



Why

For general measure theory, we need an algebra of sets closed under countable unions; we define such an object.¹

Definition

A *countably summable subset algebra* is a subset algebra for which (1) the base set is distinguished (2) the complement of a distinguished set is distinguished (3) the union of a sequence of distinguished sets is distinguished.

The name is justified, as each countably summable subset algebra is a subset algebra, because the union of A_1, \dots, A_n coincides with the union of $A_1, \dots, A_n, A_n, A_n, A_n \dots$.

We call the set of distinguished sets a *sigma algebra* (or *sigma field*) on the base set. This language is justified (as for a regular subset algebra) by the closure properties of the sigma algebra under the usual set operations. We sometimes write are σ -algebra and σ -field.

A *sub- σ -algebra* (*sub-sigma-algebra*) is a subset of a sigma algebra which is itself a sigma algebra.

Notation

We often denote a sigma algebra by \mathcal{A} or \mathcal{F} ; the former is a mnemonic for “algebra” and the second is a mnemonic for “field”. The calligraphic typeface is meant as a reminder that the object so denoted is a set of sets.

A common pattern is also to use the calligraphic font of whichever letter is being used for the base set. Thus, if A is a set, then we might choose to denote a sigma algebra on A by \mathcal{A} .

¹Future editions will make no reference to measure theory. The entire development will follow the genetic approach, and so roughly follow the historical development for handling integration.

Often, instead of saying “let (A, \mathcal{A}) be a countably summable subset algebra” we say instead “let \mathcal{A} be a sigma algebra on A .” Since the largest element of the sigma algebra is the base set, we can also say (without ambiguity): “let \mathcal{A} be a sigma algebra.” The base set is $\cup \mathcal{A}$.

Examples

Example 1. *For any set A , 2^A is a sigma algebra.*

Example 2. *For any set A , $\{A, \emptyset\}$ is a sigma algebra.*

Example 3. *Let A be an infinite set. Let \mathcal{A} the collection of finite subsets of A . \mathcal{A} is not a sigma algebra.*

Example 4. *Let A be an infinite set. Let \mathcal{A} be the collection subsets of A such that the set or its complement is finite. \mathcal{A} is not a sigma algebra.*

Proposition 1. *The intersection of a family of sigma algebras is a sigma algebra.*

Example 5. *For any infinite set A , let \mathcal{A} be the set*

$$\{B \subset A \mid |B| \leq \aleph_0 \vee |C_A(B)| \leq \aleph_0\}.$$

\mathcal{A} is an algebra; the countable/co-countable algebra.²

²Future editions will clean up and modify these examples.

