Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering $\geq$ on S^d defined by

$$A \geq B \quad \longleftrightarrow \quad A - B \geq 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_+.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \quad \longleftrightarrow \quad A - B > 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_{++}.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$
\begin{aligned}
\alpha A \geq 0 \quad &\text{for all } \delta > 0, A \geq 0, \\
A + B \geq 0 \quad &\text{for all } A, B \geq 0, \\
A \geq B \text{ and } B \geq A \longrightarrow A = B \quad &\text{for all } A, B \in \mathsf{S}^d, \\
\lim_{n \to \infty} A_n = A \longrightarrow A \geq 0 \quad &\text{for all sequences } (A_n)_n \text{ in } \mathsf{S}^d_+.
\end{aligned}
$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that $\geq$ is antisymmetric. Moreover,

$$A \geq A \quad \& \text{ for all } A \in \mathsf{S}^d, \text{ and}$$

$$A \geq B \text{ and } B \geq C \longrightarrow A \geq C \& \text{ for all } A, B, C \in \mathsf{S}^d.$$

In other words, $\geq$ is also reflexive and transitive. In other words, $\geq$ is a partial order (see Partial Orders).[1]

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[1]Future editions will include more formal accounts.

For $d = 1$, $\geq$ reduces to the familiar total order of the real line(see **Real Order**). The converse perspective is to see the positive semidefinite order as an extension of the order on **R** to the space S^d. Of course, the key difference is that two matrices may not be comparable. The order is *partial*.

For example, the matrices $A, B \in \mathsf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.

Positive Semidefinite Matrix Order

Positive Semidefinite Cone

Positive Definite Matrices

Quadratic Forms

Identity Matrices · Matrix Trace · Real Matrix-Matrix Products · Symmetric Real Matrices

Matrix Transpose · Real Matrix-Vector Products

Matrices · Vector as Matrices

Real Matrices · Real Linear Combinations · Real Inner Product

Linear Equations · Vectors · Real Norm · Space Inner Product · Multivariate Polynomials · Real Convex Cones

Arrays · Linear Functions · Real Vectors · Space Norm · Multivariate Functions · Real Convex Sets · Cones

Space Vector · Plane Vector Sums and Differences · Plane Norm · Plane Inner Product · N-Dimensional Space

Plane Vectors · Absolute Value · Distance · Right Triangle Sides Relation

Space Distance · Real Polynomials · Real Squares

Plane Distance · Real Arithmetic

Interval Length · Real Multiplicative Inverses

Intervals · Real Space · Real Products · Square Numbers

Real Plane · Real Line · Real Additive Inverses · Square

Real Summation · Real Order · Real Sums · Integral Line · Rational Arithmetic · Rational Multiplicative Inverses

Fields · Real Numbers · Rational Sums · Rational Order · Rational Products · Geometry

Rings · Rational Numbers

Integer Arithmetic

Lists

Natural Additive Identity · Natural Multiplicative Identity · Set Numbers · Integer Products

Direct Products · Identity Elements · Arithmetic · Finite Sets

Comparisons · Inverse Elements · Natural Summation · Equivalent Sets · Integer Order · Natural Powers

Family Unions and Intersections · Family Operations · Orders · Function Inverses · Element Functions · Natural Equations · Natural Sums · Integer Sums

Families · Function Composites · Function Images · Operations · Natural Order · Equations · Integer Numbers

Convex Relations · Functions

Relations · Equivalence Relations · Recursion Theorem

Set Products · Peano Axioms

Generalized Set Dualities · Set Powers · Natural Induction

Set Unions and Intersections · Set Dualities · Unordered Triples · Successor Sets · Intersection of Empty Set · Partitions · Set Symmetric Differences

Ordered Pairs · Pair Unions · Set Intersections · Set Complements

Pair Intersections · Set Unions · Set Differences

Unordered Pairs · Empty Set

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects