Why

TODO

Definition

Defining Result

PROPOSITION **1.** *Let $(X, \mathcal{A})$ be a measurable space. Let μ and ν be finite measures with $\nu \ll \mu$.*

There exists $g : X \to [0, \infty)$ such that

$$\nu(A) = \int_A g \, d\mu$$

for all $A \in \mathcal{A}$. The function g is μ-almost everywhere unique.

Proof. Define

$$\mathcal{F} = \left\{ f : X \to [0, \infty) \;\middle|\; f \text{ measurable and } \int_A f d\mu \leq \nu(A) \right\}.$$

The function $f \equiv 0$ is in $\mathcal{F}$, since it is a measurable simple function whose integral over every measureable set is zero.

If f_1 and f_2 are in $\mathcal{F}$, then $f_1 \vee f_2$ is in $\mathcal{F}$. To check, let $A \in \mathcal{A}$, and define the sets $A_1 = \{x \in A \mid f_1(x) > f_2(x)\}$ and $A_2 = \{x \in A \mid f_1(x) \leq f_2(x)\}$. A_1 and A_2 partition A, so

$$\int_A f_1 \vee f_2 = \int_{A_1} f_1 \vee f_2 + \int_{A_2} f_1 \vee f_2$$

$$= \int_{A_1} f_1 + \int_{A_2} f_2$$

$$\leq \nu(A_1) + \nu(A_2)$$

Since A_1 and A_2 partition A,

$$\nu(A_1) + \nu(A_2) = \nu(A_1 \cup A_2) = \nu(A).$$

Select a sequence of functions $(f_n)_n$ in $\mathcal{F}$ so that

$$\lim_n \int f_n = \sup\left\{ \int f \;\middle|\; f \in \mathcal{F} \right\}.$$

Toward ensuring the sequence is increasing, define $g_1 = f_1$, $g_2 = g_1 \vee f_2$, and $g_n = g_{n-1} \vee f_n$ for $n \geq 3$. Using the observation in the previous paragraph, $g_n \in \mathcal{F}$ for each n.

Let g be the pointwise limit of the $(g_n)_n$. The monotone convergence of integrals shows

$$\int_A g = \lim_n \int_A g_n.$$

for each $A \in \mathcal{A}$. Since $\int_A g_n \leq \nu(A)$, so too is the limit and thus so too is $\int_A g$. Thus, $g \in \mathcal{F}$. By construction, for $A = X$, $\int g = \sup\{\int f \mid f \in \mathcal{F}\}$. We have constructed an element of $\mathcal{F}$ attaining the supremum.

We know that the integral of g on A with respect to μ is bounded above by $\nu(A)$. We want the gap to be zero. Regardless of the gap, the function $\nu_0 : \mathcal{A} \to [0, \infty)$ defined by

$$\nu_0(A) = \nu(A) - \int (g, A, \mu).$$

for each $A \in \mathcal{A}$ is a positive measure If ν_0 is identically zero, then there is no gap.

Suppose there is a gap: then there exists a measurable set with strictly positive measure under ν_0. Since the base set contains this set, and measures are monotone, the base set must have stricty positive measure. Since μ is finite, there exists a natural number n so that

$$\nu_0(X) > \frac{1}{n}\mu(X).$$

Define a new measure $\nu_1 = \nu_0 - \frac{1}{n}\mu$. Denote a signed-set decomposition of ν_1 by (P, N). Then $\nu_1(A \cap P) \geq 0$, or equivalently,

$$\nu_0(A \cap P) - \frac{1}{n}\mu(A \cap P) \geq 0,$$

for all A so. and so

$$\nu(A) = \nu_0(A) + \int (g, A, \mu)$$
$$\geq \nu_0(A \cap P) + \int (g, A, \mu).$$

$\square$

Some call the above the *Radon-Nikodym theorem* .

Notation

Measure Derivatives

Measures

Extended Real Numbers

Real Limits

Intervals

Real Sequences

Subset Algebras

Real Numbers

Sequences

Set Operations

Cardinality

Rational Numbers

Direct Products

Algebras

Integer Numbers

Injective Functions

Natural Order

Natural Families

Subset Systems

Zero

Operations

Families

Function Composites

Set Symmetric Differences

Natural Numbers

Power Set

Equation Solutions

Functions

Set Unions

Set Intersections

Set Complements

Ordered Pairs

Relations

Equations

Empty Set

Unordered Pairs

Set Specification

Set Inclusion

Sentences

Set Equality

Sets

Identity

Objects