Why

Result

We bound below the measure that a non-negative measurable real-valued function exceeds some value by its integral.

Prop. 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $g : X \to [0, \infty]$ be measurable and square-integrable. Then for all t such that $\int t d\mu \in [0, \int g d\mu)$,*

$$\mu(\{x \in X \mid g(x) > t\}) \geq \frac{(\int (g-t) d\mu)^2}{\int g^2 d\mu}.$$

Proof. Let t such that $\int t d\mu \in [0, \int g)$. We have selected t so that $\int (g-t) d\mu \geq 0$. Define $h = (g-t)^+$ and $A = \{x \in X \mid h(x) > 0\}$. Then

$$\int (g-t) d\mu \leq \int h d\mu = \int h \chi_A d\mu \leq \sqrt{\int h^2 d\mu \int \chi_A^2 \, d\mu}$$

□

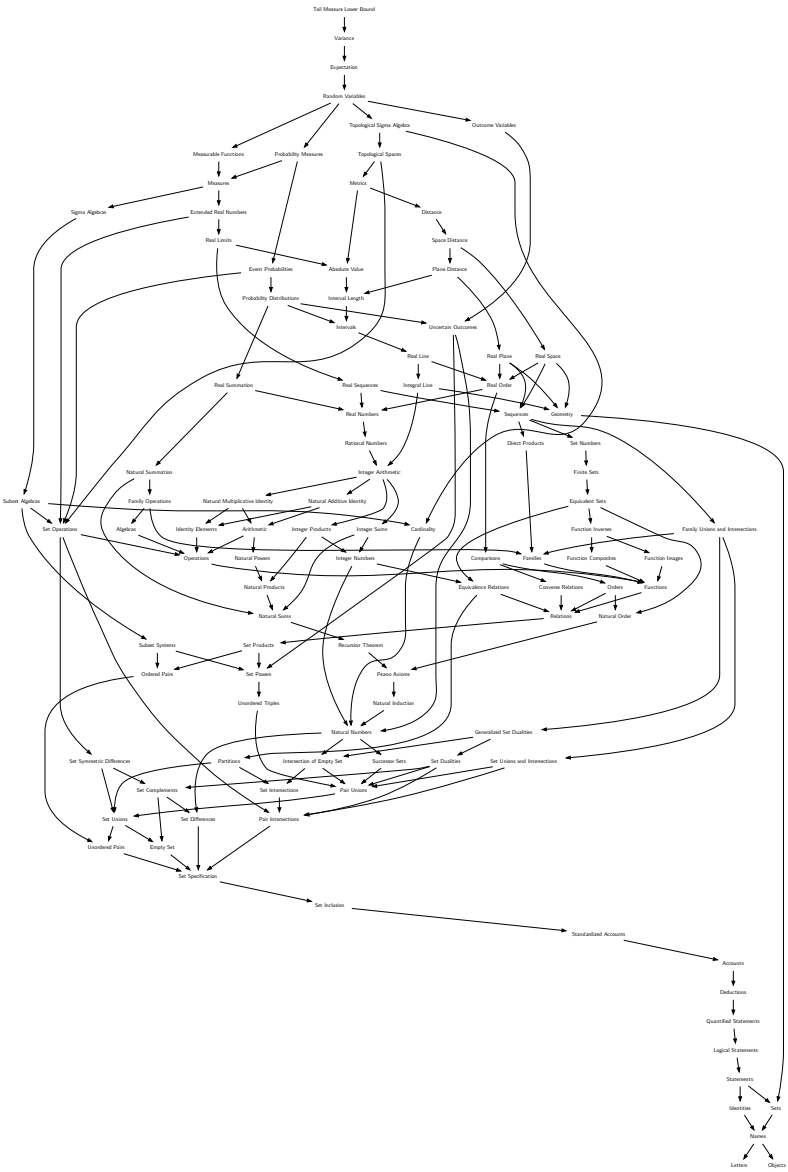
Now $g^2 > h^2$, so $\int g^2 d\mu \geq \int h^2 d\mu$. Also $\chi_A^2 = \chi_A$ so $\int \chi_A^2 = \mu(A)$. $h(x) > 0$ if and only if $g(x) \geq t$ for all x. So $A = \{x \in X \mid g(x) \geq t\}$. Combining we have:

$$\int (g-t) d\mu \leq \sqrt{\left(\int g^2 d\mu\right) \mu(A)}.$$

Prop. 2. *Let X be a random variable with $\mathbf{E}(X^2) \leq \infty$. Then for all $t \in [0, \mathbf{E}(X))$, we have*

$$P(X > t) \geq \frac{(\mathbf{E}(X) - t)^2}{\mathbf{E}X^2}.$$

The above is also called the *Paley-Zygmund Inequality*.


Tail Measure Lower Bound
Variance
Expectation
Random Variables
Outcome Variables
Topological Sigma Algebra
Measurable Functions
Probability Measures
Topological Spaces
Measures
Metrics
Sigma Algebras
Extended Real Numbers
Distance
Real Limits
Space Distance
Event Probabilities
Absolute Value
Plane Distance
Interval Length
Probability Distributions
Intervals
Uncertain Outcomes
Real Line
Real Plane
Real Space
Real Summation
Real Sequences
Integral Line
Real Order
Real Numbers
Sequences
Geometry
Rational Numbers
Direct Products
Set Numbers
Natural Summation
Integer Arithmetic
Finite Sets
Subset Algebras
Family Operations
Natural Multiplication Identity
Natural Addition Identity
Equivalent Sets
Set Operations
Algebras
Identity Elements
Arithmetic
Integer Products
Integer Sums
Cardinality
Function Inverses
Family Unions and Intersections
Operations
Natural Powers
Integer Numbers
Comparisons
Families
Function Composites
Function Images
Natural Products
Equivalence Relations
Converse Relations
Orders
Functions
Natural Sums
Relations
Natural Order
Subset Systems
Set Products
Recursion Theorem
Ordered Pairs
Set Powers
Peano Axioms
Unordered Triples
Natural Induction
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Set Specificates
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects