Study of the dynamics of the single and double pendulum and their numerical stability

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Abstract

Four distinct schemes were tested in terms of stability and accuracy on solving the single pendulum. Energy conservation provided a test for stability and Runge Kutta 4 was found to be the most "useful" method to study double pendulum dynamics. The dynamics exhibited coupling of two oscillatory modes as a function of the damping coefficient and mass ratio, while the numerical stability of the double pendulum was found to have clear dependences on parameters of the system.

1 Introduction

Physical processes cannot always be described analytically [1]. There are many different numerical methods that can be used to solve a system, each with its own drawbacks. A method needs to satisfy three criteria to be considered "useful". Specifically, it needs to be consistent and accurate to ensure that the solution generated by it indeed corresponds to the system. It also needs to be stable to ensure that the time evolution of the system is not unreasonably sensitive to initial conditions. In other words, the system should not change behaviour unpredictably during its evolution in time.

This report focuses on solving the single and double pendulum numerically under the small angle approximation and on analysing the solutions. Four different finite difference schemes are examined in the simpler case of the single pendulum. The schemes are the explicit Euler (EE), leapfrog (LP), Runge-Kutta four (RK4) and implicit Euler (IE) [2]. Additionally, EE is modified to include arbitrary angles (called CE in this case hereinafter). The methods are examined in terms of the three criteria mentioned above and RK4 is used to investigate the behaviour of a double pendulum. Finally, the stability of the double pendulum solution is analysed with respect to the parameters of the system.

2 Method

The first part of the study involved solving the single pendulum with the methods stated in the introduction. The system was nondimensionalised and solved for arbitrary initial conditions and damping. The scaled time of the system was set in units $\sqrt{\frac{l}{g}}$, thus the scaled damping coefficient was rescaled $D \to G = \frac{D}{m\sqrt{gl}}$, where m, g, l are the mass of the hanging bob, the gravitational acceleration and the length of the string respectively. The CE scheme did not assume $\sin\theta \approx \theta$ for the amplitude [1].

In order to analyse the stability of the system, the evolution of its energy was studied. The system was characterised as stable if its fractional energy remained bounded

$$\frac{E_{final}}{E_{initial}} < M \in \mathbb{R}^+. \tag{1}$$

Any damped or undamped isolated mechanical system cannot have growing energy, therefore this formula is a necessary condition for stability. Choosing the potential energy to be 0 at the height of the mass when it rests, the energy cannot be negative and therefore the energy is expected to be found between 0 and 1, i.e. behave predictably. However, in order to allow for fluctuations of the energy due to its discretised nature in time, M was chosen to be 1.1 for most schemes. It was set to 100 and 10^{15} respectively for LP and CE because these two schemes presented rapid energy fluctuations. This can be justified because they also presented rapid changes in stability captured by these large bounds.

The stability of a scheme does not depend on the initial conditions and hence may only depend on the time step it is given and the damping coefficient G. They are the only independent variables left, given that the experiment lasts long enough to allow for unpredicted behaviour. Any scheme that remains unstable for time steps $h \gtrsim 5 \times 10^{-4}$ was declared "unconditionally unstable", while any scheme that remains stable for time steps $h \lesssim 10$ (1000 for CE) was declared "unconditionally stable". These values were chosen as "extreme" values beyond which the system should not present any change in its behaviour. The cases of conditional stability were analysed as a function of damping. The critical time step was found by modifying relation (1) to an equality and solving it using a bisection method.

The RK4 scheme was chosen and used to investigate the double pendulum system. Following the same steps as above for RK4, the system was solved for constant and equal string lengths holding the bobs, but arbitrary initial conditions and damping. The conditional stability was examined as a function of the damping coefficient G and the ratio of the two masses $R = \frac{m_{lower}}{m_{upper}}$.

3 Results and Discussion

Single Pendulum

The dynamics of the single pendulum can be described by all schemes in a satisfactory way for simulations of low duration and a sufficiently short time step to ensure accuracy. However, figure (1) illustrates that, given enough time, the various schemes seem to differentiate from each other presenting their own characteristic features.

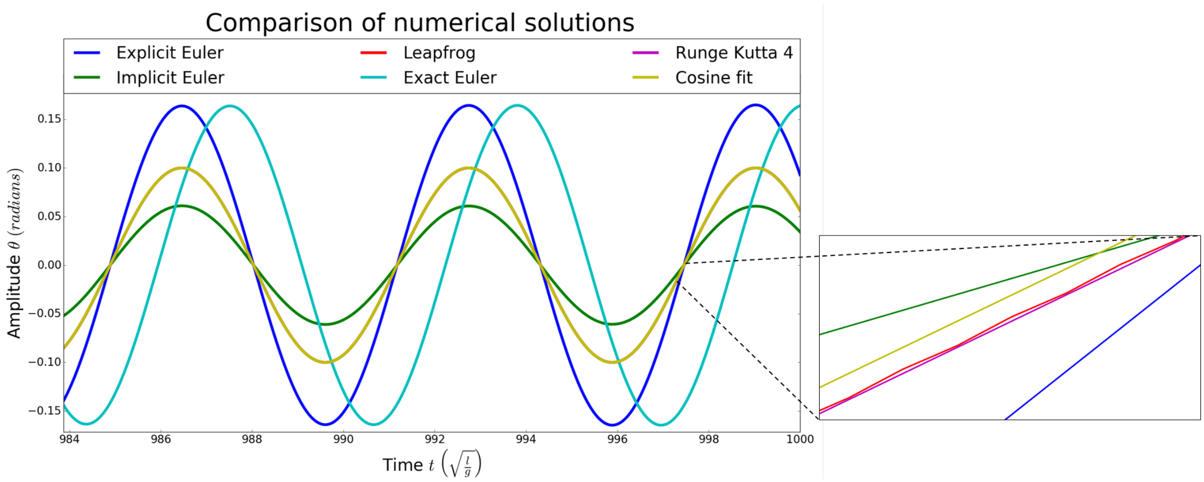

Figure 1: Single pendulum oscillation for $G = 0$, $h = 0.001$ and initial conditions $(\theta, \dot{\theta}) = (0.1, 0)$.

The cosinusoidal fit in figure (1) is the analytical solution of the system in the case of small amplitudes and no damping. It is seen that the EE and CE quickly diverge from above, while IE quickly diverges from below. Therefore, the Euler schemes are less accurate than LP and RK4, which remain close to the solution as seen by zooming in the graph. CE has gained a phase compared to the rest of the schemes and the "analytical" solution, indicating that, given enough time, the small angle approximation always breaks down. The larger the amplitude, the quicker the phase is gained.

In the case of damping, the results change and a summary of them in terms of stability is given in table (1) for a weak damping coefficient $G = 0.2$.

Table 1: Summary of stability of schemes for $G = 0.2$

Scheme	$G = 0$	$G = 0.2$
EE	unstable	$h_c = 0.20$
CE	stable	$h_c = 10.00$
IE	stable	stable
LP	$h_c = 1.00$	unstable
RK	$h_c = 2.83$	$h_c = 2.95$

Although both EE and CE seem to diverge, the latter is stable in the sense that the energy of the system does not grow indefinitely, even for large amplitudes. However the fractional energy is capped at 10^{15} as discussed in the method section, suggesting that the energy gets much higher than the analytical energy of the system. IE never returns energy that blows up, but the accuracy is always low, and worsens as the time step increases, allowing the scheme to decay more rapidly. The stability of LP breaks down immediately when damping is introduced, while RK4 always remains stable for reasonable time steps. The stability of EE, CE and RK4 can be generalised with regard to damping as illustrated in figure (2).

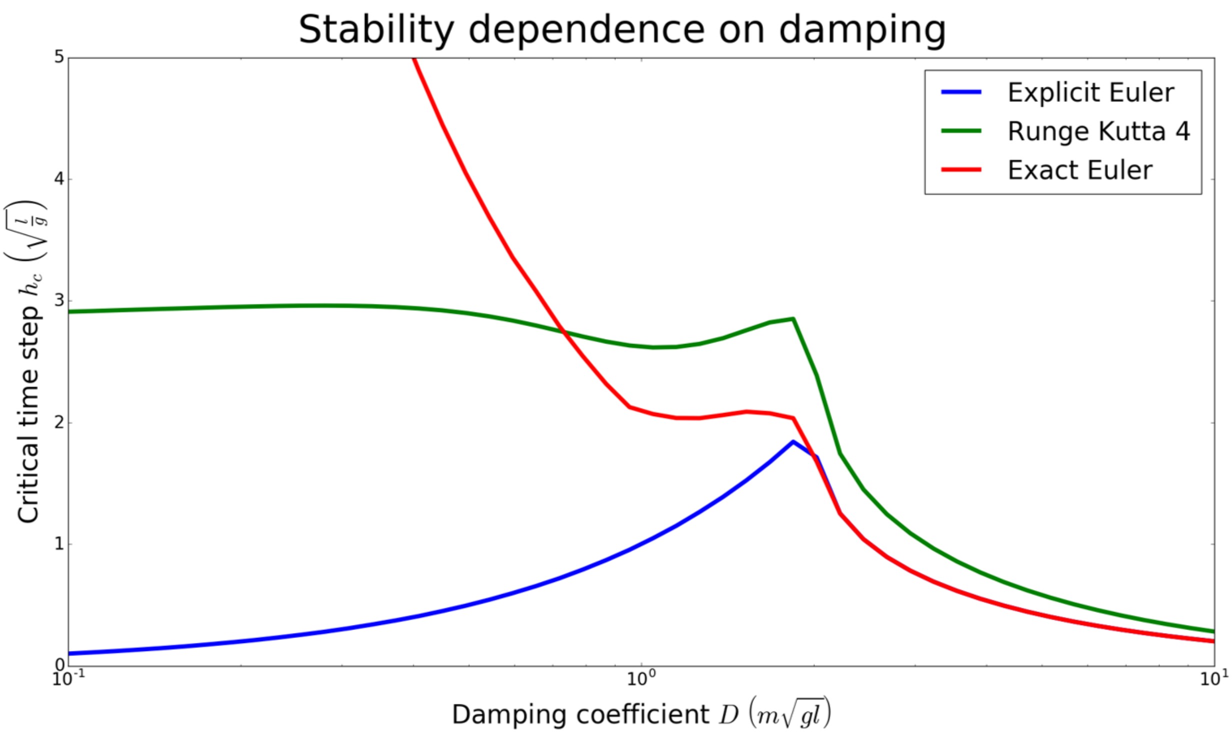

Figure 2: Critical time step h_c as a function of damping coefficient $D = m\sqrt{gl}G$. The initial conditions are $(\theta, \dot{\theta}) = (0.1, 0)$.

Considering that the horizontal axis is logarithmic, the critical time step for EE increases linearly with the damping coefficient up to $G = 2$ which corresponds to critical damping for the system. In fact, $h_c = G$ for $G \leq 2$, while h_c tends to 0 as G decreases, suggesting than EE is unstable for all time steps if there is no damping, as already stated in table (1). IE starts off from infinity, again in agreement with table (1), and becomes more unstable as G increases. After the critical point, it decays identically as EE. RK4 starts from $h_c = 2.83$ according to table (1) and remains relatively steady until it decays after the critical point, tending asymptotically to the two Euler schemes. Overall, heavy damping induces large instability to the numerical schemes.

After examining all five schemes, RK4 can be seen in figure (2) to always be stable for reasonable time steps and is relatively very accurate according to figure (1). For both reasons, it is used to study the double pendulum.

Double Pendulum

The motion of the two bobs in a double pendulum is certainly less intuitive at first. Figure (3) gives insight to the motion as a function of the mass ratio R.

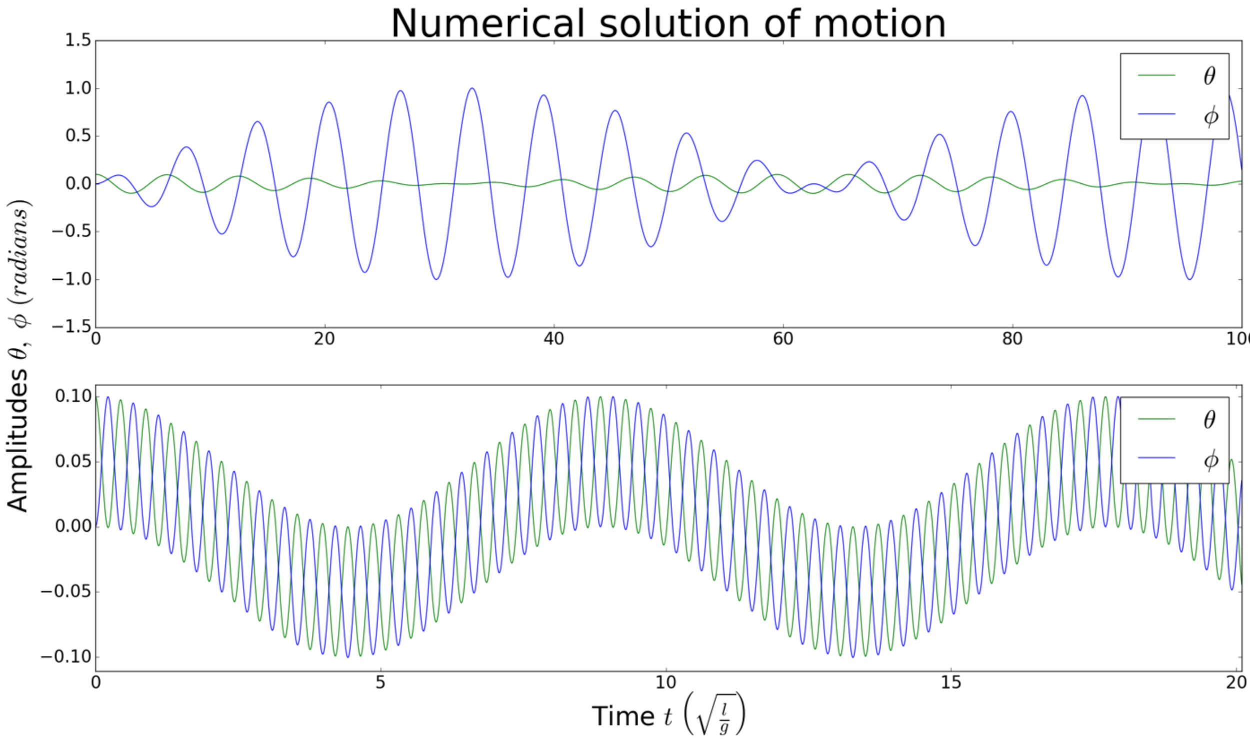

Figure 3: Double pendulum oscillations for $G = 0$, $h = 0.001$ and initial conditions $(\theta, \phi, \dot{\theta}, \dot{\phi}) = (0.1, 0, 0, 0)$. The top graph corresponds to $R = 0.01$ and the bottom graph to $R = 100$.

Looking first at the bottom figure that corresponds to a very heavy lower mass, we see that the motion illustrates long and short oscillations. The two bobs always move out of phase at short time scales due to the light upper mass, but also perform a more observable coupled motion over longer times that resembles the single pendulum motion of a bob hanging from a string with double the length. At the opposite limit, represented by the upper graph, there is more significant coupling between the two bobs. Kinetic energy is transferred periodically between them, modulating the motion of both. The lower mass, being the lighter as well, generally follows the motion of the upper mass until it gains enough energy to oscillate itself (e.g. symmetrically around time $t \approx 62\sqrt{\frac{l}{g}}$). The coupling is expected to be higher around $R \approx 1$ and the bobs would oscillate with only some approximate regularity, while it is more likely that the bobs do not have time to couple as damping increases.

The stability of the numerical solutions this time depends on both the mass ratio and the damping coefficient. The RK4 scheme remains conditionally stable in all cases as figure (4) suggests. This plot presents several familiar features from the single pendulum. At the limit of the ratio and the damping coefficient both tending to 0, the system resembles the undamped single pendulum and h_c should tend to 2.83 from table (1). This is supported by the trend of the graph which peaks at 2.77 near this limit. The opposite limit in R is very unstable. One explanation arising from the study of the dynamics of the system is that the lighter mass gains enormous amounts of energy at some points in the evolution of the system and performs unstable oscillations.

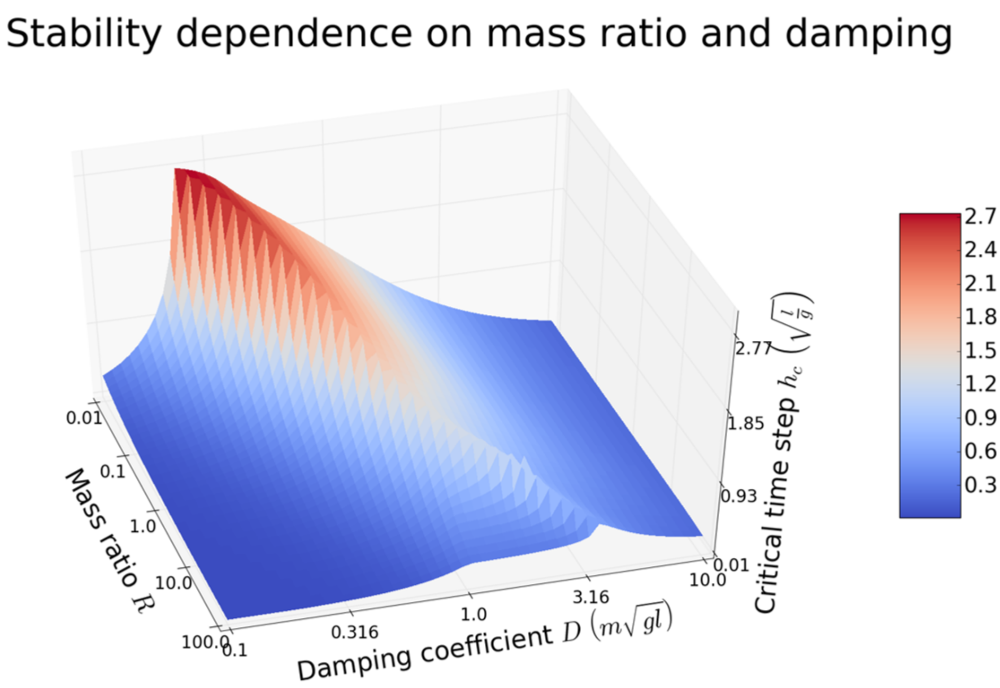

Figure 4: Critical time step h_c as a function of mass ratio R and damping coefficient D.

Figure (4) also suggests that the double pendulum experiences rapid drop at the critical time step for stability as damping increases to the heavy damping regime regardless of what the mass ratio is. Overall, at any given combination of mass ratio and damping, a time step below the surface can be chosen to ensure stability. Noticeably, there is a continuous curve connecting all maxima of the surface that indicates a regime of maximum stability. Near this curve, a wider range of time steps results in stable solutions.

4 Conclusion

The aim of this project was to analyse the dynamics of the single and double pendulum as well as the stability of their numerical solutions. The initial study of the single pendulum raised important comparisons between all five schemes used with respect to accuracy and stability. Using the fact that the single pendulum is solved analytically, the LP and RK4 schemes were shown to be more accurate. Exploiting the conservation of mechanical energy, LP was shown to be unstable in the damped case, thus making the RK4 scheme the most “useful”. Using this scheme, the dynamics of the double pendulum and their numerical stability were studied, finding similarities and differences with the single pendulum.

The most important similarities between the two systems were observed at the limiting values of the double pendulum parameters as depicted in figures (3) and (4). The similiraties could possibly be extended by relating, for example, the coefficient of critical damping as a function of mass ratio to the curve that connects all maxima in figure (4). This can be motivated by the fact that discontinuities in the time step were observed only at critical damping in the case of the single pendulum. However, this is a rough estimate and a higher resolution for the graph may be needed before drawing such conclusions. Another possible issue raised due to computational power is that arbitrary values were chosen to stop the search for stability as mentioned in the method section and thus there is a statistical chance that some features of stability are hidden away from the results obtained.

References

[1] Ochs, K. (2011) A comprehensive analytical solution of the nonlinear pendulum. *European Journal of Physics*. 32 (2).

[2] Press, W., et. al. (2007) *Numerical Recipes*. Cambridge University Press.

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