



Why

How should we modify probabilities, given that we know some aspect of the outcome (i.e., that some event has occurred).

Definition

Suppose Ω is a set of outcomes and $\mathbf{P} : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$ is a finite probability measure. Suppose $A, B \subset \Omega$ with $\mathbf{P}(B) \neq 0$. The *conditional probability* of A *given* B is fraction of the probability of $A \cap B$ over the probability of B .

Notation

In a slightly slippery but universally standard notation, we denote the conditional probability of A given B by $\mathbf{P}(A \mid B)$. In other words, we define

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

for all $A, B \subset \Omega$, whenever $\mathbf{P}(B) \neq 0$.

For example, we can express the law of total probability (see Event Probabilities) as

$$\mathbf{P}(B) = \sum_{i=1}^n \mathbf{P}(A_i) \mathbf{P}(B \mid A_i),$$

where A_1, \dots, A_n partition Ω and $B \subset \Omega$ with $\mathbf{P}(B) > 0$.

Conditional probability measure

Suppose $B \subset \Omega$ and $\mathbf{P}(B) > 0$. Then (i) $\mathbf{P}(A \mid B) \geq 0$ for all $A \subset \Omega$, since $\mathbf{P}(A \cap B) \geq 0$. Moreover, (ii)

$$\mathbf{P}(\Omega \mid B) = \mathbf{P}(\Omega \cap B) / \mathbf{P}(B) = \mathbf{P}(B) / \mathbf{P}(B) = 1$$

Similarly, $\mathbf{P}(\emptyset \mid B) = \mathbf{P}(\emptyset \cap B) / \mathbf{P}(B) = 0 / \mathbf{P}(B) = 0$.

Finally, (iii) if $A \cap C = \emptyset$, then

$$\begin{aligned}
\mathbf{P}(A \cap C \mid B) &= \mathbf{P}((A \cap C) \cap B) / \mathbf{P}(B) \\
&= \mathbf{P}((A \cap B) \cap (C \cap B)) / \mathbf{P}(B) \\
&\stackrel{(a)}{=} (\mathbf{P}(A \cap B) + \mathbf{P}(C \cap B)) / \mathbf{P}(B) \\
&= \mathbf{P}(A \cap B) / \mathbf{P}(B) + \mathbf{P}(C \cap B) / \mathbf{P}(B) \\
&= \mathbf{P}(A \mid B) + \mathbf{P}(C \mid B).
\end{aligned}$$

where (a) follows since $A \cap B$ and $C \cap B$ are disjoint.

Together, (i)-(iii) mean that $\mathbf{P}(\cdot \mid B)$ is itself a probability measure on Ω . We therefore refer to $\mathbf{P}(\cdot \mid B)$ as a *conditional probability measure*.

Induced conditional distribution

Therefore, we expect there to also correspond a new distribution on the set of outcomes. For \mathbf{P}_p , define $q : \Omega \rightarrow \mathbf{R}$ by

$$q(\omega) = \begin{cases} \frac{p(\omega)}{\mathbf{P}(B)} & \text{if } \omega \in B \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\mathbf{P}_q(A) = \mathbf{P}_p(A \mid B)$. We call q the *conditional distribution* induced by *conditioning on* the event B .

Finite intersections

As a simple repeated application of our definition, suppose $A_1, \dots, A_n \subset \Omega$. Then

$$\mathbf{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbf{P}(A_1) \mathbf{P}(A_2 \mid A_1) \dots \mathbf{P}(A_n \mid A_1 \cap \dots \cap A_{n-1})$$

Many authors call this the *chain rule*.

