Computing the Volume of a Pareto Frontier

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Given a set of Pareto optimal solutions $\mathcal{P}$ to a multi-objective mathematical programming model with a set of objectives O of cardinality $N := |O|$, this algorithm computes the volume V of the objective space bounded by the Pareto frontier defined by the solutions $x \in \mathcal{P}$. The objectives are assumed to be normalized so that the objective space is the N-dimensional unit hypercube with the nadir objective vector and the ideal objective vector defining the origin and the point $\vec{\mathbf{1}}$, respectively.

We project the objective space into $N-1$ dimensions by eliminating the dimension associated with an (arbitrarily-chosen) objective $p \in O$. We define the set of objectives $\overline{O} := O \backslash \{p\}$. It is assumed that $x \in \mathcal{P}$ are sorted in descending order according to p. The algorithm proceeds by sequentially adding solutions to the $(N-1)$-dimensional space, and calculating the contribution to the frontier volume as a product of the volume contribution in $N-1$ dimensions and its achievement in objective p.

Let $\overline{V_x}$ be the $(N-1)$-dimensional volume contribution of solution x and x_p be the achievement of solution x in objective p. Further, let F be the set of non-dominated solutions in $N-1$ dimensions. We proceed to compute the N-dimensional volume of the frontier V as follows.

Algorithm 1 Computing the volume of a Pareto frontier

$V \leftarrow 0$
$\overline{V} \leftarrow 0$
$F \leftarrow \emptyset$
for all $x \in \mathcal{P}$ **do**
    $\overline{V}_x \leftarrow \prod_{o \in \overline{O}} x_o - \overline{V}$
    for all $f \in F$ **do**
        if $f_o < x_o \forall o \in \overline{O}$ **then**
            $F \leftarrow F \backslash \{f\}$
        end if
    end for
    for all $o \in \overline{O}$ **do**
        $F_{x,o} := \{\, f \in F : f_o > x_o \,\}$
        Sort $f \in F_{x,o}$ in ascending order by their oth component, f_o
        $v_i \leftarrow x_o$
        for all $f \in F_{x,o}$ **do**
            $v_t \leftarrow f_o$
            $\delta_o := v_t - v_i$
            $\overline{V}_x \leftarrow \overline{V}_x + \delta_o \prod_{\sigma \in \overline{O} \backslash \{o\}} f_\sigma$
            $v_i \leftarrow v_t$
        end for
    end for
    $F \leftarrow F \cup \{x\}$
    $\overline{V} \leftarrow \overline{V} + \overline{V}_x$
    $V \leftarrow V + x_p \overline{V}_x$
end for