Wave Equation

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1 Discretization

We want to solve the equation

$$\frac{\partial^2 u}{\partial^2 t} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{1}$$

1.1 The general scheme for interior points

We use a set of approximations for the derivatives in (1). For the second-order derivative, we use the approximation:

$$\frac{\partial^2 u}{\partial^2 t} \approx \frac{u_{i,i}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2}$$

For the first-ordere derivative, we use the centered difference approximation:

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,i}^{n+1} - u_{i,j}^{n-1}}{2\Delta t}$$

For the terms on the right-hand side of the equations, we want to evaluate the outer derivative first. We define

$$\phi_x = q(x,y)\frac{\partial u}{\partial x}, \;\; \phi_y = q(x,y)\frac{\partial u}{\partial y}$$

We will adress ϕ_x as an example; the same process is applied to ϕ_x. We use a centered difference approximation for the derivative of ϕ_x. For simplicitys sake, we let $\phi = \phi_x$:

$$\left[\frac{\partial \phi}{\partial x}\right]_{i,j}^{n} \approx \frac{\phi_{i+\frac{1}{2},j} - \phi_{i-\frac{1}{2},j}}{\Delta x}$$

We then discretize $\phi_{i+\frac{1}{2},j}$ and $\phi_{i-\frac{1}{2},j}$:

$$\phi_{i+\frac{1}{2},j} = q_{i+\frac{1}{2},j}\left[\frac{\partial u}{\partial x}\right]^n_{i+\frac{1}{2},j} \approx q_{i+\frac{1}{2},j}\frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x}$$

$$\phi_{i-\frac{1}{2},j} = q_{i-\frac{1}{2},j}\left[\frac{\partial u}{\partial x}\right]^n_{i-\frac{1}{2},j} \approx q_{i-\frac{1}{2},j}\frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x}$$

We now combine these two to get

$$\left[\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right)\right]^n_{i,j} \approx \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j} - u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j})\right) \tag{2}$$

The corresponding appoximation for ϕ_y gives

$$\left[\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\right]^n_{i,j} \approx \frac{1}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1} - u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j} - u^n_{i,j-1})\right) \tag{3}$$

Next, we need to be able to compute the coefficient q between the mesh points. To do this, we use the arithmetic average:

$$q_{i+\frac{1}{2},j} = \frac{1}{2}(q_{i,j} + q_{i+1,j}) = [\bar{q}^x]_{i,j}$$

$$q_{i,j+\frac{1}{2}} = \frac{1}{2}(q_{i,j} + q_{i,j+1}) = [\bar{q}^y]_{i,j}$$

We can now write the discrete equations compactly using operator notation. The equation becomes:

$$[D_tD_tu + bD_tu = D_x\bar{q}^xD_xu + D_y\bar{q}^yD_yu + f]^n_{i,j} \tag{4}$$

1.2 Boundary conditions

We have the Neumann boundary condition

$$\frac{\partial u}{\partial n} = \mathbf{n}\cdot\nabla u$$

1.2.1 $x = 0$, $y = y_j$, $t = t^n$:

We have that $\mathbf{n} = \mathbf{i}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$\frac{u^n_{-1,j} - u^n_{1,j}}{2\Delta x} = 0$$

We can use this to express the ficticious value $u^n_{-1,j}$ (which is outside the mesh):

$$u^n_{-1,j} = u^n_{1,j} \tag{5}$$

1.2.2 $x = L_x$, $y = y_j$, $t = t^n$:

We have that $\mathbf{n} = -\mathbf{i}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$-\frac{u^n_{L_x-1,j} - u^n_{L_x+1,j}}{2\Delta x} = 0$$

We can use this to express the ficticious value $u^n_{L_x+1,j}$ (which is outside the mesh):

$$u^n_{L_x+1,j} = u^n_{L_x-1,j} \tag{6}$$

1.2.3 $x = x_i$, $y = 0$, $t = t^n$:

We have that $\mathbf{n} = \mathbf{j}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$\frac{u^n_{i,-1} - u^n_{i,1}}{2\Delta y} = 0$$

We can use this to express the ficticious value $u^n_{i,-1}$ (which is outside the mesh):

$$u^n_{i,-1} = u^n_{i,1} \tag{7}$$

1.2.4 $x = x_i$, $y = L_y$, $t = t^n$:

We have that $\mathbf{n} = -\mathbf{j}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$-\frac{u^n_{i,L_y-1} - u^n_{i,L_y+1}}{2\Delta y} = 0$$

We can use this to express the ficticious value u^n_{i,L_y+1} (which is outside the mesh):

$$u^n_{i,L_y+1} = u^n_{i,L_y-1} \tag{8}$$

1.3 Initial conditions

The equation (1) has two initial conditions:

$$u(x, y, 0) = I(x, y) \tag{9a}$$

$$u_t(x, y, 0) = V(x, y) \tag{9b}$$

The first IC (9a) is used to set $u^0_{i,j}$ for all the mesh points, i.e.

$$u^0_{i,j} = I(x_i, y_j)$$

for $i = 1, 2, ..., (L_x - 1)$ and $j = 1, 2, ..., (L_y - 1)$. The second IC (9b) is used to determine $u_{i,j}^{-1}$. We use the backward difference appriximation for the derivative:

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,j}^{n} - u_{i,j}^{n-1}}{\Delta t}$$

At $t = t^n$, this gives

$$\frac{u_{i,j}^{0} - u_{i,j}^{-1}}{2\Delta t} = V(x, y)$$

This gives us an expression for the ficticious value $u_{i,j}^{-1}$, which is needed to determine the initial condition for all the mesh points.

In the next section, we summarize the findings from this section.

2 The scheme

2.1 The general scheme for computing $u_{i,j}^{n+1}$ at interior spatial mesh points

Solving equation (4) with respect to $u_{i,j}^{n+1}$ gives us the general computational scheme for all interior $(i = 1, ..., (L_x - 1),\ j = 1, ..., (L_y - 1))$ spatial mesh points:

$$\begin{aligned} u_{i,j}^{n+1} = &\frac{1}{\left(1 + \frac{b\Delta t}{2}\right)} \Bigg[2u_{i,j}^{n} - u_{i,j}^{n-1}\left(1 - \frac{b\Delta t}{2}\right) + \\ &\frac{\Delta t^2}{2\Delta x^2}\big((q_{i,j} + q_{i+1,j})(u_{i+1,j}^{n} - u_{i,j}^{n}) - (q_{i-1,j} + q_{i,j})(u_{i,j}^{n} - u_{i-1,j}^{n})\big) + \\ &\frac{\Delta t^2}{2\Delta y^2}\big((q_{i,j} + q_{i,j+1})(u_{i,j+1}^{n} - u_{i,j}^{n}) - (q_{i,j-1} + q_{i,j})(u_{i,j}^{n} - u_{i,j-1}^{n})\big) + \\ &f(x_i, y_j, t^n)\Bigg] \end{aligned} \tag{10}$$

This scheme requires the current (n) and previous $(n-1)$ time steps in order to compute the next $(n + 1)$ time step.

2.2 The modified scheme at boundary points

At the boundary points, we replace the ficticious values that appear with the values calculated in section 1.2.

2.2.1 $x = 0$, $y = y_j$, $t = t^n$

NOTE: FIND OUT WHAT TO DO WITH $q_{-1,j}$! In this case, we replace $u^n_{-1,j}$ with $u^n_{1,j}$.

$$
\begin{aligned}
u^{n+1}_{0,j} =& \frac{1}{\left(1+\frac{b\Delta t}{2}\right)} \Bigg[2u^n_{0,j} - u^{n-1}_{0,j}\left(1 - \frac{b\Delta t}{2}\right) + \\
& \frac{\Delta t^2}{2\Delta x^2}\big((q_{0,j} + q_{1,j})(u^n_{1,j} - u^n_{0,j}) - (q_{i-1,j} + q_{0,j})(u^n_{0,j} - u^n_{1,j})\big) + \\
& \frac{\Delta t^2}{2\Delta y^2}\big((q_{0,j} + q_{0,j+1})(u^n_{0,j+1} - u^n_{0,j}) - (q_{0,j-1} + q_{0,j})(u^n_{0,j} - u^n_{0,j-1})\big) + \\
& f(x_0, y_j, t^n) \Bigg]
\end{aligned}
\tag{11}
$$

To make sure that the BC are correct also on the corners, we can do a test. If $j = 0$ or $j = L_y$, we have to replace the occurences of $u^n_{0,-1}$ and u^n_{0,L_y+1} with $u^n_{0,1}$ and u^n_{0,L_y-1} respectively. In a program, this can be done with a simple *if-test*.

2.2.2 $x = L_x$, $y = y_j$, $t = t^n$

NOTE: FIND OUT WHAT TO DO WITH $q_{L_x+1,j}$! In this case, we replace $u^n_{L_x+1,j}$ with $u^n_{L_x-1,j}$.

$$
\begin{aligned}
u^{n+1}_{L_x,j} =& \frac{1}{\left(1+\frac{b\Delta t}{2}\right)} \Bigg[2u^n_{L_x,j} - u^{n-1}_{L_x,j}\left(1 - \frac{b\Delta t}{2}\right) + \\
& \frac{\Delta t^2}{2\Delta x^2}\big((q_{L_x,j} + q_{L_x+1,j})(u^n_{L_x-1,j} - u^n_{L_x,j}) - (q_{L_x-1,j} + q_{L_x,j})(u^n_{L_x,j} - u^n_{L_x-1,j})\big) + \\
& \frac{\Delta t^2}{2\Delta y^2}\big((q_{L_x,j} + q_{L_x,j+1})(u^n_{L_x,j+1} - u^n_{L_x,j}) - (q_{L_x,j-1} + q_{L_x,j})(u^n_{L_x,j} - u^n_{L_x,j-1})\big) + \\
& f(x_{L_x}, y_j, t^n) \Bigg]
\end{aligned}
\tag{12}
$$

Again, to make sure that the BC are correct also on the corners, we can do a test. If $j = 0$ or $j = L_y$, we have to replace the occurences of $u^n_{L_x,-1}$ and $u^n_{L_x,L_y+1}$ with $u^n_{L_x,1}$ and $u^n_{L_y,L_y-1}$ respectively. In a program, this can be done with a simple *if-test*.

2.2.3 $x = x_i$, $y = 0$, $t = t^n$

Since we have tested to make sure the corners are correct already, we let $i = 1, ..., (L_x - 1)$. In this case, we replace $u^n_{i,-1}$ with $u^n_{i,1}$.

$$\begin{aligned}
u^{n+1}_{i,0} =& \frac{1}{\left(1+\frac{b\Delta t}{2}\right)}\Bigg[2u^n_{i,0} - u^{n-1}_{i,0}\left(1-\frac{b\Delta t}{2}\right)+\\
&\frac{\Delta t^2}{2\Delta x^2}\left((q_{i,0}+q_{i+1,0})(u^n_{i+1,0}-u^n_{i,0})-(q_{i-1,0}+q_{i,0})(u^n_{i,0}-u^n_{i-1,0})\right)+\\
&\frac{\Delta t^2}{2\Delta y^2}\left((q_{i,0}+q_{i,1})(u^n_{i,1}-u^n_{i,0})-(q_{i,j-1}+q_{i,0})(u^n_{i,0}-u^n_{i,1})\right)+\\
&f(x_i,y_0,t^n)\Bigg]
\end{aligned} \tag{13}$$

2.2.4 $x = x_i$, $y = L_y$, $t = t^n$

Again, since we have tested to make sure the corners are correct already, we let $i = 1, ..., (L_x - 1)$. In this case, we replace u^n_{i,L_y+1} with u^n_{i,L_y-1}.

$$\begin{aligned}
u^{n+1}_{i,L_y} =& \frac{1}{\left(1+\frac{b\Delta t}{2}\right)}\Bigg[2u^n_{i,L_y} - u^{n-1}_{i,L_y}\left(1-\frac{b\Delta t}{2}\right)+\\
&\frac{\Delta t^2}{2\Delta x^2}\Big((q_{i,L_y}+q_{i+1,L_y})(u^n_{i+1,L_y}-u^n_{i,L_y})-(q_{i-1,L_y}+q_{i,L_y})(u^n_{i,L_y}-u^n_{i-1,L_y})\Big)+\\
&\frac{\Delta t^2}{2\Delta y^2}\Big((q_{i,L_y}+q_{i,L_y-1})(u^n_{i,L_y-1}-u^n_{i,L_y})-(q_{i,L_y-1}+q_{i,L_y})(u^n_{i,L_y}-u^n_{i,L_y-1})\Big)+\\
&f(x_i,y_{L_y},t^n)\Bigg]
\end{aligned} \tag{14}$$

The modified scheme for the first step at interior points

Here, we use (9a) for $i = 1, 2, ..., (L_x - 1)$ and $j = 1, 2, ..., (L_y - 1)$, as stated earlier, and the modified scheme for the first time step becomes

$$u^0_{i,j} = I(x, y) \tag{15}$$

The modified scheme for the first step at boundary points

At the boundary points, the modified scheme for the first step is a little more complicated, as we have to take into consideration the boundary conditions as well. From section 1.3 we have an expression for $u^{-1}_{i,j}$:

$$u^{-1}_{i,j} = u^0_{i,j} - \Delta t V(x, y)$$