



## Sigma Algebra Event Independence

### 1 Why

TODO

### 2 Definition

The sigma algebra *generated by an event* is the sigma algebra consisting of the empty set, the event, the complement (in the base set) of the event, and the base set.

A family of events are *independent* if the sigma algebras generated by the events are independent.

#### 2.1 Notation

Let  $(X, \mathcal{A}, \mu)$  be a probability space. Let  $A \in \mathcal{A}$  be an event. The sigma algebra generated by  $A$  is  $\{\emptyset, A, X - A, X\}$ . We denote it by  $\sigma(A)$ .

Let  $B \in \mathcal{A}$ . If  $A$  is independent of  $B$  we write  $A \perp B$ .

### 3 Equivalent Condition

**Proposition 1.** *Two events are independent if and only if the measure of their intersection is the product of their measures.*

*Proof.* Let  $(X, \mathcal{A}, \mu)$  be a probability space. Let  $A, B \in \mathcal{A}$ .

( $\Rightarrow$ ) If  $A \perp B$ , then by definition  $A \in \sigma(A)$  and  $B \in \sigma(B)$  and so:

$$\mu(A \cap B) = \mu(A)\mu(B).$$

( $\Leftarrow$ ) Conversely, let  $a \in \sigma(A)$  and  $b \in \sigma(B)$ . If  $a = \emptyset$  or  $b = \emptyset$  then  $a \cap b = \emptyset$ . So

$$\mu(a \cap b) = \mu(\emptyset) = \mu(a)\mu(b),$$

since one of the two measures on the right hand side is zero. On the other hand, if  $a = X$ , then  $a \cap b = b$  and so

$$\mu(a \cap b) = \mu(b) = \mu(a)\mu(b),$$

since  $\mu(a) = \mu(X) = 1$ . Likewise if  $b = X$ .

So it remains to verify  $\mu(a \cap b) = \mu(a)\mu(b)$  for the cases  $a \in \{A, X - A\}$  and  $b \in \{B, X - B\}$ . If  $a = A$ , and  $b = B$ , then the identity follows by hypothesis. Next, observe that  $A \cap (X - B) = A - (A \cap B)$  and  $(A \cap B) \subset A$  so  $\mu(X) < \infty$  allows us to deduce:

$$\begin{aligned} \mu(A \cap (X - B)) &= \mu(A - (A \cap B)) \\ &= \mu(A) - \mu(A \cap B) \\ &= \mu(A)(1 - \mu(B)) \\ &= \mu(A)\mu(X - B). \end{aligned}$$

Similar for  $X - A$  and  $B$ . Finally, recall that  $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ . So then,

$$\begin{aligned} \mu((X - A) \cap (X - B)) &= 1 - \mu(A \cup B) \\ &= 1 - \mu(A) - \mu(B) + \mu(A \cap B) \\ &= 1 - \mu(A) - \mu(B) + \mu(A)\mu(B) \\ &= (1 - \mu(A))(1 - \mu(B)) \\ &= \mu(X - A)\mu(X - B). \end{aligned}$$

□