CSCI 5512 · HW 3

Nikki Kyllonen
3/30/19
kyllo089

① Filtering

1.1) $P(x_1 \mid e_1 = \text{"not flooded"}) = ?$

hidden $(x_1) \xrightarrow{\text{causes}} (e_1)$ evidence (observable)

* assume we know $E_{1:t}$ and X_0

* Recursive Definition of Filtering: $P(x_t \mid e_{1:t}) = \alpha P(e_t \mid x_t) \sum_{x_{t-1}} \left(P(x_t \mid x_{t-1}) \underline{\underline{P(x_{t-1} \mid e_{1:t-1})}} \right)$

previous step

@ e_1 = "not flooded":

$P(x_1 \mid e_1) = \alpha P(x_1, e_1)$ → "alpha trick"

$= \alpha \sum_{x_0} P(x_0, x_1, e_1)$

$= \alpha \sum_{x_0} P(x_0) P(x_1 \mid x_0) P(e_1 \mid x_1)$ → Bayes Net! $P(X \mid \text{Parents}(X))$

$= \alpha P(e_1 \mid x_1) \sum_{x_0 = \text{low, med, high}} P(x_0) P(x_1 \mid x_0)$

@ x_0 = low: → $P(x_0 = \text{low}) = P(x_0 = \text{med}) = P(x_0 = \text{high}) = 1/3$
∴ all same values

→ @ x_1 = low:

$P(\text{low} \mid e_1) = \alpha (1) \left[(1/3)(0.6) + (1/3)(0.2) + (1/3)(0) \right] = \underline{\underline{\alpha(0.2667)}}$

→ @ x_1 = med:

$P(\text{med} \mid e_1) = \alpha (0.95) \left[(1/3)(0.35) + (1/3)(0.6) + (1/3)(0.5) \right] = \underline{\underline{\alpha(0.4592)}}$

→ @ x_1 = high:

$P(\text{high} \mid e_1) = \alpha (0.6) \left[(1/3)(0.05) + (1/3)(0.2) + (1/3)(0.5) \right] = \underline{\underline{\alpha(0.15)}}$

* find α * solve w/ α *

$\sum_{x_1} P(x_1 \mid e_1) = 1 = \alpha \sum_{x_1} P(x_1, e_1)$

$\alpha \langle 0.2667, 0.4592, 0.15 \rangle = 1$

$\underline{\underline{\alpha = 1.1417}}$ →

$P(x_1 = \text{low} \mid e_1) = 0.3045$
$P(x_1 = \text{med} \mid e_1) = 0.5243$
$P(x_1 = \text{high} \mid e_1) = 0.1713$

1.2) $P(x_2 \mid e_1 = \text{"not flooded"}, e_2 = \text{"not flooded"}) = ?$

@ $e_1 = e_2 =$ "not flooded":

$$P(x_t \mid e_{1:t}) = P(x_t \mid e_t, e_{1:t-1}) \longrightarrow t = 2$$

$$P(x_2 \mid e_1, e_2) = \alpha\, P(e_2 \mid x_2) \sum_{x_1 = \text{low, med, high}} P(x_2 \mid x_1)\, \underbrace{P(x_1 \mid e_1)}_{\star \text{ from } 1.1}$$

@ $x_2 =$ low:

$$P(x_2 = \text{low} \mid e_1, e_2) = \alpha(1)\left[(0.6)(0.3045) + (0.2)(0.5243) + (0)\right] = \underline{\underline{\alpha(0.2876)}}$$

@ $x_2 =$ med:

$$P(x_2 = \text{med} \mid e_1, e_2) = \alpha(0.95)\left[(0.35)(0.3045) + (0.6)(0.5243) + (0.5)(0.1713)\right] = \underline{\underline{\alpha(0.4815)}}$$

@ $x_2 =$ high:

$$P(x_2 = \text{high} \mid e_1, e_2) = \alpha(0.6)\left[(0.05)(0.3045) + (0.2)(0.5243) + (0.5)(0.1713)\right] = \underline{\underline{\alpha(0.1234)}}$$

* find α values & solve *

$$\sum_{x_2} P(x_2 \mid e_1, e_2) = 1 = \alpha \sum_{x_2} P(x_2, e_1, e_2)$$

$$\alpha \langle 0.2876, 0.4815, 0.1234 \rangle = 1$$

$$\underline{\underline{\alpha = 1.1205}} \longrightarrow \boxed{\begin{aligned} P(x_2 = \text{low} \mid e_1, e_2) &= 0.3223 \\ P(x_2 = \text{med} \mid e_1, e_2) &= 0.5395 \\ P(x_2 = \text{high} \mid e_1, e_2) &= 0.1383 \end{aligned}}$$

1.3) $P(x_3 \mid e_1 = \text{"not flooded"}, e_2 = \text{"not flooded"}, e_3 = \text{"flooded"}) = ?$

@ $e_1 = e_2 = \text{"not flooded"}$, $e_3 = \text{"flooded"}$:

$$P(x_3 \mid e_1, e_2, e_3) = \alpha P(e_3 \mid x_3) \sum_{x_2 = \text{low, med, high}} P(x_3 \mid x_2) \underbrace{P(x_2 \mid e_1, e_2)}_{\text{* from } 1.2}$$

@ $x_3 = \text{low}$:

$$P(x_3 = \text{low} \mid e_{1:3}) = \alpha(0) = \boxed{0}$$

@ $x_3 = \text{med}$:

$$P(x_3 = \text{med} \mid e_{1:3}) = \alpha(0.05)\left[(0.35)(0.3223) + (0.6)(0.5395) + (0.5)(0.1383)\right]$$

$$= \alpha(0.0253)$$

@ $x_3 = \text{high}$:

$$P(x_3 = \text{high} \mid e_{1:3}) = \alpha(0.4)\left[(0.05)(0.3223) + (0.2)(0.5395) + (0.5)(0.1383)\right]$$

$$= \alpha(0.0773)$$

* find α and solve *

$$\sum_{x_3} P(x_3 \mid e_1, e_2, e_3) = 1 = \alpha \sum_{x_3} P(x_3, e_1, e_2, e_3)$$

$$\alpha \langle 0, 0.0253, 0.0773 \rangle = 1$$

$$\alpha = 9.7479 \longrightarrow \boxed{\begin{array}{l} P(x_3 = \text{low} \mid e_{1:3}) = 0 \\ P(x_3 = \text{med} \mid e_{1:3}) = 0.2466 \\ P(x_3 = \text{high} \mid e_{1:3}) = 0.7534 \end{array}}$$

(2) Smoothing : $e_1 = e_2 =$ "not flooded", $e_3 =$ "flooded"

$P(x_k \mid e_{1:t})$, $k < t$ $\longrightarrow$ smooth the function / fill in the holes using what we've observed ($e_{1:t}$)

* Definition using Filtering : $P(x_k \mid e_{1:t}) = \alpha \underbrace{P(e_{k+1:t} \mid x_k)}_{\text{* backwards message}} \underbrace{P(x_k \mid e_{1:k})}_{\text{* filtering!}}$

* $P(x_1 \mid e_{1:3}) \rightarrow k = 1, t = 3$

$= \alpha P(e_2, e_3 \mid x_1) P(x_1 \mid e_1)$

* $P(x_2 \mid e_{1:3}) \rightarrow k = 2, t = 3$

$= \alpha P(e_3 \mid x_2) P(x_2 \mid e_1, e_2)$

* $P(x_3 \mid e_{1:3}) \rightarrow k = 3, t = 3$

$= \alpha \underbrace{P(e_{4:3} \mid x_3)}_{\times \text{ b/t } k \not< t} P(x_3 \mid e_1, e_2, e_3)$

filtering, not smoothing :

$$\boxed{\begin{array}{l} P(x_3 = \text{low} \mid e_{1:3}) = 0 \\ P(x_3 = \text{med} \mid e_{1:3}) = 0.2466 \\ P(x_3 = \text{high} \mid e_{1:3}) = 0.7534 \end{array}} \; f(3)$$

* Final Smoothing recursive definition : $P(x_k \mid e_{1:t}) = \alpha f(k) * \underline{h(k)}$

$\Rightarrow \underline{h(k)} = \sum_{x_{k+1}} P(e_{k+1} \mid x_{k+1}) \underline{h(k+1)} P(x_{k+1} \mid x_k)$ } backwards message!

f(1): $P(x_1 = \text{low} \mid e_1) = 0.3045$

$P(x_1 = \text{med} \mid e_1) = 0.5243$

$P(x_1 = \text{high} \mid e_1) = 0.1713$

f(2): $P(x_2 = \text{low} \mid e_{1:2}) = 0.3223$

$P(x_2 = \text{med} \mid e_{1:2}) = 0.5395$

$P(x_2 = \text{high} \mid e_{1:2}) = 0.1383$

(a) $h(2) = \sum_{x_3 = \text{low, med, high}} P(e_3 \mid x_3) h(3) P(x_3 \mid x_2) \longrightarrow$ let $h(3) = \langle \overset{\text{low}}{1}, \underset{\text{med}}{1}, \overset{\text{high}}{1} \rangle$

$x_2 = \text{low}$: $= (0) + (0.05)(1)(0.35) + (0.4)(1)(0.05) = \underline{0.0375}$

$h(x_2 = \text{med}) = (0) + (0.05)(1)(0.6) + (0.4)(1)(0.2) = \underline{0.11}$

$h(x_2 = \text{high}) = (0) + (0.05)(1)(0.5) + (0.4)(1)(0.5) = \underline{0.225}$

} $h(2)$

$h(x_1 = \text{low}) = \sum_{x_2} P(e_2 \mid x_2) h(2) P(x_2 \mid x_1) = (1)(0.0375)(0.6) + (0.95)(0.11)(0.35) + (0.6)(0.225)(0.05) = \underline{0.0658}$

$$h(x_1=\text{med}) = (1)(0.0375)(0.2) + (0.95)(0.11)(0.6) + (0.6)(0.225)(0.2)$$
$$= \underline{\underline{0.0972}}$$

$$h(x_1=\text{high}) = (1)(0.0375)(0) + (0.95)(0.11)(0.5) + (0.6)(0.225)(0.5)$$
$$= \underline{\underline{0.11975}}$$

* Point-wise product and solve: $P(x_k \mid e_{1:t}) = \alpha\, f(k) * h(k)$

$$\alpha \langle f(x_1=\text{low}) * h(x_1=\text{low}),\ f(x_1=\text{med}) * h(x_1=\text{med}),\ f(x_1=\text{high}) * h(x_1=\text{high}) \rangle$$

$$= 1 = \alpha \langle (0.3045)(0.0658),\ (0.5243)(0.0972),\ (0.1713)(0.11975) \rangle$$

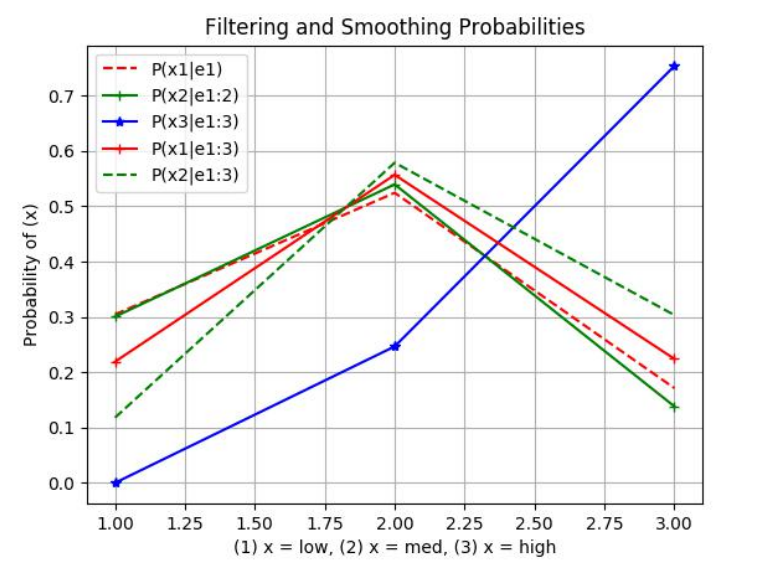
$$1 = \alpha \langle 0.0200,\ 0.05096,\ 0.02051 \rangle$$

$$\underline{\underline{\alpha = 10.9276}} \longrightarrow \boxed{\begin{aligned} P(x_1=\text{low} \mid e_{1:3}) &= 0.2190 \\ P(x_1=\text{med} \mid e_{1:3}) &= 0.5568 \\ P(x_1=\text{high} \mid e_{1:3}) &= 0.2241 \end{aligned}}$$

@ x_2: $1 = \alpha \langle (0.3223)(0.0375),\ (0.5395)(0.11),\ (0.1383)(0.225) \rangle$

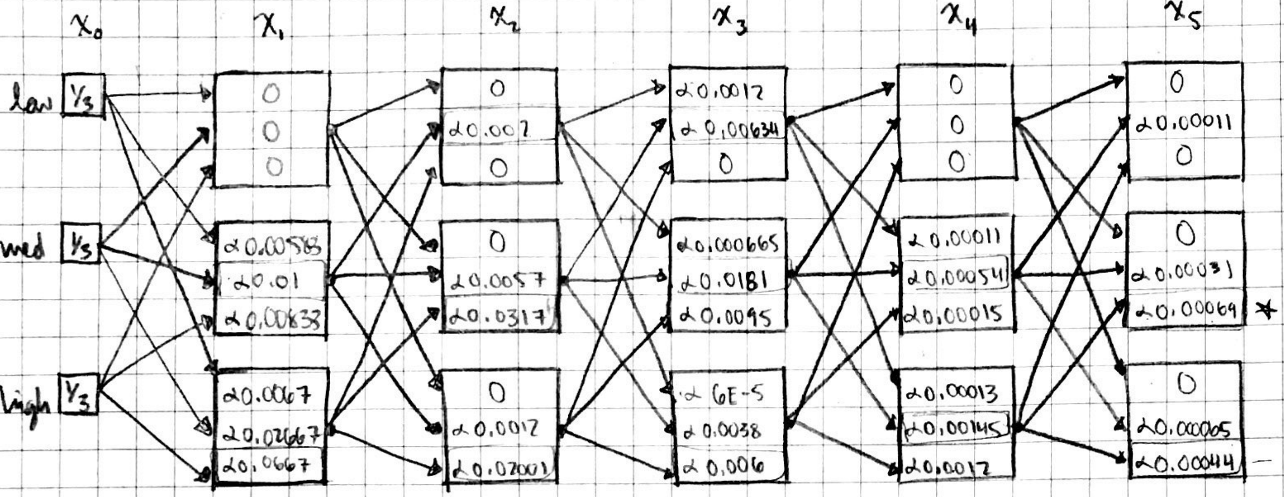
$$= \alpha \langle 0.01209,\ 0.05935,\ 0.03112 \rangle$$

$$\underline{\underline{\alpha = 9.7515}} \longrightarrow \boxed{\begin{aligned} P(x_2=\text{low} \mid e_{1:3}) &= 0.1179 \\ P(x_2=\text{med} \mid e_{1:3}) &= 0.5787 \\ P(x_2=\text{high} \mid e_{1:3}) &= 0.3034 \end{aligned}}$$



③ Most Likely Explanation

$e_1 =$ "flooded" $= e_4$ $\quad e_2 = e_3 = e_5 =$ "not flooded" $\longrightarrow P(x_{1:5} \mid e_{1:5}) = ?$



* Filtering, but with a max function instead of summation *

$$P(x_t \mid e_{1:t}) = \alpha\, P(e_t \mid x_t) \max_{x_{t-1}} \left(P(x_t \mid x_{t-1})\, P(x_{t-1} \mid e_{1:t-1}) \right)$$

x_1

@ $x_1 = \text{low}$: $P(x_1 = \text{low} \mid e_1) = \alpha\, P(e_1 \mid x_1) \max_{x_0} \left(P(x_1 \mid x_0)\, P(x_0) \right)$

$= \alpha\,(0) = \underline{0}$

@ $x_1 = \text{med}$: $P(x_1 = \text{med} \mid e_1) = \alpha\,(0.05) \max\left\{ \underbrace{(0.35)(1/3)}_{\alpha 0.00583}, \underbrace{(0.6)(1/3)}_{\alpha 0.01}, \underbrace{(0.5)(1/3)}_{\alpha 0.00833} \right\}$ *med

$= \underline{\underline{\alpha\,(0.01)}}$

@ $x_1 = \text{high}$: $P(x_1 = \text{high} \mid e_1) = \alpha\,(0.4) \max\left\{ \underbrace{(0.05)(1/3)}_{\alpha 0.00667}, \underbrace{(0.2)(1/3)}_{\alpha 0.02667}, \underbrace{(0.5)(1/3)}_{\alpha 0.0667} \right\}$ *high

$= \underline{\underline{\alpha\,(0.0667)}}$

x_2

@ $x_2 = \text{low}$: $P(x_2 = \text{low} \mid e_{1:2}) = \alpha\, P(e_2 \mid x_2) \max_{x_1} \left\{ P(x_2 \mid x_1)\, P(x_1 \mid e_1) \right\}$

$= \alpha\,(1) \max\left\{ \underbrace{(0.6)(0)}_{0}, \underbrace{(0.2)(0.01)}_{\alpha 0.002}, \underbrace{(0)}_{0} \right\}$ *med

$= \underline{\underline{\alpha\,(0.002)}}$

@ $x_2 = \text{med}$: $P(x_2 = \text{med} \mid e_{1:2}) = \alpha\,(0.95) \max\left\{ \underbrace{(0.35)(0)}_{0}, \underbrace{(0.6)(0.01)}_{\alpha 0.0057}, \underbrace{(0.5)(0.0667)}_{\alpha 0.0317} \right\}$ *high

$= \underline{\underline{\alpha\,(0.0317)}}$

@ $x_2 = \text{high}$: $P(x_2 = \text{high} \mid e_{1:2}) = \alpha\,(0.6) \max\left\{ \underbrace{(0.05)(0)}_{0}, \underbrace{(0.2)(0.01)}_{\alpha 0.0012}, \underbrace{(0.5)(0.0667)}_{\alpha 0.02001} \right\}$ *high

X_3

@ $x_3 = \text{low}$: $P(x_3 = \text{low} \mid e_{1:3}) = \alpha\, P(e_3 \mid x_3) \max_{x_2} \{ P(x_3 \mid x_2) P(x_2 \mid e_{1:2}) \}$

$= \alpha\, (1) \max \{ \underbrace{(0.6)(0.002)}_{\alpha\, 0.0012}, \underbrace{(0.2)(0.0317)}_{\alpha\, 0.00634}, \underbrace{(0)}_{0} \}$ * med

$= \alpha\, (0.00634)$

@ $x_3 = \text{med}$: $P(x_3 = \text{med} \mid e_{1:3}) = \alpha\, (0.95) \max \{ \underbrace{(0.35)(0.002)}_{\alpha\, 0.000665}, \underbrace{(0.6)(0.0317)}_{\alpha\, 0.0181}, \underbrace{(0.5)(0.02)}_{\alpha\, 0.0095} \}$ * med

$= \alpha\, (0.0181)$

@ $x_3 = \text{high}$: $P(x_3 = \text{high} \mid e_{1:3}) = \alpha\, (0.6) \max \{ \underbrace{(0.05)(0.002)}_{\alpha\, 6E-5}, \underbrace{(0.2)(0.0317)}_{\alpha\, 0.0038}, \underbrace{(0.5)(0.02)}_{0.006} \}$ * high

$= \alpha\, (0.006)$

X_4

@ $x_4 = \text{low}$: $P(x_4 = \text{low} \mid e_{1:4}) = \alpha\, P(e_4 \mid x_4) \max_{x_3} \{ P(x_4 \mid x_3) P(x_3 \mid e_{1:3}) \}$

$= \alpha\, (0)$

@ $x_4 = \text{med}$: $P(x_4 = \text{med} \mid e_{1:4}) = \alpha\, (0.05) \max \{ \underbrace{(0.35)(0.00634)}_{\alpha\, 0.00011}, \underbrace{(0.6)(0.0181)}_{\alpha\, 0.00054}, \underbrace{(0.5)(0.006)}_{\alpha\, 0.00015} \}$ * med

$= \alpha\, (0.00054)$

@ $x_4 = \text{high}$: $P(x_4 = \text{high} \mid e_{1:4}) = \alpha\, (0.4) \max \{ \underbrace{(0.05)(0.00634)}_{\alpha\, 0.00013}, \underbrace{(0.2)(0.0181)}_{\alpha\, 0.00145}, \underbrace{(0.5)(0.006)}_{\alpha\, 0.0012} \}$ * med

$= \alpha\, (0.00145)$

X_5

@ $x_5 = \text{low}$: $P(x_5 = \text{low} \mid e_{1:5}) = \alpha\, P(e_5 \mid x_5) \max_{x_4} \{ P(x_5 \mid x_4) P(x_4 \mid e_{1:4}) \}$

$= \alpha\, (1) \max \{ (0.6)(0), \underbrace{(0.2)(0.00054)}_{\alpha\, 0.00011}, (0) \}$ * med

$= \alpha\, (0.00011)$

@ $x_5 = \text{med}$: $P(x_5 = \text{med} \mid e_{1:5}) = \alpha\, (0.95) \max \{ \underbrace{(0.35)(0)}_{0}, \underbrace{(0.6)(0.00054)}_{\alpha\, 0.00031}, \underbrace{(0.5)(0.00145)}_{\alpha\, 0.00069} \}$ * high

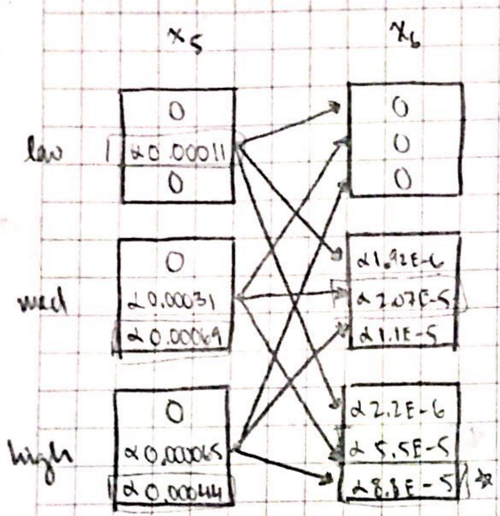
$= \alpha\, (0.00069)$

@ $x_5 = \text{high}$: $P(x_5 = \text{high} \mid e_{1:5}) = \alpha\, (0.6) \max \{ \underbrace{(0.05)(0)}_{0}, \underbrace{(0.2)(0.00054)}_{\alpha\, 0.000065}, \underbrace{(0.5)(0.00145)}_{\alpha\, 0.00044} \}$ * high

$= \alpha\, (0.00044)$

* Trace path backwards, from largest value *

$\boxed{x_5 = \text{med},\ x_4 = \text{high},\ x_3 = \text{med},\ x_2 = \text{med},\ x_1 = \text{high},\ x_0 = \text{high}}$



e_6 = "flooded"

@ $x_6 = \text{low}$: $P(x_6 \mid e_{1:6}) = \alpha P(e_6 \mid x_6) \max_{x_5} \{ P(x_6 \mid x_5) P(x_5 \mid e_{1:5}) \}$

$= \alpha(0)$

@ $x_6 = \text{med}$: $P(x_6 = \text{med} \mid e_{1:6}) = \alpha(0.05) \max \{ \overset{\alpha 1.92E-6}{(0.35)(0.00011)} + \overset{\alpha 2.07E-5}{(0.6)(0.00069)} + \overset{\alpha 1.1E-5}{(0.5)(0.00044)} \}$ med ★

$= \alpha(0.000021)$

@ $x_6 = \text{high}$: $P(x_6 = \text{high} \mid e_{1:6}) = \alpha(0.4) \max \{ \overset{\alpha 2.2E-6}{(0.05)(0.00011)} + \overset{\alpha 5.5E-5}{(0.2)(0.00069)} + \overset{\alpha 8.8E-5}{(0.5)(0.00044)} \}$ med ★

$= \alpha(0.000088)$

* Trace path backwards, from largest value *

$\boxed{x_6 = \text{high},\ x_5 = \text{high},\ x_4 = \text{high},\ x_3 = \text{med},\ x_2 = \text{med},\ x_1 = \text{high},\ x_0 = \text{high}}$

④ Particle Filtering — see particle_filter.py

$ python3 particle_filter.py 1,000,000

State for x_{10}:

low = 60,891	→	$P(x_{10} = \text{low} \mid e_{1:10}) = 0.060891$
med = 621,035	→	$P(x_{10} = \text{med} \mid e_{1:10}) = 0.621035$
high = 318,074	→	$P(x_{10} = \text{high} \mid e_{1:10}) = 0.318,074$

with 1,000,000 particles

⑤ After 10 throws, variance $\sigma^2 \le 10$

$P(x_0) = N(0,1)$ $\quad P(e_t \mid x_t) = N(x_t \mid 0.75)$

MIQUELRIUS