DS 3001 A6

Question 1:

$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \xrightarrow[\text{model}]{\text{plug in}} SSE = \sum_{i=1}^{n} (y_i - (b_0 + b_1 z_{i1} + b_2 z_{i2}))^2$

$z_1 + z_2$ are centered $b_0 = \bar{y}$

model $\hat{y}_i = \bar{y} + b_1 z_{i1} + b_2 z_{i2}$

Residual $y_i - \hat{y}_i = y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}$

Full Expression:

$SSE = \sum_{i=1}^{n} (y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2})^2$

z_1 and z_2 = Centered

$\sum z_{i1} = \sum z_{i2} = 0$

b_0

$\sum (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0 \Rightarrow \sum (y_i - b_0) = 0 \Rightarrow b_0 = \bar{y}$

$b_1 + b_2$

$\sum z_{i1}(y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = 0$

$\sum z_{i2}(y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = 0$

Question 2:

$\boxed{b_0}$ $\frac{\partial SSE}{\partial b_0} = \sum_{i=1}^{n} 2(y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) \cdot (-1)$

$\frac{\partial SSE}{\partial b_0} = -2 \sum_{i=1}^{n} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2})$

$\sum_{i=1}^{n} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0$

$\boxed{b_1}$ $\frac{\partial SSE}{\partial b_1} = \sum_{i=1}^{n} 2(y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) \cdot (-z_{i1})$

$\frac{\partial SSE}{\partial b_1} = -2 \sum_{i=1}^{n} z_{i1} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2})$

$\sum_{i=1}^{n} z_{i1} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0$

$\boxed{b_2}$ $\frac{\partial SSE}{\partial b_2} = \sum_{i=1}^{n} 2(y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) \cdot (-z_{i2})$

$\frac{\partial SSE}{\partial b_2} = -2 \sum_{i=1}^{n} z_{i2} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2})$

$\sum_{i=1}^{n} z_{i2} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0$

Question 3:

Average Residual

$\bar{e} = \frac{1}{n} \sum_{i=1}^{n} e_i = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2})$

$\sum (y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = \sum (y_i - \bar{y}) - b_1 \sum z_{i1} - b_2 \sum z_{i2}$

$\sum (y_i - \bar{y}) = 0$ $\sum e_i = 0 \Rightarrow \bar{e} = 0$ av res = 0

Optimum $e \cdot z = 0$

$\sum e_i z_{i1} = 0$ and $\sum e_i z_{i2} = 0$

$\sum z_{i1}(y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = \sum e_i z_{i1} = 0$

$\sum z_{i2}(y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = \sum e_i z_{i2} = 0$

Question 4:

Partial dev b_0: $\frac{\partial SSE}{\partial b_0} = -2 \sum_{i=1}^{n} (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2})$

$\sum (y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0$

$n b_0 = \sum y_i - b_1 \sum z_{i1} - b_2 \sum z_{i2}$

$n b_0 = \sum y_i \Rightarrow b_0^* = \bar{y}$

$b_0^* = \bar{y} = \hat{y}$ when $z_1 = z_2 = 0$

$\sum z_{i1}(y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0$ $\sum z_{i1}(y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = 0$

$\sum z_{i2}(y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}) = 0$ $\sum z_{i2}(y_i - \bar{y} - b_1 z_{i1} - b_2 z_{i2}) = 0$

Sub $b_0^* = \bar{y}$

Centered y $y_i^c = y_i - \bar{y}$ $\rightarrow \begin{cases} \sum z_{i1} y_i^c = b_1 \sum z_{i1}^2 + b_2 \sum z_{i1} z_{i2} \\ \sum z_{i2} y_i^c = b_1 \sum z_{i1} + b_2 \sum z_{i2}^2 \end{cases}$

Question 5:

Matrix form

$$\underbrace{\begin{bmatrix} \sum z_{i1}^2 & \sum z_{i1} z_{i2} \\ \sum z_{i1} z_{i2} & \sum z_{i2}^2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} \sum z_{i1} y_i^c \\ \sum z_{i2} y_i^c \end{bmatrix}}_{C}$$

Question 6:

$z_{ij} = x_{ij} - m_j$ and $y_i^c = y_i - \bar{y}$

$\frac{1}{N} \sum z_{i1}^2 = var(x_1)$, $\frac{1}{N} \sum z_{i2}^2 = var(x_2)$, $\frac{1}{N} \sum z_{i1} z_{i2} = cov(x_1, x_2)$

DIVIDE BY N $\frac{1}{N} \sum z_{i1} y_i^c = cov(x_1, y)$, $\frac{1}{N} \sum z_{i2} y_i^c = cov(x_2, y)$

$$\begin{bmatrix} var(x_1) & cov(x_1, x_2) \\ cov(x_1, x_2) & var(x_2) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} cov(x_1, y) \\ cov(x_2, y) \end{bmatrix}$$

$\frac{1}{N} \sum_{i=1}^{N} z_{i1}(y_i - \bar{y}) = b_1 \frac{1}{N} \sum_{i=1}^{N} z_{i1}^2 + b_2 \frac{1}{N} \sum_{i=1}^{N} z_{i1} z_{i2}$

$\frac{1}{N} \sum_{i=1}^{N} z_{i2}(y_i - \bar{y}) = b_1 \frac{1}{N} \sum_{i=1}^{N} z_{i1} z_{i2} + b_2 \frac{1}{N} \sum_{i=1}^{N} z_{i2}^2$

Substitute $z_{ij} = x_{ij} - m_j$

$\frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)(y_i - \bar{y}) = b_1 \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)^2 + b_2 \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)(x_{i2} - m_2)$

$\frac{1}{N} \sum_{i=1}^{N} (x_{i2} - m_2)(y_i - \bar{y}) = b_1 \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)(x_{i2} - m_2) + b_2 \frac{1}{N} \sum_{i=1}^{N} (x_{i2} - m_2)^2$

Matrix form

$$\begin{pmatrix} \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)^2 & \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)(x_{i2} - m_2) \\ \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - m_1)(x_{i2} - m_2) & \frac{1}{N} \sum_{i=1}^{N} (x_{i2} - m_2)^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{N} \sum (x_{i1} - m_1)(y_i - \bar{y}) \\ \frac{1}{N} \sum (x_{i2} - m_2)(y_i - \bar{y}) \end{pmatrix}$$

Intuition: $Ab = C$

$A \rightarrow$ summarizes predictor - predictor relationship

$C \rightarrow$ summarizes predictor - outcome relationships

Solving $Ab = C$ adjusts slopes so linear model aligns optimally