

Assessment 1

Logic,
Regular Languages,
and
Finite Automata

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Propositional Logic

1. Model as a propositional formal the poroperty that the inital state cannot be reached from c or b

Answer:

$$model \rightarrow \neg((c \rightarrow a') \vee (b \rightarrow a'))$$

2. Prove the following sequent:

$$a \rightarrow b'. b \rightarrow c'. c \rightarrow b' \vdash (a \vee b) \rightarrow (b' \vee c')$$

$$1. a \rightarrow b'. b \rightarrow c'. c \rightarrow b' \text{ [premise]}$$

$$2. (a \rightarrow b') \wedge (b \rightarrow c') \wedge (c \rightarrow b') \text{ } [\wedge_i] \text{ } 1$$

$$3. (a \rightarrow b') \wedge (c \rightarrow b') \wedge (b \rightarrow c') \text{ [commutativity of } \wedge] \text{ } 2$$

$$4. (a \wedge c \rightarrow b') \wedge (b \rightarrow c') \text{ [distributivity of } \rightarrow] \text{ } 3$$

$$5. a \wedge c \rightarrow b' \text{ } [\wedge_{e2}] \text{ } 4$$

$$6. a \rightarrow b' \text{ } [\wedge_{e2}] \text{ } 5$$

$$7. a \vee b \rightarrow b' \text{ } [\vee_{i2}] \text{ } 6$$

$$8. (a \vee b) \rightarrow (b' \vee c') \text{ } [\vee_i] \text{ } 7 \text{ [conclusion]}$$

3. Model the property that the automate can only be in one stat at a time via a defintion of a propositional function

Answer:

$$A \oplus B \equiv (A \vee B) \wedge (\neg A \vee \neg B) \text{ [definiton of the XOR operator]}$$

$$goodState(a, b, c) \equiv (a \rightarrow b') \oplus (b \rightarrow c') \oplus (c \rightarrow b')$$

4. Negate the formula $((a \rightarrow b') \wedge (a \wedge \neg b) \wedge (b' \wedge \neg a')) \rightarrow (b \rightarrow \neg a')$ and convert it to CNF.

(a) Negate the formula

$$\neg(((a \rightarrow b') \wedge (a \wedge \neg b) \wedge (b' \wedge \neg a')) \rightarrow (b \rightarrow \neg a'))$$

(b) Remove \rightarrow using $A \rightarrow B \equiv \neg A \vee B$ property

$$\neg(((\neg a \vee b') \wedge (\neg a \vee \neg b) \wedge (b' \wedge \neg a')) \rightarrow (\neg b \vee \neg a'))$$

$$\neg(\neg((\neg a \vee b') \wedge (\neg a \vee \neg b) \wedge (b' \wedge \neg a')) \vee (\neg b \vee \neg a'))$$

(c) Push \neg inwards using De Morgan's Laws

$$((\neg a \vee b') \wedge (\neg a \vee \neg b) \wedge (b' \wedge \neg a')) \wedge \neg(\neg b \vee \neg a')$$

$$((\neg a \vee b') \wedge (\neg a \vee \neg b) \wedge (b' \wedge \neg a')) \wedge (b \wedge a')$$

(d) Distribute

$$b \wedge ((\neg a \vee b') \wedge (\neg a \vee \neg b) \wedge (b' \wedge \neg a')) \wedge a' \wedge ((\neg a \vee b') \wedge (\neg a \vee \neg b) \wedge (b' \wedge \neg a'))$$

$$b \wedge (\neg a \vee b') \wedge b \wedge (\neg a \vee \neg b) \wedge b \wedge (b' \wedge \neg a') \wedge a' \wedge (\neg a \vee b') \wedge a' \wedge (\neg a \vee \neg b) \wedge a' \wedge (b' \wedge \neg a')$$

(e) Simplify (remove duplicate elements because $A \wedge A \equiv A$)

$$(\neg a \vee b') \wedge b \wedge (b' \wedge \neg a') \wedge (\neg a \vee \neg b) \wedge a'$$

Answer:

$$(\neg a \vee b') \wedge b \wedge (b' \wedge \neg a') \wedge (\neg a \vee \neg b) \wedge a'$$

5. Explain the principle of unit propagation from DPLL, and use it to show that CNF formula of the previous question is unsatisfiable.

1. **Answer:**

Unit propagation refers to a technique that is used to simplify logical formulas. When we have a clause that *only* contains a single literal such as x , in all other clauses that contain x , we can replace it with *True*.

2. **Answer:**

We can use it to show that the previous formula is unsatisfiable by demonstrating that it is a contradiction. We simplify it and continue to substitute *True* for single literal clauses. When we rearrange the formula, it becomes clear that we have $b \wedge \neg b$, this is a contradiction.

$$\begin{aligned} & (True \vee b') \wedge True \wedge \neg b \wedge b' \wedge \neg a' \wedge b \\ & (True \vee True) \wedge True \wedge \neg b \wedge True \wedge \neg a' \wedge b \\ & \neg b \wedge True \wedge \neg a' \wedge b \\ & \neg b \wedge \neg a' \wedge b \\ & \neg b \wedge b \wedge \neg a' \end{aligned}$$

6. Comment on the validity of the original formula.

Answer:

The formula is not satisfiable and therefore not valid as satisfiability is a prerequisite for validity. In other words, because that logical statement was a contradiction (*assumed something can be False and True at the same time*), there was no way in which it could have been true and because of that it is not valid.

First Order Logic

1. Write a formula in first-order logic, using the parent relation, that states that two entities x and y are siblings if they share a parent

Answer:

$$\forall i. \forall j. \text{siblings}(i, j) \equiv (\text{parent}(x, i) = \text{parent}(x, j))$$

2. Assuming there is an equality relation \equiv on P such that $x \equiv y$ means x and y are the same entity, write a formula in first order logic stating that every entity has two distinct parents.

Answer:

$$\begin{aligned} \exists! x. P(x) &\equiv \exists x (P(x) \wedge \forall y (P(y) \rightarrow y = x)) \text{ }^1 \\ \forall i. \exists! y. \exists! x (\text{parent}(x, i) \wedge \text{parent}(y, i) \wedge \neg(x \equiv y)) \end{aligned}$$

3. Prove that the following sequent is valid:

$$\text{parent}(p, q). \forall x. \forall y (\text{parent}(x, y) \rightarrow \exists z. \text{parent}(z, x)) \vdash \exists. \text{parent}(z, p)$$

1. $\text{parent}(p, q). \forall x. \forall y (\text{parent}(x, y) \rightarrow \exists z. \text{parent}(z, x))$ [premise]

2. —

3. $\exists. \text{parent}(z, p)$ [conclusion] 1 – ?

4. Interpret the following formula into simple English statement

$$\forall x. \exists y. \text{parent}(y, x) \rightarrow \forall a. \exists b. \text{parent}(a, b)$$

Answer:

If all children have a parent, then all parents have a child.

5. The sequent below is not valid. Show this by proving a counter-example: a concrete definition of the universe P (a set) and the relation parent. Explain why this is a counterexample.

$$\vdash \forall x. \exists y. \text{parent}(y, x) \rightarrow \forall a. \exists b. \text{parent}(a, b)$$

In English

if all children have a parent then all parents have a child.

1. $\forall x. \exists y. \text{parent}(y, x) \rightarrow \forall a. \exists b. \text{parent}(a, b)$ [premise]

2. $P = \{ x \mid x \}$

3. *False* [conclusion]

¹<https://math.stackexchange.com/questions/228285/how-can-i-get-the-negation-of-exists-unique-existential-quantification>

6. What is the smallest possible counterexample universe P and relation parent . Explain your reasoning.

Answer:

The smallest possible counterexample universe (a set) would consist of one element, because one counterexample is sufficient to falsify any statement.

7. Prove that the following sequent is valid.

$$\forall x.\forall y.\text{parent}(x,y) \rightarrow \neg(x \equiv y) \vdash \neg\exists x.\text{parent}(x,x)$$

ie. if someone is a parent and has a child, then that child cannot be it's parent.

1. $\forall x.\forall y.\text{parent}(x,y) \rightarrow \neg(x \equiv y)$ [premise]

2. $\text{parent}(x_i, y_j) \rightarrow \neg(x_i \equiv y_j) \quad \forall_e 1$

3. $\neg\text{parent}(x_i, y_j) \vee \neg(x_i \equiv y_j)$ because $P \rightarrow Q \equiv \neg P \vee Q$

4. $\neg(\text{parent}(x_i, y_j) \wedge (x_i \equiv y_j))$ De Morgan's Law

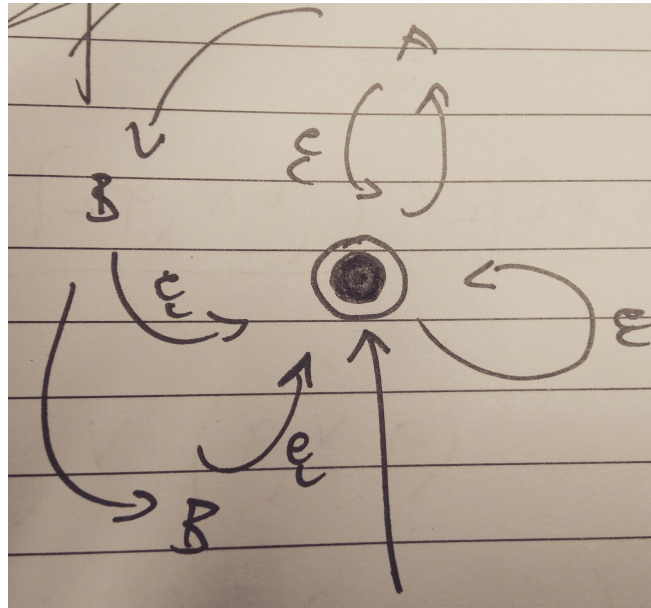
5. $\neg\exists x.\text{parent}(x,x)$ [conclusion]

Regular Languages and Finite Automata

1. For the regular expression $(A|AB|ABB)^*$ do the following steps (and justify them):

- translate it into an NFA

Answer:

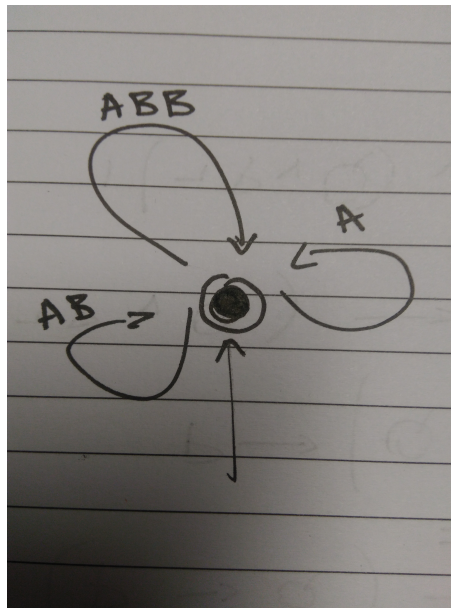


- remove (semantic preserving) the ϵ transitions

Answer:

- create the corresponding DFA

Answer:



2. Consider the following language: A word is in the language if and only if it contains an even number of A s an odd number of B s and precisely one C .

- Design a DFA for this language.

Answer:

Not possible.

- Explain your design (what information do you “store” in your states?)

Answer:

Not possible.

References

- <https://en.wikipedia.org/wiki/Negation>
- https://en.wikipedia.org/wiki/Distributive_property
- https://en.wikipedia.org/wiki/Conjunctive_normal_form
- https://en.wikipedia.org/wiki/De_Morgan%27s_laws
- https://en.wikipedia.org/wiki/Modus_ponens
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