Majorization Fragments in Resource Theories of Magic

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Magic states are essential for achieving universality in fault-tolerant schemes. Magic resource theories attempt to quantify magic via monotones and thus describe the manipulation of magic states. Here we introduce the concept of majorization fragments as a more generalised projection of such theories in discrete odd dimensions which allows for more powerful results than what monotones can provide. Fragments naturally link the symmetries of a gate sequence with conditions on the convertibility between states. We demonstrate the power of fragments by providing exact conditions for the convertibility of single-copy qutrit magic states as well as stricter distillation bounds than the established mana monotone in any odd dimension.

I. INTRODUCTION

1. Fault-tolerance [1–9]
2. Magic [10–17]
3. Bringing in majorization [18–26]
4. Section breakdown

II. MAGIC RESOURCE THEORIES

A. Introduction

Magic states are necessary for achieving universal quantum computation within fault-tolerant schemes. Identifying magic as a resource for quantum universality has led to several theories which try to provide a framework for its quantification and manipulation [CITE]. The main question that such a theory attempts to answer is:

Given two magic states ρ and ρ' is there a free operation that can convert ρ to ρ'?

We are interested in all resource theories of magic $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ in which free operations cannot generate any amount of resource. Further denote by $\mathcal{D}$ the set of states considered under the theory, that is the union of free and resource states. The structure of such theory is described by a partial order [CITE], hereinafter called a *pre-order*, $\prec_{\mathcal{R}}$ between states. We write $\rho' \prec_{\mathcal{R}} \rho$ iff there exists $\mathcal{E} \in \mathcal{O}$ such that $\mathcal{E}(\rho) = \rho'$. Naturally, states may be incomparable under the given theory, meaning that there exists no free operation that converts one to the other. We further call $\mathcal{R}' = (\mathcal{F}', \mathcal{O}')$ a *subtheory* of $\mathcal{R}$ iff $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{O}' \subseteq \mathcal{O}$. The above notation will be used for general resource theories as well.

Formally, the no resource generation condition on the theories translates into two assumptions:

I Free operations send free states into free states, $\mathcal{E} : \mathcal{F} \mapsto \mathcal{F}$, for all $\mathcal{E} \in \mathcal{O}$;

II Resource theory $\mathcal{R}$ is a completely free state preserving theory, in the sense that for any d-dimensional ancilla system and all free operations $\mathcal{E}$, $(\mathbb{1}_d \otimes \mathcal{E})\sigma \in \mathcal{F}$ whenever $\sigma \in \mathcal{F}$.

The first assumption simply states that resources cannot be generated for free and is a minimal requirement for a resource theory. An immediate consequence is that if statistical mixing is included in $\mathcal{O}$, then the resource theory is convex. Convex resource theories have attracted a lot of attention recently [CITE] and include the magic theories discussed in Section II B. The second assumption implies that resources cannot be generated even when ancillas are allowed [example of T state generation on Bell state by Campbell].

Monotones are often used [CITE] to address the question of state convertibility, although such approaches are usually generic. A monotone of any general resource theory is a projection of the theory onto the non-negative real numbers, collapsing the pre-order of the theory to the total order defined on the real line. This is the [most naive] non-trivial projection under which the images of incomparable states can be compared. Our first contribution is the introduction of a generalised notion of *resource projection* which maps a general resource theory onto a subtheory which in principle still retains a partial structure. Applying this notion on existing magic theories highlights the hidden stochasticity that governs magic state conversions. We show that a magic theory can be subdivided into *fragments* [expand]

B. Previous work

The stabilizer theory [CITE] is the first theory to introduce the idea of magic and it is discussed in sufficient de-

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tail for our purposes in Section III A. It comprises of the so-called "stabilizer" states (STAB) and operations (SO), while non-stabilizer (resource) states are called magic. The stabilizer operations can be expressed in terms of a Stinespring dilation as

$$\mathcal{E}(\rho) = \mathrm{tr}_E[U(\rho \otimes \sigma_E)U^\dagger], \tag{1}$$

for an ancilla stabilizer state σ_E. The motivation of the theory stems from the fact that stabilizer operations are experimentally straightforward to implement and they can be used to detect and correct errors on the stabilizer states due to their construction [CITE]. The Gottesman-Knill theorem however indicates that stabilizer operations need to be supplemented with magic states in order to achieve universality, justifying the term "magic".

Generalisations of the stabilizer theory appear in the literature intending to include broader classes of operations [CITE]. The class of stabilizer preserving operations (SPO) is defined as the set of CPTP maps that send stabilizer states into stabilizer states [27]. An important subclass of SPO is the set of completely stabilizer preserving operations (CSPO), which intuitively cannot induce "non-stabilizerness" even when applied to only part of a quantum state, i.e. operations $\mathcal{E}$ such that $(\mathbb{1}_d \otimes \mathcal{E})\sigma \in$ STAB for all positive dimensions d whenever $\sigma \in$ STAB.

Even though non-stabilizerness is a necessary resource for universality, it has been proven insufficient for magic state distillation [15, 28]. In fact, all states with non-negative Wigner distributions have been proven to be efficiently classically simulable in [29], a result that serves as a generalization of the Gottesman-Knill theorem. The Wigner distribution of a state in odd prime dimensions is discussed rigorously in Section III B and arises as the unique quasi-probability representation of quantum theory that identifies non-contextuality exactly with the states that are efficiently classically simulable [11, 30]. In this framework, the stabilizer states are the only pure states represented with non-negative distributions [16]. However, there exist mixed states with non-negative Wigner distributions that are not mixtures of stabilizer states [4]. Therefore, stabilizer-preserving theories have been extended to a theory that preserves state "Wigner positivity" [12], formally defined in Section III B for odd prime dimensions. Informally, it can be considered as the maximal theory of magic $\mathcal{R}_{\max} = (\mathcal{F}_{\max}, \mathcal{O}_{\max})$, where free states have non-negative Wigner distributions and free operations completely preserve this property.

III. PHASE SPACE FORMALISM

A. Stabilizer Theory

Let $\{|k\rangle\}_{k\in\mathbb{Z}_d}$ be the standard computational basis for an arbitrary fault-tolerant scheme, defined over the finite field $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, with d an odd prime. Since the field has character d, addition and multiplication on the field are always considered modulo d. The Hilbert space of any system associated with this scheme is $\mathcal{H}_d \coloneqq \mathrm{span}\,\{|k\rangle : k \in \mathbb{Z}_d\}$.

The generalised Pauli matrices X, Z can be defined by their respective roles as shift and phase operators,

$$X\,|k\rangle = |k+1\rangle \tag{2}$$

$$Z\,|k\rangle = \omega^k\,|k\rangle\,, \tag{3}$$

where $\omega \coloneqq e^{2\pi i/d}$ is the d-th root of unity.

The Hilbert space $\mathcal{H}_d$ is associated with a phase space $\mathcal{P}_d \coloneqq \mathbb{Z}_d \times \mathbb{Z}_d$, where every point $\boldsymbol{x} \coloneqq (x_0, x_1)$ corresponds to a displacement operator, defined as

$$D_{\boldsymbol{x}} \coloneqq \tau^{x_0 x_1} X^{x_0} Z^{x_1},\ \boldsymbol{x} \in \mathcal{P}_d. \tag{4}$$

The phase factor $\tau \coloneqq -\omega^{1/2}$ ensures unitarity. For a system with composite Hilbert space, $\mathcal{H}_d = \mathcal{H}_{d_A} \otimes \mathcal{H}_{d_B}$, the displacement operators are defined as

$$D_{\boldsymbol{x}_A \oplus \boldsymbol{x}_B} \coloneqq D_{\boldsymbol{x}_A} \otimes D_{\boldsymbol{x}_B}, \tag{5}$$

where $\boldsymbol{x}_A \oplus \boldsymbol{x}_B \coloneqq (x_{A0}, x_{B0}, x_{A1}, x_{B1}) \in \mathcal{P}_{d_A} \times \mathcal{P}_{d_B}$.

The displacement operators, form a group under matrix multiplication modulo phases,

$$\mathrm{GP_d} \coloneqq \{\tau^k D_{\boldsymbol{z}} : k \in \mathbb{Z}_d, \boldsymbol{z} \in \mathcal{P}_d\}. \tag{6}$$

The Clifford unitaries $\mathcal{C}_d$ can then be defined as the normaliser of this group, [Reformulate for copies of qudits: $\mathrm{GP_d},\ \mathcal{C}_d \to \mathrm{GP_d}^n,\ \mathcal{C}_d{}^n$. $C-SUM$s live in $\mathcal{C}_d{}^2$]

$$\mathcal{C}_d \coloneqq \{U \in \mathrm{SU}(d) : U\mathrm{GP_d}U^\dagger = \mathrm{GP_d}\}. \tag{7}$$

The pure stabilizer states are then the orbit of the Clifford unitaries over a computational basis state,

$$\mathrm{STAB_{pure}} \coloneqq \{U\,|0\rangle\!\langle 0|\,U^\dagger : U \in \mathcal{C}_d\}. \tag{8}$$

The free states of the stabilizer theory are mixtures of pure stabilizers,

$$\mathrm{STAB} = \mathrm{conv}\ \mathrm{STAB_{pure}}. \tag{9}$$

The free operations of the stabilizer theory is the set of stabilizer operations SO defined as any composition of:

1. Preparation in computational basis;

2. Random Clifford unitaries RCU, i.e. operations $\mathcal{E}$ such that

$$\mathcal{E}(\rho) = \sum_i p_i U_i \rho U_i^\dagger,\ U_i \in \mathcal{C}_d; \tag{10}$$

3. Measurement in computational basis.

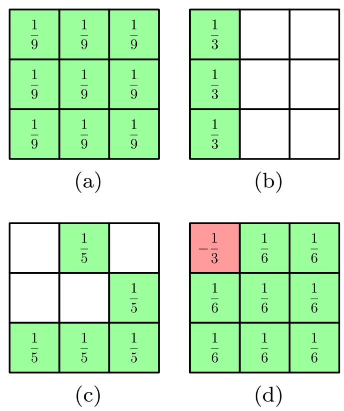

FIG. 1. **Qutrit Wigner distributions of varying magic.** (a) Maximally mixed state $\frac{1}{3}\mathbb{1}$; (b) Stabilizer zero state $|0\rangle\langle 0|$; (c) A non-stabilizer Wigner-positive state; (d) Magic strange state $|\mathrm{S}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$. [Explain what a magic / bound magic state is in intro]

B. Wigner Distribution

We can define the phase-point operators,

$$A_{\boldsymbol{x}} := \frac{1}{d} \sum_{\boldsymbol{z} \in \mathcal{P}_d} \omega^{\boldsymbol{x} \wedge \boldsymbol{z}} D_{\boldsymbol{z}}, \ \boldsymbol{x} \in \mathcal{P}_d. \tag{11}$$

[$\wedge$ has not be defined] They form an orthogonal Hermitian operator basis. Therefore, any quantum state $\rho \in \mathcal{B}(\mathcal{H}_d)$ can be expressed as a linear combination of the phase-point operators,

$$\rho = \sum_{\boldsymbol{z} \in \mathcal{P}_d} \mathrm{W}_\rho(\boldsymbol{z}) A_{\boldsymbol{z}}, \tag{12}$$

where the coefficient vector W_ρ is the Wigner distribution of state ρ,

$$\mathrm{W}_\rho(\boldsymbol{x}) := \frac{1}{d} \mathrm{tr}[A_{\boldsymbol{x}} \rho]. \tag{13}$$

It is in fact a real, bounded, d^2-dimensional quasi-probability distribution over $\mathcal{P}_d$ as shown in Appendix A.

The Wigner distributions of different types of qutrit states are illustrated in Fig. (1).

We can exploit the channel-state duality and use the normalised Choi-Jamiołkowski state

$$\frac{1}{d_A} \mathcal{J}(\mathcal{E}) := \frac{1}{d_A} (\mathbb{1} \otimes \mathcal{E}) \sum_{i,j} |ii\rangle \langle jj| \tag{14}$$

to extend the definition of the Wigner distribution to quantum CPTP operations $\mathcal{E} : \mathcal{B}(\mathcal{H}_{d_A}) \mapsto \mathcal{B}(\mathcal{H}_{d_B})$,

$$\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) := d_A^2 \mathrm{W}_{\frac{1}{d_A}\mathcal{J}(\mathcal{E})}(\bar{\boldsymbol{x}} \oplus \boldsymbol{y}) \tag{15}$$

$$= \frac{1}{d_B} \mathrm{tr}_B [A_{\boldsymbol{y}} \mathcal{E}(A_{\boldsymbol{x}})], \tag{16}$$

where $\bar{\boldsymbol{x}} := (x_0, -x_1)$.

The specific form of Eq. (15) is chosen so that Wigner distributions of operations act as transition matrices for Wigner distributions of states, $\mathrm{W}_{\mathcal{E}(\rho)} = \mathrm{W}_\mathcal{E} \mathrm{W}_\rho$. In particular, CPTP operations that map between density operators of equal dimensions and have non-negative Wigner distributions correspond to stochastic matrices, as shown in Appendix A

The single-qudit Hadamard gate H and phase gate S generate the d-dimensional Clifford group $\mathcal{C}_d$. [CITE] Their Wigner distributions are given by permutation matrices,

$$H := \frac{1}{\sqrt{d}} \sum_{j,k} \omega^{jk} |j\rangle\langle k|, \mathrm{W}_H(\boldsymbol{y}|\boldsymbol{x}) = \delta_{y_0, -x_1} \delta_{y_1, x_0}; \tag{17}$$

$$S := \sum_k \tau^{k(k+1)} |k\rangle\langle k|, \mathrm{W}_S(\boldsymbol{y}|\boldsymbol{x}) = \delta_{y_0, x_0} \delta_{y_1, x_0 + x_1 + 2^{-1}}. \tag{18}$$

IV. STOCHASTIC STRUCTURE OF MAGIC THEORIES

A. Magic fragments

Equipped with the definitions of the Wigner distribution in odd prime dimensions, we can formally recast the maximal magic theory $\mathcal{R}_{\max}$ into a stochasticity setting. The free states correspond to proper probability distributions

$$\mathcal{F}_{\max} := \{\rho : \mathrm{W}_\rho(\boldsymbol{z}) \geq 0 \text{ for all } \boldsymbol{z} \in \mathcal{P}_d\} \tag{19}$$

The free operations should send the set of free states $\mathcal{F}_{\max}$ into itself and completely preserve the non-negativity of the states, in the sense that $\mathcal{E} \in \mathcal{O}_{\max}$ iff $(\mathbb{1}_d \otimes \mathcal{E})\sigma \in \mathrm{STAB}$ for all odd prime dimensions d whenever $\sigma \in \mathcal{F}_{\max}$. It is shown by Wang *et al.* [12] that $\mathcal{O}_{\max}$ coincides with the set of operations $\mathcal{E}$ that correspond to stochastic Wigner distributions,

$$\mathcal{O}_{\max} = \{\mathcal{E} : \mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{P}_d\}. \tag{20}$$

Any magic theory $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ is a subtheory of $\mathcal{R}_{\max}$ as explained in Section I, and as such it falls under this new stochasticity setting. For technical simplicity in what follows we assume that $\mathcal{F}$ is a closed set, and note that $\mathcal{F}_{\max}$ is itself a closed set, since it is specified by a finite set of linear constraints of the form $\mathrm{tr}[L\rho] \geq 0$ with $L \in \mathcal{B}(\mathcal{H})$. [Unfortunately, this is not enough. There are positive Wigner ditributions which correspond to invalid states i.e. they are NOT positive semi-definite. Therefore, there are "holes" in the set specified by these linear constraints. We need a stricter argument.]

Given this context we now define the following key notion, that is central to our analysis.

Definition 1 (**σ-fragment**). *Given a resource theory of magic $\mathcal{R} = (\mathcal{F}, \mathcal{O})$, the σ–*fragment *of $\mathcal{R}$ is the resource* *theory $\mathcal{R}_\sigma = (\mathcal{F}, \mathcal{O}_\sigma)$, where the free operations are restricted to the ones that leave σ invariant, namely*

$$\mathcal{O}_\sigma := \{\mathcal{E} \in \mathcal{O} : \mathcal{E}(\sigma) = \sigma\}. \tag{21}$$

With this basic notion defined, we now show that any resource theory of magic can be faithfully subdivided into σ–fragments, in such a way that any problem of interconversion in the parent magic theory $\mathcal{R}$ can be analysed across the different fragments.

Theorem 2. *Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a theory of magic. Every operation in $\mathcal{O}$ leaves at least one free state invariant,*

$$\mathcal{O} = \bigcup_{\sigma \in \mathcal{F}} \mathcal{O}_\sigma. \tag{22}$$

Therefore, $\rho \longrightarrow \tau$ in $\mathcal{R}$ if and only if $\rho \longrightarrow \tau$ in a σ–fragment of $\mathcal{R}$.

Proof. Suppose $\mathcal{E}$ is in a σ–fragment $\mathcal{O}_\sigma$. Then it is also in $\mathcal{O}$, hence $\bigcup_{\sigma \in \mathcal{F}} \mathcal{O}_\sigma \subseteq \mathcal{O}$.

Conversely, suppose $\mathcal{E}$ is in $\mathcal{O}$. The free states are a closed set that is mapped one-to-one to a closed subset $\mathcal{S}$ of the (d^2-1)-dimensional probability simplex. $\mathcal{S}$ is convex, since any combination of free states is also free and the Wigner distribution is linear. Therefore, $\mathcal{S}$ is convex and compact as a closed convex subset of the bounded compact probability simplex.

We can now view $\mathrm{W}_\mathcal{E}$ as a stochastic, continuous mapping from $\mathcal{S}$ to itself, thus Brouwer's fixed point theorem [CITE] implies that there exists a probability distribution $d_{\boldsymbol{z}}$ for some $\boldsymbol{z} \in \mathcal{P}_d$ that is a fixed point of $\mathrm{W}_\mathcal{E}$. This corresponds to a free state $\sigma := \sum_{\boldsymbol{z} \in \mathcal{P}_d} d_{\boldsymbol{z}} A_{\boldsymbol{z}} \in \mathcal{F}$. Therefore $\mathcal{E} \in \mathcal{O}_\sigma$, and so $\mathcal{O} = \bigcup_{\sigma \in \mathcal{F}} \mathcal{O}_\sigma$.

The state interconversion result follows immediately. □

The zoo of all magic operation classes is summarised in Fig. (2). Completely positive-Wigner-preserving operations [12] form the operation class $\mathcal{O}_{\max}$. Therefore, σ–fragments cover this theory of magic exactly and any magic subtheory is contained within this cover. In particular, the stabilizer operations SO are contained within $\mathcal{O}_{\max}$.

The subdivision of magic theories into σ–fragments is powerful because the pre-order $\prec_{\mathcal{R}_\sigma}$ of every σ–fragment is described by well-behaved majorization tools, as we establish in Section IV B.

B. Majorization of quasi-probabilities in the σ–fragments

Majorization is a collection of powerful tools that has recently found many applications in quantum information theory [CITE]. It can describes the [disorder / non-uniformity] of distributions that undergo stochastic transformations.

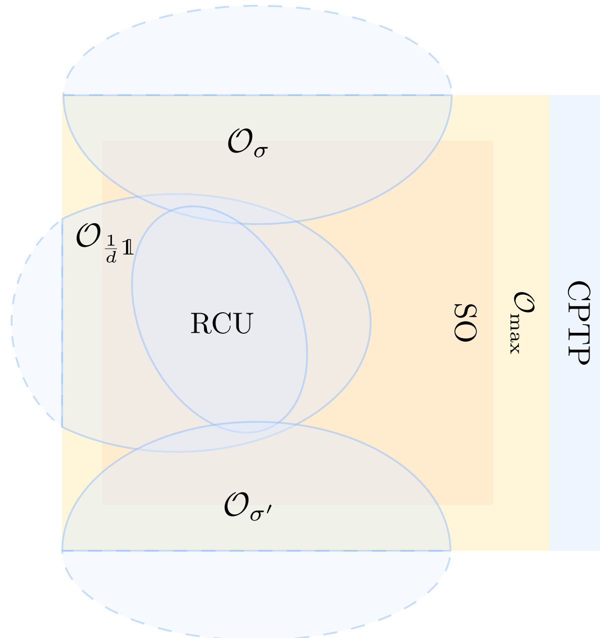

FIG. 2. **Decomposition of a magic theory $\mathcal{R}$ intro σ–fragments.** Examples of magic theories (SO: Stabilizer operations, $\mathcal{O}_{\max}$: Completely positive-Wigner-preserving operations) involve operations within the yellow regions, with all other established magic theories between these two. We introduce σ–fragments $\mathcal{O}_\sigma$ defined for all free states σ that cover $\mathcal{O}_{\max}$ with each one extendible to a set of stochastic maps outside the CPTP operations. Within each σ–fragment $\boldsymbol{d}$-majorization can be used allowing for an [in-depth approach] towards the study of magic state interconversion.

To formally state majorization results, we first denote by $\mathrm{S}_d(\boldsymbol{d})$ the set of $(d \times d)$ stochastic matrices that preserve the probability vector $\boldsymbol{d}$. [Should we introduce notation directly in the magic setting?] Specifically, for any $S \in \mathrm{S}_d(\boldsymbol{d})$, all matrix elements are non-negative, all rows sum to 1 and $S\boldsymbol{d} = \boldsymbol{d}$. The set $\mathrm{S}_d(\boldsymbol{d})$ forms a group under matrix multiplication for all $\boldsymbol{d}$ with positive components.

Majorization finds an important application on quantum thermodynamics in the absence of coherence. The use of majorization in this setting provides useful intuition for our purposes. At any given temperature β, the thermal state γ_β is thermodynamically the most disordered state. Thermal operations are defined as operations that cannot extract energy from the Gibbs state, $\mathcal{E}(\gamma_\beta) = \gamma_\beta$. Convertibility between states via thermal operations is equivalent to a stochasticity condition on the energy level populations of the states [CITE]. Roughly, the statement is that there exists a thermal operation $\mathcal{E}$ such that $\tau = \mathcal{E}(\rho)$ if and only if there exists a a matrix $S \in \mathrm{S}_d(\boldsymbol{d})$ such that $\boldsymbol{q} = S\boldsymbol{p}$, where $\boldsymbol{q}, \boldsymbol{p}$ and $\boldsymbol{d}$ and the energy level population vectors of τ, ρ, γ_β respectively.

Drawing intuition from this setting, we can define majorization as follows.

Definition 3 (**$\boldsymbol{d}$-majorization**). *Given $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d} \in \mathbb{R}^d$,* *such that the components of* $\boldsymbol{d}$ *are positive,* $\boldsymbol{y}$ *is said to* $\boldsymbol{d}$*-majorize* $\boldsymbol{x}$*, iff there exists a matrix* $S \in \mathrm{S}_d(\boldsymbol{d})$ *such that* $\boldsymbol{x} = S\boldsymbol{y}$*.*

We denote this pre-order by $\boldsymbol{x} \prec_{\boldsymbol{d}} \boldsymbol{y}$. If $\boldsymbol{d} = \frac{1}{d}\mathbf{1}$, the d-dimensional uniform distribution, then $\mathrm{S}_d(\boldsymbol{d})$ is the set of doubly stochastic matrices and we retrieve the familiar notion of majorization in entanglement theory. [CITE]

The pre-order $\prec_{\mathcal{R}_\sigma}$ of the σ–fragment $\mathcal{R}_\sigma = (\mathcal{F}, \mathcal{O}_\sigma)$ between d-dimensional states corresponds to the majorization pre-order $\prec_{\mathrm{W}_\sigma}$ between their d^2-dimensional Wigner distributions. For simplicity we shall merge this notation into $\prec_\sigma$, as there is little risk of confusion.

Theorem 4. *Let* $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ *be a theory of magic and suppose the state conversion* $\rho \longrightarrow \tau$ *is possible. Then, there exists a full-rank free state* $\sigma \in \mathcal{F}$ *such that* $\mathrm{W}_\tau \prec_\sigma \mathrm{W}_\rho$*.*

[We need to address zeros in the Wigner distribution (I think full-rank states do not have any zeros - there certainly exist non-full-rank states with no zeros).

For example, the replacement operation $\mathcal{E}(\rho) = |0\rangle\langle 0|$ *is free in* $\mathcal{R}_{\max}$ *with* $\mathcal{E} \in \mathcal{O}_{|0\rangle\langle 0|}$*, BUT* $W|0\rangle\langle 0|$*-majorization is not defined because* $W|0\rangle\langle 0|$ *contains zeros.*

In such a case we can always add some ϵ *amount of noise by mixing* σ *with a free full-rank state, e.g. a thermal state* γ_β*, to get* $\sigma' = (1-\epsilon)\sigma + \epsilon\gamma_\beta$*. This ensures that all Wigner components of* σ' *are strictly positive and* $\mathrm{W}_{\sigma'}$*-majorization can be used BUT* $\mathcal{E}$ *is NOT in* $\mathcal{O}_{\sigma'}$ *now for any* $\epsilon > 0$*.*

This could be solved if every operation preserves some thermal state. Otherwise, simply stating in this theorem that "there exists a full-rank free state $\sigma \in \mathcal{F}$ *such that* $\mathrm{W}_\tau \prec_{\mathrm{W}_\sigma} \mathrm{W}_\rho$*" is not accurate and needs reformulation to include the error* ϵ*.]*

Proof. Suppose there exists $\mathcal{E} \in \mathcal{O}$ such that $\mathcal{E}(\rho) = \tau$. The free operation belongs to a σ–fragment, $\mathcal{E} \in \mathcal{O}_\sigma$, for some $\sigma \in \mathcal{F}$ so that $\mathrm{W}_\mathcal{E}\mathrm{W}_\rho = \mathrm{W}_\tau$ with $\mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$.

If all components of W_σ are positive, we directly have $\mathrm{W}_\tau \prec_{\mathrm{W}_\sigma} \mathrm{W}_\rho$.

If W_σ contains some zero components, we can construct the full-rank state $\sigma' = (1-\epsilon)\sigma + \epsilon\gamma_\beta$ by mixing an arbitrarily small amout $\epsilon > 0$ of some thermal state γ_β [can replace with maximally mixed state], so that all components of $\mathrm{W}_{\sigma'}$ are positive. [but $\mathrm{W}_\tau \prec_{\mathrm{W}_{\sigma'}} \mathrm{W}_\rho$ is NOT true now] □

A visual representation of $\boldsymbol{d}$-majorization is provided by Lorenz curves. Let the vector $\boldsymbol{z}^\downarrow$ denote a component permutation of vector $\boldsymbol{z} \in \mathbb{R}^d$, so that its components are arranged in non-increasing order.

Definition 5 (Lorenz curve). *Let* $\boldsymbol{z} \in \mathbb{R}^d$*. Let* $\boldsymbol{d} \in \mathbb{R}^d$ *be a vector with positive components,* π *a permutation mapping* $(z_i/d_i) \mapsto (z_i/d_i)^\downarrow$ *for all* $i = 1, \dots, d$ *and* $D = \sum_{i=1}^d d_i$*. The Lorenz curve* $L(\boldsymbol{z}|\boldsymbol{d})$ *of vector* $\boldsymbol{z}$ *is the piecewise linear curve obtained by joining the points* $\{(x_k, L_k(\boldsymbol{z}|\boldsymbol{d}))\}_{k=1,\dots,d}$*, where*

$$(x_k, L_k(\boldsymbol{z}|\boldsymbol{d})) := \left(\frac{1}{D}\sum_{i=1}^k d_{\pi(i)}, \sum_{i=1}^k z_{\pi(i)}\right) \in \mathbb{R}^2. \tag{23}$$

Remark 1. The origin $(x_0, L_0(\boldsymbol{z}|\boldsymbol{d})) := (0,0)$ is usually included in the curve.

Remark 2. Components x_k are rescaled by D so that comparison of curves with unequal dimensions is possible. In fact, the Lorenz curves $L(\boldsymbol{z}|\boldsymbol{d})$ and $L(\boldsymbol{z}\otimes\boldsymbol{d}|\boldsymbol{d}\otimes\boldsymbol{d})$, where $\otimes$ denotes the Kronecker product, are identical.

Remark 3. Lorenz curves are always concave.

Remark 4. If $L_d(\boldsymbol{z}|\boldsymbol{d}) = 1$ and for all k, $L_k(\boldsymbol{z}|\boldsymbol{d}) \leq 1$, then $\boldsymbol{z}$ is a probability distribution. Lorenz curves of quasi-probability distributions in principle reach above 1.

Remark 5. We can recast the Lorenz curve into the *function* $L_{\boldsymbol{z}|\boldsymbol{d}}(x)$ defined on $[0,1]$.

A vector $\boldsymbol{y}$ is said to $\boldsymbol{d}$*–majorize* another vector $\boldsymbol{x}$ if and only if the Lorenz curve $L(\boldsymbol{y}|\boldsymbol{d})$ lies above Lorenz curve $L(\boldsymbol{x}|\boldsymbol{d})$.

Theorem 6. *Let* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d} \in \mathbb{R}^d$*, such that the components of* $\boldsymbol{d}$ *are positive. Then,* $\boldsymbol{x} \prec_{\boldsymbol{d}} \boldsymbol{y}$ *if and only if* $L_k(\boldsymbol{x}|\boldsymbol{d}) \leq L_k(\boldsymbol{y}|\boldsymbol{d})$ *for all* $k = 1, 2, \dots, d-1$ *and* $L_d(\boldsymbol{x}|\boldsymbol{d}) = L_d(\boldsymbol{y}|\boldsymbol{d})$*.*

A restatement of the theorem including more equivalent conditions, along with a proof is provided in Appendix B.

C. Lorenz curves for magic states in a σ–fragment

With this it is straightforward to construct Lorenz curves for Wigner distributions in any σ–fragment. For any σ being a full rank free state we have that $W_\sigma(\boldsymbol{x})$ is a strictly positive full-rank probability distribution, and so one can define a corresponding notion of d–majorization on *quasi*–distributions.We write $L_{\rho|\sigma}(x)$ for the Lorenz curve of $W_\rho(\boldsymbol{x})$ with respect to $W_\sigma(\boldsymbol{x})$.

We now see that $0 \leq L_{\rho|\sigma}(x) \leq 1$ for all x if and only if ρ is a positive Wigner state. Moreover, the area of the function $L_{\rho|\sigma}$ above the line $y = 1$ is a resource monotone in this fragment. [expand on this – relate to mana?] [the area depends on several positive Wigner components and cannot be written as a function of mana annoyingly - can make plot similar to 4.1 in my thesis]

An example of comparison between different Lorenz curves is illustrated in Fig. (3).

V. EXTENSION TO GENERAL QUANTUM RESOURCE THEORIES

In the previous section we introduced the notion of σ–fragments for any resource theory of magic. In this section we pause to generalise this concept to an arbitrary resource theory and explain precisely how it connects with resource monotones. The busy reader more focussed on magic may skip this section.

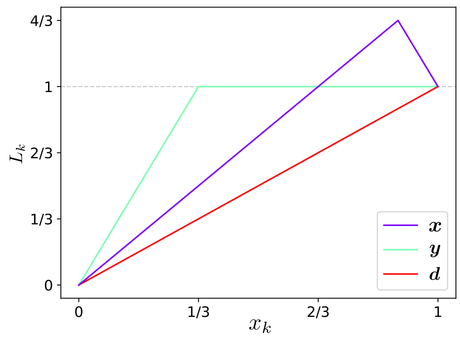

FIG. 3. Example of different Lorenz curves for quasi-probability vectors under $\boldsymbol{d}$-majorization. Vectors $\boldsymbol{y}$ and $\boldsymbol{d}$ are simply probability distributions. The curve corresponding to vector $\boldsymbol{d}$ is always the line segment connecting $(0,0)$ and $(1,1)$, so that any other Lorenz curve lies above it, for example $\boldsymbol{x} \prec_{\boldsymbol{d}} \boldsymbol{d}$. Curves $L_k(\boldsymbol{x}|\boldsymbol{d})$ and $L_k(\boldsymbol{y}|\boldsymbol{d})$ intersect, so $\boldsymbol{x} \not\prec_{\boldsymbol{d}} \boldsymbol{y}$ as well as $\boldsymbol{y} \not\prec_{\boldsymbol{d}} \boldsymbol{x}$. [Recast in terms of magic]

State convertibility within a given resource theory is often a hard question to address due to the intricate structure of the theory. In general, the structure of a theory $\mathcal{R}$ is described by a pre-order $\prec_{\mathcal{R}}$ and usually resource monotones are employed to reduce this structure into a simple real number ordering[CITE]. The subdivisions of magic theories into σ–fragments suggests a new approach towards investigating state convertibility which retains more structure of the origin theory than a measure can.

A. Monotones

We highlight the defining property of a monotone for the discussion that follows.

Definition 7 (**Resource monotone**). *Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a resource theory. A resource monotone $\mathcal{M}$ is a projection from the set of quantum states of the theory onto the real line, so that $\mathcal{M}$ is monotonically decreasing under free operations,*

$$\mathcal{M}(\rho_1) \leq \mathcal{M}(\rho_2) \text{ whenever } \rho_1 \prec_{\mathcal{R}} \rho_2. \tag{24}$$

The monotonicity condition reflects the no resource generating property of free operations, so that monotones respect the pre-order $\prec_{\mathcal{R}}$ of the theory. Furthermore, a monotone may also satisfy the additivity condition,

$$\mathcal{M}(\rho_1 \otimes \rho_2) = \mathcal{M}(\rho_1) + \mathcal{M}(\rho_2). \tag{25}$$

This optional condition is of practical importance for resource distillation, which we discuss in Section VI within the context of magic.

The most fundamental and commonly used magic monotone is the *mana* of a state [CITE], defined as

$$\mathrm{mana}\,(\rho) := \log\left(\sum_{\boldsymbol{z}\in\mathcal{P}_d} \left|\mathrm{W}_\rho\,(\boldsymbol{z})\right|\right). \tag{26}$$

It is a monotone function of the ℓ_1-norm of state negativity [CITE] that satisfies the additivity condition.

B. Fragments

Monotones reduce the structure of the resource theory $\mathcal{R}$ to a *total* order on the real numbers. Therefore, two states, even if incomparable in $\mathcal{R}$, are always mapped onto ordered real numbers. We now generalise this idea of a theory projection that preserves comparability between states.

Definition 8 (**Covariant projection**). *Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a resource theory with pre-order $\prec_{\mathcal{R}}$. Then a* covariant resource projection *of $\mathcal{R}$ to a resource theory $\mathcal{R}'$ with pre-order $\prec_{\mathcal{R}'}$, is a pair of mappings (Π_s, Π_o), where Π_s maps quantum states in $\mathcal{R}$ to quantum states in $\mathcal{R}'$, and Π_o maps free operations in $\mathcal{R}$ to free operations in $\mathcal{R}'$. Moreover, these obey*

1. *$\Pi_s(\rho_1) \prec_{\mathcal{R}'} \Pi_s(\rho_2)$ whenever $\rho_1 \prec_{\mathcal{R}} \rho_2$;*
2. *$\Pi_o(\mathcal{E}) = \Pi_o(\mathcal{E}_1) \circ \Pi_o(\mathcal{E}_2)$ whenever $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$.*

We call $\mathcal{R}'$ a covariant fragment *of $\mathcal{R}$.*

Resource monotones can now be clearly seen as a special case of covariant resource projections.

Proposition 9 (**Totally ordered covariant theories**). *Any resource monotone $\mathcal{M}$ of a resource theory $\mathcal{R}$ is a covariant projection for which $\prec_{\mathcal{R}'}$ is a total order. Conversely, any such covariant projection corresponds to a resource monotone $\mathcal{M}$.*

Proof. Consider a monotone $\mathcal{M}$ in the context of a general resource theory $\mathcal{R} = (\mathcal{F}, \mathcal{O})$. State order is covariantly preserved due to the defining property of a monotone, stated in Definition 7, where the pre-order $\prec_{\mathcal{R}'}$ is simply the total order $\leq$ on $\mathbb{R}$.

Operational composition is covariantly preserved when we simply choose $\Pi_o(\mathcal{E}) = 1_\times$, namely the 'multiplication by 1' operation on real numbers. The definition of a resource monotone then automatically implies covariance.

Conversely, given any covariant projection of $\mathcal{R}$ for which $\prec_{\mathcal{R}'}$ is a total order, we may map the totally ordered set of elements $\Pi_s(\rho)$ via an injective, non-decreasing function f into $\mathbb{R}$. Then, $\mathcal{M}(\rho) := f(\Pi_s(\rho))$ provides a numerical value for each ρ that obeys the definition of a monotone.

$\square$

We can also view σ–fragments as an example of reducing the structure of a magic theory $\mathcal{R}$ to a subtheory with a tractable pre-order. However, states which are incomparable in $\mathcal{R}$ remain incomparable and conversions between states which are comparable in $\mathcal{R}$ may no longer be possible.

Definition 10 (Contravariant projection). *Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a resource theory with pre-order $\prec_{\mathcal{R}}$. Then a* contravariant resource projection *of $\mathcal{R}$ onto a resource theory $\mathcal{R}'$ with pre-order $\prec_{\mathcal{R}'}$, is a pair of mappings (Π_s, Π_o), where Π_s maps quantum states in $\mathcal{R}$ onto quantum states in $\mathcal{R}'$, and Π_o maps free operations in $\mathcal{R}$ onto free operations in $\mathcal{R}'$. Moreover, these obey*

1. *$\rho_1 \prec_{\mathcal{R}} \rho_2$ whenever $\Pi_s(\rho_1) \prec_{\mathcal{R}'} \Pi_s(\rho_2)$;*
2. *$\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$ whenever $\Pi_o(\mathcal{E}) = \Pi_o(\mathcal{E}_1) \circ \Pi_o(\mathcal{E}_2)$.*

We call $\mathcal{R}'$ a contravariant fragment *of $\mathcal{R}$.*

Note that a contravariant projection is always a surjective mapping for both states and operations, so that the conditions in Definition 10 make sense. The use of covariant and contravariant in Definitions 8 and 10 refers to the direction of implication between the two pre-orders and operation compositions[1].

Proposition 11. *Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a resource theory, and let $\mathcal{O}' \subseteq \mathcal{O}$ be a non-empty subset of the free operations that is closed under composition, and moreover $\mathcal{O}'$ is the largest such subset, in the sense that for any $\mathcal{E}_1 \notin \mathcal{O}'$ and any $\mathcal{E}_2 \in \mathcal{O}'$ we have that both $\mathcal{E}_1 \circ \mathcal{E}_2$ and $\mathcal{E}_2 \circ \mathcal{E}_1$ are not in $\mathcal{O}'$. Then $\mathcal{R}' = (\mathcal{F}, \mathcal{O}')$ of $\mathcal{R}$ defines a contravariant fragment of $\mathcal{R}$.*

Proof. We first define $\Pi_s(\rho) = \rho$ for all ρ. It is clear that since $\mathcal{O}'$ is a subset of $\mathcal{O}$ any operation in $\mathcal{O}'$ will map the set of free states into itself. Moreover the identity channel id is necessarily in $\mathcal{O}'$, due to the maximality assumption. For Π_o we let $\Pi_s(\mathcal{E}) = \mathcal{E}$ if $\mathcal{E} \in \mathcal{O}'$ and otherwise $\Pi_s(\mathcal{E}) = id$. Now consider $\Pi_o(\mathcal{E}_1 \circ \mathcal{E}_2)$. Either the triple $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_1 \circ \mathcal{E}_2\}$ are all in $\mathcal{O}'$ or they are all outside of $\mathcal{O}'$. For the former case $\Pi_o(\mathcal{E}_1 \circ \mathcal{E}_2) = \mathcal{E}_1 \circ \mathcal{E}_2 = \Pi_o(\mathcal{E}_1) \circ \Pi_o(\mathcal{E}_2)$, while for the latter $\Pi_o(\mathcal{E}_1 \circ \mathcal{E}_2) = id = \Pi_o(\mathcal{E}_1) \circ \Pi_o(\mathcal{E}_2)$, which proves that compositions are respected under the map. Finally, $\rho \prec_{R'} \sigma$ implies there exists $\mathcal{E} \in \mathcal{O}' \subseteq \mathcal{O}$ such that $\mathcal{E}(\rho) = \sigma$, and since $\mathcal{E} \in \mathcal{O}$ this implies $\rho \prec_{\mathcal{R}} \sigma$, as required, which completes the proof. □

[The above proposition is not relevant to σ-fragments where "for any $\mathcal{E}_1 \notin \mathcal{O}_\sigma$ and any $\mathcal{E}_2 \in \mathcal{O}_\sigma$" we sometimes have $\mathcal{E}_1 \circ \mathcal{E}_2 \in \mathcal{O}_\sigma$ as well as $\mathcal{E}_2 \circ \mathcal{E}_1 \in \mathcal{O}_\sigma$ and sometimes $\mathcal{E}_1 \circ \mathcal{E}_2 \notin \mathcal{O}_\sigma$ as well as $\mathcal{E}_2 \circ \mathcal{E}_1 \notin \mathcal{O}_\sigma$] As an immediate corollary of Proposition 11, a σ–fragment of any magic theory $\mathcal{R}$ is a contravariant fragment of $\mathcal{R}$.

Proposition 12. *Let $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ be a resource theory, and let $\mathcal{D} \in \mathcal{O}$ be a free operation, which is reversible by $\mathcal{D}_{\text{rev}} \in \mathcal{O}$, so that $\mathcal{D}_{\text{rev}} \circ \mathcal{D} = 1_{\text{C}}$.*

Then, we can define a contravariant projection of $\mathcal{R}$, by acting on all quantum states with $\mathcal{D}$.

Proof. We show that the theory $\mathcal{R}' = (\mathcal{F}', \mathcal{O})$, with $\mathcal{F}' = \{\mathcal{D}(\rho) : \rho \in \mathcal{F}\}$, is a contravariant fragment of $\mathcal{R}$.

Let Π_s map every state ρ to $\mathcal{D}(\rho)$ and suppose $\mathcal{D}(\rho_1) \prec \mathcal{D}(\rho_2)$. Then, there exists $\mathcal{E} \in \mathcal{O}$ such that $\rho_1 = (\mathcal{D}_{\text{rev}} \circ \mathcal{E} \circ \mathcal{D})(\rho_2)$, so $\rho_1 \prec \rho_2$.

Finally, let Π_o map every free operation to itself, so that composition of operations is trivially preserved.

[If $\mathcal{D}$ is a recovery map, so that $\mathcal{D} \circ \mathcal{D}_{\text{rev}} = 1_{\text{C}}$, then this is a covariant projection instead.

If $\mathcal{D}$ is not reversible, this mapping is in general NOT contravariant (consider the replacement map $\mathcal{D}(\rho) = \frac{1}{d}\mathbb{1}$ for a strange state and stabilizer state - surely there is such a counterexample in thermodynamics theory if we consider a highly coherent state and one with the same energy population but no coherences.] □

Important examples of resource fragments appear in several established resource theories. [Need to check if the thermodynamics example works, include magic theories as fragments of $\mathcal{R}_{\max}$, include Nielsen's bipartite entanglement.] [Don't worry about these things now – let's get the computations section improved]

[1] Note that strictly these are not projections in the sense of $\Pi^2 = \Pi$, but are instead morphisms. Here our use of the term projection is motivated by the idea that one one generally loses information about $\mathcal{R}$ under the mapping.

VI. MAGIC STATE INTERCONVERSION AND LORENZ CURVES

Any quantum circuit aiming at a given magic state conversion $\rho \longrightarrow \tau$ possesses certain symmetries according to Theorem 17 that allow us to study the conversion within only certain σ-fragments. As a simple example, the dephasing channel

$$\Delta(\rho) = \sum_{k \in \mathbb{Z}_d} |k\rangle\langle k| \, \rho \, |k\rangle\langle k| \tag{27}$$

removes coherent phases in the computational basis and therefore only leaves invariant mixtures of computational basis states. If the circuit simply consists of such dephasing channels, the whole analysis could be restricted in the σ–fragments for pure computational basis states σ.

Lorenz curves provide a very efficient method of numerically checking if a certain state conversion is impossible within a σ–fragment by exploiting the equivalence with d–majorization stated in Theorem 6. Such methods are often more conclusive than magic monotones as we discuss in section Section VI A.

A. Mana and majorization in magic theories

We now discuss majorization features common to all fragments, before specialising to particular σ–fragments and how analysis in them proceeds. Mana is the fundamental resource monotone for magic. Here we show that its properties are in fact special cases of more general majorization features. More precisely, its properties can be viewed as majorization-based, and independent of the particular σ–fragment one works in.

As discussed above, we know that every magic interconversion problem can be analysed across all σ–fragments that reflect symmetries of the circuit and moreover the pre-order in each such fragment is exactly specified by d–majorization.

Consider a general magic state distillation process, [Generalise discussion to any state conversion?]

$$\rho^{\otimes k} \longrightarrow \tau, \tag{28}$$

where k noisy copies of magic state ρ are converted to a single-copy magic state τ.

It is shown in Appendix B that mana is an additive monotone in every σ–fragment under d–majorization, so it provides a bound for the process,

$$\mathrm{mana}\,(\rho) \geq \frac{1}{k} \mathrm{mana}\,(\tau). \tag{29}$$

A new bound can be obtained in every σ–fragment by comparing the Lorenz curves of the initial and target states,

$$L_k(\rho^{\otimes k}|\sigma) \geq L_k(\tau|\sigma), \; k = 1, \ldots, d^2, \tag{30}$$

If the Lorenz curve of the initial state is below the target curve at any point, the process is not possible. In general, the Wigner components of a k-copy state $\rho^{\otimes k}$ are calculated, along with their multiplicities, by expanding the terms in the multinomial expansion $\left(\sum_{\boldsymbol{z} \in \mathcal{P}_d} W_\rho(\boldsymbol{z})\right)^n$. This follows from the multiplicativity of the Wigner distribution.

The Strange state $|\mathrm{S}\rangle\langle\mathrm{S}|$ depicted in Fig. (1(d)) is the simplest to analyse, since it only has two distinct components $\{-\frac{1}{3}, \frac{1}{6}\}$, the latter with a multiplicity of 8. Calculating the binomial expansion for the components of $|S\rangle\langle S|^{\otimes k}$ gives $\{(-1)^j 2^{j-k} 3^{-k}\}_{0 \leq j \leq k}$ with multiplicity $8^{k-j}\binom{k}{j}$ for the j-th term. This allows analytical calculation of all Lorenz curve points, hence the maximum of the k-copy state is

$$\max_k L_k\left(|S\rangle\langle S|^{\otimes k} \mid \sigma\right) = 1 + \left(\frac{4}{3}\right)^k \sum_{j: 1 \leq 2j+1 \leq k} 4^{-(2j+1)} \binom{k}{2j+1}. \tag{31}$$

[We can probably put upper and lower bounds on this value fairly readily. E.g. using known sum rules. Also – bad choice of variable k, using it as index and also copy number. I think $L_{\rho|\sigma}$ is a neater formulation, since you then package things into a single object – a function.]

Consider the noisy Strange state,

$$\rho_{\mathrm{S}}(\epsilon) = (1-\epsilon)\, |\mathrm{S}\rangle \langle \mathrm{S}| + \epsilon\sigma, \tag{32}$$

in the σ–fragment $\mathcal{O}_\sigma$. At noise level $\epsilon \leq \frac{3}{4}$, the Wigner distribution $\mathrm{W}_{\rho_{\mathrm{S}}(\epsilon)}$ contains negativities and the state can be purified so as to obtain a single-copy state with sufficiently low ϵ. In Fig. (4), we examine the purifying process

$$\rho_{\mathrm{S}}^{\otimes k}(\epsilon_{\mathrm{th}}) \xrightarrow{\mathcal{E} \in \mathcal{O}_\sigma} \rho_{\mathrm{S}}(0.05), \;\; \sigma = (1-p)\, |0\rangle\langle 0| + p\frac{1}{3}\mathbb{1} \tag{33}$$

with ϵ_{th} being the noise level threshold that does not prohibit the process for given number of copies k and σ–fragment, parametrised by p as a mixture of the zero and the maximally mixed states.

Thresholds provided by Lorenz curve comparison are always much stricter than mana thresholds [threshold/bound? need to define the notion of a bound precisely]. In fact, it is clear than this is the case in any general distillation process.

Theorem 13. *Consider the distillation process in Eq. (28). In any σ–fragment, W_σ-majorization provides a stricter bound than mana.*

Proof. The maximum of the Lorenz curve of a state ρ is independent of the σ–fragment

The Lorenz curve maximum can then be expressed monotonically in terms of mana,

$$\max_k L_k(\rho|\sigma) = 1 + \sum_{\boldsymbol{z}: \mathrm{W}_\rho(\boldsymbol{z}) < 0} |\mathrm{W}_\rho(\boldsymbol{z})| = \frac{1}{2}\left(1 + e^{\mathrm{mana}(\rho)}\right). \tag{34}$$

Therefore, the majorization condition stated in Eq. (30) implies that $\mathrm{mana}\left(\rho^{\otimes k}\right) \geq \mathrm{mana}\,(\tau)$. [This is a bit too short] □

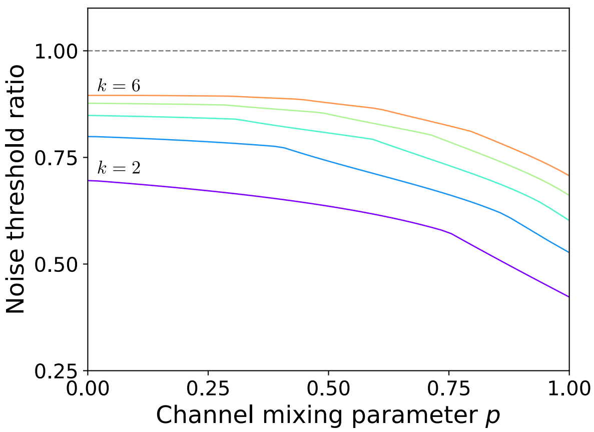

FIG. 4. **Noise threshold ratios.** Mana and Lorenz curve ratios for the Strange state purifying process in Eq. (33). The ratios are calculated for different numbers of initial state copies and different σ–fragments parametrised by p such that $\sigma = (1-p)\,|0\rangle\langle 0| + p\frac{1}{3}\mathbb{1}$. Lorenz curve comparison consistently gives stricter bounds as proven in Theorem 13. [Consider set of pure stabilizer states instead of $(1-p)\,|0\rangle\langle 0| + p\frac{1}{3}\mathbb{1}$ perhaps]

VII. CONCLUSION

1. Introduced fragments
2. Identify symmetries of the setup
3. Combined single-shot thermodynamics with magic
4. Can we solve other cases exactly? (apart from single qutrit)

[1] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[2] D. Gross and J. Eisert, Phys. Rev. Lett. **98**, 220503 (2007).
[3] I. L. Markov and Y. Shi, SIAM J. Comput. **38**, 963 (2008).
[4] D. Gross, S. T. Flammia, and J. Eisert, Phys. Rev. Lett. **102**, 190501 (2009).
[5] M. Van den Nest, A. Miyake, W. Dür, and H. J. Briegel, Phys. Rev. Lett. **97**, 150504 (2006).
[6] M. Van den Nest, W. Dür, G. Vidal, and H. J. Briegel, Phys. Rev. A **75**, 012337 (2007).
[7] G. Vidal, Phys. Rev. Lett. **98**, 070201 (2007).
[8] K. Fujii and M. Hayashi, Phys. Rev. A **96**, 030301 (2017).
[9] D. Gottesman, *Stabilizer codes and quantum error correction*, Ph.D. thesis, California Institute of Technology (1997).
[10] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, New Journal of Physics **14**, 113011 (2012).
[11] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, New Journal of Physics **16**, 013009 (2014).
[12] X. Wang, M. M. Wilde, and Y. Su, arXiv e-prints , arXiv:1903.04483 (2019), arXiv:1903.04483.
[13] X. Wang, M. M. Wilde, and Y. Su, arXiv e-prints , arXiv:1812.10145 (2018), arXiv:1812.10145 [quant-ph].
[14] M. Howard and E. Campbell, Phys. Rev. Lett. **118**, 090501 (2017).
[15] E. T. Campbell and D. E. Browne, Phys. Rev. Lett. **104**, 030503 (2010).
[16] D. Gross, Journal of Mathematical Physics **47**, 122107 (2006).
[17] D. Gross, Applied Physics B **86**, 367 (2007).
[18] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Phys. Rev. Lett. **115**, 210403 (2015).
[19] M. Lostaglio, D. Jennings, and T. Rudolph, Nature Communications **6** (2015), 10.1038/ncomms7383.
[20] M. Lostaglio, arXiv e-prints , arXiv:1807.11549 (2018).
[21] G. Gour, D. Jennings, F. Buscemi, R. Duan, and I. Marvian, Nature Communications **5352** (2018), 10.1038/s41467-018-06261-7.
[22] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, International Journal of Theoretical Physics **39**, 2717 (2000).
[23] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Phys. Rev. Lett. **111**, 250404 (2013).
[24] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens, and N. Yunger Halpern, Physics Reports **583** (2015), 0.1016/j.physrep.2015.04.003.
[25] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications* (Springer, 2011).
[26] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, United Kingdom, 2010).
[27] M. Ahmadi, H. B. Dang, G. Gour, and B. C. Sanders, Phys. Rev. A **97**, 062332 (2018).
[28] S. Bravyi and A. Kitaev, Phys. Rev. A **71**, 022316 (2005).
[29] A. Mari and J. Eisert, Phys. Rev. Lett. **109**, 230503 (2012).
[30] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Nature **510** (2014), 10.1038/nature13460.

Appendix A: Properties of the Wigner distribution

Here we present important properties of the Wigner distribution that are used throughout the paper.

Proposition 14. *The Wigner distribution of a state ρ is*

1. *Real valued:* $\mathrm{W}_\rho \in \mathbb{R}^{d^2}$;
2. *Normalised:* $\sum_{\boldsymbol{z}\in\mathcal{P}_d} \mathrm{W}_\rho(\boldsymbol{z}) = 1$;
3. *Bounded:* $|\mathrm{W}_\rho(\boldsymbol{x})| \le \frac{1}{d}$.
4. *Additive under mixing:* $\mathrm{W}_{\sum_i p_i\rho_i}(\boldsymbol{x}) = \sum_i p_i \mathrm{W}_{\rho_i}(\boldsymbol{x})$;
5. *Multiplicative under tensor products:* $\mathrm{W}_{\rho_A\otimes\rho_B}(\boldsymbol{x}_A \oplus \boldsymbol{x}_B) = \mathrm{W}_{\rho_A}(\boldsymbol{x}_A)\mathrm{W}_{\rho_B}(\boldsymbol{x}_B)$.

A distribution satisfying the first three properties does not necessarily correspond to a positive semi-definite state.

Proposition 15. *The Wigner distribution of a* CPTP *operation $\mathcal{E} : \mathcal{B}(\mathcal{H}_{d_A}) \mapsto \mathcal{B}(\mathcal{H}_{d_B})$ is:*

1. *Real-valued:* $\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) \in \mathbb{R}$;
2. *Normalised:* $\sum_{\boldsymbol{z}\in\mathcal{P}_{d_B}} \mathrm{W}_\mathcal{E}(\boldsymbol{z}|\boldsymbol{x}) = 1$ *for any* $\boldsymbol{x} \in \mathcal{P}_{d_A}$;
3. *Bounded:* $|\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x})| \le \frac{d_A}{d_B}$;
4. *[Transitive]:* $\mathrm{W}_{\mathcal{E}(\rho)}(\boldsymbol{y}) = \sum_{\boldsymbol{z}\in\mathcal{P}_{d_A}} \mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{z})\mathrm{W}_\rho(\boldsymbol{z})$ *for any* $\boldsymbol{y} \in \mathcal{P}_{d_B}$.

If $d_A = d_B$, and in particular if operation $\mathcal{E}$ maps a Hilbert space onto itself, then the stochasticity condition $|\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x})| \le 1$ is satisfied.

Appendix B: Properties of majorization

1. Equivalent conditions for majorization

Theorem 16. *Given $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d} \in \mathbb{R}^n$, such that the components of $\boldsymbol{d}$ are positive, the following statements are equivalent:*

1. $\boldsymbol{x} \prec_{\boldsymbol{d}} \boldsymbol{y}$;
2. $\Gamma_{\boldsymbol{d}}(\boldsymbol{x}) \prec \Gamma_{\boldsymbol{d}}(\boldsymbol{y})$;
3. $\sum_{i=1}^n |x_i - td_i| \le \sum_{i=1}^n |y_i - td_i|$ *for all* $t \in \mathbb{R}$;
4. $\sum_{i=1}^n (x_i - td_i)^+ \le \sum_{i=1}^n (y_i - td_i)^+$ *for all* $t \in \mathbb{R}$ *and* $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$;
5. $\forall k, L_{\boldsymbol{x}|\boldsymbol{d}}(k) \le L_{\boldsymbol{y}|\boldsymbol{d}}(k)$ *and* $L_{\boldsymbol{x}|\boldsymbol{d}}(k = n) = L_{\boldsymbol{y}|\boldsymbol{d}}(k = n)$.

Proof. 1↔ 2 Suppose now there exists a stochastic S such that $\boldsymbol{x} = S\boldsymbol{y}$ with $\boldsymbol{d} = S\boldsymbol{d}$ and let $B = \Gamma_{\boldsymbol{d}} \circ S \circ \Gamma_{\boldsymbol{d}}^{-1}$. B is a D-dimensional bistochastic matrix, since composition of stochastic matrices is stochastic and $(\Gamma_{\boldsymbol{d}} \circ S \circ \Gamma_{\boldsymbol{d}}^{-1})(\frac{1}{D}\mathbf{1}) = (\Gamma_{\boldsymbol{d}} \circ S)(\boldsymbol{d}) = \Gamma_{\boldsymbol{d}}(\boldsymbol{d}) = \frac{1}{D}\mathbf{1}$. Then, B maps $\Gamma_{\boldsymbol{d}}(\boldsymbol{y})$ to $\Gamma_{\boldsymbol{d}}(\boldsymbol{x})$. Conversely, given B, let $S = \Gamma_{\boldsymbol{d}}^{-1} \circ B \circ \Gamma_{\boldsymbol{d}}$. Similarly, S is the stochastic matrix that preserves $\boldsymbol{d}$ and maps $\boldsymbol{y}$ to $\boldsymbol{x}$.

2 ↔ 3, 2 ↔ 4, 2 ↔ 5 These three statement are equivalent to [blah] respectively for the embedded vectors $\Gamma_{\boldsymbol{d}}(\boldsymbol{x}), \Gamma_{\boldsymbol{d}}(\boldsymbol{y})$. This is clear by rewriting

$$\sum_{i=1}^n |x_i - td_i| = \sum_{i=1}^n d_i \left|\frac{x_i}{d_i} - t\right| = \sum_{i=1}^D |\Gamma_{\boldsymbol{d}}(\boldsymbol{x})_i - t|, \tag{B1}$$

$$\sum_{i=1}^n (x_i - td_i)^+ = \sum_{i=1}^D (\Gamma_{\boldsymbol{d}}(\boldsymbol{x})_i - t)^+, \tag{B2}$$

$$L_{\boldsymbol{x}|\boldsymbol{d}}(k) = L_{\Gamma_{\boldsymbol{d}}(\boldsymbol{x})}(k'), \tag{B3}$$

$$\text{with } k = 1, \dots, n \text{ and } k' = 1, \dots, D$$

and similarly for the right hand side. □

2. Mana properties

Mana monotonicity can be directly seen due to statement 3 in Theorem 6 for $t = 0$. Furthermore, mana is additive due to the multiplicative property 4 of Proposition 14.

Appendix C: Properties of σ-fragments

Theorem 17. *Let $\mathcal{R} = (\mathcal{O}, \mathcal{F})$ be a magic theory. The following statements hold:*

1. *No σ-fragment is empty.*
2. *If a free operation leaves two states invariant, then it also leaves their mixtures invariant,*

$$\mathcal{O}_\sigma \cap \mathcal{O}_{\sigma'} \subseteq \mathcal{O}_{p\sigma+(1-p)\sigma'} \text{ for all } p \in [0, 1]. \tag{C1}$$

3. *Let $\mathcal{E}$ be a* CPTP *operation with Wigner distribution $\mathrm{W}_\mathcal{E}$. For $\mathcal{R} = \mathcal{R}_{\max}$ $\mathcal{E} \in \mathcal{O}_\sigma$ iff $\mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$.*

Proof.

1. The identity channel $1_{\mathrm{C}} : \mathcal{D} \mapsto \mathcal{D}$ belongs to every σ-fragment, as $1_{\mathrm{C}} \in \mathcal{O}$ and $1_{\mathrm{C}}\sigma = \sigma$ for all $\sigma \in \mathcal{F}$.

2. Let $\mathcal{E} \in \mathcal{O}_\sigma \cap \mathcal{O}_{\sigma'}$. Then $\mathcal{E} \in \mathrm{CPTP}$ and corresponds to stochastic Wigner distribution $\mathrm{W}_\mathcal{E}$ such that $\mathrm{W}_\mathcal{E}\mathrm{W}_\sigma = \mathrm{W}_\sigma$ and $\mathrm{W}_\mathcal{E}\mathrm{W}_{\sigma'} = \mathrm{W}_{\sigma'}$. Then, $\mathrm{W}_\mathcal{E}\mathrm{W}_{p\sigma+(1-p)\sigma'} = \mathrm{W}_{p\sigma+(1-p)\sigma'}$ for any $p \in [0,1]$ due to the additive property 4 of the Wigner distribution, implying that state $p\sigma + (1-p)\sigma'$ is also left invariant by $\mathcal{E}$.

3. Let $\mathcal{O}'_\sigma \coloneqq \{\mathcal{E} \in \mathrm{CPTP} : \mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)\}$ be the described set of operations.

   Suppose $\mathcal{E}$ is in $\mathcal{O}_\sigma$, then $\mathcal{E} \in \mathrm{CPTP}$ and $\mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$ due to property 4 of Proposition 15, hence $\mathcal{O}_\sigma \subseteq \mathcal{O}'_\sigma$.

   Conversely, suppose $\mathcal{E} \in \mathrm{CPTP}$ with $\mathrm{W}_\mathcal{E} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma)$. Then, $\mathrm{W}_\mathcal{E}(\boldsymbol{y}|\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{y}$, hence $\mathcal{E} \in \mathcal{O}$. Furthermore, $\mathrm{W}_\mathcal{E}\mathrm{W}_\sigma = \mathrm{W}_\sigma$ implies $\mathcal{E}(\sigma) = \sigma$ using Eq. (15) defined for any CPTP operation $\mathcal{E}$. Hence, $\mathcal{O}'_\sigma \subseteq \mathcal{O}_\sigma$.

$\square$

Any free state $\sigma \in \mathcal{B}(\mathcal{H}_d)$ corresponds to a d^2-dimensional probability distribution W_σ and any free operation $\mathcal{E} : \mathcal{B}(\mathcal{H}_d) \mapsto \mathcal{B}(\mathcal{H}_d)$ corresponds to a $d^2 \times d^2$ stochastic matrix (or conditional probability distribution) $\mathrm{W}_\mathcal{E}$. Note that these mappings are one-to-one due to the orthogonality of the phase-point operators as an operator basis.

Remark 1. Note that free states $\mathcal{F}$ are mapped onto a *strict subset* of the set of probability distributions. As a counterexample, the sharp d^2-dimensional probability distribution $(1, 0, \ldots, 0)$ does not correspond to any qudit Wigner distribution because of the boundedness condition 3 in Proposition 14.

Remark 2. Similarly, not all stochastic matrices correspond to completely positive operations.

As an example, consider the permutation matrix

$$\Pi_X = \begin{pmatrix} 0&1&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \end{pmatrix} \otimes \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \end{pmatrix} \in \mathrm{S}_{d^2}(\mathrm{W}_\sigma),\ d = 5. \tag{C2}$$

It preserves the uniform distribution $\mathrm{W}_{\frac{1}{5}\mathbb{1}}$, but it does not correspond to any CP operation, hence any $\mathcal{E} \in \mathcal{O}$ due to Theorem 17.