Why

Result

We bound below the measure that a non-negative measurable real-valued function exceeds some value by its integral.

Prop. 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $g : X \to [0, \infty]$ be measurable and square-integrable. Then for all t such that $\int t d\mu \in [0, \int g d\mu)$,*

$$\mu(\{x \in X \mid g(x) > t\}) \geq \frac{(\int (g-t) d\mu)^2}{\int g^2 d\mu}.$$

Proof. Let t such that $\int t d\mu \in [0, \int g)$. We have selected t so that $\int (g-t) d\mu \geq 0$. Define $h = (g-t)^+$ and $A = \{x \in X \mid h(x) > 0\}$. Then

$$\int (g-t) d\mu \leq \int h d\mu = \int h \chi_A d\mu \leq \sqrt{\int h^2 d\mu \int \chi_A^2 \, d\mu}$$

□

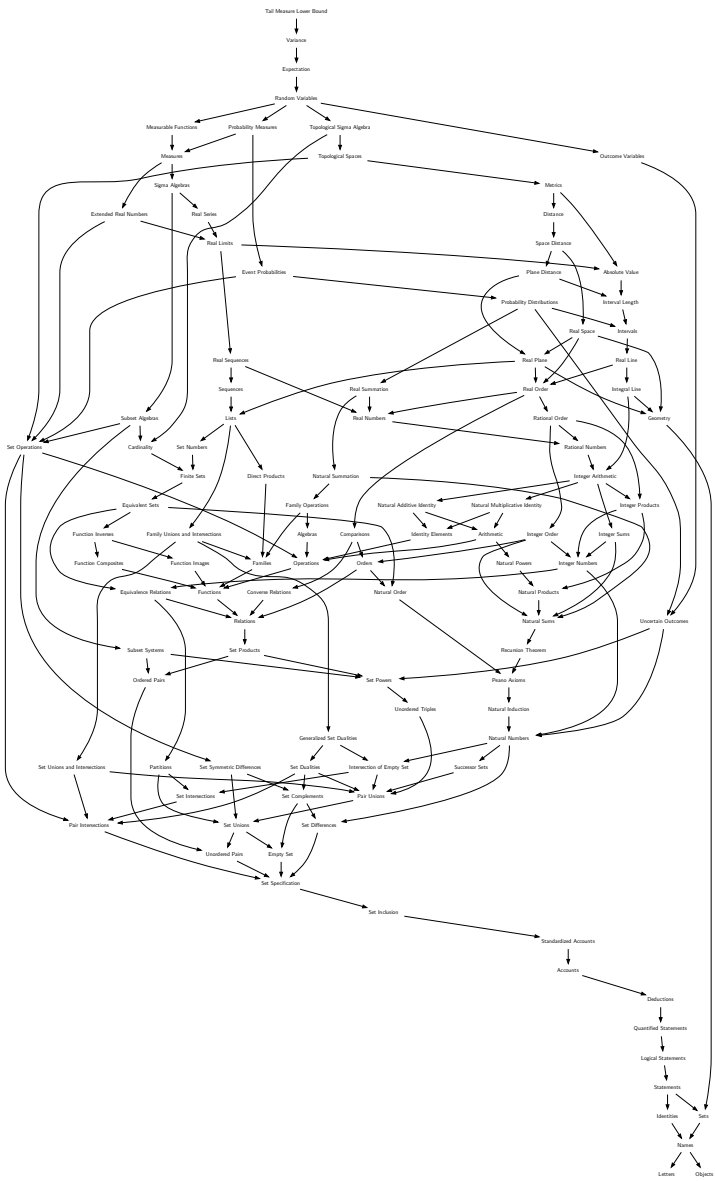
Now $g^2 > h^2$, so $\int g^2 d\mu \geq \int h^2 d\mu$. Also $\chi_A^2 = \chi_A$ so $\int \chi_A^2 = \mu(A)$. $h(x) > 0$ if and only if $g(x) \geq t$ for all x. So $A = \{x \in X \mid g(x) \geq t\}$. Combining we have:

$$\int (g-t) d\mu \leq \sqrt{\left(\int g^2 d\mu\right) \mu(A)}.$$

Prop. 2. *Let X be a random variable with $\mathbf{E}(X^2) \leq \infty$. Then for all $t \in [0, \mathbf{E}(X))$, we have*

$$P(X > t) \geq \frac{(\mathbf{E}(X) - t)^2}{\mathbf{E}X^2}.$$

The above is also called the *Paley-Zygmund Inequality*.


Tail Measure Lower Bound
Variance
Expectation
Random Variables
Measurable Functions
Probability Measures
Topological Sigma Algebra
Outcome Variables
Measures
Topological Spaces
Sigma Algebras
Metrics
Extended Real Numbers
Real Series
Distance
Real Limits
Space Distance
Absolute Value
Plane Distance
Event Probabilities
Interval Length
Probability Distributions
Real Space
Intervals
Real Sequences
Real Plane
Real Line
Sequences
Real Summation
Real Order
Integral Line
Subset Algebras
Lists
Real Numbers
Rational Order
Geometry
Set Operations
Cardinality
Set Numbers
Rational Numbers
Finite Sets
Direct Products
Natural Summation
Integer Arithmetic
Equivalent Sets
Family Operations
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Function Inverses
Family Unions and Intersections
Algebras
Comparisons
Identity Elements
Arithmetic
Integer Order
Integer Sums
Function Composites
Function Images
Families
Operations
Orders
Natural Powers
Integer Numbers
Equivalence Relations
Functions
Converse Relations
Natural Order
Natural Products
Relations
Natural Sums
Uncertain Outcomes
Subset Systems
Set Products
Recursion Theorem
Ordered Pairs
Set Powers
Peano Axioms
Unordered Triples
Natural Induction
Generalized Set Dualities
Natural Numbers
Set Unions and Intersections
Partitions
Set Symmetric Differences
Set Dualities
Intersection of Empty Set
Successor Sets
Set Intersections
Set Complements
Pair Unions
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
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Objects