Why

We want to modify ordinary row reduction to handle the case in which a pivot is zero by selecting another suitable pivot.

Example

Let $A \in \mathbf{R}^{5\times5}$. If $A_{11} \neq 0$, we may subtract multiples of row 1 from row $2, \ldots, 5$ to eliminate variable x_1 from those equations. If A reduces to $C \in \mathbf{R}^{5\times5}$ and $C_{22} \neq 0$, then step 2 moves from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & C_{22} & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \times & \times & \times & \times & \times \\ & C_{22} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \end{bmatrix}.$$

What if $C_{22} = 0$? In this case suppose we pick a different row. For example, if $C_{42} \neq 0$ we can move from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & C_{42} & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \times & \times & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & C_{42} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \end{bmatrix}.$$

Alternatively, we could introduce zeros in column 3 rather than column 2. For example, if we pick the pivot C_{43} we move from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & C_{43} & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \mathbf{0} & \times & \times \\ & \times & \mathbf{0} & \times & \times \\ & \times & C_{43} & \times & \times \\ & \times & \mathbf{0} & \times & \times \end{bmatrix}.$$

We can choose any nonzero entry in $C_{k:m,k:m}$ as the pivot.

Suppose we pick pivot $C_{st} \neq 0$ for $k \leq s, t \leq m$. Define $\tilde{C}$ by swapping row s of C with row k of C and column t of C

with column k of C. Then $\tilde{C}_{kk} = C_{st} \neq 0$ and there exists an ordinary row reduction for $\tilde{C}$. We call this reduction of $(\tilde{C}, \tilde{d})$ a *pivoted row reduction* of C or the *st-reduction* of C.

If all remaining pivots are zero, then there is no viable pivot. In this case, at least one variable is free and we do not have a unique solution. For convenience, in this case, we still call the system an *st*-reduction of itself.

Definition

At step k of ordinary elimination, multiples of row k are subtracted from rows $k+1, \ldots, m$ to introduce zeros in entry k of the rows. If we denote the matrix at the beginning of that step by X, then row k of X, column k of X and especially the pivot X_{kk} play a role. Ordinarily, we subtract from every entry in the submatrix $X_{k+1:m,k:m}$ the product of a number in row k and a number in column k, divided by the pivot X_{kk}. Generally, however, we can choose as pivot any nonzero entry of $X_{k:m,k:m}$.

An m-variable system (A, b) is *pivot reducible* (or *reducible*) if there exists a sequence of systems $S_1, \ldots, S_{m-1}$ so that S_1 is a reduction of (A, b) and S_i is a reduction of S_{i-1} for $i = 1, \ldots, m-1$. We call S_{m-1} the *final reduction* (or *reduction*) of (A, b). An immediate consequence of our definition is

Proposition 1. *All systems are reducible.*

Pivoted Row Reductions

Ordinary Row Reductions

Linear System Row Reductions

Linear Equation Solutions

Linear Equations

Linear Combinations

Vectors

Arrays

N-Dimensional Space

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Line — Real Plane — Real Space

Real Summation — Integral Line — Real Order — Sequences

Real Numbers — Fields — Geometry — Direct Products

Rational Numbers — Groups — Set Numbers

Integer Arithmetic — Integer Additive Inverses — Finite Sets

Integer Sums — Natural Multiplicative Identity — Natural Additive Identity — Inverse Elements — Equivalent Sets

Natural Summation — Arithmetic — Identity Elements — Family Operations — Element Functions — Comparisons

Integer Products — Natural Powers — Natural Equations — Algebra — Orders — Function Inverses — Families — Family Unions and Intersections

Natural Products — Integer Numbers — Equations — Operations — Natural Order — Function Composites — Function Images

Natural Sums — Functions — Converse Relations — Equivalence Relations

Recursion Theorem — Relations

Peano Axioms

Natural Induction — Set Products

Natural Numbers — Set Powers — Generalized Set Dualities

Successor Sets — Unordered Triples — Set Dualities — Intersection of Empty Set — Partitions — Set Unions and Intersections

Ordered Pairs — Set Complements — Pair Unions — Set Intersections

Set Differences — Set Unions — Pair Intersections

Unordered Pairs — Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities — Sets

Names

Letters — Objects