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On the multi-domain compact finite difference relaxation method for high dimensional chaos: The nine-dimensional Lorenz system

J.N. Kouagou [a], P.G. Dlamini [b,*], S.M. Simelane [b]

[a] *African Institute of Mathematical Sciences, Muizenberg, South Africa*
[b] *Department of Mathematics and Applied Mathematics, University of Johannesburg, P.O. Box 17011, Doornfontein 2028, South Africa*

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Abstract In this paper, we implement the multidomain compact finite difference method to numerically study high dimensional chaos by considering the nine-dimensional Lorenz system. Most of the existing numerical methods converge slowly for this kind of problems and this results in inaccurate approximations. Though highly accurate, the compact finite difference method becomes less accurate for problems characterized by chaotic solutions, even with an increase in the number of grid points. As a result, in this work, we adopt the multidomain approach. This approach remarkably improves the results as well as the efficiency of the method.

1. Introduction

The theory of chaos is an area of mathematics that studies dynamical systems that are characterized by highly oscillating solutions and high sensitivity to small perturbations of the initial conditions. It was first discovered by Edward Lorenz in 1963through his work on weather prediction [1]. Since then, several researchers have developed many chaotic dynamical systems arising from science and engineering [2–5]. The complexity of the chaotic behaviour of these systems is often reliant on the number of positive Lyapunov exponents in the system. A chaotic system has got only one positive Lyapunov exponent, whereas systems with at least two positive Lyapunov exponents are said to be hyperchaotic. The concept of hyperchaos was first studied by Rössler [7].

The nature of the solutions of chaotic systems means that it is not possible to compute them exactly. It is, therefore, necessary to develop numerical methods that give a very good approximation of the exact solution. Several numerical methods have been used to solve chaotic dynamical systems [8–13]. Due to the high sensitivity to initial conditions, numerical solutions of chaotic systems are sensitive to numerical errors. Some common numerical methods, such as the higher-order explicit Runge–Kutta methods, become less efficient when large time intervals are considered. This led to the idea that chaotic systems could only be solved numerically over short time intervals. As a result, many researchers have introduced the idea of domain decomposition to circumvent this limitation. It involves splitting the time interval into several non-

* Corresponding author.
E-mail address: phgdlamini@gmail.com (P.G. Dlamini).
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overlapping smaller intervals and solving efficiently over each subinterval. The advantage of multi-domain techniques over single-domain computations is that they use small-sized and well-conditioned matrices. This enhances the efficiency of the methods and improves the accuracy of the results.

Examples of numerical methods where the multi-domain technique has successfully been used include: the multi-stage Adomian decomposition method [14], the multi-stage homotopy analysis method [15], the multistage differential transform method [16], the multi-stage variational iteration method [17], the multi-stage homotopy perturbation method [18], the multi-stage spectral relaxation method [19,20], the piecewise quasi-linearization method [21] and the piecewise spectral parametric iteration method [22]. Most of these methods are semi-analytic. The drawback with the multi-domain methods based on analytical approximations is that it becomes tedious and time-consuming to analytically integrate in each of the several subdomains.

In this article, we present a method based on blending the ideas of the Gauss–Seidel method, sixth-order compact finite difference schemes (CFD6) and the multi-domain technique to solve high dimensional chaotic systems [23]. The method is called the multidomain compact finte difference relaxation method (MD-CFDRM). The advantage of the compact schemes is their high accuracy, which is achieved with a relatively small number of grid points. Compact finite difference schemes have been used to solve different types of differential equations. These include two-point boundary value problems [24], integrodifferential equations [25], and partial differential equations [26–29]. We examine the applicability of the MD-CFDRM in solving high dimensional chaotic systems. In particular, we consider the nine-dimensional (9D) Lorenz system derived by Reiterer et al. [30]. The system was derived by applying a triple Fourier expansion to the Boussinesq-Oberbeck equations governing thermal convection in a 3D spatial domain by using an approach similar to the well known 3D Lorenz's system. The 9D Lorenz is given by

$$\begin{cases} \dot{x}_1 = -\sigma b_1 x_1 - \sigma b_2 x_7 - x_2 x_4 + b_4 x_4^2 + b_3 x_3 x_5 \\ \dot{x}_2 = -\sigma x_2 - 0.5\sigma x_9 + x_1 x_4 - x_2 x_5 + x_4 x_5 \\ \dot{x}_3 = -\sigma b_1 x_3 + \sigma b_2 x_8 + x_2 x_4 - b_4 x_2^2 - b_3 x_1 x_5 \\ \dot{x}_4 = -\sigma x_4 + 0.5\sigma x_9 - x_2 x_3 - x_2 x_5 + x_4 x_5 \\ \dot{x}_5 = -\sigma b_5 x_5 + 0.5 x_2^2 - 0.5 x_4^2 \\ \dot{x}_6 = -b_6 x_6 + x_2 x_9 - x_4 x_9 \\ \dot{x}_7 = -r x_1 - b_1 x_7 + 2 x_5 x_8 - x_4 x_9 \\ \dot{x}_8 = r x_3 - b_1 x_8 - 2 x_5 x_7 + x_2 x_9 \\ \dot{x}_9 = -r x_2 + r x_4 - x_9 - 2 x_2 x_6 + 2 x_4 x_6 + x_4 x_7 - x_2 x_8 \end{cases} \tag{1}$$

where the constant parameters b_i are defined by

$$b_1 = 4\frac{1+a^2}{1+2a^2}, \qquad b_2 = \frac{1+2a^2}{2(1+a^2)}, \qquad b_3 = 2\frac{1-a^2}{1+a^2}$$

$$b_4 = \frac{a^2}{1+a^2}, \qquad b_5 = \frac{8a^2}{1+2a^2}, \qquad b_6 = \frac{4}{1+2a^2}.$$

The article is organized as follows: In Section 2, we show the construction of the MD-CFDRM to solve system (1). In Section 3, we discuss the convergence of the method. In Section 4, we present and discuss the results obtained. Lastly, final remarks and the conclusion are presented in Section 5.

2. Description of the MD-CFDRM for the 9D Lorenz system

In this section, we outline the implementation of the multidomain compact relaxation method (MD-CFDRM) to solve the 9D Lorenz system. Without loss of generality, system (1) can be expressed in the form

$$\dot{x} + \Lambda x + \mathbf{F}(x) = 0, \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_9(t)]^T, \Lambda$ is a 9×9 matrix with entries $\alpha_{i,j}$, $(i,j = 1,2,\ldots 9)$ given by

$$\Lambda = \begin{pmatrix} \sigma b_1 & 0 & 0 & 0 & 0 & 0 & \sigma b_2 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0.5\sigma \\ 0 & 0 & \sigma b_1 & 0 & 0 & 0 & 0 & -\sigma b_2 & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 & 0 & 0 & -0.5\sigma \\ 0 & 0 & 0 & 0 & \sigma b_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6 & 0 & 0 & 0 \\ r & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 \\ 0 & 0 & -r & 0 & 0 & 0 & 0 & b_1 & 0 \\ 0 & r & 0 & -r & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\mathbf{F}(x)$ is a vector of the nonlinear components of system (1) given by

$$\mathbf{F}(x) = \begin{pmatrix} x_2 x_4 - b_4 x_4^2 - b_3 x_3 x_5 \\ -x_1 x_4 + x_2 x_5 - x_4 x_5 \\ -x_2 x_4 + b_4 x_2^2 + b_3 x_1 x_5 \\ x_2 x_3 + x_2 x_5 - x_4 x_5 \\ -0.5 x_2^2 + 0.5 x_4^2 \\ -x_2 x_9 + x_4 x_9 \\ -2 x_5 x_8 + x_4 x_9 \\ 2 x_5 x_7 - x_2 x_9 \\ 2 x_2 x_6 - 2 x_4 x_6 - x_4 x_7 + x_2 x_8 \end{pmatrix}.$$

Next, we present the derivation of the CFD6 schemes used to approximate the first derivative of a function $v(\tau)$ over a given interval $[a,b]$. We discretize $[a,b]$ into a uniform partition of N equally spaced grid points $a = \tau_1 < \ldots < \tau_N = b$ given by $x = a + (i-1)h \quad (i = 1,2,3,\ldots,N)$, where $h = (b-a)/(N-1)$is the step size.

The first derivative is approximated using the following compact finite difference scheme

$$\alpha_{-1}\dot{v}_{i-1} + \dot{v}_i + \alpha_1 \dot{v}_{i+1} = \frac{1}{h}(\beta_{-2}v_{i-2} + \beta_{-1}v_{i-1} + \beta_0 v_i + \beta_1 v_{i+1} + \beta_2 v_{i+2}),$$
$$i = 3,4,\ldots,N-2, \tag{3}$$

where $v_i = v(\tau_i), \dot{v}_i = \dot{v}(\tau_i)$and $v_{i+p} = v(\tau_i + ph)$ for all $(i,p) \in \mathbb{N}^* \times \mathbb{Z}$.

The objective is to find the coefficients α_j, β_jso that the equality holds with accuracy $O(h^6)$. To achieve this, we expand both sides by Taylor series about the point $\tau_i$with respect to $h$to obtain:

$$\begin{cases} \dot{v}_{i-1} = \dot{v}(\tau_i) - h\ddot{v}(\tau_i) + \frac{h^2}{2} v^{(3)}(\tau_i) - \frac{h^3}{3!} v^{(4)}(\tau_i) + \frac{h^4}{4!} v^{(5)}(\tau_i) - \frac{h^5}{5!} v^{(6)}(\tau_i) + O\left(h^6\right) \\ \dot{v}_{i+1} = \dot{v}(\tau_i) + h\ddot{v}(\tau_i) + \frac{h^2}{2} v^{(3)}(\tau_i) + \frac{h^3}{3!} v^{(4)}(\tau_i) + \frac{h^4}{4!} v^{(5)}(\tau_i) + \frac{h^5}{5!} v^{(6)}(\tau_i) + O\left(h^6\right) \\ v_{i-2} = v(\tau_i) - 2h\dot{v}(\tau_i) + \frac{4h^2}{2} \ddot{v}(\tau_i) - \frac{8h^3}{3!} v^{(3)}(\tau_i) + \frac{16h^4}{4!} v^{(4)}(\tau_i) - \frac{32h^5}{5!} v^{(5)}(\tau_i) + \frac{64h^6}{6!} v^{(6)}(\tau_i) + O\left(h^7\right) \\ v_{i-1} = v(\tau_i) - h\dot{v}(\tau_i) + \frac{h^2}{2} \ddot{v}(\tau_i) - \frac{h^3}{3!} v^{(3)}(\tau_i) + \frac{h^4}{4!} v^{(4)}(\tau_i) - \frac{h^5}{5!} v^{(5)}(\tau_i) + \frac{h^6}{6!} v^{(6)}(\tau_i) + O\left(h^7\right) \\ v_{i+1} = v(\tau_i) + h\dot{v}(\tau_i) + \frac{h^2}{2} \ddot{v}(\tau_i) + \frac{h^3}{3!} v^{(3)}(\tau_i) + \frac{h^4}{4!} v^{(4)}(\tau_i) + \frac{h^5}{5!} v^{(5)}(\tau_i) + \frac{h^6}{6!} v^{(6)}(\tau_i) + O\left(h^7\right) \\ v_{i+2} = v(\tau_i) + 2h\dot{v}(\tau_i) + \frac{4h^2}{2} \ddot{v}(\tau_i) + \frac{8h^3}{3!} v^{(3)}(\tau_i) + \frac{16h^4}{4!} v^{(4)}(\tau_i) + \frac{32h^5}{5!} v^{(5)}(\tau_i) + \frac{64h^6}{6!} v^{(6)}(\tau_i) + O\left(h^7\right) \end{cases} \tag{4}$$

Substituting Eq. (4) into Eq. (3), we obtain the following:

$$\begin{aligned} &(\alpha_{-1} + 1 + \alpha_1)\dot{v}(\tau_i) + h(\alpha_1 - \alpha_{-1})\ddot{v}(\tau_i) + \tfrac{h^2}{2}(\alpha_{-1} + \alpha_1)v^{(3)}(\tau_i) + \tfrac{h^3}{3!}(\alpha_1 - \alpha_{-1})v^{(4)}(\tau_i) \\ &+ \tfrac{h^4}{4!}(\alpha_{-1} + \alpha_1)v^{(5)}(\tau_i) + \tfrac{h^5}{5!}(\alpha_1 - \alpha_{-1})v^{(6)}(\tau_i) = \tfrac{1}{h}(\beta_{-2} + \beta_{-1} + \beta_0 + \beta_1 + \beta_2)v(\tau_i) \\ &+ (-2\beta_{-2} - \beta_{-1} + \beta_1 + 2\beta_2)\dot{v}(\tau_i) + h\left(2\beta_{-2} + \tfrac{1}{2}\beta_{-1} + \tfrac{1}{2}\beta_1 + 2\beta_2\right)\ddot{v}(\tau_i) \\ &+ \tfrac{h^2}{2}\left(-\tfrac{8}{3}\beta_{-2} - \tfrac{1}{3}\beta_{-1} + \tfrac{1}{3}\beta_1 + \tfrac{8}{3}\beta_2\right)v^{(3)}(\tau_i) + \tfrac{h^3}{3!}\left(4\beta_{-2} + \tfrac{1}{4}\beta_{-1} + \tfrac{1}{4}\beta_1 + 4\beta_2\right)v^{(4)}(\tau_i) \\ &+ \tfrac{h^4}{4!}\left(-\tfrac{32}{5}\beta_{-2} - \tfrac{1}{5}\beta_{-1} + \tfrac{1}{5}\beta_1 + \tfrac{32}{5}\beta_2\right)v^{(5)}(\tau_i) + \tfrac{h^5}{5!}\left(\tfrac{64}{6}\beta_{-2} + \tfrac{1}{6}\beta_{-1} + \tfrac{1}{6}\beta_1 + \tfrac{64}{6}\beta_2\right)v^{(6)}(\tau_i) + O\left(h^6\right) \end{aligned}$$

By equating coefficients of h^n, we obtain 7 equations which we solve to obtain the constants:

$$\alpha_{-1} = \frac{1}{3}, \alpha_1 = \frac{1}{3},\ \beta_{-2} = \frac{-1}{36},\ \beta_{-1} = \frac{-7}{9},\ \beta_0 = 0,\ \beta_1 = \frac{7}{9},\ \beta_2 = \frac{1}{36}.$$

As a result, Eq. (3) becomes

$$\frac{1}{3}\dot{v}_{i-1} + \dot{v}_i + \frac{1}{3}\dot{v}_{i+1} = \frac{1}{h}\left(\frac{-1}{36}v_{i-2} - \frac{7}{9}v_{i-1} + \frac{7}{9}v_{i+1} + \frac{1}{36}v_{i+2}\right)$$

for $i = 3, 4, \ldots, N-2$with accuracy $O\left(h^6\right)$.To maintain the sixth order accuracy at the end points, we adjust the formula (using more points in the right hand side expression). By the same process detailed above, we find

$$A = \begin{pmatrix} 1 & \frac{1}{3} & & & & \\ \frac{1}{3} & 1 & \frac{1}{3} & & & \\ & \frac{1}{3} & 1 & \frac{1}{3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{3} & 1 & \frac{1}{3} \\ & & & & \frac{1}{3} & 1 \end{pmatrix}_{(N-1)\times(N-1)}, \quad K = \frac{1}{h}\begin{pmatrix} -\frac{7}{45}v_1 \\ -\frac{1}{36}v_1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{(N-1)\times 1},$$

$$\begin{aligned} \dot{v}_2 + \tfrac{1}{3}\dot{v}_3 = {} & \tfrac{1}{h}\left(-\tfrac{7}{45}v_1 - \tfrac{17}{12}v_2 + \tfrac{83}{36}v_3 - \tfrac{11}{9}v_4 + \tfrac{2}{3}v_5 - \tfrac{37}{180}v_6 + \tfrac{1}{36}v_7\right) \quad \text{for } i = 2, \\ \tfrac{1}{3}\dot{v}_{N-2} + \dot{v}_{N-1} + \tfrac{1}{3}\dot{v}_N = {} & \tfrac{1}{h}\left(\tfrac{35}{36}v_N - \tfrac{7}{12}v_{N-1} + \tfrac{7}{36}v_{N-2} - v_{N-3} + \tfrac{7}{12}v_{N-4} - \tfrac{7}{36}v_{N-5} + \tfrac{1}{36}v_{N-6}\right) \quad \text{for } i = N-1 \text{ and} \\ \tfrac{1}{3}\dot{v}_{N-1} + \dot{v}_N = {} & \tfrac{1}{h}\left(\tfrac{451}{180}v_N - \tfrac{1003}{180}v_{N-1} + \tfrac{20}{3}v_{N-2} - \tfrac{55}{9}v_{N-3} + \tfrac{125}{36}v_{N-4} - \tfrac{67}{60}v_{N-5} + \tfrac{7}{45}v_{N-6}\right) \quad \text{for } i = N. \end{aligned}$$

In matrix form, we can write the above as

$$A\dot{V} = BV + K,$$

where

$$V = \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix}_{(N-1)\times 1} \text{ and } B = \frac{1}{h}\begin{pmatrix} -\frac{17}{12} & \frac{83}{36} & -\frac{11}{9} & \frac{2}{3} & -\frac{37}{180} & \frac{1}{36} & & \\ -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & & & & \\ -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & \\ & & & -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} \\ & \frac{1}{36} & -\frac{7}{36} & \frac{7}{12} & -1 & \frac{7}{36} & -\frac{7}{12} & \frac{35}{36} \\ & \frac{7}{45} & -\frac{67}{60} & \frac{125}{36} & -\frac{55}{9} & \frac{20}{3} & -\frac{1003}{180} & \frac{451}{180} \end{pmatrix}_{(N-1)\times(N-1)}.$$

Since $A$is invertible (linearly independent rows), we have equivalently,

$$\dot{V} = EV + H, \tag{5}$$

where $E = A^{-1}B$and $H = A^{-1}K$.

To solve system (2), we first decompose the time domain $I = [0, T]$ into M subdomains of uniform length $\frac{T}{M}$, with each subdomain

$$I_j = [t_j, t_{j+1}], \quad j = 0, 1, \ldots, M-1$$

and having the property

$$\bigcup_{j=0}^{M-1} [t_j, t_{j+1}] = [0, T].$$

Using Eq. (5) to approximate the first derivatives in system (2), with $\dot{X} = EX + H$ as defined above, we get the following system of algebraic equations

$$(\mathbf{E} + \alpha_{r,r}\mathbf{I})X_r^j + H_r^j + \sum_{i=1,\ i\neq r}^{9} \alpha_{r,i}X_i^j + F_r^j(X) = 0,$$
$$r = 1, 2, \ldots, 9 \tag{6}$$

where τ_i^j and $X_r^j = \left[x_r(\tau_1^j), x_r(\tau_2^j), \ldots, x_r(\tau_N^j)\right]^T$ represent the grid points and the approximate solution in each subdomain, respectively, and

$$X = [X_1, X_2, X_3, \ldots, X_9],$$
$$F_r(X) = [F_r(X(\tau_1)), F_r(X(\tau_2)), \ldots, F_r(X(\tau_N))]^T.$$

To solve system (6), we use the Gauss–Seidel idea of solving algebraic equations to get the following iteration scheme

$$(\mathbf{E} + \alpha_{r,r}\mathbf{I})X_{r,s+1}^j = -H_r^j - \sum_{i=1}^{r-1}\alpha_{r,i}X_{i,s+1}^j - \sum_{i=r+1}^{9}\alpha_{r,i}X_{i,s}^j - F_{r,s,s+1}^j, \tag{7}$$

where $X_{r,s+1}$ is the approximation of each x_r at the $(s+1)^{th}$ iteration and

$$F_{r,s,s+1} = F_r(X_{1,s+1}, X_{2,s+1}, \ldots, X_{r-1,s+1}, X_{r,s}, \ldots, X_{9,s}).$$

We remark that the nonlinear terms are evaluated using values from previous iteration. Therefore

$$X_{r,s+1} = \mathscr{A}_r^{-1}\mathscr{B}_r, \tag{8}$$

where

$$\mathscr{A}_r = \mathbf{E} + \alpha_{r,r}\mathbf{I}, \tag{9}$$

$$\mathscr{B}_r = -H_r^j - \sum_{i=1}^{r-1}\alpha_{r,i}X_{i,s+1}^j - \sum_{i=r+1}^{9}\alpha_{r,i}X_{i,s}^j - F_{r,s,s+1}^j. \tag{10}$$

The solution of system (2) on $[0, T]$, is then given by

$$x_r = \bigcup_{j=1}^{M} X_r^j(\tau^j).$$

3. Convergence

System (6), can be written as the block matrix system

$$AX = B, \tag{11}$$

where

$$A = \begin{pmatrix} (\mathbf{E} + \alpha_{1,1}\mathbf{I}) & \alpha_{1,2}\mathbf{I} & \alpha_{1,3}\mathbf{I} & \ldots & \alpha_{1,9}\mathbf{I} \\ \alpha_{2,1}\mathbf{I} & (\mathbf{E} + \alpha_{2,2}\mathbf{I}) & \alpha_{2,3}\mathbf{I} & \ldots & \alpha_{2,9}\mathbf{I} \\ & & & \ddots & \\ \alpha_{9,1}\mathbf{I} & \alpha_{9,2}\mathbf{I} & \alpha_{9,3}\mathbf{I} & \ldots & (\mathbf{E} + \alpha_{9,9}\mathbf{I}) \end{pmatrix},$$

$$X^j = \begin{pmatrix} X_1^j \\ X_2^j \\ \vdots \\ X_9^j \end{pmatrix} \text{and} B^j = \begin{pmatrix} -H_1^j - F_1^j \\ -H_2^j - F_2^j \\ \vdots \\ -H_9^j - F_9^j \end{pmatrix}$$

Theorem 1. *For any $X^{(0)} \in \mathscr{R}^n$, the sequence $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X = A^{-1}B$for each $k \geqslant 1$, converge to the unique solution $\overline{X}$ if the matrix A is strictly diagonally dominant.*

To prove convergence of the method, we show that matrix A in system (11) is diagonally dominant. Since we are dealing with the block matrix A, and instead of considering the modulus of the coefficients, we consider the norm of the matrices inside A. For any $m \times n$ matrix $A = (a_{i,j})_{i,j}$ with real entries, we have the norm equivalence

$$\|A\|_{max} \leqslant \|A\|_2 \leqslant \sqrt{mn}\|A\|_{max}, \tag{12}$$

where $\|A\|_{max} = max_{\{i,j\}}|a_{i,j}|$and $\|.\|_2$ is the spectral norm.

From the definition of $\mathbf{E}$, if h is sufficiently small, then

$$\mathbf{E} \gg \alpha_{r,r}I, \quad r = 1, 2, \ldots, 9.$$

This implies that

$$\|\mathbf{E} + \alpha_{r,r}\mathbf{I}\|_{max} \approx |\frac{1}{h} \times C|,$$

where C is a non-zero constant, independent of h.

Similarly,

$$\sum_{j=1, j\neq r}^{9} \|\alpha_{r,j}\mathbf{I}\|_{max} = \sum_{j=1, j\neq r}^{9} |\alpha_{r,j}|.$$

Thus, for h sufficiently small, we have

$$\|\mathbf{E} + \alpha_{r,r}\mathbf{I}\|_{max} \approx |\frac{1}{h} \times C| \geqslant \sum_{j=1, j\neq r}^{9} |\alpha_{r,j}|_{max} = \sum_{j=1, j\neq r}^{9} \|\alpha_{r,j}I\|. \tag{13}$$

Therefore, the block matrix A is diagonally dominant in the sense of matrix norm. Hence, the method converges.

4. Results and discussion

In this section, we present the MD-CFDRM results for the 9D Lorenz system (1). We then compare the results with those of the Runge–Kutta (RK4) method. When the parameter value r is greater than 43.3, the system is hyperchaotic. Otherwise it is chaotic [30]. Therefore, we solve system (1) for $r = 14.1$ (chaotic cases) and $r = 55$ (hyperchaotic cases), with the initial condition $x(0) = \{0.01, 0, 0.01, 0, 0, 0, 0, 0, 0.01\}$ and $\sigma = 0.5$. In Figs. 1 and 2, we plot the phase projections on the $x_6 - x_7$ and $x_6 - x_9$ planes for varying values of r, respectively. Our phase portraits are consistent with those of Reiterer et al. [30], which further attest that MD-CFDRM can handle high dimensional chaotic systems.

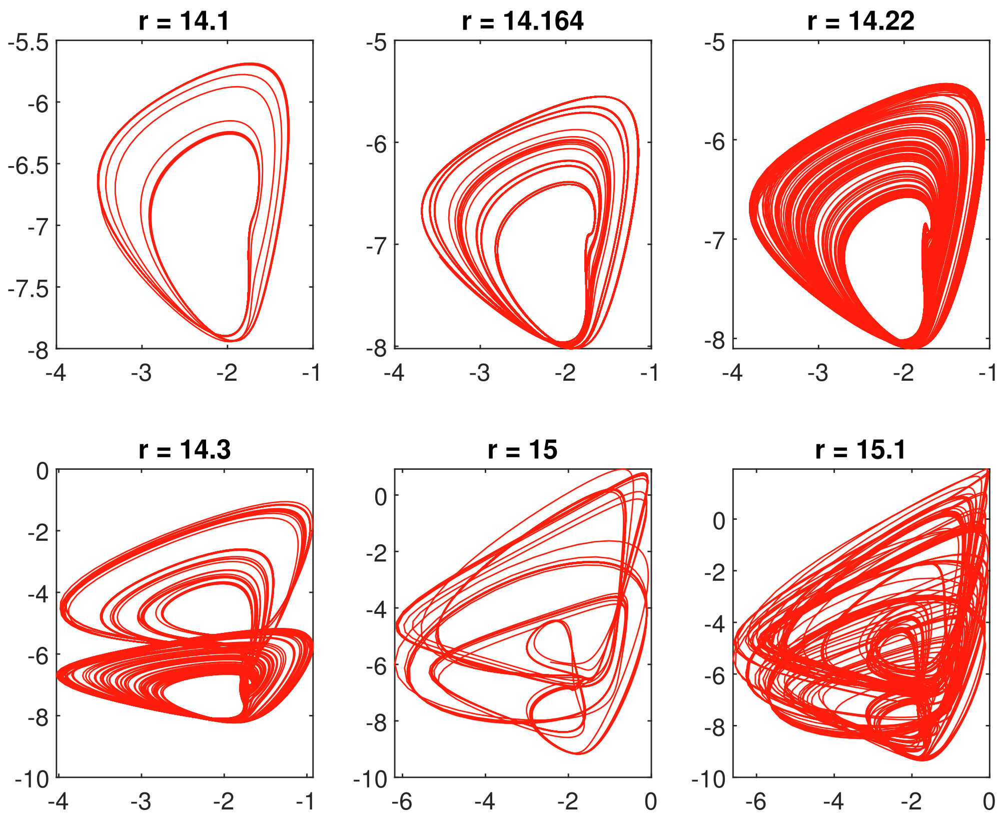

Fig. 1 Phase portraits for the 9D attractor on the $x_6 - x_7$ plane for various values of r.

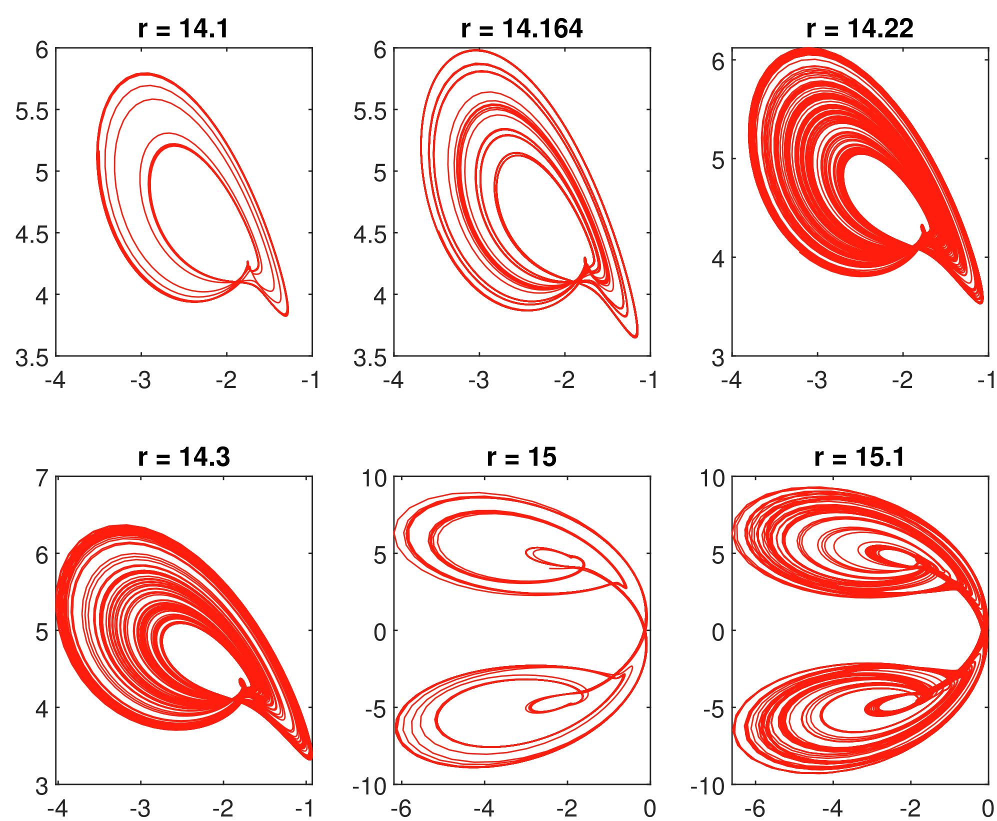

Fig. 2 Phase portraits for the 9D attractor on the $x_6 - x_9$ plane for various values of r.

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Table 1 Comparison of the MD-CFDRM and the RK4 solutions for $r = 14.1$.

t	x_1		x_2		x_3	
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
20	1.7794616020	1.7794616020	−0.7204522789	−0.7204522789	−0.6362110296	−0.6362110296
40	1.5609534149	1.5609534149	−0.2347187741	−0.2347187741	−1.2023505190	−1.2023505190
60	1.4855822940	1.4855822940	−0.1307250241	−0.1307250241	−0.2401679408	−0.2401679408
80	1.7958197536	1.7958197536	−0.7477525490	−0.7477525490	−0.9536794241	−0.9536794241
100	1.6599843283	1.6599843283	−0.1433734338	−0.1433734338	−0.9914698728	−0.9914698728
t	x_4		x_5		x_6	
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
20	0.7245177009	0.7245177009	−0.3269195959	−0.3269195959	−2.6127239408	−2.6127239408
40	0.8327202872	0.8327202872	−0.1787710447	−0.1787710447	−1.6858331785	−1.6858331785
60	0.9777946806	0.9777946806	−0.6137459610	−0.6137459610	−1.7451784330	−1.7451784330
80	0.5724218362	0.5724218362	−0.1220058785	−0.1220058785	−2.3524652982	−2.3524652982
100	0.8524360371	0.8524360371	−0.2153136081	−0.2153136081	−1.5321979302	−1.5321979302
t	x_7		x_8		x_9	
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
20	−7.3469555719	−7.3469555719	−4.9185099929	−4.9185099929	4.5741481236	4.5741481236
40	−7.0111342856	−7.0111342856	−6.2312240135	−6.2312240135	4.1679032967	4.1679032967
60	−6.3019708209	−6.3019708209	−3.7653381446	−3.7653381446	4.5200856239	4.5200856239
80	−7.7542533122	−7.7542533122	−5.4472879550	−5.4472879550	4.1959347196	4.1959347196
100	−7.4421079409	−7.4421079409	−5.4831875435	−5.4831875435	3.9713031504	3.9713031504
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
CPU time(s)	1.277981	5.281342				

Table 2 Comparison of the MD-CFDRM and the RK4 solutions for $r = 55$

t	x_1		x_2		x_3	
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
4	2.264802	2.264802	5.374258	5.374258	1.663534	1.663534
8	−0.254814	−0.254814	−4.806631	−4.806631	5.133758	5.133758
12	−4.177500	−4.177500	3.236132	3.236132	4.609824	4.609824
16	5.971705	5.971705	−3.474095	−3.474095	−2.599292	−2.599292
20	−0.458099	−0.458099	1.194282	1.194282	1.946396	1.946396
t	x_4		x_5		x_6	
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
4	−5.680465	−5.680465	3.295077	3.295077	−44.575344	−44.575344
8	−6.166781	−6.166781	2.900768	2.900768	−23.072529	−23.072529
12	1.230726	1.230726	−0.997081	−0.997081	−8.179131	−8.179131
16	−3.455944	−3.455944	0.276657	0.276657	−8.746022	−8.746022
20	−4.536999	−4.536999	−0.265394	−0.265394	−23.120074	−23.120074
t	x_7		x_8		x_9	
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
4	1.558516	1.558516	−0.874860	−0.874860	−8.160863	−8.160863
8	19.065008	19.065008	9.809716	9.809716	3.606984	3.606984
12	25.129832	25.129832	36.175991	36.175991	−26.497018	−26.497018
16	−55.935017	−55.935017	−17.578250	−17.578250	35.292195	35.292195
20	−2.946509	−2.946509	17.728235	17.728235	−39.708821	−39.708821
	MD-CFDRM	RK4	MD-CFDRM	RK4	MD-CFDRM	RK4
CPU time(s)	181.612003	818.470979				

Table 1 shows the comparison between the MD-CFDRM and the RK4 results for the chaotic case ($r = 14.1$), computed over the interval $[0, 100]$. For the MD-CFDRM, we used $N = 20$ grid points and $M = 500$ subdomains, and for the RK4, we used a stepsize of 10^{-4}. The results of the two methods are in good agreement up to at least 10 decimal digits. In terms of computation speed, the RK4 method is very slow compared to the MD-CFDRM as seen in Table 1. Table 2 gives the comparison between the MD-CFDRM and the RK4 results for the hyperchaotic case, $r = 55$, on the interval $[0, 20]$. For the MD-CFDRM, we used $N = 200$ grid points and $M = 500$ subdomains, and for the RK4, we used a stepsize of 10^{-5}. The results given by the two methods are again in good agreement up to at least 6 decimal digits. Table 2 also gives the computational times of the methods, which further reveals the superiority of the MD-CFDRM over the RK4 method, in terms of efficiency. The efficiency of the MD-CFDRM, in computation time, compared to the RK4 can be attributed to the fact that a very small step size is required for the RK4 to gain the same level of accuracy as the MD-CFDRM. On the other hand, we have shown, especially for large domain problems, that the accuracy of the compact finite difference schemes is enhanced significantly if the problem is solved over multiple domains than using many grid points. This is because the multi-domain approaches allow the use of large step sizes and hence resulting in better-conditioned matrices. (see Figs. 3 and 4).

5. Conclusion

In this paper, we successfully computed solutions of a chaotic nine-dimensional Lorenz system, using a method based on blending the Gauss–Seidel relaxation method and compact finite difference schemes. The method herein referred to as the multidomain compact finite difference relaxation method (MD-CFDRM), is based on the decomposition of the main domain into smaller subdomains. The problem is then solved in each of these subdomains. We adapted it to solve complex dynamical systems like the chaotic and hyperchaotic systems. The multidomain approach allows for the use of few grid points across subdomains. This has a significant effect on the speed of the numerical computation, as seen in the results. The results presented in tabular and graphical forms are comparable to results obtained using the RK4 method. The results also show that the proposed MD-CFDRM is accurate, computationally efficient, and a reliable method for solving complex dynamical systems with both chaotic and hyperchaotic behaviour.

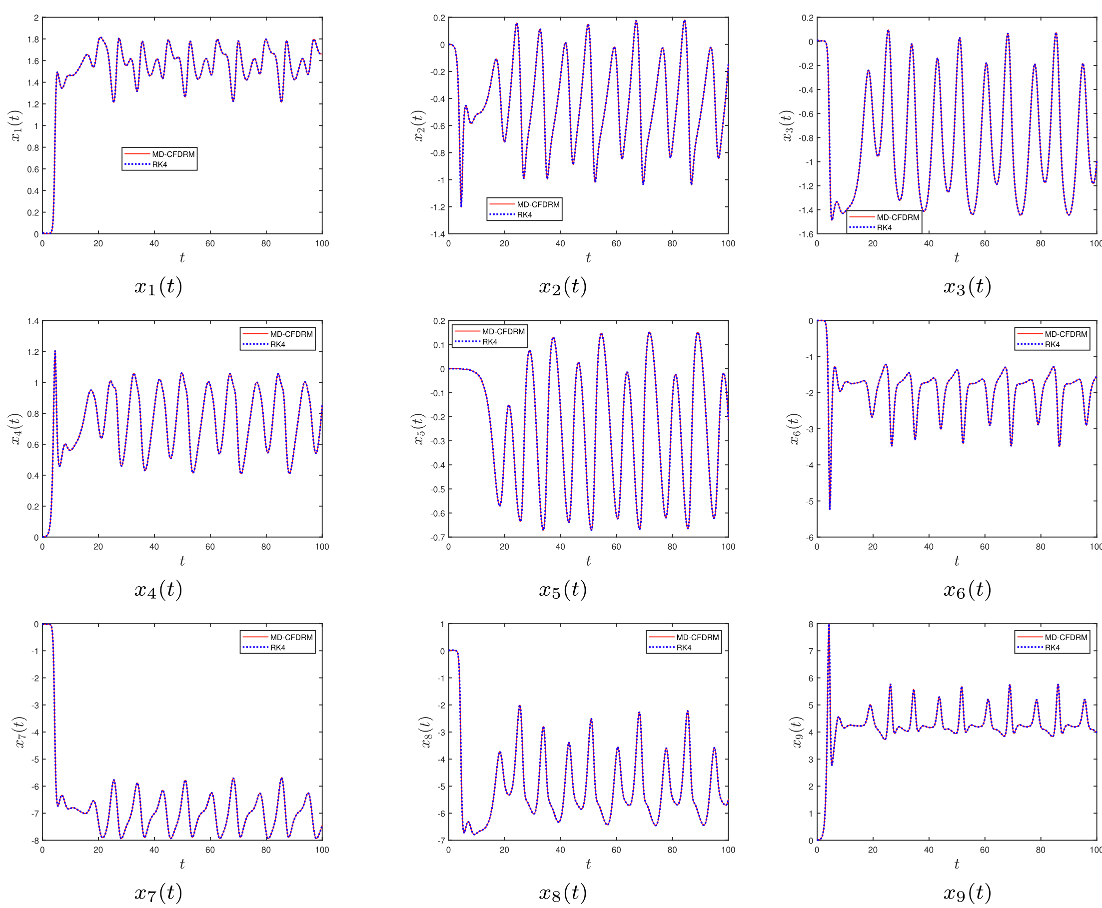

Fig. 3 Time series solution for the 9D attractor for $r = 14.1$ (the chaotic case).

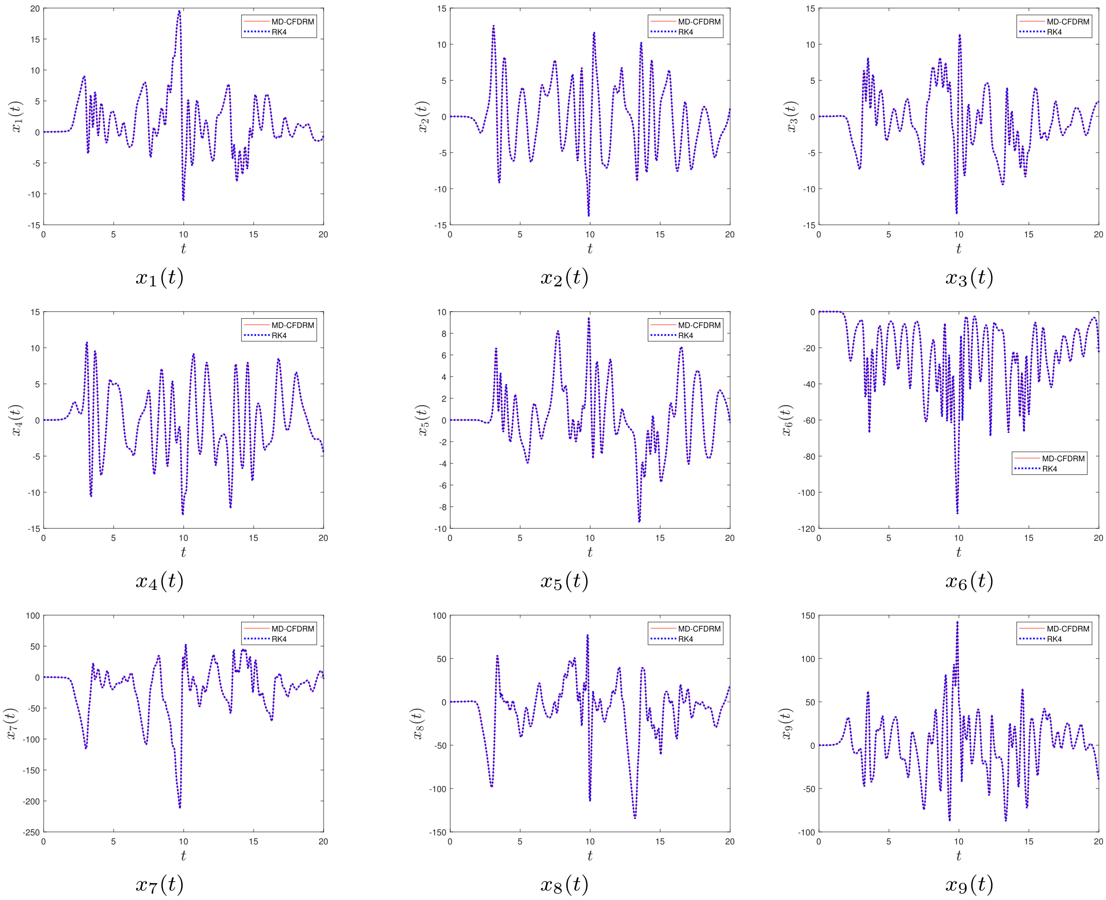

Fig. 4 Time series solution for the 9D attractor for $r = 55$ (the hyperchaotic case).

Declaration of Competing Interest

The authors declare that they have no conflict of interest.

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Further reading

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