



Why

A simple way to obtain a sigma algebra, is to start with some sets, and then to add all the sets needed to make the starting set closed under the various operations.

Definition

The *generated sigma algebra* for a set of subsets is the smallest sigma algebra containing the set of subsets. We must prove the existence and uniqueness of this sigma algebra.

Proposition 1. *The intersection of a non-empty set of sigma algebras over the same set is a sigma algebra.*

Proof. Given a family of sigma algebras $\{(A, \mathcal{A}_\alpha)\}_{\alpha \in I}$ over some set, define $\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$.

1. For all $\alpha \in I$, $A \in \mathcal{A}_\alpha$, thus $A \in \mathcal{A}$; condition (a).
2. For all $B \in \mathcal{A}$, for all $\alpha \in I$, $B \in \mathcal{A}_\alpha$. Thus, for all $\alpha \in I$, $C_A(B) \in \mathcal{A}_\alpha$. And so $C_A(B) \in \mathcal{A}$; condition (b).
3. For all sequences $\{B_n\} \subset \mathcal{A}$, $\{B_n\} \subset \mathcal{A}_\alpha$ for all α . Thus $\bigcup_n B_n \in \mathcal{A}_\alpha$ for all α and so $\bigcup_n B_n \in \mathcal{A}$; condition (c).

□

On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

Proposition 2. *If A is a set and $\mathcal{A} \subset 2^A$, then there is a unique a smallest sigma algebra containing \mathcal{A} .*

Proof. We know of one sigma algebra containing \mathcal{A} : the power set of A . Thus, the set of sigma algebras containing \mathcal{A} is not empty. Proposition 1 implies the intersection of all such sigma algebras (containing \mathcal{A}) is a sigma algebra. The intersection contains \mathcal{A} , and is contained in all other sigma algebras with this property, so is a smallest sigma algebra containing \mathcal{A} . If \mathcal{B}, \mathcal{C} were two smallest sigma algebras, then $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C} \subset \mathcal{B}$, but then $\mathcal{B} = \mathcal{C}$; thus the smallest sigma algebra is unique. \square

Notation

Let A be a set and $\mathcal{A} \subset \mathcal{P}(A)$. We denote the sigma algebra generated by \mathcal{A} by $\sigma(\mathcal{A})$.

