Contents

1 Week 1: 8/26- 8/30

1. The following identities are true:

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ A \cap (B \cap C) &= (A \cap B) \cap C, \\ A \backslash (B \cap C) &= (A \backslash B) \cup (A \backslash C). \end{aligned}$$

Hint. Let us show

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{1.1}$$

Indeed, $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and ($x \in B$ or $x \in C$)$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) $\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$.

Answer. The equality $A \cap (B \cap C) = (A \cap B) \cap C$ is obvious. In order to prove

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

we show that the complements of the sets on both sides are equal. By DeMorgan's law and (1.1),

$$\begin{aligned} (A \cup (B \cap C))^c &= A^c \cap \left(B^c \cup C^c\right) \\ &= \left(A^c \cap B^c\right) \cup \left(A^c \cap C^c\right), \end{aligned}$$

and

$$\begin{aligned} ((A \cup B) \cap (A \cup C))^c &= (A \cup B)^c \cup (A \cup C)^c \\ &= \left(A^c \cap B^c\right) \cup \left(A^c \cap C^c\right). \end{aligned}$$

Now, by de Morgan's law and (1.1) again,

$$\begin{aligned} A \backslash (B \cap C) &= A \cap (B \cap C)^c = A \cap \left(B^c \cup C^c\right) \\ &= (A \cap B^c) \cup (A \cap C^c) = (A \backslash B) \cup (A \backslash C). \end{aligned}$$

2. Let $\{A_i, i \in I\}$ be a collection of sets. Prove De Morgan's Laws:

$$(\cup_i A_i)^c = \cap_i A_i^c,\ (\cap_i A_i)^c = \cup_i A_i^c.$$

3. a) Let $\mathcal{F}$ be a σ-field, $A, B \in \mathcal{F}$. Show that $A \backslash B$ and $A \triangle B \in \mathcal{F}$.

b) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of Ω. Show that $\mathcal{F}_1 \cap \mathcal{F}_2$, the collection of subsets of Ω that belong to both $\mathcal{F}_1$ and $\mathcal{F}_2$, is σ-field.

4. Let $\mathcal{A}$ be a collection of subsets of Ω, and let $\mathcal{F}_i, i \in I$, be all σ-fields that contain $\mathcal{A}$. Show that $\mathcal{F} = \cap_i \mathcal{F}_i$ is a σ-field.

Comment. Note that $\mathcal{P}(\Omega)$, the σ-field of all subsets of Ω, is among $\mathcal{F}_i$. The collection $\mathcal{F} = \cap_i \mathcal{F}_i$ is called the smallest σ-field containing $\mathcal{A}$.

5. Describe the sample spaces for the following experiments:

(a) Two balls were drawn without replacement from an urn which originally contained two red and two black balls.

Answer. Mark balls in the urn: $B_1, B_2, R_1.R_2$. If drawn one by one, and going with ordered pairs, then Ω consists of $4 \cdot 3 = 12$ ordered pairs:

$$\{R_1R_2, R_2R_1, R_1B_1, R_1B_2, R_2B_1, R_2B_2, B_1B_2, B_2B_1, B_1R_1, B_1R_2, B_2R_1, B_2R_2\}.$$

If drawn loosely as group of two, then Ω consists of $\binom{4}{2} = 6$ unordered pairs:

$$\{\{R_1, R_2\}, \{R_1, B_1\}, \{R_2, B_1\}, \{R_1, B_2\}, \{R_2, B_2\}, \{B_1, B_2\}\}.$$

(b) A coin is tossed three times.

Answer. $\Omega = \{HHH, HHT, HTT, HTH, TTT, TTH, THT, THH\}$ consists of $2^3 = 8$ different outcomes.

6. A fair die is thrown twice. What is the probability that:

(a) a six turns up exactly once?

Answer. The event A consists of all $(6, i)$ and $(j, 6)$ with $1 \leq i, j \leq 5$. Hence $\mathbf{P}(A) = 10/36 = 5/18$.

(b) both numbers are odd?

Answer. The event A consists of all (i, j) with $i, j \in \{1, 3, 5\}$; $\#A = 3^2 = 9$ and $\mathbf{P}(A) = 9/36 = 1/4$.

(c) the sum of the scores is 4?

Answer. $A = \{(1, 3), (2, 2), (3, 1)\}$, and $\mathbf{P}(A) = 3/36 = 1/12$.

(d) the sum of the scores is divisible by 3?

Answer. $A = \{(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6.6)\}$, and

$$\mathbf{P}(A) = 12/36 = 1/3.$$

7. A fair coin is thrown repeatedly. What is the probability that on the nth throw:

(a) a head appears for the first time?

Answer. The sample space Ω consists of all "words" of length n made by H and T: $\#\Omega = 2^n$. The event $A =$" H appears 1st on the nth throw" is a single word with $n - 1$ T before H at the end:

$$\mathbf{P}(A) = 1/2^n = 2^{-n}.$$

(b) the number of heads and tails to date are equal?

Answer. If n is odd the probability is zero. If n is even, then there are $\binom{n}{n/2}$ different "words" with $n/2$ H:

$$\mathbf{P}(A) = \binom{n}{n/2} 2^{-n}.$$

(c) exactly two heads have appeared altogether to date?

Answer. There are $\binom{n}{2}$ different words with exactly two H :

$$\mathbf{P}(A) = \binom{n}{2} 2^{-n}.$$

(d) at least two heads have appeared to date?

Answer. Let B="at most one H" consists of one word with all T in it and n words with a single H in them: $\mathbf{P}(B) = (n+1)\,2^{-n}$ and $\mathbf{P}(A) = 1 - (n+1)\,2^{-n}$.

8. Show that the probability that exactly one of the events A and B occurs is

$$\mathbf{P}(A) + \mathbf{P}(B) - 2\mathbf{P}(A \cap B).$$

(note: it is the event $A \triangle B = (A \backslash B) \cup (B \backslash A)$)

Answer. Since $A \triangle B$ is the union of two disjoint sets,

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \backslash (A \cap B)) \cup (B \backslash (A \cap B)),$$

and $A \cap B \subseteq A$, $A \cap B \subseteq B$, we have

$$\begin{aligned} \mathbf{P}(A \triangle B) &= \mathbf{P}(A \backslash (A \cap B)) + \mathbf{P}(B \backslash (A \cap B)) \\ &= \mathbf{P}(A) - \mathbf{P}(A \cap B) + \mathbf{P}(B) - \mathbf{P}(A \cap B). \end{aligned}$$

9. a) A fair coin is tossed n times. What is the probability of H in the last toss?

Answer. The sample space Ω consists of all "words" of length n made by H and T: $\#\Omega = 2^n$. The event A="H in the last toss" consists of all words of length n formed by H and T with H at the end; $\#A = 2^{n-1}$ and

$$\mathbf{P}(A) = \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

Similarly, probability of H in any toss is 1/2.

b) An urn contains 9 whites and one red ball. All ten balls are randomly drawn out without replacement one by one. What is the probability that:

(i) the red ball is taken out first?

Answer. The event A="red ball is taken out first" consists of all orderings of 10 balls with red ball first: $\#A = (n-1)!$. Hence

$$\mathbf{P}(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(ii) the red ball is taken out last?

Answer. The event A="red ball is taken out last" consists of all orderings of 10 balls with red ball last: $\#A = (n-1)!$. Hence

$$\mathbf{P}(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(ii) the red ball is taken in the kth draw $(1 \le k \le 10)$?

Answer. The event A="red ball is taken out last" consists of all orderings of 10 balls with red ball in the kth position: $\#A = (n-1)!$. Hence

$$\mathbf{P}(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

10. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with stars on. If the cups are placed randomly onto the saucers (one each), find the probability that no cup is upon a saucer of the same pattern.

Hint. Put the saucers in a row. For instance, $RRWWSS$. The sample space Ω is the space of all cup $(R_1, R_2, W_1, W_2, S_1, S_2)$ arrangements along the row of saucers $RRWWSS$.

Answer. Put the saucers in a row. For instance, $RRWWSS$. The sample space Ω is the space of all cup $(R_1, R_2, W_1, W_2, S_1, S_2)$ arrangements along the row of saucers $RRWWSS$: $\#\Omega = 6!$ Let A="no cup is upon a saucer of the same pattern". Then $A = A_1 \cup A_2$, where A_1 are all arrangements where one pattern cups go to the same different pattern of saucers, and A_2 are all arrangements where one pattern cups go to the different patterns of saucers.

Note A_1 consists of $W_1W_2S_1S_2R_1R_2$ and $S_1S_2R_1R_2W_1W_2$ and their arrangements with switching R_1 and R_2, W_1 and W_2, S_1 and S_2 : $\#A_1 = 2 \cdot 2! \cdot 2!2! = 16$.

Counting A_2. For R_1 we have 4 choices (4 positions in $WWSS$ part), for R_2 we have remaining 2 choices: total number of R arrangements is $4 \cdot 2 = 8$. After both R are placed we have 4 ways to arrange both W. After R and W are placed there are only 2 ways to arrange remaining S. Hence $\#A_2 = 8 \cdot 4 \cdot 2 = 64$, and $\#A = \#A_1 + \#A_2 = 16 + 64 = 80$. Thus $\mathbf{P}(A) = 80/6! = 1/9$.

11. You choose k of the first n positive integers, and a lottery chooses a random subset L of the same size (k numbers in it). What is the probability that:

(a) L includes no consecutive integers? *Hint about counting nonconsecutive integers*. Let $n = 7, k = 3$. (i) Put $7 - 3 = 4$ white balls in a row with spaces between them, in the beginning and at the end (there are 5 spaces). (ii) Choose 3 spaces and put 3 black balls there. Number all balls from the left to the right. For instance, if the 2nd, fourth and fifth space were chosen, then L consists of the numbers $2, 5, 7$. Realize that number of ways to have non consecutive integers in L equals to the number of ways to choose 3 spaces among 5 available.

Answer. As a sample space Ω take all possible groups of k numbers from $\{1, 2, \ldots, n\}$: $\#\Omega = \binom{n}{k}$. Let A="L includes no consecutive integers". In order to count outcomes in A, imagine we put $n-k$ white balls in a row with spaces between them, also one space before the first ball and one space after the last ball: there are $n - k + 1$ spaces. Choose k spaces and place black balls there: there are $\binom{n-k+1}{k}$ different ways to do it. Write the numbers $1, 2, \ldots, n$ from the left to the right on the balls in the row. The numbers written on the black balls are numbers in L (no consecutive integers). Thus $\#A = \binom{n-k+1}{k}$, and

$$\mathbf{P}(A) = \frac{\binom{n-k+1}{k}}{\binom{n}{k}}.$$

Comment. We were choosing a group of k numbers. The counting of the outcomes in A goes with a row of the balls because we are choosing from the set of numbers $\{1, 2, 3, \ldots, n\}$ that has an order (one number is bigger than the other and A cannot contain consecutive numbers).

(b) L includes exactly one pair of consecutive integers? Hint. Like (a) but think about one pair of consecutive integers as one entity.

Answer. The sample space Ω is the same as in (a): $\#\Omega = \binom{n}{k}$. Let $A =$"L includes exactly one pair of consecutive integers". In order to count the outcomes in A, imagine we put $n - k$ white balls in a row with spaces between them, also one space before the first ball and one space after the last ball: there are $n - k + 1$ spaces. Take $k - 2$ black balls and a narrow box with two black balls. First put the box into one of the spaces: there are $n - k + 1$ ways to do it. Then choose $k - 2$ spaces from the remaining $n - k$ spaces and put the black balls there: there are $\binom{n-k}{k-2}$ different ways. Write the numbers from 1 to n on the balls including those in the box. The k numbers on the black balls are the numbers in L (there is exactly one pair of consecutive numbers). Thus

$$\#A = (n-k+1)\binom{n-k}{k-2} = (k-1)\binom{n-k+1}{k-1}$$

and

$$\mathbf{P}(A) = \frac{(k-1)\binom{n-k+1}{k-1}}{\binom{n}{k}}.$$

Question. If you play lottery, (a) or (b) strategy would be better?

(c) the numbers in L are drawn in increasing order? Hint. Any ordering of k numbers is equally likely.

Answer. Since any ordering of k numbers is equally likely, the probability is $1/k!$.

(d) your choice of numbers is the same as L?

Answer. The probability is, obviously,

$$\frac{1}{\binom{n}{k}}.$$

(e) there are exactly l of your numbers matching members of L?

Answer. In your collection of k numbers there are $\binom{k}{l}$ different groups of size l that could match with l numbers in L; you particular group of l numbers must be in L but the remaining $k-l$ numbers in L do not match; you remove from $\{1,2,\ldots,n\}$ the whole your group of k numbers and choose $k-l$ numbers from the remaining $n-k$ numbers in $\binom{n-k}{k-l}$ different ways. The probability is

$$\frac{\binom{k}{l}\binom{n-k}{k-l}}{\binom{n}{k}}.$$

12. 10% of the surface of a sphere is colored blue, the rest is red. Show that it is possible to inscribe a cube in S with all its vertices red. Hint: select a random inscribed cube and think what is the probability that a kth vertex is blue. Then estimate from above the probability that at least one vertex is blue.

Answer. The inscribed cube has 8 vertices. Let B_k be the event that kth vertex is blue, $1 \le k \le 8$. Then

$$\mathbf{P}(\text{at least one vertex is blue}) = \mathbf{P}\left(\cup_{k=1}^{8} B_k\right) \le \sum_{k=1}^{8} \mathbf{P}(B_k) \le 8 \cdot 0.1 = 0.8.$$

Hence $\mathbf{P}$ (all vertices red) ≥ 0.2.

2 Week 2: 9/4- 9/6

1. A rare disease affects one person in 10^3. A test for the disease shows positive with probability 0.99 when applied to an ill person, and with probability 0.01 when applied to a healthy person. What is the probability that you have the disease given that the test shows negative?

Answer. Notation: $A =$ "you have a disease", $B =$ "test shows negative". By Bayes,

$$\begin{aligned}\mathbf{P}(A|B) &= \frac{\mathbf{P}(B|A)\,\mathbf{P}(A)}{\mathbf{P}(B|A)\,\mathbf{P}(A) + \mathbf{P}(B|A^c)\,\mathbf{P}(A^c)} \\ &= \frac{0.01 \cdot 0.001}{0.01 \cdot 0.001 + 0.01 \cdot 0.999} = \frac{1}{1+999} = 0.001.\end{aligned}$$

2. English and American spellings are *rigour* and *rigor*, respectively. At a certain hotel, 40% of guests are from England and the rest are from America. A guest at the hotel writes the word (as either *rigour* or *rigor*), and a randomly selected letter from that word turns out to be a vowel. Compute the probability that the guest is from England.

Answer. Denoting $B =$ "selected letter is a vowel", $A =$ "guest from England", by Bayess,

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(B|A)\,\mathbf{P}(A)}{\mathbf{P}(B)}.$$

By total probability law,

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(B|A)\,\mathbf{P}(A) + \mathbf{P}\left(B|A^c\right)\mathbf{P}\left(A^c\right) \\ &= \frac{3}{6} \cdot 0.4 + \frac{2}{5} \cdot 0.6 = 0.44.\end{aligned}$$

So,

$$\mathbf{P}(A|B) = \frac{\frac{3}{6} \cdot 0.4}{0.44} = 0.454\,55.$$

3. In a certain community, 36% of all the families have a dog and 30% have a cat. Of those families with a dog, 22% also have a cat. Compute the probability that a randomly selected family

(a) has both a dog and a cat;

Answer. Notation: $D =$" family has a dog", $C =$"family has a cat". We have $\mathbf{P}(D) = 0.36, \mathbf{P}(C) = 0.3, \mathbf{P}(C|D) = 0.22.$

By multiplication rule, $0.22 \cdot 0.36 = 0.079\,2$

$$\mathbf{P}(DC) = \mathbf{P}(C|D)\,\mathbf{P}(D) = 0.22 \cdot 0.36 = 0.079\,2.$$

(b) has a dog given that it has a cat.

Answer. By definition,

$$\mathbf{P}(D|C) = \frac{\mathbf{P}(CD)}{\mathbf{P}(C)} = \frac{0.079\,2}{0.3} = 0.264\,.$$

4. (Galton's paradox) You flip three fair coins. At least two are alike, and it is an even chance that the third is a head or a tail. Therefore $\mathbf{P}$(all alike)=$\frac{1}{2}$. Do you agree? (all alike means all heads or all tails).

Answer. No. The sample space Ω of all words of length 3 made of H and T: $\#\Omega = 2^3 = 8$ and all outcomes are equally likely. Since "all alike"=$\{HHH, TTT\}$,

$$\mathbf{P}(\text{all alike}) = \frac{2}{2^3} = \frac{1}{4}.$$

5. The event A is said to be repelled by the event B if $\mathbf{P}(A|B) < \mathbf{P}(A)$, and to be attracted by B if $\mathbf{P}(A|B) > \mathbf{P}(A)$. Show that if B attracts A, then A attracts B, and B^c repels A. If A attracts B, and B attracts C, does A attract C?

Answer. If B attracts A, then

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} > \mathbf{P}(A). \tag{2.2}$$

Hence

$$\mathbf{P}(B|A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} > \mathbf{P}(B),$$

i.e., A attracts B. On the other hand, by (2.2),

$$\mathbf{P}\left(A|B^c\right) = \frac{\mathbf{P}(A \cap B^c)}{\mathbf{P}(B^c)} = \frac{\mathbf{P}(A) - \mathbf{P}(A \cap B)}{\mathbf{P}(B^c)} < \mathbf{P}(A),$$

i.e., B^c repells A.

If A attracts B, and B attracts C, does A attract C? Not necessarily. For example, a fair die is rolled, A=" score is 3 or 4 or 5", B=" score is ≥ 4", C=" score is even". Then

$$\begin{aligned}
\mathbf{P}(B|A) &= \frac{2}{3} > \mathbf{P}(B) = \frac{1}{2}, \\
\mathbf{P}(C|B) &= \frac{2}{3} > \mathbf{P}(C) = \frac{1}{2}, \\
\mathbf{P}(C|A) &= \frac{1}{3} < \frac{1}{2} = \mathbf{P}(C).
\end{aligned}$$

6. Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

Answer. $\mathbf{P}(\text{exactly one ace and two kings}) = \frac{\binom{4}{2}\binom{4}{1}\binom{44}{10}}{\binom{52}{13}} = \frac{301\,587}{3215\,975} = 9.377\,8 \times 10^{-2}$.

$\mathbf{P}(\text{exactly one ace| exactly two kings}) = \frac{\binom{4}{2}\binom{4}{1}\binom{44}{10}}{\binom{4}{2}\binom{48}{11}} = \frac{2849}{6486} = 0.439\,25$.

7. A woman has n keys, of which two will open her door. (a) If she tries the keys at random, discarding those that do not work,what is the probability that she will open the door on her kth try? (b) What if she does not discard previously tried keys? What is the probability of no right key in k tries ($k \geq 1$)? If $n = 9$, how many tries are needed to be 90% sure that the door is opened?

Answer. (a) There are $n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$ equally likely orderings of k keys out of n. There are

$$\begin{aligned}
&(n-2)(n-3)\ldots(n-2-(k-1)+1) \\
= &\frac{(n-2)!}{(n-2-(k-1))!} = \frac{(n-2)!}{(n-k-1)!}
\end{aligned}$$

different orderings of $n-2$ wrong keys in the first $k-1$ tries, and there are $2!$ orderings of two right keys. Thus

$$\begin{aligned}
\mathbf{P}\,(k\text{th try opens}) &= \frac{2!\frac{(n-2)!}{(n-k-1)!}}{\frac{n!}{(n-k)!}} = \frac{2}{n}\cdot\frac{(n-k)}{(n-1)} \\
&= \frac{2}{n}\cdot\left(1-\frac{k-1}{n-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}\,(\text{door closed after } k \text{ tries}) &= \frac{\frac{(n-2)!}{(n-2-k)!}}{\frac{n!}{(n-k)!}} = \frac{(n-k)\,(n-k-1)}{n\,(n-1)} \\
&= \left(1-\frac{k}{n}\right)\left(1-\frac{k}{n-1}\right).
\end{aligned}$$

Comment. If $n=9$, she can be 90% sure that the door will be opened in 6 tries.

(b) The sample space Ω is the set of all distinct results of k experiments with n outcomes: $\#\Omega = n^k$. Now,

$$\begin{aligned}
&\#\,(\text{the first } k-1 \text{ tries fail, } kth \text{ succeeds}) \\
=\ & (n-2)^{k-1}\cdot 2.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbf{P}\,(k\text{th try opens}) &= \frac{(n-2)^{k-1}\cdot 2}{n^k} = \frac{(n-2)^{k-1}}{n^{k-1}}\frac{2}{n} \\
&= \left(1-\frac{2}{n}\right)^{k-1}\frac{2}{n},
\end{aligned}$$

and

$$\mathbf{P}\,(\text{door closed after } k \text{ tries}) = \left(1-\frac{2}{n}\right)^k,$$

$\mathbf{P}(\text{door opened in k tries}) = 1-\left(1-\frac{2}{n}\right)^k$. With $n=9$,

$$1-\left(\frac{7}{9}\right)^k = 0.9, \left(\frac{7}{9}\right)^k = 0.1, k = \frac{\ln 0.1}{\ln\frac{7}{9}} = 10.$$

She can be 90% sure that the door will be opened in 10 tries.

Comment. All computations can be done by assuming independence (without counting) in the part (b).

8. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information because he already knows that at least one of the two will go free. The jailer refuses to answer the question, pointing out that if A knew which of his fellow prisoners were to be set free, then his own probability of being executed would rise from 1/3 to 1/2 because he would then be one of two prisoners. What do you think of the jailer's reasoning?

Answer. It is the "same" as the class problem about an award behind 3 closed doors. Consider A="death sentence for A", B="death sentence for B", C="death sentence for C",

F$_B$=" jailer says B goes free", F$_C$ =" jailer says C goes free".

We know $P(A) = P(B) = P(C) = \frac{1}{3}$, $P(F_B|A) = P(F_C|A) = \frac{1}{2}$, $P(F_B|B) = 0$, $P(F_C|B) = 1$, $P(F_B|B) = 0$, $P(F_C|C) = 0$, $P(F_B|C) = 1$.

Jailer is wrong: by Bayes,

$$\begin{aligned} P(A|F_B) &= \frac{\frac{1}{2}\frac{1}{3}}{\frac{1}{2}\frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{3} = \mathbf{P}(A), \\ P(A|F_C) &= \frac{1}{3} = \mathbf{P}(A). \end{aligned}$$

As we found in class with the award, A and F_B, also A and F_C, are independent.

3 Week 3: 9/9-9/13

1. Let $A_k, k \geq 1$, be a sequence of events. Show that

$$\mathbf{P}\left(\cup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbf{P}\left(A_n\right).$$

2. Let $A_k, k \geq 1$, be events such that $\mathbf{P}\left(A_k\right) = 1$ for all k. Show that $\mathbf{P}\left(\cap_{k=1}^{\infty} A_k\right) = 1$.

3. At least one of the events $A_k, 1 \leq k \leq n$, is certain to occur, but certainly no more than two occur. If $\mathbf{P}\left(A_k\right) = p, \mathbf{P}\left(A_k \cap A_j\right) = q, k \neq j$, show that $p \geq 1/n$ and $q \leq 2/n$.

4. Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$B_n = \cup_{m=n}^{\infty} A_m, C_n = \cap_{m=n}^{\infty} A_m,$$

that is $B_n =$ "at least one A_m after n happens", $C_n =$ "all $A_m, m \geq n$ happen. Note that $C_n \subseteq A_n \subseteq B_n$, the sequence B_n is decreasing and the sequence C_n is increasing with limits

$$\begin{aligned} \lim_n B_n &= B = \cap_{n=1}^{\infty} B_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m, \\ \lim_n C_n &= C = \cup_{n=1}^{\infty} C_n = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m. \end{aligned}$$

We denote

$$B = \limsup_n A_n, C = \liminf_n A_n.$$

Show that

a) $B = \limsup_n A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\} =$ "infinitely many A_n happen" and

$$\mathbf{P}\left(\limsup_n A_n\right) = \mathbf{P}(B) = \lim_n \mathbf{P}\left(\cup_{m=n}^{\infty} A_m\right) \geq \lim_n \sup_{m \geq n} \mathbf{P}\left(A_m\right) =: \limsup_n \mathbf{P}\left(A_n\right).$$

Here $\limsup_n \mathbf{P}\left(A_n\right) = \lim_n \sup_{m \geq n} \mathbf{P}\left(A_m\right)$ is the upper limit of the sequence of numbers $\mathbf{P}\left(A_n\right)$

Recall for a sequence of numbers a_n,

$$\limsup_n a_n = \lim_{n \to \infty} \sup_{k \geq n} a_k = \inf_{n \geq 1} \sup_{k \geq n} a_k, \quad \liminf_n a_n = \lim_{n \to \infty} \inf_{k \geq n} a_k = \sup_{n \geq 1} \inf_{k \geq n} a_k.$$

Limit of a sequence of numbers a_n exists iff $\limsup_n a_n = \liminf_n a_n$.

b) $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\} =$ "all A_n except a finite number of them happen" and

$$\mathbf{P}\left(\liminf_n A_n\right) = \mathbf{P}(C) = \lim_n \mathbf{P}\left(\cap_{m=n}^{\infty} A_m\right) \leq \lim_n \inf_{m \geq n} \mathbf{P}\left(A_m\right) =: \liminf_n \mathbf{P}\left(A_n\right).$$

c)

$$\mathbf{P}\left(\liminf_n A_n\right) \leq \mathbf{P}\left(\limsup_n A_n\right),$$

and if $\liminf_n A_n = \limsup_n A_n = A$, then

$$\mathbf{P}(A) = \lim_n \mathbf{P}\left(A_n\right).$$

5. A coin with $\mathbf{P}(H) = p, \mathbf{P}(T) = q = 1 - p$, is tossed repeatedly (indefinitely). Let $H_k =$ "H in the kth toss", $T_k =$ "T in the kth toss". Assume all tosses are independent.

(a) Find $\mathbf{P}$ (at least one H after n) $= \mathbf{P}\left(\cup_{m=n}^{\infty} H_m\right) = 1 - \mathbf{P}\left(\cap_{m=n}^{\infty} T_m\right)$.

Hint. Recall, by continuity of probability, $\mathbf{P}\left(\cap_{m=n}^{\infty} T_m\right) = \lim_{l\to\infty} \mathbf{P}\left(\cap_{m=n}^{n+l} T_m\right)$.

(b) Find probability of infinitely many $H's$.

Hint. $\mathbf{P}$ (infinitely many $H's$) $= \mathbf{P}\left(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} H_m\right)$: use the previous part (a) and #4 above.

6. Independent trials, consisting of rolling a pair of fair dice, are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice? Hint. Let $A =$"5 appears before 7". Apply total probability law to the partition $\Omega = B_1 \cup B_2 \cup B_3$, where B_1= "the first trial results in a 5", B_2= "1st trial results in a 7", B_3= "first trial results in neither a 5 nor a 7". Then solve equation for $\mathbf{P}(A)$.

7. Some form of prophylaxis is said to be 90 per cent effective at prevention during one year's treatment. If the degrees of effectiveness in different years are independent, show that the treatment is more likely than not to fail within 7 years.

8. The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one of them is a blue-eyed gene and the other a brown-eyed gene, then the person will have brown eyes. (Because of the latter fact, we say that the brown-eyed gene is dominant over the blue-eyed one.) A newborn child independently receives one eye gene from each of its parents, and the gene it receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose that Smith and both of his parents have brown eyes, but Smith's sister has blue eyes.

(a) What is the probability that Smith possesses a blueeyed gene?

(b) Suppose that Smith's wife has blue eyes. What is the probability that their first child will have blue eyes?

(c) If their first child has brown eyes, what is the probability that their next child will also have brown eyes?

9. A pack contains m cards, labelled 1, 2,..., m. The cards are dealt out in a random order, one by one. Given that the label of the kth card dealt is the largest of the first k cards dealt, what is the probability that it is also the largest in the pack? Hint. All orders in which first k cards are arranged are equally likely.

10. a) Show that if F and G are distribution functions and $0 < \lambda < 1$, then $\lambda F + 1 - \lambda G$ is a distribution function. Is the product FG a distribution function? Hint. A function $F : \mathbf{R} \to [0, 1]$ is a distribution function if the rpoperties a), b), c) of Lemma (6) on p. 28 hold.

b) A random variable X has distribution function F. What is the distribution function of $Y = aX + b$, where a and b are real constants?

11. A fair coin is tossed n times. Show that, under reasonable assumptions, the probability of exactly k heads is $\binom{n}{k}2^{-n}$. What is the corresponding quantity when head appears with probability p on each toss?

12. Let X have a distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2}x & \text{if } 0 \le x \le 2, \\ 1 & \text{if } x > 2, \end{cases}$$

and let $Y = X^2$. Find

(a) $\mathbf{P}\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right)$; (b) $\mathbf{P}(1 \leq X < 2)$; (c) $\mathbf{P}(Y \leq X)$; (d)$\mathbf{P}(X \leq 2Y)$.

(e) the distribution function of $Z = \sqrt{X}$.

13. Let X have a distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{if } x < -1, \\ 1 - p & \text{if } -1 \leq x < 0, \\ 1 - p + \frac{1}{2}xp & \text{if } 0 \leq x \leq 2, \\ 1 & \text{if } x > 2. \end{cases}$$

Sketch the function and find: (a) $\mathbf{P}(X = -1)$, (b) $\mathbf{P}(X = 0)$, (c) $\mathbf{P}(X \geq 1)$.

14. Each toss of a coin results in a head with probability p. The coin is tossed until the first H appears. Let X be the total number of tosses. What is $\mathbf{P}(X > m)$? Find the distribution function (cdf) of X.

15. Buses arrive at ten minutes intervals starting at noon. A man arrives at the bus stop a random number X minutes after noon, where X has distribution function (cdf)

$$F(x) = \mathbf{P}(X \leq x) = \begin{cases} 0 & \text{if } x < 0, \\ x/60 & \text{if } 0 \leq x \leq 60, \\ 1 & \text{if } x > 60. \end{cases}$$

What is the probability that he waits less than 5 minutes for a bus?