



**Why**

We want to capture the useful properties of the standard basis vectors.

**Definition**

A set of vectors  $\{v_1, \dots, v_k\} \subset \mathbf{R}^n$  is *independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0.$$

Notice that independence is a property of a set of vectors, not of any vector in particular. Another way of saying this is that no vector can be represented as a linear combination of another.

**Unique representation**

Suppose  $v_1, \dots, v_k$  are independent and we have

$$x = \sum_{i=1}^k \alpha_i v_i \quad \text{and} \quad x = \sum_{i=1}^k \beta_i v_i.$$

Then

$$0 = x - x = \sum_{i=1}^n (\alpha_i - \beta_i) v_k.$$

Using the definition of independence, we conclude  $\alpha_i - \beta_i = 0$  for  $i = 1, \dots, k$ . Consequently,  $\alpha_i = \beta_i$ . In other words, if  $x$  can be represented as a linear combination of the vectors  $v_1, \dots, v_k$ , that representation is *unique*. We have shown that independence implies uniqueness? What of the converse?

We show that lack of independence gives a lack of uniqueness. Suppose there exists  $\alpha_1, \dots, \alpha_k$ , not all zero, so that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0.$$

In particular, suppose  $\alpha_i \neq 0$ . Then we have

$$v_i = (1/\alpha_i) \sum_{j \neq i} \alpha_j v_j.$$

Suppose  $x$  can be written as a linear combination of  $v_1, \dots, v_k$ . In other words, there are  $\beta_1, \dots, \beta_k$  so that

$$x = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$



