UNIVERSITY OF WATERLOO | IQC Institute for Quantum Computing

QSYS Quantum School for Young Students

QUANTUM TELEPORTATION

JOHN DONOHUE

Sending Quantum States

$$|\psi\rangle$$

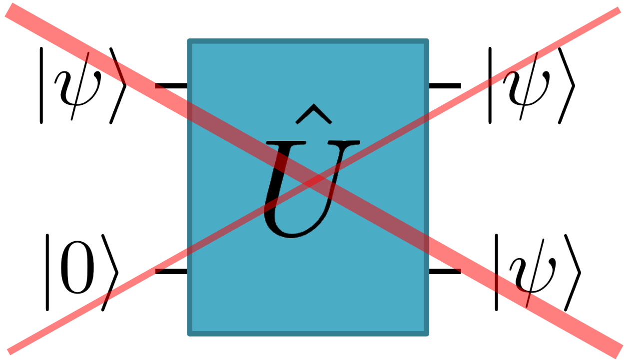
Alice wants to send a quantum state to Bob

But what if:

Alice can’t just re-prepare the state if it gets lost on the way?

Alice’s state is in a superconducting qubit, and Bob only has an ion trap?

The No-Cloning Theorem



There exists no unitary which can create a **perfect** copy of an **unknown** quantum state

W.K. Wootters & W.H. Zurek, Nature 299, 802 (1982)

$$\text{GOAL}: U|\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\psi\rangle$$

$$\text{REALITY}: \hat{U}|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle$$

$$\hat{U}|1\rangle \otimes |0\rangle = |1\rangle \otimes |1\rangle$$

$$\hat{U}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\Big)$$

$$\neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

We can’t just make a copy to send it

Solution: Just teleport it instead!

Quick Entanglement Review

Two-Player Quantum Systems

Separate quantum systems are described by the **tensor product** of the separate, individual Hilbert spaces.

Alice's Qubit

$$|0\rangle_A$$

Alice & Bob's Two-Qubit System

$$|0\rangle_A \otimes |1\rangle_B = |01\rangle$$

$$\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

Bob's Qubit

$$|1\rangle_B$$

Two-Player Quantum Systems

Alice & Bob's
Two-Qubit System

$$|\psi\rangle_A \otimes |\phi\rangle_B = |\Psi\rangle_{AB}$$

New Computational Basis

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Two-Player Quantum Systems

We can use bra-ket notation to simplify composite system problems.

Find the state
in the computational basis
$$|+_i\rangle_A \otimes |-\rangle_B$$

$$\frac{|00\rangle - |01\rangle + i|10\rangle - i|11\rangle}{2}$$

What is the quantum state vector for $|+\rangle_A \otimes |-\rangle_B$?

A. $\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$

B. $\frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2}$

C. $\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$

D. $\frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2}$

E. None of the above

Two-Player Born's Rule

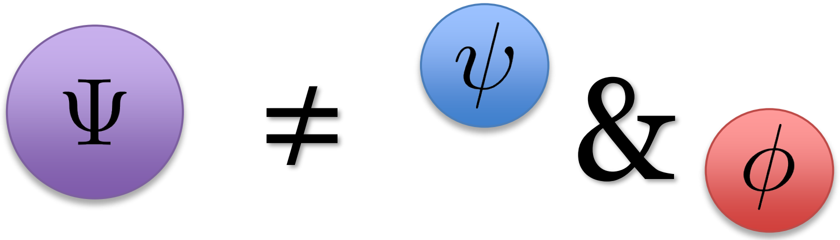
We can make measurements on both Alice and Bob's systems individually, with outcome probabilities defined by the two-qubit version of Born's rule.

$$|\Psi\rangle_{AB} = |0\rangle_A \otimes \frac{|0\rangle_B + |1\rangle_B}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$

$\mathrm{P}(0_A 0_B) = \lvert\langle 00\lvert\Psi\rangle_{AB}\rvert^2 = \frac{1}{2}$	$\mathrm{P}(1_A 0_B) = \lvert\langle 10\lvert\Psi\rangle_{AB}\rvert^2 = 0$
$\mathrm{P}(0_A 1_B) = \lvert\langle 01\lvert\Psi\rangle_{AB}\rvert^2 = \frac{1}{2}$	$\mathrm{P}(1_A 1_B) = \lvert\langle 11\lvert\Psi\rangle_{AB}\rvert^2 = 0$

Entanglement

Entangled   Separable



$$|\Psi\rangle_{AB} \neq |\phi\rangle_A \otimes |\psi\rangle_B$$

Entanglement is defined in reverse

If a state is **not** separable, it is entangled

Are they entangled?

Is this state entangled?

$$|\Psi\rangle = \frac{|00\rangle - |01\rangle - |10\rangle + |11\rangle}{2}$$

No

$$|\Psi\rangle = |-\rangle \otimes |-\rangle$$

Is this state entangled?

$$|\Psi\rangle = \frac{|00\rangle - |01\rangle + |10\rangle + |11\rangle}{2}$$

Yes!

Can show by contradiction

$$|\Psi\rangle = \frac{|0\rangle|-\rangle + |1\rangle|+\rangle}{\sqrt{2}}$$

Which of the following states are entangled?

A. $\frac{|00\rangle + i|01\rangle}{\sqrt{2}}$

B. $|+\rangle \otimes |+_i\rangle$

C. $\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$

D. $\frac{|01\rangle + i|10\rangle}{\sqrt{2}}$

E. All of the above

The Bell State

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

This is a **maximally entangled** state for two qubits.

Bell State Correlations

What are the two-qubit measurement probabilities for the entangled states?

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$P(0_A 0_B) = |\langle 00|\Psi^-\rangle|^2 = 0$$

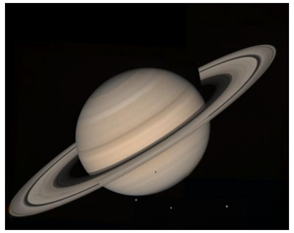
$$P(1_A 0_B) = |\langle 10|\Psi^-\rangle|^2 = \frac{1}{2}$$

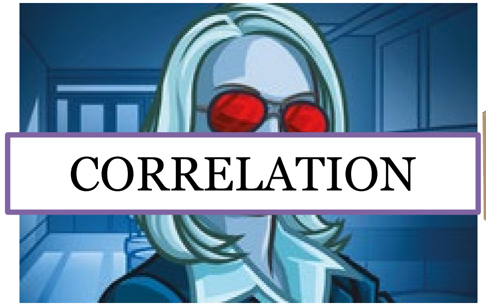
$$P(0_A 1_B) = |\langle 01|\Psi^-\rangle|^2 = \frac{1}{2}$$

$$P(1_A 1_B) = |\langle 11|\Psi^-\rangle|^2 = 0$$

Alice and Bob's measurement results are individually random, but **correlated**

Entanglement vs. Correlation





CORRELATION

Correlation is nothing “spooky” at all

Entanglement is correlation **plus** superposition

Entanglement vs. Correlation

What separates

$$|01\rangle \textbf{ OR } |10\rangle$$

from

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Entanglement is only distinguishable from classical correlations when measuring in multiple bases

What is the following state?

$$U_{\text{cNOT}}\left(|+\rangle \otimes |0\rangle\right)$$

A. $\frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$

B. $\frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$

C. $\frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$

The cNOT is an entangling gate

D. $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Recall:
$$U_{\text{cNOT}}|00\rangle = |00\rangle$$
$$U_{\text{cNOT}}|01\rangle = |01\rangle$$
$$U_{\text{cNOT}}|10\rangle = |11\rangle$$
$$U_{\text{cNOT}}|11\rangle = |10\rangle$$

$$U_{\text{cNOT}}\left(|+\rangle \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}\left(U_{\text{cNOT}}|00\rangle + U_{\text{cNOT}}|10\rangle\right)$$

Quantum Teleportation

Teleportation, really?

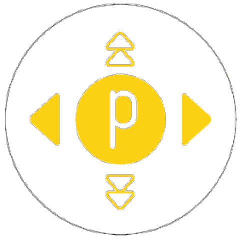
Remember!

A quantum state is not an object!

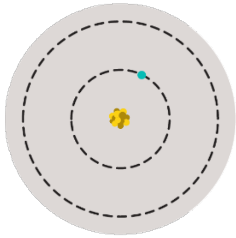
It describes a property of an object!

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

Polarized photon



$c_0|\rightarrow\rangle + c_1|\uparrow\rangle$



Trapped-ion energy

$c_0|g\rangle + c_1|e\rangle$

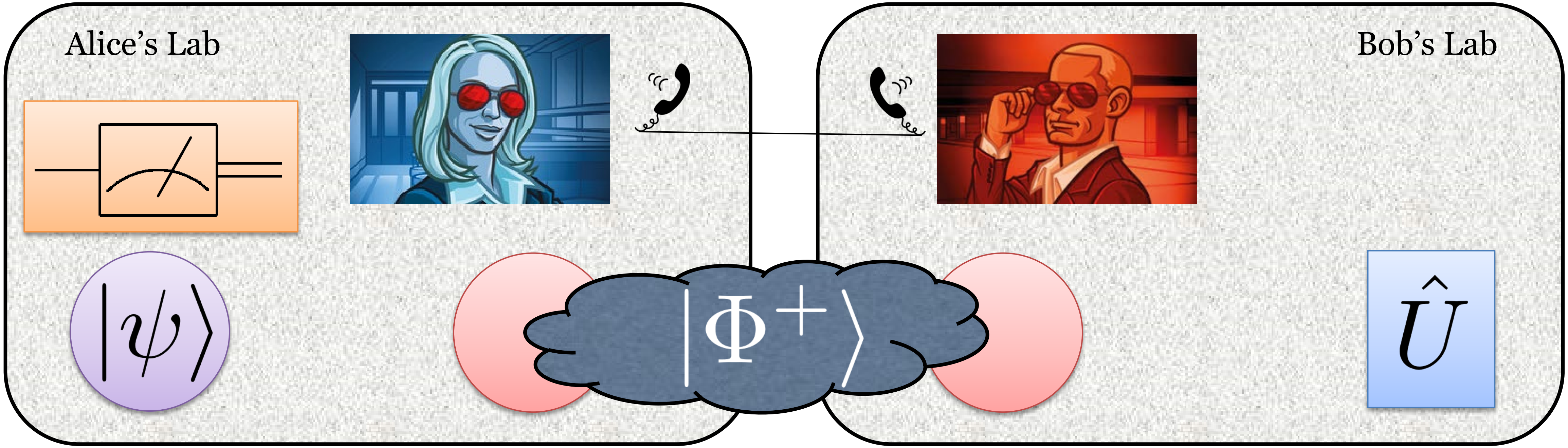
The Tools of Teleportation

Alice has the state $|\psi\rangle$ and the ability to make **<u>any</u>** measurement

Alice and Bob can communicate classical information

Alice and Bob share **<u>one</u>** entangled pair

Bob can make any single-qubit rotation



A Recipe for Teleportation

1. Alice makes a two-qubit measurement between the unknown state and her half of the entangled state

2. Alice communicates the result of the measurement to Bob

3. Bob applies a rotation to his state, depending on the measurement result

But what measurement to make?

The Bell Basis

Remember:
The Two-Qubit Computational Basis
$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

Complete set of
two-qubit entangled states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

We can rewrite the original
basis states in this basis...

$$|00\rangle = \boxed{?}$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle)$$

$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$$

The Bell Basis

We can also perform measurements in this basis

$\downarrow$

Joint measurement

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|+\rangle \otimes |-\rangle \, ?$$

In computational basis?

In Bell basis?

We can rewrite the original basis states in this basis…

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right)$$

$$|01\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right)$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle - |\Psi^-\rangle\right)$$

$$|11\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right)$$

The Bell Basis

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\begin{aligned}|+\rangle \otimes |-\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\\ &= \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(|\Phi^-\rangle - |\Psi^-\rangle\right)\end{aligned}$$

How to Build Bell States

$$U_{\mathrm{cNOT}}\left(|+\rangle \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

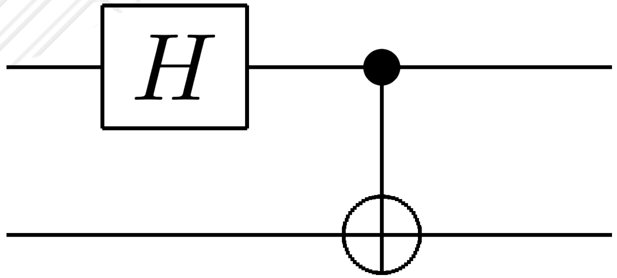
$$U_{\mathrm{cNOT}}\left(|+\rangle \otimes |1\rangle\right) = \ ?$$

$$U_{\mathrm{cNOT}}\left(|-\rangle \otimes |0\rangle\right) = \ ?$$

$$U_{\mathrm{cNOT}}\left(|-\rangle \otimes |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

This circuit maps the computational basis to the Bell basis

$$U_{cNOT}\left(H \otimes \mathbb{1}\right)$$

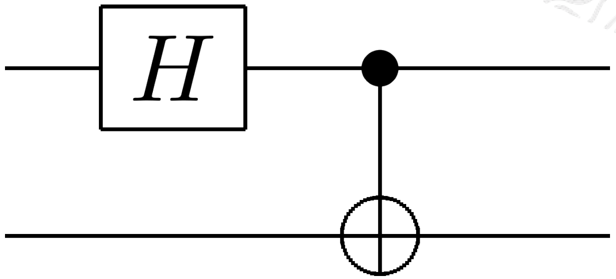


Input	Output
$\|00\rangle$	$\|\Phi^+\rangle$
$\|01\rangle$	$\|\Psi^+\rangle$
$\|10\rangle$	$\|\Phi^-\rangle$
$\|11\rangle$	$\|\Psi^-\rangle$

How to Build Bell *Measurements*

This circuit maps
the computational basis
to the Bell basis

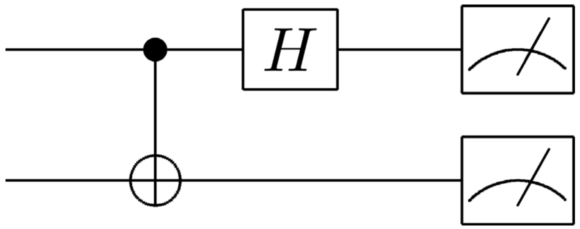
$$U_{cNOT}\,(H \otimes \mathbb{1})$$



Input	Output
$\|00\rangle$	$\|\Phi^+\rangle$
$\|01\rangle$	$\|\Psi^+\rangle$
$\|10\rangle$	$\|\Phi^-\rangle$
$\|11\rangle$	$\|\Psi^-\rangle$

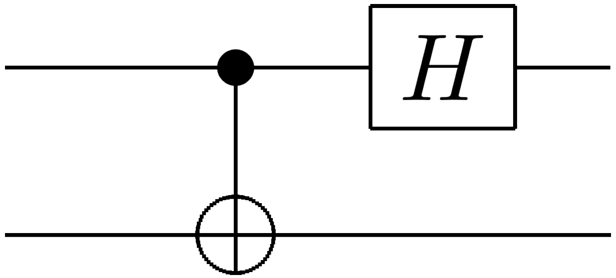
What could do
the reverse
operation?

Bell State Measurement



This circuit maps
the Bell basis
to the computational basis

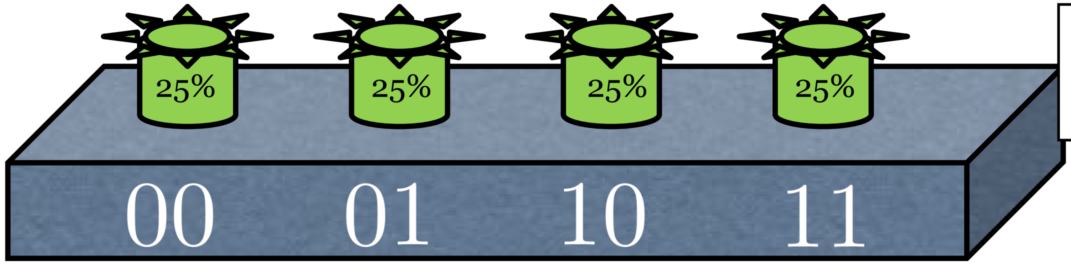
$$(H \otimes \mathbb{1})\,U_{cNOT}$$



Input	Output
$\|\Phi^+\rangle$	$\|00\rangle$
$\|\Psi^+\rangle$	$\|01\rangle$
$\|\Phi^-\rangle$	$\|10\rangle$
$\|\Psi^-\rangle$	$\|11\rangle$

Computational Basis Measurement

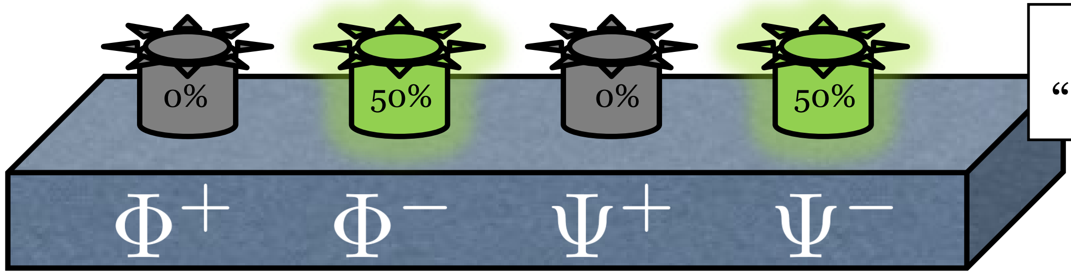
$$|+\rangle \otimes |-\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$



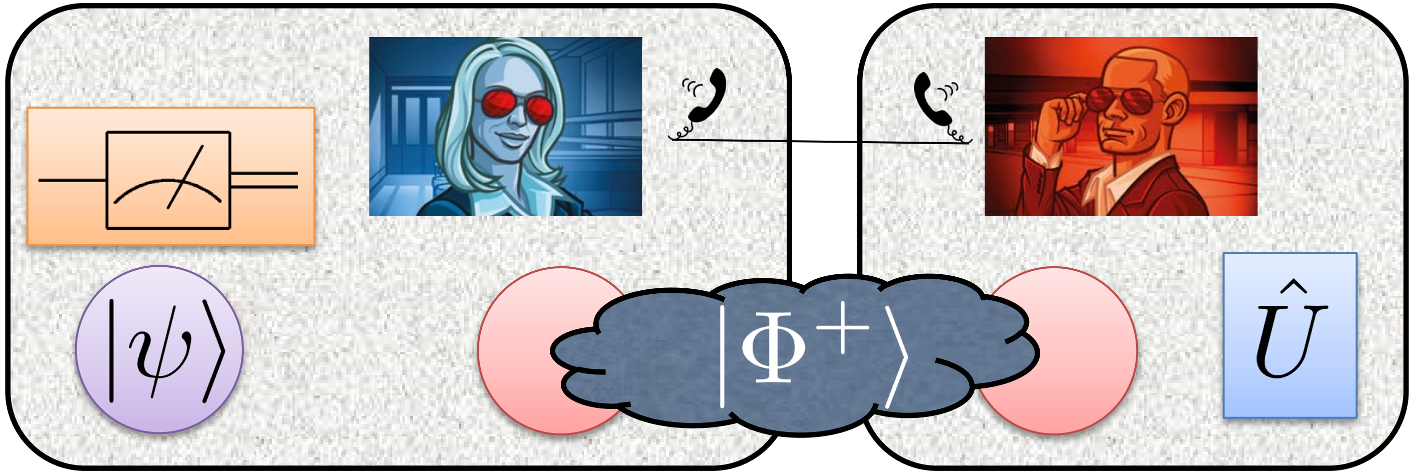
Asks the question
"Are the qubits zero or one?"

Bell Basis Measurement

$$|+\rangle \otimes |-\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^-\rangle - |\Psi^-\rangle\right)$$



Asks the question
"How are these qubits related?"



Alice's State

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Global State:

$$|\psi\rangle_A \otimes |\Phi^+\rangle_{AB} = \overbrace{(\alpha|0\rangle_A + \beta|1\rangle_A)}^{\text{Alice's Qubit}} \otimes \frac{1}{\sqrt{2}} \overbrace{(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)}^{\text{Shared Entangled State}}$$

$$\downarrow$$

expand

$$\downarrow$$

$$= \frac{\alpha}{\sqrt{2}}|\underline{00}\rangle_A|0\rangle_B + \frac{\alpha}{\sqrt{2}}|\underline{01}\rangle_A|1\rangle_B + \frac{\beta}{\sqrt{2}}|\underline{10}\rangle_A|0\rangle_B + \frac{\beta}{\sqrt{2}}|\underline{11}\rangle_A|1\rangle_B$$

Next: Rewrite Alice's terms in Bell basis

$$|\psi\rangle_A \otimes |\Phi^+\rangle_{AB} = (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

$$= \frac{\alpha}{\sqrt{2}}|00\rangle_A|0\rangle_B + \frac{\alpha}{\sqrt{2}}|01\rangle_A|1\rangle_B + \frac{\beta}{\sqrt{2}}|10\rangle_A|0\rangle_B + \frac{\beta}{\sqrt{2}}|11\rangle_A|1\rangle_B$$

Write Alice's state in the Bell basis using...

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle)$$

$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$$

$$\begin{aligned} &|\psi\rangle_A \otimes |\Phi^+\rangle_{AB} \\ = \quad & \frac{1}{2}|\Phi^+\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) \\ &+ \frac{1}{2}|\Phi^-\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B) \\ &+ \frac{1}{2}|\Psi^+\rangle_A(\beta|0\rangle_B + \alpha|1\rangle_B) \\ &+ \frac{1}{2}|\Psi^-\rangle_A(-\beta|0\rangle_B + \alpha|1\rangle_B) \end{aligned}$$

Alice's Bell state measurement collapses the state to one of:

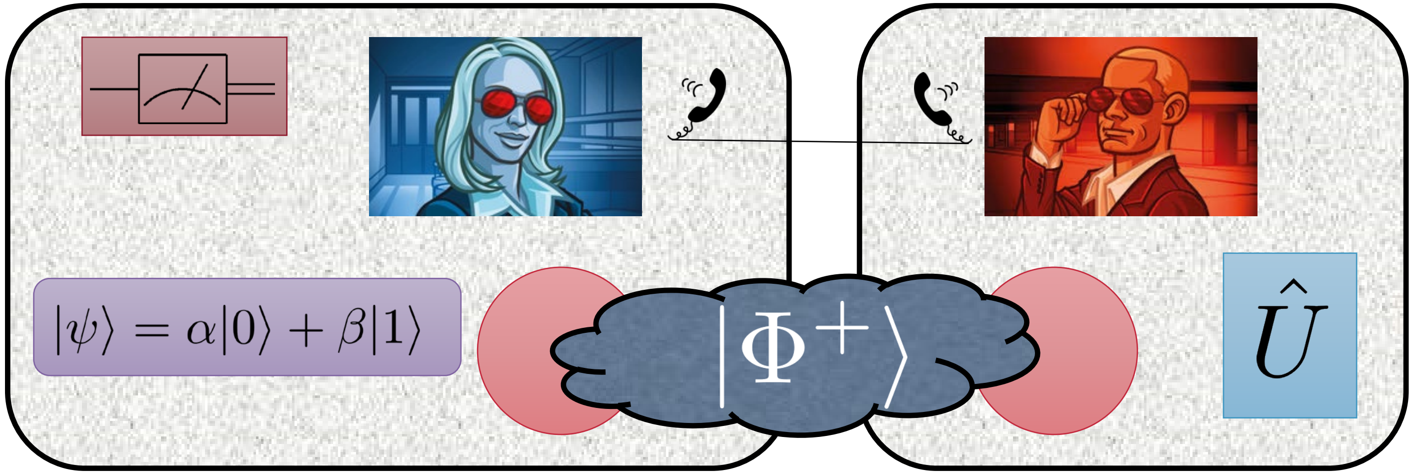
$$|\Phi^+\rangle_A \otimes \underbrace{(\alpha|0\rangle_B + \beta|1\rangle_B)}_{|\psi\rangle}$$

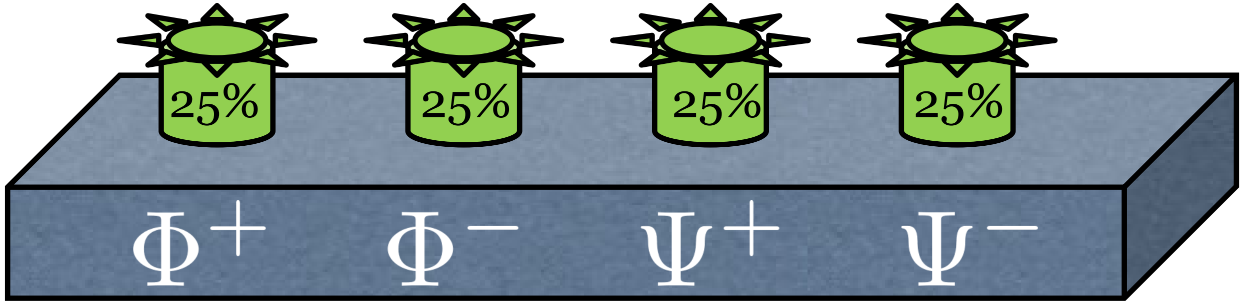
$$|\Phi^-\rangle_A \otimes \underbrace{(\alpha|0\rangle_B - \beta|1\rangle_B)}_{Z|\psi\rangle}$$

$$|\Psi^+\rangle_A \otimes \underbrace{(\beta|0\rangle_B + \alpha|1\rangle_B)}_{X|\psi\rangle}$$

$$|\Psi^-\rangle_A \otimes \underbrace{(-\beta|0\rangle_B + \alpha|1\rangle_B)}_{ZX|\psi\rangle}$$

Each with equal probability





Depending on which state Alice measures, Bob can rotate his qubit to obtain a perfect copy of the original state

Alice measures:	Bob has:	Bob applies:	Bob ends up with:
$\|\Phi^+\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	$\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$
$\|\Phi^-\rangle$	$\alpha\|0\rangle - \beta\|1\rangle$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$
$\|\Psi^+\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$
$\|\Psi^-\rangle$	$-\beta\|0\rangle + \alpha\|1\rangle$	$ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$

How many copies of the original state $|\psi\rangle$ exist after the teleportation?

A. None

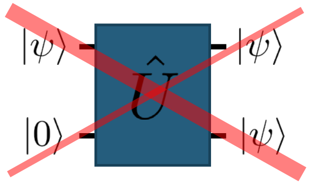
Alice destroyed her state
with the Bell state measurement

B. One, and Alice has it

C. One, and Bob has it

D. Two, one each for Alice and Bob

E. An infinite number



The no-cloning theory
remains valid

What state does Bob have if he never receives information about Alice's measurement?

A. The Bell state $|\Phi^+\rangle$

B. The intended state $|\psi\rangle$

C. The rotated version, $ZX|\psi\rangle$

D. A completely random state

E. The intended state with a random rotation

Quantum information is lost with classical communication!

No faster-than-light communication!

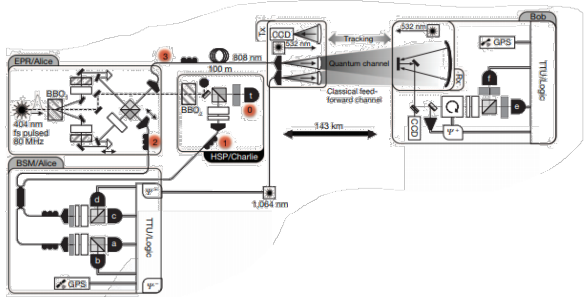
Experiments!

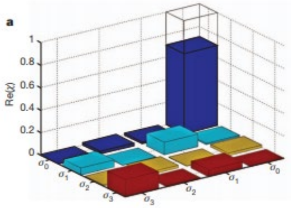
Quantum teleportation over 143 kilometres using active feed-forward

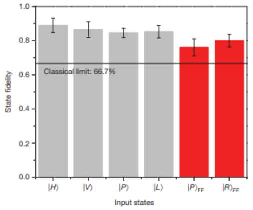
Xiao-Song Ma[1,2]†, Thomas Herbst[1,2], Thomas Scheidl[1], Daqing Wang[1], Sebastian Kropatschek[1], William Naylor[1], Bernhard Wittmann[1,2], Alexandra Mech[1,2], Johannes Kofler[1,3], Elena Anisimova[4], Vadim Makarov[4], Thomas Jennewein[1,4], Rupert Ursin[1] & Anton Zeilinger[1,2]

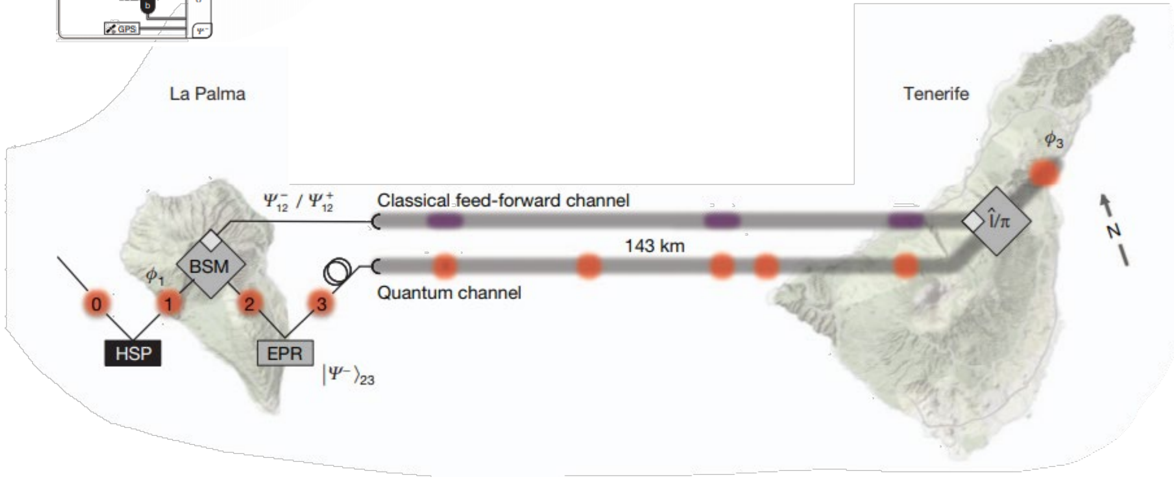
Nature **489**, 269–273 (13 September 2012)

IQC





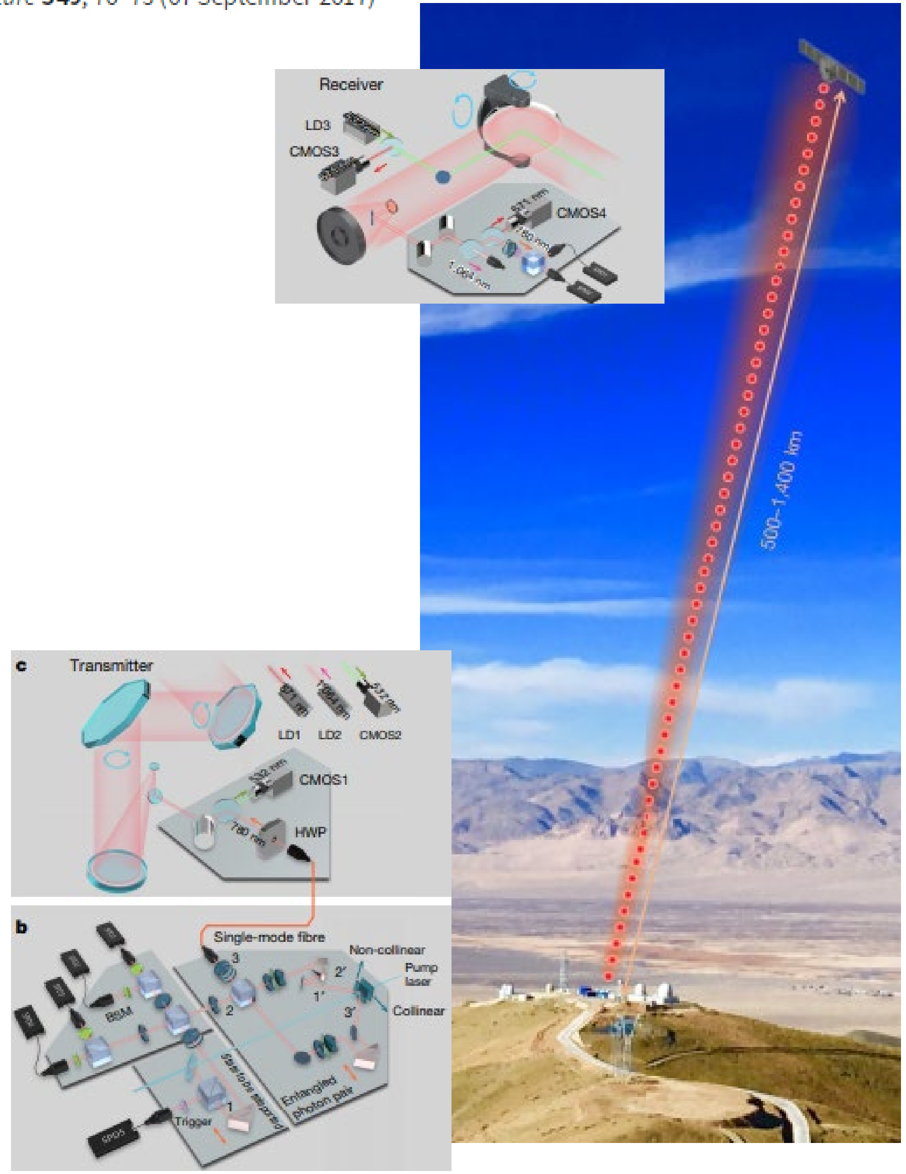




Ground-to-satellite quantum teleportation

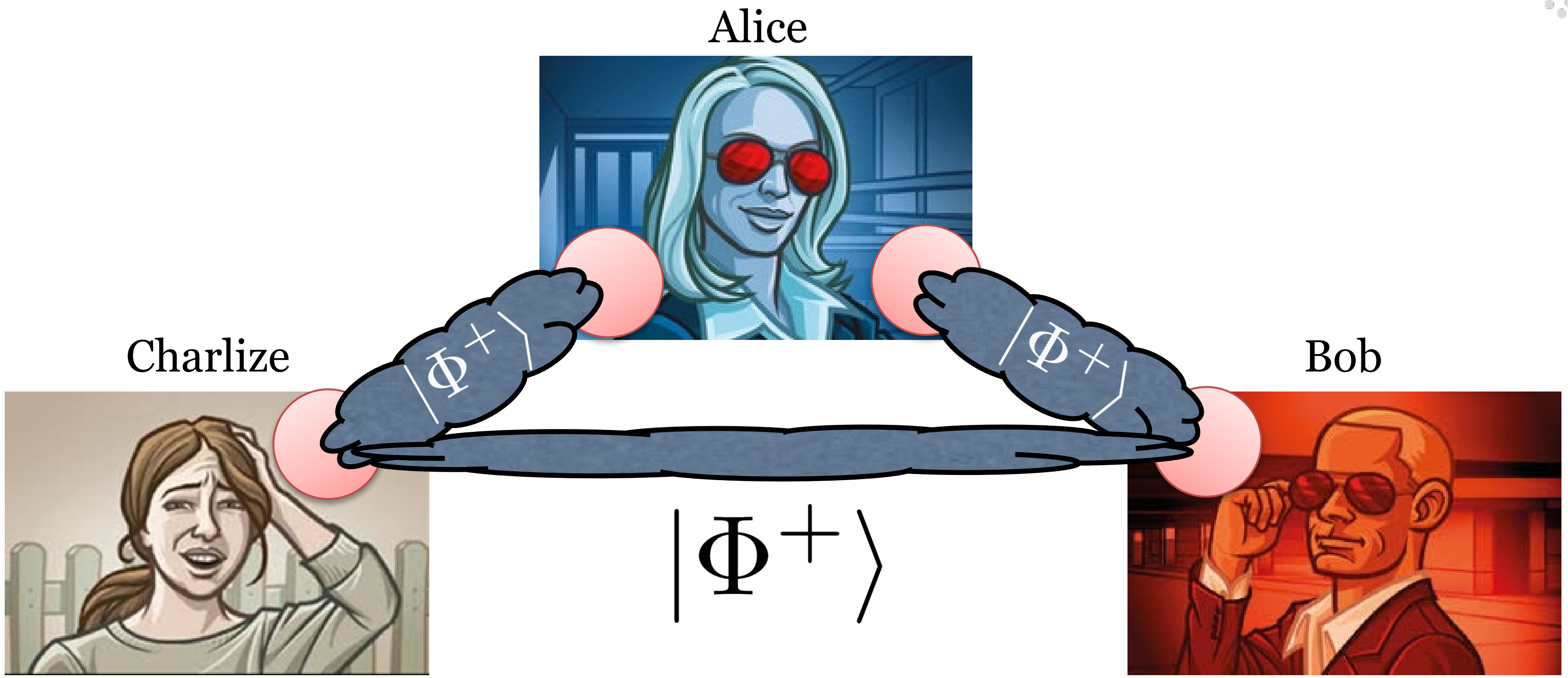
Ji-Gang Ren[1,2], Ping Xu[1,2], Hai-Lin Yong[1,2], Liang Zhang[2,3], Sheng-Kai Liao[1,2], Juan Yin[1,2], Wei-Yue Liu[1,2], Wen-Qi Cai[1,2], Meng Yang[1,2], Li Li[1,2], Kui-Xing Yang[1,2], Xuan Han[1,2], Yong-Qiang Yao[4], Ji Li[5], Hai-Yan Wu[5], Song Wan[6], Lei Liu[6], Ding-Quan Liu[3], Yao-Wu Kuang[3], Zhi-Ping He[3], Peng Shang[1,2], Cheng Guo[1,2], Ru-Hua Zheng[7], Kai Tian[8], Zhen-Cai Zhu[6], Nai-Le Liu[1,2], Chao-Yang Lu[1,2], Rong Shu[2,3], Yu-Ao Chen[1,2], Cheng-Zhi Peng[1,2], Jian-Yu Wang[2,3] & Jian-Wei Pan[1,2]

Nature **549**, 70–73 (07 September 2017)



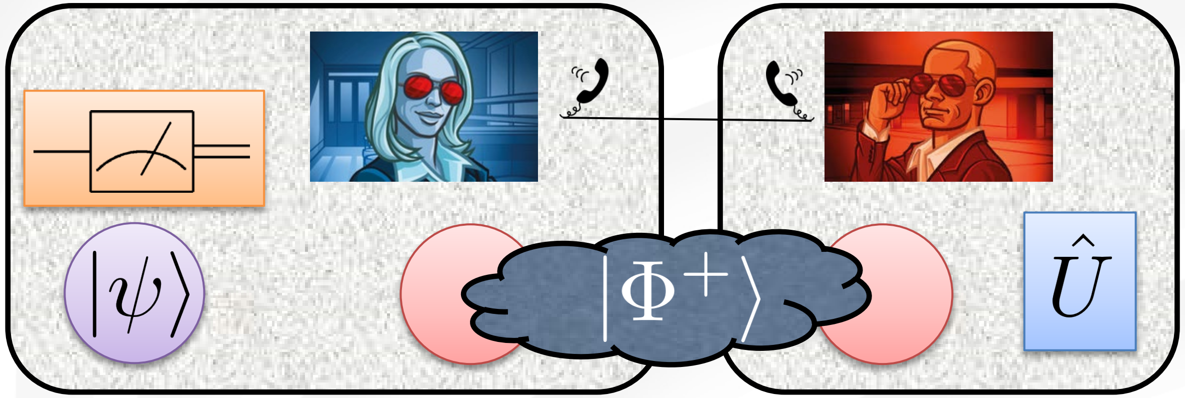
Three's a Crowd

What happens if Alice attempts to teleport an entangled qubit?



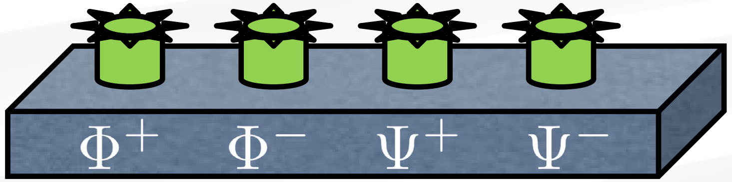
Entanglement Swapping!
Key technique for quantum repeaters, some quantum computing protocols, and entangling different quantum systems!

Quantum Teleportation



Using one entangled photon and classical communication, Alice can teleport an unknown quantum state to Bob

No-cloning and special relativity restrictions remain valid



An essential tool for quantum communication and computing

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