



## Why

We want to talk about making decisions over time.<sup>1</sup>

## Definition

Let  $\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_T$  and  $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_{T-1}$  be sets. For  $t = 0, \dots, T - 1$ , let  $f_t : \mathcal{X}_t \times \mathcal{U}_t \rightarrow \mathcal{X}_{t+1}$ . We call the sequence

$$((\mathcal{X}_t)_{t=0}^T, (\mathcal{U}_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1})$$

a *deterministic discrete-time dynamical system*. We call the index  $t$  the *epoch*, the *stage* or the *period*.

Let  $x_0 \in \mathcal{X}_0$ . Let  $u_0 \in \mathcal{U}_0, \dots, u_{T-1} \in \mathcal{U}_{T-1}$ . Define a state sequence  $x_1 \in \mathcal{X}_1, \dots, x_T \in \mathcal{X}_T$  by

$$x_{t+1} = f_t(x_t, u_t).$$

In this case we call  $x_0$  the *initial state*. We call the  $x_t$  the *states*. We call the  $u_t$  a sequence of *inputs* (or *actions*, *decisions*, *choices*, or *controls*). We call  $f_t$  the *transition function* or *dynamics function*.

We call  $T$  the *horizon*. In the case that we have an infinite sequence of state sets, input sets, and dynamics, then we refer to a *infinite-horizon* dynamical system. To use language in contrast with this case, we refer to the dynamical system when  $T$  is finite as a *finite-horizon* dynamical system.

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<sup>1</sup>Future editions will expand, and may develop dynamic systems via the genetic approach by appealing to their classical use in Newtonian physics for modeling celestial mechanics.

## State

The current action  $u_t$  affects future states  $x_s$  for  $s > t$ , but not the current or past states. The current state  $x_t$  depends on the initial state  $x_0$  and the sequence of past actions  $u_0, \dots, u_{t-1}$ . So the state is a “link” between the past and the future. Given  $x_t$  and  $u_t, \dots, u_{s-1}$ , for  $s > t$ , we can compute  $x_s$ . In other words, the prior actions  $u_0, \dots, u_{t-1}$  are not relevant.

This nonrelevancy of prior actions and prior states simplifies the sequential decision problem (see **Sequential Decisions**).

## Variations

The dynamical system is called *time-invariant* if  $\mathcal{X}_t$ ,  $\mathcal{U}_t$  and  $f_t$  do not depend on  $t$ . A simple variation is that  $\mathcal{U}_t$  depends on  $x_t$ .<sup>2</sup>

## Examples

### Finite dynamical system

A dynamical system is finite if the state and action sets are finite. For example,  $\mathcal{X} = \{1, \dots, n\}$  and  $\mathcal{U} = \{1, \dots, m\}$ . Then  $f_t : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{U}$  is called a *transition map*.

Or else, let  $(V, E)$  be a directed graph, then  $\mathcal{X} = V$ ,  $\mathcal{U}_{x_t} = \{(u, v) \in E \mid u = x_t\}$  and  $f_t(x_t, u_t) = v$  when  $u_t = (x_t, v)$  is a dynamical system. Roughly this system models “moving” on the graph.

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<sup>2</sup>Future editions will say more here.

### Discrete-time linear dynamical system

Let  $\mathcal{X} = \mathbf{R}^n$  and  $\mathcal{U} = \mathbf{R}^m$ . Define  $f_t : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$  by

$$x_{t+1} = f_t(x_t, u_t) = A_t x_t + B_t u_t + c_t$$

for  $y = 1, \dots, T - 1$ .<sup>3</sup>

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<sup>3</sup>This very special form of dynamics arises in many applications. Future editions will say more.

