Computing the Volume of a Pareto Frontier

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Given a set of Pareto optimal solutions $\mathcal{P}$ to a multi-objective mathematical programming model with a set of objectives O of cardinality $|O| = N$, this algorithm computes the volume V of the objective space bounded by the Pareto frontier defined by the solutions $x \in \mathcal{P}$. The objectives are assumed to be normalized so that the objective space is the N-dimensional unit hypercube with the nadir objective vector and the ideal objective vector defining the origin and the point $\vec{\mathbf{1}}$, respectively.

We project the objective space into $N-1$ dimensions by eliminating the dimension associated with an (arbitrarily-chosen) objective $p \in O$. It is assumed that $x \in \mathcal{P}$ are sorted in descending order according to p. The algorithm proceeds by sequentially adding solutions to the $(N-1)$-dimensional space, and calculating the contribution to the frontier volume as a product of the volume contribution in $N-1$ dimensions and its achievement in objective p.

Let $\overline{V_x}$ be the $(N-1)$-dimensional volume contribution of solution x and x_p be the achievement of solution x in objective p. Then the volume of the frontier is given by

$$V = \sum_{x \in \mathcal{P}} \overline{V_x} x_p \tag{1}$$

where

$$\overline{V_x} = \left(\prod_{o \in \overline{O}} x_o \right) - \left(\sum_{y \in P_x} \overline{V_y} \right) + \sum_{o \in \overline{O}} \sum_{f \in \overline{F_{x,o}}} \Delta^o_{f+,f} \prod_{o' \in \overline{O} \setminus \{o\}} f_{o'}$$

where $\overline{O} = O \setminus \{p\}$ is the set of objectives sans p; $P_x = \{\, y \in \mathcal{P} : y_p \geq x_p, y \neq x \,\}$ is the set of solutions with pth component no less than x_p; and $\overline{F_{x,o}}$ is defined as

$$\begin{aligned} \overline{F_{x,o}} = \{ y \in P_x : \big(\exists o' \in \overline{O} : y_{o'} \geq z_{o'} \forall z \in P_x \big) \\ \cap \left(y_o \geq x_o \right) \} \end{aligned} \tag{2}$$

That is, the set $\overline{F_{x,o}}$ is comprised of solutions with greater or equal o and p components than x and are not dominated in the $(N-1)$-dimensional space by any solution in P_x.

Finally, for solutions $f \in \overline{F_{x,o}}$, we define $\Delta^o_{f^+,f}$ as

$$\Delta^o_{f^+,f} = \left(\max_{z \in \overline{F^{(y)}_{x,o}}} z_o \right) - f_o \tag{3}$$

where

$$\overline{F^{(y)}_{x,o}} = \{\, z \in \{x\} \cup \overline{F_{x,o}} : z_o < y_o \,\} \tag{4}$$

In other words, $\Delta^o_{f^+,f}$ is the difference in the oth component of adjacent members in the set of solutions $\overline{F_{x,o}}$, ordered by their oth components.