Why

What is the optimal tree approximator of a multivariate normal density?

Result

Proposition 1. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a normal density with mean $\mu \in \mathbf{R}^d$ and covariance $\Sigma \in \mathbf{S}^d_{++}$. The normal density $f^*_T : \mathbf{R}^d \to \mathbf{R}$ with mean μ and precision matrix P defined by*

- $P_{11} = \Sigma_{11}^{-1} + \sum_{\mathrm{pa}\, j=1} \Sigma_{j1}^2 \Sigma_{11}^{-2} \Sigma_{j|1}^{-1}$
- *for* $i = 2, \ldots, d$, $P_{ii} = \Sigma_{i|\mathrm{pa}\, i}^{-1} + \sum_{\mathrm{pa}\, j=i} \Sigma_{ji}^2 \Sigma_{ii}^{-2} \Sigma_{j|i}^{-1}$
- $i, j = 1, \ldots d$ *and* $i = \mathrm{pa}\, j$, $P_{ij} = P_{ji} = -\Sigma_{ji} \Sigma_{jj}^{-1} \Sigma_{j|i}^{-1}$

where $\mathrm{pa}\, i$ *is the parent of* i *in an optimal approximator tree* T *(*$i = 2, \ldots, n$*) is an optimal tree approximator of* g*.*

Proof. Using Proposition 1 of Best Tree Density Approximators, express an optimal tree approximator of g by

$$(1/c) \exp\left(-\frac{1}{2}\left(\Sigma_{11}^{-1} \bar{x}_1^2 + \sum_{i \neq 1} \left(\bar{x}_i - \Sigma_{i,\mathrm{pa}\, i} \Sigma_{\mathrm{pa}\, i,\mathrm{pa}\, i}^{-1} \bar{x}_{\mathrm{pa}\, i}\right)^2 \Sigma_{i|\mathrm{pa}\, i}^{-1}\right)\right)$$

where $\bar{x}_i = x_i - \mu_i$ and $c = \sqrt{(2\pi)^d \Sigma_{11} \prod_{i \neq 1} \Sigma_{i|\mathrm{pa}\, i}}$.

Second, express the quadratic in the exponential as

$$\Sigma_{11}^{-1} \bar{x}_1^2 + \sum_{i \neq 1} \left[\Sigma_{i|\mathrm{pa}\, i}^{-1} \bar{x}_i^2 - 2\Sigma_{i,\mathrm{pa}\, i} \Sigma_{\mathrm{pa}\, i,\mathrm{pa}\, i}^{-1} \Sigma_{i|\mathrm{pa}\, i}^{-1} \bar{x}_i \bar{x}_{\mathrm{pa}\, i} + \Sigma_{i,\mathrm{pa}\, i}^2 \Sigma_{\mathrm{pa}\, i,\mathrm{pa}\, i}^{-2} \Sigma_{i|\mathrm{pa}\, i}^{-1} \bar{x}_{\mathrm{pa}\, i}^2\right]$$

With P defined as earlier, we can express the above as $\bar{x}^\top P \bar{x}$.

Third, note that c is $\sqrt{(2\pi)^d \det P^{-1}}$ since f^*_T is a density and so integrates to one. □

Notice that f^*_T is a tree normal density.

Empirical normal

In particular, notice that we can approximate the empirical normal density of a dataset with a density that factors according to a tree.

Tree Approximators of a Normal
Optimal Tree Density Approximators
Tree Normal
Tree Density Approximators
Normal Conditionals
Tree Densities
Multivariate Normal
Rooted Tree Densities
Differential Relative Entropy
Positive Definite Matrices
Conditional Densities
Differential Cross Entropy
Quadratic Forms
Marginal Densities
Differential Entropy
Real Matrix Inverse
Identity Matrices
Matrix Trace
Multivariate Real Densities
Normal Densities
Real Matrix-Matrix Products
Matrix Transpose
Exponential Function
Probability Densities
Real Matrix-Vector Products
Symmetric Real Matrices
Real Integrals
Matrix Determinants
Matrices
Vectors as Matrices
Tree Distribution Approximators
Real Linear Combinations
Real Matrices
Real Sums Products
Simple Integrals
Tree Distributions
Distribution Approximators
Real Sums
Multivariate Polynomials
Linear Equations
Vectors
Real Vectors
Square Sums Products
Multivariate Functions
Conditional Distributions
Entropy Functions
Cross Entropy
Divergence
Linear Functions
Square Roots
Sigma Algebra
Sets
Marginal Distributions
Conditional Event Probabilities
Space Vectors
Plane Vector Sums and Differences
Square Sums Products
Euclidean Norm
Real Sums Roots
Real Series
Estimated Real Sums
Joint Distributions
Event Probabilities
Empirical Distribution
Plane Sums
Right Triangle Sides Squares
Distance
Plane Lines
Space Distances
Real Polynomials
Real Squares
Real Inverses
Probability Distributions
Plane Distances
Real Lines
Real Derivatives
Sample Functions
Tree
Real Jacobians
Absolute Value
Rooted Trees
Real Polynomials
Real Matrix
Real Matrix Inverses
Natural Length
Subset Algebra
Tangents
Real Functions
Undirected Paths
Connected Graphs
Real Products
Natural Square Roots
Cartesian
Intervals
Undirected Graphs
Pythagorean
Square Roots
Real Sums
Real Addition Inverses
Set Products
Real Derivatives
Pythagoras
Squares
Cross Sum
Rational Numbers
Natural Multiplication Inverses
Integer Line
Real Series
Real Derivatives
Integers
Integer Sums
Rational Sums
Natural Products
Real Sums
Real Derivatives
Integer Sums
Natural Sums
Natural Numbers
Direct Products
Element Functions
Integer Sums
Optimum
Natural Addition Inverses
Natural Multiplication Inverses
Integer Products
Natural Exponentiation
Factorials
Natural Exponents
Natural Sums
Integer Sums
Natural Products
Finite Elements
Finite Operations
Finite Sums and Intersections
Natural Sums
Natural Products
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