Wave Equation

The Author

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Discretization

We want to solve the equation

$$\frac{\partial^2 u}{\partial^2 t} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) \tag{1}$$

The general scheme for interior points

We use a set of approximations for the derivatives in (1). For the second-order derivative, we use the approximation:

$$\frac{\partial^2 u}{\partial^2 t} \approx \frac{u_{i,i}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2}$$

For the first-ordere derivative, we use the centered difference approximation:

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,i}^{n+1} - u_{i,j}^{n-1}}{2\Delta t}$$

For the terms on the right-hand side of the equations, we want to evaluate the outer derivative first. We define

$$\phi_x = q(x,y)\frac{\partial u}{\partial x}, \;\; \phi_y = q(x,y)\frac{\partial u}{\partial y}$$

We will adress ϕ_x as an example; the same process is applied to ϕ_x. We use a centered difference approximation for the derivative of ϕ_x. For simplicitys sake, we let $\phi = \phi_x$:

$$\left[\frac{\partial \phi}{\partial x}\right]_{i,j}^{n} \approx \frac{\phi_{i+\frac{1}{2},j} - \phi_{i-\frac{1}{2},j}}{\Delta x}$$

We then discretize $\phi_{i+\frac{1}{2},j}$ and $\phi_{i-\frac{1}{2},j}$:

$$\phi_{i+\frac{1}{2},j} = q_{i+\frac{1}{2},j}\left[\frac{\partial u}{\partial x}\right]^n_{i+\frac{1}{2},j} \approx q_{i+\frac{1}{2},j}\frac{u^n_{i+1,j}-u^n_{i,j}}{\Delta x}$$
$$\phi_{i-\frac{1}{2},j} = q_{i-\frac{1}{2},j}\left[\frac{\partial u}{\partial x}\right]^n_{i-\frac{1}{2},j} \approx q_{i-\frac{1}{2},j}\frac{u^n_{i,j}-u^n_{i-1,j}}{\Delta x}$$

We now combine these two to get

$$\left[\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right)\right]^n_{i,j} \approx \frac{1}{\Delta x^2}\Big(q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})\Big) \tag{2}$$

The corresponding appoximation for ϕ_y gives

$$\left[\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\right]^n_{i,j} \approx \frac{1}{\Delta y^2}\Big(q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})\Big) \tag{3}$$

Next, we need to be able to compute the coefficient q between the mesh points. To do this, we use the arithmetic average:

$$q_{i+\frac{1}{2},j} = \frac{1}{2}(q_{i,j}+q_{i+1,j}) = [\bar{q}^x]_{i,j}$$
$$q_{i,j+\frac{1}{2}} = \frac{1}{2}(q_{i,j}+q_{i,j+1}) = [\bar{q}^y]_{i,j}$$

We can now write the discrete equations compactly using operator notation. The equation becomes:

$$[D_tD_tu + bD_tu = D_x\bar{q}^xD_xu + D_y\bar{q}^yD_yu + f]^n_{i,j} \tag{4}$$

Solving this with respect to $u^{n+1}_{i,j}$ gives us the general computational scheme for the inner points:

$$u^{n+1}_{i,j} = 2u^n_{i,j} - u^{n-1}_{i,j} \tag{5}$$
$$\frac{\Delta t^2}{\Delta x^2}\left(\frac{1}{2}(q_{i,j}+q_{i+1,j})(u^n_{i+1,j}-u^n_{i,j}) - \frac{1}{2}(q_{i-1,j}+q_{i,j})(u^n_{i,j}-u^n_{i-1,j})\right) + \tag{6}$$
$$\frac{\Delta t^2}{\Delta y^2}\left(\frac{1}{2}(q_{i,j}+q_{i,j+1})(u^n_{i,j+1}-u^n_{i,j}) - \frac{1}{2}(q_{i,j-1}+q_{i,j})(u^n_{i,j}-u^n_{i,j-1})\right) + \tag{7}$$
$$f(x_i,y_j,t_n) \tag{8}$$

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This scheme requires the current and previous time steps in order to compute the next time step.

Initial conditions

The equation (1) has two initial conditions:

$$u(x, y, 0) = I(x, y)$$
$$u_t(x, y, 0) = V(x, y)$$

The first IC is used to set $u^0_{i,j}$ for all inner points, i.e.

$$u^0_{i,j} = I(x_i, y_j)$$

for

Boundary conditions

We have the Neumann boundary condition

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$$

$x = 0$, $y = y_j$, $t = t_n$:

We have that $\mathbf{n} = \mathbf{i}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$\frac{u^n_{-1,j} - u^n_{1,j}}{2\Delta x} = 0$$

We can use this to express the ficticious value $u^n_{-1,j}$ (which is outside the mesh):

$$u^n_{-1,j} = u^n_{1,j} \tag{9}$$

$x = L_x$, $y = y_j$, $t = t_n$:

We have that $\mathbf{n} = -\mathbf{i}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$-\frac{u^n_{L_x-1,j} - u^n_{L_x+1,j}}{2\Delta x} = 0$$

We can use this to express the ficticious value $u^n_{L_x+1,j}$ (which is outside the mesh):

$$u^n_{L_x+1,j} = u^n_{L_x-1,j} \tag{10}$$

$x = x_i,\ y = 0,\ t = t_n$:

We have that $\mathbf{n} = \mathbf{j}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$\frac{u^n_{i,-1} - u^n_{i,1}}{2\Delta y} = 0$$

We can use this to express the ficticious value $u^n_{i,-1}$ (which is outside the mesh):

$$u^n_{i,-1} = u^n_{i,1} \tag{11}$$

$x = x_i,\ y = L_y,\ t = t_n$:

We have that $\mathbf{n} = -\mathbf{j}$. Discretizing the boundary condition using a centered difference at this boundary gives

$$-\frac{u^n_{i,L_y-1} - u^n_{i,L_y+1}}{2\Delta y} = 0$$

We can use this to express the ficticious value u^n_{i,L_y+1} (which is outside the mesh):

$$u^n_{i,L_y+1} = u^n_{i,L_y-1} \tag{12}$$

By using (9), (10), (11) and (12) in the discretized scheme, we have a modified scheme at the boundary points.