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Compolsory project 1

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The two-dimensional wave equation

Consider the standard two-dimensional linear wave equation including damping,

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t), \qquad (1)$$

imposed to the boundary condition $\frac{\partial u}{\partial n} = 0$ in a rectangular spatial domain $[0, L_x]\times[0, L_y]$ with the initial conditions $u(x, y, 0) = I(x, y)$ and $u_t(x, y, 0) = V(x, y)$.

Now, let the temporal domain $[0, T]$ be represented by a finite number of mesh points $\{t^n\}_{n=0}^{N}$, where

$$t^0 = 0 < t^1 < t^2 < ... < t^{N-1} < tN = T.$$

Similarly, the spatial domains $[0, L_x]$ and $[0, L_y]$ are replaced by $\{x_i\}_{i=0}^{N_x}$ and $\{y_j\}_{j=0}^{N_y}$, respectively, where

$$0 = x_0 < x_1 < x_2 < ... < x_{N_x-1} < x_{N_x} = L_x,$$
$$0 = y_0 < y_1 < y_2 < ... < y_{N_y-1} < y_{N_y} = L_y.$$

For uniformly distributed mesh points, the constant mesh spacings Δt, Δx, and Δy may be introduced, such that

$$t^n = n\Delta t, \qquad n = 0, ..., N,$$
$$x_i = i\Delta x, \qquad i = 0, ..., N_x,$$
$$y_j = j\Delta y, \qquad j = 0, ..., N_y.$$

The solution $u(x,y,t)$ is sought at the mesh points, that is $u(x_i,y_j,t^n)$ for $i=0,...,N_x$, $j=0,...,N_y$, and $n=0,...,N$. The numerical approximation at the mesh point (x_i,y_j,t^n) is denoted as $u^n_{i,j}$. For a numerical solution by the finite difference method, the condition that (1) holds for all points in the domain $(0,L_x)\times(0,L_y)\times(0,T]$ is therefore is reduces to the requirement that the PDE is fulfilled at the mesh points.

By means of centered differences, the first- and second-order derivative with respect to time can be expressed as

$$\frac{\partial}{\partial t}u(x_i,y_j,t^n)\approx\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\Delta t}=[D_{2t}u]^n_{i,j},$$

and

$$\frac{\partial^2}{\partial t^2}u(x_i,y_j,t^n)\approx\frac{u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}}{\Delta t^2}=[D_tD_tu]^n_{i,j}\,,$$

respectively. Furthermore, the principal idea when discretizing a variable coefficient is to first discretize the outer derivative by means of the centered derivative, and thereafter similarly discretize the inner derivative. Defining $\phi_x=q(x,y)\frac{\partial u}{\partial x}$, $\phi_y=q(x,y)\frac{\partial u}{\partial y}$, and following through with the abovementioned description,

$$\left[\frac{\partial\phi_x}{\partial x}\right]^n_{i,j}\approx\frac{(\phi_x)^n_{i+\frac{1}{2},j}-(\phi_x)^n_{i-\frac{1}{2},j}}{\Delta x}=[D_x\phi_x]^n_{i,j}.$$

The contributions from the inner derivative are thus

$$(\phi_x)^n_{i+\frac{1}{2},j}=q_{i+\frac{1}{2},j}\left[\frac{\partial u}{\partial x}\right]^n_{i+\frac{1}{2},j}\approx q_{i+\frac{1}{2},j}\frac{u^n_{i+1,j}-u^n_{i,j}}{\Delta x}=[qD_xu]^n_{i+\frac{1}{2},j},$$

and

$$(\phi_x)^n_{i-\frac{1}{2},j}=q_{i-\frac{1}{2},j}\left[\frac{\partial u}{\partial x}\right]^n_{i-\frac{1}{2},j}\approx q_{i-\frac{1}{2},j}\frac{u^n_{i,j}-u^n_{i-1,j}}{\Delta x}=[qD_xu]^n_{i-\frac{1}{2},j}.$$

These intermediate results combined yield that

$$\left[\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right)\right]^n_{i,j}\approx\frac{1}{\Delta x^2}\Big(q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j})-q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})\Big)=[D_xqD_xu]^n_{i,j}\,.$$

Similar reasoning may be applied to ϕ_y, such that

$$\left[\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\right]^n_{i,j}\approx\frac{1}{\Delta y^2}\Big(q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j})-q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})\Big)=[D_yqD_yu]^n_{i,j}\,.$$

Hence, a discretization of (1) in terms of centered differences reads

$$\big[D_tD_tu + bD_{2t}u = c^2(D_xD_xu + D_yD_yu) + f\big]_{i,j}^n .$$

Written out in full detail, this is equivalent to

$$\begin{aligned}
&\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} + b\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} \\
&= \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^n - u_{i,j}^n) - q_{i-\frac{1}{2},j}(u_{i,j}^n - u_{i-1,j}^n)\right) \\
&+ \frac{1}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^n - u_{i,j}^n) - q_{i,j-\frac{1}{2}}(u_{i,j}^n - u_{i,j-1}^n)\right) + f_{i,j}^n.
\end{aligned}$$

Due to the arithmetic mean, values for q may be determined even though the function might be discrete, that is

$$\begin{aligned}
q_{i+\frac{1}{2},j} &= \frac{q_{i,j} + q_{i+1,j}}{2}, \\
q_{i-\frac{1}{2},j} &= \frac{q_{i,j} + q_{i-1,j}}{2}, \\
q_{i,j+\frac{1}{2}} &= \frac{q_{i,j} + q_{i,j+1}}{2}, \\
q_{i,j-\frac{1}{2}} &= \frac{q_{i,j} + q_{i,j-1}}{2}.
\end{aligned}$$

Finally, solving for the unknown $u_{i,j}^{n+1}$ result in the scheme

$$\begin{aligned}
u_{i,j}^{n+1} = (1 + \frac{b}{2}\Delta t)^{-1}\Bigg[&2u_{i,j}^n - (1 - \frac{b}{2}\Delta t)u_{i,j}^{n-1} + \Delta t^2 f_{i,j}^n \\
&+ \frac{\Delta t^2}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^n - u_{i,j}^n) - q_{i-\frac{1}{2},j}(u_{i,j}^n - u_{i-1,j}^n)\right) \\
&+ \frac{\Delta t^2}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^n - u_{i,j}^n) - q_{i,j-\frac{1}{2}}(u_{i,j}^n - u_{i,j-1}^n)\right)\Bigg].
\end{aligned} \tag{2}$$

Due to the imposed Neumann condition on the boundary,

$$\frac{\partial u}{\partial n} = \mathbf{n}\cdot\nabla u = n_x\frac{\partial u}{\partial x} + n_y\frac{\partial u}{\partial y} = 0.$$

On the edge $x = 0$ the normal vector pointing out of the spatial domain is $-\hat{i} = (-1, 0)$, so consequently $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}$. Discretizing this for $\{y_j\}_{j=0}^{N_y}$ and $\{t^n\}_{n=0}^{N}$ provide the relation

$$-\frac{u_{1,j}^n - u_{-1,j}^n}{2\Delta x} = 0 \implies u_{-1,j}^n = u_{1,j}^n.$$

Inserting this condition into the scheme provide a modified scheme for the boundary points at $x = 0$, that is

$$
\begin{aligned}
u_{0,j}^{n+1} = (1 + \frac{b}{2}\Delta t)^{-1} \Big[& 2u_{0,j}^{n} - (1 - \frac{b}{2}\Delta t)u_{0,j}^{n-1} + \Delta t^2 f_{0,j}^{n} \\
& + \frac{\Delta t^2}{\Delta x^2} \Big(q_{\frac{1}{2},j}(u_{1,j}^{n} - u_{0,j}^{n}) - q_{-\frac{1}{2},j}(u_{0,j}^{n} - u_{1,j}^{n}) \Big) \\
& + \frac{\Delta t^2}{\Delta y^2} \Big(q_{0,j+\frac{1}{2}}(u_{0,j+1}^{n} - u_{0,j}^{n}) - q_{0,j-\frac{1}{2}}(u_{0,j}^{n} - u_{0,j-1}^{n}) \Big) \Big].
\end{aligned} \tag{3}
$$

On the edge $x = L_x$ the normal vector pointing out of the spatial domain is $\hat{i} = (1, 0)$, so consequently $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}$. Discretizing this for $\{y_j\}_{j=0}^{N_y}$ and $\{t^n\}_{n=0}^{N}$ provide the relation

$$
\frac{u_{N_x+1,j}^{n} - u_{N_x-1,j}^{n}}{2\Delta x} = 0 \implies u_{N_x+1,j}^{n} = u_{N_x-1,j}^{n}.
$$

Inserting this condition into the scheme provide a modified scheme for the boundary points at $x = L_x$, that is

$$
\begin{aligned}
u_{N_x,j}^{n+1} = (1 + \frac{b}{2}\Delta t)^{-1} \Big[& 2u_{N_x,j}^{n} - (1 - \frac{b}{2}\Delta t)u_{N_x,j}^{n-1} + \Delta t^2 f_{N_x,j}^{n} \\
& + \frac{\Delta t^2}{\Delta x^2} \Big(q_{N_x+\frac{1}{2},j}(u_{N_x-1,j}^{n} - u_{N_x,j}^{n}) - q_{N_x-\frac{1}{2},j}(u_{N_x,j}^{n} - u_{N_x-1,j}^{n}) \Big) \\
& + \frac{\Delta t^2}{\Delta y^2} \Big(q_{N_x,j+\frac{1}{2}}(u_{N_x,j+1}^{n} - u_{N_x,j}^{n}) - q_{N_x,j-\frac{1}{2}}(u_{N_x,j}^{n} - u_{N_x,j-1}^{n}) \Big) \Big].
\end{aligned} \tag{4}
$$

On the edge $y = 0$ the normal vector pointing out of the spatial domain is $-\hat{j} = (0, -1)$, so consequently $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y}$. Discretizing this for $\{x_i\}_{i=0}^{N_x}$ and $\{t^n\}_{n=0}^{N}$ provide the relation

$$
-\frac{u_{i,1}^{n} - u_{i,-1}^{n}}{2\Delta y} = 0 \implies u_{i,-1}^{n} = u_{i,1}^{n}.
$$

Inserting this condition into the scheme provide a modified scheme for the boundary points at $y = 0$, that is

$$
\begin{aligned}
u_{i,0}^{n+1} = (1 + \frac{b}{2}\Delta t)^{-1} \Big[& 2u_{i,0}^{n} - (1 - \frac{b}{2}\Delta t)u_{i,0}^{n-1} + \Delta t^2 f_{i,0}^{n} \\
& + \frac{\Delta t^2}{\Delta x^2} \Big(q_{i+\frac{1}{2},0}(u_{i+1,0}^{n} - u_{i,0}^{n}) - q_{i-\frac{1}{2},0}(u_{i,0}^{n} - u_{i-1,0}^{n}) \Big) \\
& + \frac{\Delta t^2}{\Delta y^2} \Big(q_{i,\frac{1}{2}}(u_{i,1}^{n} - u_{i,0}^{n}) - q_{i,-\frac{1}{2}}(u_{i,0}^{n} - u_{i,1}^{n}) \Big) \Big].
\end{aligned} \tag{5}
$$

On the edge $y = L_y$ the normal vector pointing out of the spatial domain is $\hat{j} = (0, 1)$, so consequently $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial y}$. Discretizing this for $\{x_i\}_{i=0}^{N_x}$ and $\{t^n\}_{n=0}^{N}$ provide the relation

$$\frac{u^n_{i,N_y+1} - u^n_{i,N_y-1}}{2\Delta y} = 0 \implies u^n_{i,N_y+1} = u^n_{i,N_y-1}.$$

Inserting this condition into the scheme provide a modified scheme for the boundary points at $y = L_y$, that is

$$\begin{aligned} u^{n+1}_{i,N_y} = (1 + \frac{b}{2}\Delta t)^{-1} \Big[& 2u^n_{i,N_y} - (1 - \frac{b}{2}\Delta t)u^{n-1}_{i,N_y} + \Delta t^2 f^n_{i,N_y} \\ & + \frac{\Delta t^2}{\Delta x^2}\Big(q_{i+\frac{1}{2},N_y}(u^n_{i+1,N_y} - u^n_{i,N_y}) - q_{i-\frac{1}{2},N_y}(u^n_{i,N_y} - u^n_{i-1,N_y})\Big) \\ & + \frac{\Delta t^2}{\Delta y^2}\Big(q_{i,N_y+\frac{1}{2}}(u^n_{i,N_y-1} - u^n_{i,N_y}) - q_{i,N_y-\frac{1}{2}}(u^n_{i,N_y} - u^n_{i,N_y-1})\Big)\Big]. \end{aligned} \tag{6}$$

The initial conditions are $u(x, y, 0) = I(x, y)$ and $u_t(x, y, 0) = V(x, y)$. This implies that

$$u^0_{i,j} = I(x_i, y_j), \tag{7}$$

and

$$\frac{u^1_{i,j} - u^{-1}_{i,j}}{2\Delta t} = V(x_i, y_j),$$

for all $i = 1, ..., N_x - 1$ and $j = 1, ..., N_y - 1$. Now, evaluating the scheme at t_1 one may eliminate the contribution from the point outside the mesh, that is

$$\begin{aligned} u^1_{i,j} = (1 + \frac{b}{2}\Delta t)^{-1} \Big[& 2u^0_{i,j} - (1 - \frac{b}{2}\Delta t)u^{-1}_{i,j} + \Delta t^2 f^0_{i,j} \\ & + \frac{\Delta t^2}{\Delta x^2}\Big(q_{i+\frac{1}{2},j}(u^0_{i+\frac{1}{2},j} - u^0_{i,j}) - q_{i-\frac{1}{2},j}(u^0_{i,j} - u^0_{i-\frac{1}{2},j})\Big) \\ & + \frac{\Delta t^2}{\Delta y^2}\Big(q_{i,j+\frac{1}{2}}(u^0_{i,j+\frac{1}{2}} - u^0_{i,j}) - q_{i,j-\frac{1}{2}}(u^0_{i,j} - u^0_{i,j-\frac{1}{2}})\Big)\Big]. \end{aligned}$$

Inserting the initial Neumann condition $u^{-1}_{i,j} = u^1_{i,j} - 2\Delta t V_{i,j}$ provides the necessary second initial condition

$$
\begin{aligned}
u^1_{i,j} &= I_{i,j} + \Delta t(1 - \frac{b}{2}\Delta t)V_{i,j} + \Delta t^2 f^0_{i,j} \\
&\quad + \frac{1}{2}\Bigg[\frac{\Delta t^2}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u^0_{i+\frac{1}{2},j} - u^0_{i,j}) - q_{i-\frac{1}{2},j}(u^0_{i,j} - u^0_{i-\frac{1}{2},j})\right) \\
&\quad + \frac{\Delta t^2}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u^0_{i,j+\frac{1}{2}} - u^0_{i,j}) - q_{i,j-\frac{1}{2}}(u^0_{i,j} - u^0_{i,j-\frac{1}{2}})\right)\Bigg].
\end{aligned} \tag{8}
$$