Measures

Why

A measurable function from a first measure space to a second measurable space induces a measure on the latter.

Definition

Consider two measurable spaces and a measurable function between them. The *image measure* of a measure on the first space *under* the measurable function is the measure on the second space which assigns to each measurable set (of that space) the measure of the inverse image of that set.

We say that the function *induces* the image measure on the codomain. Alternatively, we say that we *push forward* the measure to the codomain, and so call the image measure a *push forward measure*.

Notation

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. Let $f : X \to Y$ be a measurable function. Let $\mu : \mathcal{A} \to [0, \infty]$ be a measure. We denote the image measure of μ under f by $\mu \circ f^{-1}$, for the reason that it

$$\mu \circ f^{-1}(B) = \mu(f^{-1}(B))$$

for every $B \in \mathcal{B}$.

Change of Variables

The main property we would like to hold is that integration on the new measure space is the same as integration on the old.

Prop. 1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let R denote the real numbers. Let $f : X \to Y$ be a measurable function and let $\mu : \mathcal{A} \to [0, \infty]$ be a measure.*

Then $g : Y \to R$ is integrable with respect to $\mu \circ f^{-1}$ if and only if $g \circ f$ is integrable with respect to μ. In this case,

$$\int g d(\mu \circ f^{-1}) = \int g \circ f d\mu.^1$$

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[1]Future editions will include the proof.

Image Measures

Measures

Sigma Algebras

Extended Real Numbers

Real Series

Real Limits

Absolute Value

Interval Length

Intervals

Real Line

Real Order

Real Sequences

Real Numbers

Rational Order

Sequences

Integral Line

Rational Numbers

Subset Algebras

Integer Arithmetic

Lists

Natural Additive Identity

Natural Multiplicative Identity

Cardinality

Set Numbers

Identity Elements

Arithmetic

Finite Sets

Integer Products

Operations

Comparisons

Function Inverses

Equivalent Sets

Natural Powers

Direct Products

Function Composites

Function Images

Orders

Natural Products

Integer Order

Integer Sums

Family Unions and Intersections

Converse Relations

Functions

Natural Order

Equivalence Relations

Natural Sums

Integer Numbers

Families

Relations

Recursion Theorem

Peano Axioms

Subset Systems

Set Products

Ordered Pairs

Set Powers

Natural Induction

Generalized Set Dualities

Set Symmetric Differences

Partitions

Unordered Triples

Natural Numbers

Successor Sets

Intersection of Empty Set

Set Dualities

Set Unions and Intersections

Set Complements

Set Intersections

Pair Unions

Set Unions

Set Differences

Pair Intersections

Unordered Pairs

Empty Set

Geometry

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects

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