



## N-DIMENSIONAL SPACE

### Why

If  $\mathbf{R}$  corresponds to a line, and  $\mathbf{R}^2$  to a plane, and  $\mathbf{R}^3$  to space, does  $\mathbf{R}^4$  correspond to anything? What of  $\mathbf{R}^5$ ?

### Definition

Let  $n$  be a natural number. We call the set  $\mathbf{R}^n$  *n-dimensional space* (or *Euclidean n-space*). We call elements of  $\mathbf{R}^n$  *points*. We identify  $\mathbf{R}^1$  with  $\mathbf{R}$  in the obvious way.

We call the point associated with  $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$  with  $x_i = 0$  for  $1 \leq i \leq n$  the *origin*. When clear from context, we denote the origin by 0. Similarly, we denote the point  $x$  with  $x_i = 1$  for all  $i = 1, \dots, n$  by 1.

### Visualization

We can not visualize  $n$ -dimensional space. Thus, our intuition for it comes from real space (see Real Space).

### Distance

There is a natural notion of distance for  $\mathbf{R}^n$ , which is generalize the definitions of distance in  $\mathbf{R}^2$  and  $\mathbf{R}^3$ . We define the *distance* (*Euclidean distance*) between  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbf{R}^n$  as

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

Does this have the properties that distance has in the plane and in space? We discussed these properties It does. Denote

the function which associates to  $x, y \in \mathbf{R}^n$  their distance  $d : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$ . So  $d(x, y)$  is the distance between the points corresponding to  $x$  and  $y$ .

**Proposition 1.**  *$d$  is non-negative, symmetric, and the distance between two points is no larger than the sum of the distances with any third object.*<sup>1</sup>

## Order

Let  $x, y \in \mathbf{R}^n$ . If  $x_i < y_i$  for all  $i = 1, \dots, n$  then we say  $x$  is *less than*  $y$ . Likewise, if  $x_i \leq y_i$  for all  $i = 1, \dots, n$  then we say  $x \leq y$ . Likewise for  $>$  and  $\geq$ .

## Notation

If  $x \in \mathbf{R}^n$  is less than  $y \in \mathbf{R}^n$  then we write  $x < y$ . Similarly for  $x \leq y$ ,  $x > y$  and  $x \geq y$ .

---

<sup>1</sup>Future editions will include an account.

