



Result

This result is called sometimes called the *probability inverse transform*.

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $X : \Omega \rightarrow \mathbf{R}$ be a random variable with cumulative distribution function $F_X : \mathbf{R} \rightarrow [0, 1]$. Suppose $F_X^{-1} : [0, 1] \rightarrow \mathbf{R}$ exists, then $Y = F_X^{-1} \circ X$ is a random variable with cumulative distribution function $F_Y : [0, 1] \rightarrow [0, 1]$ satisfying $F_Y(y) = y$.*

Remark 1. *The conclusion is equivalent to the following: Y has a density and that density is the the standard uniform density (see *Uniform Densities*).*

Proof. Express $F_Y(\gamma) = \mathbf{P}[Y \leq \gamma] = \mathbf{P}(Y^{-1}([0, \gamma]))$ Notice¹

$$\begin{aligned} Y^{-1}([0, \gamma]) &= \{\omega \in \Omega \mid Y(\omega) \leq \gamma\} \\ &= \{\omega \in \Omega \mid F_X(X(\omega)) \leq \gamma\} \\ &= \{\omega \in \Omega \mid X(\omega) \leq F_X^{-1}(\gamma)\} = X^{-1}(\dots). \end{aligned}$$

□

Remark 2. *Using different notation the above can be expressed succinctly as*

$$\begin{aligned} F_Y(\gamma) &= \mathbf{P}[Y \leq \gamma] = \mathbf{P}[F_X \circ X \leq \gamma] \\ &= \mathbf{P}[X \leq F_X^{-1}(\gamma)] = F_X(F_X^{-1}(\gamma)) = \gamma. \end{aligned}$$

Future editions will discuss *inverse transform sampling*.

¹Future editions will complete.

