Majorization Fragments in Resource Theories of Magic

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Magic states are essential for achieving universality in fault-tolerant schemes. Magic resource theories attempt to quantify magic via monotones and thus describe the manipulation of magic states. Here we introduce the concept of majorization fragments as a more generalised projection of such theories in discrete odd dimensions which allows for more powerful results than what monotones can provide. Fragments naturally link the symmetries of a gate sequence with conditions on the convertibility between states. We demonstrate the power of fragments by providing exact conditions for the convertibility of single-copy qutrit magic states as well as stricter distillation bounds than the established mana monotone in any odd dimension.

I. INTRODUCTION

1. Fault-tolerance [1–9]
2. Magic [10–17]
3. Bringing in majorization [18–26]
4. Section breakdown

II. MAGIC RESOURCE THEORIES

A. Introduction

Magic states are necessary for achieving universal quantum computation within fault-tolerant schemes. Identifying magic as a resource for quantum universality has led to several theories which try to provide a framework for its quantification and manipulation [CITE]. The main question that such a theory attempts to answer is:

Given two magic states ρ and ρ' is there a free operation that can convert ρ to ρ'?

Magic monotones are often used [CITE] to partially address this question, although such approaches do not exploit much of the structure of the theory. Therefore, we introduce a generalised notion of a measure, the *fragment*, applicable in general resource theories and in particular in all theories of magic we consider. [expand]

We are interested in all resource theories of magic $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ that obey two assumptions:

I Free operations send free states into free states, $\mathcal{E} : \mathcal{F} \mapsto \mathcal{F}$, for all $\mathcal{E} \in \mathcal{O}$;

II Resource theory $\mathcal{R}$ is a completely free state preserving theory, in the sense that for any d-dimensional ancilla system and all free operations $\mathcal{E}$, $(\mathbb{1}_d \otimes \mathcal{E})\sigma \in \mathcal{F}$ whenever $\sigma \in \mathcal{F}$.

The first assumption simply states that resources cannot be generated for free and is a minimal requirement for a resource theory. An immediate consequence is that if statistical mixing is included in $\mathcal{O}$, then the resource theory is convex. Convex resource theories have attracted a lot of attention recently [CITE] and include the magic theories discussed in Section II B. The second assumption implies that resources cannot be generated even when ancillas are allowed [example of T state generation on Bell state by Campbell].

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The main result of this paper is that any magic theory $\mathcal{R}$ satisfying the above assumptions can be recast as a subtheory of stochasticity. This reformulation grants additional power to the theory as the structure of stochastic process can be exploited to add insight to the main questions of theories as stated above. [expand]

B. Previous work

The stabilizer theory [CITE] is the first theory to introduce the idea of magic and it is discussed in sufficient detail for our purposes in Section III A. It comprises of the so-called "stabilizer" states (STAB) and operations (SO), while non-stabilizer (resource) states are called magic. The stabilizer operations can be expressed in terms of a Stinespring dilation as

$$\mathcal{E}(\rho) = \mathrm{tr}_E[U(\rho \otimes \sigma_E)U^\dagger], \tag{1}$$

for an ancilla stabilizer state σ_E. The motivation of the theory stems from the fact that stabilizer operations are experimentally straightforward to implement and they can be used to detect and correct errors on the stabilizer states due to their construction [CITE]. The Gottesman-Knill theorem however indicates that stabilizer operations need to be supplemented with magic states in order to achieve universality, justifying the term "magic".

Generalisations of the stabilizer theory appear in the literature intending to include broader classes of operations [CITE]. The class of stabilizer preserving operations (SPO) is defined as the set of CPTP maps that send stabilizer states into stabilizer states [27]. An important subclass of SPO is the set of completely stabilizer preserving operations (CSPO), which intuitively cannot induce "non-stabilizerness" even when applied to only part of a quantum state, i.e. operations $\mathcal{E}$ such that $(\mathbb{1}_d \otimes \mathcal{E})\sigma \in \mathrm{STAB}$ for all positive dimensions d whenever $\sigma \in \mathrm{STAB}$.

Even though non-stabilizerness is a necessary resource for universality, it has been proven insufficient for magic state distillation [15, 28]. In fact, all states with non-negative Wigner distributions have been proven to be efficiently classically simulable in [29], a result that serves as a generalization of the Gottesman-Knill theorem. The Wigner distribution of a state in odd prime dimensions is discussed rigorously in Section III B and arises as the unique quasi-probability representation of quantum theory that identifies non-contextuality exactly with the states that are efficiently classically simulable [11, 30]. In this framework, the stabilizer states are the only pure states represented with non-negative distributions [16]. However, there exist mixed states with non-negative Wigner distributions that are not mixtures of stabilizer states [4]. Therefore, stabilizer-preserving theories have been extended to a theory that preserves state "Wigner positivity" [12], formally defined in Section III B for odd prime dimensions. Informally, it can be considered as the maximal theory of magic $\mathcal{R}_{\max} = (\mathcal{F}_{\max}, \mathcal{O}_{\max})$, where free states have non-negative Wigner distributions and free operations completely preserve this property.

III. PHASE SPACE FORMALISM

A. Stabilizer Theory

Let $\{|k\rangle\}_{k\in\mathbb{Z}_d}$ be the standard computational basis for an arbitrary fault-tolerant scheme, defined over the finite field $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$, with d an odd prime. Since the field has character d, addition and multiplication on the field are always considered modulo d. The Hilbert space of any system associated with this scheme is $\mathcal{H}_d \coloneqq \mathrm{span}\{|k\rangle : k \in \mathbb{Z}_d\}$.

The generalised Pauli matrices X, Z can be defined by their respective roles as shift and phase operators,

$$X|k\rangle = |k+1\rangle \tag{2}$$
$$Z|k\rangle = \omega^k |k\rangle, \tag{3}$$

where $\omega \coloneqq e^{2\pi i/d}$ is the d-th root of unity.

The Hilbert space $\mathcal{H}_d$ is associated with a phase space $\mathcal{P}_d \coloneqq \mathbb{Z}_d \times \mathbb{Z}_d$, where every point $\boldsymbol{x} \coloneqq (x_0, x_1)$ corresponds to a displacement operator, defined as

$$D_{\boldsymbol{x}} \coloneqq \tau^{x_0 x_1} X^{x_0} Z^{x_1}, \ \boldsymbol{x} \in \mathcal{P}_d. \tag{4}$$

The phase factor $\tau \coloneqq -\omega^{1/2}$ ensures unitarity. For a system with composite Hilbert space, $\mathcal{H}_d = \mathcal{H}_{d_A} \otimes \mathcal{H}_{d_B}$, the displacement operators are defined as

$$D_{\boldsymbol{x}_A \oplus \boldsymbol{x}_B} \coloneqq D_{\boldsymbol{x}_A} \otimes D_{\boldsymbol{x}_B}, \tag{5}$$

where $\boldsymbol{x}_A \oplus \boldsymbol{x}_B \coloneqq (x_{A0}, x_{B0}, x_{A1}, x_{B1}) \in \mathcal{P}_{d_A} \times \mathcal{P}_{d_B}$.

The displacement operators, form a group under matrix multiplication modulo phases,

$$\mathrm{GP_d} \coloneqq \{\tau^k D_{\boldsymbol{z}} : k \in \mathbb{Z}_d, \boldsymbol{z} \in \mathcal{P}_d\}. \tag{6}$$

The Clifford unitaries $\mathcal{C}_d$ can then be defined as the normaliser of this group, [Reformulate for copies of qudits: $\mathrm{GP_d},\ \mathcal{C}_d \rightarrow \mathrm{GP_d}^n,\ \mathcal{C}_d{}^n.\ C-SUMs$ live in $\mathcal{C}_d{}^2$]

$$\mathcal{C}_d \coloneqq \{U \in \mathrm{SU}(d) : U\mathrm{GP_d}U^\dagger = \mathrm{GP_d}\}. \tag{7}$$

The pure stabilizer states are then the orbit of the Clifford unitaries over a computational basis state,

$$\mathrm{STAB_{pure}} \coloneqq \{U|0\rangle\langle 0|U^\dagger : U \in \mathcal{C}_d\}. \tag{8}$$

The free states of the stabilizer theory are mixtures of pure stabilizers,

$$\mathrm{STAB} = \mathrm{conv}\ \mathrm{STAB_{pure}}. \tag{9}$$

The free operations of the stabilizer theory is the set of stabilizer operations SO defined as any composition of:

1. Preparation in computational basis;

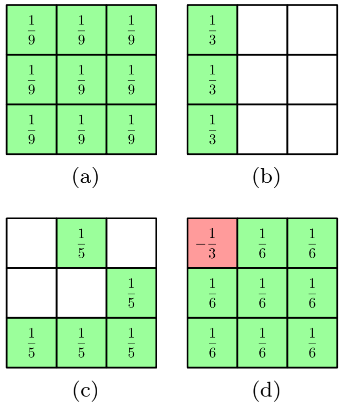

FIG. 1. Wigner distributions for qutrit states of varying magic. (a) Maximally mixed state $\frac{1}{3}\mathbb{1}$; (b) Stabilizer zero state $|0\rangle\langle 0|$; (c) A non-stabilizer Wigner-positive state; (d) Magic strange state $|\mathrm{S}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$. [Explain what a magic / bound magic state is in intro]

2. Random Clifford unitaries RCU, i.e. operations $\mathcal{E}$ such that

$$\mathcal{E}(\rho) = \sum_i p_i U_i \rho U_i^\dagger, \ U_i \in \mathcal{C}_d; \tag{10}$$

3. Measurement in computational basis.

B. Wigner Distribution

We can define the phase-point operators,

$$A_{\boldsymbol{x}} := \frac{1}{d} \sum_{\boldsymbol{z} \in \mathcal{P}_d} \omega^{\boldsymbol{x} \wedge \boldsymbol{z}} D_{\boldsymbol{z}}, \ \boldsymbol{x} \in \mathcal{P}_d. \tag{11}$$

[$\wedge$ has not be defined] They form an orthogonal Hermitian operator basis. Therefore, any quantum state $\rho \in \mathcal{B}(\mathcal{H}_d)$ can be expressed as a linear combination of the phase-point operators,

$$\rho = \sum_{\boldsymbol{z} \in \mathcal{P}_d} \mathrm{W}_\rho(\boldsymbol{z}) A_{\boldsymbol{z}}, \tag{12}$$

where the coefficient vector W_ρ is the Wigner distribution of state ρ,

$$\mathrm{W}_\rho(\boldsymbol{x}) := \frac{1}{d} \mathrm{tr}[A_{\boldsymbol{x}} \rho]. \tag{13}$$

It is in fact a bounded d^2-dimensional quasi-probability distribution over $\mathcal{P}_d$ as shown in Appendix A.

The Wigner distributions of different types of qutrit states are illustrated in Fig. (1).

We can exploit the channel-state duality and use the normalised Choi-Jamiołkowski state

$$\frac{1}{d_A} \mathcal{J}_\mathcal{E} := \frac{1}{d_A} (\mathbb{1} \otimes \mathcal{E}) \sum_{i,j} |ii\rangle \langle jj| \tag{14}$$

to extend the definition of the Wigner distribution to quantum CPTP operations $\mathcal{E} : \mathcal{B}(\mathcal{H}_{d_A}) \mapsto \mathcal{B}(\mathcal{H}_{d_B})$,

$$\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) := d_A^2 \mathrm{W}_{\frac{1}{d_A}\mathcal{J}_\mathcal{E}}(\bar{\boldsymbol{x}} \oplus \boldsymbol{y}) \tag{15}$$

$$= \frac{1}{d_B} \mathrm{tr}_B[A_{\boldsymbol{y}} \mathcal{E}(A_{\boldsymbol{x}})], \tag{16}$$

where $\bar{\boldsymbol{x}} := (x_0, -x_1)$.

The particular form of Eq. (15) was chosen so that Wigner distributions of operations act as transition matrices for Wigner distribution of states, $\mathrm{W}_{\mathcal{E}(\rho)} = \mathrm{W}_\mathcal{E} \mathrm{W}_\rho$. In particular, CPTP operations that map a Hilbert space to itself and have non-negative Wigner distributions correspond to stochastic matrices. All these properties are shown in Appendix A

The single-qudit Hadamard gate H and phase gate S generate the d-dimensional Clifford group $\mathcal{C}_d$. [CITE] Their Wigner distributions are given by permutation matrices,

$$H := \frac{1}{\sqrt{d}} \sum_{j,k} \omega^{jk} |j\rangle\langle k|, \mathrm{W}_H(\boldsymbol{y}|\boldsymbol{x}) = \delta_{y_0,-x_1} \delta_{y_1,x_0}; \tag{17}$$

$$S := \sum_k \tau^{k(k+1)} |k\rangle\langle k|, \mathrm{W}_S(\boldsymbol{y}|\boldsymbol{x}) = \delta_{y_0,x_0} \delta_{y_1,x_0+x_1+2^{-1}}. \tag{18}$$

Equipped with the definitions of the Wigner distribution, we can formally define the Wigner positivity theory $\mathcal{R}_{\max}$ in odd prime dimensions. The free states are all states with non-negative Wigner distributions

$$\mathcal{F}_{\max} := \{\rho : \mathrm{W}_\rho(\boldsymbol{z}) \geq 0 \text{ for all } \boldsymbol{z} \in \mathcal{P}_d\} \tag{19}$$

The free operations should send the set of free states $\mathcal{F}_{\max}$ into itself and completely preserve the non-negativity of the states, in the sense that $\mathcal{E} \in \mathcal{O}_{\max}$ iff $(\mathbb{1}_d \otimes \mathcal{E})\sigma \in \mathrm{STAB}$ for all odd prime dimensions d whenever $\sigma \in \mathcal{F}_{\max}$.

It is shown by Wang *et al.* [12] that $\mathcal{O}_{\max}$ coincides with the set of operations $\mathcal{E}$ with a non-negative Wigner distributions,

$$\mathcal{O}_{\max} = \{\mathcal{E} : \mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{P}_d\}. \tag{20}$$

IV. STOCHASTIC STRUCTURE OF MAGIC THEORIES

A. Fragments

Magic is commonly quantified via monotones [CITE]. A monotone is a projection from the d-dimensional set of density states of the theory to the real line. It is monotonically decreasing under free operations, reflecting the no resource generating property of free operations and thus respecting the pre-order $\prec$ of the theory. A popular magic monotone is the *mana* of a state [CITE], defined

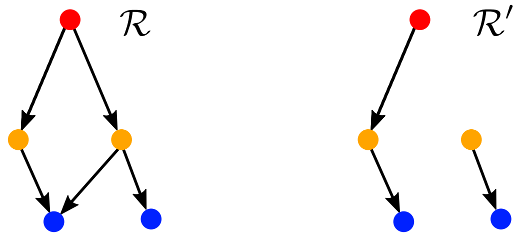

FIG. 2. Fragments [Split into subfigures]

as

$$\operatorname{mana}(\rho) := \log\left(\sum_{\boldsymbol{z}\in\mathcal{P}_d} \left|\mathrm{W}_\rho(\boldsymbol{z})\right|\right). \tag{21}$$

However, a single monotone does not provide information on whether two states are incomparable in the theory. Motivated by this restrictive nature of the monotone, we introduce the idea of a *fragment* as a less restrictive way of comparing the resource of states in an arbitrary theory $\mathcal{R} = (\mathcal{F}, \mathcal{O})$. [explain exactly why fragments > monotones] Let $\mathcal{R}' = (\mathcal{F}', \mathcal{O}')$ be a subtheory of $\mathcal{R}$ so that $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{O}' \subseteq \mathcal{O}$. Then, we define a fragment as follows.

Definition 1. *Let a resource theory $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ have pre-order $\prec_\mathcal{R}$ and operational composition rule $\circ_\mathcal{R}$. We call a resource fragment of $\mathcal{R}$ any theory $\mathcal{R}' = (\mathcal{F}', \mathcal{O}')$ with pre-order $\prec_{\mathcal{R}'}$ and operational composition rule $\circ_{\mathcal{R}'}$, if there exists a surjective projection $\Pi \equiv (\Pi_\mathrm{s}, \Pi_\mathrm{o}) : \mathcal{R} \mapsto \mathcal{R}'$ that satisfies the following two conditions.*

1. *$\Pi_\mathrm{s} : \mathcal{F} \mapsto \mathcal{F}'$ and $\Pi_\mathrm{s}(\rho_1) \prec_{\mathcal{R}'} \Pi_\mathrm{s}(\rho_2)$ whenever $\rho_1 \prec_\mathcal{R} \rho_2$ for any states $\rho_1, \rho_2 \in \mathcal{F}$;*

2. *$\Pi_\mathrm{o} : \mathcal{O} \mapsto \mathcal{O}'$ and $\Pi_\mathrm{o}(\mathcal{E}_1) \circ_{\mathcal{R}'} \Pi_\mathrm{o}(\mathcal{E}_2) = \Pi_\mathrm{o}(\mathcal{E}_1 \circ_\mathcal{R} \mathcal{E}_2)$ for any free operations $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{O}$.*

The subtheory $\mathcal{R}'$ is essentially the image of the surjection Π. The point of constructing a resource fragment with such a surjective map would be to achieve a more convenient set of free states or operations, or importantly a more convenient pre-order.

Any monotone is a trivial example of a resource fragment in the sense of the following Proposition 2.

Proposition 2. *Any monotone $\mathcal{M}$ of a resource theory $\mathcal{R}$ is a resource projection that maps the set of free states to 0 and the pre-order $\prec_\mathcal{R}$ to a total order.*

Proof. Trivial. □

Mapping free states to real numbers and free operations to simple addition, the two conditions in the definition are equivalent to monotonicity and additivity respectively. The "free state" for real numbers is 0 and all free states of the resource theory are mapped onto it.

The most general magic theory, the theory of Wigner negativity [or positivity?], can be expressed as $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ with

$$\mathcal{F} = \{\rho : \mathrm{W}_\rho(\boldsymbol{z}) \geq 0\} \text{ and} \tag{22}$$
$$\mathcal{O} = \{\mathcal{E} : \mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) \geq 0\} \tag{23}$$

for all points $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}_d$. Therefore, any free state σ corresponds to a d^2-dimensional probability distribution W_σ and any free operation $\mathcal{E} : \mathcal{B}(\mathcal{H}_d) \mapsto \mathcal{B}(\mathcal{H}_d)$ corresponds to a $d^2 \times d^2$ stochastic matrix (or conditional probability distribution) $\mathrm{W}_\mathcal{E}$. Note that these mappings are one-to-one due to the orthogonality of the phase-point operators as an operator basis.

Our goal is to give insight into magic state transformations. We first break up the theory into fragments that allow for the incorporation of $\boldsymbol{d}$-majorization in its framework.

Definition 3. *A subtheory $\mathcal{R}'$ of the Wigner negativity theory $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ is called a* σ-fragment *iff $\mathcal{R}' = (\mathcal{F}, \mathcal{O}_\sigma)$, where the free operations are restricted to the ones that preserve σ,*

$$\mathcal{O}_\sigma := \{\mathcal{E} \in \mathcal{O} : \mathcal{E}(\sigma) = \sigma\}. \tag{24}$$

State σ is thus a fixed point of all operations in $\mathcal{O}_\sigma$ and we provide the following theorem which characterises the σ-fragments as well as how they make up the whole set of free operations. [move below]

B. Majorization

Majorization is a fundamental tool that has recently found many applications in quantum information theory [CITE]. It describes the [disorder / non-uniformity] of vectors via stochastic transformations that are possible between them.

In order to discuss such stochastic transformations, we first denote by $\mathrm{S}_d(\boldsymbol{g})$ the set of $(d \times d)$ stochastic matrices that preserve vector $\boldsymbol{g}$. Specifically, any $S \in \mathrm{S}_d(\boldsymbol{g})$ satisfies:

1. $S_{ij} \geq 0$ for all $i, j \in \mathbb{Z}_d$;

2. $\sum_{j=1}^{n} S_{ij} = 1$ for all $i \in \mathbb{Z}_d$;

3. $S\boldsymbol{g} = \boldsymbol{g}$.

It forms a group under matrix multiplication because it contains the identity and $S^{-1} \in \mathrm{S}_d(\boldsymbol{g})$ for any $S \in \mathrm{S}_d(\boldsymbol{g})$.

We can motivate majorization very well for our purposes via its application on quantum thermodynamics in the absence of coherence. At any given temperature β, the thermal state γ_β is intuitively the most ordered state. Thermal operations are defined as operations that cannot extract energy from the Gibbs state, $\mathcal{E}(\gamma_\beta) = \gamma_\beta$. Convertibility between states via thermal operations is equivalent to a stochasticity condition on the energy level populations of the states [CITE]. Roughly, the statement is that there exists a thermal operation $\mathcal{E}$ such that $\tau = \mathcal{E}(\rho)$ if and only if there exists a a matrix $S \in \mathrm{S}_d(\boldsymbol{g})$ such that $\boldsymbol{q} = S\boldsymbol{p}$, where $\boldsymbol{q}, \boldsymbol{p}$ and $\boldsymbol{g}$ and the energy level population vectors of τ, ρ, γ_β respectively. [expand or remove]

We can define majorization based on this definition.

Definition 4. *Given $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{g} \in \mathbb{R}^d$, such that the components of $\boldsymbol{g}$ are positive, $\boldsymbol{y}$ is said to $\boldsymbol{g}$-majorize $\boldsymbol{x}$, iff there exists a matrix $S \in \mathrm{S}_d(\boldsymbol{g})$ such that $\boldsymbol{x} = S\boldsymbol{y}$.*

We denote this partial order by $\boldsymbol{x} \prec_{\boldsymbol{g}} \boldsymbol{y}$.

If $\boldsymbol{g} = \frac{1}{d}\mathbf{1}$, the d-dimensional uniform distribution, then $\mathrm{S}_d(\boldsymbol{g})$ is the set of bistochastic matrices and we retrieve the familiar notion of majorization in entanglement theory. [CITE]

A visual representation of $\boldsymbol{g}$-majorization is provided by the Lorenz curve of a vector $\boldsymbol{z} \in \mathbb{R}^d$. Let the vector $\boldsymbol{z}^\downarrow$ denote $\boldsymbol{z}$, but with its components arranged in non-increasing order.

Definition 5. *Let $\boldsymbol{z} \in \mathbb{R}^n$. Let $\boldsymbol{g} \in \mathbb{R}^d$ be a vector with positive components, π a permutation mapping $(z_i/g_i) \mapsto (z_i/g_i)^\downarrow$ for all $i = 1, \ldots, d$ and $D = \sum_{i=1}^d g_i$. The Lorenz curve $L(\boldsymbol{z}|\boldsymbol{g})$ of vector $\boldsymbol{z}$ is the piecewise linear curve obtained by joining the points*

$$\left\{ L_k(\boldsymbol{z}|\boldsymbol{g}) \coloneqq \left(\frac{1}{D}\sum_{i=1}^{k} g_{\pi(i)}, \sum_{i=1}^{k} z_{\pi(i)} \right) \in \mathbb{R}^2 : k = 1, \ldots, d \right\}. \tag{25}$$

Remark 1. The origin $L_0(\boldsymbol{z}|\boldsymbol{g}) \coloneqq (0,0)$ is usually included in the curve.

Remark 2. The first components of $L_k(\boldsymbol{z}|\boldsymbol{g})$ are rescaled by D so that comparison of curves with unequal dimensions is possible. In fact, the Lorenz curves $L(\boldsymbol{z}|\boldsymbol{g})$ and $L(\boldsymbol{z}\otimes\boldsymbol{g}|\boldsymbol{g}\otimes\boldsymbol{g})$, where $\otimes$ denotes the Kronecker product, are identical.

Remark 3. Lorenz curves are always concave.

Remark 4. If $L_d(\boldsymbol{z}|\boldsymbol{g}) = 1$ and for all k, $L_k(\boldsymbol{z}|\boldsymbol{g}) \leq 1$, then $\boldsymbol{z}$ is a probability distribution. Lorenz curves of quasi-probability distributions in principle reach above 1.

An example of comparison between different Lorenz curves is illustrated in Fig. (3).

We now present the important majorization result needed in our analysis of magic.

Theorem 6. *Given $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{g} \in \mathbb{R}^d$, such that the components of $\boldsymbol{g}$ are positive, the following statements are equivalent:*

1. *$\boldsymbol{x} \prec_{\boldsymbol{g}} \boldsymbol{y}$;*

2. *$L_k(\boldsymbol{x}|\boldsymbol{g}) \leq L_k(\boldsymbol{y}|\boldsymbol{g})$ for all $k = 1, 2, \ldots, d-1$ and $L_d(\boldsymbol{x}|\boldsymbol{g}) = L_d(\boldsymbol{y}|\boldsymbol{g})$.*

A restatement of the theorem including more equivalent conditions, along with a proof is provided in the [appendix].

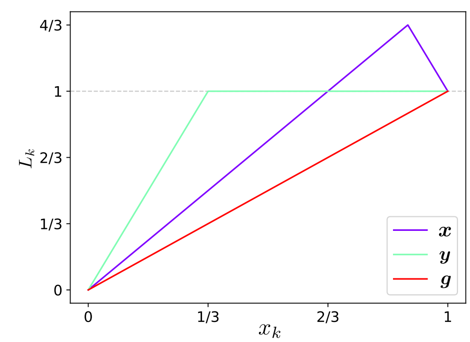

FIG. 3. Example of different Lorenz curves for quasi-probability vectors under $\boldsymbol{g}$-majorization. Vectors $\boldsymbol{y}$ and $\boldsymbol{g}$ are simply probability distributions. The curve corresponding to vector $\boldsymbol{g}$ is always the straight line connecting $(0,0)$ and $(1,1)$, so that any other Lorenz curve lies above it, for example $\boldsymbol{x} \prec_{\boldsymbol{g}} \boldsymbol{g}$. Curves $L_k(\boldsymbol{x}|\boldsymbol{g})$ and $L_k(\boldsymbol{y}|\boldsymbol{g})$ intersect, so neither $\boldsymbol{x} \prec_{\boldsymbol{g}} \boldsymbol{y}$ nor $\boldsymbol{y} \prec_{\boldsymbol{g}} \boldsymbol{x}$.

C. Magic fragments

Theorem 7. *In the resource theory of Wigner negativity $\mathcal{R} = (\mathcal{O}, \mathcal{F})$, the following statements hold:*

1. *The σ-fragment $\mathcal{O}_\sigma$ is equal to the set of all* CPTP *operations with stochastic Wigner distributions that leave W_σ invariant. [Assuming maximal resource theory]*

2. *Every free operation leaves at least one free state invariant such that $\mathcal{O} = \bigcup_{\sigma\in\mathcal{F}} \mathcal{O}_\sigma$.*

3. *If a free operation leaves two states invariant, then it also leaves every mixture of them invariant, $\mathcal{O}_\sigma \cap \mathcal{O}_{\sigma'} \subseteq \mathcal{O}_{p\sigma+(1-p)\sigma'}$ for all $p \in [0,1]$.*

Proof. 1. Let $\mathcal{O}'_\sigma \coloneqq \{\mathcal{E} \in \mathrm{CPTP} : \mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)\}$ be the described set of operations.

Suppose $\mathcal{E}$ is in $\mathcal{O}_\sigma$, then $\mathcal{E} \in \mathrm{CPTP}$ and $\mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$ due to property 4 of Proposition 11, hence $\mathcal{O}_\sigma \subseteq \mathcal{O}'_\sigma$.

Conversely, suppose $\mathcal{E} \in \mathrm{CPTP}$ with $\mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$. Then, $\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{y}$, hence $\mathcal{E} \in \mathcal{O}$. Furthermore, $\mathrm{W}_\mathcal{E}\mathrm{W}_\sigma = \mathrm{W}_\sigma$ implies $\mathcal{E}(\sigma) = \sigma$ using Eq. (15) defined for any CPTP operation $\mathcal{E}$. Hence, $\mathcal{O}'_\sigma \subseteq \mathcal{O}_\sigma$.

2. Suppose $\mathcal{E}$ is in $\mathcal{O}_\sigma$, then it is also in $\mathcal{O}$, hence $\bigcup_{\sigma\in\mathcal{F}} \mathcal{O}_\sigma \subseteq \mathcal{O}$.

Conversely, suppose $\mathcal{E}$ is in $\mathcal{O}$. The free states are mapped one-to-one to a subset $\mathcal{S}$ of the (d^2-1)-dimensional probability simplex. $\mathcal{S}$ is convex, since any combination of free states is also free and the Wigner distribution is linear. Therefore, $\mathcal{S}$ is convex and compact as a convex subset of the bounded compact probability simplex. Then, $\mathrm{W}_{\mathcal{E}}$ is a stochastic, thus continuous, mapping from $\mathcal{S}$ to itself and Brouwer's fixed point theorem [CITE] implies that there exists a probability distribution $g_{\boldsymbol{z}}, \boldsymbol{z} \in \mathcal{P}_d$ that is preserved by $\mathrm{W}_{\mathcal{E}}$. This corresponds one-to-one to a state $\sigma \coloneqq \sum_{\boldsymbol{z}\in\mathcal{P}_d} g_{\boldsymbol{z}} A_{\boldsymbol{z}}$ and so $\mathcal{O} \subseteq \bigcup_{\sigma\in\mathcal{F}} \mathcal{O}_\sigma$.

3. Let $\mathcal{E} \in \mathcal{O}_\sigma \cap \mathcal{O}_{\sigma'}$. Then $\mathcal{E} \in$ CPTP and corresponds to stochastic Wigner distribution $\mathrm{W}_{\mathcal{E}}$ such that $\mathrm{W}_{\mathcal{E}}\mathrm{W}_\sigma = \mathrm{W}_\sigma$ and $\mathrm{W}_{\mathcal{E}}\mathrm{W}_{\sigma'} = \mathrm{W}_{\sigma'}$. Then, $\mathrm{W}_{\mathcal{E}}\mathrm{W}_{p\sigma+(1-p)\sigma'} = \mathrm{W}_{p\sigma+(1-p)\sigma'}$ for any $p \in [0,1]$ due to the additive property 4 of the Wigner distribution, implying that state $p\sigma + (1-p)\sigma'$ is also left invariant by $\mathcal{E}$.

□

The structure of the σ-fragment $(\mathcal{F}, \mathcal{O}_\sigma)$ for any d-dimensional σ in $\mathcal{F}$ admits the pre-order of $\boldsymbol{g}$-majorization with $\boldsymbol{g} = \mathrm{W}_\sigma$, a d^2-dimensional probability vector. If any component of W_σ is zero, then we can always add some ϵ amount of unital noise by mixing σ with the maximally mixed state $\frac{1}{d}\mathbb{1}$. This ensures that all components are strictly positive and $\boldsymbol{d}$-majorization can be used.

Proposition 8. *A state conversion $\rho \xrightarrow{\mathcal{E}\in\mathcal{O}} \tau$ is possible only if there exists a free state σ such that for [an infinitesimal ϵ], $\mathrm{W}_\tau \prec_{\mathrm{W}_{\sigma'}} \mathrm{W}_\rho$, where $\sigma' = (1-\epsilon)\sigma + \epsilon\frac{1}{d}\mathbb{1}$.*

Remark 1. Note that free states $\mathcal{F}$ map onto a *strict subset* of the set of probability distributions. As a counterexample, consider the 9-dimensional probability distribution with non-zero components $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. It does not correspond to any qutrit Wigner distribution because the component with value $\frac{1}{2}$ does not satisfy the boundedness condition 3 of Proposition 10.

Remark 2. Similarly, any $\mathcal{O}_\sigma$ may be mapped onto a *strict subset* of the set $\mathrm{S}_{d^2}(\mathrm{W}_\sigma)$ of stochastic matrices that preserve W_σ.

As an example, consider the permutation matrix

$$\Pi_X = \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix} \otimes \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma),\ d = 5. \tag{26}$$

It preserves the uniform distribution $\mathrm{W}_{\frac{1}{5}\mathbb{1}}$, but it does not correspond to any completely positive operation and therefore to any $\mathcal{E} \in \mathcal{O}$, due to Theorem 7.

Remark 3. No σ-fragment is empty. In fact, an example of a stabilizer operation $\mathcal{E} \in \mathrm{SO} \cap \mathcal{O}_\sigma$, for any σ-fragment, is the completely depolarising [replacement] map

$$\mathcal{E}(\rho) = \sigma \mathrm{tr}[\rho], \text{ with} \tag{27}$$
$$\mathrm{W}_{\mathcal{E}}(\boldsymbol{y}|\boldsymbol{x}) = \mathrm{W}_\sigma(\boldsymbol{y})\mathrm{tr}[\rho], \tag{28}$$

which can be thought as a sequence of tracing out state ρ and preparing the stabilizer σ.

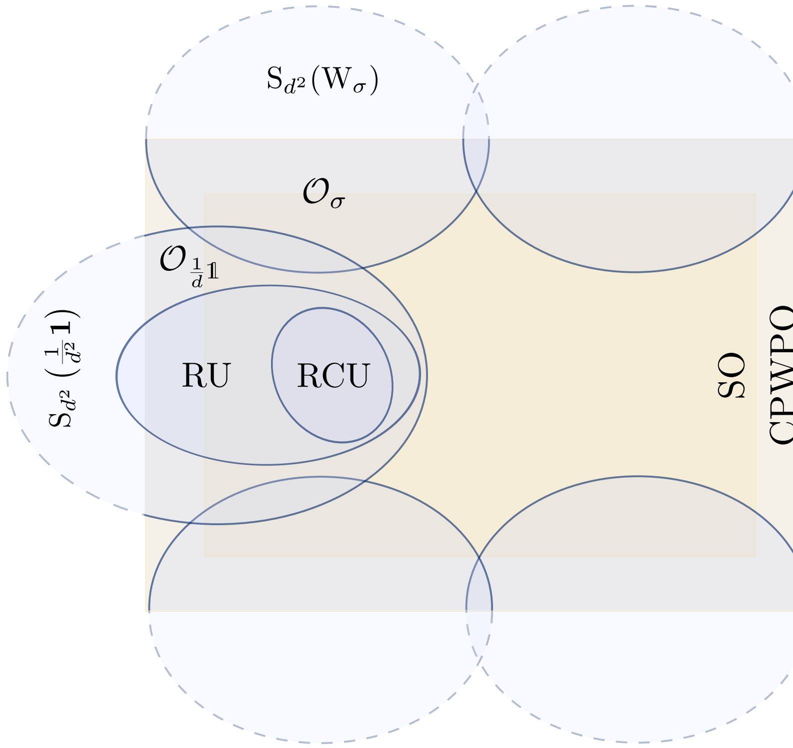

FIG. 4. Zoo of allowed operations for magic resource theories. Established theories involve operations within the yellow regions, following the hierarchy SO ⊂ CSPO ⊂ SPO ⊂ CPWPO. We introduce fragments $\mathcal{O}_\sigma \subset \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$, $\sigma \in \mathcal{F}$ that cover CPWPO with each one extending to a set of stochastic maps that allows for $\boldsymbol{d}$-majorization to be used. [fragments are the intersections of blue and orange, not the whole bubble - technically $\mathrm{S}_{d^2}(\mathrm{W}_\sigma)$ should be replaced by its operational pre-image.]

The zoo of all the operation classes is summarised in Fig. (4). Completely positive-Wigner-preserving operations (CPWPO) form the largest operation class in the literature. Therefore, σ-fragments cover this theory of magic exactly and any magic subtheory is contained within this cover. For example, the stabilizer-preserving (SPO) [27] and completely stabilizer-preserving (CSPO) [31] operation subclasses follow the hierarchy SO ⊂ CSPO ⊂ SPO ⊂ CPWPO and therefore are described by our σ-fragments.

V. DISTILLATION BOUNDS

Consider a general magic state distillation process,

$$\rho^{\otimes k} \xrightarrow{\mathcal{E}\in\mathcal{O}} \tau, \tag{29}$$

where n noisy copies of magic state ρ are converted to a single-copy magic state τ. Identifying a symmetry of the distillation process responsible for leaving a state σ invariant is equivalent with restricting the process to the σ-fragment $\mathcal{O}_\sigma$.

Mana is monotonic and additive in all σ-fragments as seen in Appendix B, so it provides a bound for distillation processes

$$\operatorname{mana}(\rho) \geq \frac{1}{k}\operatorname{mana}(\tau). \tag{30}$$

.

A new bound can be obtained in any σ-fragment by comparing the Lorenz curves of the initial and target states,

$$L_k(\rho^{\otimes k}|\sigma) \geq L_k(\tau|\sigma),\ k = 1, \ldots, d^2, \tag{31}$$

If the Lorenz curve of the initial state is below the target curve at any point, the process is not possible. In general, the Wigner components of a k-copy state $\rho^{\otimes k}$ are calculated, along with their multiplicities, by expanding the terms in the multinomial expansion $\left(\sum_{\boldsymbol{z}\in\mathcal{P}_d} W_\rho(\boldsymbol{z})\right)^n$. This follows from the multiplicativity of the Wigner distribution.

The Strange state $|\mathrm{S}\rangle\langle\mathrm{S}|$ depicted in Fig. (1(d)) is the simplest to analyse, since it only has two distinct components $\{-\frac{1}{3}, \frac{1}{6}\}$, the latter with a multiplicity of 8. Calculating the binomial expansion for the components of $|S\rangle\langle S|^{\otimes k}$ gives $\{(-1)^j 2^{j-k} 3^{-k}\}_{0\leq j\leq k}$ with multiplicity $8^{k-j}\binom{k}{j}$ for the j-th term. This allows analytical calculation of all Lorenz curve points, hence the maximum of the k-copy state is

$$\max_k L_k\left(|S\rangle\langle S|^{\otimes k} \mid \sigma\right) = 1 + \left(\frac{4}{3}\right)^k \sum_{j:1\leq 2j+1\leq k} 4^{-(2j+1)}\binom{k}{2j+1}. \tag{32}$$

Consider the noisy Strange state,

$$\rho_{\mathrm{S}}(\epsilon) = (1-\epsilon)\,|\mathrm{S}\rangle\,\langle\mathrm{S}| + \epsilon\sigma, \tag{33}$$

in the σ-fragment $\mathcal{O}_\sigma$. At noise level $\epsilon \leq \frac{3}{4}$, the Wigner distribution $\mathrm{W}_{\rho_{\mathrm{S}}(\epsilon)}$ contains negativities and the state can be purified so as to obtain a single-copy state with sufficiently low ϵ. In Fig. (5), we examine the purifying process

$$\rho_{\mathrm{S}}^{\otimes k}(\epsilon_{\mathrm{th}}) \xrightarrow{\mathcal{E}\in\mathcal{O}_\sigma} \rho_{\mathrm{S}}(0.05),\ \sigma = (1-p)\,|0\rangle\langle 0| + p\frac{1}{3}\mathbb{1} \tag{34}$$

with ϵ_{th} being the noise level threshold that does not prohibit the process for given number of copies k and σ-fragment, parametrised by p as a mixture of the zero and the maximally mixed states.

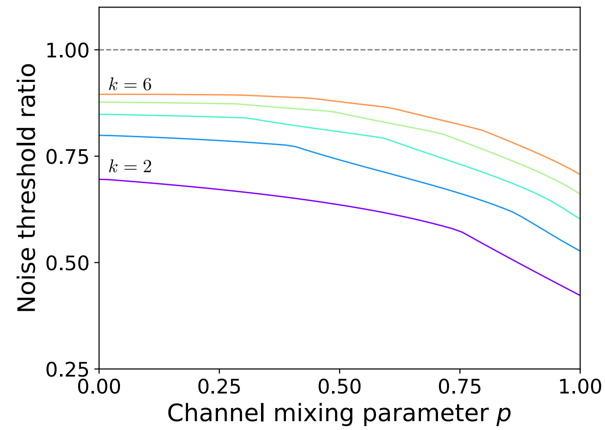

FIG. 5. Plot of the noise level threshold ratios between mana and Lorenz curve for the Strange state purifying process in Eq. (34). The ratios are calculated for different numbers of initial state copies and different σ-fragments parametrised by p such that $\sigma = (1-p)\,|0\rangle\langle 0| + p\frac{1}{3}\mathbb{1}$. Lorenz curve comparison consistently gives stricter bounds as proven in Theorem 9.

Thresholds provided by Lorenz curve comparison are always much stricter than mana thresholds [threshold/bound? need to define the notion of a bound precisely]. In fact, it is clear than this is the case in any general distillation process.

Theorem 9. *Consider the distillation process in Eq. (29). In any σ-fragment, W_σ-majorization provides a stricter bound than mana.*

Proof. The maximum of the Lorenz curve of a state ρ can be expressed monotonically in terms of mana, independently of the σ-fragment,

$$\max_k L_k(\rho|\sigma) = 1 + \sum_{\boldsymbol{z}:\mathrm{W}_\rho(\boldsymbol{z})<0} |\mathrm{W}_\rho(\boldsymbol{z})| = \frac{1}{2}\left(1 + e^{\operatorname{mana}(\rho)}\right). \tag{35}$$

Therefore, the majorization condition stated in Eq. (31) implies that $\operatorname{mana}\left(\rho^{\otimes k}\right) \geq \operatorname{mana}(\tau)$. □

VI. CONCLUSION

1. Introduced fragments

2. Can we solve other cases exactly? (apart from single qutrit)

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Appendix A: Properties of the Wigner distribution

Here we present important properties of the Wigner distribution that are used throughout the paper.

Proposition 10. *The Wigner distribution of a state* ρ *is*

1. *Real valued:* $\mathrm{W}_\rho \in \mathbb{R}^{d^2}$*;*
2. *Normalised:* $\sum_{\boldsymbol{z}\in\mathcal{P}_d} \mathrm{W}_\rho\left(\boldsymbol{z}\right) = 1$*;*
3. *Bounded:* $|\mathrm{W}_\rho\left(\boldsymbol{x}\right)| \le \frac{1}{d}$*.*
4. *Additive under mixing:*
$$\mathrm{W}_{\sum_i p_i\rho_i}\left(\boldsymbol{x}\right) = \sum_i p_i \mathrm{W}_{\rho_i}\left(\boldsymbol{x}\right);$$
5. *Multiplicative under tensor products:*
$$\mathrm{W}_{\rho_A\otimes\rho_B}\left(\boldsymbol{x}_A \oplus \boldsymbol{x}_B\right) = \mathrm{W}_{\rho_A}\left(\boldsymbol{x}_A\right)\mathrm{W}_{\rho_B}\left(\boldsymbol{x}_B\right).$$

Proposition 11. *The Wigner distribution of a* CPTP *operation* $\mathcal{E} : \mathcal{B}(\mathcal{H}_{d_A}) \mapsto \mathcal{B}(\mathcal{H}_{d_B})$ *is:*

1. *Real-valued:* $\mathrm{W}_\mathcal{E}\left(\boldsymbol{y}|\boldsymbol{x}\right) \in \mathbb{R}$*;*
2. *Normalised:* $\sum_{\boldsymbol{z}\in\mathcal{P}_{d_B}} \mathrm{W}_\mathcal{E}\left(\boldsymbol{z}|\boldsymbol{x}\right) = 1$ *for any* $\boldsymbol{x} \in \mathcal{P}_{d_A}$*;*
3. *Bounded:* $|\mathrm{W}_\mathcal{E}\left(\boldsymbol{y}|\boldsymbol{x}\right)| \le \frac{d_A}{d_B}$*;*
4. *[Transitive]:* $\mathrm{W}_{\mathcal{E}(\rho)}\left(\boldsymbol{y}\right) = \sum_{\boldsymbol{z}\in\mathcal{P}_{d_A}} \mathrm{W}_\mathcal{E}\left(\boldsymbol{y}|\boldsymbol{z}\right)\mathrm{W}_\rho\left(\boldsymbol{z}\right)$ *for any* $\boldsymbol{y} \in \mathcal{P}_{d_B}$*.*

If $d_A = d_B$, and in particular if operation $\mathcal{E}$ maps a Hilbert space onto itself, then the stochasticity condition $|\mathrm{W}_\mathcal{E}\left(\boldsymbol{y}|\boldsymbol{x}\right)| \le 1$ is satisfied.

Appendix B: Properties of majorization

1. Equivalent conditions for majorization

Theorem 12. *Given* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d} \in \mathbb{R}^n$*, such that the components of* $\boldsymbol{d}$ *are positive, the following statements are equivalent:*

(TM1) $\boldsymbol{x} \prec_{\boldsymbol{d}} \boldsymbol{y}$*;*

(TM2) $\Gamma_{\boldsymbol{d}}(\boldsymbol{x}) \prec \Gamma_{\boldsymbol{d}}(\boldsymbol{y})$*;*

(TM3) $\sum_{i=1}^{n} |x_i - td_i| \leq \sum_{i=1}^{n} |y_i - td_i|$ *for all* $t \in \mathbb{R}$*;*

(TM4) $\sum_{i=1}^{n} (x_i - td_i)^+ \leq \sum_{i=1}^{n} (y_i - td_i)^+$ *for all* $t \in \mathbb{R}$ *and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$*;*

(TM5) $\forall k, L_{\boldsymbol{x}|\boldsymbol{d}}(k) \leq L_{\boldsymbol{y}|\boldsymbol{d}}(k)$ *and* $L_{\boldsymbol{x}|\boldsymbol{d}}(k = n) = L_{\boldsymbol{y}|\boldsymbol{d}}(k = n)$*.*

Proof. 1$\leftrightarrow$ 2 Suppose now there exists a stochastic S such that $\boldsymbol{x} = S\boldsymbol{y}$ with $\boldsymbol{d} = S\boldsymbol{d}$ and let $B = \Gamma_{\boldsymbol{d}} \circ S \circ \Gamma_{\boldsymbol{d}}^{-1}$. B is a D-dimensional bistochastic matrix, since composition of stochastic matrices is stochastic and $(\Gamma_{\boldsymbol{d}} \circ S \circ \Gamma_{\boldsymbol{d}}^{-1})(\frac{1}{D}\mathbf{1}) = (\Gamma_{\boldsymbol{d}} \circ S)(\boldsymbol{d}) = \Gamma_{\boldsymbol{d}}(\boldsymbol{d}) = \frac{1}{D}\mathbf{1}$. Then, B maps $\Gamma_{\boldsymbol{d}}(\boldsymbol{y})$ to $\Gamma_{\boldsymbol{d}}(\boldsymbol{x})$. Conversely, given B, let $S = \Gamma_{\boldsymbol{d}}^{-1} \circ B \circ \Gamma_{\boldsymbol{d}}$. Similarly, S is the stochastic matrix that preserves $\boldsymbol{d}$ and maps $\boldsymbol{y}$ to $\boldsymbol{x}$.

$2 \leftrightarrow 3$, $2 \leftrightarrow 4$, $2 \leftrightarrow 5$ These three statement are equivalent to [blah] respectively for the embedded vectors $\Gamma_{\boldsymbol{d}}(\boldsymbol{x}), \Gamma_{\boldsymbol{d}}(\boldsymbol{y})$. This is clear by rewriting

$$\sum_{i=1}^{n} |x_i - td_i| = \sum_{i=1}^{n} d_i \left| \frac{x_i}{d_i} - t \right| = \sum_{i=1}^{D} |\Gamma_{\boldsymbol{d}}(\boldsymbol{x})_i - t|, \tag{B1}$$

$$\sum_{i=1}^{n} (x_i - td_i)^+ = \sum_{i=1}^{D} (\Gamma_{\boldsymbol{d}}(\boldsymbol{x})_i - t)^+, \tag{B2}$$

$$L_{\boldsymbol{x}|\boldsymbol{d}}(k) = L_{\Gamma_{\boldsymbol{d}}(\boldsymbol{x})}(k'), \quad \text{with } k = 1, \ldots, n \text{ and } k' = 1, \ldots, D \tag{B3}$$

and similarly for the right hand side.

□

2. Mana properties

Mana monotonicity can be directly seen due to statement 12 in Theorem 6 for $t = 0$. Furthermore, mana is additive due to the multiplicative property 4 of Proposition 10.