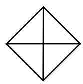


Why

We generalize the algebraic structure of *addition* and *multiplication* over the integers.[1]

Definition

A *ring* (or *ring with identity*) (R, f, g) is a set A and two binary operations on R satisfying the following set of conditions.

(A) (i) f is associative. (ii) f is *commutative*, (iii) A has an identity for f (i.e., is $e \in R$ with $f(r, e) = f(e, r) = r$ for all $r \in R$ (iv) R has inverse elements for f (i.e., for any $r \in R$, there is $\tilde{r}$ satisfying $f(r, \tilde{r}) = f(\tilde{r}, r) = e$)

(B) (i) g is associative; (ii) R has an identity element for g (i.e., for any $r \in R$, there is $\tilde{e} \in A$ satisfying $g(r, \tilde{e}) = g(\tilde{e}, r) = r$)

(C) (i) g left distributes:

$$g(f(x, y), \alpha) = f(g(\alpha, x), g(\alpha, y)) \quad \text{for all } x, y, \alpha \in R$$

(ii) g right distributes:

$$g(\alpha, f(x, y)) = f(g(\alpha, x), g(\alpha, y)) \quad \text{for all } x, y, \alpha \in R$$

Conditions (A) concern f, conditions (B) concern g, and conditions (C) relate the two.

Clearly, $\mathbf{Z}$ with addition and multiplication is a ring. The element referred to in (A.2) is $0 \in \mathbf{Z}$, so we refer to this element in any ring as the *additive identity*. That referred to (A.3) is $1 \in \mathbf{Z}$, so we refer to this element in any ring as the *multiplicative identity*. We refer to the elements mentioned in (A.4) as *additive inverses*. We call to f *ring addition* and g *ring multiplication*.

[1] Future editions will likely modify this sheet, and give a genetic treatment involving the solution of polynomial equations by Galois.

A ring which for which multiplication is commutative is called a *commutative ring*. Note that a ring is *always* commutative with respect to addition, here the term commutative refers to multiplication. A ring for which there are inverse elements, excepting 0, is called a *division ring*).

Notation

The notation commonly adopted in discussing rings relies on analogy with the set of integers $\mathbf{Z}$. We denote the ring addition by $+$ and ring multiplication by $\cdot$. Moreover, we denote the ring's additive identity by 0 and the ring's multiplicative identity by 1. Finally, we denote the additive inverse of $a \in A$ by $-a$.

Rewriting the conditions (A), (B), (C) in this notation gives familiar-looking relations, from when the objects involved were integers. (A) (1) $a + (b + c) = (a + b) + c$; (2) $a + b = b + a$; (3) $a + 0 = 0 + a = a$; (4) $a + (-a) = 0$. (B) (1) $a(bc) = (ab)c$; (2) $1a = a1 = a$. (C) (1) $(a + b)c = ac + bc$; (2) $c(a + b) = ca + cb$.

Immediate consequences

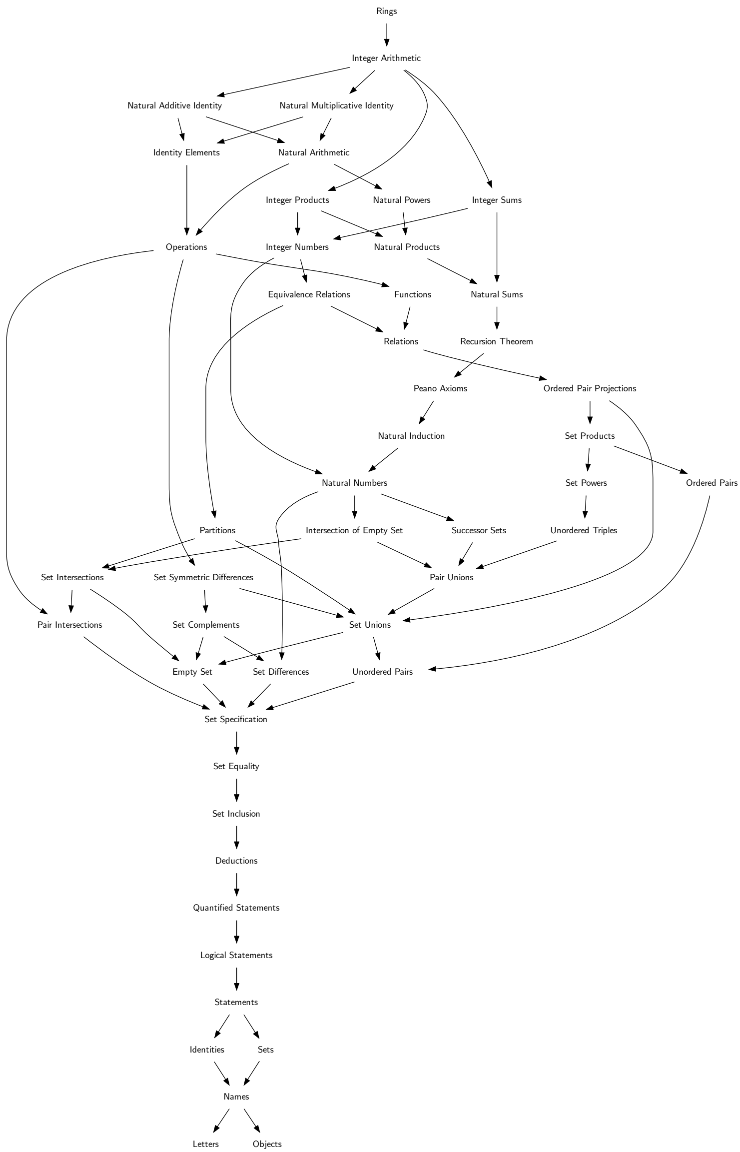
We need not require that $0x = 0$, because we can deduce it:

$$0x + x = (0 + 1)x = 1x = x.$$

Similarly, $(-a)b = -(ab)$ since

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

Other familiar relations among the integers, e.g. $(-a)(-b) = ab$, may be deduced.


Rings
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Identity Elements
Natural Arithmetic
Integer Products
Natural Powers
Integer Sums
Operations
Integer Numbers
Natural Products
Equivalence Relations
Functions
Natural Sums
Relations
Recursion Theorem
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects