



Norms

1 Why

We want to measure the size of an element in a vector space.

2 Definition

A **norm** is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisfies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms.

2.1 Notation

Let (X, F) be a vector space where F is the field of real numbers or the field of complex numbers. Let R denote the set of real numbers. Let $f : X \rightarrow R$. The functional f is a norm if

1. $f(v) \geq 0$ for all $x \in V$
2. $f(v) = 0$ if and only if $x = 0 \in X$.

3. $f(\alpha x) = |\alpha|f(x)$ for all $\alpha \in F, x \in X$
4. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.

In this case, for $x \in X$, we denote $f(x)$ by $|x|$, read aloud “norm x”. The notation follows the notation of absolute value as a norm. When we wish to distinguish the norm from the absolute value function, we may write $\|x\|$. In some cases, we go further, and for a norm indexed by some parameter α or set A we write $\|x\|_\alpha$ or $\|x\|_A$.