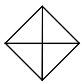


Why

When is a linear transformation between V and W one-to-one? In other words, when does

$$Tx = Ty \Rightarrow x = y$$

Rearranging, and using additivity, we ask when

$$T(x - y) = 0 \Rightarrow x - y = 0$$

Clearly we are interested in vectors z for which $Tz = 0$.

Definition

Suppose $T \in \mathcal{L}(V, W)$. The *null space* (or *kernel*) of T is the set of vectors in V which are mapped to 0 under T. In symbols, the null space of T is the set

$$T = \{v \in V \mid Tv = 0\}$$

The word "null" means "zero" in German.

A subspace

Why use the term *space*? Well, T is a *subspace* of V.

Proposition 1. *Suppose $T \in \mathcal{L}(V, W)$. Then (T) is a subspace of V.*

Proof. We verify that (T) contains 0 and is closed under vector addition and scalar multiplication. First, $0 \in T$ since $T0 = 0$ by homogeneity. Second, by additivity, if $x, y \in T$, then

$$T(x + y) = Tx + Ty = 0 + 0 = 0$$

Third, if $u \in T$ and $\alpha \in \mathbf{F}$, then

$$T(\lambda u) = \lambda(Tu) = \lambda 0 = 0$$

$\square$

Characterization of injectivity

Proposition 2. *Suppose $T \in \mathcal{L}(V, W)$. Then*

$$T = \{0\} \longleftrightarrow T \textit{ is one-to-one}$$

If $T = \{0\}$ we say that T has *zero nullspace* or *trivial nullspace.*

Examples

Zero map. Suppose T is the zero map from V to W. In other words,

$$Tv = 0 \quad \text{for all } v \in V$$

Then $T = V$. I.e., the null space is the whole space.

Simple function on $\mathbf{C}^3$. Define $\phi \in \mathcal{L}(\mathbf{C}^3, \mathbf{C})$ by

$$\phi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$$

Then ϕ is

$$\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1 + 2z_2 + 3z_3 = 0\}$$

This is the *solution set* of a linear equation.

Polynomial differentiation. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ is the linear map defined by

$$Dp = p' \quad \text{for all } p \in \mathcal{P}(\mathbf{R})$$

In other words, Dp is the derivative of the polynomial p. Then (D) is the set of constant functions.

Multiplication by x^2. Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbf{R} \text{ and } p \in \mathcal{P}(\mathbf{R})$$

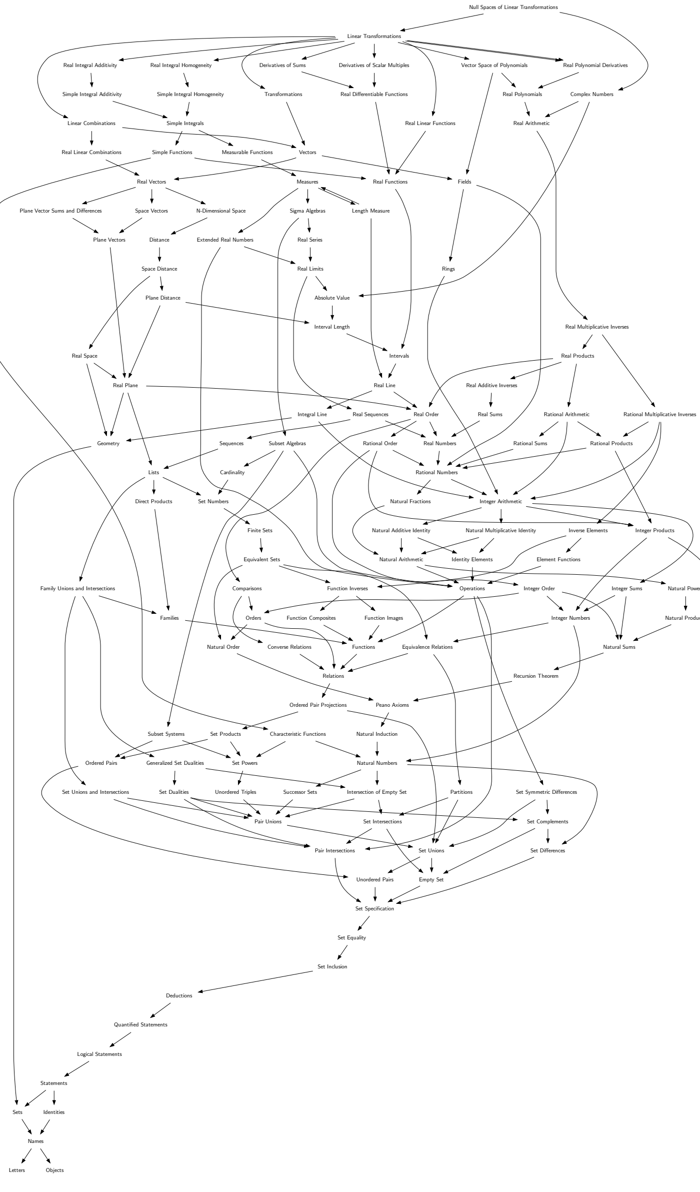
Then $(T) = \{0\}$, since no other polynomial satisfies $x^2 p(x) = 0$ for all $x \in \mathbf{R}$.

Backward shift. Define $T \in \mathcal{L}(\mathbf{F}^{\mathbf{N}}, \mathbf{F}^{\mathbf{N}})$ by

$$T(x_1, x_2, x_3, \ldots,) = (x_2, x_3, \ldots,)$$

so that T is the backward shift Then $T(x_1, x_2, x_3, \dots) = 0$ if and only $x_2 = x_3 = \cdots = 0$. So

$$T = \{(\alpha, 0, 0, \dots,) \mid \alpha \in \mathsf{F}\}$$


Null Spaces of Linear Transformations
Linear Transformations
Real Integral Additivity
Real Integral Homogeneity
Derivatives of Sums
Derivatives of Scalar Multiples
Vector Space of Polynomials
Real Polynomial Derivatives
Simple Integral Additivity
Simple Integral Homogeneity
Transformations
Real Differentiable Functions
Real Polynomials
Complex Numbers
Linear Combinations
Simple Integrals
Real Linear Functions
Real Arithmetic
Real Linear Combinations
Simple Functions
Measurable Functions
Vectors
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Measures
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Fields
Plane Vector Sums and Differences
Space Vectors
N-Dimensional Space
Sigma Algebras
Length Measure
Plane Vectors
Distance
Extended Real Numbers
Real Series
Space Distance
Real Limits
Rings
Plane Distance
Absolute Value
Interval Length
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Integral Line
Real Sequences
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Geometry
Sequences
Subset Algebras
Rational Order
Real Numbers
Rational Sums
Rational Products
Lists
Cardinality
Rational Numbers
Direct Products
Set Numbers
Natural Fractions
Integer Arithmetic
Finite Sets
Natural Additive Identity
Natural Multiplicative Identity
Inverse Elements
Integer Products
Equivalent Sets
Natural Arithmetic
Identity Elements
Element Functions
Family Unions and Intersections
Comparisons
Function Inverses
Operations
Integer Order
Integer Sums
Natural Powers
Families
Orders
Function Composites
Function Images
Integer Numbers
Natural Products
Natural Order
Converse Relations
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Subset Systems
Set Products
Characteristic Functions
Natural Induction
Ordered Pairs
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects