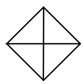


Why

How many ways can we select k chairs from a set of size n? Here, and throughout this sheet, $1 \leq k \leq n$.

Seating k guests in n chairs

First, we ask: how many ways are there to seat k guests in n chairs? We start by numbering the guests. Then, the first guest can be seated in any of the n chairs—or in n ways. Having seated this first guest, the next guest can be seated in any of the $n-1$ remaining chairs. Having seated these first two guests, the third can be seated in any of the remaining $n-2$ chairs. And so on. We conclude that the number of ways of seating k guests in n chairs is

$$n(n-1)(n-2)\cdots(n-k+1)$$

Factorial notation. We can express this number can be expressed as

$$\frac{n!}{(n-k)!}$$

For example, with $n = 7$ and $k = 3$,

$$7 \cdot 6 \cdot 5 \cdot 4 = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$$

If we agree that the number of ways to seat no people in n chairs is 1, then the ration of fraction on the right also makes sense for $k = 0$. In other words $n!/n! = 1$.

Selecting k chairs out of n

Now we ask our original question: How many ways are there to select k chairs out of n? Observe that our previous discussion involved seating k guests in n chairs. We could break this down in a different way—first we select the k chairs, and *then* we select how to place people in the chairs. Denote the number of ways to do the first task by x. We have seen that there are $k!$ ways to do the second task. So by the fundmental principle the number of ways to do this task is $x \cdot k!$, which must be the same as our expression above

$$x \cdot k! = \frac{n!}{(n-k)!}$$

We conclude that

$$x = \frac{n!}{k!(n-k)!} = \frac{(n)(n-1)\cdots(n-k+1)}{k!}$$

Notation

We denote this number by

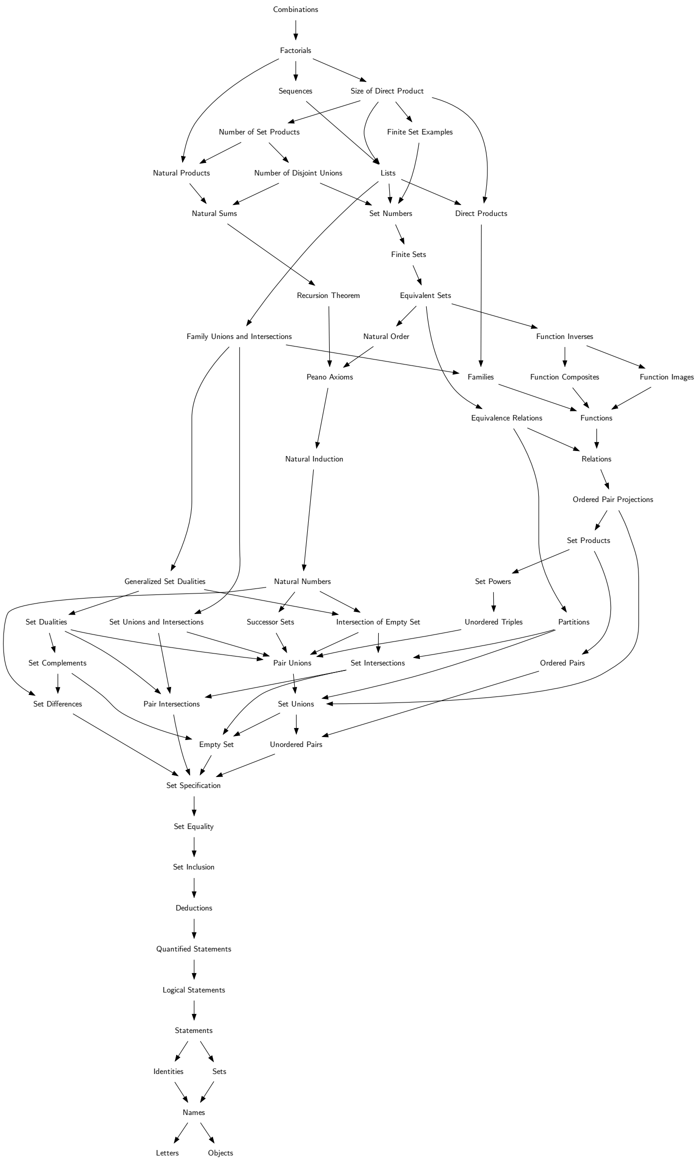
$$\binom{n}{k}$$

read aloud as "n *choose* k". Another notation is $C(n,k)$, where C is meant to stand for *combination* or *choice*.

Number of subsets

The number of subsets of size k from a finite set of n elements is $\binom{n}{k}$. Since there is one subset of size 0 from a set of size n (the empty set) and one subset of size n (the whole set), it is a pleasant validation of our convention that $0! = 1$ that

$$\binom{n}{0} = \binom{n}{n} = 1.$$


Combinations
Factorials
Sequences
Size of Direct Product
Number of Set Products
Finite Set Examples
Natural Products
Number of Disjoint Unions
Lists
Natural Sums
Set Numbers
Direct Products
Finite Sets
Recursion Theorem
Equivalent Sets
Family Unions and Intersections
Natural Order
Function Inverses
Peano Axioms
Families
Function Composites
Function Images
Equivalence Relations
Functions
Natural Induction
Relations
Ordered Pair Projections
Set Products
Generalized Set Dualities
Natural Numbers
Set Powers
Set Dualities
Set Unions and Intersections
Successor Sets
Intersection of Empty Set
Unordered Triples
Partitions
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Set Differences
Pair Intersections
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects