Bourbaki

Contents

1 Objects

1.1 Why

We want to talk about things.

1.2 Definition

We use the word **object** with its usual sense in the English language. An object may be tangible, in that we can hold or touch it, or an object may be abstract, in that we can do neither.

1.2.1 Notation

To aid in discussing and denoting objects, let us tend to give them short names. A single Latin letter regularly suffices: for example, a, b or c. To aid our memory, we tend to choose the letter mnemonically.

We denote that the object a and the object b are the same object by $a = b$, read aloud as "a is b." We denote that the object a and b are different by $a \neq b$, read aloud as "a is not b."

2 Sets

2.1 Why

We want to talk about none, one, or several objects considered as a whole, for which we will use the word *set*.

2.2 Definition

A **set** is an abstract object which we think of as several objects considered at once. We say that the set **contains** the objects so considered. We call these the **elements** of the set.

We call the set which contains no objects the **empty set**. We call a set which contains only a single object a **singleton**. A singleton is not the same as the object it contains. Besides these two cases, we think of sets as containing two or more objects.

The objects a set contains may be other sets. This may be subtle at first glance, but becomes familiar with experience.

2.2.1 Notation

Let us tend to denote sets by upper case Latin letters: for example, A, B, and C. To aid our memory, let us tend to use the lower case form of the letter for an element of the set. For example, let A and B be non-empty sets. Let us tend to denote by a an element of A, and likewise, by b an element of B

Let us denote that an object a is an element of a set A by

$a \in A$. We read the notation $a \in A$ aloud as "a in A." The $\in$ is a stylized lower case Greek letter: ϵ. It is read aloud "ehp-sih-lawn" and is a mnemonic for "element of". We write $a \notin A$, read aloud as "a not in A," if a is not an element of A.

If we have named the elements of a set, and can list them, let us do so between braces. For example, let a, b, and c be three distinct objects. Denote by $\{a, b, c\}$ the set containing theses three objects and only these three objects. We can further compress notation, and denote this set of three objects by A: so, $A = \{a, b, c\}$. Then $a \in A$, $b \in A$, and $c \in A$. Moreover, if d is an object and $d \in A$, then $d = a$ or $d = b$ or $d = c$.

We denote the empty set by $\varnothing$. Note that $\varnothing \neq \{\varnothing\}$. The left hand side, $\varnothing$, is the empty set. The right hand side, $\{\varnothing\}$, is the singleton whose element is the empty set. We distinguish the set containing one element from the element itself.

3   Set Examples

3.1   Why

We give some examples of objects and sets.

3.2 Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of

hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, ..., x, y, z. This set has twenty-six elements.

$$\diamondsuit\!\!\!\!\!+$$

4   Set Extension

4.1   Why

When are two sets the same?

4.2   Definition

Two sets are **equal** if and only if they have the same elements.

4.2.1   Notation

Let A and B be two sets. As with any objects, we denote that A and B are equal by $A = B$.

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5   Subsets

5.1   Why

We want to speak of sets which contain all the elements of other sets.

5.2   Two Sets

A **subset** of a set A is any set B for which each element of the set B is an element of the set A. In this case, we say that B is a subset of A. Conversely, we say that A is a **superset** of B.

Every set is a subset of itself. So if the set A is the set B, then A is a subset of B and B is a subset of A. Conversely, if A is a subset of B and B is a subset of A, then A is B. To argue that A is B, we argue that membership in A implies membership in B and second, we argue that membership in B implies membership in A.

The **power set** of a set is the set of all subsets of that set. It includes the set itself and the empty set. We call these two sets **improper subsets** of the set. We call all other sets **proper subsets**.

5.2.1   Notation

Let A and B be sets. We denote that A is a subset of B by $A \subset B$. We read the notation $A \subset B$ aloud as "A subset B".

If $A \subset B$ and $B \subset A$, then $A = B$. The converse also holds.

We denote the power set of A by 2^A, read aloud as "two to the A." $A \in 2^A$ and $\varnothing \in 2^A$. However, $A \subset 2^A$ is false.

5.2.2   Examples

Let a, b, c be distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in 2^A$. As always, $\varnothing \in 2^A$ and $A \in 2^A$ as well. In this case, we can list the elements (which are sets) of the power set:

$$2^A = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$

6   Set-Builder Notation

6.1   Why

We specify a subset of a known set via a property.

6.2   Definition

Let A be a nonempty set. We use the notation

$$\{a \in A \mid \text{———}\}$$

to indicate a subset of A that satisfies some property specified after the |. We read the symbol | aloud as "such that." We read the whole notation aloud as "a in A such that..."

We call the notation **set-builder notation**. Set-builder notation avoids enumerating elements.

6.3 Example

For example, let L be the set of Latin letters and V the set of Latin vowels. A first notation for V is $\{a, e, i, o, u\}$. A second notation for V is $\{l \in L \mid l \text{ is a vowel}\}$. We may prefer the second, in cases when it saves time. This notation is really indispensable for sets which have many members, too many to reasonably write down.

7 Ordered Pairs

7.1 Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

7.2 Definition

Let A and B be non-empty sets. Let $a \in A$ and $b \in B$. The **ordered pair** of a and b is the set $\{\{a\}, \{a, b\}\}$. The **first element** of $\{\{a\}, \{a, b\}\}$ is a and the **second element** is b.

The **cartesian product** of A and B is the set of all ordered pairs. If $A \neq B$, the ordering causes the cartesian product of A and B to differ from the cartesian product of B with A. If $A = B$, however, the symmetry holds.

7.2.1 Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the cartesian product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

8   Relations

8.1   Why

How can we relate the elements of two sets?

8.2 Definition

A **relation** between two non-empty sets A and B is a subset of $A \times B$. A relation on a single set C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The pair (a, b) may or may not be in a relation on A and B. If $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

8.2.1 Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as "a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

9 Functions

9.1 Why

We want a notion for a correspondence between two sets.

9.2 Definition

A **functional** relation on two sets relates each element of the first set with a unique element of the second set. A **function** is a functional relation.

The **domain** of the function is the first set and **codomain** of the function is the second set. The function **maps** elements **from** the domain **to** the codomain. We call the codomain element associated with the domain element the **result** of **applying** the function to the domain element.

9.2.1 Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in A \to B$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

10   Operations

10.1   Why

We want to "combine" elements of a set.

10.2   Definition

Let A be a non-empty set. An **operation** on A is a function from ordered pairs of elements in the set to the same set. We use operations to combine the elements. We operate on pairs.

10.2.1   Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\,g\,b$ instead. We call this **infix notation**.

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

11   Algebras

11.1   Why

We name a set together with an operation.

11.2   Definition

An **algebra** is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The **ground set** of the algebra is the set on which the operation is defined.

11.2.1   Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.