Pull-Push Method: A new approach to Edge-Isoperimetric Problems

Sergei L. Bezrukov Nikola Kuzmanovski Jounglag Lim

Abstract

We prove a generalization of the Ahlswede-Cai local-global principle. A new technique to handle edge-isoperimetric problems is introduced which we call the pull-push method. Our main result includes all previously published results in this area as special cases with the only exception of the edge-isoperimetric problem for grids. With this we partially answer a question of Harper on local-global principles. We also describe a strategy for further generalization of our results so that the case of grids would be covered, which would completely settle Harper's question.

1 Introduction

For a finite simple graph $G = (V, E)$, sets $A, B \subseteq V$ and integer $m \geq 0$ denote

$$\begin{aligned} I_G(A, B) &= \{\{u, v\} \in E \mid u \in A, v \in B\}, \\ I_G(A) &= I_G(A, A), \\ I_G(m) &= \max_{S \subseteq V, |S|=m} |I_G(S)|, \\ \Theta(A) &= \{\{u, v\} \in E \mid u \in A, v \notin A\}, \\ \Theta(m) &= \min_{S \subseteq V, |S|=m} |\Theta(m)|. \end{aligned}$$

In the sequel the index G will be omitted whenever the graphs in question are clear from the context. The following versions of the edge-isoperimetric problem on graphs have been intensively studied in the literature:

Induced Edges Problem: For a given $m \in \{1, \ldots, |V|\}$ find a set $A \subseteq V$ such that $|A| = m$ and $I(|A|) = |I(A)|$.

Cut Edges Problem: For a given $m \in \{1, \ldots, |V|\}$ find a set $A \subseteq V$ such that $|A| = m$ and $\Theta(|A|) = |\Theta(A)|$.

Many authors have previously realized that these two problems are equivalent for regular graphs, which follows from the next assertion.

Lemma 1.1. *If $G = (V, E)$ is regular of degree d and $A \subseteq V$ then*

$$|\Theta(A)| + 2|I(A)| = d|A|.$$

A set $A \subseteq V$ is called *optimal* if $I_G(|A|) = |I_G(A)|$. We say that $G = (V, E)$ admits *nested solutions* if there exists a chain of optimal subsets $A_1 \subset A_2 \subset \cdots \subset A_{|V|}$. In this case we call the graph G *isoperimetric.*

A total order on a graph G is a bijection $\eta_G : V \to \{1, \dots, |V|\}$. For positive integers k, l with $k < l$, and $u, v \in V$, we define

$$\eta_G[k, l] = \eta_G^{-1}(\{k, \cdots, l\}),$$
$$\eta_G[k] = \eta_G^{-1}(\{1, \cdots, k\}),$$
$$u <_{\eta_G} v \text{ iff } \eta_G(u) < \eta_G(v).$$

We call $\eta_G[k]$ *initial segment* of size k of the order η_G. For an isoperimetric graph $G = (V, E)$ and its chain of nested solutions $\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{|V|}$ there is a natural total order η_G, which we call an *optimal order* on G, defined by $\eta_G^{-1}(i) = u_i$ for $\{u_i\} = A_i \setminus A_{i-1}$, $i \in \{1, \dots, |V|\}$. Note that an optimal order depends not only on G, but also on the chain of nested solutions which is not unique, in general.

We study edge-isoperimetric problems on Cartesian products of graphs. For graphs G and H their *Cartesian product* is a graph $G \Box H$ defined as follows:

$$V_{G\Box H} = V_G \times V_H$$
$$E_{G\Box H} = \{((v_G, v_H), (u_G, u_H)) \bigm| v_G = u_G \text{ and } (v_H, u_H) \in E_H, \text{ or } v_H = u_H \text{ and } (v_G, u_G) \in E_G\}.$$

Denote $G^d = G \Box \cdots \Box G$ (d times), where G^0 is a simple graph with one vertex.

We define a *Lexicographic order* of dimension d, $\mathcal{L}^d$ on $\mathbb{R}^d$, such that for tuples of real numbers $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ we say that $x <_{\mathcal{L}^d} y$ iff $x_1 = y_1, \dots, x_i = y_i$ and $x_{i+1} < y_{i+1}$ for some $i \in \{0, 1, \dots, d-1\}$.

Suppose that $G_1, \dots, G_d$ are isoperimetric graphs with optimal orders $\eta_1, \dots, \eta_d$, respectively, and let $G = G_1 \Box \cdots \Box G_d$. The following total order $\mathcal{L}_G^d$ on the Cartesian product of graphs, called *Lexicographic order on* G of dimension d, plays an important role in various extremal problems. For $v = (v_1, \dots, v_d) \in V_G$ and $u = (u_1, \dots, u_d) \in V_G$, we write $v <_{\mathcal{L}_G^d} u$ iff $(\eta_1(v_1), \dots, \eta_d(v_d)) <_{\mathcal{L}^d} (\eta_1(u_1), \dots, \eta_d(u_d))$. The next theorem is one of the earliest results on edge-isoperimetric problems.

Theorem 1.2 (Harper [14], Bernstein [4], Hart [16])**.** *The order $\mathcal{L}_G^d$ is optimal for $G = K_2^d$.*

A similar result was obtained later for the powers of larger cliques and products of some other graphs, where the lexicographic order was proved to be optimal. At the end of the twentieth century Ahlswede and Cai established the optimality of the lexicographic order in the most general form, called by them the *local-global principle*, from which Theorem 1.2 and many similar results follow.

Theorem 1.3 (Ahlswede-Cai [3])**.** *Let $G_1, \dots, G_d$ be isoperimetric graphs and $G = G_1 \Box \cdots \Box G_d$. If the order $\mathcal{L}^2_{G_i \Box G_j}$ is optimal for all i, j with $1 \le i < j \le d$, then $\mathcal{L}_G^d$ is optimal for $d \geq 3$.*

Remark 1.4. Actually, Theorem 1.3 was originally stated in a more general form for sub-modular and super-modular functions over finite sets (see [3] and [2]), and the functions I_G and Θ_G belong to this category. Although our results can be also generalized for sub-modular and super-modular functions, we will only be dealing with functions I_G and Θ_G relevant to the edge-isoperimetric problems.

Despite many applications of Theorem 1.3, there are graphs for whose Cartesian products the lexicographic order is not optimal. One of such graphs is a grid, i.e. the Cartesian product of paths, which was later generalized to the product of arbitrary trees.

Theorem 1.5 (Bollobás-Leader [10])**.** *If G is a path then G^d has nested solutions for the Induced Edges problem.*

Theorem 1.6 (Ahlswede-Bezrukov [1]). *If $G_1, \ldots, G_d$ are trees then $G_1 \square \cdots \square G_d$ has nested solutions for the Induced Edges problem.*

Theorems 1.5 and 1.6 are proved combinatorially by induction on d. In [10] the authors also solved the Cut Edges problem, but they used calculus for this. The Cut Edges problem does not have nested solutions for the grid. Ahlswede and Bezrukov generalized Theorem 1.5 and gave a simpler proof based on a new approach. Later, some other graphs were found for which the optimal order is different from the Lexicographic one.

Theorem 1.7 (Bezrukov-Das-Elsässer [8]). *If G is the Petersen graph then G^d has nested solutions.*

Theorem 1.8 (Bezrukov-Das-Elsässer [8]). *If G_1 is the Petersen graph and $G_2 = K_2$ then $G_1^{d_1} \square G_2^{d_2}$ has nested solutions.*

Theorem 1.9 (Carlson [12]). *If G is the cycle on 5 vertices then G^d has nested solutions.*

The proof technique used in Theorems 1.7, 1.8 and 1.9 is also based on induction on d, where the base case $d = 2$ was considered specially. This approach can also be considered as a local-global principle. The proof of the induction steps involved a large number of cases, subcases and subsubcases, along with a decent amount of computations, and is based on specific properties of the considered graphs.

Our result is the most general local-global principle. It is valid for a variety of total orders, out of which the lexicographic order and the other orders appearing in theorems 1.7, 1.8 and 1.9 are special cases. The proof of our result is purely geometric and involves just a few cases or computations. We use a new technique that we call the pull-push method. This technique is somewhat different from the one used to prove the earlier theorems and does not depend on the structure of the involved graphs. Harper in [15] asked if Theorem 1.3 can be extended to prove Theorems 1.7 and 1.5. Our main result answers this affirmatively for Theorem 1.7. In the last section of the paper, we lay out a strategy on how to further generalize the main result to handle Theorem 1.5 as well.

The edge-isoperimetric problems have a lot of applications, some of which can be found in [15] and [6]. The applications include the wirelength problem, the bisection width and the edge congestion problem of graph embedding, modeling the brain, the cutwidth problem and graph partitioning. The graphs in Theorem 1.7 are called *folded Petersen networks* and have been studied in [20, 22, 21, 18, 13] as a communication-efficient interconnection network topology for multiprocessors. The graphs in Theorem 1.8 are called *folded Petersen cubes* and have been studied in [13, 19].

The paper has 7 sections. In the next section we introduce the necessary definitions to formulate the main result. In section 3 we present the geometric structure of the problem. In section 4 we go over some well-known results on compression which we use in the paper. The main result is proved in section 5. Section 6 is devoted to some corollaries of our main result and can be considered as a short survey of results in the area. Concluding remarks and possible directions for generalizing our results are put in section 7.

2 δ-sequences, partitions and statement of the main result

Let $\mathcal{O}^d$ be a total order on $\mathbb{R}^d$ and let $G_1, \ldots, G_d$ be isoperimetric graphs with optimal orders $\eta_1, \ldots, \eta_d$. Consider $G = G_1 \square \cdots \square G_d$ and define the total order $\mathcal{O}^d_G$ on G so that $v <_{\mathcal{O}^d_G} u$ for $v = (v_1, \ldots, v_d) \in V_G$ and $u = (u_1, \ldots, u_d) \in V_G$, iff $(\eta_1(v_1), \ldots, \eta_d(v_d)) <_{\mathcal{O}^d} (\eta_1(u_1), \ldots, \eta_d(u_d))$. We call $\mathcal{O}^d_G$ the *induced by $\mathcal{O}^d$ order* on G.

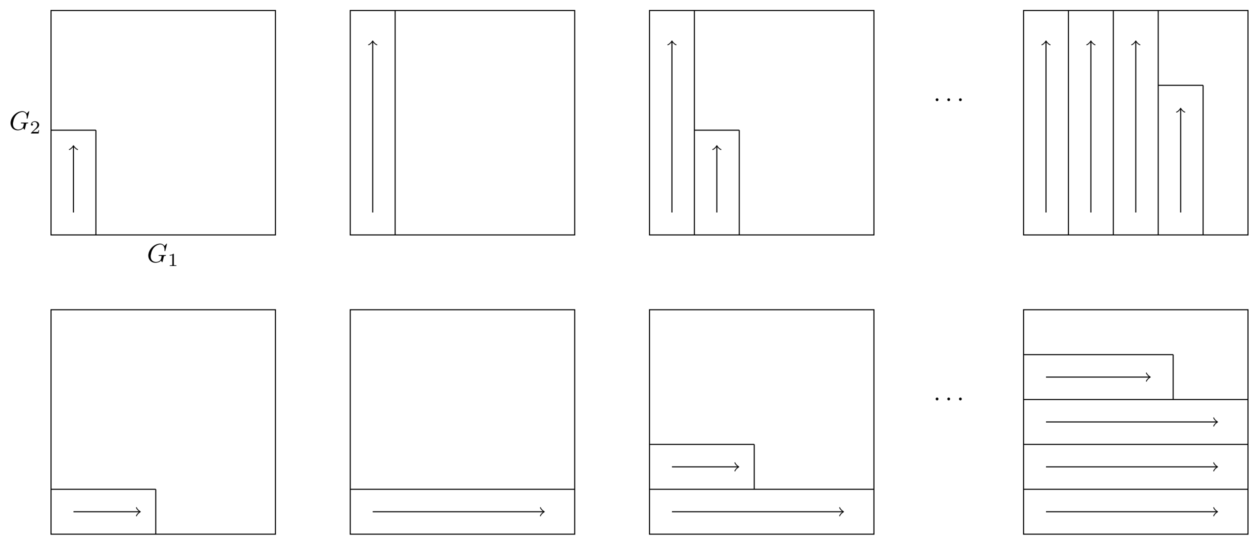

Figure 2.1: Geometric interpretation of lexicographic (top) and colexicographic (bottom) orders in two dimensions

Denote by $\mathfrak{S}_d$ the symmetric group of degree d, i.e., the set of all permutations on $\{1,\dots,d\}$. For $\pi \in \mathfrak{S}_d$ we define the *domination order* $\mathcal{D}^{\pi,d}$ of dimension d so that $x <_{\mathcal{D}^{\pi,d}} y$ for $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ and $y = (y_1,\dots,y_d) \in \mathbb{R}^d$ iff $(x_{\pi(1)},\dots,x_{\pi(d)}) <_{\mathcal{L}^d} (y_{\pi(1)},\dots,y_{\pi(d)})$. Respectively, for $G = G_1 \Box \cdots \Box G_d$ we obtain the domination order $\mathcal{D}_G^{\pi,d}$ on G induced by $\mathcal{D}^{\pi,d}$. For example, $\mathcal{D}^{\mathrm{id},d} = \mathcal{L}^d$, where id is the identity permutation on $\{1,\dots,d\}$. If $\pi(i) = (d-i+1)$ then $\mathcal{D}^{\pi,d}$ is the *colexicographic order*. For $d = 2$ we have just two domination orders, the lexicographic and colexicographic ones. A geometric interpretation of these orders is given in Figure 2.1. For $d = 3$ there are 6 dominations orders including the lexicographic and colexicographic ones. A geometric interpretation of these orders is given in Figure 2.2.

Corollary 2.1. *(of Theorem 1.3) Let $G_1,\dots,G_d$ be isoperimetric graphs, $G = G_1 \Box \cdots \Box G_d$, and $\pi \in \mathfrak{S}_d$. If for all $i,j \in \{1,\dots,d\}$ with $i < j$ the order $\mathcal{L}^2_{G_{\pi(i)} \Box G_{\pi(j)}}$ is optimal, then $\mathcal{D}_G^{\pi,d}$ is optimal for $d \geq 3$.*

Proof. Denote $H = G_{\pi(1)} \Box \cdots \Box G_{\pi(d)}$ and define $\psi : G \to H$ such that

$$\psi((v_1,\dots,v_d)) = (v_{\pi(1)},\dots,v_{\pi(d)}).$$

It is easily seen that ψ is a graph isomorphism. For $x, y \in V_G$ one has $x <_{\mathcal{D}_G^{\pi,d}} y$ iff $\psi(x) <_{\mathcal{L}_H^d} \psi(y)$. Since $\mathcal{L}_H^d$ is optimal for H we conclude $\mathcal{D}_G^{\pi,d}$ is optimal for G. $\square$

For a graph $G = (V,E)$ and integer $m \in \{1,\dots,|V|\}$ denote

$$\delta_G(m) = I_G(m) - I_G(m-1)$$

with $\delta_G(1) = 0$. If G is isoperimetric with optimal order $\mathcal{O}_G$ and $v \in V$ denote

$$\Delta_G(v) = \delta_G(\mathcal{O}_G(v))$$

Lemma 2.2 (Bezrukov [7])**.** *If $G = (V,E)$ is isoperimetric then $\delta(i+1) - \delta(i) \leq 1$ for $i = 1,\dots,|V|-1$.*

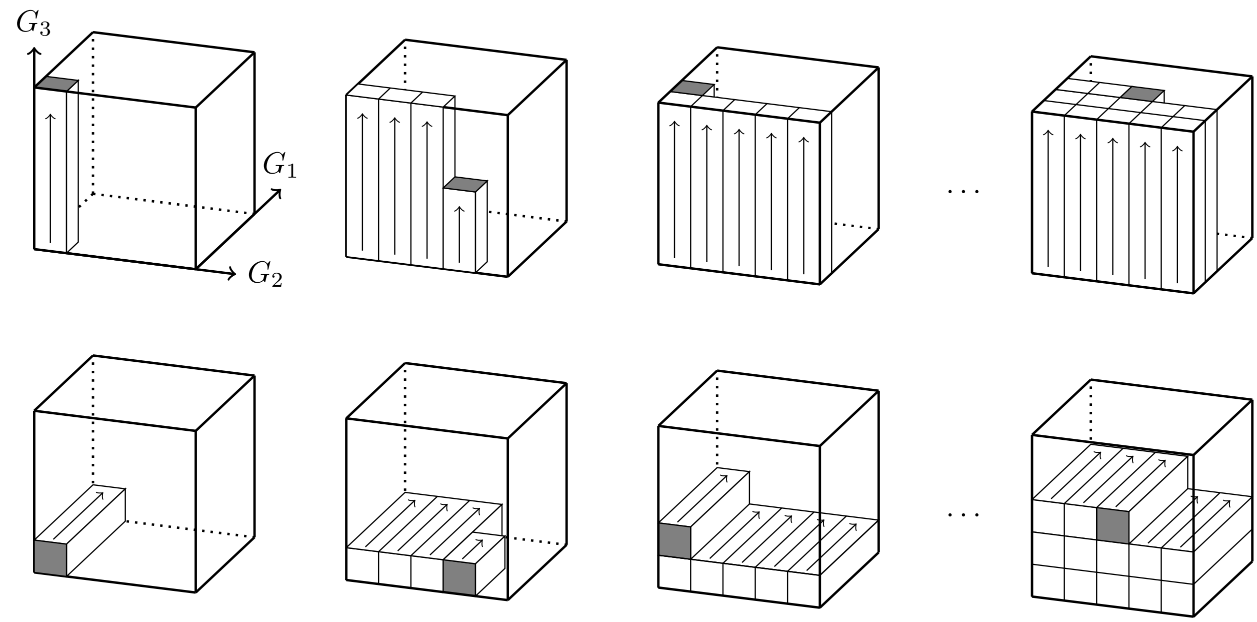

Figure 2.2: Geometric interpretation of lexicographic (top) and colexicographic (bottom) orders in three dimensions

Let $G = (V, E)$ be isoperimetric with optimal order $\mathcal{O}_G$. For integers $a, b \in \{1, \ldots, |V|\}$ with $a < b$ denote by $\delta_G[a, b] = (\delta(a), \ldots, \delta(b))$, a *monotonic segment*, where

1. For all $i \in \{a, \ldots, b-1\}$ we have $\delta(i+1) - \delta(i) = 1$.

2. If $a > 1$ then $\delta(a) - \delta(a-1) < 1$.

3. If $b < |V|$ then $\delta(b+1) - \delta(b) < 1$.

In other words, a monotonic segment is a longest increasing sequence of δ-values. We say that $\mathcal{O}_G[a, b] \subseteq V$ is a *monotonic set* from a to b. For two monotonic sets $\mathcal{O}_G[a_1, b_1]$ and $\mathcal{O}_G[a_2, b_2]$, we write $\mathcal{O}_G[a_1, b_1] <_{\mathcal{O}_G} \mathcal{O}_G[a_2, b_2]$ iff $b_1 < a_2$. It is easily seen that V is uniquely partitioned into monotonic sets, hence, δ_G - into monotonic segments. We call such a partition the *standard monotonic partition* of G, and we denote it by $\mathfrak{M}_G = \{\mathcal{O}_G[a_1, b_1], \ldots, \mathcal{O}_G[a_k, b_k]\}$. Here are a few examples, where monotonic segments are underlined:

$$\begin{aligned}
\delta_{K_n} &= (\underline{0, 1, 2, 3, \ldots, n-1}), \\
\delta_{P_n} &= (\underline{0, 1}, \underline{1}, \underline{1}, \ldots, \underline{1}), \\
\delta_{\text{Petersen}} &= (\underline{0, 1}, \underline{1}, \underline{1, 2}, \underline{1, 2}, \underline{2}, \underline{2, 3}).
\end{aligned}$$

Thus, $|\mathfrak{M}_{K_n}| = 1$, $|\mathfrak{M}_{P_n}| = n - 1$, and $|\mathfrak{M}_{\text{Petersen}}| = 6$. It turns out that $\mathfrak{M}_G$ has interesting properties.

Theorem 2.3 (Bezrukov-Bulatovic-Kuzmanovski [5]). *Let G be isoperimetric and consider its standard monotonic partition $\mathfrak{M}_G = \{\mathcal{O}_G[a_1, b_1] <_{\mathcal{O}_G} \cdots <_{\mathcal{O}_G} \mathcal{O}_G[a_k, b_k]\}$. For $i \in \{1, \ldots, k\}$ one has*

1. *The graph $(\mathcal{O}_G[a_i, b_i], I_G(\mathcal{O}_G[a_i, b_i]))$ is a clique, and hence it is isoperimetric with induced order $\mathcal{O}_i$, such that $u <_{\mathcal{O}_i} v$ for $u, v \in \mathcal{O}_G[a_i, b_i]$ iff $u <_{\mathcal{O}_G} v$.*

2. *For all* $v \in \mathcal{O}_G[a_i, b_i]$ *we have* $|I(\mathcal{O}_G[a_1, b_{i-1}], \{v\})| = \delta(a_i)$.

We extend the concept of monotonic partitions to more general partitions $\mathfrak{P}_G = \{\mathcal{O}_G[a_1, b_1] <_{\mathcal{O}_G} \cdots <_{\mathcal{O}_G} \mathcal{O}_G[a_k, b_k]\}$ of V, where $\mathcal{O}_G[a_i, b_i]$ are not necessarily monotonic sets. We say that $\mathfrak{P}_G$ is an *isoperimetric partition* if

1. The graph $(\mathcal{O}_G[a_i, b_i], I_G(\mathcal{O}_G[a_i, b_i]))$ is isoperimetric with induced order $\mathcal{O}_i$, such that $u <_{\mathcal{O}_G} v$ for $u, v \in \mathcal{O}_G[a_i, b_i]$ iff $u <_{\mathcal{O}_i} v$.
2. For every $v \in \mathcal{O}_G[a_i, b_i]$ it holds $|I(\mathcal{O}_G[a_1, b_{i-1}], \{v\})| = \delta(a_i)$.

For example, for the Petersen graph we can partition the δ-sequence in two parts

$$\delta_{\text{Petersen}} = (\underline{0, 1, 1, 1, 2}, \underline{1, 2, 2, 2, 3}).$$

Each of the parts induces a cycle of length 5 studied in [12].

Lemma 2.4. *Let* $G = (V, E)$ *be an isoperimetric graph with an isoperimetric partition* $\mathfrak{P}_G = \{\mathcal{O}_G[a_1, b_1] <_{\mathcal{O}_G} \cdots <_{\mathcal{O}_G} \mathcal{O}_G[a_k, b_k]\}$. *Then for the graph* $H_i = (\mathcal{O}_G[a_i, b_i], I_G(\mathcal{O}_G[a_i, b_i]))$, $i = 1, \ldots, k$, *and* $x, y \in \mathcal{O}_G[a_i, b_i]$ *with* $y <_{\mathcal{O}_G} x$ *it holds that*

$$\Delta_G(x) - \Delta_G(y) = \Delta_{H_i}(x) - \Delta_{H_i}(y).$$

Proof. Indeed, $\Delta_G(x) - \Delta_G(y) = \Delta_G(a_i) + \Delta_{H_i}(x) - \Delta_G(a_i) - \Delta_{H_i}(y) = \Delta_{H_i}(x) - \Delta_{H_i}(y)$. □

We call the first and last vertex of $\mathcal{O}[a, b] \in \mathfrak{P}_G$ the *start* and *end* of $\mathcal{O}[a, b]$. Further denote by $\mathfrak{T}_G = \{\mathcal{O}_G^{-1}(a_1), \ldots, \mathcal{O}_G^{-1}(a_k)\}$ the *start set* of the partition $\mathfrak{P}_G$. We say that $\mathfrak{P}_G$ is *non-decreasing* if for every $i \in \{1, \ldots, k\}$ the sequence $\delta_{(\mathcal{O}_G[a_i, b_i], I_G(\mathcal{O}_G[a_i, b_i]))}$ is non-decreasing. We say that $\mathfrak{P}_G$ is *regular* if

$$\delta_{(\mathcal{O}_G[a_1, b_1], I_G(\mathcal{O}_G[a_1, b_1]))} = \delta_{(\mathcal{O}_G[a_k, b_k], I_G(\mathcal{O}_G[a_k, b_k]))}$$

Note that for any isoperimetric graph G, the standard monotonic partition $\mathfrak{M}_G$ is an isoperimetric and non-decreasing partition. However, $\mathfrak{M}_G$ is not always regular, as the next example shows. Consider the graph G which is the union of two disjoint cliques K_5 and K_4. Then

$$\delta_G = (0, 1, 2, 3, 4, 0, 1, 2, 3).$$

For $G = G_1 \square \cdots \square G_d$ and nonempty subset $S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ we define the *sub-product* of G of dimension k as $G_S = G_{i_1} \square \cdots \square G_{i_k}$. Let $\mathcal{O}_G$ and $\mathcal{O}_{G_S}$ be total orders on G and G_S, respectively. We say that $\mathcal{O}_G$ is *consistent* with $\mathcal{O}_{G_S}$ if $x <_{\mathcal{O}_G} y$ for $x = (x_1, \ldots, x_d) \in V_G$ and $y = (y_1, \ldots, y_d) \in V_G$ with $x_j = y_j$ for $j \notin S$ implies $(x_{i_1}, \ldots, x_{i_k}) <_{\mathcal{O}_{G_S}} (y_{i_1}, \ldots, y_{i_k})$.

Suppose $G_i = (V_i, E_i)$ for $i = 1, \ldots, d$ is an isoperimetric graph with an optimal order $\mathcal{O}_{G_i}$, and let $\mathfrak{P}_{G_i}$ be its isoperimetric partition with the start set $\mathfrak{T}_{G_i}$. Define a *block* of G to be an element of

$$\{Z_1 \times \cdots \times Z_d \mid Z_1 \in \mathfrak{P}_{G_i}, \ldots, Z_d \in \mathfrak{P}_{G_d}\}$$

and a *start* of G to be an element of

$$\{(z_1, \ldots, z_d) \mid z_1 \in \mathfrak{T}_{G_1}, \ldots, z_d \in \mathfrak{T}_{G_d}\}.$$

Note that blocks of G and starts of G are in a bijective correspondence. We say that the partitions $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$ compose a *domination collection* if for each block $B = Z_1 \times \cdots \times Z_d$ of G one has:

1. For each nonempty $S = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, d\}$ there is an optimal domination order $\mathcal{D}_{H_S}^{\pi_S,k}$ on graph $H_S = (Z_{i_1} \times \cdots \times Z_{i_k}, I_G(Z_{i_1} \times \cdots \times Z_{i_k}))$.

2. For any nonempty sets $S_1 = \{i_1 < \cdots < i_{k_1}\} \subseteq \{1, \ldots, d\}$ and $S_2 = \{i_1 < \cdots < i_{k_2}\} \subseteq \{1, \ldots, d\}$ with $S_1 \subset S_2$, the order $\mathcal{D}_{H_{S_2}}^{\pi_{S_2},k_2}$ is consistent with $\mathcal{D}_{H_{S_1}}^{\pi_{S_1},k_1}$.

We can obtain a domination collection by defining domination orders for each block of G. Note that for any $S \subset \{1, \ldots, d\}$ the vertices of H_S form a block in the subproduct G_S. Also note that $\mathfrak{P}_{G_{i_1}}, \ldots, \mathfrak{P}_{G_{i_k}}$ is a domination collection on G_S. Hence, each of $2^d - 1$ subproducts has a start set, blocks, and domination collection. For brevity, we denote the domination order on a block B of some k-dimensional subproduct by $\mathcal{D}_B$.

Now, we introduce a new total order for which we prove a local-global principle in the next sections. For nonempty $S = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, d\}$ define the *block lexicographic order* $\mathcal{BL}_{G_S}^k$ of dimension k on G_S such that for $u, v \in V_G$ we have $u <_{\mathcal{BL}_{G_S}^k} v$ iff

1. If u and v are in the same block B, then $u <_{\mathcal{D}_B} v$.

2. If u and v are in different blocks, say B_u and B_v, with respective starts z_u and z_v, then $z_u <_{\mathcal{L}_G^k} z_v$.

We abbreviate $\mathcal{BL}_{G_{\{1,\ldots,d\}}}^d$ to $\mathcal{BL}_G^d$. Just one more definition is needed to state our main result below. Suppose that for $d \geq 3$ and $i = 1, \ldots, d$ we have an isoperimetric graph $G_i = (V_i, E_i)$ with optimal order $\mathcal{O}_{G_i}$ and isoperimetric partition $\mathfrak{P}_{G_i} = \{\mathcal{O}_{G_i}[a_{i,1}, b_{i,1}] <_{\mathcal{O}_{G_i}} \cdots <_{\mathcal{O}_{G_i}} \mathcal{O}_{G_i}[a_{i,n_i}, b_{i,n_i}]\}$. We say that $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$ is a *regular domination collection* if the following hold:

1. The partition $\mathfrak{P}_{G_i}$ is regular for $i = 2, \ldots, d-1$.

2. With

$$B_1 = O_{G_2}[a_{2,1}, b_{2,1}] \times \cdots \times O_{G_{d-1}}[a_{d-1,1}, b_{d-1,1}],$$
$$B_2 = O_{G_2}[a_{2,n_2}, b_{2,n_2}] \times \cdots \times O_{G_{d-1}}[a_{d-1,n_{d-1}}, b_{d-1,n_{d-1}}].$$

   the domination order on B_1 is the same as the domination order on B_2. That is, if $\mathcal{D}_{B_1}$ is induced by a permutation π_1 and $\mathcal{D}_{B_2}$ is induced by a permutation π_2, then $\pi_1 = \pi_2$.

Theorem 2.5. *Let $G_1, \ldots, G_d$ be isoperimetric graphs and let their corresponding isoperimetric partitions be $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$. Denote $G = G_1 \square \cdots \square G_d$ and suppose that the following hold:*

1. *For $i = 1, \ldots, d-1$ the partition $\mathfrak{P}_{G_i}$ is non-decreasing.*

2. *The collection of partitions $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$ is a regular domination collection.*

If for $i, j \in \{1, \ldots, d\}$ with $i < j$ the order $\mathcal{BL}_{G_i \square G_j}^2$ is optimal, then the order $\mathcal{BL}_G^d$ is optimal for $d \geq 3$.

3 Geometry of the problem

The objects introduced here play a key role in understanding transformations and proof techniques used in the paper. Throughout this section we assume that $G_1, \ldots, G_d$ are isoperimetric graphs with domination collection $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$ and start sets $\mathfrak{T}_{G_1}, \ldots, \mathfrak{T}_{G_d}$. Denote $G = G_1 \square \cdots \square G_d =$

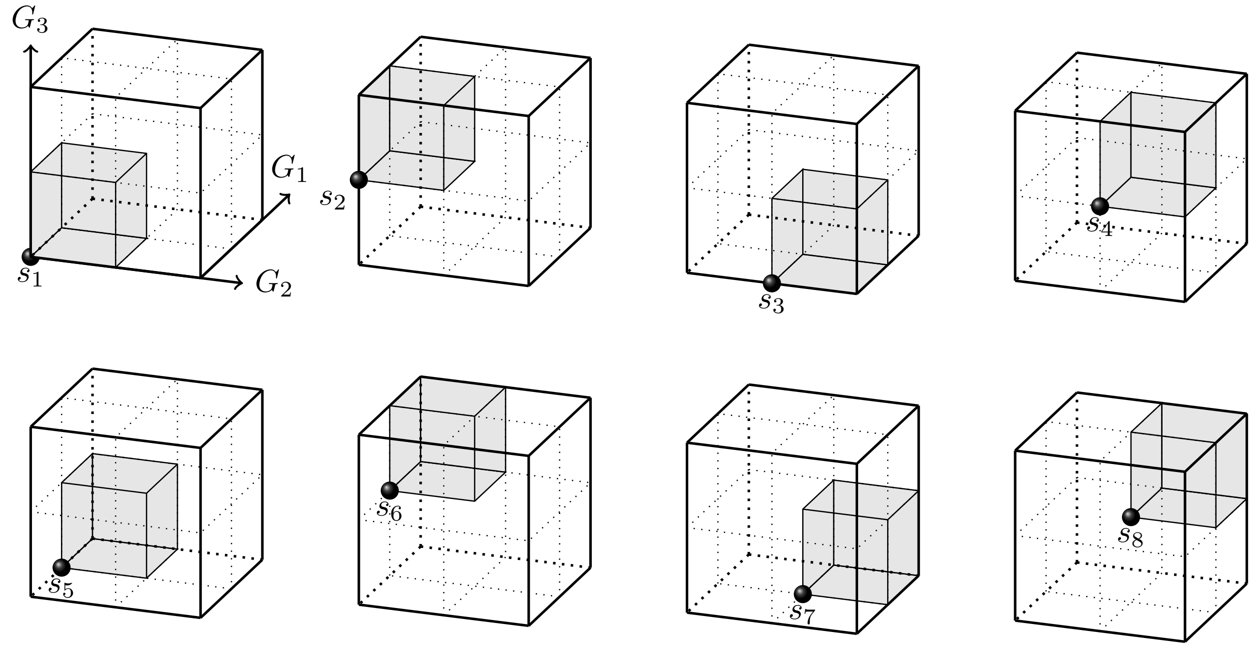

Figure 3.1: Ordering of blocks of G for $d = 3$, $|\mathfrak{P}_{G_1}| = |\mathfrak{P}_{G_2}| = |\mathfrak{P}_{G_3}| = 2$, and $\mathfrak{T}_{G_1} \times \mathfrak{T}_{G_2} \times \mathfrak{T}_{G_3} = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ listed in increasing order.

(V, E). We can view G as a d-dimensional rectangular body where the vectors with integer coordinates (see Figure 3.1) correspond to the vertices of G, and the vector coordinates along each coordinate axis are ordered according to the isoperimetric order on G_i.

For $u \in V$ denote by $\mathrm{Block}_G(u)$ the d-dimensional block containing u. Since the blocks of G partition V, for every $u \in V$ its containing block is defined uniquely. Denote by $\mathrm{Start}_G(B)$ the start vertex of block B and by $\mathrm{Start}_G(u)$ the start vertex of the block $\mathrm{Block}_G(u)$. Figure 3.1 shows some blocks and their starts.

For $i \in \{1, \ldots, d\}$ define the i-th *bone* and the *skeleton* of a block $B = Z_1 \times \cdots \times Z_d$ as

$$\begin{aligned} \mathrm{Bone}_G(B, i) &= \left(\prod_{j=1}^{i-1} \{\mathrm{Start}_{G_j}(Z_j)\} \right) \times Z_i \times \left(\prod_{j=i+1}^{d} \{\mathrm{Start}_{G_j}(Z_j)\} \right) \\ \mathrm{Skeleton}(B) &= \bigcup_{i=1}^{d} \mathrm{Bone}_G(B, i). \end{aligned}$$

Figure 3.2 shows a visualization of the bones and skeleton of a block.

If $C = Y_1 \times \cdots \times Y_d$ is some block other than B with $Y_i = Z_i$, we say that blocks B and C *share* the i-th bone in the *product decomposition* of C and B. Figure 3.3 shows examples of bone sharing.

For $i \in \{1, \ldots, d\}$, $\sigma \in \mathfrak{T}_{G_1} \times \cdots \times \mathfrak{T}_{G_{i-1}}$, $\tau \in \mathfrak{T}_{G_{i+1}} \times \cdots \times \mathfrak{T}_{G_d}$, and $s \in \mathfrak{T}_{G_i}$ we consider the vertices of the form $(\sigma, s, \tau) \in V$. Then define the *stack* in direction i at $\alpha = (\sigma, i, \tau)$ as

$$\mathrm{Stack}_G(\alpha) = \bigcup_{s \in \mathfrak{T}_{G_i}} \mathrm{Block}((\sigma, s, \tau)).$$

Some stacks are visualized in Figure 3.4. For the left stack one has $\sigma = (s_1, s_2)$ where s_1 is the

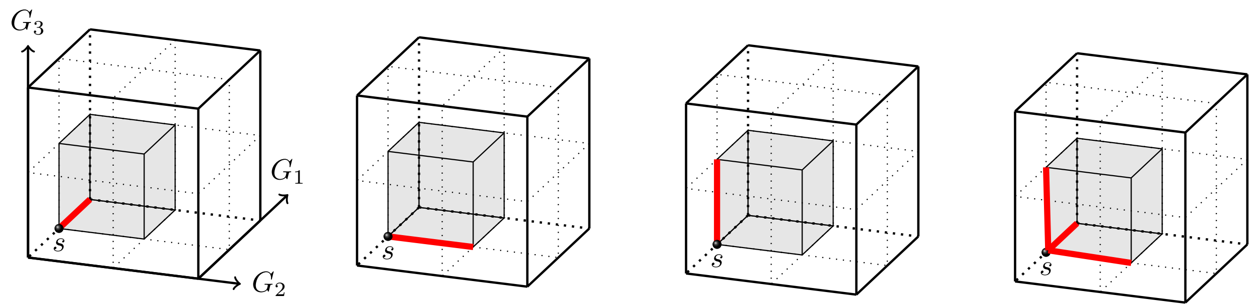

Figure 3.2: Bones and the skeleton of the block with start s.

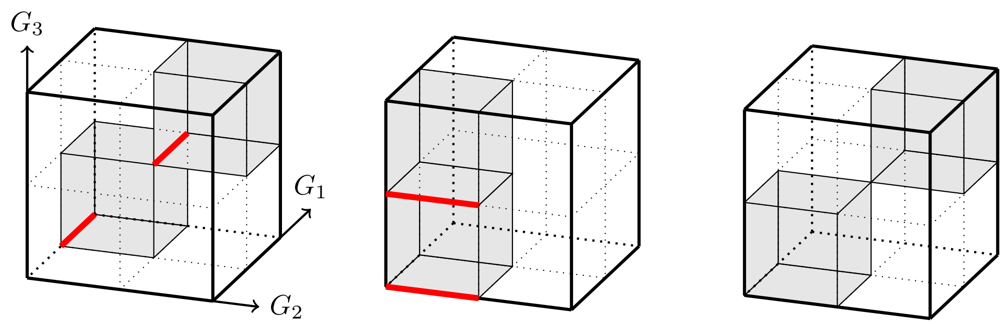

Figure 3.3: Blocks sharing 1st bone, 2nd bone, and no bone.

fourth start of G_1 and s_2 is the first start of G_2, and there is no τ. For the other stack $\tau = (s_3, s_4)$ where s_3 is the second start of G_2 and s_4 is the first start of G_3, and there is no σ.

The last objects we will need are called *slices*. For $i \in \{1, \dots, |\mathfrak{T}_{G_1}|\}$ denote by s_i the i-th start of G_1. The i-th *slice* of G is defined as the union of all blocks B whose first coordinate of $\mathrm{Start}_G(B)$ is s_i, and denoted by $\mathrm{Slice}_G(i)$. In other terms a slice is the union of all blocks that share the 1st bone. Some slices are shown in Figure 3.5.

For blocks B_1 and B_2 of G we say that $B_1 <_{\mathcal{BL}^d_G} B_2$ iff $\mathrm{Start}_G(B_1) <_{\mathcal{BL}^d_G} \mathrm{Start}_G(B_2)$. This way we obtain a total order on the set of blocks of G, which is illustrated in Figure 3.1

Since a stack is a disjoint union of blocks, all the blocks of a stack become totally ordered. For an example, where blocks are ordered according to the indices of their starts, see the left part of

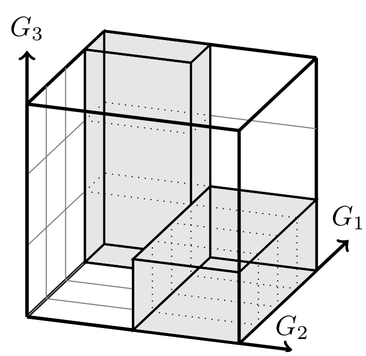

Figure 3.4: Visualization of stacks of G for $d = 3$, the left stack is in direction 3, the other one is in direction 1.

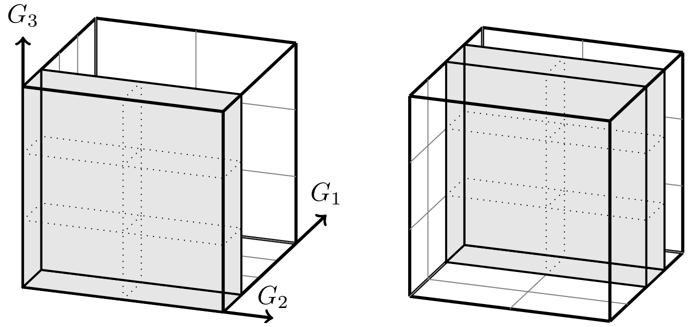

Figure 3.5: $\mathrm{Slice}_G(1)$ and $\mathrm{Slice}_G(3)$.

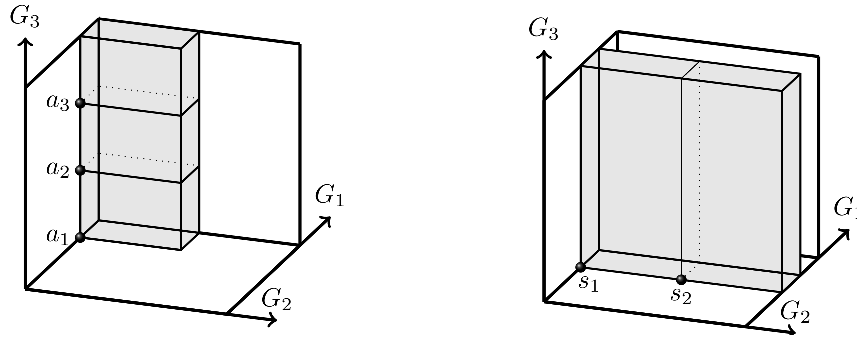

Figure 3.6: Ordering of blocks in a stack and ordering of stacks in a slice.

Figure 3.6. For stacks $\mathrm{Stack}_G(\alpha)$ and $\mathrm{Stack}_G(\beta)$ in direction d we write $\mathrm{Stack}_G(\alpha) <_{\mathcal{BL}_G^d} \mathrm{Stack}_G(\beta)$ iff the first block of $\mathrm{Stack}_G(\alpha)$ is less (in the above defined order) than the first block of $\mathrm{Stack}_G(\beta)$. This ordering can be observed in the right part of Figure 3.6.

Finally, we need to define an order on slices. One can view a slice as disjoint union of blocks. Alternatively, a slice can be viewed as disjoint union of stacks in the d-th direction (see Figure 3.6). We write $\mathrm{Slice}_G(i) <_{\mathcal{BL}_G^d} \mathrm{Slice}_G(j)$ iff $i < j$.

Summing up, V_G is partitioned by slices, each slice is partitioned by stacks in the d-th direction, and each stack is partitioned by blocks. Furthermore, there is a total order of slices, stacks, and blocks induced by the order $\mathcal{BL}_G^d$.

4 Compression

Let $G_1, \ldots, G_d$ be graphs with some total orders $\mathcal{O}_{G_1}, \ldots, \mathcal{O}_{G_d}$ on their vertex sets, and let $G = (V_G, E_G) = G_1 \square \cdots \square G_d$. For $S = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, d\}$ denote by $\mathcal{O}_{G_S}$ the induced order on $G_S = G_{i_1} \square \cdots \square G_{i_k}$. Denote $\overline{S} = \{j_1 < \cdots < j_{d-k}\} = \{1, \ldots, d\} \backslash S$. For $x = (x_{j_1}, \ldots, x_{j_{d-k}}) \in V_{G_{\overline{S}}}$ we define the *cut* or *section* of G_S at x to be the graph $G_S(x) = (V_{G_S(x)}, E_{G_S(x)})$, where

$$\begin{aligned} V_{G_S(x)} &= \{(v_1, \ldots, v_d) \in V_G \bigm| \text{with } v_q = x_q \text{ for } q \in \overline{S}\}, \\ E_{G_S(x)} &= I_G(V_{G_S(x)}). \end{aligned}$$

Note that the graphs $G_S(x)$ are isomorphic to G_S for all x. This isomorphism provides a total order $\mathcal{O}_{G_S(x)}$ on $G_S(x)$ induced by the order $\mathcal{O}_{G_S}$ on G_S.

For a set $A \subseteq V_G$ we define the *compression* $\mathrm{Comp}_{G,\mathcal{O}_{G_S}}(A)$ of G with respect to $\mathcal{O}_{G_S}$ as the operation that replaces the vertices in each $A \cap G_S$ with an initial segment (of order $\mathcal{O}_{G_S(x)}$ inside the cut $G_S(x)$) of the same size. More formally we use the definition from [15],

$$\mathrm{Comp}_{G,\mathcal{O}_{G_S}}(A) = \bigcup_{x \in V_{G_{\{1,\dots,d\}\setminus S}}} \mathcal{O}_{G_S(x)}[|A \cap V_{G_S(x)}|].$$

The following lemmas 4.1 - 4.3 have been discovered and used by many authors, see, e.g. [15], so we present them without a proof here.

Lemma 4.1. *If $\mathcal{O}_{G_S}$ is optimal then for any $A \subseteq V_G$ it holds:*

1. $|\mathrm{Comp}_{G,\mathcal{O}_{G_S}}(A)| = |A|$.
2. *If $B \subseteq A$ then* $\mathrm{Comp}_{G,\mathcal{O}_{G_S}}(B) \subseteq \mathrm{Comp}_{G,\mathcal{O}_{G_S}}(A)$.
3. $|I_G(A)| \leq |I_G(\mathrm{Comp}_{G,\mathcal{O}_{G_S}}(A))|$.

The next lemma informally says that if the orders in question are consistent, then after a finite time of applying the compression one gets a *stable* set.

Lemma 4.2. *Let $S_0, \dots, S_{p-1} \subset \{1, \dots, d\}$ and $\mathcal{S} = (S_0, \dots, S_{p-1})$. For $A \subseteq V_G$ and $n \geq 1$ define*

$$\mathrm{Comp}^n_{G,\mathcal{S}}(A) = \begin{cases} \mathrm{Comp}_{G,\mathcal{O}_{G_{S_0}}}(A) & \text{if } n = 1, \\ \mathrm{Comp}_{G,\mathcal{O}_{G_{S_{n \bmod p}}}}(\mathrm{Comp}^{n-1}_{G,\mathcal{S}}(A)) & \text{if } n \geq 2. \end{cases}$$

If the order $\mathcal{O}_{G_{S_q}}$ is consistent with $\mathcal{O}_G$ for $q = 1, \dots, n$ then the sequence $(\mathrm{Comp}^n_{G,\mathcal{S}}(A))_{n=1}^{\infty}$ is eventually constant. In other words, there is n_0 such that for all $n \geq n_0$ one has $\mathrm{Comp}^{n+1}_{G,\mathcal{S}}(A) = \mathrm{Comp}^n_{G,\mathcal{S}}(A)$.

Denote by $\mathrm{Comp}_{G,\mathcal{S}}(A)$ the resulting stable set in Lemma 4.2. We say that $A \subseteq V_G$ is *compressed* if for $\mathcal{S} = (\{1\}, \dots, \{d\})$ we have $\mathrm{Comp}_{G,\mathcal{S}}(A) = A$. Furthermore, we say that A is *strongly compressed* if $\mathrm{Comp}_{G,S}(A) = A$ for any proper subset S of $\{1, \dots, d\}$. In the sequel we will be looking for solutions to the edge-isoperimetric problem that are compressed sets.

For optimal orders $\mathcal{O}_{G_1}, \dots, \mathcal{O}_{G_d}$ and $A \subseteq V_G$ define the weight of A as

$$\omega_G(A) = \sum_{(i_1,\dots,i_n) \in A} \left(\sum_{j=1}^{n} \Delta_{G_j}(i_j) \right).$$

Lemma 4.3. *If $\mathcal{O}_{G_1}, \dots, \mathcal{O}_{G_d}$ are optimal orders and $A \subseteq V_G$ is a compressed set then $|I_G(A)| = \omega_G(A)$.*

This lemma immediately implies the following assertion.

Corollary 4.4. *Let $A \subseteq V_G$ be compressed, $T_1 \subseteq V_G \setminus A$, and $T_2 \subseteq A$. If $(A \cup T_1) \setminus T_2$ is compressed then*

$$|I_G(A)| - |I_G((A \cup T_1) \setminus T_2)| = \omega_G(A) - \omega_G((A \cup T_1) \setminus T_2) = \omega_G(T_2) - \omega_G(T_1).$$

We extend compression to the geometric objects introduced in the previous section. Suppose that $A \subseteq V_G$ is compressed and there is a domination collection $\mathfrak{P}_{G_1}, \dots, \mathfrak{P}_{G_d}$. Let $B_1, \dots, B_p$ be all blocks of G ordered so that $B_a <_{\mathcal{BL}^d_G} B_b$ whenever $a < b$. Denote by r be the largest block number such that $A \cap B_r \neq \emptyset$. We say that A is *block compressed* if $B_i \subseteq A$ for $i < r$.

Consider a slice $\mathrm{Slice}_G(q)$ and let $B_1, \dots, B_{p_q}$ be its blocks ordered so that $B_a <_{\mathcal{BL}^d_G} B_b$ whenever $a < b$. Denote by r_q be the largest block number such that $A \cap B_{r_q} \neq \emptyset$. We say that A is *slice compressed* if for each $q = 1, \dots, |\mathfrak{P}_{G_1}|$ and $i < r_q$ it holds $B_i \subseteq A$. Thus, if A is slice compressed then it is block compressed.

5 Proof of the Main result

Let $G_1, \ldots, G_d$ be isoperimetric graphs with a regular domination collection of non-decreasing isoperimetric partitions $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$. We also assume that the order $\mathcal{BL}^2_{G_i \square G_j}$ is optimal for all $i, j \in \{1, \ldots, d\}$ with $i < j$ and let $A \subseteq V_G$ be an optimal set.

The theorem is proved by introducing a series of operations that transform the set A into the initial segment of order $\mathcal{BL}^d_G$ of the same size without reducing the number of induced edges. First, we make A slice-compressed (Theorem 5.6), then block-compressed (Theorem 5.10), and finally use a special transformation of the resulting set. We assume that the theorem holds for all $d' < d$ and proceed by induction on d for $d \geq 3$.

Lemma 5.1. *The order $\mathcal{BL}^d_G$ is consistent with $\mathcal{BL}^k_{G_S}$ for any $S = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, d\}$.*

Proof. Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be vertices of V_G with $x <_{\mathcal{BL}^d_G} y$ and $x_j = y_j$ for $j \notin S$. Denote $x' = (x_{i_1}, \ldots, x_{i_k})$ and $y' = (y_{i_1}, \ldots, y_{i_k})$. If $\mathrm{Block}_G(x) <_{\mathcal{BL}^d_{G_S}} \mathrm{Block}_G(y)$ then $\mathrm{Block}_{G_S}(x') <_{\mathcal{BL}^k_{G_S}} \mathrm{Block}_{G_S}(y')$ since lexicographic order is consistent. This implies $x' <_{\mathcal{BL}^k_{G_S}} y'$.

If $\mathrm{Block}_G(x) = \mathrm{Block}_G(y)$ then $\mathrm{Block}_{G_S}(x') = \mathrm{Block}_{G_S}(y')$, and the domination order $\mathcal{D}_{\mathrm{Block}_G(x)}$ is consistent with the domination order $\mathcal{D}_{\mathrm{Block}_{G_S}(x')}$, since the partitions $\mathfrak{P}_{G_1}, \ldots, \mathfrak{P}_{G_d}$ form a domination collection. Thus, $x' <_{\mathcal{BL}^k_{G_S}} y'$. $\square$

This lemma along with lemmas 4.1, 4.2 implies that there is no loss of generality to assume that $A \subseteq V_G$ is strongly compressed. To make A slice-compressed, several auxiliary results are needed. We say that two blocks B_1 and B_2 in the same stack $\mathrm{Stack}_G(\alpha)$ are *consecutive*, if there is no block $B_3 \subseteq \mathrm{Stack}_G(\alpha)$ such that $B_1 <_{\mathcal{BL}^d_G} B_3 <_{\mathcal{BL}^d_G} B_2$.

Lemma 5.2. *Let $A \subseteq V_G$ be strongly compressed and $B_1 <_{\mathcal{BL}^d_G} B_2$ be consecutive blocks of a stack in some direction i. If $B_2 \cap A \neq \emptyset$ then* $\mathrm{Skeleton}_G(B_1) \subseteq A$.

Proof. Note that $\mathrm{Start}_G(B_2) \in A$, since $B_2 \cap A \neq \emptyset$ and A is strongly compressed. We show that all bones of B_1 are in A. Let $\mathrm{Start}_G(B_1) = (s_1, \ldots, s_d)$ and $x = (s_1, \ldots, s_{j-1}, x_j, s_{j+1}, \ldots, s_d) \in \mathrm{Bone}_G(B_1, j)$ for some $j \in \{1, \ldots, d\}$. So, at least $d-1$ coordinates of x and $\mathrm{Start}_G(B_1)$ are the same. Also, all coordinates except the i-th one of $\mathrm{Start}_G(B_1)$ and $\mathrm{Start}_G(B_2)$ match because the blocks are in the same stack in direction i. Hence, $\mathrm{Start}_G(B_2)$ and x share at least $d - 2 \geq 1$ equal coordinates. Since A is strongly compressed, $x <_{\mathcal{BL}^d_G} \mathrm{Start}_G(B_2)$, and $\mathrm{Start}_G(B_2)$ and x match in at least 1 coordinate, we conclude $x \in A$. $\square$

Lemma 5.3. *Let $A \subseteq V_G$ be strongly compressed and blocks B_1 and B_2 with $B_1 <_{\mathcal{BL}^d_G} B_2$ share the i-th bone for some $i \in \{1, \ldots, d\}$ and* $\mathrm{Bone}_G(B_2, i) \subseteq A$. *Then $B_1 \subseteq A$.*

Proof. Let $\mathrm{Start}_G(B_2) = (s_1, \ldots, s_d)$ and $x = (x_1, \ldots, x_i, \ldots, x_d) \in B_1$. Then we have

$$y = (s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_d) \in \mathrm{Bone}_G(B_2, i).$$

Since x and y match in the i-th coordinate and $x <_{\mathcal{BL}^d_G} y$ we conclude $x \in A$. This implies $B_1 \subseteq A$. $\square$

Corollary 5.4. *Let $A \subseteq V_G$ be strongly compressed and blocks $B_1 <_{\mathcal{BL}^d_G} B_2$ share a bone. Let* $\mathrm{Stack}_G(\alpha)$ *be a stack containing B_2 and $B_2 <_{\mathcal{BL}^d_G} B_3$ for some block $B_3 \subseteq$* $\mathrm{Stack}_G(\alpha)$. *If $B_1 \not\subseteq A$ and $B_2 \cap A \neq \emptyset$ then $B_3 \cap A = \emptyset$.*

Proof. For the contrary, assume $B_3 \cap A \neq \emptyset$. Then $\mathrm{Skeleton}_G(B_2) \subseteq A$, by Lemma 5.2. Since B_1 and B_2 share a bone, Lemma 5.3 implies $B_1 \subseteq A$, which is a contradiction. $\square$

For a slice $\mathrm{Slice}_G(q)$ and blocks $B_1 <_{\mathcal{BL}_G^d} B_2$ in it we say B_1 and B_2 are *consecutive* in slice $\mathrm{Slice}_G(q)$ if there is no block B_3 in $\mathrm{Slice}_G(q)$ with $B_1 <_{\mathcal{BL}_G^d} B_3 <_{\mathcal{BL}_G^d} B_2$.

Lemma 5.5. *Let blocks $B_1 <_{\mathcal{BL}_G^d} B_2$ be in slice $\mathrm{Slice}_G(q)$. If $B_1 \not\subseteq A$ and $A \cap B_2 \neq \emptyset$, then B_1 and B_2 are consecutive in $\mathrm{Slice}_G(q)$.*

Proof. Let $\mathrm{Stack}_G(\alpha)$ and $\mathrm{Stack}_G(\beta)$ be stacks of $\mathrm{Slice}_G(q)$ in direction d that contain B_1 and B_2, respectively. If $\mathrm{Stack}_G(\alpha) = \mathrm{Stack}_G(\beta)$ then the statement follows from lemmas 5.2 and 5.3. So assume $\mathrm{Stack}_G(\alpha) \neq \mathrm{Stack}_G(\beta)$.

Let B_1' be the last block of $\mathrm{Stack}_G(\alpha)$ and B_2' be the first block of $\mathrm{Stack}_G(\beta)$. We show that $B_1' = B_1$ and $B_2' = B_2$. Indeed, if $B_2' \neq B_2$ then since $\mathrm{Skeleton}_G(B_2') \subseteq A$ by Lemma 5.2, we get $B_1 \subseteq A$ by Lemma 5.3, a contradiction. Also, if $B_1' \neq B_1$ then by the definition of slices, B_2 and B_1' share the first bone. Since $B_2 \cap A \neq \emptyset$ and A is strongly compressed, for all $i \in \{2, \ldots, d\}$ we have $\mathrm{Bone}_G(B_1', i) \subseteq A$. Thus, $B_1 \subseteq A$ by Lemma 5.3, since $d \geq 3$ and B_1 and B_1' share the j-th bone for all $j \in \{1, \ldots, d-1\}$. This implies $B_1 \subseteq A$, a contradiction.

It remains to show is that there is no stack $\mathrm{Stack}_G(\gamma)$ with $\mathrm{Stack}_G(\alpha) <_{\mathcal{BL}_G^d} \mathrm{Stack}_G(\gamma) <_{\mathcal{BL}_G^d} \mathrm{Stack}_G(\beta)$. Assume for the contrary that this is not the case and let B_3 be the last block in $\mathrm{Stack}_G(\gamma)$. We get $\mathrm{Bone}_G(B_3, j) \subseteq A$ for all $j \in \{2, \ldots, d\}$, since A is strongly compressed and B_3 and B_2 share the first bone. If B_3 is the only block in $\mathrm{Stack}_G(\gamma)$, then B_1 and B_3 share the d-th bone. Hence, $B_1 \subseteq A$ by Lemma 5.3, a contradiction. So, suppose that there is another block $B_4 \subseteq \mathrm{Stack}_G(\gamma)$. Then B_3 and B_4 share the j-th bone for all $j \in \{1, \ldots, d-1\}$. Lemma 5.3 implies $B_4 \subseteq A$, since $d \geq 3$. However, Lemma 5.3 implies $B_1 \subseteq A$, since B_1 and B_4 share the first bone. The obtained contradiction completes the proof. $\square$

The above lemmas are used as a basis for establishing the next results by using the pull-push method.

Theorem 5.6. *For any strongly compressed set $A \subseteq V_G$ there exist a strongly compressed and slice-compressed set $B \subseteq V_G$ such that $|A| = |B|$ and $|I_G(A)| \leq |I_G(B)|$.*

Remark 5.7. The proof of Theorem 5.6 has three steps. First, we use Lemma 5.5 to consider two consecutive cubes. Then we are going to move vertices to the earlier block from the later one. This is done in two steps, we first do a pull and then a push. The pull introduces a new set A' that is compressed similar to A, but not strongly compressed. We then use compression on A' to pull vertices to the earlier block from the later one and obtain a new set D'. The push deals with how we transfer information gained from the pull on A' to the set A. The push compares the pulled vertices with corresponding vertices that come later in the block lexicographic order.

Properties of strong compression imply that the number of induced edges by a set obtained after pulling cannot decrease. Similarly, the non-decreasing property of isoperimetric partitions of graphs in the product guarantee that the pushing operation also does not decrease the number of induced edges. This way, applying both operations to an optimal set results in another optimal set satisfying some structural properties which make it looking closer to an initial segment of the block lexicographic order.

We use the pull-push method three more times in Theorem 5.10.

Proof. Let $B_1 <_{\mathcal{BL}_G^d} \cdots <_{\mathcal{BL}_G^d} B_p$ be the blocks of some slice $\mathrm{Slice}_G(q)$ and let r be the largest index for which $A \cap B_r \neq \emptyset$. Lemma 5.5 implies $B_1, \ldots, B_{r-2} \subseteq A$. Omitting trivial cases we assume $r \geq 2$ and $B_{r-1} \not\subseteq A$. We will pump vertices from B_r to B_{r-1} by using the pull-push method.

Let $S = \{2, \ldots, d\}$ and $B_r = Z_1 \times \cdots \times Z_d$. Note that Z_1 is the partition of G_1 that all blocks in $\mathrm{Slice}_G(q)$ share. Denote by $x \in Z_1$ the first vertex in the order $\mathcal{O}_{G_1}$, such that $V_{G_S(x)} \cap B_{r-1} \not\subseteq A$ and by $y \in Z_1$ the last vertex in the order $\mathcal{O}_{G_1}$ such that $V_{G_S(y)} \cap B_r \cap A \neq \emptyset$. Therefore, for all $v \in Z_1$ with $v <_{\mathcal{O}_{G_1}} x$ we have $V_{G_S(v)} \cap B_{r-1} \subseteq A$ and for all $v \in Z_1$ with $v >_{\mathcal{O}_{G_1}} y$ we have $V_{G_S(v)} \cap B_r \cap A = \emptyset$. Note that x is defined with respect to B_{r-1} and y with respect to B_r. Also note that $x >_{\mathcal{O}_{G_1}} y$, since otherwise $V_{G_S(x)} \cap B_{r-1} \subseteq A$ because A is strongly compressed. We are going to pump vertices from $V_{G_S(y)} \cap B_r$ to $G_S(x) \cap B_{r-1}$ in the two following steps.

Pull: Let W be the projection of $V_{G_S(x)} \cap B_{r-1} \cap A$ to $V_{G_S(y)} \cap B_{r-1}$, that is

$$W = \{(y, v_2, \ldots, v_d) \mid (x, v_2 \ldots, v_d) \in G_S(x) \cap B_{r-1} \cap A\}.$$

Denote by E the set of all $d-1$ dimensional blocks in B_{r-1} such that

$$E = \bigcup_{\substack{v \in Z_1 \\ v \geq_{\mathcal{O}_{G_1}} y}} V_{G_S(v)} \cap B_{r-1},$$

and denote by R the set of all slices greater than $\mathrm{Slice}_G(q)$, that is

$$R = \bigcup_{t > q} \mathrm{Slice}_G(t).$$

Consider the set

$$A' = (A \setminus (E \cup R)) \cup W.$$

This set has the following properties:

1. A' is compressed. To prove this we show that if $v = (v_1, \ldots, v_d) \in A'$ and $u = (u_1, \ldots, u_d) \in V_G$ with $u_1 \leq_{\mathcal{O}_{G_1}} v_1, \ldots, u_d \leq_{\mathcal{O}_{G_d}} v_d$, then $u \in A'$. First, note that B_r and B_{r-1} are the last blocks that have a nonempty intersection with A', since we removed R from A to get A'. If $v \notin B_r$ or $v \notin B_{r-1}$ then $u \in A'$, since A is strongly compressed and we only modified B_{r-1} to get A'. If $v \in B_{r-1}$ then $u \in A'$, since $W \subseteq G_S(y) \cap B_{r-1} \cap A$. So, it remains to consider the case $v \in B_r$ and $u \in B_{r-1}$. Actually, it is sufficient to consider only the case when B_r and B_{r-1} are in the same stack, since otherwise there is some $i \in \{1, \ldots, d\}$ for which $u_i >_{\mathcal{O}_{G_i}} v_i$. So, suppose that B_r and B_{r-1} are in the same stack in direction i. Note that $i \neq 1$ because B_r and B_{r-1} are in the same slice. Without loss of generality, assume that $u_1 = v_1,\ \ldots,\ u_{i-1} = v_{i-1},\ u_{i+1} = v_{i+1},\ \ldots,\ u_d = v_d$. Since A is strongly compressed, we get $u \in A$. If $u_1 <_{\mathcal{O}_{G_1}} y$ then $u \in A'$, by the definition of A' because $u \notin E$. If $u_1 = y$ then $(x, u_2, \ldots, u_d) \in A$ because A is strongly compressed and $d \geq 3$. Therefore, in all cases $u \in A'$ by the definition of W and A', implying A' is compressed.

2. The set $A' \cap V_{G_S(y)} \cap B_{r-1}$ forms an initial segment of the domination order in block $V_{G_S(y)} \cap B_{r-1}$.

3. The set $A' \cap V_{G_S(y)} \cap B_r$ forms an initial segment of the domination order in block $V_{G_S(y)} \cap B_r$.

4. For all $l < r-1$ it holds that $B_l \subseteq A'$.

5. For all $l > r$ it holds that $B_l \cap A' = \emptyset$.

Denote $D' = \mathrm{Comp}_{G,\mathcal{O}_{G_S}}(A')$. The set D' is obtained from A' by moving n vertices from B_r to B_{r-1}, where

$$a = \min\{|A' \cap (V_{G_S(y)} \cap B_r)|, |(V_{G_S(y)} \cap B_{r-1}) \setminus (A' \cap G_S(y) \cap B_{r-1})|\}$$

Note that D' is compressed, since A' is compressed. Denote by T_r the set of the last a vertices of $A' \cap V_{G_S(y)} \cap B_r$ in the domination order on $V_{G_S(y)} \cap B_r$, and denote by T_{r-1} the set of the first a vertices of $(V_{G_S(y)} \cap B_{r-1}) \setminus A'$ in the domination order on $V_{G_S(y)} \cap B_{r-1}$. In these terms, $D' = (A' \setminus T_r) \cup T_{r-1}$. Taking into account Corollary 4.4 one has

$$0 \leq |I_G(D')| - |I_G(A')| = \omega_G(T_{r-1}) - \omega_G(T_r).$$

Push: Denote

$$\begin{aligned} T_x &= \{(x, v_2, \ldots, v_d) \mid (y, v_2, \ldots, v_d) \in T_{r-1}\} \\ D &= (A \setminus T_r) \cup T_x. \end{aligned}$$

Note that block B_r, as well as any other block, belongs to d stacks of G (in different directions). If there is a block B in one of those stacks such that $B_r <_{\mathcal{BL}_G^d} B$ then $B \cap A = \emptyset$ by Corollary 5.4 because B_{r-1} and B_r share a bone. Therefore, $A \setminus T_r$ is compressed. By definition of x we get that D is also compressed.

Now, Corollary 4.4, Lemma 2.4, and the non-decreasing property of $\mathfrak{P}_{G_1}$ imply

$$\begin{aligned} |I_G(D)| - |I_G(A)| &= \omega_G(T_x) - \omega_G(T_r) \\ &= \omega_G(T_x) - \omega_G(T_{r-1}) + \omega_G(T_{r-1}) - \omega_G(T_r) \\ &\geq a(\Delta_{G_1}(x) - \Delta_{G_1}(y)) + 0 \\ &= a(\Delta_{(Z_1, I_{G_1}(Z_1))}(x) - \Delta_{(Z_1, I_{G_1}(Z_1))}(y)) \\ &\geq 0. \end{aligned}$$

Apply to D the strong compression operation and denote by F the resulting set. One has $|F| = |D|$ and $|I_G(F)| \geq |I_G(D)|$. Applying the described transformation over and over we obtain a stable set B of the same size, which is strongly compressed and slice compressed, and for which $|I_G(A)| \leq |I_G(B)|$. $\square$

We now apply a similar approach for reducing the problem of constructing optimal sets to block compressed sets.

Lemma 5.8. *Let $A \subseteq V_G$ be a strongly compressed set and let $(s_1, \ldots, s_d)$ be the start of the first block of some slice* $\mathrm{Slice}_G(q)$. *If for $i \in \{2, \ldots, d\}$ it holds*

$$\{s_1\} \times \cdots \times \{s_{i-1}\} \times V_{G_i} \times \{s_{i+1}\} \times \cdots \times \{s_d\} \subseteq A,$$

then $\mathrm{Slice}_G(p) \subseteq A$ *for all $p < q$.*

Proof. Indeed, if $x = (x_1, \ldots, x_i, \ldots, x_d) \in \mathrm{Slice}_G(p)$ then $(s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_d) \in A$ and $x_1 <_{\mathcal{O}_{G_1}} s_1$. Hence, $x \in A$, since A is strongly compressed. $\square$

Lemma 5.9. *Let $A \subseteq V_G$ be strongly compressed and slice-compressed and slices* $\mathrm{Slice}_G(p) <_{\mathcal{BL}_G^d} \mathrm{Slice}_G(q)$ *be such that* $\mathrm{Slice}_G(q) \cap A \neq \emptyset$ *and* $\mathrm{Slice}_G(p) \not\subseteq A$. *If* $\mathrm{Stack}_G(\alpha_q)$ *and* $\mathrm{Stack}_G(\alpha_p)$ *are the first and last stacks of* $\mathrm{Slice}_G(q)$ *and* $\mathrm{Slice}_G(p)$*, respectively, then*

1. *For any stack* $\mathrm{Stack}_G(\beta) \subseteq \mathrm{Slice}_G(q)$ *different from* $\mathrm{Stack}_G(\alpha_q)$ *it holds* $\mathrm{Stack}_G(\beta) \cap A = \emptyset$.

2. *For any stack* $\mathrm{Stack}_G(\beta) \subseteq \mathrm{Slice}_G(p)$ *different from* $\mathrm{Stack}_G(\alpha_p)$*, it holds* $\mathrm{Stack}_G(\beta) \subseteq A$.

3. *The slices* $\mathrm{Slice}_G(p)$ *and* $\mathrm{Slice}_G(q)$ *are consecutive, that is,* $q = p + 1$.

Proof. Assume to the contrary that the first claim does not hold. Since A is slice compressed, one has $\mathrm{Stack}_G(\alpha_q) \subseteq A$. By Lemma 5.8, $\mathrm{Slice}_G(p) \subseteq A$, a contradiction. For the second one note that $\mathrm{Slice}_G(q) \cap A \neq \emptyset$ implies $\mathrm{Stack}_G(\alpha_p) \cap A \neq \emptyset$, since A is strongly compressed. Thus, all stacks preceding $\mathrm{Stack}_G(\alpha_p)$ must be in A. For the last statement, assume that $q > p + 1$. Then $\mathrm{Slice}_G(p) <_{\mathcal{BL}^d_G} \mathrm{Slice}_G(q-1)$. Let $(s_1, \dots, s_d)$ be the start of the first block in $\mathrm{Slice}_G(q-1)$. Since A is strongly compressed we have

$$\{s_1\} \times \cdots \times \{s_{d-1}\} \times V_{G_d} \subseteq A.$$

For $x = (x_1, \dots, x_d) \in \mathrm{Slice}_G(p)$ one has $(s_1, \dots, s_{d-1}, x_d) \in A$ and $x_1 < s_1$. Hence, $x \in A$, since A is strongly compressed. Therefore, $\mathrm{Slice}_G(p) \subseteq A$, a contradiction. $\square$

Theorem 5.10. *For any strongly and slice compressed set* $A \subseteq V_G$ *there exists a strongly and block compressed set* $B \subseteq V_G$ *such that* $|A| = |B|$ *and* $|I_G(A)| \leq |I_G(B)|$.

Proof. Let q be the largest integer such that $\mathrm{Slice}_G(q) \cap A \neq \emptyset$. Omitting trivial cases we assume $q > 1$ and $\mathrm{Slice}_G(q-1) \not\subseteq A$. By Lemma 5.9, if $q \geq 3$ we have $\mathrm{Slice}_G(1), \dots, \mathrm{Slice}_G(q-2) \subseteq A$. Let $\mathrm{Stack}_G(\alpha_q)$ be the first stack of $\mathrm{Slice}_G(q)$, and $\mathrm{Stack}_G(\alpha_{q-1})$ be the last stack in $\mathrm{Slice}_G(q-1)$. Note that we always take stacks in the d-th direction when talking about them in slices. Lemma 5.9 tells that for every stack $\mathrm{Stack}_G(\beta) \subseteq \mathrm{Slice}_G(q)$ different from $\mathrm{Stack}_G(\alpha_q)$ one has $\mathrm{Stack}_G(\beta) \cap A = \emptyset$. By the same lemma, $\mathrm{Stack}_G(\beta) \subseteq A$ for every stack $\mathrm{Stack}_G(\beta) \subseteq \mathrm{Slice}_G(q-1)$ different from $\mathrm{Stack}_G(\alpha_{q-1})$. We will apply pull-push approach to pump vertices from $\mathrm{Stack}_G(\alpha_q)$ to $\mathrm{Stack}_G(\alpha_{q-1})$.

Let $B_q = Z_1 \times \cdots \times Z_d$ and $B_{q-1} = Y_1 \times \cdots \times Y_d$ be the first blocks of $\mathrm{Stack}_G(\alpha_q)$ and $\mathrm{Stack}_G(\alpha_{q-1})$, respectively. Denote

$$H_Z = (Z_2, I_{G_2}(Z_2)) \square \cdots \square (Z_{d-1}, I_{G_{d-1}}(Z_{d-1})),$$
$$H_Y = (Y_2, I_{G_2}(Y_2)) \square \cdots \square (Y_{d-1}, I_{G_{d-1}}(Y_{d-1})).$$

It follows that Z_i and Y_i are, respectively, the first and the last parts of the partition $\mathfrak{P}_{G_i}$ for $i \in \{2, \dots, d-1\}$. These notations are illustrated in Figure 5.1. Remember that according to our assumption, the partitions $\mathfrak{P}_{G_1}, \dots, \mathfrak{P}_{G_d}$ form a regular domination collection, which means that:

1. For $i = 2, \dots, d-1$ the partition $\mathfrak{P}_{G_i}$ is regular, i.e., $\delta_{(Z_i, I_{G_i}(Z_i))} = \delta_{(Y_i, I_{G_i}(Y_i))}$.

2. There is a $\tau \in \mathfrak{S}_{d-2}$ such that the domination orders on H_Z and H_Y are both induced by $\mathcal{D}^{\tau, d-2}$. Denote $\mathcal{D} = \mathcal{D}^{\tau, d-2}$ for brevity.

Set $S = \{1, d\}$ and let $y \in V_{H_Y}$ be the first vertex in the order $\mathcal{D}_{H_Y}$ for which $V_{G_{S(y)}} \cap \mathrm{Stack}_G(\alpha_{q-1}) \not\subseteq A$. Also, let $z \in V_{H_Z}$ be the last vertex in the order $\mathcal{D}_{H_Z}$ such that $V_{G_{S(z)}} \cap \mathrm{Stack}_G(\alpha_q) \cap A \neq \emptyset$. It follows that $V_{G_{S(v)}} \cap \mathrm{Stack}_G(\alpha_{q-1}) \subseteq A$ for every $v \in V_{H_Y}$ with $v <_{\mathcal{D}_{H_Y}} y$ and $V_{G_{S(v)}} \cap \mathrm{Stack}_G(\alpha_q) \cap A = \emptyset$ for every $v \in V_{H_Z}$ with $v >_{\mathcal{D}_{H_Z}} z$. Denote $k = \mathcal{D}_{H_Y}(y)$ and $l = \mathcal{D}_{H_Z}(z)$ and let $\mathrm{Stack}_G(\psi)$ be the first stack of the slice $\mathrm{Slice}_G(q-1)$.

Case 1: Assume $k < l$. Let W be the projection of $A \cap \mathrm{Stack}_G(\alpha_{q-1})$ to $\mathrm{Stack}_G(\psi)$, i.e.,

$$W = \{(v_1, \mathcal{D}^{-1}_{H_Z}(\mathcal{D}_{H_Y}(u)), v_d) \mid (v_1, u, v_d) \in \mathrm{Stack}_G(\alpha_{q-1}) \cap A\}.$$

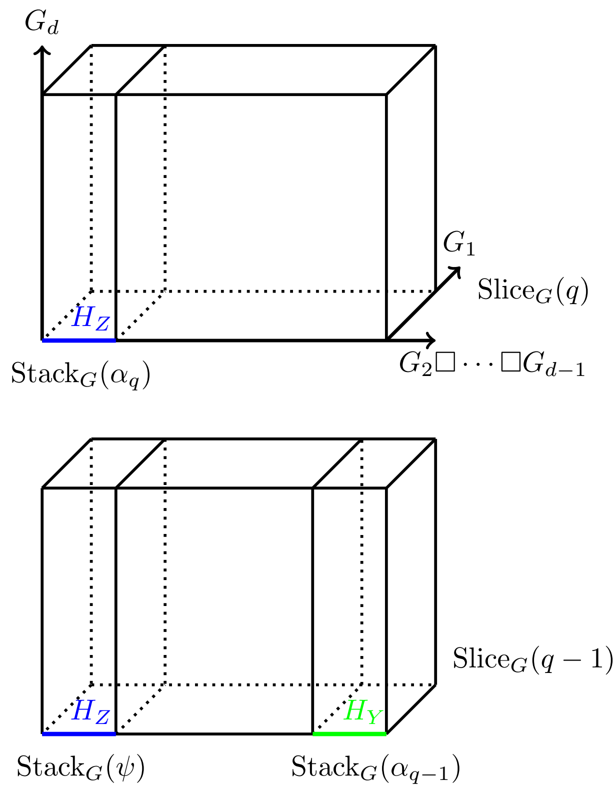

Figure 5.1: Two consecutive slices and stacks that appear in the proof of Theorem 5.10.

Consider the set A',

$$A' = (A \setminus \text{Slice}_G(q-1)) \cup W.$$

Set A' has the following:

1. A' is compressed. To show this, let $v = (v_1, v_2, \ldots, v_d) \in A'$ and $u = (u_1, u_2, \ldots, u_d) \in V_G$ such that $\exists i \in \{1, 2, \ldots, d\}$ with $u_i <_{\mathcal{O}_{G_i}} v_i$ and $u_j = v_j$ for all $j \neq i$. Without loss of generality we assume that $u \notin \text{Slice}_G(p)$ for $p < q-1$ since $\text{Slice}_G(p) \subseteq A'$. First, suppose $v \in \text{Slice}_G(q-1)$. If $u \in \text{Slice}_G(q-1)$, then $u, v \in \text{Stack}_G(\psi)$, hence $u \in A'$ as A is compressed. Now suppose $v \in \text{Slice}_G(q)$. If $u \in \text{Slice}_G(q)$, then $u \in A'$ since A is compressed. If $u \in \text{Slice}_G(q-1)$ then $i = 1$, hence $u = (u_1, v_2, v_3, \ldots, v_d)$. Since $\text{Slice}_G(q) \cap A' = \text{Slice}_G(q) \cap A$, we have $v \in A$. So, any vertex $(u'_1, u'_2, \ldots, u'_{d-1}, v_d) \in V_G \cap \text{Slice}_G(q-1)$ with $u'_1 <_{\mathcal{O}_{G_1}} v_1$ is in A as A is strongly compressed. In particular, $(u_1, \mathcal{D}^{-1}_{H_Y}(\mathcal{D}_{H_Z}(u_2, u_3, \ldots, u_{d-1})), v_d) \in A$. This implies $u \in A'$.

2. For every $v \in V_{H_Z}$ the set $V_{G_{S(v)}} \cap A' \cap \text{Stack}_G(\alpha_q)$ is an initial segment of order $\mathcal{BL}^2_{G_{S(v)}}$ restricted to $\text{Stack}_G(\alpha_q)$, since A is strongly compressed.

3. For every $v \in V_{H_Z}$ the set $V_{G_{S(v)}} \cap A' \cap \text{Stack}_G(\psi)$ is an initial segment of order $\mathcal{BL}^2_{G_{S(v)}}$ restricted to $\text{Stack}_G(\psi)$, since A is strongly compressed and by the definition of W.

Construct $D' = \text{Comp}_{G,S}(A')$ (see Figure 5.2) and denote by $y' \in V_{H_Z}$ the vertex corresponding to y, i.e., $y' = \mathcal{D}^{-1}_{H_Z}(k)$. The set D' can be constructed by moving vertices from $V_{G_{S(v)}} \cap A' \cap$

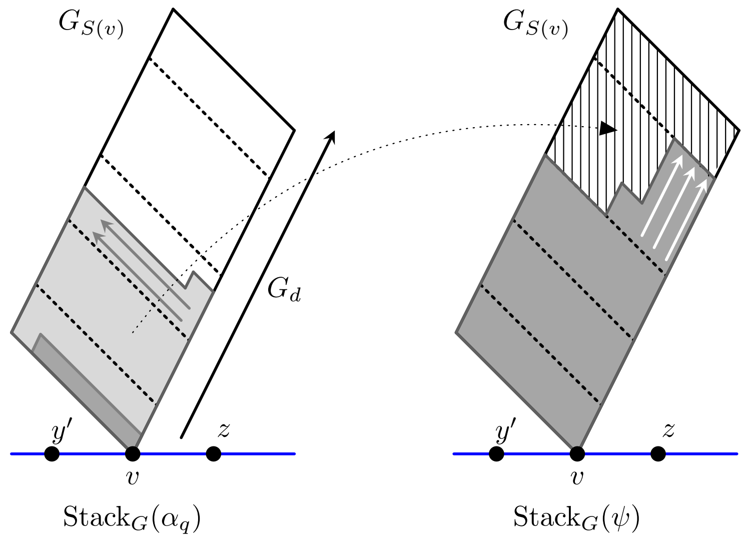

Figure 5.2: Construction of D' in case 1 of the proof.

$\mathrm{Stack}_G(\alpha_q)$ to $(V_{G_{S(v)}} \cap \mathrm{Stack}_G(\psi)) \setminus A'$, for all $v \in R_{H_Z}$, where $R_{H_Z} = \{v \in V_{H_Z} \mid y' \leq_{\mathcal{D}_{H_Z}} v \leq_{\mathcal{D}_{H_Z}} z\}$. For $v \in R_{H_Z}$ denote by a_v the number of vertices moved,

$$a_v = \min\{|V_{G_{S(v)}} \cap A' \cap \mathrm{Stack}_G(\alpha_q)|, |(V_{G_{S(v)}} \cap \mathrm{Stack}_G(\psi)) \setminus A'|\}.$$

The set D' can be constructed from A' by applying the following two steps for all $v \in R$:

1. Remove the set $T_{q,v}$ consisting of the last a_v vertices of $V_{G_{S(v)}} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ in the order $\mathcal{BL}^2_{G_{S(v)}}$ restricted to $\mathrm{Stack}_G(\alpha_q)$.
2. Add the set $T_{q-1,v}$ consisting of the first a_v vertices in $(V_{G_{S(v)}} \cap \mathrm{Stack}_G(\psi)) \setminus A'$ in the order $\mathcal{BL}^2_{G_{S(v)}}$ restricted to $\mathrm{Stack}_G(\psi)$.

Corollary 4.4 implies

$$0 \leq |I_G(D')| - |I_G(A')| = \sum_{v \in R_{H_Z}} \omega_G(T_{q-1,v}) - \omega_G(T_{q,v}).$$

For $v \in R_{H_Z}$ denote

$$T_v = \{(v_1, \mathcal{D}^{-1}_{H_Y}(\mathcal{D}_{H_Z}(u)), v_d) \mid (v_1, u, v_d) \in T_{q-1,v}\}$$

and construct the set

$$D = \left(A \setminus \bigcup_{v \in R_{H_Z}} T_{q,v}\right) \cup \bigcup_{v \in R_{H_Z}} T_v.$$

We show that D is compressed. Indeed, let $v = (v_1, v_2, \ldots, v_d) \in D$ and $u = (u_1, u_2, \ldots, u_d) \in V_G$ such that $\exists i \in \{1, 2, \ldots, d\}$ with $u_i <_{\mathcal{O}_{G_i}} v_i$ and $u_j = v_j$ for all $j \neq i$. Without loss of generality we assume $u \notin \mathrm{Slice}_G(p)$ for all $p < q-1$ since $\mathrm{Slice}_G(p) \subseteq D$. If $v \in \mathrm{Slice}_G(q-1)$, then $u \in \mathrm{Slice}_G(q-1)$ by the previous assumption.

If $u \notin \mathrm{Stack}_G(\alpha_{q-1})$, then $u \in D$ since for any stack $\mathrm{Stack}_G(\beta) \subseteq \mathrm{Slice}_G(q-1)$ different from $\mathrm{Stack}_G(\alpha_{q-1})$ we have $\mathrm{Stack}_G(\beta) \subseteq D$. So suppose $u \in \mathrm{Stack}_G(\alpha_{q-1})$. Then $u \in D$ since $\mathrm{Stack}_G(\alpha_{q-1}) \cap D$ is a projection of $\mathrm{Stack}_G(\psi) \cap D'$ which is in the compressed set D'. If $v \in \mathrm{Slice}_G(q)$ and $u \in \mathrm{Slice}_G(q)$, then $u \in D$ as $D \cap \mathrm{Slice}_G(q) = D' \cap \mathrm{Slice}_G(q)$ and D' is compressed. Finally, if $v \in \mathrm{Slice}_G(q)$ and $u \in \mathrm{Slice}_G(q-1)$ we may assume $u \in \mathrm{Stack}_G(\psi)$. Then $i = 1$, hence $u = (u_1, v_2, v_3, \dots, v_d)$. Since $u \in \mathrm{Stack}_G(\psi) \subseteq D$, we get $u \in D$. Therefore, D is compressed.

One has,

$$\begin{aligned}
|I_G(D)| - |I_G(A)| &= \sum_{v \in R_{H_Z}} \omega_G(T_v) - \omega_G(T_{q,v}), \\
&= \sum_{v \in R_{H_Z}} \omega_G(T_v) - \omega_G(T_{q-1,v}) + \omega_G(T_{q-1,v}) - \omega_G(T_{q,v}), \\
&\geq \sum_{v \in R_{H_Z}} \omega_G(T_v) - \omega_G(T_{q-1,v}) \\
&= \sum_{v \in R_{H_Z}} a_v(\Delta_{G_{\{1,\dots,d\}\setminus S}}(\mathrm{Start}_{G_{\{1,\dots,d\}\setminus S}}(H_Y)) + \delta_{H_Y}(\mathcal{D}_{H_Z}(v)) - \Delta_{H_Z}(v)) \\
&\geq \sum_{v \in R_{H_Z}} a_v(\delta_{H_Y}(\mathcal{D}_{H_Z}(v)) - \Delta_{H_Z}(v)) \\
&= 0.
\end{aligned}$$

Case 2: Assume $k \geq l$. The reader might find it helpful to recall Figure 5.1 throughout this case. Denote $F = \{\pi(d-2)\}$. Note that F consists of the "most dominating direction" of H and J under the order $\mathcal{D}$. We can write $z = (z_1, \dots, z_{d-2})$ and $y = (y_1, \dots, y_{d-2})$, and denote

$$\begin{aligned}
Z^* &= Z_2 \times \cdots Z_{\pi(d-1)-1} \times Z_{\pi(d-1)+1} \times \cdots \times Z_{d-1}, \\
Y^* &= Y_2 \times \cdots Y_{\pi(d-1)-1} \times Y_{\pi(d-1)+1} \times \cdots \times Y_{d-1}, \\
z^* &= (z_1, \dots, z_{\pi(d-2)-1}, z_{\pi(d-2)+1}, \dots, z_{d-2}), \\
y^* &= (y_1, \dots, y_{\pi(d-2)-1}, y_{\pi(d-2)+1}, \dots, y_{d-2}).
\end{aligned}$$

Further denote by $\mathcal{Y}$ and $\mathcal{Z}$ the domination orders on $H_{Y_F}(y^*)$ and $H_{Z_F}(z^*)$, respectively. Note that the orders $\mathcal{Y}$ and $\mathcal{Z}$ are just the restrictions of $\mathcal{D}_{H_Y}$ to $H_{Y_F}(y^*)$ and $\mathcal{D}_{H_Z}$ to $H_{Z_F}(z^*)$ respectively. Denote $m = \mathcal{Y}(Y)$ and $n = \mathcal{Z}(z)$. Note that $H_{Y_F}(y^*)$ and $H_{Z_F}(z^*)$ are one dimensional structures. Furthermore, the graph $(H_{Y_F}(y^*), I_G(H_{Y_F}(y^*))$ is isomorphic to the graph induced by the last part of the partition $\mathfrak{P}_{G_{\pi(d-2)+1}}$, and the graph $(H_{Z_F}(z^*), I_G(H_{Z_F}(z^*))$ is isomorphic to the graph induced by the first part of the partition $\mathfrak{P}_{G_{\pi(d-2)+1}}$. More formally,

$$\begin{aligned}
(H_{Y_F}(y^*), I_G(H_{Y_F}(y^*)) &\cong (Y_{\pi(d-2)+1}, I_{G_{\pi(d-2)+1}}(Y_{\pi(d-2)+1})) \\
(H_{Z_F}(z^*), I_G(H_{Z_F}(z^*)) &\cong (Z_{\pi(d-2)+1}, I_{G_{\pi(d-2)+1}}(Z_{\pi(d-2)+1}))
\end{aligned}$$

Thus, there are orders $\overline{\mathcal{Y}}$ and $\overline{\mathcal{Z}}$ in these graphs, corresponding to the orders $\mathcal{Y}$ and $\mathcal{Z}$.

Case 2.1: Assume $m < n$ (see Figure 5.3).

Denote

$$\begin{aligned}
y' &= (z_1, \dots, z_{\pi(d-2)-1}, \overline{\mathcal{Z}}^{-1}(m), z_{\pi(d-2)+1}, \dots, , z_{d-2}), \\
z' &= (y_1, \dots, y_{\pi(d-2)-1}, \overline{\mathcal{Y}}^{-1}(n), y_{\pi(d-2)+1}, \dots, , y_{d-2}).
\end{aligned}$$

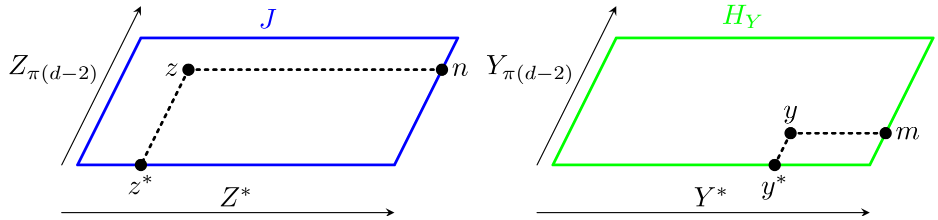

Figure 5.3: General setup for case 2. Case 2.1 is shown, since $m < n$.

Denote by E the set of all two-dimensional sections under $\mathcal{D}_{H_Z}$ and $S = \{1, d\}$ of $\mathrm{Stack}_G(\psi)$ preceding $\mathrm{Stack}_G(\psi) \cap V_{G_S(y')}$, i.e.,

$$E = \bigcup_{\substack{v \in V_{H_Z} \\ v <_{\mathcal{D}_{H_Z}} y'}} (\mathrm{Stack}_G(\psi) \cap V_{G_S(v)}).$$

For $v = (y_1, \ldots, y_{\pi(d-2)-1}, u, y_{\pi(d-2)+1}, \ldots, , y_{d-2}) \in R_{H_Y}$ with $R_{H_Y} = \{v \in V_{H_Y} \mid y \leq_{\mathcal{D}_{H_Y}} v \leq_{\mathcal{D}_{H_Y}} z'\}$ denote

$$v' = (z_1, \ldots, z_{\pi(d-2)-1}, \overline{\mathcal{Z}}^{-1}(\overline{\mathcal{Y}}(u)), z_{\pi(d-2)+1}, \ldots, , z_{d-2}).$$

For $v \in R_{H_Y}$ denote by W_v the projection of $V_{G_S(v)} \cap \mathrm{Stack}_G(\alpha_{q-1}) \cap A$ to $V_{G_S(v')} \cap \mathrm{Stack}_G(\psi)$, i.e.,

$$W_v = \{(v_1, v', v_d) \mid (v_1, v, v_d) \in V_{G_S(v)} \cap \mathrm{Stack}_G(\alpha_{q-1}) \cap A\},$$

and put

$$W = \bigcup_{v \in R_{H_Y}} W_v.$$

Finally, define

$$A' = (A \setminus \mathrm{Slice}_G(q-1)) \cup E \cup W.$$

Set A' has the following properties:

1. A' is compressed. The proof is similar to case 1 and is left to the reader.

2. For each $v \in V_{H_Z}$ the set $V_{G_S(v)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ (see Figure 5.4 for the general picture) is an initial segment of $\mathcal{BL}^2_{G_S(v)}$ restricted to $\mathrm{Stack}_G(\alpha_q)$, since A is strongly compressed.

3. For each $v \in V_{H_Z}$ the set $V_{G_S(v)} \cap A' \cap \mathrm{Stack}_G(\psi)$ (see Figure 5.4 for the general picture) is an initial segment of $\mathcal{BL}^2_{G_S(v)}$ restricted to $\mathrm{Stack}_G(\psi)$, since A is strongly compressed and by the definition of W.

Denote $D' = \mathrm{Comp}_{G,S}(A')$. The set D' can be constructed by moving a_v vertices from $V_{G_S(v)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ to $(V_{G_S(v)} \cap \mathrm{Stack}_G(\psi)) \setminus A'$, for all $v \in R_{H_Z}$ where $R_{H_Z} = \{v \in V_{H_Z} \mid y' \leq_{\mathcal{D}_{H_Z}} v \leq_{\mathcal{D}_{H_Z}} z\}$ and

$$a_v = \min\{|V_{G_S(v)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)|, |(V_{G_S(v)} \cap \mathrm{Stack}_G(\psi)) \setminus A'|\}.$$

The set D' can be obtained from A' by applying the following steps for every $v \in R_{H_Z}$:

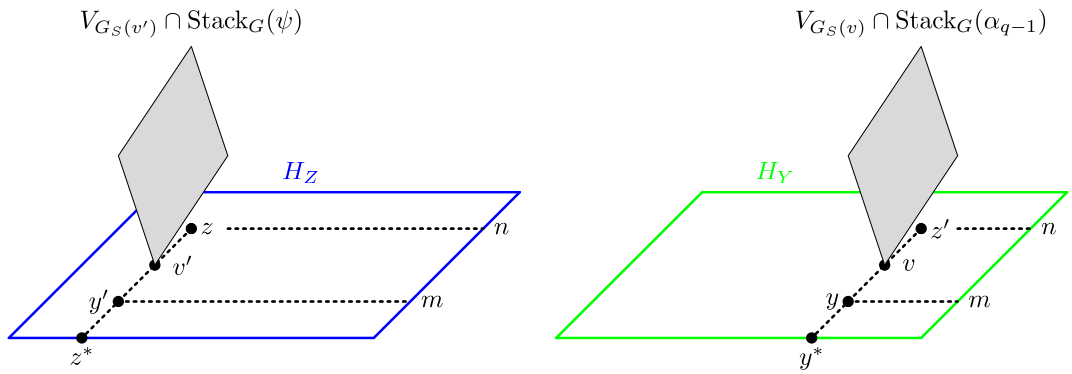

Figure 5.4: Setup for the pull part in Case 2.1

1. Remove the set $T_{q,v}$ consisting of the last a_v vertices of $V_{G_S(v)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ in the order $\mathcal{BL}^2_{G_S(v)}$ restricted to $\mathrm{Stack}_G(\alpha_q)$.
2. Add the set $T_{q-1,v}$ consisting of the first a_v vertices of $(V_{G_S(v)} \cap \mathrm{Stack}_G(\psi)) \setminus A'$ in the order $\mathcal{BL}^2_{G_S(v)}$ restricted to $\mathrm{Stack}_G(\psi)$.

Corollary 4.4 implies

$$0 \leq |I_G(D')| - |I_G(A')| = \sum_{v \in R_{H_Z}} \omega_G(T_{q-1,v}) - \omega_G(T_{q,v}).$$

For $v \in R_{H_Y}$ denote

$$T_v = \{(v_1, v, v_d) \bigm| (v_1, v', v_d) \in T_{q-1,v'}\}.$$

and consider the set

$$D = \left(A \setminus \bigcup_{v \in R_{H_Z}} T_{q,v} \right) \cup \bigcup_{v \in R_{H_Y}} T_v.$$

Similarly to case 1 it can be shown that D is compressed. One has,

$$\begin{aligned}
|I_G(D)| - |I_G(A)| &= \sum_{v \in R_{H_Y}} \omega_G(T_v) - \omega_G(T_{q,v'}), \\
&= \sum_{v \in R_{H_Y}} \omega_G(T_v) - \omega_G(T_{q-1,v'}) + \omega_G(T_{q-1,v'}) - \omega_G(T_{q,v'}), \\
&\geq \sum_{v \in R_{H_Y}} \omega_G(T_v) - \omega_G(T_{q-1,v'}) \\
&= \sum_{v \in R_{H_Y}} a_v(\Delta_{G_{\{1,\dots,d\}\setminus S}}(\mathrm{Start}_{G_{\{1,\dots,d\}\setminus S}}(H_Y)) + \Delta_{H_Y}(v) - \Delta_{H_Z}(v')) \\
&\geq \sum_{v \in R_{H_Y}} a_v(0 + \Delta_{H_Y}(v) - \Delta_{H_Z}(v')) \\
&\geq 0.
\end{aligned}$$

Note that the last inequality follows from the non-decreasing property of $\mathfrak{P}_{G_2}, \ldots, \mathfrak{P}_{G_{d-1}}$.

Case 2.2: Now assume $m \geq n$. Denote by E the set of all sections under $\mathcal{D}_{H_Z}$ and S of $\mathrm{Stack}_G(\psi)$ preceding $\mathrm{Stack}_G(\psi) \cap V_{G_S(z)}$, i.e.,

$$E = \bigcup_{\substack{v \in V_{H_Z} \\ v <_{\mathcal{D}_{H_Z}} z}} \mathrm{Stack}_G(\psi) \cap V_{G_S(v)}.$$

Let W be the projection of $V_{G_S(y)} \cap \mathrm{Stack}_G(\alpha_{q-1}) \cap A$ to $V_{G_S(z)} \cap \mathrm{Stack}_G(\psi)$, i.e.,

$$W = \{(v_1, z, v_d) \mid (v_1, y, v_d) \in V_{G_S(y)} \cap \mathrm{Stack}_G(\alpha_{q-1}) \cap A\}.$$

Consider the set

$$A' = (A - \mathrm{Slice}_G(q-1)) \cup E \cup W.$$

Set A' has the following properties:

1. A' is compressed. The proof is similar to case 1 and is left to the reader.

2. The set $V_{G_S(z)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ is an initial segment of $\mathcal{BL}^2_{G_S(z)}$ restricted to $\mathrm{Stack}_G(\alpha_q)$, since A is strongly compressed.

3. The set $V_{G_S(z)} \cap A' \cap \mathrm{Stack}_G(\psi)$ is an initial segment of $\mathcal{BL}^2_{G_S(z)}$ restricted to $\mathrm{Stack}_G(\psi)$, since A is strongly compressed and by the definition of W.

Construct set $D' = \mathrm{Comp}_{G,S}(A')$ by moving a vertices from $V_{G_S(z)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ to $(V_{G_S(z)} \cap \mathrm{Stack}_G(\psi)) \setminus A'$, where

$$a = \min\{|V_{G_S(z)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)|, |(V_{G_S(z)} \cap \mathrm{Stack}_G(\psi)) \setminus A'|\}.$$

More exactly, the set D' can be obtained from A' in two following steps:

1. Remove the set T_q consisting of the last a vertices of $V_{G_S(z)} \cap A' \cap \mathrm{Stack}_G(\alpha_q)$ in the order $\mathcal{BL}^2_{G_S(z)}$ restricted to $\mathrm{Stack}_G(\alpha_q)$.

2. Add the set T_{q-1} consisting of first a vertices of $(V_{G_S(z)} \cap \mathrm{Stack}_G(\psi)) \setminus A'$ in the order $\mathcal{BL}^2_{G_S(z)}$ restricted to $\mathrm{Stack}_G(\psi)$.

Corollary 4.4 implies

$$0 \leq |I_G(D')| - |I_G(A')| = \omega_G(T_{q-1}) - \omega_G(T_q).$$

Denote

$$T_y = \{(v_1, y, v_d) \mid (v_1, z, v_d) \in T_{q-1}\}$$

and let

$$D = (A \setminus T_q) \cup T_y.$$

Similarly to case 1 it can be shown that the set D is compressed, One has:

$$\begin{aligned}
|I_G(D)| - |I_G(A)| &= \omega_G(T_y) - \omega_G(T_q), \\
&= \omega_G(T_y) - \omega_G(T_{q-1}) + \omega_G(T_{q-1}) - \omega_G(T_q), \\
&\geq \omega_G(T_y) - \omega_G(T_{q-1}) \\
&= a(\Delta_{G_{\{1,\dots,d\}\setminus S}}(\mathrm{Start}_{G_{\{1,\dots,d\}\setminus S}}(H_Y)) + \Delta_{H_Y}(y) - \Delta_{H_Z}(z)) \\
&\geq a(0 + \Delta_{H_Y}(y) - \Delta_{H_Z}(z)) \\
&\geq 0.
\end{aligned}$$

The last inequality follows from the non-decreasing property of $\mathfrak{P}_{G_2}, \dots, \mathfrak{P}_{G_{d-1}}$.

In cases 1 and 2 we showed how move vertices from $\mathrm{Slice}_G(q)$ to $\mathrm{Slice}_G(q-1)$ to transform A into set D such that $|A| = |D|$ and $|I_G(A)| \leq |I_G(D)|$. Make D strongly compressed and denote the resulting set by E. By Theorem 5.6 there is a strongly compressed and sliced compressed set F with

$$\begin{gathered}
|A| = |D| = |E| = |F|, \\
|I_G(A)| \leq |I_G(D)| \leq |I_G(E)| \leq |I_G(F)|.
\end{gathered}$$

Repeat the transformations described above until we get a stable set B. This will be the case because we replace some vertices with the ones that come earlier in the order $\mathcal{BL}_G^d$. The set B is strongly compressed and block compressed, $|A| = |B|$, and $|I_G(A)| \leq |I_G(B)|$. $\square$

We are almost done with the proof of our main result. Here is a summary of what we have established:

1. Reduced the problem to strongly compressed sets.

2. Reduced the problem to slice compressed sets.

3. Reduced the problem to block compressed sets.

However, it could be the case that we get a block compressed set which has only one partially filled block that is not ordered according to the order $\mathcal{BL}_G^d$. But this is not a problem, since we assumed that the domination order on this block is optimal. In this case we just replace the set of vertices in this block with an initial segment of the same size in order $\mathcal{BL}_G^d$. The resulting set will be optimal because of the second property of isoperimetric partitions. Namely, for the partition $\mathfrak{P}_{G_j} = \{\mathcal{O}_{G_j}[a_1, b_1] <_{\mathcal{O}_{G_j}} \cdots <_{\mathcal{O}_{G_j}} \mathcal{O}_{G_j}[a_k, b_k]\}$ one has $|I(\mathcal{O}_{G_j}[a_1, b_{i-1}], \{v\})| = \delta(a_i) = \Delta_{G_j}(\mathcal{O}_{G_j}(a_i))$ for all $v \in \mathcal{O}_{G_j}[a_i, b_i]$. With this concluding remark the proof of the main result is complete.

6 Applications of the Main result

In this section we show that most previously known results on edge-isoperimetric problems are corollaries of our main Theorem 2.5. Let us start with the theorem of Ahlswede-Cai as the most general one. For this we define the *atomic partition* of an isoperimetric graph G with optimal order $\mathcal{O}$, as $\mathfrak{A}_G = \{\{\mathcal{O}_G(1)\}, \dots, \{\mathcal{O}_G(|V|)\}\}$ with $\{\mathcal{O}_G(1)\} <_{\mathcal{O}_G} \cdots <_{\mathcal{O}_G} \{\mathcal{O}_G(|V|)\}$.

Corollary 6.1. *Let $G_1, \dots, G_d$ be isoperimetric graphs and for $S \subset \{1, \dots, d\}$ let $\mathcal{BL}_{G_S}^{|S|}$ be the block-lexicographic order based on atomic partitions $\mathfrak{A}_{G_1}, \dots, \mathfrak{A}_{G_d}$. If for all $i, j \in \{1, \dots, d\}$ with $i < j$ the order $\mathcal{BL}_{G_i \square G_j}^2$ is optimal, then $\mathcal{BL}_G^d$ is optimal for $d \geq 3$.*

Proof. Atomic partitions are regular and non-decreasing. So, all the conditions of Theorem 2.5 are satisfied. $\square$

The block-lexicographic order of Corollary 6.1 is just the lexicographic order. Every block of this order consists of only one vertex. The blocks are ordered according to their starts which, in turn, are ordered lexicographically. Thus, there is a unique domination order of each block and unique block lexicographic order for each sub-product. This way slices are $(d-1)$-dimensional sub-products and stacks are 1-dimensional ones. Also, the proofs of most statements in section 5 can be merely simplified and deduced from the properties of the strong compression.

Evidently, every application of Theorem 1.3 based on the local-global principle follows from Corollary 6.1. Let us mention just a few of them. In all these results the 2-dimensional case must be handled separately. In some cases, e.g., the hypercube it is rather trivial, whereas in other cases like cliques or Petersen graphs it requires more work. There are not any general methods known to us that eliminate multiple sub-cases by working with compressed sets in products of two graphs.

Corollary 6.2. *The order $\mathcal{L}^d_{K_2^d}$ is optimal for every $d \geq 2$.*

Corollary 6.3 (Lindsey [17])**.** *Suppose that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_d$. The lexicographic order is optimal for $K_{n_1} \square \cdots \square K_{n_d}$.*

Corollary 6.4 (Bezrukov-Bulatovic-Kuzmanovski [5])**.** *Suppose that $1 \leq n_1 \leq \cdots \leq n_d$ and consider the complete bipartite graphs $K_{n_1,n_1}, \ldots, K_{n_d,n_d}$. The lexicographic order is optimal for $K_{n_1,n_1} \square \cdots \square K_{n_d,n_d}$.*

Corollary 6.5 (Bezrukov and Elsässer [9] for $s = 2$, Bezrukov, Bulatovic and Kuzmanovski for arbitrary $s \geq 2$)**.** *Suppose that $s \geq 2$, $p \geq 3$ and $1 \leq i \leq p - i$ and let G be an isoperimetric graph such that*

$$\delta_G = (\underline{0, \ldots, p-1}, \underline{p-i, \ldots, p-i+(p-1)}, \ldots, \underline{(s-1)(p-i), \ldots, (s-i)(p-1)+p-1}).$$

The lexicographic order is optimal for G^d, for any $d \geq 2$.

It is important to note that graphs with δ-sequences specified in Corollary 6.5 exist. Such graphs are constructed in [5] by a method involving the join operation on graphs. In case $s = 2$ one can construct such a graph by removing i disjoint perfect matchings form K_{2p}. There are even more graphs studied in [9] and [5] to which the local-global principle is applicable. Below we present several results that do not follow from the theorem of Ahlswede-Cai. For this we need to define the standard block-lexicographic order.

Let $G_1, \ldots, G_d$ be isoperimetric graphs along with their standard partitions $\mathfrak{M}_{G_1}, \ldots, \mathfrak{M}_{G_d}$. Recall from Theorem 2.3 that the subgraphs induced by monotonic sets are cliques. Hence, a subgraph of $G = G_1 \square \cdots \square G_d$ induced by every block is isomorphic to the product of cliques which admits nested solutions according to the result of Lindsey (cf. Corollary 6.3). In fact, the optimal order is a domination order! We make it more explicit now.

Let $S \subseteq \{1, \ldots d\}$ and $B = V_{K_{n_1}} \times \cdots \times V_{K_{n_{|S|}}}$ be a block of G_S. Define a total order $\eta : \{0, 1 \ldots, |S|\} \to \{0, 1, \ldots |S|\}$ as follows: for $i, j \in \{1, \ldots, |S|\}$ we say $i <_\eta j$ iff $n_i < n_j$ or if $n_i = n_j$ and $i < j$. Therefore, since $1 \leq n_{\eta(1)} \leq \cdots \leq n_{\eta(d)}$ the lexicographic order is optimal for $K_{n_{\eta(1)}} \square \cdots \square K_{n_{\eta(d)}}$. Note that $\eta \in \mathfrak{S}_d$ and there is a unique domination order $\mathcal{D}^{\eta,|S|}$. So, for $K = K_{n_1} \square \cdots \square K_{n_{|S|}}$ we have the induced domination order $\mathcal{D}^{\eta,|S|}_K$ on K. Furthermore, the order $\mathcal{D}^{\eta,|S|}_K$ is optimal, since lexicographic order is optimal for $K_{n_{\eta(1)}} \square \cdots \square K_{n_{\eta(d)}}$.

Hence, for any block B we can construct a permutation η for which there is an optimal domination order on the graph induced by a block. It is easily seen that $\mathfrak{M}_{G_1}, \ldots, \mathfrak{M}_{G_d}$ is a domination collection with these domination orders. We call $\mathcal{D}_K^{\eta,|S|}$ the *standard block domination order* on B. Furthermore, we call the block lexicographic orders formed by the standard partitions $\mathfrak{M}_{G_1}, \ldots, \mathfrak{M}_{G_d}$ and the standard block domination orders, the *standard block lexicographic orders* and for $S \subseteq \{1, \ldots, d\}$ denote them by $\mathcal{SBL}_{G_S}^{|S|}$.

Let us talk on the regularity of partitions for regular graphs. Many authors in the past have noticed the following result.

Lemma 6.6. *If $G = (V, E)$ is regular then $A \subseteq V$ is optimal iff $V \setminus A$ is optimal.*

The above lemma provides a relationship between regular graphs and their regular partitions.

Corollary 6.7. *Let $G = (V, E)$ be a regular isoperimetric graph with optimal order $\mathcal{O}_G$ and let $\mathcal{O}'_G$ be its reverse order. Then the order $\mathcal{O}'_G$ is optimal and the partition $\mathfrak{M}_G$ is regular.*

Proof. Note that for any $i \in \{1, \ldots, V\}$ the set $\mathcal{O}_G[1, i]$ is optimal. Thus, $\mathcal{O}'_G[1, |V| - i + 1]$ is optimal for all $i \in \{1, \ldots, |V|\}$ by Lemma 6.6. Therefore, the first and last monotonic segments of the order $\mathcal{O}_G$ are of the same size. □

By Corollary 6.7, if the graph G is isoperimetric and regular, then $\mathfrak{M}_G$ is an isoperimetric, non-decreasing and regular partition. See [11] for properties of the δ-sequences of regular graphs.

Corollary 6.8. *Let $G_1, \ldots, G_d$ be regular isoperimetric graphs with their respective standard partitions $\mathfrak{M}_{G_1}, \ldots, \mathfrak{M}_{G_d}$ and $G = G_1 \Box \cdots \Box G_d$. If $\mathcal{SBL}_{G_i \Box G_j}$ is optimal for all $i < j$ then $\mathcal{SBL}_G^d$ is optimal for $d \geq 3$.*

Proof. We have that $\mathfrak{M}_{G_i}$ is non-decreasing and regular. Thus, $\mathfrak{M}_{G_1}, \ldots, \mathfrak{M}_{G_d}$ is a regular domination collection by the definition of the standard block-domination order. The statement then follows from Theorem 2.5. □

Taking all this into account, we show how to obtain some results which are not covered by the theorem of Ahlswede-Cai.

Corollary 6.9 (Bezrukov-Das-Elsässer [8])**.** *Let G_1 be the Petersen graph, $G_2 = K_2$, and $d_1, d_2 \geq 0$. Then $G = G_1^{d_1} \Box G_2^{d_2}$ has nested solutions and the standard block-lexicographic order is optimal.*

Proof. One just needs to show that the standard block-lexicographic order is optimal for G_1^2, $G_1 \Box G_2$ and G_2^2. This is trivial for G_2^2 and the other cases are covered in Chapter 9 of [15], and in [8]. Therefore, the statement follows from Corollary 6.8. □

The authors of [8] first proved Corollary 6.9 for $d_1 \geq 0$ and $d_2 = 0$. The general case required more work. The authors had to define a new order and handle multiple new cases. In the next results the two-dimensional cases needed for Corollary 6.8 are covered in [12].

Corollary 6.10 (Carlson [12])**.** *If $d_1, d_2, d_3 \geq 0$ then the standard block-lexicographic order is optimal for $C_5^{d_1} \Box C_4^{d_2} \Box C_3^{d_3}$.*

Corollary 6.11 (Carlson [12])**.** *If $d_1, d_2, d_3 \geq 0$ and $n \geq 6$ then the standard block-lexicographic order is optimal for $C_n \Box C_5^{d_1} \Box C_4^{d_2} \Box C_3^{d_3}$.*

It is worse to note that in [12] Carlson first proved Theorem 1.9. Then after several auxiliary results and multiple statements about higher dimensions he finally established the results outlined in Corollaries 6.10 and 6.11. Throughout these steps Carlson needed to define some special orders. In contrast to this, in our case we need just one standard block-lexicographic order. A similar situation occurred in [8] by proving Corollary 6.9. Our methods allow us to further extend the results of Carlson.

Corollary 6.12. *If $d_1, d_2, d_3, d_4 \geq 0$ and $n \geq 6$ then the standard block-lexicographic order is optimal for $C_5^{d_1} \square C_4^{d_2} \square K_2^{d_3} \square C_3^{d_4}$ and $C_n \square C_5^{d_1} \square C_4^{d_2} \square K_2^{d_3} \square C_3^{d_4}$.*

Proof. We just need to check the two-dimensional cases of Corollary 6.8 for the product of every graph in question with K_2. They, however, follow from the two-dimensional cases of [12] since an initial segment of size 2 in any connected graph induces K_2. Also, note that C_3 is isomorphic to K_3. Therefore, the statement follows from 6.8. $\square$

7 Concluding remarks and future directions

In this paper we generalized the Ahlswede-Cai Theorem and proved that almost all results in the area of edge-isoperimetric problems on Cartesian products of graphs are consequences of our main result. In turn, our result can be generalized like it is done in the original local-global principle paper [3, 2]. In particular, Theorem 2.5 has an analog for Θ_G. In general, there is an analog of Theorem 2.5 for any sub-modular function defined in [3, 2]. We only worked with I_G to make the paper easier to read.

Block-lexicographic orders are made up of two parts: the block part and the lexicographic part. One can think of defining block-domination orders similarly to how we handled domination orders by starting with lexicographic order. A natural question is if we can go even further. If we have some d-dimensional order $\mathcal{O}$, can we say anything on the optimality of a block order where the blocks are ordered according to the order $\mathcal{O}$? What properties does $\mathcal{O}$ have to satisfy? Note that the optimal orders in Theorems 1.5 and 1.6 are block orders, but not lexicographic ones. The orders use the standard partitions for the blocks. We would be surprised if the local-global principles could not be extended to such orders. Answering these questions would settle Harper's question completely. Here is a conjecture that we believe would help to find answers to the posed questions.

Conjecture 7.1. *If $d_1, d_2 \geq 0$ and $n_1, n_2 \geq 2$ then $P_{n_1}^{d_1} \square K_{n_2}^{d_2}$ has nested solutions.*

A lot of work has been done for the products of 3 or more graphs and very powerful methods have been developed. However, the 2-dimensional case still requires a special treatment and no general methods are known for it. One possible direction could be to improve the upper bound for i in Corollary 6.5. Examples show that it can be much higher. For $s = 2$ the following conjecture is proved in [9].

Conjecture 7.2. *Statement of Corollary 6.5 is valid for $s \geq 2$, $p \geq 3$ and $1 \leq i \leq p - p/s$. For $i > p - p/s$ there are no nested solutions.*

The optimal order for the product of two Petersen graphs was established by using a computer in [8]. Harper developed a method in [15] which in many cases can be used without computers. It would be interesting to develop general methods for the products of two graphs to handle cases like the powers of Petersen graph. To stimulate this we present another conjecture.

Conjecture 7.3. *If $d_1, d_2, d_3, d_4 d_5 \geq 0$, $n \geq 6$ and G is the Petersen graph, then the standard block lexicographic order is optimal for $C_5^{d_1} \square G^{d_2} \square C_4^{d_3} \square K_2^{d_4} \square C_3^{d_5}$ and $C_n \square C_5^{d_1} \square G^{d_2} \square C_4^{d_3} \square K_2^{d_4} \square C_3^{d_5}$.*

References

[1] Ahlswede, R., and Bezrukov, S. L. Edge isoperimetric theorems for integer point arrays. *Appl. Math. Lett. 8*, 2 (1995), 75–80.

[2] Ahlswede, R., and Cai, N. General edge-isoperimetric inequalities. I. Information-theoretical methods. *European J. Combin. 18*, 4 (1997), 355–372.

[3] Ahlswede, R., and Cai, N. General edge-isoperimetric inequalities. II. A local-global principle for lexicographical solutions. *European J. Combin. 18*, 5 (1997), 479–489.

[4] Bernstein, A. J. Maximally connected arrays on the n-cube. *SIAM J. Appl. Math. 15* (1967), 1485–1489.

[5] Bezrukov, S., Bulatovic, P., and Kuzmanovski, N. New infinite family of regular edge-isoperimetric graphs. *Theoret. Comput. Sci. 721* (2018), 42–53.

[6] Bezrukov, S. L. Edge isoperimetric problems on graphs. In *Graph theory and combinatorial biology (Balatonlelle, 1996)*, vol. 7 of *Bolyai Soc. Math. Stud.* János Bolyai Math. Soc., Budapest, 1999, pp. 157–197.

[7] Bezrukov, S. L. On an equivalence in discrete extremal problems. *Discrete Math. 203*, 1-3 (1999), 9–22.

[8] Bezrukov, S. L., Das, S. K., and Elsässer, R. An edge-isoperimetric problem for powers of the Petersen graph. *Ann. Comb. 4*, 2 (2000), 153–169.

[9] Bezrukov, S. L., and Elsässer, R. Edge-isoperimetric problems for cartesian powers of regular graphs. *Theoretical computer science 307*, 3 (2003), 473–492.

[10] Bollobás, B., and Leader, I. Edge-isoperimetric inequalities in the grid. *Combinatorica 11*, 4 (1991), 299–314.

[11] Bonnet, E., and Sikora, F. A note on edge isoperimetric numbers and regular graphs. *Internat. J. Found. Comput. Sci. 27*, 6 (2016), 771–774.

[12] Carlson, T. A. The edge-isoperimetric problem for discrete tori. *Discrete Math. 254*, 1-3 (2002), 33–49.

[13] Das, S. K., Hohndel, D. H., Ibel, M., and Öhring, S. R. Efficient communication in folded petersen networks. *International Journal of Foundations of Computer Science 08*, 02 (1997), 163–185.

[14] Harper, L. H. Optimal assignments of numbers to vertices. *J. Soc. Indust. Appl. Math. 12* (1964), 131–135.

[15] Harper, L. H. *Global methods for combinatorial isoperimetric problems*, vol. 90 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.

[16] Hart, S. A note on the edges of the n-cube. *Discrete Math. 14*, 2 (1976), 157–163.

[17] Lindsey, II, J. H. Assignment of numbers to vertices. *Amer. Math. Monthly 71* (1964), 508–516.

[18] Öhring, S., and Das, S. Folded petersen cube networks: new competitors for the hypercubes. In *Proceedings of 1993 5th IEEE Symposium on Parallel and Distributed Processing* (1993), pp. 582–589.

[19] Öhring, S., and Das, S. Folded petersen cube networks: new competitors for the hypercubes. *IEEE Transactions on Parallel and Distributed Systems 7*, 2 (1996), 151–168.

[20] Öhring, S., and Das, S. K. The folded petersen network : A new communication-efficient multiprocessor topology. In *1993 International Conference on Parallel Processing - ICPP'93* (1993), vol. 1, pp. 311–314.

[21] Öhring, S. R., and Das, S. K. Efficient communication in the folded petersen interconnection network. In *Proceedings of the 6th International PARLE Conference on Parallel Architectures and Languages Europe* (Berlin, Heidelberg, 1994), PARLE '94, Springer-Verlag, p. 25–36.

[22] Öhring, S. R., and Das, S. K. The folded Petersen network: a new versatile multiprocessor interconnection topology. In *Graph-theoretic concepts in computer science (Utrecht, 1993)*, vol. 790 of *Lecture Notes in Comput. Sci.* Springer, Berlin, 1994, pp. 301–314.