



# Tree Approximators of a Normal

## 1 Why

What is the optimal tree approximator of a multivariate normal density?

## 2 Result

**Proposition 1.** *Let  $g : \mathbf{R}^n \rightarrow \mathbf{R}$  be a normal density with mean  $\mu \in \mathbf{R}^d$  and covariance  $\Sigma \in \mathbf{S}_{++}^d$ . The normal density  $f_T^* : \mathbf{R}^d \rightarrow \mathbf{R}$  with mean  $\mu$  and precision matrix  $P$  defined by*

- $P_{11} = \Sigma_{11}^{-1} + \sum_{\mathbf{pa}_j=1} \Sigma_{j1}^2 \Sigma_{11}^{-2} \Sigma_{j|1}^{-1}$
- for  $i = 2, \dots, d$ ,  $P_{ii} = \Sigma_{i|\mathbf{pa}_i}^{-1} + \sum_{\mathbf{pa}_j=i} \Sigma_{ji}^2 \Sigma_{ii}^{-2} \Sigma_{j|i}^{-1}$
- $i, j = 1, \dots, d$  and  $i \neq \mathbf{pa}_j$ ,  $P_{ij} = P_{ji} = -\Sigma_{ji} \Sigma_{jj}^{-1} \Sigma_{j|i}^{-1}$

where  $\mathbf{pa}_i$  is the parent of  $i$  in an optimal approximator tree  $T$  ( $i = 2, \dots, n$ ) is an optimal tree approximator of  $g$ .

*Proof.* Using Proposition 1 of Best Tree Density Approximators, express an optimal tree approximator of  $g$  by

$$(1/c) \exp \left( -\frac{1}{2} \left( \Sigma_{11}^{-1} \bar{x}_1^2 + \sum_{i \neq 1} \left( \bar{x}_i - \Sigma_{i, \mathbf{pa}_i} \Sigma_{\mathbf{pa}_i, \mathbf{pa}_i}^{-1} \bar{x}_{\mathbf{pa}_i} \right)^2 \Sigma_{i|\mathbf{pa}_i}^{-1} \right) \right)$$

where  $\bar{x}_i = x_i - \mu_i$  and  $c = \sqrt{(2\pi)^d \Sigma_{11} \prod_{i \neq 1} \Sigma_{i|\mathbf{pa}_i}}$ .

Second, express the quadratic in the exponential as

$$\Sigma_{11}^{-1} \bar{x}_1^2 + \sum_{i \neq 1} \left[ \Sigma_{i|\mathbf{pa}_i}^{-1} \bar{x}_i^2 - 2 \Sigma_{i, \mathbf{pa}_i} \Sigma_{\mathbf{pa}_i, \mathbf{pa}_i}^{-1} \Sigma_{i|\mathbf{pa}_i}^{-1} \bar{x}_i \bar{x}_{\mathbf{pa}_i} + \Sigma_{i, \mathbf{pa}_i}^2 \Sigma_{\mathbf{pa}_i, \mathbf{pa}_i}^{-2} \Sigma_{i|\mathbf{pa}_i}^{-1} \bar{x}_{\mathbf{pa}_i}^2 \right]$$

With  $P$  defined as earlier, we can express the above as  $\bar{x}^\top P \bar{x}$ .

Third, note that  $c$  is  $\sqrt{(2\pi)^d \mathbf{det} P^{-1}}$  since  $f_T^*$  is a density and so integrates to one.

□

Notice that  $f_T^*$  is a tree normal density.

### 3 Empirical Normal

In particular, notice that we can approximate the empirical normal density of a dataset with a density that factors according to a tree.