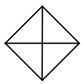


Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Suppose p is a distribution on a *finite* set of outcomes Ω. Given an event $E \subset \Omega$, we define the *event probability* of E *under* p as the sum of the probabilities of the outcomes in E.

Notation

It is common to define a function $P : \mathcal{P}(\Omega) \to \mathbf{R}$ by

$$P(A) = \sum_{a \in A} p(a) \quad \text{for all } A \subset \Omega$$

We call this function P *the event probability function* (or *the probability measure*) associated with p. Since it depends on the sample space Ω and the distribution p, we occasionally denote this dependence by $P_{\Omega,p}$ or P_p.

Example: a single six-sided die

We model rolling a single die with the set of outcomes $\Omega = \{1, \dots, 6\}$ as usual. We $p : \Omega \to \mathbf{R}$ by $p(\omega) = 1/6$ for $\omega = 1, \dots, 6$. In other words, p is the constant function at value $1/6$ on Ω. Now, we model the event that the number of pips showing is an even number by the set E defined by $E = \{2, 4, 6\}$. Given all this modeling, the probability of the event E is

$$\textstyle\sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of event probabilities

As a result of the conditions on p, $\mathbf{P}$ satisfies

1. $\mathbf{P}(A) \geq 0$ for all $A \subset \Omega$;

2. $\mathbf{P}(\Omega) = 1$ (and $\mathbf{P}(\varnothing) = 0$);

3. $\mathbf{P}(A) + \mathbf{P}(B)$ for all $A, B \subset \Omega$ and $A \cap B = \varnothing$. This statement follows from the more general identity

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

for $A, B \subset \Omega$, by using $\mathbf{P}(\varnothing) = 0$ of (2) above.

These three conditions are sometimes called the *axioms of probability for finite sets.* Do all such $\mathbf{P}$ satisfying (1)-(3) have a corresponding underlying probability distribution?

In other words, suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satisfies (1)-(3). Define $q : \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an *event probability function* (or a *(finite) probability measure*).

Other basic consequences

Probability by cases

Let $\mathbf{P}$ be a probability event function. Suppose $A_1, \ldots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$\mathbf{P}(B) = \textstyle\sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

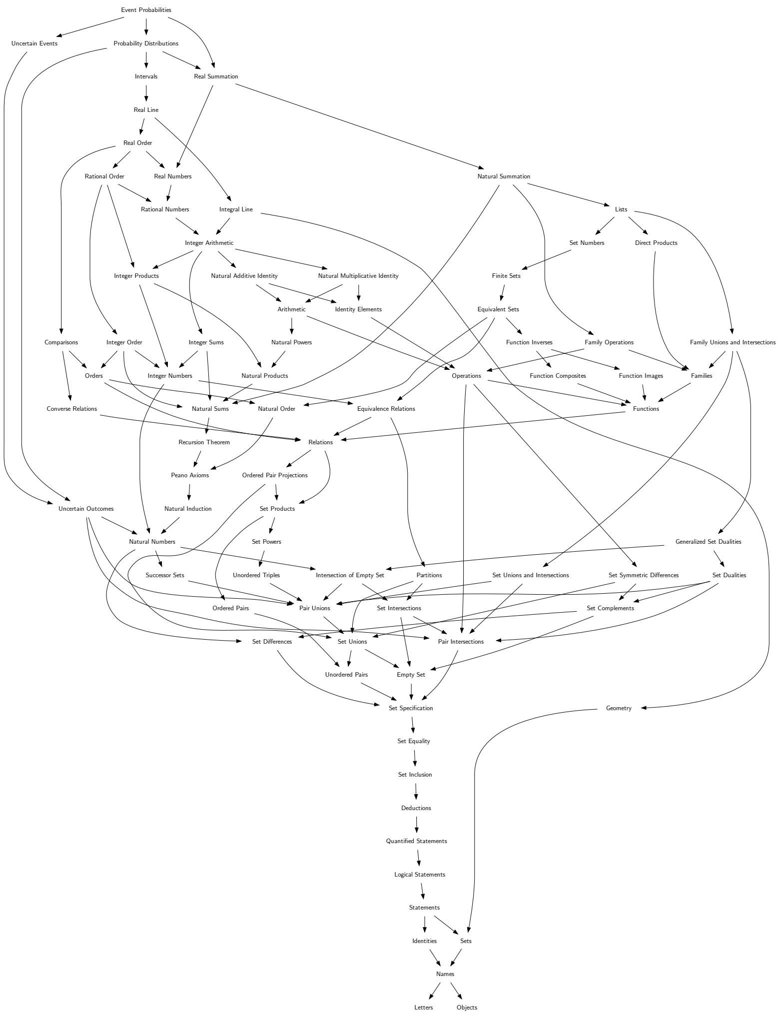
Some authors call this the *law of total probability.*

Monotonicity

If $A \subseteq B$, then $\mathbf{P}(A) \leq P(B)$. This is easy to see by splitting B into $A \cap B$ and $B - A$, and applying (1) and (3).

Subadditivity

For $A, B \subset \Omega$, $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$. This is easy to see from the more general identity in (3) above. This is sometimes referred to as a *union bound*, in reference to *bounding* the quantity $\mathbf{P}(A \cup B)$.



Event Probabilities
Uncertain Events
Probability Distributions
Intervals
Real Summation
Real Line
Real Order
Rational Order
Real Numbers
Natural Summation
Rational Numbers
Integral Line
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Comparisons
Integer Order
Integer Sums
Natural Powers
Function Inverses
Family Operations
Family Unions and Intersections
Orders
Integer Numbers
Natural Products
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Generalized Set Dualities
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Symmetric Differences
Set Dualities
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Set Differences
Set Unions
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