



Why

We can extend the notion of independence beyond pairs of uncertain events, to sets of events.

Definition

Suppose P is an event probability function on a finite sample space Ω . The events A_1, \dots, A_n are *independent* (or *mutually independent*), if for all k between 1 and n , and distinct indices i_1, \dots, i_k between 1 and n ,

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \dots P(A_{i_k}).$$

Similar to the case of pairs of events, one can show that this condition is equivalent to the statement that for any *distinct* indices $i_1, \dots, i_k, j_1, \dots, j_l$,

$$P(A_{j_1} \cap \dots \cap A_{j_l} \mid A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{j_1} \cap \dots \cap A_{j_l})$$

Examples

n tosses of a coin. As usual, model n tosses of a coin with $\{0, 1\}^n$ and put a distribution $p : \Omega \rightarrow [0, 1]$ so that

$$p(\omega) = 1/2^n \quad \text{for all } \omega \in \Omega$$

Now, for $i = 1, \dots, n$, define the event A_i by

$$A_i = \{\omega \in \Omega \mid \omega(i) = 1\}$$

We claim that the set $\{A_1, \dots, A_n\}$ is mutually independent. To see this, notice that for any distinct indices i_1, \dots, i_k ,

$$|A_{i_1} \cap \dots \cap A_{i_k}| = 2^{n-k}$$

This holds because, once k of the coin flips, there are 2^{n-k} ways for the remaining coins to land (using the fundamental principle of counting). Consequently,

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = \frac{2^{n-k}}{2^n} = 2^{-k}$$

We can use this result with one set $P(A_i) = 1/2$, and so we obtain

$$P(A_{i_1} \cap \cdots A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}),$$

as desired.

Basic implications

It can be shown¹ that if A_1, \dots, A_n are independent events, and B_1, \dots, B_n are events such that B_i is either A_i or A_i^c , then B_1, \dots, B_n are mutually independent.

¹Future editions will.

