1. If we take $\epsilon = 0$, then $E_{surf} = -\sigma \sum_{i=1}^{L} n_i$.

   (i) Then

$$\begin{aligned}\Xi_s &= \sum_{\nu} e^{-\beta E_\nu + \beta\mu N_\nu} \\ &= \sum_{n_i} e^{\beta\sigma \sum_{i=1}^{L} n_i + \beta\mu \sum_{i=1}^{L} n_i} \\ &= \left[1 + e^{(\sigma+\mu)/k_B T}\right]^L\end{aligned}$$

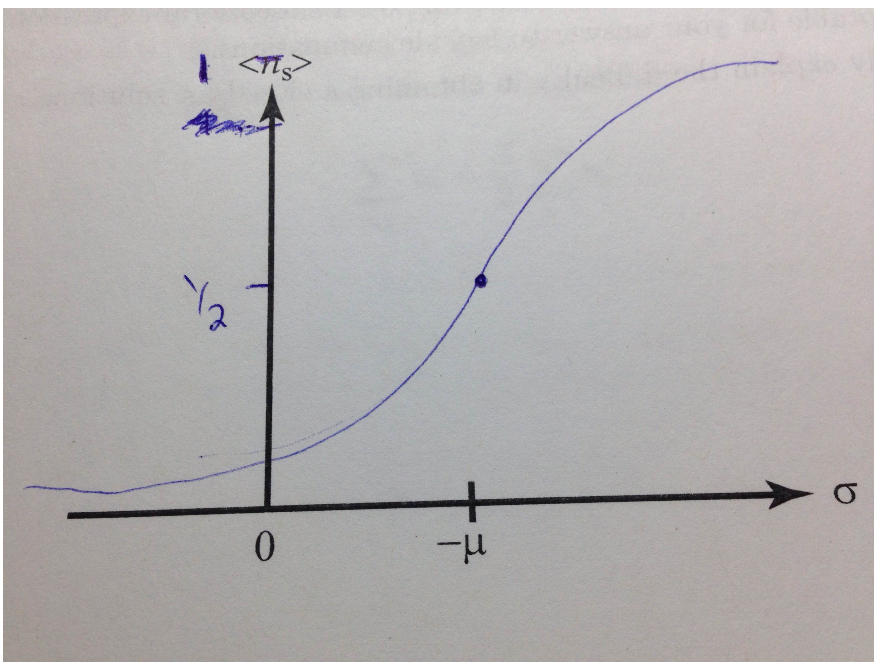
   (ii)

$$\begin{aligned}\langle N_s \rangle &= \left(\frac{d \ln \Xi_s}{d\beta\mu}\right)_\beta \\ &= \frac{L e^{\beta(\sigma+\mu)}}{1 + e^{\beta(\sigma+\mu)}}\end{aligned}$$

   So then

$$\langle n_s \rangle = \frac{e^{\beta(\sigma+\mu)}}{1 + e^{\beta(\sigma+\mu)}}$$

   (iii) At $\sigma = -\mu$, $\langle n_s \rangle = 1/2$.



2. Now $\epsilon \neq 0$, so $E_{surf} = -\sigma \sum_{i=1}^{L} n_i - \epsilon \sum_{ij}{}' n_i n_j$.

(i)

$$\begin{aligned}\Xi_s &= \sum_{\nu} e^{-\beta E_\nu + \beta\mu N_\nu} \\ &= \sum_{n_i} e^{\beta\sigma \sum_{i=1}^{L} n_i + \beta\epsilon \sum_{ij}{}' n_i n_j + \beta\mu \sum_{i=1}^{L} n_i}\end{aligned}$$

The second term in the exponential is not factorizable, so it is highly non-trivial to find a closed form representation of Ξ_s.

(ii)

$$Q_{Ising} = \sum_{\nu} e^{-\beta E_\nu} = \sum_{s_i} e^{\beta h \sum_i s_i + \beta J \sum_{ij}{}' n_i n_j}$$

(iii) We know that $s_i = 2n_i - 1$, so performing the isomorphism yields

$$\begin{aligned}Q_{Ising} &= \sum_{n_i} e^{\beta h \sum_i (2n_i - 1) + \beta J \sum_{ij}{}' (2n_i - 1)(2n_j - 1)} \\ &\sim \sum_{n_i} e^{\beta 2(h - Jz) \sum_i n_i + 4\beta J \sum_{ij}{}' n_i n_j}\end{aligned}$$

Thus we have that $\epsilon = 4J$ and $(\sigma + \mu) = 2(h - Jz) = 2(h - \epsilon z/4)$.

3. (i)

$$z_1 = \sum_{n_1=0}^{1} e^{\beta\sigma_{MF} n_1 + \beta\mu n_1} = 1 + e^{\beta(\sigma_{MF} + \mu)}$$

(ii) Note that

$$e^{-x/2} e^{x/2} (1 + e^x) = e^{x/2} 2\cosh x.$$

That said,

$$z_1 = e^{\frac{\beta}{2}(\sigma_{MF} + \mu)} 2\cosh\left[\frac{\beta}{2}(\sigma_{MF} + \mu)\right].$$

(iii)

$$\langle n_1 \rangle = \left(\frac{d \ln z_1}{d\beta\mu}\right)_{\beta} = \frac{1}{2}\left(1 + \tanh\left[\frac{\beta}{2}(\sigma_{MF} + \mu)\right]\right)$$

(iv) Recall that $\sigma_{MF} = \sigma + \epsilon z \bar{n}$. Plugging this into the above result yields

$$\bar{n} = \frac{1}{2}\left(1 + \tanh\left[\frac{\beta}{2}(\sigma + \epsilon z \bar{n} + \mu)\right]\right).$$

(v) Setting $h = 0$ in the isomorphism to the Ising model gives

$$\mu_{coex} = -(2Jz + \sigma) = -(\epsilon z/2 + \sigma).$$

(vi)

$$\bar{n} = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{\beta}{2}z\epsilon(\bar{n} - \frac{1}{2})\right]$$
$$\bar{n} - \frac{1}{2} = \frac{1}{2}\tanh\left[\frac{\beta}{2}z\epsilon(\bar{n} - \frac{1}{2})\right]$$

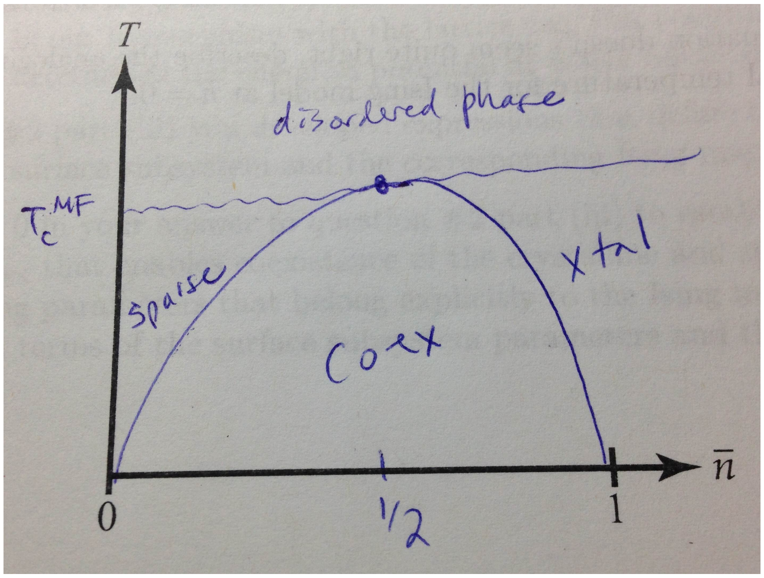
(vii) For small x, $\tanh x \approx x$, so

$$\bar{n} - \frac{1}{2} = \frac{\beta}{4}z\epsilon(\bar{n} - \frac{1}{2}).$$

Thus

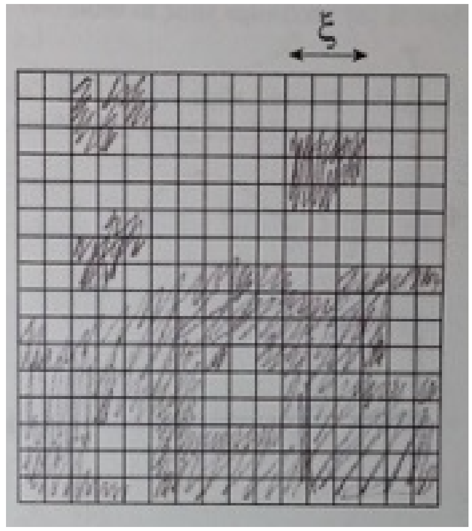
$$T_c^{MF} = \frac{\epsilon z}{4k_B}.$$

(viii) The coexistence diagram is similar to those we have drawn previously. Here the critical point occurs at $\bar{n} = 1/2$, where T_c is approximated as T_c^{MF}.



(ix) A larger ϵ suggests a higher temperature for crystallization. The mean field approximation should lead to an overestimated critical temperature, since it neglects fluctuations due to correlations with nearest neighbors.

(x) Here we are in the coexistence regime, so both phases will be present.



(xi) The average occupation number $\bar{n}$ could be expected to be larger when $\epsilon \neq 0$, as it is more energetically favorable to have higher occupation numbers when $\epsilon \neq 0$.