STUDY OF SHOCK FRONT AND SOLITARY PROFILE OF ELECTRON ACOUSTIC WAVE IN WEAK RELATIVISTIC PLASMA IN CRITICAL LIMIT

A Dissertation Presented by

Nilayan Paul

Rajabazar Science College

(University of Calcutta)

Under the guidance of

Dr. Swarniv Chandra

Assistant Professor of Physics

Government General Degree College at Kushmandi,

West Bengal 733121, India

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OATH

Oath of the Scientist

As a part of the scientific community, I earnestly assert that

- I will apply my scientific skills and principles to benefit society;
- I will continue to practice and support a scientific process that is based on logic, intellectual rigor, personal integrity, and an uncompromising respect for truth;
- I will treat my colleagues’ work with respect and objectivity;
- I will convey these scientific principles in my chosen profession, in Mentoring, and in public debate;
- I will seek to increase public understanding of the principles of science and its humanitarian goals.
- These things I do promise.

The pledge

India is my country. All Indians are my brothers and sisters
I love my country, and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall give respect to my parents, teachers and all the elders, and treat everyone with courtesy.
To my country and my people, I pledge my devotion.
In their wellbeing and prosperity alone, lies my happiness.

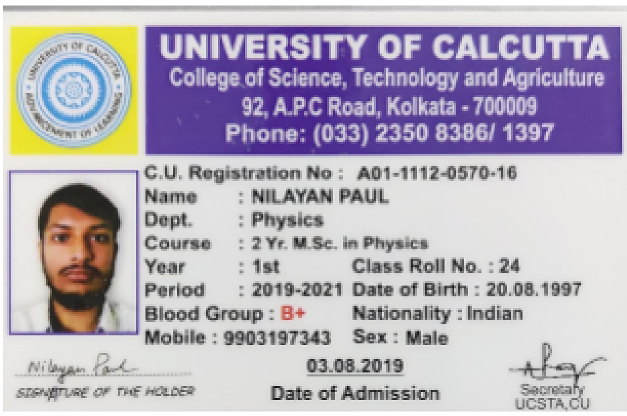
ভারত আমার দেশ।
সব ভারতবাসী আমার ভাই বোন।
আমি আমার দেশ কে ভালোবাসি।
আমার দেশের বিবিধ সংস্কৃতিতে আমি গর্বিত।
আমি, আমার দেশের সুযোগ্য অধিকারী হওয়ার জন্য সদা প্রচেষ্টায় থাকবো।
আমি নিজের মা, বাবা, শিক্ষক এবং গুরুজনদের সদা সম্মান করবো।
এবং বিনীত থাকবো।
আমি আমার দেশ ও দেশবাসীদের প্রতি সত্যনিষ্ঠার প্রতিজ্ঞাবদ্ধ হলাম।
এঁদের কল্যাণ এবং সমৃদ্ধি তেই আমার সুখ বিলীন।

DECLARATION

I would like to declare that this research project titled "STUDY OF SHOCK FRONT AND SOLITARY PROFILE OF ELECTRON ACOUSTIC WAVE IN WEAK RELATIVISTIC PLASMA IN CRITICAL LIMIT" has been prepared by me under the guidance of Dr. Swarniv Chandra during the four week long "Summer Workshop on Plasma Physics 2020", Organized by PhysicsJoint , An Association of Physicists, located at Kolkata during 6th July to 8th August, 2020. This work is the outcome of pure academic interest.

Nilayan Paul

Nilayan Paul,
Regn. ID: B1P36,
Rajabazar Science College,
University of Calcutta.



UNIVERSITY OF CALCUTTA
College of Science, Technology and Agriculture
92, A.P.C Road, Kolkata - 700009
Phone: (033) 2350 8386/ 1397

C.U. Registration No : A01-1112-0570-16
Name : NILAYAN PAUL
Dept. : Physics
Course : 2 Yr. M.Sc. in Physics
Year : 1st Class Roll No. : 24
Period : 2019-2021 Date of Birth : 20.08.1997
Blood Group : B+ Nationality : Indian
Mobile : 9903197343 Sex : Male

Nilayan Paul
SIGNATURE OF THE HOLDER

03.08.2019
Date of Admission

Secretary
UCSTA,CU

CERTIFICATE

This is to certify that this research project titled "STUDY OF SHOCK FRONT AND SOLITARY PROFILE OF ELECTRON ACOUSTIC WAVE IN WEAK RELATIVISTIC PLASMA IN CRITICAL LIMIT" submitted by Nilayan Paul(Regn. ID: B1P36) of Rajabazar Science College(University of Calcutta) is the outcome of the four week long "Summer Workshop on Plasma Physics 2020", Organized by PhysicsJoint , An Association of Physicists , Kolkata during 6th July to 8th August, 2020 under the guidance of Dr. Swarniv Chandra. It is further certified that the work is purely academic in nature and an exercise towards the strengthening of the scientific community of India.

Dr. Swarniv Chandra
Assistant Professor of Physics
Government General Degree College at Kushmandi,
West Bengal 733121, India

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Kolkata, 02/08/2020,

Nilayan Paul

Study of shock front and solitary profile of electron acoustic wave in a weakly relativistic plasma in critical limit

Nilayan Paul

2/08/2020

1 Abstract

Using the quantum hydrodynamic model, the Korteweg–de Vries Burgers(KdVB) type solitary structures of electron acoustic waves(EAWs) have been analytically studied, in a homogeneous, isotropic, non-magnetised plasma consisting of two electron temperatures with a weak relativistic degeneracy. The dispersion relation for such waves has been derived, and the effect of various plasma parameters(Quantum diffraction H, density related parameter δ, relativity parameter β, viscosity η) has been thoroughly investigated. The KdVB equation is derived in a critical regime using the reductive perturbation technique, and the effect of the above parameters on the resulting travelling shock wave solution have been studied by simulation models and hence an observation on the formation and propagation of shock waves is made.

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2 Introduction

A plasma is a quasi-neutral gas of charged and neutral particles which exhibits collective behaviour. While individual single particle like behaviour is dominant in low density plasma, the fluid like behaviour is visible in high density plasma, such as those present in the upper atmosphere. Hence, the motion of fluid elements is taken into account for such cases, although it is much different from the usual fluid dynamics due to the involvement of electric and/or magnetic fields. In the fluid approximation, the plasma is composed of two or more interpenetrating fluids, one for each species, such as ions, electrons, or neutrals. Even multiple species of electrons or ions are possible depending upon the energy(temperature equivalent). Due to the presence of charged particles, a plasma couples to electric and magnetic fields and because of the fluid like behaviour plasma supports a wide variety of wave phenomena.

The electromagnetic fields in a plasma are assumed to have two parts, one static/equilibrium part and one oscillating/perturbation part. Waves in plasmas can be classified as electromagnetic or electrostatic according to whether or not there is an oscillating magnetic field. Acoustic waves in plasmas are primarily electrostatic. This means the magnetic filed is constant, or zero. In this paper, the propagation of electron acoustic waves are explored. For such a wave, the ions are considered to be stationary and infinitely heavy while electrons oscillate. Moreover these waves can be formed only when two different electron pooulations exist, typically referred to as hot and cold electrons. These are high frequency oscillations, but less than the plasma frequency. The restoring force is due to the pressure of the hot electrons, while the cold electron component provides the inertia. The phase velocity of these electron acoustic waves is much higher than the thermal velocities of both cold electrons and ions, yet much smaller than the thermal velocity of the hot electrons[1].

Plasmas which are not in thermal equilibrium exhibit two-temperature electrons. Such plasmas are found in laboratory experiments[19, 7] i.e., fusion devices, sputtering magnetron plasma, and laser-plasma interaction experiments and space environments[6, 16] i.e., solar wind, earth's bow shock, near interplanetary shocks, planetary and neutron star magnetosphere etc[11]. The importance of EAW related structures in different astrophysical situations has been reported by various spacecraft missions[12](e.g., FAST at the auroral regions, GEOTAIL and POLAR missions in the magnetosphere). In a classical plasma where the de-Broglie wavelength associated with the plasma particles is very much less than the size of the system, they can be treated as point-like objects but in quantum plasma the de-Broglie wavelength of the charged carriers become comparable to the spatial scale of the plasma system and then quantum effects become crucial and hence the Quantum Hydrodynamic(QHD) model is introduced, which is actually the extension of the classical fluid model in plasma. A quantum correction term, the so-called "Bohm potential," appears in the equation of motion for the charged particles.[5, 11]. Also in high density plasma the thermal pressure of the high velocity electrons become negligible compared to the Fermi degeneracy pressure, which arises due to the Pauli's exclusion principle. Under such extreme conditions, electrons can attain the speeds close to that of light in vacuum. So, in such cases both degeneracy and relativity have to be considered[3].

Perturbation techniques which have been first developed in 19^{th} century for astronomical applications is now considered to be a standard tool for the analysis of non-linearity of dynamical systems. In this technique, the system variables are expanded in terms of a

small parameter ϵ and the unperturbed value of the variable. The reductive perturbation technique(RPT developed in [13]) was first used in determining the solitary wave characteristics of ion acoustic wave[17] .With re-scaling of the coordinate system, some "stretched" variables are introduced in terms of a small parameter ϵ and the phase velocity V_0 of the solitary acoustic wave. Along with the different types of "stretching", the expansion of flow variables and taking terms up-to small order(2^{nd}, 3^{rd}) perturbations, KdV, KdVB or modified forms of these equations are obtained.
The purpose of the present paper is to look into the effect of different plasma parameters on the dispersion relation and to graphically investigate and comment on the formation of shock waves(ripples) and the time evolution of a stationary shock wavefront, considering a the stretching of coordinates in a critical limit[3]. Also the effect of the plasma parameters on the wave is shown graphically. The results may be used in understanding behaviour of EAWs in dense quantum plasma, treated in a critical region.

3 Mathematical Treatment

3.1 Basic Equations

The following basic equations have been used. Assuming that the region of treatment is closed, i.e, no source or sink is present; The continuity equations for hot and cold electrons respectively, are as follows:

$$\frac{\partial n_h}{\partial t} + \frac{\partial (n_h u_h)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial n_c}{\partial t} + \frac{\partial (n_c u_c)}{\partial x} = 0 \tag{2}$$

The hot electrons have enough energy such that they are more mobile compared to the cold electrons and hence are not bound by the typical fluid equations. Hence,

$$0 = \frac{1}{m_e}[e\frac{\partial \phi}{\partial x} - \frac{1}{n_h}\frac{\partial P_h}{\partial x} + \frac{\hbar^2}{2m_e}\frac{\partial}{\partial x}[\frac{1}{\sqrt{n_h}}\frac{\partial^2 \sqrt{n_h}}{\partial x^2}]] \tag{3}$$

The cold electrons are moving slow enough(inertial) that they are under the influence of oscillations. Hence,

$$(\frac{\partial}{\partial t} + u_c\frac{\partial}{\partial x})u_c = \frac{1}{m_e}[e\frac{\partial \phi}{\partial x} + \frac{\hbar^2}{2m_e}\frac{\partial}{\partial x}[\frac{1}{\sqrt{n_c}}\frac{\partial^2 \sqrt{n_c}}{\partial x^2}] + \eta_c\frac{\partial^2 u_c}{\partial x^2}] \tag{4}$$

And finally since acoustic waves are being considered, the Poisson equation is still valid. So, the Poisson equation for cold, hot electrons and ions is as follows:

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_c + n_h - Z_i n_i) \tag{5}$$

The equation of state is basically a side product of weak relativistic degeneracy. Thus the pressure depends non linearly on the density of hot electrons and is as follows[3]:

$$P_j = \frac{1}{20}(\frac{3}{\pi})^{\frac{2}{3}}(\frac{h^2}{m_e}){n_j}^{\frac{5}{3}} \tag{6}$$

3.2 Normalization

The quantities in the equations need to be normalized, in order to make the Mathematical Treatment easier. This means using certain fixed quantities to make the equations dimensionless. The following Normalization procedure is used: $t \to t\omega_{pc}$, $x \to x\frac{\omega_{pc}}{V_{Feh}}$, $u_j \to \frac{u_j}{V_{Feh}}$, $n_j \to \frac{n_j}{n_{j0}}$, $\phi \to \frac{e\phi}{2k_BT_{Feh}}$, $\eta_c \to \eta_c\frac{\omega_{pc}}{V_{Feh}{}^2m_e}$

Where ω_{pc} denotes the plasma frequency of cold electrons, and V_{Feh} denotes the Fermi velocity of the hot electrons. So, $V_{Feh} = \sqrt{\frac{2k_BT_{Feh}}{m_e}}$ and thus, $\omega_{pc} = \sqrt{\frac{4\pi e^2 n_{ec0}}{m_e}}$. The quantum diffraction term is denoted by $H = \frac{\hbar\omega_{pc}}{2k_bT_{Feh}}$. This is solely present because of the treatment being for a quantum plasma. Using the above scheme, the normalized equations end up as[1]:

$$\frac{\partial n_h}{\partial t} + \frac{\partial n_h u_h}{\partial x} = 0 \tag{7}$$

$$\frac{\partial n_c}{\partial t} + \frac{\partial n_c u_c}{\partial x} = 0 \tag{8}$$

$$0 = \frac{\partial\phi}{\partial x} - \frac{\beta}{n_h{}^{\frac{1}{3}}}\frac{\partial n_h}{\partial x} + \frac{H^2}{2}\frac{\partial}{\partial x}[\frac{1}{\sqrt{n_h}}\frac{\partial^2\sqrt{n_h}}{\partial x^2}] \tag{9}$$

$$(\frac{\partial}{\partial t} + u_c\frac{\partial}{\partial x})u_c = \frac{\partial\phi}{\partial x} + \frac{H^2}{2}\frac{\partial}{\partial x}[\frac{1}{\sqrt{n_c}}\frac{\partial^2\sqrt{n_c}}{\partial x^2}] + \eta_c\frac{\partial^2 u_c}{\partial x^2} \tag{10}$$

$$\frac{\partial^2\phi}{\partial x^2} = n_c + \frac{n_h}{\delta} - \frac{\delta_1 n_i}{\delta} \tag{11}$$

Where $\delta = \frac{n_{c0}}{n_{h0}}$ and $\delta_1 = \frac{Z_i n_{i0}}{n_{h0}}$ are the ratios of the equilibrium concentrations of cold electrons, hot electrons and ions respectively.
And $\beta = \frac{1}{12}(\frac{3n_{h0}}{\pi})^{\frac{2}{3}}(\frac{h^2}{2k_Bm_eT_{Feh}})$

3.3 Dispersion Relation

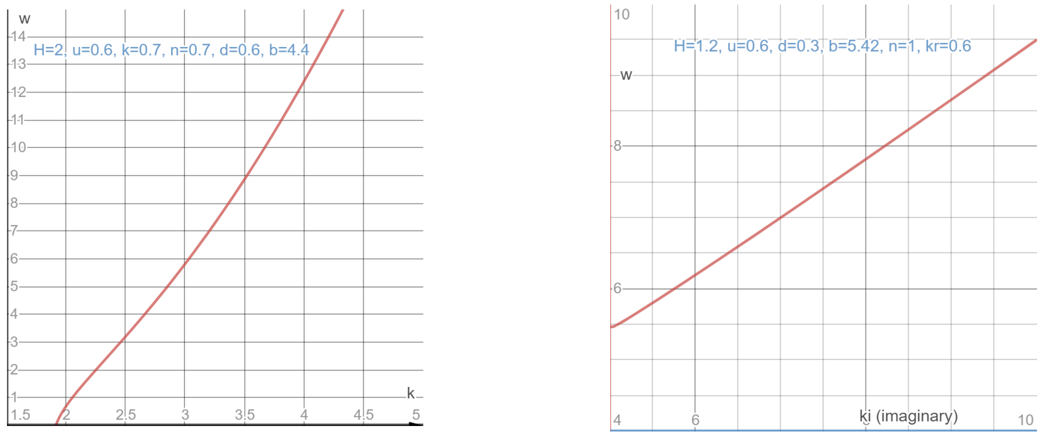
The equilibrium values of all electron/ion concentrations and ϕ are taken to be uniform(denoted by the subscript 0), so that the study of the wave like behaviour is done through perturbations about these equilibrium values. However this is not applicable for ions since they are treated as infinitely heavy and form a “uniform background”. Consequently, the following perturbation scheme is used.

$$\begin{aligned} n_j &= 1 + \epsilon n_j{}^{(1)} + \epsilon^2 n_j{}^{(2)} + \ldots && (12)\\ u_j &= u_0 + \epsilon u_j{}^{(1)} + \epsilon^2 u_j{}^{(2)} + \ldots && (13)\\ \phi &= \phi_0 + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \ldots && (14) \end{aligned}$$

Hence, to find the linear dispersion relation, the relations among the first order terms are obtained ...
The continuity equation gives (for hot and cold electrons respectively):

$$(\omega - u_ok)n_j{}^{(1)} = ku_j{}^{(1)} \tag{15}$$

[1]Relevant calculations in 5



(a) Real part of the dispersion relation

(b) Imaginary part of the dispersion relation

Figure 1: Plots showing the real and imaginary part of the dispersion relation: eq.(19)

Now considering the momentum equation for the hot electrons, only the first order terms in the Taylor expansion of $\sqrt{n_h}$ and $n_h^{\frac{-1}{3}}$ are retained, giving the following:

$$\phi^{(1)} = (\beta + \frac{H^2k^2}{4}){n_h}^{(1)} \tag{16}$$

Carrying out the same thing for the momentum equation of the cold electrons,

$$-(\omega - u_0k){u_c}^{(1)} = k\phi^{(1)} - \frac{H^2k^3}{4}{n_c}^{(1)} + ik^2\eta_c{u_c}^{(1)} \tag{17}$$

where, $i = \sqrt{-1}$. This indicates that the dispersion relation is complex, i.e, k is a complex quantity and in general damping is to be expected. Finally, the Poisson equation for the hot and cold electrons reveals,

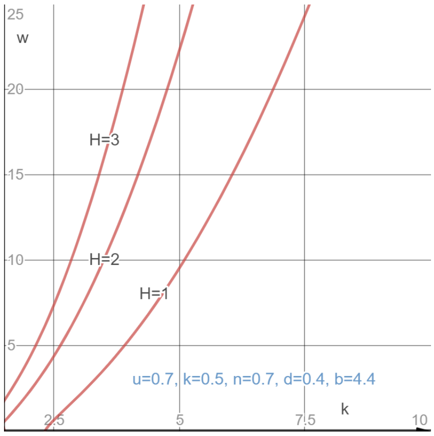
$$-k^2\phi^{(1)} = {n_c}^{(1)} + \frac{{n_h}^{(1)}}{\delta} \tag{18}$$

Solving these linear relations, the following dispersion relation[2] is obtained:

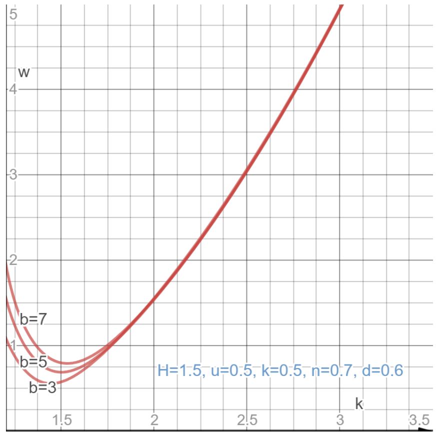
$$\frac{\delta(\beta k^2 + \frac{H^2k^4}{4})}{1 + \delta(\beta k^2 + \frac{H^2k^4}{4})} = (w - u_0k)^2 - \frac{H^2k^4}{4} + i\eta_c k^2(w - u_0k) \tag{19}$$

Since the factor $i = \sqrt{-1}$ is present in the above equation, it implies the presence of a complex dispersion relation between ω and k. As such, the wave vector k is complex and can be written as $k_r + i\kappa$. Hence the dispersion relation has a real and an imaginary part. The imaginary part implies damping. Figures 2 show the variation of the real part of the dispersion relation with respect to the plasma parameters.

[2]Relevant calculations provided in Appendix A



(a) Variation with H.



(b) Variation with β, identified by b in the graph.

Figure 2: Variation of the dispersion relation with H(Fig. a) and β(Fig. b).

4 Shock Waves and the KdV Burgers equation

4.1 Reductive perturbation technique

This technique is used to obtain the shock wave equation in plasma. It involves scaling or "stretching" of the space and time coordinates and then examining the resulting equations in various orders of perturbations of smaller magnitudes. The stretching of the coordinates helps in amplification of these perturbations for sake of the treatment. What entails in the following is a case of extreme stretching such that the space coordinates are stretched by a factor of ϵ while time coordinate is stretched by a factor of ϵ^3.

$$\xi = \epsilon(x - V_0 t) \tag{20}$$

$$\tau = \epsilon^3 t \tag{21}$$

$$\eta = \epsilon \eta_0 \tag{22}$$

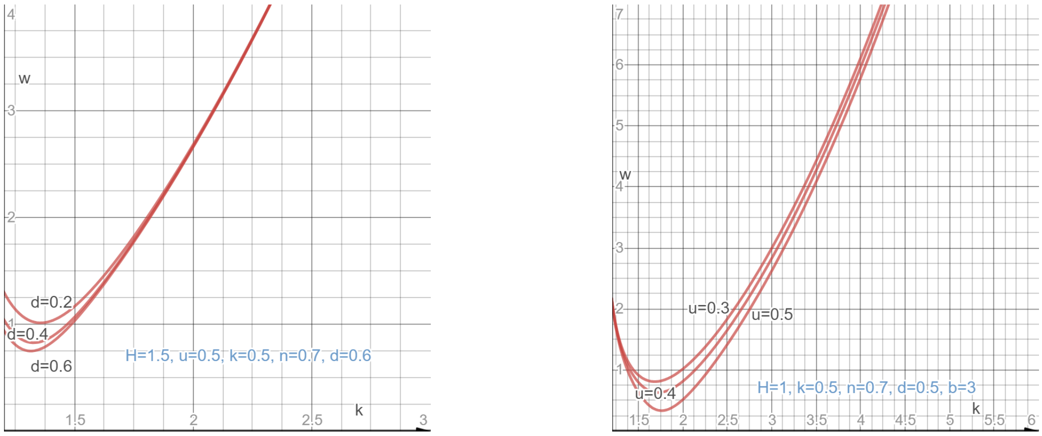
The reason for stretching η is because, of the way it is normalized. The normalization of η is through velocity and frequency. Since the velocity and frequency become stretched, η is stretched, quite similar to the process presented in the Appendix section of [8]. Due to the re-scaling, differentials with respect to time and space change as follows:

$$\frac{\partial}{\partial t} = \epsilon^3 \frac{\partial}{\partial \tau} - \epsilon V_0 \frac{\partial}{\partial \xi} \tag{23}$$

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial \xi} \tag{24}$$

Substituting equations 24 and 23 in equations 7 - 11, the following are obtained...
The continuity equations become:

$$\begin{aligned} &(\epsilon^3 \frac{\partial}{\partial \tau} - \epsilon V_0 \frac{\partial}{\partial \xi})(\epsilon {n_j}^{(1)} + \epsilon^2 {n_j}^{(2)} + \dots) + \\ &\epsilon \frac{\partial}{\partial \xi}(1 + \epsilon {n_j}^{(1)} + \epsilon^2 {n_j}^{(2)} + \dots)(u_0 + \epsilon {u_j}^{(1)} + \epsilon^2 {u_j}^{(2)} + \dots) = 0 \end{aligned} \tag{25}$$



(a) Variation with δ, identified by b in the graph.

(b) Variation with u_0, identified by u in the graph.

Figure 3: Variation of the dispersion relation with δ(Fig. a) and u_0(Fig. b).

Taking the terms with varying powers of ϵ

$$\epsilon^2 : \qquad (u_0 - V_0)\frac{\partial {n_j}^{(1)}}{\partial \xi} + \frac{\partial}{\partial \xi} u_j^{(1)} = 0 \tag{26}$$

$$\epsilon^3 : \qquad (u_0 - V_0)\frac{\partial {n_j}^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}({n_j}^{(1)}{u_j}^{(1)} + {u_j}^{(2)}) = 0 \tag{27}$$

$$\epsilon^4 : \qquad \frac{\partial {n_j}^{(1)}}{\partial \tau} + (u_0 - V_0)\frac{\partial {n_j}^{(3)}}{\partial \xi} + \frac{\partial}{\partial \xi}({u_j}^{(2)}{n_j}^{(1)} + {u_j}^{(1)}{n_j}^{(2)} + {u_j}^{(3)}) = 0 \tag{28}$$

The momentum equation for the hot electrons (taking up-to first order in Taylor expansion of $\sqrt{n_h}$ or ${n_h}^{-\frac{1}{3}}$) becomes...

$$\begin{aligned} 0 = \epsilon\frac{\partial}{\partial \xi}(\epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \dots) - \epsilon\beta(1 - \epsilon\frac{{n_h}^{(1)}}{3} - \epsilon^2\frac{{n_h}^{(2)}}{3}\dots)\frac{\partial}{\partial \xi}(\epsilon {n_h}^{(1)} + \epsilon^2 {n_h}^{(2)} + \dots) \\ + \epsilon^3\frac{H^2}{2}\frac{\partial}{\partial \xi}[(1 - \epsilon\frac{{n_h}^{(1)}}{2} - \epsilon^2\frac{{n_h}^{(2)}}{2}\dots)\frac{\partial^2}{\partial \xi^2}(1 + \epsilon\frac{{n_h}^{(1)}}{2} + \epsilon^2\frac{{n_h}^{(2)}}{2}\dots)] \end{aligned} \tag{29}$$

Taking the terms with varying powers of ϵ

$$\epsilon^2 : \qquad 0 = \frac{\partial \phi^{(1)}}{\partial \xi} - \beta\frac{\partial {n_h}^{(1)}}{\partial \xi} \tag{30}$$

$$\epsilon^3 : \qquad 0 = \frac{\partial \phi^{(2)}}{\partial \xi} - \beta\frac{\partial {n_h}^{(2)}}{\partial \xi} + \beta\frac{n_h^{(1)}}{3}\frac{\partial n_h^{(1)}}{\partial \xi} \tag{31}$$

$$\epsilon^4 : \qquad 0 = \frac{\partial \phi^{(3)}}{\partial \xi} - \beta\frac{\partial n_h^{(3)}}{\partial \xi} + \frac{\beta}{3}\frac{\partial}{\partial \xi}(n_h^{(1)}n_h^{(2)}) + \frac{H^2}{4}\frac{\partial^3 n_h^{(1)}}{\partial \xi^3} \tag{32}$$

The momentum equation for the cold electrons becomes...

$$(\epsilon^3\frac{\partial}{\partial\tau}-\epsilon V_0\frac{\partial}{\partial\xi})(\epsilon n_c{}^{(1)}+\epsilon^2 n_c{}^{(2)}+\dots)+\epsilon(u_0+\epsilon u_c^{(1)}+\dots)\frac{\partial}{\partial\xi}(\epsilon u_c^{(1)}+\dots)$$
$$=\epsilon\frac{\partial}{\partial\xi}(\epsilon\phi^{(1)}+\dots)+\epsilon^3\frac{H^2}{2}\frac{\partial}{\partial\xi}[(1-\epsilon\frac{n_c^{(1)}}{2}+\dots)\frac{\partial^2}{\partial\xi^2}(1+\epsilon\frac{n_c^{(1)}}{2})]+\epsilon^3\eta_0\frac{\partial^2}{\partial\xi^2}(\epsilon u_c^{(1)}+\dots) \tag{33}$$

Taking the terms with varying powers of ϵ

$$\epsilon^2: \qquad (u_0-V_0)\frac{\partial u_c^{(1)}}{\partial\xi}=\frac{\partial\phi^{(1)}}{\partial\xi} \tag{34}$$
$$\epsilon^3: \qquad (u_0-V_0)\frac{\partial u_c^{(2)}}{\partial\xi}+u_c^{(1)}\frac{\partial u_c^{(1)}}{\partial\xi}=\frac{\partial\phi^{(2)}}{\partial\xi} \tag{35}$$
$$\epsilon^4: \qquad \frac{\partial u_c^{(1)}}{\partial\tau}+(u_0-V_0)\frac{\partial u_c^{(3)}}{\partial\xi}+\frac{\partial}{\partial\xi}(u_c^{(1)}u_c^{(2)})=\frac{\partial\phi^{(3)}}{\partial\xi}+\frac{H^2}{4}\frac{\partial^3 n_c^{(1)}}{\partial\xi^3}+\eta_0\frac{\partial^2 u_c^{(1)}}{\partial\xi^2} \tag{36}$$

And finally, the Poisson equation becomes...

$$\epsilon^2\frac{\partial^2}{\partial\xi^2}(\epsilon\phi^{(1)}+\epsilon^2\phi^{(2)}+\dots)=(1+\frac{1}{\delta}-\frac{\delta_1 n_i}{\delta})+\epsilon(n_c^{(1)}+\frac{n_h^{(1)}}{\delta})+\epsilon^2(n_c^{(2)}+\frac{n_h^{(2)}}{\delta})+\dots \tag{37}$$

Taking terms with varying powers of ϵ

$$\epsilon: \qquad n_c^{(1)}+\frac{n_h^{(1)}}{\delta}=0 \tag{38}$$
$$\epsilon^2: \qquad n_c^{(2)}+\frac{n_h^{(2)}}{\delta}=0 \tag{39}$$
$$\epsilon^3: \qquad \frac{\partial^2\phi^{(1)}}{\partial\xi^2}=n_c^{(3)}+\frac{n_h^{(3)}}{\delta} \tag{40}$$
$$\epsilon^4: \qquad \frac{\partial^2\phi^{(2)}}{\partial\xi^2}=n_c^{(4)}+\frac{n_h^{(4)}}{\delta} \tag{41}$$
$$\tag{42}$$

Since the second derivative of $\phi^{(2)}$ is related to the fourth order perturbation terms which are already very small,

$$\frac{\partial\phi^{(2)}}{\partial\xi}=\int(n_c^{(4)}+\frac{n_h^{(4)}}{\delta})d\xi$$

This $\frac{\partial\phi^{(2)}}{\partial\xi}$ term is therefore very small and can be safely neglected in the upcoming equations.
Now, using equations 26, 30, 34, 38, all first order perturbations $n_h^{(1)}$, $n_c^{(1)}$, $u_h^{(1)}$ and $u_c^{(1)}$

are obtained in terms of $\phi^{(1)}$.

$$n_h^{(1)} = \frac{\phi^{(1)}}{\beta} \tag{43}$$

$$n_c^{(1)} = -\frac{\phi^{(1)}}{(V_0 - u_0)^2} \tag{44}$$

$$u_h^{(1)} = \frac{(V_0 - u_0)}{\beta}\phi^{(1)} \tag{45}$$

$$u_c^{(1)} = -\frac{1}{(V_0 - u_0)}\phi^{(1)} \tag{46}$$

The terms $u_c^{(2)}$ and $n_c^{(2)}$ may be written in terms of $n_c^{(1)}$ as

$$u_c^{(2)} = \frac{(V_0 - u_0)^2}{2}(n_c^{(1)})^2 \tag{47}$$

$$n_c^{(2)} = \frac{3}{2}(n_c^{(1)})^2 \tag{48}$$

By elimination of the third order perturbation terms, and dropping the superscript from $\phi^{(1)}$ the non-linear KdV-Burgers equation for the EAWs is obtained[3].

$$\frac{\partial \phi}{\partial \tau} + p\phi^2 \frac{\partial \phi}{\partial \xi} + r\frac{\partial^3 \phi}{\partial \xi^3} - q\frac{\partial^2 \phi}{\partial \phi^2} = 0 \tag{49}$$

Where,

$$\begin{aligned}
p &= \frac{3}{(V_0 - u_0)^4} + \frac{\delta^2}{12}\frac{1}{(V_0 - u_0)^3} + \frac{1}{2(V_0 - u_0)^3} \\
q &= \frac{\eta}{2} \\
r &= \frac{\delta H^2}{8\beta} + \frac{(V_0 - u_0)^3}{2} + \frac{H^2}{8}\frac{1}{(V_0 - u_0)}
\end{aligned}$$

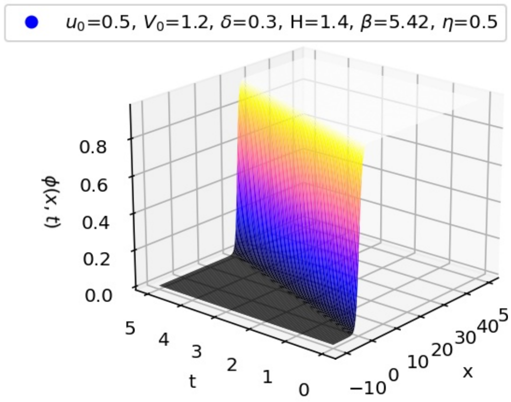
V_0 is also referred to as the *Mach number* M.

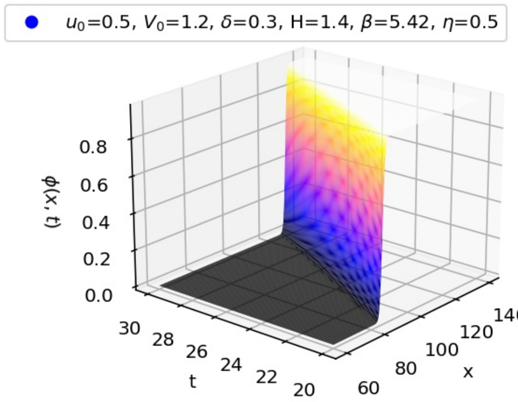
4.2 Solution to the modified KdV-Burgers equation

Travelling wave solutions are desired from equation (49), since these are physically relevant to the investigation. There are multiple possible ways to obtain such solutions, such as the tanh-sech method[18], homogeneous balance method[4], extended tanh method[10].

While the tanh method introduced by Malfiet[18] is a reliable way to treat nonlinear wave equations, the extended tanh method is a more direct method. Not going through the entire process of derivation of the solutions, the results obtained by Bekir[2] are directly stated for the modified KdV-Burgers equation.

[3] Relevant calculations provided in Appendix A





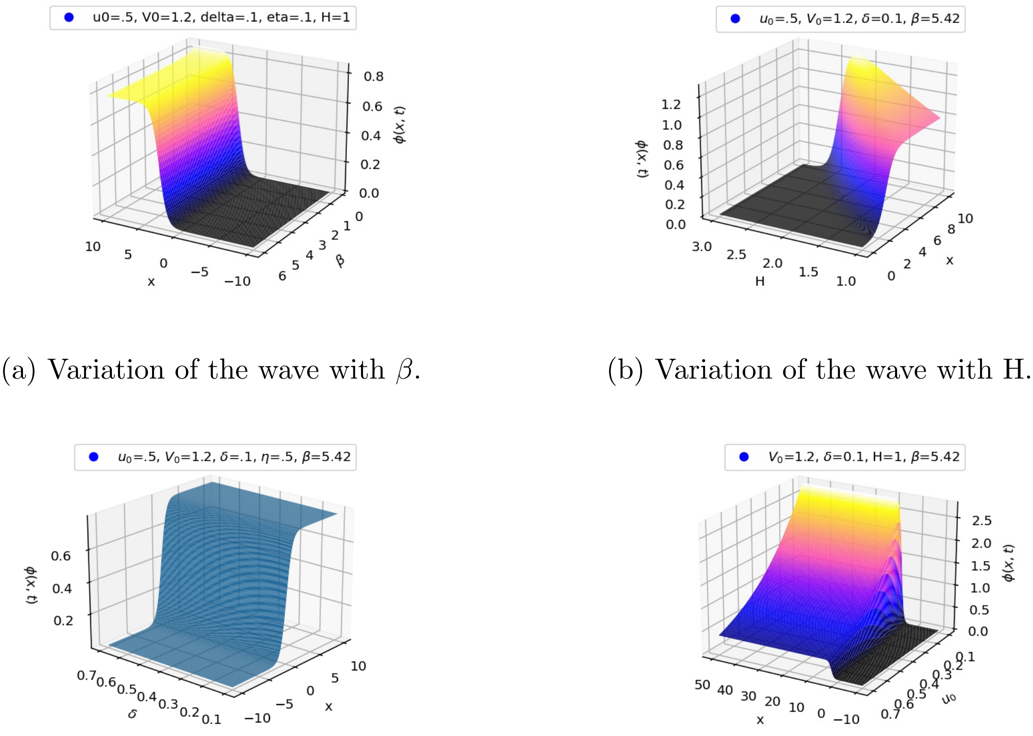
(a) Positions of wave in time interval 0 to 5 (b) Positions of wave in time interval 20 to 30

Figure 4: This is the plot of ϕ_2 with respect to space and time, showing the propagation of the wave with time. The parameters of the plasma are given in the graphs

There are three possible solutions to the equation.

$$\begin{aligned}
\phi_1 &= \pm\sqrt{\frac{6r}{p}}[1 + coth(x - 8rt)] \\
\phi_2 &= \pm\sqrt{\frac{6r}{p}}[1 + tanh(x - 8rt)] \\
\phi_3 &= \pm 3\sqrt{\frac{6r}{p}}[2 + tanh(x - 32rt) + coth(x - 32rt)]
\end{aligned}$$

From these three possible solutions, ϕ_2 is used from now on. Plots of positive values of ϕ_2 are shown at the top in Fig. 4. The two graphs show how the wave is travelling through space and time. The KdV-Burgers equation obtained in eq.(49) is a special kind since the solution to this in not like the solitons obtained from a normal KdV equation. Here the wave progresses unaltered with a fixed velocity. So the location of the wave can simply be shown on the x-t plane, as a straight line whose slope gives the speed of the travelling wave. Judging by this nature, it is imperative that the plasma parameters mainly affect the amplitude of the wave, when a single soliton is involved. Same occurs for the negative solution of ϕ, which is why it is omitted. The plots in Fig. 5 and Fig. 5c describe describe the variation of the waveform with respect to the different plasma parameters, all evaluated at $t = 1$. The parameters are given along with the plots. Fig. 5a describes the variation of the amplitude with changing degeneracy pressure, and it is clear that it does not have much effect on the wave. Same goes for Fig. 5c showing variation with respect to δ, the ratio of cold to hot electron density. However, Fig. 5b, 5d, and 6 show how critically these affect the waveform.



(a) Variation of the wave with β.

(b) Variation of the wave with H.

(c) Variation of wave with δ.

(d) Variation of wave with u_0.

Figure 5: Variation of the amplitude of the travelling wave with change of plasma parameters.

4.3 Numerical simulation of the modified KdV-Burgers equation

Given how powerful computers have become in the last couple of years, it would be a shame to not carry out a numerical simulation of the nonlinear equation (49). There are a plethora of options available for numerically solving differential equations, the simplest being Euler's method, using forward/backward/central difference methods. A more sophisticated approach is through the use of RK4 algorithm. The first computational solution to this equation was given by Zabusky and Krushkal[9] in 1965, using a periodic boundary condition which was again justified in another paper by Zabusky[15]. However, there are also other convergence criteria involved in its solution, which implies that just applying a difference equation won't work, specially when third order derivatives are involved along with extreme non-linearities. The approach used here is a Fourier transform - split step method[14], which is well applicable for nonlinear equations like the KdV or mKdV. The algorithm goes as follows...

Eq.(49) can be written in a compact form as:

$$\partial_\tau \phi + p\phi^2 \partial_\xi \phi + r\partial_{\xi\xi\xi}\phi - q\partial_{\xi\xi}\phi = 0 \tag{50}$$

Taking a Fourier transform

$$d_\tau \tilde{\phi} + \frac{p}{3}(ik)\widetilde{\phi^3} + r(ik)^3\tilde{\phi} - q(ik)^2\tilde{\phi} = 0 \tag{51}$$

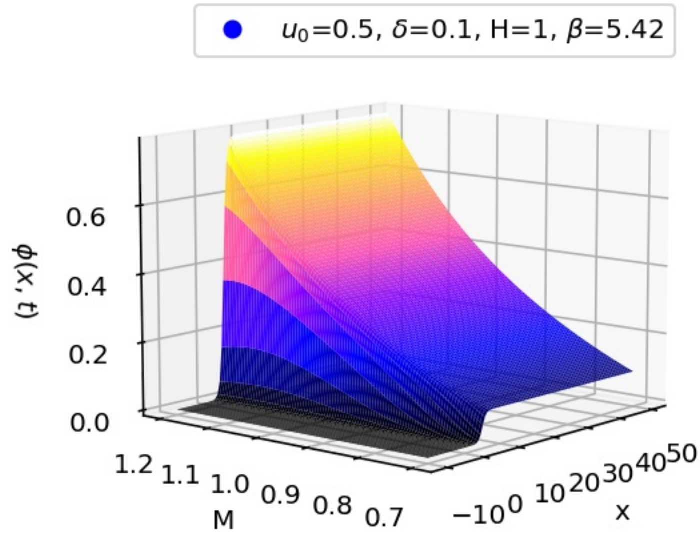


Figure 6: Variation of wave with Mach number M.

Where $\tilde{\phi}$ is the Fourier transform of ϕ while $\widetilde{\phi^3}$ is the Fourier transform of ϕ^3.
Now Simplifying...

$$d_\tau\tilde{\phi} = -\frac{p}{3}(ik)\widetilde{\phi^3} + irk^3\tilde{\phi} + qk^2\tilde{\phi} \tag{52}$$

The linear and nonlinear part of this equation is treated separately:

$$d_\tau\tilde{\phi}(k,t) = (irk^3 - qk^2)\tilde{\phi}(k,t) \tag{53}$$

$$\implies \tilde{\phi}(k,t) = \tilde{\phi}(k)e^{(irk^3-qk^2)t} \tag{54}$$

$$d_\tau\tilde{\phi}(k,t) = -\frac{p}{3}(ik)\widetilde{\phi^3}(k,t) \tag{55}$$

Equation (53) handles the linear part of eq52, while 55 handles the nonlinear part. Now, from equation (54), let

$$\tilde{\phi_1}(k,t+\Delta t) = \tilde{\phi}(k,t)e^{(irk^3-qk^2)\Delta t} \tag{56}$$

Using a similar expansion as the principle of differentiation on equation (55)...

$$\tilde{\phi}(k,t+\Delta t) = \tilde{\phi_1}(k,t+\Delta t) - \frac{ipk}{3}\left[F\left(F^{-1}\left(\tilde{\phi_1}(k,t+\Delta t)\right)\right)^3\right] \tag{57}$$

Here, F denotes the Fourier transform operator, while F^{-1} denotes the inverse-Fourier transform operator. Application of equations (56) and (57) iteratively within a given time range and taking its real value, the value of ϕ is obtained[4]. Starting with the same solution used for the graphs(refer to section4.2), the following is obtained[5].
The following conditions were placed: $V_0 = 1.2, u_0 = 0.5, \delta = 0.3, H = 2, \beta = 5.42, \eta = 0.5$

[4]A matlab script is provided in appendix C
[5]For an animation, goto https://github.com/nlACh/waves-plasma.git

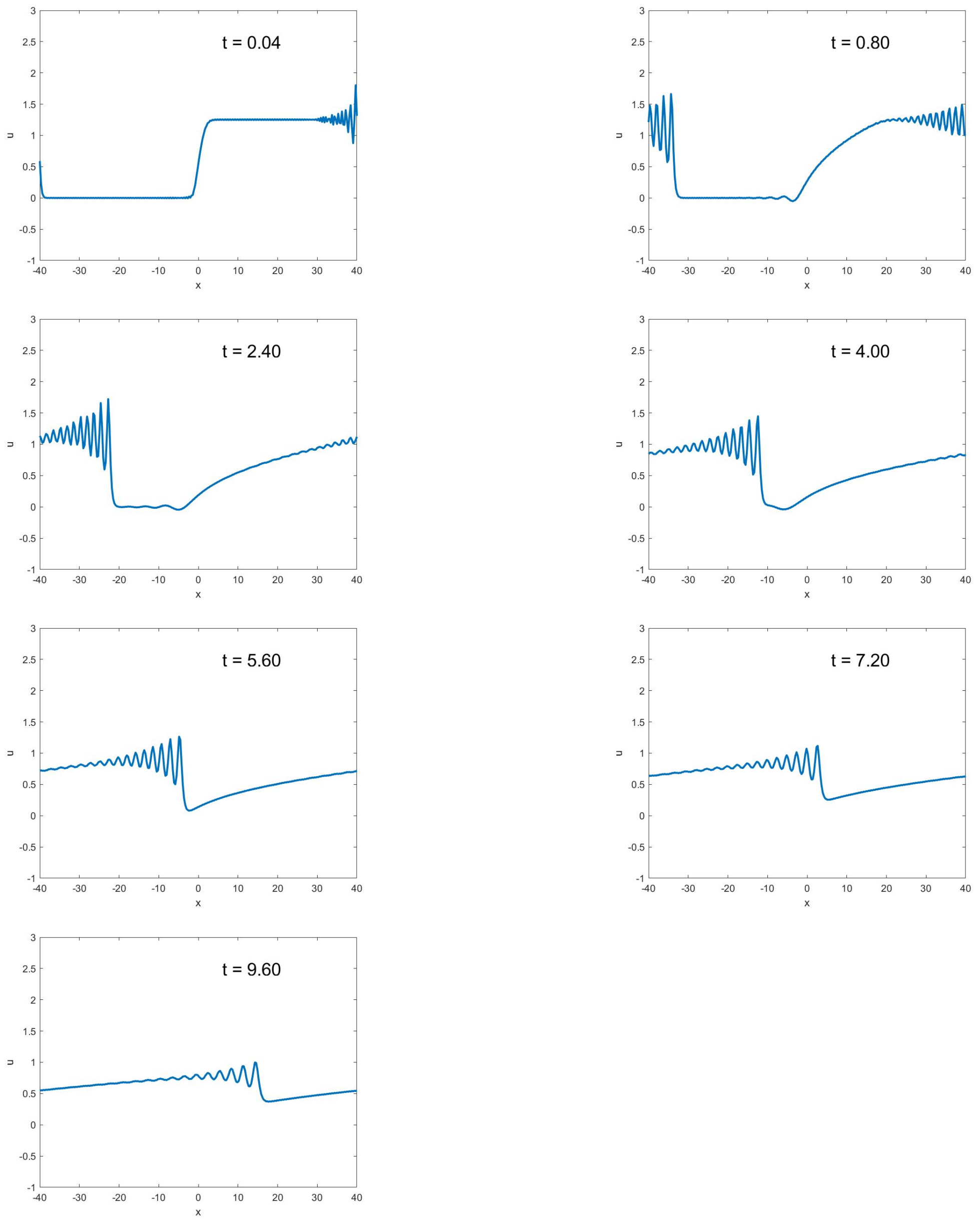


Figure 7: Propagation of travelling wave in plasma, starting with $\sqrt{\frac{6r}{p}}(1+tanh(x-8rt))$ as the initial condition. The time scale is from $t = 0.04$ to $t = 10.12$.

5 Appendix A

Calculations for normalization, dispersion relation and KdV-Burgers equation

The ~~eqn~~ Governing equation set (G_1)

$$\frac{\partial n_h}{\partial t} + \frac{\partial}{\partial x}(n_h u_h) = 0 \quad — (1)$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0 \quad — (2)$$

$$\frac{e}{m_e}\frac{\partial \phi}{\partial x} - \frac{1}{m_e n_h}\frac{\partial P_h}{\partial x} + \frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_h}}\frac{\partial^2 \sqrt{n_h}}{\partial x^2}\right] = 0 \quad — (3)$$

$$\left(\frac{\partial}{\partial t} + u_e\frac{\partial}{\partial x}\right)u_e = \frac{e}{m_e}\frac{\partial \phi}{\partial x} + \frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_e}}\frac{\partial^2 \sqrt{n_e}}{\partial x^2}\right] + \frac{\eta_e}{m_e}\frac{\partial^2 u_e}{\partial x^2} \quad — (4)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e\left(n_e + n_h - Z_i n_i\right) \quad — (5)$$

The given normalization scheme

$$\bar{x} \to x\frac{\omega_{pe}}{V_{Fe}};\ \bar{t} \to t\omega_{pe},\ \bar{\phi} \to \frac{e\phi}{2K_B T_{Fe}},\ \bar{n}_j \to \frac{n_j}{n_{j0}},\ \bar{u}_j \to \frac{u_j}{V_{Fe}}$$

$$\bar{\eta} = \frac{\omega_{pe}}{m_e V_{Fe}^2}\eta$$

Here $V_{Fe} = \sqrt{\dfrac{2K_B T_{Fe}}{m_e}}$ is the fermi velocity of electron

$H =$ non dimensional quantum diffraction parameter
$= \dfrac{\hbar\omega_{pe}}{2K_B T_{Fe}}$ where T_{Fe} is the fermi temperature.

Now, $P_h = \dfrac{1}{20}\left(\dfrac{3}{\pi}\right)^{2/3}\dfrac{h^2}{m_e}n_h^{5/3} = A n_h^{5/3}$ [Weak Relativistic degeneracy]

$$\bar{P}_h = A\cdot(\bar{n}_h n_0)^{5/3}$$

$\omega_{pe} = \sqrt{\dfrac{4\pi n_{e0} e^2}{m_e}}$ is the cold plasma frequency.

Now eqn (1) becomes :-

$$\frac{\partial \bar{n}_h}{\partial \bar{t}} n_{e0}\omega_{pe} + \frac{\partial}{\partial \bar{x}}(\bar{n}_h \bar{u}_h)\cdot\frac{\cancel{V_{Fe}}}{\cancel{\omega_{pe}}}\, n_0 \frac{\omega_{pe}}{V_{Fe}} V_{Fe} = 0$$

$$\Rightarrow \frac{\partial \bar{n}_h}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{n}_h \bar{u}_h) = 0 \quad — (1a)$$

Similarly eqn (2) becomes

$$\frac{\partial \bar{n}_e}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{n}_e \bar{u}_e) = 0 \quad — (2a)$$

Now eqn ③ becomes :-

$$\frac{e}{m_e}\frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\omega_{pe}}{v_{Fe}}\frac{2K_BT_{Fe}}{e} - \frac{1}{m_e n_h}\frac{\partial \bar{n}_h^{5/3}}{\partial\bar{x}} A n_0^{5/3} + \frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial\bar{x}}\left[\frac{1}{\sqrt{\bar{n}_h}}\frac{\partial^2\sqrt{\bar{n}_h}}{\partial\bar{x}^2}\right]\cdot\frac{\omega_{pe}^3}{v_{Fe}^3} = 0$$

— (3a)

$$\Rightarrow \omega_{pe}v_{Fe}\frac{\partial\bar{\phi}}{\partial\bar{x}} - \frac{\omega_{pe}\, 5 n_0^{5/3} A}{v_{Fe}\, 3\cdot m_e n_{c0}}\bar{n}_h^{-1/3}\frac{\partial\bar{n}_h}{\partial\bar{x}} + \left(\frac{\hbar\omega_{pe}}{2K_BT_{Fe}}\right)^2\cdot\frac{\omega_{pe}}{\cancel{v_{Fe}}}\frac{m_e^2 v_{Fe}}{2m_e^2}\frac{\partial}{\partial\bar{x}}\left[\frac{1}{\sqrt{\bar{n}_h}}\frac{\partial^2\sqrt{\bar{n}_h}}{\partial\bar{x}^2}\right] = 0$$

$$\Rightarrow \frac{\partial\bar{\phi}}{\partial\bar{x}} - \beta\bar{n}_h^{-1/3}\frac{\partial\bar{n}_h}{\partial\bar{x}} + \frac{H^2}{2}\frac{\partial}{\partial\bar{x}}\left[\frac{1}{\sqrt{\bar{n}_h}}\frac{\partial^2\sqrt{\bar{n}_h}}{\partial\bar{x}^2}\right] = 0 \quad \text{— (3a)}$$

where $\beta = \dfrac{5A}{3m_e}\dfrac{n_{c0}^{5/3}}{n_{c0}}\dfrac{1}{v_{Fe}^2}$

eqn ④ becomes :-

$$\left(\frac{\partial}{\partial\bar{t}}\omega_{pe} + \bar{u}_e v_{Fe}\frac{\partial}{\partial\bar{x}}\frac{\omega_{pe}}{v_{Fe}}\right)v_{Fe}\bar{u}_e = \frac{e}{m_e}\frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\omega_{pe}}{v_{Fe}}\frac{2K_BT_{Fe}}{e}$$

$$+\omega_{pe}v_{Fe}\frac{H^2}{2}\frac{\partial}{\partial\bar{x}}\left[\frac{1}{\sqrt{\bar{n}_e}}\frac{\partial^2\sqrt{\bar{n}_e}}{\partial\bar{x}^2}\right] + \frac{m_e v_{Fe}^2\,\bar{n}_e}{\omega_{pe}\, m_e}v_{Fe}\frac{\partial^2\bar{u}_e}{\partial\bar{x}^2}\frac{\omega_{pe}^2}{v_{Fe}^2}$$

$$\Rightarrow\left(\frac{\partial}{\partial\bar{t}} + \bar{u}_e\frac{\partial}{\partial\bar{x}}\right)\bar{u}_e = \frac{\partial\bar{\phi}}{\partial\bar{x}} + \frac{H^2}{2}\frac{\partial}{\partial\bar{x}}\left[\frac{1}{\sqrt{\bar{n}_e}}\frac{\partial^2\sqrt{\bar{n}_e}}{\partial\bar{x}^2}\right] + \bar{n}_e\frac{\partial^2\bar{u}_e}{\partial\bar{x}^2} \quad \text{— (4a)}$$

Eqn ⑤ becomes :-

$$\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}\frac{2K_BT_{Fe}}{e}\frac{\omega_{pe}^2}{v_{Fe}^2} = 4\pi e\left[n_{c0}\bar{n}_e + n_{h0}\bar{n}_h - Z_i\bar{n}_i n_{i0}\right]$$

$$\Rightarrow \frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}\frac{4\pi n_{c0}e^2}{m_e}\cdot\frac{2K_BT_{Fe}}{e}\cdot\frac{m_e}{2K_BT_{Fe}} = 4\pi e n_{c0}\left[\bar{n}_e + \frac{\bar{n}_h}{\delta_1} - \frac{Z_i\bar{n}_i n_{i0}}{n_{c0}}\right]$$

$$\Rightarrow \frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} = \bar{n}_e + \frac{1}{\delta}\bar{n}_h - \frac{\bar{n}_i\delta_i}{\delta} \quad \text{— (5a)}$$

where $\delta = \dfrac{n_{c0}}{n_{h0}}$

$\delta_i = \dfrac{n_{i0}Z_i}{n_{h0}}$

1. The normalisation equations are

$$\frac{\partial n_h}{\partial t} + \frac{\partial}{\partial x}(n_h u_h) = 0 \quad -⑥$$

$$\frac{\partial n_c}{\partial t} + \frac{\partial}{\partial x}(n_c u_c) = 0 \quad -⑦$$

$$\frac{\partial \phi}{\partial x} - \beta n_h^{-1/3} \frac{\partial n_h}{\partial x} + \frac{H^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_h}} \frac{\partial^2 \sqrt{n_h}}{\partial x^2}\right] = 0 \quad -⑧$$

$$\left(\frac{\partial}{\partial t} + u_c \frac{\partial}{\partial x}\right) u_c = \frac{\partial \phi}{\partial x} + \frac{H^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_c}} \frac{\partial^2 \sqrt{n_c}}{\partial x^2}\right] + \eta_e \frac{\partial^2 u_c}{\partial x^2} \quad -⑨$$

$$\frac{\partial^2 \phi}{\partial x^2} = n_c + \frac{n_h}{\delta} - \frac{\delta_1 n_i}{\delta} \quad ⑩$$

Now the perturbation expansion (E1)

$$n_h = 1 + \epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)}$$

$$n_c = 1 + \epsilon n_c^{(1)} + \epsilon^2 n_c^{(2)}$$

$$u_c = \epsilon u_c^{(1)} + \epsilon^2 u_c^{(2)} + u_{c0}$$

$$u_h = \epsilon u_h^{(1)} + \epsilon^2 u_h^{(2)} + u_{h0}$$

$$\phi = \phi_0 + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)}$$

and $\frac{\partial}{\partial x} \equiv ik$; $\frac{\partial}{\partial t} = -i\omega$

From eqn ⑥

$$-i\omega n_h^{(1)} + ik\left[u_h^{(1)} + u_{h0} n_h^{(1)}\right] = 0$$

$$\Rightarrow u_h^{(1)} = \frac{(\omega - k u_{h0})}{k} n_h^{(1)}$$

similarly from ⑦

$$u_c^{(1)} = \frac{\omega - k u_{c0}}{k} n_c^{(1)}$$

From eqn ⑧

$$\frac{\partial \phi}{\partial x} = ik\epsilon\phi^{(1)}$$

$$\frac{\partial n_h}{\partial x} = ik\epsilon n_h^{(1)}$$

$$n_h^{-1/3} = \left[1 + \epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)}\right]^{-1/3} \cong 1 - \frac{\epsilon}{3} n_h^{(1)}$$

$$\therefore n_h^{-1/3} \frac{\partial n_h}{\partial x} = ik\epsilon n_h^{(1)}$$

$$\frac{H^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_h}}\frac{\partial^2\sqrt{n_h}}{\partial x^2}\right] = -\frac{iH^2k^3}{4} n_h^{(1)}$$

$$\therefore ik\epsilon\phi^{(1)} - \beta ik\epsilon n_h^{(1)} - ik\cdot\frac{H^2k^2}{4} n_h^{(1)} = 0$$

$$\Rightarrow \phi^{(1)} = \left(\beta + \frac{H^2k^2}{4}\right) n_h^{(1)}$$

From (9)

$$-i\omega u_c^{(1)} + ik\epsilon u_0 u_c^{(1)} = ik\phi^{(1)} - \frac{\epsilon H^2k^2}{4}\cdot ikn_c^{(1)} - \frac{\epsilon\eta_c k^2}{\cancel{m_e}} u_c^{(1)}$$

$$\Rightarrow -u_c^{(1)}\left[\omega - ku_0 - \frac{\eta_c k^2}{i\cancel{m_e}}\right] = k\phi^{(1)} - \frac{H^2k^3}{4} n_c^{(1)}$$

$$\Rightarrow -k\phi^{(1)} = \frac{\omega-ku_0}{k}\left[(\omega-ku_0) + i\frac{\eta_c k^2}{\cancel{m_e}}\right] n_c^{(1)} - \frac{H^2k^3}{4}n_c^{(1)}$$

$$\Rightarrow -k^2\phi^{(1)} = (\omega-ku_0)\left[(\omega-ku_0)+i\frac{\eta_c k^2}{\cancel{m_e}}\right]n_c^{(1)} - \frac{H^2k^4}{4}n_c^{(1)}$$

$$\Rightarrow n_c^{(1)} = \frac{-k^2\phi^{(1)}}{(\omega-ku_0)\left[(\omega-ku_0)+i\frac{\eta_c k^2}{\cancel{m_e}}\right] - \frac{H^2k^4}{4}}$$

From (10)

$$-k^2\phi^{(1)} = n_e^{(1)} + \frac{1}{\delta} n_h^{(1)}$$

$$\Rightarrow -k^2\phi^{(1)} = \frac{-k^2\phi^{(1)}}{(\omega-ku_0)^2 + i\frac{\eta_e k^2}{m_e}(\omega-ku_0) - \frac{H^2k^4}{4}} + \frac{1}{\delta}\frac{\phi^{(1)}}{\beta+\frac{H^2k^2}{4}}$$

$$\Rightarrow 1 = \frac{1}{\left[(\omega-ku_0)^2 - \frac{H^2k^4}{4}\right] + i\frac{\eta_e k^2}{m_e}(\omega-ku_0)} - \frac{1}{\delta\left(\beta k^2 + \frac{H^2k^4}{4}\right)}$$

~~Equating the real part both side we get~~

$$\cancel{\Rightarrow 1 + \frac{1}{\delta\left(\beta k^2+\frac{H^2k^4}{4}\right)} = \frac{(\omega-ku_0)^2 - \frac{H^2k^4}{4}}{\left[(\omega-ku_0)^2-\frac{H^2k^4}{4}\right]^2 - \left[\frac{\eta_e k^2(\omega-ku_0)}{m_e}\right]^2}}$$

$$\Rightarrow \frac{\delta\left(\beta k^2+\frac{H^2k^4}{4}\right)}{1+\delta\left(\beta k^2+\frac{H^2k^4}{4}\right)} = (\omega-ku_0)^2 - \frac{H^2k^4}{4} + i\frac{\eta_e k^2}{m_e}(\omega-ku_0)$$

This is the dispersion relation

Now put $k = k_r + i k_i$

$$\frac{\delta\left(\beta K^2 + \frac{H^2K^4}{4}\right)}{1+\delta\left(\beta K^2 + \frac{H^2K^4}{4}\right)} = (\omega - KU_0)^2 - \frac{H^2K^4}{4} + i\eta_e K^2(\omega - KU_0)$$

Now $\left[1+\delta\left(\beta K^2 + \frac{H^2K^4}{4}\right)\right] = 1+\delta\beta(K_r^2 - K_i^2) + \frac{\delta H^2}{4}(K_r^2-K_i^2)^2 - \delta H^2 K_r^2 K_i^2$
$+ i\left[2\delta\beta K_r K_i + \delta H^2 K_r K_i (K_r^2 - K_i^2)\right]$

$$\cancel{\frac{\delta\left(\beta K^2 + \frac{H^2K^4}{4}\right)\left[1 - \delta\left(\beta K^2 + \frac{H^2K^4}{4}\right)\right]}{1 - \delta^2\left(\beta K^2 + \frac{H^2K^4}{4}\right)^2}}$$

$\therefore$ ~~1+δ(βK~~

$$\frac{\left\{\underbrace{\left[1+\delta\beta(K_r^2-K_i^2) + \frac{\delta H^2}{4}(K_r^2-K_i^2)^2 - \delta H^2 K_r^2 K_i^2\right]}_{f_1(K_r,K_i)} - i\underbrace{\left[2\delta\beta K_r K_i + \delta H^2 K_r K_i \times (K_r^2-K_i^2)\right]}_{f_2(K_r,K_i)}\right\} \times \left\{\delta\beta(K_r^2-K_i^2) + \frac{\delta H^2}{4}(K_r^2-K_i^2)^2 - \delta H^2 K_r^2 K_i^2 + i\left[2\delta\beta K_r K_i + \delta H^2 K_r K_i \cdot (K_r^2-K_i^2)\right]\right\}}{\left[1+\delta\beta(K_r^2-K_i^2) + \frac{\delta H^2}{4}(K_r^2-K_i^2)^2 - \delta H^2 K_r^2 K_i^2\right]^2 - \left[2\delta\beta K_r K_i + \delta H^2 K_r K_i (K_r^2-K_i^2)\right]^2}$$

$$\frac{\left[\{1+f_1(K_r,K_i)\} - i f_2(K_r,K_i)\right]\left[f_1(K_r,K_i) + i f_2(K_r,K_i)\right]}{\left[1+f_1(K_r,K_i)\right]^2 - \left(f_2(K_r,K_i)\right)^2}$$

Re part of this is $f_1 = \dfrac{[1+f_1(K_r,K_i)]\, f_1(K_r,K_i) + [f_2(K_r,K_i)]^2}{[1+f_1(K_r,K_i)]^2 - [f_2(K_r,K_i)]^2}$

Imaginary part of this is $\dfrac{-f_1 f_2 + f_2 + f_1 f_2}{(1+f_1)^2 - f_2^2}$

Now $(\omega - ku_0)^2 - \frac{H^2 k^4}{4} + i\eta_e k^2 (\omega - ku_0)$

Real part of this :-

$$\omega^2 - 2\omega u_0 k_r + u_0^2(k_r^2 - k_i^2) - \frac{H^2}{4}(k_r^2 - k_i^2)^2 + H^2 k_r^2 k_i^2$$

$$- 2k_r k_i \eta_0 (\omega - k_r u_0) + \eta_e k_i u_0 (k_r^2 - k_i^2)$$

$$= \omega^2 - \omega\left[2u_0 k_r - 2k_r k_i \eta_0\right] + f_3(k_r, k_i)$$

where $f_3(k_r, k_i) = u_0^2(k_r^2 - k_i^2) - \frac{H^2}{4}(k_r^2 - k_i^2)^2 + H^2 k_r^2 k_i^2$

$$+ 2k_r^2 k_i \eta_0 u_0 + \eta_0 k_i u_0 (k_r^2 - k_i^2)$$

Comparing this with previous real part

$$\omega^2 - \omega(2u_0 k_r - 2k_r k_i \eta_0) + f_3(k_r, k_i) = f_4$$

$$\omega^2 - \omega(2u_0 k_r - 2k_r k_i \eta_0) + f_5 = 0$$

$$\omega = \frac{(2u_0 k_r - 2k_r k_i \eta_0) + \sqrt{(u_0 k_r - k_r k_i \eta_0)^2 - f_5}}{2}$$

$\boxed{\text{for real part vs } k_r}$ — $k_i = 0.5,\ u_0 = 0.5,\ \eta_0 = 2,\ H = 2$

$\delta = 0.1,\ \beta = 5.42$

Imaginary part of this :-

$$2k_r k_i u_0 - 2\omega u_0 k_i + \eta (k_r^2 - k_i^2)(\omega - k_r u_0) + 2\eta_e k_r k_i^2 u_0 - H^2 k_r k_i (k_r^2 - k_i^2)$$

$$- \frac{f_2}{(1+f_1)^2 - f_2^2} = 0$$

The stretching (S2)

$$\xi = \epsilon(x - v_0 t), \quad \tau = \epsilon^3 t, \quad \eta = \epsilon \eta_0$$

$$\therefore \frac{\partial}{\partial x} f(\xi, \tau) = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial x}$$

$$\therefore \frac{\partial}{\partial x} \equiv \epsilon \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} = -\epsilon v_0 \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau}$$

From equ (6)

$$\left[-\epsilon v_0 \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau}\right]\left[1 + \epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \epsilon^3 n_h^{(3)}\right] +$$
$$\epsilon \frac{\partial}{\partial \xi}\Big[u_0 + \epsilon\left(n_h^{(1)} u_0 + u_h^{(1)}\right) + \epsilon^2\left(u_0 n_h^{(2)} + n_h^{(1)} u_h^{(1)} + u_h^{(2)}\right) +$$
$$\epsilon^3\left(n_h^{(1)} u_h^{(2)} + n_h^{(2)} u_h^{(1)} + u_0 n_h^{(3)} + u_h^{(3)}\right)\Big] = 0$$

equating both side the coefficients of $\epsilon^2, \epsilon^3, \epsilon^4$

$$\epsilon^2: \; -v_0 \frac{\partial n_h^{(1)}}{\partial \xi} + \frac{\partial}{\partial \xi}\left[n_h^{(1)} u_0 + u_h^{(1)}\right] = 0$$

$$\Rightarrow \boxed{u_h^{(1)} = (V_0 - u_0)\, n_h^{(1)}} \quad \text{— (12)}$$

$$\epsilon^3: \; -v_0 \frac{\partial n_h^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}\left[u_0 n_h^{(2)} + n_h^{(1)} u_h^{(1)} + u_h^{(2)}\right] = 0$$

$$\Rightarrow (-v_0 + u_0)\frac{\partial n_h^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}\left[n_h^{(1)} u_h^{(1)}\right] + \frac{\partial}{\partial \xi} u_h^{(2)} = 0 \quad \text{— (13)}$$

$$\epsilon^4: \; -v_0 \frac{\partial n_h^{(3)}}{\partial \xi} + \frac{\partial n_h^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left[n_h^{(1)} u_h^{(2)} + n_h^{(2)} u_h^{(1)} + u_0 n_h^{(3)} + u_h^{(3)}\right] = 0$$

$$\Rightarrow (-v_0 + u_0)\frac{\partial n_h^{(3)}}{\partial \xi} + \frac{\partial n_h^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left[n_h^{(1)} u_h^{(2)} + n_h^{(2)} u_h^{(1)} + u_h^{(3)}\right] = 0 \quad \text{— (14)}$$

Similarly from eqn (7)

$$\epsilon^2 : \boxed{u_c^{(1)} = (v_0 - u_0)\, n_c^{(1)}} \quad - (15)$$

$$\epsilon^3 : (-v_0 + u_0)\frac{\partial n_c^{(2)}}{\partial \xi} + n_c^{(1)}\frac{\partial n_c^{(1)}}{\partial \xi}\cdot 2(v_0 - u_0) + \frac{\partial u_c^{(2)}}{\partial \xi} = 0 \quad - (16)$$

$$\epsilon^4 : (-v_0 + u_0)\frac{\partial n_c^{(3)}}{\partial \xi} + \frac{\partial n_c^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}\left[n_c^{(1)} u_c^{(2)} + n_c^{(2)} u_c^{(1)} + u_c^{(3)} \right] = 0 \quad - (17)$$

From eqn (8)

i) $$\frac{\partial \phi}{\partial x} = \epsilon \frac{\partial \phi}{\partial \xi} = \epsilon \frac{\partial}{\partial \xi}\left[\phi_0 + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \epsilon^3 \phi^{(3)} \right]$$

ii) $$n_h^{-1/3} = \left[1 + \epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \ldots \right]^{-1/3}$$

$$= 1 - \frac{1}{3}\left(\epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \epsilon^3 n_h^{(3)} \right) + \cancel{\frac{4}{9\cdot 2!}\epsilon^2 {n_h^{(1)}}^2}$$

$$-\beta \frac{\partial n_h}{\partial x} = -\beta \epsilon \frac{\partial}{\partial \xi}\left[1 + \epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \epsilon^3 n_h^{(3)} \right]$$

From ii)

$$\epsilon^2 : -\beta \frac{\partial n_h^{(1)}}{\partial \xi}$$

$$\epsilon^3 : +\beta \frac{1}{3} n_h^{(1)} \frac{\partial n_h^{(1)}}{\partial \xi} + \frac{\partial n_h^{(2)}}{\partial \xi}(-\beta)$$

$$\epsilon^4 : -\beta\left[n_h^{(2)} \frac{\partial n_h^{(1)}}{\partial \xi} + \frac{\partial n_h^{(3)}}{\partial \xi} \right] \cancel{\frac{2}{9} {n_h^{(1)}}^2 \frac{\partial}{\partial \xi} n_h^{(1)}}$$

iii) $$\frac{1}{\sqrt{n_h}} = 1 - \frac{1}{2}\left[\epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \epsilon^3 n_h^{(3)} \right]$$

$$\frac{\partial^2}{\partial x^2} = \epsilon^2 \frac{\partial^2}{\partial \xi^2}$$

$$\sqrt{n_h} = 1 + \frac{1}{2}\left[\epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \epsilon^3 n_h^{(3)} \right]$$

$$\frac{\partial^2 \sqrt{n_h}}{\partial x^2} = \frac{\epsilon^3}{2}\frac{\partial^2 n_h^{(1)}}{\partial \xi^2} + \frac{\epsilon^4}{2}\frac{\partial^2 n_h^{(2)}}{\partial \xi^2}$$

$$\frac{1}{\sqrt{n_h}}\frac{\partial^2 \sqrt{n_h}}{\partial x^2} \cong \frac{\epsilon^3}{2}\frac{\partial^2 n_h^{(1)}}{\partial \xi^2} + \frac{\epsilon^4}{2}\frac{\partial^2 n_h^{(2)}}{\partial \xi^2}$$

$$\frac{H^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_h}}\frac{\partial^2 \sqrt{n_h}}{\partial x^2} \right] \cong \frac{\epsilon^4 H^2}{4}\cdot \frac{\partial^3 n_h^{(1)}}{\partial \xi^3}$$

$\therefore$ equating the coefficient of ϵ^2, ϵ^3 & ϵ^4 of equ (8)

$$\epsilon^2: \quad \frac{\partial \phi^{(1)}}{\partial \xi} - \beta \frac{\partial n_h^{(1)}}{\partial \xi} = 0$$

$$\Rightarrow \boxed{\phi^{(1)} = \beta n_h^{(1)}} \quad \text{--- (18)}$$

$$\epsilon^3: \quad \frac{\partial \phi^{(2)}}{\partial \xi} + \frac{\beta}{3} n_h^{(1)} \frac{\partial n_h^{(1)}}{\partial \xi} - \beta \frac{\partial n_h^{(2)}}{\partial \xi} = 0 \quad \text{--- (19)}$$

$$\epsilon^4: \quad \frac{\partial \phi^{(3)}}{\partial \xi} - \beta n_h^{(2)} \frac{\partial n_h^{(1)}}{\partial \xi} - \beta \frac{\partial n_h^{(3)}}{\partial \xi} + \frac{H^2}{4} \frac{\partial^3 n_h^{(1)}}{\partial \xi^3} = 0 \quad \text{--- (20)}$$

From equ (9)

$$\frac{\partial}{\partial t} u_e = \left[-\epsilon v_0 \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau}\right]\left[u_0 + \epsilon u_e^{(1)} + \epsilon^2 u_e^{(2)} + \epsilon^3 u_e^{(3)} + \cdots\right]$$

$$u_e \frac{\partial u_e}{\partial x} = \left[u_0 + \epsilon u_e^{(1)} + \epsilon^2 u_e^{(2)} + \cdots\right] \epsilon \frac{\partial}{\partial \xi}\left[u_0 + \epsilon u_e^{(1)} + \epsilon^2 u_e^{(2)} + \cdots\right]$$

$$\frac{\eta_e}{m_e} \frac{\partial^2 u_e}{\partial x^2} = \frac{\epsilon \eta_0}{m_e} \frac{\epsilon^2 \partial^2}{\partial \xi^2}\left[u_0 + \epsilon u_e^{(1)} + \epsilon^2 u_e^{(2)} + \cdots\right]$$

$$\frac{H^2}{2} \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_e}} \frac{\partial^2 \sqrt{n_e}}{\partial x^2}\right] = \frac{\epsilon^4 H^2}{4} \frac{\partial^3 n_e^{(1)}}{\partial \xi^3}$$

$$\frac{\partial \phi}{\partial x} = \epsilon \frac{\partial}{\partial \xi}\left[\phi_0 + \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots\right]$$

$\therefore$ equating both side the coefficient of ϵ^2, ϵ^3 & ϵ^4

$$\epsilon^2: \quad -v_0 \frac{\partial u_e^{(1)}}{\partial \xi} + u_0 \frac{\partial u_e^{(1)}}{\partial \xi} = \frac{\partial \phi^{(1)}}{\partial \xi}$$

$$\Rightarrow -\phi^{(1)} = (v_0 - u_0) u_e^{(1)}$$

$$\Rightarrow \boxed{\phi^{(1)} = -(v_0 - u_0) u_e^{(1)}} \quad \text{--- (21)}$$

$$\epsilon^3: \quad -v_0 \frac{\partial u_e^{(2)}}{\partial \xi} + u_0 \frac{\partial u_e^{(2)}}{\partial \xi} + u_e^{(1)} \frac{\partial u_e^{(1)}}{\partial \xi} = \frac{\partial \phi^{(2)}}{\partial \xi}$$

$$\Rightarrow (-v_0 + u_0) \frac{\partial u_e^{(2)}}{\partial \xi} + (v_0 - u_0)^2 n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} = \frac{\partial \phi^{(2)}}{\partial \xi} \quad \text{--- (22)}$$

$$\epsilon^4: \quad -v_0 \frac{\partial u_e^{(3)}}{\partial \xi} + \frac{\partial u_e^{(1)}}{\partial \tau} + u_0 \frac{\partial u_e^{(3)}}{\partial \xi} + u_e^{(1)} \frac{\partial u_e^{(2)}}{\partial \xi} = \frac{\partial \phi^{(3)}}{\partial \xi} + \frac{H^2}{4} \frac{\partial^3 n_e^{(1)}}{\partial \xi^3} + \frac{\eta_0}{m_e} \frac{\partial^2 u_e^{(1)}}{\partial \xi^2} \quad \text{--- (23)}$$

From eqn (10)

$$\epsilon^2 \frac{\partial^2}{\partial \xi^2}\left[\phi_0 + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)}\right] = 1 + \epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)} + \epsilon^3 n_e^{(3)} + \cdots$$
$$\frac{\delta_1}{\delta} n_i + \frac{1}{\delta}\left[1 + \epsilon n_h^{(1)} + \epsilon^2 n_h^{(2)} + \cdots\right]$$

$\therefore$ ϵ : $n_e^{(1)} + \frac{1}{\delta} n_h^{(1)} = 0 \Rightarrow \boxed{n_e^{(1)} = -\frac{1}{\delta} n_h^{(1)}}$

ϵ^2 : $n_e^{(2)} + \frac{1}{\delta} n_h^{(2)} = 0 \Rightarrow \boxed{n_e^{(2)} = -\frac{1}{\delta} n_h^{(2)}}$

ϵ^3 : $\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = n_e^{(3)} + \frac{1}{\delta} n_h^{(3)}$

ϵ^4 : $\frac{\partial^2 \phi^{(2)}}{\partial \xi^2} = n_e^{(4)} + \frac{1}{\delta} n_h^{(4)} \Rightarrow \frac{\partial \phi^{(2)}}{\partial \xi} = \int\left[n_e^{(4)} + \frac{1}{\delta} n_h^{(4)}\right] d\xi$

as $n_e^{(4)}$ & $n_h^{(4)}$ are higher order term and $\xi \to \infty$ $\therefore$ we can neglect this term

$\therefore$ $\frac{\partial \phi^{(2)}}{\partial \xi} \to 0$ Similarly $\frac{\partial \phi^{(3)}}{\partial \xi} \to 0$

$$\therefore \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = \frac{\partial n_e^{(3)}}{\partial \xi} + \frac{1}{\delta}\frac{\partial n_h^{(3)}}{\partial \xi} \quad \text{--- (24)}$$

From eqn (22)

$$(v_0 - u_0)\frac{\partial u_e^{(2)}}{\partial \xi} = (v_0 - u_0)^2 n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi}$$

$$\Rightarrow \boxed{\frac{\partial u_e^{(2)}}{\partial \xi} = (v_0 - u_0) n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi}} \Rightarrow u_e^{(2)} = \frac{(v_0 - u_0)}{2} n_e^{(1)^2}$$

Now put this value in eqn (16)

$$(-v_0 + u_0)\frac{\partial n_e^{(2)}}{\partial \xi} + n_e^{(1)}\frac{\partial n_e^{(1)}}{\partial \xi} \cdot 2(v_0 - u_0) + (v_0 - u_0) n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} = 0$$

$$\Rightarrow \boxed{\frac{\partial n_e^{(2)}}{\partial \xi} = 3 n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi}} \Rightarrow n_e^{(2)} = \frac{3}{2} n_e^{(1)^2}$$

$\Rightarrow$

From eqn (17)

$$\frac{\partial}{\partial \xi}\left[n_e^{(1)}u_e^{(2)} + n_e^{(2)}u_e^{(1)} + u_e^{(3)}\right]$$

$$= \frac{\partial n_e^{(1)}}{\partial \xi}\,\frac{(V_0-u_0)}{2}\,n_e^{(1)^2} + n_e^{(1)}(V_0-u_0)\,n_e^{(1)}\frac{\partial n_e^{(1)}}{\partial \xi}$$

$$+\ 3n_e^{(1)}\frac{\partial n_e^{(1)}}{\partial \xi}(V_0-u_0)\,n_e^{(1)} + \frac{3}{2}\,n_e^{(1)^2}(V_0-u_0)\frac{\partial n_e^{(1)}}{\partial \xi}$$

$$+\ \frac{\partial u_e^{(3)}}{\partial \xi}$$

$$= n_e^{(1)^2}\frac{\partial n_e^{(1)}}{\partial \xi}\,6(V_0-u_0) + \frac{\partial u_e^{(3)}}{\partial \xi}$$

Now $\dfrac{\partial n_e^{(3)}}{\partial \xi} = \dfrac{\partial^3 \phi^{(1)}}{\partial \xi^3} - \dfrac{1}{\delta}\dfrac{\partial n_h^{(3)}}{\partial \xi}$ [from eqn 24]

Now put those value in eqn (17)

$$\therefore\ \cancel{(-V_0+u_0)\frac{\partial n_e^{(3)}}{\partial \xi}} + \frac{\partial n_e^{(1)}}{\partial \tau} + 6n_e^{(1)^2}(V_0-u_0)\frac{\partial n_e^{(1)}}{\partial \xi} + \frac{\partial u_e^{(3)}}{\partial \xi}$$

$$+\ (-V_0+u_0)\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{(V_0-u_0)}{\delta}\frac{\partial n_h^{(3)}}{\partial \xi} = 0$$

$$-\frac{\partial u_e^{3}}{\partial \xi} = \frac{\partial n_e^{(1)}}{\partial \tau} + 6n_e^{(1)^2}(V_0-u_0)\frac{\partial n_e^{(1)}}{\partial \xi} + \frac{(V_0-u_0)}{\delta}\frac{\partial n_h^{(3)}}{\partial \xi} - \frac{(V_0-u_0)}{1}\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} \quad (25)$$

From equ (19)

$$\frac{\beta}{3} n_h^{(1)} \frac{\partial n_h^{(1)}}{\partial \xi} = \beta \frac{\partial n_h^{(2)}}{\partial \xi}$$

$$\Rightarrow \boxed{n_h^{(2)} = \frac{1}{6} n_h^{(1)^2}}$$

From equ (20)

$$-\frac{\beta}{6} n_h^{(1)^2} \frac{\partial n_h^{(1)}}{\partial \xi} - \beta \frac{\partial n_h^{(3)}}{\partial \xi} + \frac{H^2}{4} \frac{\partial^3 n_h^{(1)}}{\partial \xi^3} = 0$$

$$\frac{\delta^3 \cancel{\beta}}{6} n_e^{(1)^2} \frac{\partial n_e^{(1)}}{\partial \xi} + \cancel{\beta} \frac{H^2}{4\beta} \frac{\partial^3 n_h^{(1)}}{\partial \xi^3} = \cancel{\beta} \frac{\partial n_h^{(3)}}{\partial \xi} \quad -(26)$$

put this value of $\frac{\partial n_h^{(3)}}{\partial \xi}$ in equ (25)

$$\therefore -\frac{\partial u_e^{(3)}}{\partial \xi} = \frac{\partial n_e^{(1)}}{\partial \tau} + 6 n_e^{(1)^2} (V_0 - u_0) \frac{\partial n_e^{(1)}}{\partial \xi} + \frac{(V_0 - u_0)}{\delta} \left[\frac{\delta^3}{6} n_e^{(1)^2} \frac{\partial n_e^{(1)}}{\partial \xi} + \frac{H^2}{4\beta} \frac{\partial^3 n_h^{(1)}}{\partial \xi^3} \right]$$
$$- (V_0 - u_0) \frac{\partial^3 \phi^{(1)}}{\partial \xi^3}$$

put this value of $-\frac{\partial u_e^{(3)}}{\partial \xi}$ in equ (23)

$$\Rightarrow (V_0 - u_0) \left[\frac{\partial n_e^{(1)}}{\partial \tau} + 6 n_e^{(1)^2} (V_0 - u_0) \frac{\partial n_e^{(1)}}{\partial \xi} + \frac{(V_0 - u_0)}{\delta} \left\{ \frac{\delta^3}{6} n_e^{(1)^2} \frac{\partial n_e^{(1)}}{\partial \xi} \right.\right.$$
$$\left.\left. + \frac{H^2}{4\beta} \frac{\partial^3 n_h^{(1)}}{\partial \xi^3} \right\} - (V_0 - u_0) \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} \right]$$
$$+ \frac{\partial u_e^{(1)}}{\partial \tau} + u_e^{(1)} \cancel{\frac{\partial}{\partial \xi}} (V_0 - u_0) n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} = \frac{H^2}{4} \frac{\partial^3 n_e^{(1)}}{\partial \xi^3} + \frac{\eta_0}{m_e} \frac{\partial^2 u_e^{(1)}}{\partial \xi^2}$$

$$\Rightarrow \frac{\partial \phi^{(1)}}{\partial \tau} \left[-\frac{1}{(V_0 - u_0)} - \frac{1}{(V_0 - u_0)} \right] + \cancel{\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} \left[-\frac{H^2}{4} + \frac{H^2}{4} \frac{V_0 - u_0}{\beta} \right.} -$$

$$+ \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} \left[-(V_0 - u_0) \frac{H^2}{4\beta} \cdot \frac{\delta}{(V_0 - u_0)^2} - (V_0 - u_0)^2 - \frac{H^2}{4} \frac{1}{(V_0 - u_0)^2} \right]$$

$$+ \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} \left[\frac{\eta_0}{m_e (V_0 - u_0)} \right]$$

$$+ \phi^{(1)^2} \frac{\partial \phi^{(1)}}{\partial \xi} \left[-\frac{6}{(V_0 - u_0)^5} - \frac{\delta^2}{6} \frac{1}{(V_0 - u_0)^4} - \frac{1}{(V_0 - u_0)^4} \right] = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial \tau} + A\phi^2 \frac{\partial \phi}{\partial \xi} + B \frac{\partial^3 \phi}{\partial \xi^3} - C \frac{\partial^2 \phi}{\partial \xi^2} = 0$$

where

$$B = \frac{\frac{H^2}{4\beta}\frac{\delta}{(V_0 - u_0)} + (V_0 - u_0)^2 + \frac{H^2}{4}\frac{1}{(V_0 - u_0)^2}}{\frac{2}{V_0 - u_0}}$$

$$= \frac{\delta H^2}{8\beta} + \frac{(V_0 - u_0)^3}{2} + \frac{H^2}{8}\frac{1}{(V_0 - u_0)}$$

$$A = \frac{3}{(V_0 - u_0)^4} + \frac{\delta^2}{12}\frac{1}{(V_0 - u_0)^3} + \frac{1}{2(V_0 - u_0)^3}$$

$$C = \frac{\eta_0}{2}$$

6 Appendix B

Code for the graphs

```
"""
@author: nlpl9
Plot the value of the function as beta is varied
"""

import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
import math as m

# Plasma parameters
#u0 = [0.2, 0.4, 0.7]
#V0 = [0.8, 1.0, 1.2]
#delta = [0.1, 0.3, 0.5, 0.7]
#eta = [0.5, 1.0, 2.5]
#beta = [0.33, 2.618, 5.42 ]
#H = [1,2,3]
# Define the functions for the diff eq. params...

u0=0.5
V0=1.2
delta=0.1
eta=1
#beta=5.42
H=1
p = 3/(V0 - u0)**4 + ((delta**2)/12)/(V0 - u0)**3 + 1/(2*(V0 - u0)**3)
q = eta/2
t = 1 # Fix the time

# Define the solution you wish to use... and vectorize it
def r(b):
    r = delta*H**2/(8*b) + 0.5*(V0-u0)**3 + ((H**2)/8)/(V0 - u0)
    return r

@np.vectorize
def g(x, beta):
    R = r(beta)
    g = m.sqrt(6*R/p)*(1 + m.tanh(x - 8*R*t))
    return g


x = np.linspace(-10, 10, 200)
beta = np.linspace(0.2, 6, 100)

X,B = np.meshgrid(x, beta)
```

```
Z = g(X, B)

fig = plt.figure()
ax = plt.gca(projection="3d")

ax.set_xlabel('x')
ax.set_ylabel(r'$\beta$')
ax.set_zlabel(r'$\phi(x,t)$')
surf = ax.plot_surface(X, B, Z, rstride=1,
            cstride=1, cmap='gnuplot2')
fake2Dline = mpl.lines.Line2D([0],[0], linestyle="none",
            c='b', marker = 'o')
plt.legend([fake2Dline], ['u0=.5, V0=1.2, delta=.1,
            eta=.1, H=1'], numpoints = 1)
ax.view_init(20, 120)
plt.savefig('C:\\Users\\nlpl9\\Desktop\\plasma-proj
            \\Graphs\\tanh_phivsBeta.jpg', transperant=True, dpi=144)
plt.show()
```

```
"""
@author: nlpl9
Variation of the solution with delta parameter...
"""
import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
import math as m

# Plasma parameters
#u0 = [0.2, 0.4, 0.7]
#V0 = [0.8, 1.0, 1.2]
#delta = [0.1, 0.3, 0.5, 0.7]
#eta = [0.5, 1.0, 2.5]
#beta = [0.33, 2.618, 5.42 ]
#H = [1,2,3]
# Define the functions for the diff eq. params...

u0=0.5
V0=1.2
#delta=0.1
eta=1
beta=5.42
H=1

q = eta/2
t = 1 # Fix the time

# Define the solution you wish to use... and vectorize it
def r(d):
    r = d*H**2/(8*beta) + 0.5*(V0-u0)**3 + ((H**2)/8)/(V0 - u0)
    return r

def p(d):
    p = 3/(V0 - u0)**4 + ((d**2)/12)/(V0 - u0)**3 + 1/(2*(V0 - u0)**3)
    return p

@np.vectorize
def g(x,D):
    R = r(D)
    P = p(D)
    g = m.sqrt(6*R/P)*(1 + m.tanh(x))# - 8*R*t))
    return g

x = np.linspace(-10, 10, 200)
d = np.linspace(0.1, 0.7, 200)

X,D = np.meshgrid(x, d)
```

```
Z = g(X, D)

fig = plt.figure()
ax = plt.gca(projection="3d")
#ax.plot_wireframe(X, T, Z, color='green')
ax.set_xlabel('x')
ax.set_ylabel(r'$\delta$')
ax.set_zlabel(r'$\phi(x,t)$')
surf = ax.plot_surface(X, D, Z, rstride=1,
                    cstride=1, cmap="gnuplot2")
fake2Dline = mpl.lines.Line2D([0],[0], linestyle="none",
                    c='b', marker = 'o')
plt.legend([fake2Dline], [r'$u_0$=.5, $V_0$=1.2, $\delta$=.1,
                    $\eta$=.5, $\beta$=5.42'], numpoints = 1)
ax.view_init(20, 220)
plt.savefig('C:\\Users\\nlpl9\\Desktop\\plasma-proj\\Graphs
                    \\tanh_phivsDelta', transperant=True, dpi=144)
plt.show()
```

```
"""
@author: nlpl9
"""
import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
import math as m

# Plasma parameters
#u0 = [0.2, 0.4, 0.7]
#V0 = [0.8, 1.0, 1.2]
#delta = [0.1, 0.3, 0.5, 0.7]
#eta = [0.5, 1.0, 2.5]
#beta = [0.33, 2.618, 5.42 ]
#H = [1,2,3]
# Define the functions for the diff eq. params...

u0=0.5
#V0=1.2
delta=0.1
eta=1
beta=5.42
H=1

q = eta/2
t = 1 # Fix the time

# Define the solution you wish to use... and vectorize it
def r(V):
    r = delta*H**2/(8*beta) + 0.5*(V-u0)**3 + ((H**2)/8)/(V - u0)
    return r

def p(V):
    p = 3/(V - u0)**4 + ((delta**2)/12)/(V - u0)**3 + 1/(2*(V - u0)**3)
    return p

@np.vectorize
def g(x,V):
    g = m.sqrt(6*r(V)/p(V))*(1 + m.tanh(x - 8*r(V)*t))
    return g

x = np.linspace(-10,50, 100)
v = np.linspace(0.7,1.2, 100)

X,V = np.meshgrid(x, v)
Z = g(X, V)

fig = plt.figure()
```

```
ax = plt.gca(projection="3d")
#ax.plot_wireframe(X, T, Z, color='green')
ax.set_xlabel('x')
ax.set_ylabel('M')
ax.set_zlabel(r'$\phi(x,t)$')
surf = ax.plot_surface(X, V, Z, rstride=1, cstride=1, cmap='gnuplot2')
fake2Dline = mpl.lines.Line2D([0],[0], linestyle="none",
                    c='b', marker = 'o')
plt.legend([fake2Dline], [r'$u_0$=0.5, $\delta$=0.1, H=1,
                    $\beta$=5.42'], numpoints = 1)
ax.view_init(10, 220)
plt.savefig('C:\\Users\\nlpl9\\Desktop\\plasma-proj\\Graphs
                    \\tanh_phivsV_0.jpg', transperant=True, dpi=144)
plt.show()
```

```
"""
@author: nlpl9
"""
import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
import math as m

# Plasma parameters
#u0 = [0.2, 0.4, 0.7]
#V0 = [0.8, 1.0, 1.2]
#delta = [0.1, 0.3, 0.5, 0.7]
#eta = [0.5, 1.0, 2.5]
#beta = [0.33, 2.618, 5.42 ]
#H = [1,2,3]
# Define the functions for the diff eq. params...

u0=0.5
V0=1.2
delta=0.1
eta=1
beta=5.42

p = 3/(V0 - u0)**4 + ((delta**2)/12)/(V0 - u0)**3 + 1/(2*(V0 - u0)**3)
q = eta/2
t = 1 # Fix the time

# Define the solution you wish to use... and vectorize it
def r(H):
    r = delta*H**2/(8*beta) + 0.5*(V0-u0)**3 + (H**2)/(8*(V0-u0))
    return r

@np.vectorize
def g(x,H):
    g = m.sqrt(6*r(H)/p)*(1 + m.tanh(x - 8*r(H)*t))
    return g

x = np.linspace(0,10, 200)
h = np.linspace(1,3, 100)
X,H = np.meshgrid(x, h)
Z = g(X, H)

fig = plt.figure()
ax = plt.gca(projection="3d")
ax.set_xlabel('x')
ax.set_ylabel('H')
ax.set_zlabel(r'$\phi(x,t)$')
surf = ax.plot_surface(X, H, Z, rstride=1, cstride=1,
```

```
                    cmap='gnuplot2')
fake2Dline = mpl.lines.Line2D([0],[0], linestyle="none",
                    c='b', marker = 'o')
plt.legend([fake2Dline], [r'$u_0$=.5, $V_0$=1.2, $\delta$=0.1,
                    $\beta$=5.42'], numpoints = 1)
ax.view_init(20, 210)
plt.savefig('C:\\Users\\nlpl9\\Desktop\\plasma-proj\\Graphs
                    \\tanh_phivsH.jpg', transperant=True, dpi=144)
plt.show()
```

```
"""
@author: nlpl9
"""
import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
import math as m

# Plasma parameters
#u0 = [0.2, 0.4, 0.7]
#V0 = [0.8, 1.0, 1.2]
#delta = [0.1, 0.3, 0.5, 0.7]
#eta = [0.5, 1.0, 2.5]
#beta = [0.33, 2.618, 5.42 ]
#H = [1,2,3]
# Define the functions for the diff eq. params...

V0=1.2
delta=0.1
eta=1
beta=5.42
H=1

q = eta/2
t = 1 # Fix the time

# Define the solution you wish to use... and vectorize it
def r(u):
    r = delta*H**2/(8*beta) + 0.5*(V0-u)**3 + ((H**2)/8)/(V0 - u)
    return r

def p(u):
    p = 3/(V0 - u)**4 + ((delta**2)/12)/(V0 - u)**3 + 1/(2*(V0 - u)**3)
    return p

@np.vectorize
def g(x,U0):
    g = m.sqrt(6*r(U0)/p(U0))*(1 + m.tanh(x - 8*r(U0)*t))
    return g


x = np.linspace(-10,50, 100)
u = np.linspace(0.1,0.7, 100)
X,U0 = np.meshgrid(x, u)
Z = g(X, U0)

fig = plt.figure()
ax = plt.gca(projection="3d")
```

```
ax.set_xlabel('x')
ax.set_ylabel('$u_0$')
ax.set_zlabel(r'$\phi(x,t)$')
surf = ax.plot_surface(X, U0, Z, rstride=1, cstride=1,
                    cmap='gnuplot2')
fake2Dline = mpl.lines.Line2D([0],[0], linestyle="none",
                    c='b', marker = 'o')
plt.legend([fake2Dline], [r'$V_0$=1.2, $\delta$=0.1,
                    H=1, $\beta$=5.42'], numpoints = 1)
ax.view_init(20, 120)
plt.savefig('C:\\Users\\nlpl9\\Desktop\\plasma-proj\\Graphs
                    \\tanh_phivsU0.jpg', transperant=True, dpi=144)
plt.show()
```

7 Appendix C

Code for the simulation of the differential equation. It is a MATLAB script.

```
close all hidden
clear all
clc
set(gca,'FontSize',18)
set(gca,'LineWidth',2)
%% Place variables for eqn
V0 = 1.2;
u0 = 0.5;
delta = 0.3;
H = 1;
beta = 5.42;
eta = 0.5;
% Set the parameters for the differential equation, namely: b, r, q
b = 3/(V0 - u0)^4 + (delta^2/12)/((V0 - u0)^3) + 0.5/(V0 - u0)^3;
r = delta*(H^2)/(8*beta) + 0.5*(V0 - u0)^3 + H^2/(8*(V0 - u0));
q = 0.5*eta;


N = 256;
x = linspace(-40,40,N);
delta_x = x(2) - x(1);
delta_k = 2*pi/(N*delta_x);

k = [0:delta_k:(N/2-1)*delta_k,0,-(N/2-1)*delta_k:delta_k:-delta_k];
t=0;

u = sqrt(6*r/b)*(1 + tanh(x - 8*r*t));

delta_t = 0.4/N^2;
plot(x,u,'LineWidth',2)
axis([-40 40 -1 3])
xlabel('x')
ylabel('u')
text(6,2.5,['t = ',num2str(t,'%1.2f')],'FontSize',18)
drawnow

tmax = 40; nplt = floor((1/100)/delta_t); nmax = round(tmax/delta_t);
udata = u.'; tdata = 0;
U = fft(u);
j = 1;

for n = 1:nmax
    t = n*delta_t;
    % first do the linear part
    U = U.*exp(1i*r*k.^3*delta_t - q*delta_t*k.^2);
```

```
    % then solve the nonlinear part

    U = U  - ((b/3)*1i*k*delta_t).*fft((real(ifft(U))).^3);

    if mod(n,nplt) == 0
        u = real(ifft(U));
        udata = [udata u.']; tdata = [tdata t];
        if mod(n,4*nplt) == 0
            plot(x,u,'LineWidth',2)
            axis([-40 100 -1 3])
            xlabel('x')
            ylabel('u')
            text(6,2.5,['t = ',num2str(t,'%1.2f')],'FontSize',18)
            drawnow
        end
    end
end

figure

waterfall(x,tdata(1:4:end),udata(:,1:4:end)')
colormap(1e-6*[1 1 1]); view(-20,25)
xlabel x, ylabel t, axis([-40 40 0 tmax -1 3]), grid off
zlabel('u')
set(gca,'ztick',[-1 0 1 2 3]), pbaspect([1 1 .13])
drawnow
```

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