# On the Interpretation of the Intergenerational Elasticity and the Rank-Rank Coefficients for Cross Country Comparison

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**Abstract**

This paper investigates the Intergenerational Elasticity (IGE) and Rank-Rank coefficients, employing Yitzhaki's theorem (Yitzhaki [1996]) to express them as weighted averages of underlying causal mechanisms driving mobility. We highlight the challenges of interpreting cross-country comparisons using either the IGE or the Rank-Rank coefficient due to the regression weighting scheme and show that while the Rank-Rank coefficient is more interpretable for positional mobility, it lacks insights into the underlying mechanisms driving mobility across countries. The analysis demonstrates potential drawbacks of using linear regression coefficients as summary statistics in the context of intergenerational mobility comparisons.

## Introduction

Numerous studies have delved into the relationship between parental income and child's income. Two prominent methods for summarizing the joint distribution of these incomes are the Intergenerational Elasticity (IGE) coefficient and the Rank-Rank slope coefficient (Mogstad and Torsvik [2021]). This paper focuses on deriving how these measures summarize the joint income distribution and their connections to the underlying mechanisms that links parental and child's income. Let $I_c$ and $I_p$ denote the child's and parental income, respectively. The IGE coefficient represents the slope coefficient obtained by regressing the logarithm of the child's income on the logarithm of the parent's income

$$\log I_c = \alpha_{IGE} + \beta_{IGE} \log I_p + \epsilon \tag{1}$$

This regression coefficient captures the persistence between the child's log income and the parent's log income, with higher values indicating stronger persistence.[1] A popular alternative to this method is the Rank-Rank regression, which assesses the correlation between the parent's and child's ranks within their respective income distributions. Assuming continuous income distribution for both parents and children, let $R_c = F_c(I_c)$ and $R_p = F_c(I_p)$ represent the parent's and child's ranks in their respective income distributions, where $F_c(x)$ and $F_p(x)$ are the cumulative distribution functions of child and parental income. Researchers then measure the Rank-Rank relationship by estimating the following regression

$$R_c = \alpha_r + \beta_r R_p + \varepsilon \tag{2}$$

This regression slope coefficient quantifies how a child's position in the income distribution relates to their parent's position in the corresponding income distribution. The IGE coefficient has been extensively employed in empirical studies to describe intergenerational persistence, dating back to the 1980s (Becker and Tomes [1986],Atkinson [1980]). The Rank-Rank coefficient, however, gained popularity more recently, following Chetty et al. [2014b] using it to measure social mobility over time in the United States . While both coefficients are utilized to describe intergenerational mobility, they convey distinct information about the joint distribution of parental and child's income. As demonstrated below, the IGE provides a weighted average of the expected change in the child's logarithmic income in relation to a change in the parent's logarithmic income.[2] Consequently, the IGE coefficient is influenced by both the marginal distributions and the dependency structure between parental and child's income. In contrast, the Rank-Rank coefficient measures positional mobility across generations, summarizing only the copula while isolating the dependency structure between the incomes and disregarding changes in the marginal distributions (Deutscher and Mazumder [2021], Mogstad and Torsvik [2021], Aloni and Krill [2017]). From a practical standpoint, the Rank-Rank coefficient has proven more robust to sample restrictions (Chetty et al. [2014a],Chetty et al. [2014b],Dahl and DeLeire [2008]) and, in some countries (although not all, Bratberg et al. [2017], Acciari et al. [2022]), the Rank-Rank coefficient exhibits an almost perfectly linear relationship between the parent's and child's rank, while the conditional expectation function $E[\log I_c | \log I_p]$ has shown significant non-linearities (Chetty et al. [2014a],Deutscher and Mazumder [2021]). Moreover, the Rank-Rank coefficient allows researchers to include individuals with no income. This could be important since, as observed by

[1] In many cases, the level of intergenerational mobility is reported using (1-$\beta_{IGE}$)

[2] Mitnik and Grusky [2020] illustrates that the IGE can be considered as the elasticity of the conditional geometric mean, i.e., the expected percentage change in the geometric mean of child's income with respect to the percentage change in the parental income.

Chetty et al. [2014a], the IGE, demonstrates significant sensitivity to the substitution of zeros with ones or 1000s.

This paper examines the challenges of using IGE and Rank-Rank coefficients for cross-country mobility comparisons. Utilizing Yitzhaki's theorem (Yitzhaki [1996]), we express these coefficients as weighted averages of causal factors affecting intergenerational mobility. We demonstrate that these coefficients assign varying weights across the parental income distribution, which helps explain certain properties identified in existing literature. We further explore how the parental income distribution influences the IGE and Rank-Rank slopes, complicating cross-country comparisons—especially when mobility occurs in different segments of the parental income distribution in each country. A related paper Maasoumi et al. [2022] also employs Yitzhaki's theorem, framing the IGE coefficient weighting scheme as a special case within a broader class of intergenerational mobility measures that captures different preference relations over income distributions. They show that the IGE coefficient corresponds to a specific case of a preference relations that places higher weight on the mobility of wealthier households. Compared to this work, our paper focuses on interpreting the coefficients as a weighted average of the underlying causal mechanisms and discusses how these interpretations are important for cross-country comparison.

## Decomposing the IGE coefficient

We begin by examining the $\beta_{IGE}$ coefficient. Let's assume that $(I_c, I_p)$ are i.i.d, $E[|\log I_c|], E[|\log I_p|] < \infty$, and $E[\log I_c | \log I_p = t]$ exist and are differentiable for all $t$. According to Yitzhaki's theorem (Yitzhaki [1996]), we can express $\beta_{IGE}$ as a weighted average of the derivative of the conditional expectations:[3]

$$\beta_{IGE} = \frac{\text{Cov}(\log I_c, \log I_p)}{\text{Var}(\log I_p)} = \int_{-\infty}^{\infty} \frac{\partial E[\log I_c | \log I_p = t]}{\partial t} w(t) dt$$

where

$$w(t) = \frac{E\left[\log I_p - \mu_{I_p} | \log I_p > t\right] \text{P}(\log I_p > t)}{\text{Var}(\log I_p)}, \quad \int_{-\infty}^{\infty} w(t)dt = 1, \quad \mu_{I_p} = E[\log I_p]$$

We can then interpret the IGE coefficient as a summary statistic of the underlying function $E[\log I_c | \log I_p]$, where the weights depend on the distribution of parental income. Specifically, these weights are maximized at $E[\log I_p]$ and approach zero at the boundary of the support (Yitzhaki [1996], Heckman et al. [2006]). Thus, $\beta_{IGE}$ assigns higher

[3] A proof for the theorem and subsequent claims is provided in the appendix

weight to households with the mean parental log income[4] and lower weights to households at the top and bottom of the parental log income distribution.

The fact that the IGE coefficient assigns lower weights to households at the extremes may be concerning in cases where a significant portion of mobility occurs for children from very poor or very rich families. This can occur potentially as a result of policies aimed at reducing poverty, or simply regression to the mean. The fact that the weights depend on the underlying parental log income distribution implies that cross-country or over-time comparisons of the IGE coefficient can be difficult to interpret. For instance, without knowing the exact weights, differences between two countries may arise simply from differences in the weighting schemes used by the IGE coefficient, even if the conditional expectation function $E[\log I_c | \log I_p]$ is the same across both countries.

Importantly, the fact the IGE coefficient assigns higher weights to mobility around the mean may explain why the IGE is considered sensitive to sample definitions and restrictions. Sample restrictions, particularly those that omit observations at the top and bottom of the distribution, such as households with zero income or those with extremely high income, can significantly impact the mean of the distribution. As a result, households that receive higher weights change, and the IGE coefficient changes as well.

To better understand how the IGE coefficient relates to the Rank-Rank coefficient and the underlying income elasticity,[5] we aim to decompose the integrand into the expected parent-child income elasticity and additional correlative mechanisms. Let the causal model governing children's income be given by:

$$I_c = h(I_p, u) \tag{3}$$

where $u$ represents other unobserved factors that affect the child's earnings. Let $\epsilon_{I_c,I_p}(u) = \frac{\partial \log I_c}{\partial \log I_p}$ be the elasticity of the child's income with respect to the parent's income for a given unobserved shock $u$. We can then rewrite the integrand

$$\begin{aligned} \frac{\partial E[\log I_c | \log I_p = t]}{\partial t} &= \int_{-\infty}^{\infty} \frac{\partial \log h(I_p, u) P(u | \log I_p = t)}{\partial t} du \\ &= \underbrace{E\left[\epsilon_{I_c,I_p}(u) \middle| \log I_p = t\right]}_{\text{Causal IGE}} + \underbrace{\int_{-\infty}^{\infty} \log I_c(u) \frac{\partial P(u | \log I_p = t)}{\partial t} du}_{\text{Other Factors}} \end{aligned} \tag{4}$$

[4] Note that this is generally not the same as families with the mean income

[5] As noted by Mitnik and Grusky [2020], the IGE coefficient does not actually provide information about the parent-child income elasticity, which is evident in this setup, as $\frac{\partial E[\log I_c | \log I_p]}{\partial \log I_p} \neq E\left[\frac{\partial \log I_c}{\partial \log I_p} | \log I_p\right]$

where the second equality follows from the product rule. The first component captures the conditional expected IGE, while the second component captures how changes in income are associated with changes in other factors that affect income.[6] Therefore, we can see that the $\beta_{IGE}$ can be represented by a summation of the weighted causal intergenerational elasticities (causal IGE) and an additional term that captures parental income is correlated with other factors that affect child's income. In most studies of intergenerational mobility, both terms are important, as researchers are interested in measuring how parental income is associated with the child's income, either through the causal effect of parental income or through the association between parental income and other factors, such as neighborhood quality, quality of schools, inherited human capital or peer effects.

## Decomposing the Rank-Rank Coefficient

We now turn our attention to the Rank-Rank coefficient. Using Yitzhaki's theorem once more, we have:

$$\beta_r = \frac{\mathrm{Cov}(R_c, R_p)}{\mathrm{Var}(R_p)} = \int_{t=0}^{1} \frac{\partial E[R_c|R_p = t]}{\partial t} w(t)dt$$

where

$$w(t) = \frac{12(1-t)t}{2}, \quad \int_0^1 w(t)dt = 1$$

After transforming the marginal distributions to uniform distributions, we can derive the exact weighting scheme. Comparing the weights of the IGE coefficient to the Rank-Rank coefficient, we observe that the Rank-Rank weights place most of the weight on households at the median of the parental income distribution, in contrast to the IGE that assigns most of the weight to households closer to the mean of the distribution. In addition, weights decline symmetrically as we move further away from the median and towards the extremes. Thus, similar to the $\beta_{IGE}$ weights, the Rank-Rank coefficient assigns lower weights to households closer to the top and bottom of the parental income distribution. It is worth noting that due to the fact that the median is usually less sensitive to changes in sample restrictions at the top and bottom of the distributions, this weighting scheme might explain why the Rank-Rank coefficient has been documented to be more robust for different sample restrictions (Dahl and DeLeire [2008], Chetty et al. [2014b]). Finally, compared to the IGE coefficient, the weights for cross-country comparisons are more

[6] This decomposition of the $\beta_{IGE}$ can be thought of as an omitted variable bias. In this case, bias is taken with respect to the OLS weighted causal effects of log parental income, as implied by Yitzhaki's theorem.

consistent, assigning similar weights to households at the same rank of the income distribution. Note that if the marginal distributions differ across countries, this implies that the Rank-Rank weighting scheme assigns different weights to households with the same income levels. Whether this is desired or not depends on the questions and objectives of the researcher.

As we did for the IGE coefficient, with a slight abuse of notation, we can express the Rank-Rank coefficients in terms of the underlying parent-child income elasticities. Let $\epsilon_c(u)$ and $\epsilon_p(u)$ be the elasticities of rank with respect to income for the child and parent, respectively, at a fixed $u$. Let $R_c\epsilon_c(u) = F_c(I_c)\frac{\partial F_c(I_c)}{\partial I_c}\frac{I_c}{R_c}$ and $R_p\epsilon_p(u) = F_p(I_p)\frac{\partial F_p(I_p)}{\partial I_p}\frac{I_p}{R_p}$ represent the semi-elasticities of rank with respect to income. These quantities measure how the rankings of the parent and child change in response to a percentage variation in their respective incomes, given $u$. We then rewrite the integrand as:[7]

$$
\begin{aligned}
\frac{\partial \mathrm{E}[R_c|R_p = t]}{\partial t} &= \frac{\partial \mathrm{E}[F_c(h(F_p^{-1}(t), u))|R_p = t]}{\partial t} \\
&= \int_{-\infty}^{\infty} \frac{\partial F_c(h(F_p^{-1}(t), u))P(u|R_p = t)}{\partial t} du \\
&= \mathrm{E}\left[\frac{\partial F_c}{\partial h(F_p^{-1}(t), u))}\frac{\partial h(F_p^{-1}(t), u))}{\partial F_p^{-1}(t)}\frac{1}{\frac{\partial F_p}{\partial I_p}}\middle| R_p = t\right] + \int_{\infty}^{\infty} F_c(h(F_p^{-1}(t), u)\frac{\partial P(u|R_p = t)}{\partial t} du \\
&= \mathrm{E}\left[\frac{\partial I_c}{\partial I_p}\frac{\frac{\partial R_c}{\partial I_c}}{\frac{\partial R_p}{\partial I_p}}\frac{I_c}{I_c}\frac{I_p}{I_p}\frac{R_c}{R_c}\frac{R_p}{R_p}\middle| R_p = t\right] + \int_{\infty}^{\infty} F_c(h(F_p^{-1}(t), u)\frac{\partial P(u|R_p = t)}{\partial t} du \\
&= \underbrace{E\left[\frac{R_c}{R_p}\frac{\epsilon_c(u)}{\epsilon_p(u)}\epsilon_{I_c,I_p}(u)\middle| R_p = t\right]}_{\text{The causal IGE}} + \underbrace{\int_{\infty}^{\infty} F_c(h(F_p^{-1}(t), u)\frac{\partial P(u|R_p = t)}{\partial t} du}_{\text{Other factors}}
\end{aligned}
\tag{5}
$$

Where the third equality is due both the product rule and the chain rule. The fourth equality results from dividing and multiplying by the parent's and child's income and ranks, and the definition of the parent's and child's ranks. The final equality follows from the definition of semi-elasticities. By expressing the integrand in this way, we can see the similarities and differences between the IGE coefficient and the Rank-Rank slope. First, as the child's income cumulative distribution function is monotonic, similar to the log function, we can see that the effects of other factors on the income have remained the same, only now we use the children's marginal income distribution to transform the

[7] Maasoumi et al. [2022] describes the rank-rank slope as a weighted average of $\frac{\partial E[\log I_c|\log I_p=t]}{\partial t}$, with weights that are generally positive but do not necessarily sum to 1. In our decomposition, the weights of $\frac{\partial \mathrm{E}[R_c|R_p=t]}{\partial t}$ do sum to 1. We investigate how the integrand is linked to the causal IGE

income , instead of log. Likewise, the Rank-Rank coefficient is also affected by the causal effects of the IGE, but now the IGE is multiplied by a "translation" term that converts the income elasticities to rank-elasticities.

If we care about the Rank-Rank coefficient for cross-country comparisons, the decomposition we derived above explicitly demonstrates that the Rank-Rank slope is only useful for comparisons of positional mobility. It cannot, however, speak to how similar or different the mechanisms driving this mobility are across countries.[8] For example, consider two countries with the same underlying causal mechanisms $h(I_p, u)$. Furthermore, assume that $I_p \perp\!\!\!\perp u$, which implies that the second term is zero. If the parental income distributions are different across the two countries, the Rank-Rank coefficient would still be different for two reasons. The first reason is that although the weighting scheme is the same for households with the same income rank, the regression weighting scheme puts different weights on households with the same income level. The second more substantial reason is that the way that the causal mechanisms affect rank would differ between the two countries, as the semi-elasticities are different in the causal IGE term in equation 5. Therefore, although we might motivate the usage of the Rank-Rank coefficient as a way to abstract away from the marginals, we can't avoid considering the marginals if we want to use the Rank-Rank coefficient to think about differences in the driving mechanisms of mobility between two countries.

## Discussion

This paper employs Yitzhaki's theorem (Yitzhaki [1996]) to express the Intergenerational Elasticity (IGE) and Rank-Rank coefficients as weighted averages of the causal mechanisms driving income and positional mobility. We demonstrate that interpreting cross-country comparisons using the IGE coefficient can be challenging due to the regression weighting scheme. Additionally, we establish that the Rank-Rank coefficient is readily interpretable only when researchers focus on positional mobility, without providing insights into the similarities or differences in the underlying mechanisms driving mobility across countries.

Throughout our discussion, we highlight the potential drawbacks of using linear regression coefficients as summary statistics. Linear regression may be preferred in certain cases for its efficiency and stability, even with a small number of observations. It appears, however, that in the context of intergenerational mobility comparisons, where recent research has shifted towards utilizing large administrative datasets, the choice to report regression coefficients over

[8] In theory, the Rank-Rank coefficient can be more informative on causal mechanisms that operate directly from the parental rank to the child's income rank, bypassing income levels. We refrain from discussing this interpretation here, as we were unable to identify any papers postulating such a mechanism.

estimates from more flexible and transparent methods may not always be well justified.

## 0.1 Appendix

In this section we follow Heckman et al. [2006] to show the proof of Yitzhaki's theorem (Yitzhaki [1996]).

**Theorem 1.** (Yitzhaki's theorem) Let $(Y, X)$ be i.i.d, assume $E[|X|], E[|Y|] < \infty$ assume that $E[Y|x]$ exists and differentiable for every $x \in supp(X)$. Denote $\mu = E[X]$ and let $f(x)$ be the probability density function of $X$. Then

$$\frac{\text{Cov}(Y, X)}{\text{Var}(X)} = \int_{-\infty}^{\infty} \frac{\partial E[Y|x = t]}{\partial t} w(t)dt$$

where

$$w(t) = \frac{1}{\text{Var}(X)} \int_{t}^{\infty} (x - \mu)f(x)dx = \frac{1}{\text{Var}(X)} E[X - \mu|X > t]\text{P}(X > t)dx$$

and the weights satisfy $\int_{-\infty}^{\infty} w(t) = 1$, $\lim_{t\to\infty} w(t) = 0$, $\lim_{t\to-\infty} w(t) = 0$ ,$\mu = arg \max w(t)$ and are increasing for $t < \mu$ and decreasing for $t > \mu$.

*Proof.* We follow Heckman et al. [2006]

$$\begin{aligned} \text{Cov}(Y, X) &= \text{Cov}(E[Y|x], X) \\ &= \int_{-\infty}^{\infty} E[Y|x = t](t - \mu)f_x(t)dt \end{aligned}$$

using integration by parts we have

$$\begin{aligned} \int_{-\infty}^{\infty} E[Y|x = t](t - \mu)f_x(t)dt &= \left[E[Y|x = t] \int_{-\infty}^{t} (u - \mu)f_x(u)du\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial E[Y|x = t]}{\partial t} \int_{-\infty}^{t} (u - \mu)f_x(u)dudt \\ &= -\int_{-\infty}^{\infty} \frac{\partial E[Y|x = t]}{\partial t} E[X - \mu|X < t]\text{P}(X < t)dt \end{aligned}$$

using the fact that

$$E[X - \mu] = 0 = E[X - \mu|X < t]\text{P}(X < t) + E[X - \mu|X < t]\text{P}(X > t)\text{P}(X > t)$$

we get

$$\mathrm{Cov}(Y,X) = \int_{-\infty}^{\infty} \frac{\partial E[Y|x=t]}{\partial t} E[X-\mu|X>t]\mathrm{P}(X>t)dt$$

Therefore the weights are given by

$$w(t) = \frac{1}{\mathrm{Var}(X)} \int_{t}^{\infty} (u-\mu)f(u)du = \frac{1}{\mathrm{Var}(X)} E[X-\mu|X>t]\mathrm{P}(X>t)$$

To see that the weights integerate to one, we can once more use integration by parts

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty} (t-\mu)(t-\mu)f(t)dt = \left[(t-\mu)\int_{-\infty}^{t} (u-\mu)f(u)du\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\int_{-\infty}^{t} (u-\mu)f(u)dudt \\ &= \int_{-\infty}^{\infty}\int_{t}^{\infty} (u-\mu)f(u)dudt\end{aligned}$$

which implies $\int_{-\infty}^{\infty} w(t) = 1$. The fact that the weights go to zero at the boundary of the support is obvious from the definition of the weights. To see that the weights are maximized at $t=\mu$, notice that for any $t<\mu$, we have

$$\int_{t}^{\infty} (x-\mu)f(x)dx - \int_{\mu}^{\infty} (x-\mu)f(x)dx = \int_{t}^{\mu} (x-\mu)f(x)dx < 0$$

similarly we have for any $t>\mu$

$$\int_{t}^{\infty} (x-\mu)f(x)dx - \int_{\mu}^{\infty} (x-\mu)f(x)dx = -\int_{\mu}^{t} (x-\mu)f(x)dx < 0$$

Finlay to see that the weights are increasing to the left of the mean and decreasing to it's right, we have that the first derivative is given by

$$\frac{\partial w(t)}{\partial t} = -(t-\mu)f(t)$$

which is decreasing for every $t>\mu$ and increasing for every $t<\mu$ □

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