1. (a) The probability of observing heads for a given flip is $1/2$, which is equivalent to the probability of observing tails for a single flip. The probability of observing a particular sequence for N flips is then

$$\left(\frac{1}{2}\right)^N$$

(b)

$$\frac{N!}{(N - N_{\mathrm{H}})! \times N_{\mathrm{H}}!}$$

(c) We know the number of sequences that yield N_{H} heads results, due to part (b), and we know the number of possible flip sequences from part (a), as each flip sequence is equally likely. The probability for observing N_{H} heads is then

$$\frac{N!}{2^N (N - N_{\mathrm{H}})! \times N_{\mathrm{H}}!}$$

(d) Let's work with $\ln P(N_{\mathrm{H}})$ instead.

$$
\begin{aligned}
\ln P(N_{\mathrm{H}}) &= \ln \frac{N!}{2^N (N - N_{\mathrm{H}})! \times N_{\mathrm{H}}!} \\
&= \ln N! - \ln (N - N_{\mathrm{H}})! - \ln N_{\mathrm{H}}! - N \ln 2
\end{aligned}
$$

Applying Stirling's equation yields

$$
\begin{aligned}
\ln P(N_{\mathrm{H}}) &= N \ln N - N - (N - N_{\mathrm{H}}) \ln (N - N_{\mathrm{H}}) + (N - N_{\mathrm{H}}) - N_{\mathrm{H}} \ln N_{\mathrm{H}} + N_{\mathrm{H}} \\
&= N(\ln N - \ln (N - N_{\mathrm{H}})) - N_H(\ln N_{\mathrm{H}} - \ln (N - N_{\mathrm{H}})) - N \ln 2 \\
&= -N \ln \left(\frac{N - N_{\mathrm{H}}}{N}\right) + N_{\mathrm{H}} \ln \left(\frac{N - N_{\mathrm{H}}}{N_{\mathrm{H}}}\right) - N \ln 2 \\
&= -N \left(\ln (1 - f) - f \ln \left(\frac{1}{f} - 1\right) + \ln 2\right)
\end{aligned}
$$

Therefore

$$I(f) = \ln (1 - f) - f \ln \left(\frac{1}{f} - 1\right) + \ln 2$$

The plot is shown below.
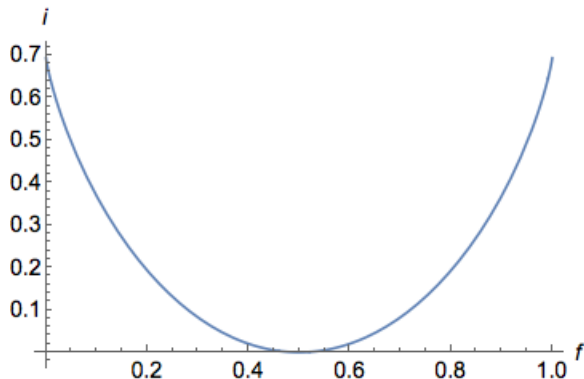
Figure 1: I(f) vs f from 1d

(e) This problem is isomorphic to the above scenario. Note the equivalence between the time average for the coin, and the ensemble average for the spins.

$$P(N_{\text{up}}) = e^{-NI(f)}$$

where $I(f)$ is defined as in part (d).

(f) As in part (d), it is easier to work with the logarithm of the probabilities. We want to calculate

$$
\begin{aligned}
\ln \Gamma & \equiv & \ln \frac{P(f = 0.5 + \delta)}{P(f = 0.5)} \\
& = & -N(I(f = 0.5 + \delta) - I(f = 0.5)) \\
& = & -N\left(I(f = 0.5) + \frac{dI(f)}{df}|_{f=0.5}\delta + \frac{d^2I(f)}{df^2}|_{f=0.5}\delta^2 - I(f = 0.5) \right)
\end{aligned}
$$

where we have taken the suggested Taylor expansion. Now, we just need to calculate the derivatives. The first derivative is 0, as shown in part (c), so we just need to find the second derivative.

$$\frac{d^2I(f)}{df^2}|_{f=0.5} = \frac{1}{f - f^2}|_{f=0.5} = 4$$

Substituting this in we get

$$\ln \Gamma = -4N\delta^2$$

Exponentiating the log gives

$$\Gamma = e^{-4*10^{12}} \sim 10^{-10^{12}}$$

(g) This indicates that measurements on macroscopic systems should be quite reproducible. The fluctuations are very small.

2. (a) For an isolated system, there are Ω microstates ν, each of which are equally probable. Therefore $P_\nu = 1/\Omega$. S_B is simple to compute:

$$S_{\text{B}} = k_{\text{B}} \ln \Omega$$

Calculating the Gibbs entropy gives:

$$
\begin{aligned}
S_{\text{G}} & = & -k_{\text{B}} \sum_\nu P(\nu) \ln P(\nu) \\
& = & -k_{\text{B}} \sum_{\nu=1}^{\Omega} \frac{1}{\Omega} \ln \frac{1}{\Omega} \\
& = & -k_{\text{B}} \frac{\Omega}{\Omega} \ln \frac{1}{\Omega} \\
& = & k_{\text{B}} \ln \Omega \\
\therefore S_{\text{G}} & = & S_{\text{B}}
\end{aligned}
$$

(b) Given N indistinguishable particles and M cells for a lattice gas, we can quickly derive the number of possible configurations. Since the particles are indistinguishable, we do not care about order so we want to use a combination.

$$\Omega = \binom{M}{N} = \frac{M!}{(M - N)!N!}$$

(c) As in Question 1, it is easier to work with $\ln \Omega$. Using Stirling's approximation, we get

$$
\begin{aligned}
\ln \Omega &= M \ln M - (M - N) \ln (M - N) - N \ln N \\
&= M \ln \left(\frac{M}{M - N} \right) + N \ln \left(\frac{M}{N} - 1 \right) \\
\frac{S_B}{M} &= k_B \left(-\ln (1 - f) + f \ln \left(\frac{1}{f} - 1 \right) \right)
\end{aligned}
$$

(d) From the definition of the Gibbs entropy, we get:

$$
s_{\text{cell}} = -k_B \left(p_1 \ln p_1 + p_0 \ln p_0 \right)
$$

(e) The average of a discrete distribution is

$$
\langle A \rangle = \sum_\nu = p_\nu A_\nu
$$

Substituting in, we get

$$
\langle n \rangle = p_1 (1) + p_0 (0) = p_1
$$

(f) As $p_1 = \langle n \rangle = f$, and $p_0 + p_1 = 1$,

$$
\begin{aligned}
s_{\text{cell}}^{\text{G}} &= -k_B (f \ln f + (1 - f) \ln (1 - f)) \\
&= -k_B (f \ln f - \ln (1 - f) + \ln (1 - f)) \\
&= k_B \left(f \ln \left(\frac{1}{f} - 1 \right) - \ln (1 - f) \right) \\
&= s_{\text{cell}}^{\text{B}} \\
\therefore S_G &= S_B
\end{aligned}
$$

3. (a) We know that

$$
\langle N \rangle = \left\langle \sum_{i=1}^{M} n_i \right\rangle = \sum_{i=1}^{M} \langle n_i \rangle
$$

Since each cell is the same on average, $\langle N \rangle = M \langle n_1 \rangle$, where $\langle n_1 \rangle$ is the average number of particles in cell 1.

Additionally, we have $\rho = \dfrac{\langle N \rangle}{V} = \dfrac{M \langle n_1 \rangle}{V}$. Thus, $\langle n_1 \rangle = \dfrac{V \rho}{M} = v\rho$. This leads us to

$$
\langle N \rangle = M v \rho
$$

(b) $\delta N = N - \langle N \rangle = \sum_i n_i - \langle N \rangle \implies (\delta N)^2 = (\sum_i n_i - \langle N \rangle)(\sum_j n_j - \langle N \rangle)$. There are two indices for suming so that we don't confuse them, however they go over the same

range of cells ($i = 1$ to M). Now let's simplify this expression.

$$
\begin{aligned}
(\delta N)^2 &= \left(\sum_i n_i - \langle N \rangle\right)\left(\sum_j n_j - \langle N \rangle\right) \\
&= \sum_i n_i \sum_j n_j - 2\langle N \rangle \sum_i n_i + \langle N \rangle^2 \\
&= \sum_i \sum_j n_i n_j - 2\langle N \rangle \sum_i n_i + \langle N \rangle^2 \\
\langle (\delta N)^2 \rangle &= \langle \sum_i \sum_j n_i n_j \rangle - 2\langle N \rangle^2 + \langle N \rangle^2 \text{since } \langle N \rangle = \langle \sum_i n_i \rangle \\
&= \langle \sum_i \sum_j n_i n_j \rangle - \langle N \rangle^2
\end{aligned}
$$

We know that different cells are uncorrelated because the gas is dilute. So, $\langle \sum_i \sum_j n_i n_j \rangle = \langle \sum_i n_i^2 \rangle = \langle \sum_i n_i \rangle$ as $n_i^2 = n_i$ for $n_i = 0$ or 1.

$$
\begin{aligned}
\langle (\delta N)^2 \rangle &= \langle N \rangle - \langle N \rangle^2 \\
&= \langle N \rangle (1 - (\sum_i \langle n_i \rangle)^2) = Mv\rho(1 - v\rho) \\
\implies \sigma &= \sqrt{Mv\rho(1 - v\rho)}
\end{aligned}
$$

When $\rho v \ll 1$, $\sigma \sim \sqrt{Mv\rho} = \sqrt{N}$. Thus σ goes as $\sqrt{N}$ for a dilute gas.

When the volume is macroscopic and N is in the range of Avogadro's number (10^{24}), the ratio $\frac{\sigma}{N} \sim \frac{1}{\sqrt{N}} = 10^{-12}$. This shows that fluctuations are negligible at the macroscopic scale, and hence, we only observe average properties.

(c) $\langle n_i \rangle = p_1(1) + p_0(0) = p_1$ where p_j is the probability that j particles occupy a cell i. Thus,

$$
p_1 = \rho v
$$

(d) In a specific configuration N out of M cells are occupied with a probability p_1 and the rest are empty with probability $p_0 = 1 - p_1$. Hence, the probability of observing a specific configuration is

$$
P(n_1, n_2, ..., n_M) = p_1^N (1 - p_1)^{M-N}
$$

(e) If there $\Omega(N)$ possible configurations, then the total probability of observing N particles in volume V is

$$
P(N) = \Omega(N)P(n_1, n_2, ..., n_M) = \Omega(N)p_1^N (1 - p_1)^{M-N}
$$

(f) We previously found that $\Omega(N) = \binom{M}{N} = \frac{M!}{N!(M-N)!}$. Taking the logarithm of this expression and using Stirling's approximation, we can find that

$$
\begin{aligned}
\ln \Omega(N) &= M \ln M - M - N \ln N + N - (M - N) \ln M - N + M - N \\
&= M \ln M - N \ln N - (M - N) \ln M - N \\
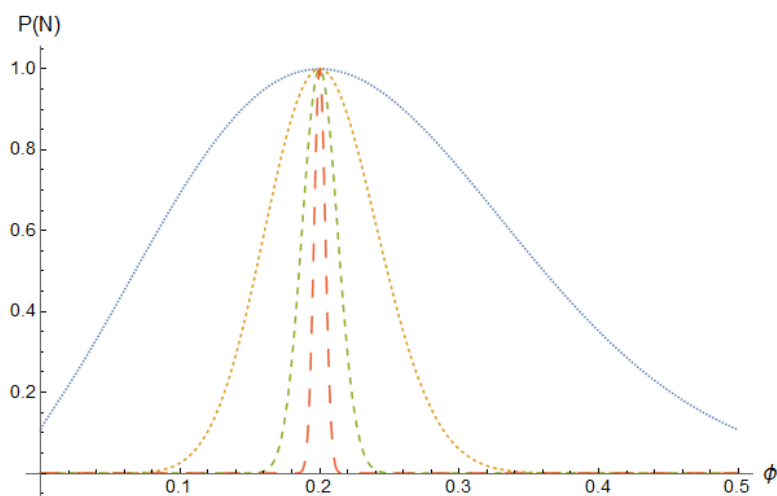&= -M(\phi \ln N + (1 - \phi) \ln M - N - \phi \ln M - (1 - \phi) \ln M)
\end{aligned}
$$

Combining logs, we get

$$\ln \Omega(N) = -M(\phi \ln \phi + (1 - \phi) \ln (1 - \phi))$$

(g) From part (g), we can plug in for finding $\ln P(N)/M$:

$$\frac{\ln P(N)}{M} = \frac{\ln \Omega(N)}{M} + \frac{N}{M} \ln p_1 + \frac{M - N}{N} \ln (1 - p_1)$$

$$\frac{\ln P(N)}{M} = \phi \ln p_1 + (1 - \phi) \ln 1 - p_1 - \phi \ln \phi - (1 - \phi) \ln 1 - \phi$$

Thus, we can find $P(N) = \exp\left(-M(\phi \ln p_1 + (1 - \phi) \ln 1 - p_1 - \phi \ln \phi - (1 - \phi) \ln 1 - \phi)\right)$ and plot this as a funciton of ϕ for $p_1 = 0.2$ and different values of M.

Figure 2: $P(N)$ versus ϕ for different values of M. $M = 10$ (blue), $M = 100$ (orange), $M = 1000$ (green), $M = 10000$ (red).

As you can see in this plot, as M increases the probability distribution becomes sharper. This is the same thing we saw in part (b) - as we increase the system size, the distribution becomes narrow and fluctuations are small as compared to the mean. For a mole of particles, you would expect almost a delta-function type behavior for $P(N)$ peaking at p_1!