



Why

Can we characterize positive (semi-)definite matrices in terms of their eigenvalues?

Main Result

Using eigenvalue decompositions, we can answer in the affirmative.

Proposition 1. *Suppose $A \in \mathbf{S}^d$ has smallest eigenvalue $\lambda_{\min}(A)$. Then*

$$\begin{aligned} A \in \mathbf{S}_+^d &\iff \lambda_{\min}(A) \geq 0 \\ &\iff \operatorname{tr} AB \geq 0 \text{ for all } B \in \mathbf{S}_+^d. \end{aligned}$$

and

$$\begin{aligned} A \in \mathbf{S}_{++}^d &\iff \lambda_{\min}(A) > 0 \\ &\iff \operatorname{tr} AB > 0 \text{ for all nonzero } B \in \mathbf{S}_{++}^d. \end{aligned}$$

