CS 412 Homework 2

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1. MLE and MAP

1.a Maximum Likelihood Estimate (MLE) of parameter α:

Given the dataset $X = \{x_1, x_2, ..., x_n\}$ and the Flutter Distribution $p(x|\alpha) = \alpha(1-x)^{\alpha-1}$, the likelihood function is:

$$L(\alpha|X) = \prod_{i=1}^{n} p(x_i|\alpha) = \prod_{i=1}^{n} \alpha(1-x_i)^{\alpha-1}$$

To maximize $L(\alpha|X)$, we can maximize its logarithm:

$$\log L(\alpha|X) = \sum_{i=1}^{n} \log(\alpha(1-x_i)^{\alpha-1}) = \sum_{i=1}^{n} (\log(\alpha) + (\alpha-1)\log(1-x_i))$$

$$= n\log(\alpha) + (\alpha-1)\sum_{i=1}^{n} \log(1-x_i)$$

To find the MLE of α, we differentiate $\log L(\alpha|X)$ with respect to α and set it to zero:

$$\frac{d}{d\alpha}(\log L(\alpha|X)) = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1-x_i) = 0$$

$$\Rightarrow \frac{n}{\alpha} = -\sum_{i=1}^{n} \log(1-x_i)$$

$$\Rightarrow \alpha = \frac{n}{-\sum_{i=1}^{n} \log(1-x_i)}$$

So, the Maximum Likelihood Estimate (MLE) of parameter α is:

$$\hat{\alpha}_{MLE} = \frac{n}{-\sum_{i=1}^{n} \log(1-x_i)}$$

1.b Maximum a Posteriori (MAP) estimation of parameter α:

Given the prior $p(\alpha) = \lambda\alpha^{\lambda-1}e^{-\lambda\alpha}$, the posterior distribution is:

$$p(\alpha|X) \propto p(X|\alpha)p(\alpha) \propto \prod_{i=1}^{n} \alpha(1-x_i)^{\alpha-1} \cdot \lambda\alpha^{\lambda-1}e^{-\lambda\alpha}$$

$$= \lambda\alpha^{\lambda+n-1}e^{-\lambda\alpha} \prod_{i=1}^{n}(1-x_i)^{\alpha-1}$$

To find the MAP estimation of α, we maximize this posterior distribution. Taking logarithm:

$$\log p(\alpha|X) = \log(\lambda) + (\lambda+n-1)\log(\alpha) - \lambda\alpha + \sum_{i=1}^{n}(\alpha-1)\log(1-x_i)$$

To maximize $\log p(\alpha|X)$, we differentiate it with respect to α and set it to zero:

$$\frac{d}{d\alpha}(\log p(\alpha|X)) = \frac{\lambda+n-1}{\alpha} - \lambda + \sum_{i=1}^{n}\log(1-x_i) = 0$$

$$\Rightarrow \frac{\lambda+n-1}{\alpha} = \lambda - \sum_{i=1}^{n}\log(1-x_i)$$

$$\Rightarrow \alpha = \frac{\lambda+n-1}{\lambda - \sum_{i=1}^{n}\log(1-x_i)}$$

So, the Maximum a Posteriori (MAP) estimation of parameter α is:

$$\hat{\alpha}_{MAP} = \frac{\lambda+n-1}{\lambda - \sum_{i=1}^{n}\log(1-x_i)}$$