Why

Is a sufficient condition for mutual independence of a set of events the fact that each pair of events of the set are independent? The surprising answer is no.

Definition

Suppose P is a event probability function on a finite sample space Ω. The events $A_1, \ldots, A_n$ are *independent* (or *mutually independent*), if for all indices $i \neq j$.

$$P(A_i \cap A_j) = P(A_i)P(A_j)$$

Clearly, if $A_1, \ldots, A_n$ are mutually independent, then they are pairwise independent. The converse, however, is not true.

Counterexample: two tosses

As usual, model tossing a coin twice with the sample space $\Omega = \{0, 1\}^2$. Put a distribution $p : \Omega \to [0, 1]$ so that $p(\omega) = 1/4$ for all $\omega \in \Omega$. Define $A = \{(1, 0), (1, 1)\}$ (the first toss is heads) , $B = \{(0, 1), (1, 1)\}$ (the second toss is heads), and $C = \{(0, 0), (1, 1)\}$. Then $P(A) = P(B) = P(C) = 1/2$, and

$$P(A \cap B) = P(B \cap C) = P(A \cap C) = 1/4.$$

Hence A, B, C are pairwise independent. However,

$$P(A \cap B \cap C) = 1/4 \neq P(A)P(B)P(C).$$

So A, B, C are *not* mutually independent.

Pairwise Independent Events

Mutually Independent Events

Independent Events

Conditional Event Probabilities

Event Probabilities

Outcome Probabilities

Real Functions

Intervals

Real Line

Real Order — Real Summation — Size of Direct Product

Rational Order — Real Numbers — Natural Summation — Number of Set Products

Integral Line — Rational Numbers — Number of Disjoint Unions — Finite Set Examples — Lists

Integer Arithmetic — Set Numbers — Direct Products

Natural Additive Identity — Natural Multiplicative Identity — Natural Fractions

Integer Products — Identity Elements — Natural Arithmetic — Finite Sets

Integer Sums — Natural Powers — Integer Order — Comparison — Uncertain Events — Family Operations — Function Inverse — Equivalent Sets — Family Unions and Intersections

Integer Numbers — Natural Products — Orders — Operations — Function Images — Function Composites — Families

Natural Sums — Converse Relations — Equivalence Relations — Natural Order — Functions

Recursion Theorem — Relations

Ordered Pair Projections — Uncertain Outcomes — Peano Axioms

Set Products — Natural Induction

Set Powers — Natural Numbers — Generalized Set Dualities

Unordered Triples — Successor Sets — Set Symmetric Differences — Intersection of Empty Set — Set Dualities — Partitions — Set Unions and Intersections

Ordered Pairs — Pair Unions — Set Complements — Set Intersections

Set Differences — Set Unions — Pair Intersections

Geometry — Unordered Pairs — Empty Set

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects