Why

A simple way to obtain a sigma algebra, is to ask it to obtain some sets, and then to ask it to contain all the sets it needs to fulfill the properties.

Definition

The *generated sigma algebra* for a set of subsets is the smallest sigma algbera containing the set of subsets. We must prove the existence and uniqueness of this sigma algebra.

PROPOSITION **1.** *The intersection of a non-empty set of sigma algebras on the same base set is a sigma algebra.*

Proof. Let $\{(A, \mathcal{A}_\alpha\}_{\alpha \in I}$ a familiy of sigma algebras on the same base set. Define $\mathcal{A}$ as $\cap_{\alpha \in I} \mathcal{A}_\alpha$.

1. For all $\alpha \in I$, $A \in \mathcal{A}_\alpha$, thus $A \in \mathcal{A}$; condition (a).

2. For all $B \in \mathcal{A}$, for all $\alpha \in I$, $B \in \mathcal{A}_\alpha$. Thus, for all $\alpha \in I$, $C_A(B) \in \mathcal{A}_\alpha$. And so $C_A(B) \in \mathcal{A}$; condition (b).

3. For all sequences $\{B_n\} \subset \mathcal{A}$, $\{B_n\} \subset \mathcal{A}_\alpha$ for all α. Thus $\cup_n B_n \in \mathcal{A}_\alpha$ for all α and so $\cup_n B_n \in \mathcal{A}$; condition (c).

$\square$

On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

PROPOSITION **2.** *If A is a set and $\mathcal{A} \subset 2^A$, then there is a unique a smallest sigma algebra containing $\mathcal{A}$.*

Proof. We know of one sigma algebra containing $\mathcal{A}$: the power set of A. Thus, the set of sigma algebras containing $\mathcal{A}$ is not empty. Proposition 1 implies the intersection of all such sigma algebras (containing $\mathcal{A}$) is a sigma algebra. The intersection contains $\mathcal{A}$, and is contained in all other sigma algebras with this property, so is a smallest sigma algebra containing $\mathcal{A}$. If $\mathcal{B}, \mathcal{C}$ were two smallest sigma algebras, then $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C} \subset \mathcal{B}$, but then $\mathcal{B} = \mathcal{C}$; thus the smallest sigma algebra is unique. $\square$

Notation

Let A be a set and $\mathcal{A} \subset 2^A$. We denote the sigma algebra generated by $\mathcal{A}$ by $\sigma(\mathcal{A})$.

Generated Sigma Algebra

Sigma Algebras

Subset Algebras

Cardinality

Set Operations

Operations

Functions

Relations

Subset Systems

Cartesian Products

Natural Numbers

Set Powers

Ordered Pairs

Successor Sets

Intersection of Empty Set

Unordered Triples

Pair Unions

Set Symmetric Differences

Set Intersections

Set Unions

Set Complements

Pair Intersections

Set Differences

Unordered Pairs

Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects