



Why

We want to sum infinitely many real numbers.

Definition

Let $(a_k)_{k \in \mathbf{N}}$ be a sequence in \mathbf{R} . Define $(s_n)_{n \in \mathbf{N}}$ by

$$s_n = \sum_{k=1}^n a_k.$$

We call s_n the *n*th partial sum of (x_k) . In other words, the first partial sum s_1 is a_1 , the second partial sum s_2 is $a_1 + a_2$, the third partial sum s_3 is $a_1 + a_2 + a_3$ and so on.

We call (s_n) the *sequence of partial sums* or *series* of (a_k) . If the *series* converges, then we say that (a_k) is *summable*. Clearly not every series is summable: consider, for example, $a_k = 1$ for all k . It has the divergent series $(1, 2, 3, 4, 5, \dots)$.

Notation

If the sequence is summable, then there exists $s \in \mathbf{R}$ (the limit), which we denote

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k. \quad (1)$$

We read these relations aloud as “ s is the limit as n goes to infinity of s_n ” and “ s is the limit as n goes to infinity of the sum of a_k from k equals 1 to n .” We often avoid referencing

s_n by abbreviating Equation (1) by

$$\sum_{k=1}^{\infty} a_k = s.$$

We read this notation aloud as “the sum from 1 to infinity of a k is s.” The notation is subtle, and requires justification by the algebra of series.¹

Existence and uniqueness

Since there exist sequences which do not converge, there exist sequences which are not summable. Consider the sequence which alternates between $+1$ and -1 , and starts with $+1$. Its series alternates between $+1$ and 0 , and so does not converge. Since limits are unique, so is $\sum_{k=1}^{\infty} a_k$.

¹Future editions will include such justification.

