Bounds for the spectral radius of nonnegative matrices

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Definitions

Let $A = (a_{ij})_{i,j=1}^n \in \mathscr{M}_n(\mathbb{R}^+)$.

- For every $i = 1, 2, \ldots, n$

$$r_i = \sum_{j=1}^n a_{ij},$$

  i-th row sum of A.

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$$\zeta_i = \begin{cases} \frac{-b_i + \sqrt{b_i^2 + 4a_{i\lambda}(r_\lambda - a_{\lambda\lambda})}}{2(r_\lambda - a_{\lambda\lambda})}, & \text{if } r_\lambda > a_{\lambda\lambda} \\ i, & \text{if } r_\lambda = a_{\lambda\lambda} \end{cases} \quad \text{and}$$

$$\omega_j = \begin{cases} \frac{-c_j + \sqrt{c_j^2 + 4a_{j\mu}(r_\mu - a_{\mu\mu})}}{2(r_\mu - a_{\mu\mu})}, & \text{if } r_\mu > a_{\mu\mu} \\ 1/j, & \text{if } r_\mu = a_{\mu\mu} \end{cases}$$

Definition

Proposition

Let

$$\begin{aligned} \zeta &= \min\{\zeta_i : 1 \leq i \leq n,\ i \neq \lambda\} \\ \omega &= \max\{\omega_j : 1 \leq j \leq n,\ j \neq \mu\}. \end{aligned}$$

Then,

$$\zeta_i \geq 1,\ \ \zeta \geq 1 \ \text{ and } \ 0 \leq \omega_j \leq 1,\ \ 0 \leq \omega \leq 1.$$

Moreover, if $\omega > 0$, then

$$\zeta(r_\lambda - a_{\lambda\lambda}) + a_{\lambda\lambda} \leq \rho(A) \leq \omega(r_\mu - a_{\mu\mu}) + a_{\mu\mu}.$$

Frobenius Bounds

Theorem(Frobenius, 1912)

Let $A = (a_{ij})_{i,j=1}^{n} \in \mathscr{M}_n(\mathbb{R}^+)$. Then

$$\max_{1\le i\le n}\{a_{ii}\} \le \rho(A)$$

$$r_\lambda = \min_{1\le i\le n}\{r_i\} \le \rho(A) \le \max_{1\le i\le n}\{r_i\} = r_\mu$$

$$\rho(A) = \max_{1\le i\le n}\{r_i\} \Longleftrightarrow r_1(A) = r_2(A) = \ldots = r_n(A).$$

Definition

Definition

Let $A = (a_{ij})_{i,j=1}^{n} \in \mathscr{M}_n(\mathbb{R}^+)$.

- Let $k \in \mathbb{N}$ with $k \geq 1$. The nonnegative quantity

$$w_i^{(k+1)} = \frac{1}{r_i} \underbrace{\sum_{j=1}^{n} a_{ij} \sum_{u=1}^{n} a_{ju} \cdots \sum_{p=1}^{n} a_{vp}}_{k\text{-sums}} r_p = \frac{1}{r_i} \sum_{\tau=1}^{n} a_{i\tau}^{(k)} r_\tau,$$

is called average $(k+1)$-row sum of A.

Generalized Frobenius Bounds

Theorem (Adam, Oikonomou, & Aretaki, 2023)

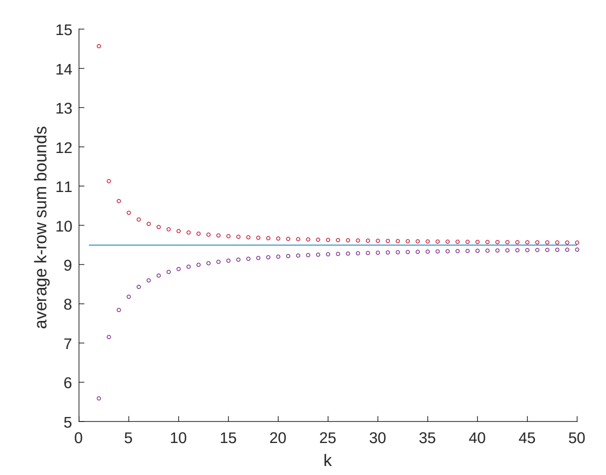
If $A \in \mathscr{M}_n(\mathbb{R}^+)$ with $r_i > 0$, $1 \le i \le n$, for every $k \ge 1$.

$$\min_{1\le i\le n} \{\sqrt[k]{w_i^{(k+1)}}\} \le \rho(A) \le \max_{1\le i\le n} \{\sqrt[k]{w_i^{(k+1)}}\}.$$

If A has a positive eigenvector, then for every $i = 1, 2, \ldots, n$

$$\rho(A) = \lim_{k\to\infty} \sqrt[k]{w_i^{(k+1)}}.$$

Generalized Frobenius Bounds



Convergence to spectral value of 9×9 nonnegative matrices

Generalized Bounds

Theorem (Adam, Oikonomou, & Aretaki, 2023)

Assume $w_1^{(k+1)} \geq \cdots \geq w_n^{(k+1)}$ with $w_1^{(k+1)} \geq \gamma$, when $b = 1$, and $w_1^{(k+1)} > \gamma$, when $b > 1$, and $w_n^{(k+1)} > \delta$ for $A \geq 0$, $k \geq 1$ and the quantities γ, δ as defined above. Then,

$$\sqrt[k]{z_n^{(k+1)}} \leq \rho(A) \leq \min\left\{ \sqrt[k]{Z_\ell^{(k+1)}} : \ 1 \leq \ell \leq n \right\}, \tag{0.1}$$

where for $\ell = 1, \ldots, n$

Generalized Bounds

Theorem (Adam, Oikonomou, & Aretaki, 2023)

Let $A \in \mathscr{M}_n(\mathbb{R})$, $A \geq 0$.

$$\sqrt[k]{w_n^{(k+1)}} \leq \sqrt[k]{z_n^{(k+1)}} \leq \rho(A).$$

Moreover, if A has a positive eigenvector, then $\rho(A) = \lim_{k\to\infty} \sqrt[k]{z_n^{(k+1)}}$.

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Thank you!

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