 0.$$

That last equation assserts that for any volume in space, there must necessarily be the same number of magnetic field lines exiting and entering the region. That is, we're not allowed to have, say, a north pole by itself generating magnetic field lines. (By contrast to electric charge, where Gauss's law tells us that free charges generate $\vec{E}$-field lines.)

Worked Example — Magnetic Flux through a Box

► A region of space contains a non-uniform magnetic field given by $\vec{B} = x^2\ \hat{\boldsymbol{\imath}} + 3xz^2\ \hat{\boldsymbol{\jmath}} - 2xz\ \hat{\boldsymbol{k}}$ (assume the coefficients on those carry the appropriate units). Suppose a box with side lengths a, b, and c is situated as shown below.



Determine the magnetic flux through the shaded side.

Worked Example — Magnetic Flux through a Box

Solution. Magnetic flux is given by

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

The $\vec{B}$-field is given. The area vector, for the face in question, has magnitude $dA = dz\ dx$ (since it's parallel to the xz-plane). Moreover, area vectors are always normal to the surface that they describe, so the area vector points in the $\hat{\boldsymbol{\jmath}}$ direction:

$$d\vec{A} = dz\ dx\ \hat{\boldsymbol{\jmath}}$$

Computing the dot-product,

$$\begin{aligned}\vec{B} \cdot d\vec{A} &= \left(x^2\ \hat{\boldsymbol{\imath}} + 3xz^2\ \hat{\boldsymbol{\jmath}} - 2xz\ \hat{\boldsymbol{k}}\right) \cdot (dz\ dx\ \hat{\boldsymbol{\jmath}}) \\ &= 3xz^2\ dz\ dx\end{aligned}$$

Worked Example — Magnetic Flux through a Box

Putting everything together,

$$\begin{aligned} \Phi_B &= \int \vec{B} \cdot d\vec{A} \\ &= \int_0^a \int_0^c 3xz^2 \; dz \; dx \\ &= 3 \underbrace{\left(\int_0^a x \; dx \right)}_{= \, a^2/2} \underbrace{\left(\int_0^c z^2 \; dz \right)}_{= \, c^3/3} \end{aligned}$$

So the flux through that face of the box is

$$\boxed{\Phi_B = \frac{a^2 \; c^3}{2}} \blacktriangleleft$$

Magnetic Forces

The force exerted on a particle of charge q moving with velocity $\vec{v}$ moving in a magnetic field $\vec{B}$ is governed by

$$\vec{F} = q\vec{v} \times \vec{B}$$

Using the Right-Hand Rule

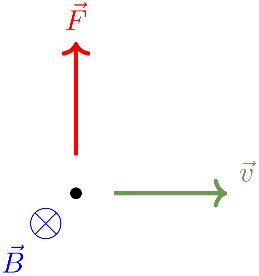
$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\underbrace{\vec{t}}_{\text{thumb}} = \underbrace{\vec{i}}_{\text{index}} \times \underbrace{\vec{m}}_{\text{middle}}$$

Motion of a Charged Particle in a Magnetic Field

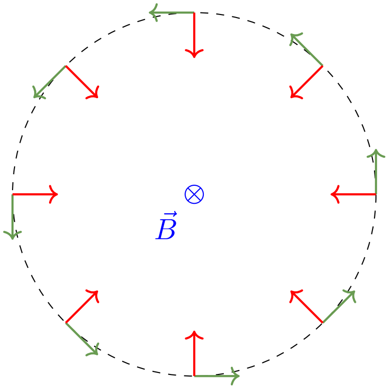
Say a particle with charge q is moving to the right in empty space with velocity $\vec{v}$.

Then, a uniform magnetic field pointing directly into the page is turned on. What happens to the particle? Remember, the motion of a charged particle in a magnetic field is governed by $\vec{F} = q\vec{v} \times \vec{B}$.



Motion of a Charged Particle in a Magnetic Field

In fact, the force exerted by the magnetic field on the particle is a *centripetal* force; it steers the particle in a circle.



Motion of a Particle in a Magnetic Field

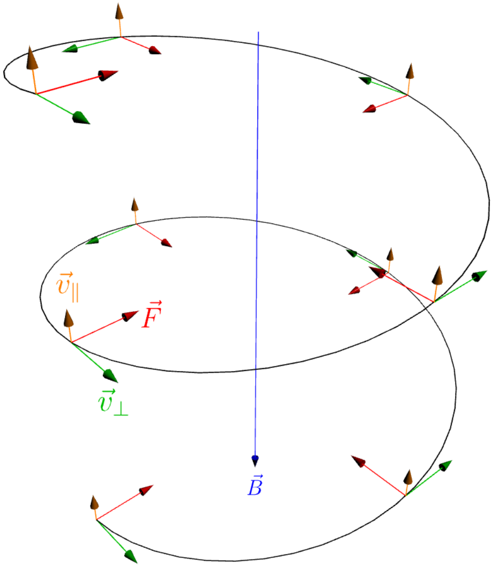
What if the velocity isn't strictly perpendicular to the magnetic field? That is, suppose the charged particle moves with some component of its velocity *parallel* to the magnetic field.

We can express the particle's velocity as

$$\vec{v} = \vec{v}_{\|} + \vec{v}_{\perp}$$

where $\vec{v}_{\|}$ and $\vec{v}_{\perp}$ are the components of the particle's velocity that are parallel and perpendicular to magnetic field, respectively. Since the motion of a charged particle in a magnetic field is governed by $\vec{F} = q\vec{v} \times \vec{B}$, it must be the case that the component of the particle's velocity that's *parallel* to the magnetic field, $\vec{v}_{\|}$, **doesn't change**.

Motion of a Particle in a Magnetic Field



Motion of a Particle in a Magnetic Field

To summarize: a charged particle moving in a magnetic field will travel in circular path if its velocity is strictly perpendicular to the magnetic field.

A charged particle will travel in a helical trajectory if some components of its velocity is parallel to the magnetic field (but not completely parallel to the magnetic field).

If the particle's velocity is completely parallel to the magnetic field (or if there's no charge at all), then there's no force on the particle — its trajectory won't change.

Magnetic Fields Do No Work on Free Charges

Recall the definition of work:

$$\text{Work } \equiv \int \vec{F} \cdot d\vec{\ell}$$

The magnetic force is given by $\vec{F} = q\vec{v} \times \vec{B}$. Moreover, we can express the differential displacement $d\vec{\ell}$ in terms of the velocity as

$$d\vec{\ell} = \vec{v}\ dt$$

(This follows from the definition of velocity as $\vec{v} \equiv \frac{d\vec{\ell}}{dt}$.) Then the work done by the magnetic field is

$$\text{Work } = \int \vec{F} \cdot d\vec{\ell} = \int \left(q\vec{v} \times \vec{B}\right) \cdot (\vec{v}\ dt)$$

Magnetic Fields Do No Work on Free Charges

The work done by the magnetic field is

$$\text{Work } = \int \vec{F} \cdot d\vec{\ell} = \int \left(q\vec{v} \times \vec{B}\right) \cdot (\vec{v}\ dt)$$

Note that the quantity $\left(q\vec{v} \times \vec{B}\right)$ is mutually perpendicular to both $\vec{v}$ and $\vec{B}$ (this is a property of cross-products). Then the quantity

$$\underbrace{\left(q\vec{v} \times \vec{B}\right)}_{\vec{F}} \cdot \underbrace{(\vec{v}\ dt)}_{d\vec{\ell}}$$

is zero, since the two vectors are perpendicular. Thus, *magnetic fields never do any work on free particles.* In consequence, a magnetic field cannot alter the kinetic energy of a free charge, since the work-energy theorem asserts that $\Delta E = \text{Work}$.

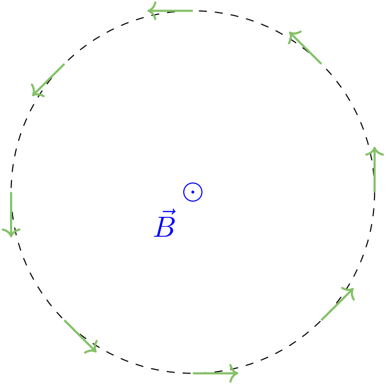
Worked Example — Mystery Particle

- A magnetic field of $150\,\mathrm{mT}$ points out of the page (for context, this is the typical magnetic field strength of a sunspot). An unknown particle moves in a counterclockwise circular path of radius $1.117 \times 10^{-5}\,\mathrm{m}$ perpendicular to the magnetic field. The imposition of an electric field of $44.2\,\mathrm{kV/m}$ makes the path straight. What is the charge-to-mass ratio of the particle?

Solution. There are two regimes governing the motion of the particle: one in which the particle moves in a circular path perpendicular to a magnetic field, and the other in which the particle moves in a straight line.

Worked Example — Mystery Particle

Let's first consider the particle's circular path. The magnetic force is the only force acting on the particle, so it must be a centripetal force. Recall that centripetal acceleration is v^2/R, where v is the speed of the particle and R is the radius of its circular path. Then Newton's second law gives



$$\overbrace{\left|q\vec{v}\times\vec{B}\right|}^{F_{\mathrm{mag}}} = \overbrace{\frac{m\ v^2}{R}}^{ma}$$

$$q\cancel{v}B = \frac{m\ v^{\cancel{2}}}{R}$$

$$qB = \frac{m\ v}{R}$$

$$\frac{q}{m} = \frac{v}{BR}$$

Worked Example — Mystery Particle

Now let's consider the other regime: the one in which the particle moves in a straight path. Newton's first law asserts that in the absence of forces, a particle moves in a straight path with constant speed. That's exactly what's happening here, so it must be the case that the magnetic force is balanced by the electric force:

$$\overbrace{\left|q\vec{v} \times \vec{B}\right|}^{F_{\text{mag}}} = \overbrace{\left|q\vec{E}\right|}^{F_{\text{elec}}}$$

$$\cancel{q}vB = \cancel{q}E$$

$$E = vB \quad \Longrightarrow \quad v = \frac{E}{B}$$

Putting this into the expression obtained earlier gives

$$\frac{q}{m} = \frac{v}{B\ R} = \boxed{\frac{E}{B^2R}}$$

Worked Example — Mystery Particle

Now we're ready to put in some numbers.

$$\frac{q}{m} = \frac{\left(44.2 \times 10^{3}\,\mathrm{V/m}\right)}{\left(150 \times 10^{-3}\,\mathrm{T}\right)^{2}\left(1.117 \times 10^{-5}\,\mathrm{m}\right)} = 1.759 \times 10^{11}\,\mathrm{C/kg}$$

But we're not done yet! Charge can be positive or negative, and the process we've used to get our answer doesn't tell us what we're dealing with here. Thankfully, we've been told that the particle moves in a *counterclockwise* path perpendicular to a magnetic field that points *out of the page*. The equation $\vec{F} = q\vec{v} \times \vec{B}$ then demands that q be negative:

$$\boxed{\frac{q}{m} = -1.759 \times 10^{11}\,\mathrm{C/kg}}$$

This particle is, in fact, an electron:

$$\frac{-e_f}{m_e} = \frac{-1.602 \times 10^{-19}\,\mathrm{C}}{9.11 \times 10^{-31}\,\mathrm{kg}} = -1.759 \times 10^{11}\,\mathrm{C/kg} \blacktriangleleft$$

Force on Current-Carrying Wires

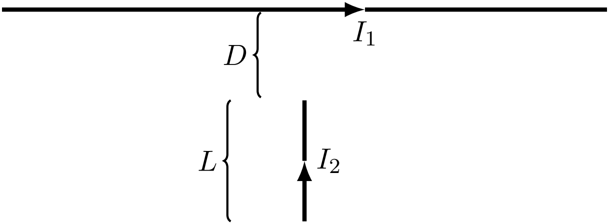
The differential amount of force, $d\vec{F}$, exerted on a wire carrying a current I in a magnetic field $\vec{B}$ is

$$d\vec{F} = I d\vec{\ell} \times \vec{B}$$

where $d\vec{\ell}$ is an infinitesimal segment of the wire.

Worked Example — Forces on Current-Carrying Wires

- Consider two perpendicular wires as shown. There's an infinite wire with current I_1 flowing from left to right, and a wire stub of length L with current I_2 flowing towards the top of the page. There's a gap of length D between the two wires.

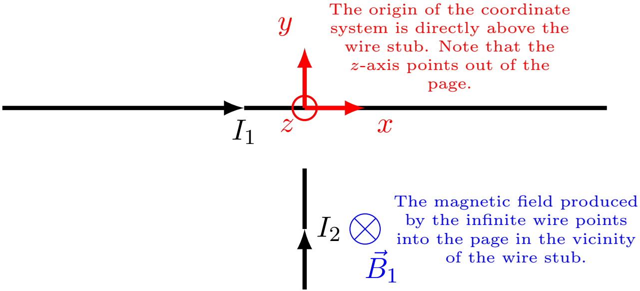


Calculate the force on wire 2 (the wire with current I_2) due to wire 1.

Solution. We want the force *on wire 2* from *due to wire 1*. We start by noting that the magnitude of the magnetic field produced by an infinite wire is

$$B_{\text{inf. wire}} = \frac{\mu_o I}{2\pi\ r}$$

Furthermore, let's impose the following coordinate system:



Worked Example — Forces on Current-Carrying Wires

In this new coordinate system, the magnetic field produced by the infinite wire, in the vicinity of the wire stub, is

$$\vec{B}_1 = \frac{\mu_o \; I_1}{2\pi \; (-y)} \; \left(-\hat{\boldsymbol{k}}\right) = \frac{\mu_o \; I_1}{2\pi \; y} \; \hat{\boldsymbol{k}}$$

Nifty. Now observe that the wire stub points up on the page. As such, we can simply take $d\vec{\ell} = dy \; \hat{\boldsymbol{\jmath}}$. As such, the differential force exerted on the wire stub (the wire carrying current I_2) is

$$d\vec{F} = I_2 \; d\vec{\ell} \times \vec{B}_1 = \overbrace{I_2 \;\; (dy \; \hat{\boldsymbol{\jmath}})}^{I \; d\vec{\ell}} \times \overbrace{\left(\frac{\mu_o \; I_1}{2\pi \; y} \; \hat{\boldsymbol{k}}\right)}^{\vec{B}} = \frac{\mu_o \; I_1 \; I_2}{2\pi \; y} \; dy \; \hat{\boldsymbol{\imath}}$$

(Note that $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$ — Think about the right-hand rule).

Worked Example — Forces on Current-Carrying Wires

Now we just need to integrate to find the total force on wire 2. We're integrating with respect to y, so the lower and upper bounds of integration are the smallest and largest values of y, respectively.

$$\begin{aligned}
\vec{F} = \int d\vec{F} &= \int_{-(D+L)}^{-D} \frac{\mu_o\ I_1\ I_2}{2\pi\ y}\ dy\ \hat{\boldsymbol{\imath}} \\
&= \frac{\mu_o\ I_1\ I_2}{2\pi} \int_{-(D+L)}^{-D} \frac{dy}{y}\ \hat{\boldsymbol{\imath}} \\
&= \frac{\mu_o\ I_1\ I_2}{2\pi} \ln\left(\frac{D}{D+L}\right)\hat{\boldsymbol{\imath}}
\end{aligned}$$

By properties of logarithms, we can re-write this result as

$$\vec{F} = \frac{\mu_o\ I_1\ I_2}{2\pi} \ln\left(\frac{D+L}{D}\right)(-\hat{\boldsymbol{\imath}})$$

So the force on wire 2 is in the negative x-direction, as expected using the right-hand rule in conjunction with $d\vec{F} = I\ d\vec{\ell} \times \vec{B}$. ◀

A Refresher on Torque

Torque is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

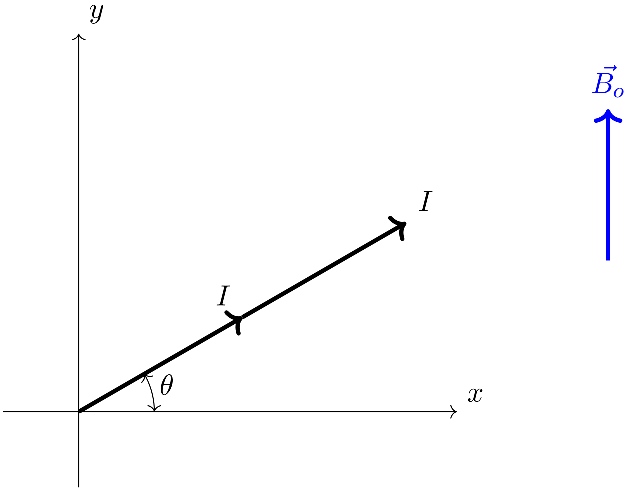
where $\vec{r}$ is the vector that points from the axis of rotation to the point of force application. If the applied force is non-uniform, we can instead consider the differential torque that comes from a differential force:

$$d\vec{\tau} = \vec{r} \times d\vec{F}$$

(Recall that just like a linear force $\vec{F}$ can be thought of a push, torque can be thought of as a twist about some axis).

Worked Example — Torque on a Slanted Wire

- A wire carrying a current I is free to rotate about the y-axis in the presence of a uniform magnetic field $\vec{B} = B_o\ \hat{\boldsymbol{\jmath}}$. What is the torque on the wire segment?



Worked Example — Torque on a Slanted Wire

Solution. Here's our strategy: we'll start by first finding the differential amount of force with

$$d\vec{F} = I \; d\vec{\ell} \times \vec{B}$$

and then we'll get the $\vec{r}$-vector and get the differential amount of torque with

$$d\vec{\tau} = \vec{r} \times d\vec{F}.$$

Finally, we'll intergrate $d\vec{\tau}$ to find the total torque $\vec{\tau}$.

Worked Example — Torque on a Slanted Wire

The differential amount of force is given by

$$d\vec{F} = I\ d\vec{\ell} \times \vec{B}$$

We have the magnetic field ($\vec{B} = B_o\ \hat{\boldsymbol{\jmath}}$). The infinitesimal length of the slanted wire segment is

$$d\vec{\ell} = dx\ \hat{\boldsymbol{\imath}} + dy\ \hat{\boldsymbol{\jmath}}$$

We want everything in terms of x. So we leave dx alone. To get dy in terms of dx, recall that the segment in question is a line, with slope $= \frac{\text{rise}}{\text{run}} = \frac{L\sin\theta}{L\cos\theta} = \tan\theta$

Worked Example — Torque on a Slanted Wire

Then it follows that $dy = (\text{slope})\, dx = \tan\theta\ dx$. So the infinitesimal length of the slanted wire segment is

$$d\vec{\ell} = dx\ \hat{\boldsymbol{\imath}} + \tan\theta\ dx\ \hat{\boldsymbol{\jmath}}$$

Taking the cross-product gives us

$$\begin{aligned} d\vec{\ell} \times \vec{B} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ dx & \tan\theta\ dx & 0 \\ 0 & B_o & 0 \end{vmatrix} \\ &= B_o dx\ \hat{\boldsymbol{k}} \end{aligned}$$

So the differential force vector is

$$d\vec{F} = I\ d\vec{\ell} \times \vec{B} = I B_o\ dx\ \hat{\boldsymbol{k}}$$

Worked Example — Torque on a Slanted Wire

The differential torque vector is given by $d\vec{\tau} = \vec{r} \times d\vec{F}$. We already have $d\vec{F}$:

$$d\vec{F} = I\ d\vec{\ell} \times \vec{B} = IB_o\ dx\ \hat{\boldsymbol{k}}$$

The $\vec{r}$-vector points from the axis of rotation (the y-axis) to the point where force is being applied on the wire (so the $\vec{r}$-vector points straight to the right in the $\hat{\boldsymbol{\imath}}$ direction). Its magnitude is given by the x-coordinate of the wire:

$$\vec{r} = x\ \hat{\boldsymbol{\imath}}$$

So the final cross-product is

$$\begin{aligned} d\vec{\tau} = \vec{r} \times d\vec{F} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ x & 0 & 0 \\ 0 & 0 & IB_o\ dx \end{vmatrix} \\ &= -IB_o\ x\ dx\ \hat{\boldsymbol{\jmath}} \end{aligned}$$

Worked Example — Torque on a Slanted Wire

We determined the differential torque vector to be

$$d\vec{\tau} = -IB_o\, x\, dx\, \hat{\boldsymbol{\jmath}}$$

We integrate to find the total torque. Since we're integrating with respect to x, the lower and upper bounds on the integral inherit the smallest and largest values of x, respectively.

$$\begin{aligned} \vec{\tau} = \int d\vec{\tau} &= \int_0^{L\cos\theta} -IB_o\, x\, dx \\ &= \boxed{\frac{-IB_oL^2\cos^2\theta}{2}\hat{\boldsymbol{\jmath}}} \blacktriangleleft \end{aligned}$$

Torque on a Current Loop

If we have some current-carrying loop in a magnetic field (and only then), we can think of the torque on the loop in terms of the magnetic dipole instead. The magnetic dipole is given by

$$\vec{\mu} = NI\vec{A},$$

where N is the number of turns, I is the current, and $\vec{A}$ is the area vector. Recall that area vectors always point normal to the surface that they describe. Then the torque on the current-carrying loop is

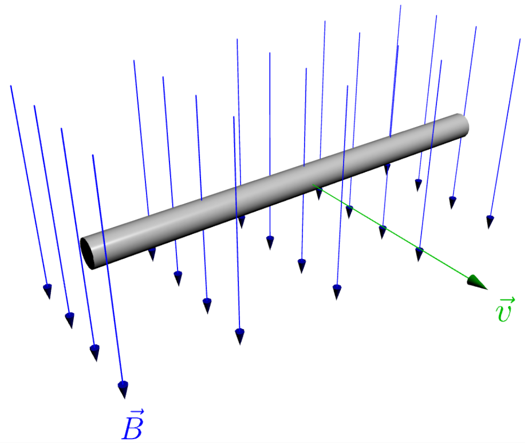
$$\vec{\tau} = \underbrace{\left(NI\vec{A}\right)}_{\vec{\mu}} \times \vec{B}$$

Motional EMF

Sometimes, the *motion* of charges through a magnetic field can *induce* an emf. Let's take a look at the mechanics behind this.

Motional EMF

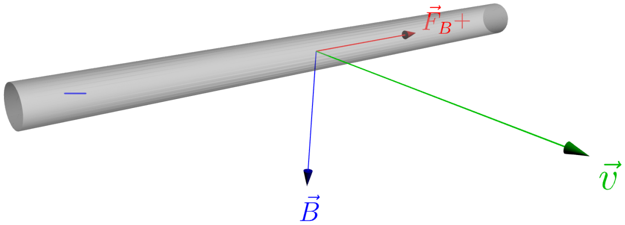
Suppose we have a conducting bar moving with velocity $\vec{v}$ in the presence of a magnetic field $\vec{B}$, as shown.



Motional EMF

There are charges in the conducting bar. And those charges are moving perpendicular to a magnetic field. So those charges are subject to a force given by

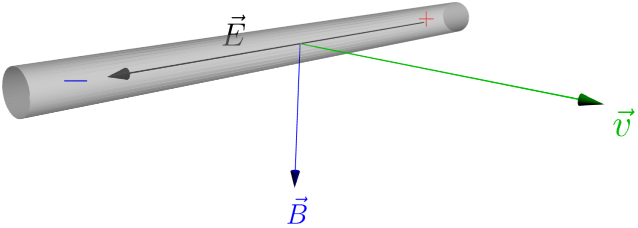
$$\vec{F} = q\vec{v} \times \vec{B}$$



Consequently, positive charges are pushed to one end of the rod, leaving a residual negative charge at the other end. (The force $\vec{F}_B$ shown in the picture above is the force exerted on a positive charge — the force on a negative charge is in the opposite direction.)

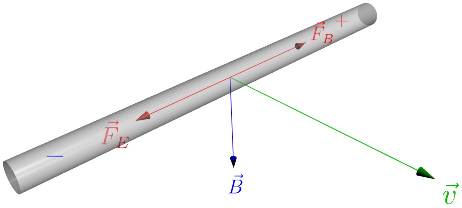
Motional EMF

This separation of charges sets up an electric field. Remember that electric fields always point from positive charges to negative charges.



Motional EMF

So an equilibrium is established, with equal and opposite electric and magnetic forces.

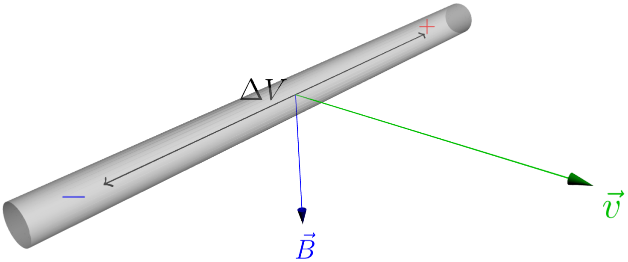


$$\overbrace{\left|q\vec{v}\times\vec{B}\right|}^{F_B} = \overbrace{\left|q\vec{E}\right|}^{F_E} \quad\Longrightarrow\quad \cancel{q}vB = \cancel{q}E \quad\Longrightarrow\quad E = vB$$

(As before, the forces $\vec{F}_E$ and $\vec{F}_B$ in the picture above are the electric and magnetic forces on a *positive* charge, respectively. Both of these forces point in the opposite direction for negative charges.)

Motional EMF

As such, the charges are fixed in place and there's a voltage difference across the rod. This is known as a *motional emf*.



To determine this voltage difference, we integrate across the length of the bar:

$$|\Delta V| = \int \vec{E} \cdot d\vec{\ell} = \int E \, d\ell = \int (vB) \, d\ell = vBL$$

Worked Example — Motional EMF

▶ A Boeing 787-8, with a wingspan of 60.12 m, departed from the South Carolina final assembly and delivery facility. Once it reached its cruising speed of Mach 0.85 (903 km/h), it transversed a region where the Earth's magnetic field of 6.0×10^{-5} T was directed 10° downward from the horizontal. What is voltage difference across the aircraft's wingtips?

Solution. This is a motional emf problem, so let's consider what happens to the charges in the aircraft's wings.

Worked Example — Motional EMF

The charges in the aircraft's wings are subject to a magnetic force:

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

As the charges are pushed towards the ends of the wings, they set up an electric field. Thus, there's a competing electric force:

$$\vec{F}_E = q\vec{E}$$

An equilibrium will be established, which insinuates that the magnitudes of these two forces are equal:

$$\left| q\vec{v} \times \vec{B} \right| = \left| q\vec{E} \right|$$

$$\cancel{q}vB\sin\theta = \cancel{q}E$$

where θ is the angle between the velocity $\vec{v}$ and the magnetic field $\vec{B}$ (this is the given angle of 10°).

Worked Example — Motional EMF

Thus, we have

$$E = vB\ \sin\theta$$

Integrating along the wingspan to get the voltage difference gives us

$$|\Delta V| = \int \vec{E} \cdot d\vec{\ell} = \int_0^L vB\ \sin\theta\ d\ell = vBL\ \sin\theta$$

Worked Example — Motional EMF

Now we're just about ready to plug in numbers. Note that we have to convert the given velocity to SI-units:

$$903\,\mathrm{km/h} \cdot \frac{1000\,\mathrm{m}}{1\,\mathrm{km}} \cdot \frac{1\,\mathrm{h}}{3600\,\mathrm{s}} = 250.83\,\mathrm{m/s}$$

So the expected voltage difference is

$$|\Delta V| = (250.83\,\mathrm{m/s})\left(6.0 \times 10^{-5}\,\mathrm{T}\right)(60.12\,\mathrm{m})\sin(10^\circ)$$
$$\Longrightarrow \boxed{\left|\Delta V\right| = 0.157\,\mathrm{V}} \blacktriangleleft$$

Faraday's Law

Faraday's law reads

$$\oint \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{\ell} = -\frac{d}{dt}\left(\int \vec{B} \cdot d\vec{A}\right)$$

which is scary. Let's break it apart.

Right-Hand Side of Faraday's Law

The right-hand side of Faraday's law has the time-derivative of the magnetic flux through some surface:

$$\frac{d}{dt}\underbrace{\left(\int \vec{B}\cdot d\vec{A}\right)}_{\Phi_B} = \frac{d\Phi_B}{dt}$$

That surface could be *any open surface*. A loop, for instance.

Left-Hand Side of Faraday's Law

The left-hand side of Faraday's law is a *closed* line integral:

$$\oint \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{\ell}$$

Let's break this integral apart at the addition:

$$\underbrace{\oint \vec{E} \cdot d\vec{\ell}}_{\substack{\text{This term asserts that} \\ \text{integrals of } \textit{curly} \text{ electric} \\ \text{fields (those produced by} \\ \text{changing magnetic fields)} \\ \text{over a closed loop gives} \\ \text{some voltage. Physically,} \\ \text{this corresponds to, say,} \\ \text{wiggling a magnet in a} \\ \text{stationary loop.}}} \quad + \quad \underbrace{\oint \left(\vec{v} \times \vec{B}\right) \cdot d\vec{\ell}}_{\substack{\text{This term asserts that the} \\ \text{motion of the loop itself also} \\ \text{creates voltage. Physically,} \\ \text{this corresponds to say,} \\ \text{wiggling a loop near a} \\ \text{stationary magnet.}}}$$

Faraday's Law

Altogether, it doesn't matter whether those voltages come from moving a magnet near a stationary loop or moving a loop near a stationary magnet. Either way, those are *induced* voltages:

$$\oint \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{\ell} = \text{emf}$$

We can therefore express Faraday's law in a much simpler form:

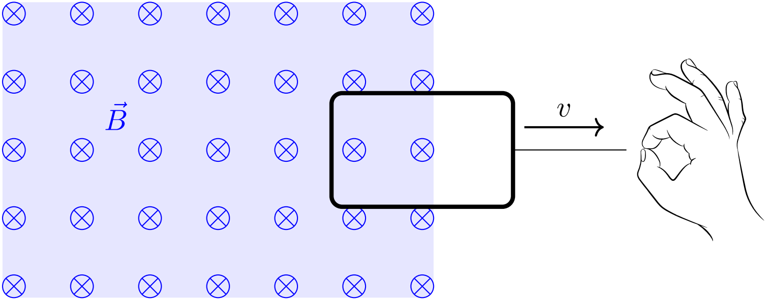
$$\oint \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\Downarrow$$

$$\text{emf} = -\frac{d\Phi_B}{dt}$$

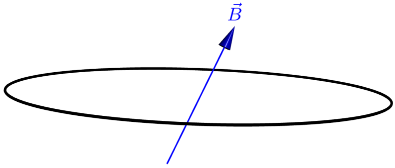
Here, Faraday's law asserts that whenever the magnetic flux through an open surface changes *for any reason*, there'll be an induced emf.

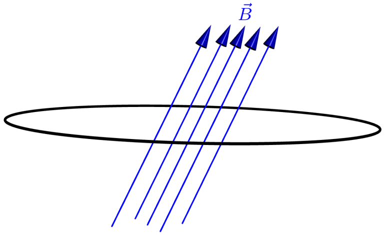
For instance, the magnetic flux through the loop below is changing with time because of the *motion of the loop*, even if the magnetic field itself is unchanged:



Faraday's Law for Electromagnetic Induction

Another possibility is when the magnetic field itself changes with time:





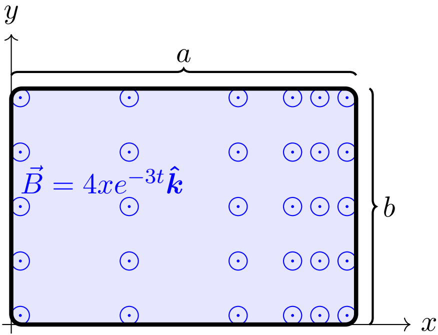
The changing magnetic flux through the loop induces an emf.

Lenz's Law

When a thing (current, emf, or a $\vec{B}$-field) is being induced by Faraday's law, that thing will act in such a way as to oppose the change that caused it.

Worked Example — A Rectangular Coil

A rectangular coil sits in the xy-plane as shown below. It has width a and height b and 32 turns. There is a spatially-varying magnetic field pointing out of the page (in the positive z-direction) whose magnitude is $B = 4x\ e^{-3t}$.



Find the magnitude of the induced emf in the coil and find the direction of the induced current.

Worked Example — A Rectangular Coil

Solution. We will apply Faraday's law, which asserts that the induced emf (through one turn of the loop) is

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

So the magnitude of the induced emf in the entire coil is

$$|\mathcal{E}| = N \cdot \frac{d\Phi_B}{dt}$$

So let's start by finding the flux through one turn of the coil.

Worked Example — A Rectangular Coil

The most general expression for the magnetic flux is

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

We know that the magnetic field is

$$\vec{B} = 4xe^{-3t}\ \hat{\boldsymbol{k}}$$

What about the differential area vector dA? Since the rectangular coil seems to lie in the xy-plane, the differential area vector is

$$d\vec{A} = dx\ dy\ \hat{\boldsymbol{k}}$$

Why the $\hat{\boldsymbol{k}}$? Recall that area vectors are always perpendicular to the area that they describe, and the $\hat{\boldsymbol{k}}$ vector is perpendicular to the xy-plane.

Worked Example — A Rectangular Coil

Now let's take the dot product. We have

$$\vec{B} \cdot d\vec{A} = \left(4xe^{-3t}\ \hat{\boldsymbol{k}}\right) \cdot \left(dx\ dy\ \hat{\boldsymbol{k}}\right) = 4x\ e^{-3t}\ dx\ dy$$

(Note that $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$.) Now we're ready to integrate! The flux through one turn of the rectangular coil is

$$\begin{aligned} \Phi_B = \int \vec{B} \cdot d\vec{A} &= \int_0^b \int_0^a 4x\ e^{-3t}\ dx\ dy \\ &= 4e^{-3t} \underbrace{\left(\int_0^b dy\right)}_{b} \underbrace{\left(\int_0^a x dx\right)}_{a^2/2} \\ &= 2ba^2 e^{-3t} \end{aligned}$$

Worked Example — A Rectangular Coil

Finally, we can apply Faraday's law (that is, take a time derivative) to find the induced emf. The magnitude of the induced emf in the entire coil is therefore

$$|\mathcal{E}| = \left| -N \frac{d\Phi_B}{dt} \right| = \left| -N \frac{d}{dt} \left(2ba^2 e^{-3t} \right) \right| = 6Nba^2 e^{-3t}$$

So the magnitude of the induced emf, as a function of time, is

$$\boxed{\mathcal{E}(t) = 6Nba^2 e^{-3t}}$$

Worked Example — A Rectangular Coil

Now we need to determine the direction of the induced current. This is where Lenz's law becomes incredibly useful. In fact, Lenz's law asserts that the induced current will be in a direction opposite to the change that caused it.

Note that the magnetic field initially points *out of the page*. And since the magnetic field assumes the form $\vec{B} = 4xe^{-3t}\ \hat{\boldsymbol{k}}$, as time progresses the magnetic field points *less out of the page* — so the change that caused the induction was a magnetic field that points *less out out the page*.

As such, the induced current must oppose this change. In fact, the induced current must be *counterclockwise*. On its own, a counterclockwise current will produce a magnetic field that points *more out the page*.

Thus, the induced current is *counterclockwise*. ◀

Mutual Inductance

For a pair of coils, the mutual inductance is the quantity that relates how the the current in one coil affects the flux in the other:

$$M = \overbrace{N_2\ \Phi_{B,\text{ one turn of 2}}}^{\text{total flux in 2}} / I_1$$

Inductance is measured in **Henries** (H):

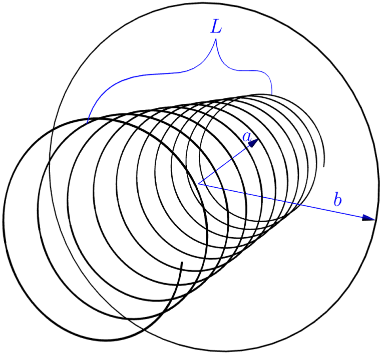
$$1\,\mathrm{H} = 1\,\frac{\mathrm{T}\cdot\mathrm{m}^2}{\mathrm{A}} = 1\,\mathrm{J}/\mathrm{A}^2$$

And Faraday's law tells us that we can take the time-derivative of magnetic flux to find the induced emf. As such, the induced emf is

$$\mathcal{E}_1 = -M\frac{dI_2}{dt} \qquad \Longleftrightarrow \qquad \mathcal{E}_2 = -M\frac{dI_1}{dt}$$

Worked Example — Solenoid Through a Coil

- A long solenoid of length $L = 18.5\,\text{cm}$ has $N_a = 25$ windings and a cross-sectional radius $a = 1.90\,\text{cm}$. The solenoid goes through the center of a circular coil of wire with $N_b = 31$ windings and radius $b = 5.50\,\text{cm}$. The current in the circular coil changes according to $I(t) = 7.0t^2 + 1.0t$. What is the magnitude of the induced emf in the solenoid at $t = 3.00\,\text{s}$?



Worked Example — Solenoid Through a Coil

Solution. We'll start by finding the mutual inductance for this pair of coils. Then we'll use Faraday's law to determine the induced emf in the solenoid at some time t.

Worked Example — Solenoid Through a Coil

The mutual inductance of a pair of coils is given as

$$M = N_2 \; \Phi_{B,\text{ one turn of 2}}/I_1$$

This means either treating the coil as producing the magnetic field and finding the flux through the solenoid:

$$M = \frac{N_{\text{solenoid}}\Phi_{B,\text{ one turn of solenoid}}}{I_{\text{coil}}}$$

Or treating the solenoid as producing the magnetic field and finding the flux through the coil:

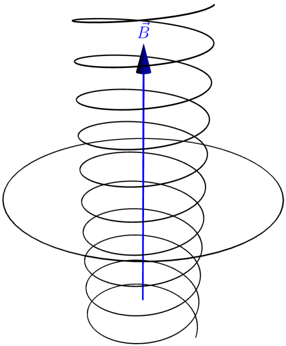
$$M = \frac{N_{\text{coil}}\Phi_{B,\text{ one turn of coil}}}{I_{\text{solenoid}}}$$

Worked Example — Solenoid Through a Coil

By far the easier approach is to find the flux through the coil. Recall that a solenoid has a nearly-uniform magnetic field within itself:

$$B_{\text{solenoid}} = \mu_o n I$$

and nearly zero magnetic field outside itself:



Worked Example — Solenoid Through a Coil

The flux through one turn of the coil is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = B \cdot A = (\mu_o n I_{\text{solenoid}}) \cdot \left(\pi a^2\right)$$

Be super careful here! Even though we're considering the flux through the coil — which has an area of πb^2 — the magnetic field is zero outside the solenoid! That is, the only part of the coil's area through which there is a non-zero flux is the part that containts the solenoid.

The mutual inductance is then

$$M = \frac{N_b \overbrace{\left(\mu_o \frac{N_a}{L} I_{\text{solenoid}}\right)\left(\pi a^2\right)}^{\Phi_{B,\text{ one turn of coil}}}}{I_{\text{solenoid}}} = \frac{\mu_o N_a N_b \pi a^2}{L} = 5.97 \times 10^{-6}\,\text{H}$$

Worked Example — Solenoid Through a Coil

Now that we have the mutual inductance, we can find the induced emf in the solenoid at some time t via

$$\mathcal{E}_{\text{solenoid}} = -M\frac{dI_{\text{coil}}}{dt}$$

The current in the coil is given as $I(t) = 7.0t^2 + 1.0t$, so $\frac{dI}{dt} = 14.0t + 1.0$. And so the magnitude of the induced emf in the coil at some time t is

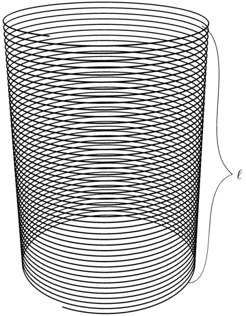
$$\mathcal{E}_{\text{solenoid}} = \overbrace{\left(\frac{\mu_o N_a N_b \pi a^2}{L}\right)}^{M} \underbrace{(14.0t + 1.0)}_{\frac{dI}{dt}}$$

In particular, at $t = 3.00\,\text{s}$, the induced emf in the solenoid is

$$\boxed{\mathcal{E}_{\text{solenoid}}\Big|_{t=3.00} = 2.57 \times 10^{-4}\,\text{V}} \blacktriangleleft$$

Self Inductance

Solenoids or coils of wire can even create a magnetic flux through itself! Consider, for instance, a current-carrying solenoid with radius R, length ℓ, and N number of turns:



The magnetic field inside a solenoid is

$$B_{\text{solenoid}} = \mu_o n I = \mu_o \left(\frac{N}{\ell} \right) I$$

Self Inductance

The total flux through the solenoid is therefore

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = B \cdot A = \left[\mu_o \left(\frac{N}{\ell}\right) I\right] \cdot \left[N\pi R^2\right]$$

$$\implies \Phi_B = \underbrace{\left(\frac{\mu_o N^2 \pi R^2}{\ell}\right)}_{L} I$$

That quantity L is known as the *self-inductance* of the solenoid, and it plays a similar role to mutual inductance (without the other coil, of course).

Self Inductance

In fact, we can apply Faraday's law (that is, take the time-derivative of the flux) to find the induced emf:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(LI) = -L\frac{dI}{dt}$$

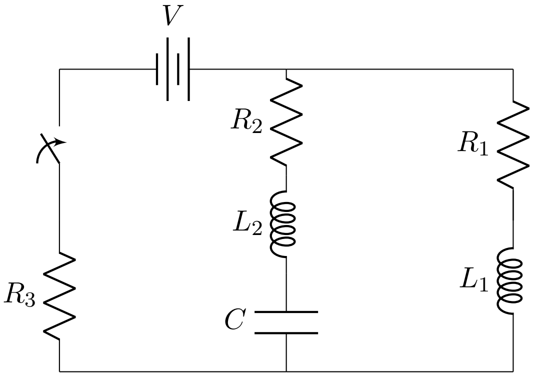
So inductors *oppose changes in current* through them.

Un-energized inductors (when current gets turned on suddenly) act like open switches

Energized inductors (when current has been flowing for a long time) act like ideal wires.

Worked Example — Inductors in a Circuit

- Consider the following circuit:



Assume that $R_1 > R_2 > R_3$. Rank the currents in through the three resistors after the switch has been closed for a very long time.

Worked Example — Inductors in a Circuit

Solution. This problem isn't going to require any math — instead, we can just recall that inductors oppose changes in current through them to solve this problem.

So in this case, the current has been on for a long time, meaning that an inductor would act like an ideal wire. However, we also have a capacitor. And capacitors do the opposite — a charged capacitor acts like a open switch, meaning there is no current in that entire middle segment.

So resistors 1 and 3 have the same current, and resistor 2 has zero current after the switch has been closed for a long time. ◄

LR Circuits

Inductor-Resistor circuits, also known as LR circuits, are similar to the RC circuits we studied earlier:

$$\underbrace{I(t) = I_{\text{final}}\left(1 - e^{-t/(L/R)}\right)}_{\text{There's some emf source in the circuit and the current reaches some final value as time } t \text{ approaches infinity}}$$

$$\underbrace{I(t) = I_{\text{initial}}e^{-t/(L/R)}}_{\text{There is no emf source in the circuit and the current decays to zero as time } t \text{ approaches infinity}}$$