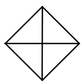


Why

We extend our notion of length, area, and volume beyond the Lebesgue measure on the product spaces of real numbers.

Definition

Suppose $\mathcal{A}$ is an algebra of sets. A function $f : \mathcal{A} \to \bar{\mathbf{R}}_+$ is *finitely additive* if

$$f(\cup_{i=1}^n A_i) = \sum_{i=1}^n f(A_i) \quad \text{for all } A_1, \dots, A_n \in \mathcal{A}$$

Similarly, suppose $\mathcal{F}$ is a σ-algebra. Then f is *countably additive* if

$$f(\cup_{i=1}^\infty F_i) = \sum_{i=1}^\infty f(F_i) \quad \text{for all sequences } \{F_i\}_{i \in \mathbf{N}} \text{ in } \mathcal{F}$$

If, in addition, $f(\varnothing) = 0$, then f is called a *finitely additive measure* or *countably additive measure* respectively. Since a countably additive measure is finitely additive (the converse is false!), when we speak of a *measure* we mean a countable additive one.

When $(X, \mathcal{F})$ is a countably unitable subset algebra and $\mu : \mathcal{F} \to \bar{\mathbf{R}}_+$, then we call $(X, \mathcal{F})$ a *measurable space* and call $(X, \mathcal{F}, \mu)$ a *measure space*. We often call $\mathcal{F}$ the *measurable sets*. In other words, a measure space is a triple: a base set, a sigma algebra, and a measure.

Notation

We often use μ for a measure since it is a mnemonic for "measure". We often also us ν and λ since these letters are near μ in the Greek alphabet.

Examples

Example 1. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure. We call μ the* counting measure.

Example 2. *Let* $(A, \mathcal{A})$ *measurable. Fix* $a \in A$*. Let* $\mu : \mathcal{A} \to [0, +\infty]$ *such that* $\mu(A)$ *is* 1 *if* $a \in A$ *and* $\mu(A)$ *is* 0 *otherwise. Then* μ *is a measure. We call* μ *the* point mass *concentrated at* a*.*

Example 3. *The Lebesgue measure on the measurable space* $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ *is a measure.*

Example 4. *Let* $\mathcal{A}$ *the co-finite algebra on* N*. Let* $\mu : \mathcal{A} \to [0, +\infty]$ *be such that* $\mu(A)$ *is 1 if* A *is infinite or 0 otherwise. Then* μ *is a finitely additive measure. However it is impossible to extend* μ *to be a countably additive measure. Observe that if* $A_n = \{n\}$ *the* $\mu(\cup_n A_n) = 1$ *but* $\sum_n \mu(A_n) = 0$*.*

Example 5. *Let* $(A, \mathcal{A})$ *a measurable space. Let* $\mu : \mathcal{A} \to [0, +\infty]$ *be* 0 *if* $A = \varnothing$ *and* $\mu(A)$ *is* $+\infty$ *otherwise. Then* μ *is a measure.*

Example 6. *Let* A *be set with at least two elements* ($|A| \geq 2$)*. Let* $\mathcal{A} = \mathcal{P}(A)$*. Let* $\mu : \mathcal{A} \to [0, +\infty]$ *such that* $\mu(A)$ *is* 0 *if* $A = \varnothing$ *and* $\mu(A) = 1$ *otherwise. Then* μ *is not a measure, nor is* μ *finitely additive.*

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \varnothing$ then using finite additivity We obtain a contradiction

$$1 = \mu(B \cup C) \neq \mu(B) + \mu(C) = 2$$

$\square$

Properties

Proposition 1 (monotonicity)**.** *Suppose* $(A, \mathcal{A}, \mu)$ *is measure space. Then*

$$\mu(B) \leq \mu(C) \quad \text{for all } B \subset C \subset A$$

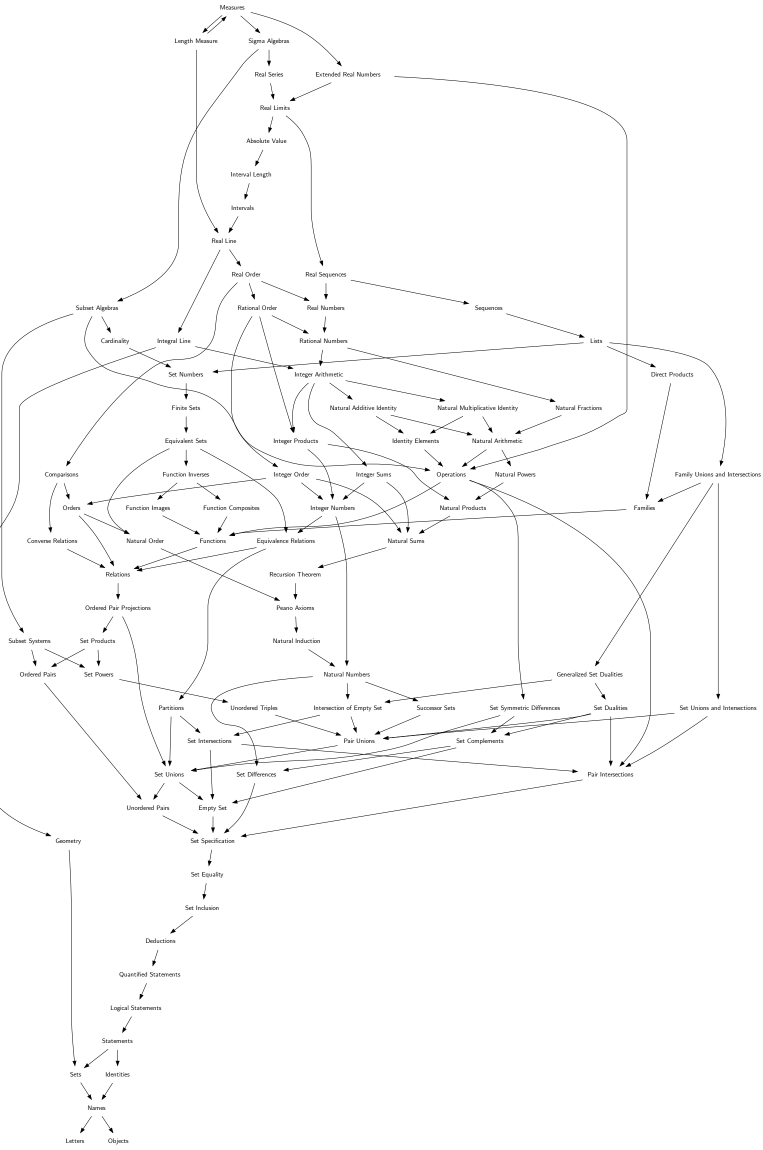
Proposition 2 (subaddivity)**.** *Suppose* $(A, \mathcal{A}, m)$ *is a measure space and* $\{A_n\} \subset \mathcal{A}$ *is a countable family. Then* $m(\cup A_n) \leq \sum_i m(A_i)$*.*

Proposition 3. *For a measure space* $(A, \mathcal{A}, m)$*.*

$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

Proposition 4. *For a measure space $(A, \mathcal{A}, m)$.*

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$


Measures
Length Measure
Sigma Algebras
Real Series
Extended Real Numbers
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Subset Algebras
Rational Order
Real Numbers
Sequences
Cardinality
Integral Line
Rational Numbers
Lists
Set Numbers
Integer Arithmetic
Direct Products
Finite Sets
Natural Additive Identity
Natural Multiplicative Identity
Natural Fractions
Equivalent Sets
Integer Products
Identity Elements
Natural Arithmetic
Comparisons
Function Inverses
Integer Order
Integer Sums
Operations
Natural Powers
Family Unions and Intersections
Orders
Function Images
Function Composites
Integer Numbers
Natural Products
Families
Converse Relations
Natural Order
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Generalized Set Dualities
Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Symmetric Differences
Set Dualities
Set Unions and Intersections
Set Intersections
Pair Unions
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects