



Why

Take a vector space, and consider the set of continuous linear functionals on that space. Given a suitable norm, this space is a complete normed space.

Defining result

Proposition 1. *Let $(V, \|\cdot\|)$ be a normed space. The set V^* of all continuous linear functionals on V is a complete normed space with respect to pointwise algebraic operations and norm $\|\cdot\|_* : V \rightarrow \mathbf{R}$ defined by*

$$\|F\|_* = \sup_{x \in V, \|x\| \leq 1} |F(x)|.$$

Proof. We argue (1) V^* is a vector space, (2) $\|\cdot\|_*$ is a norm, and (3) $(V, \|\cdot\|_*)$ is complete.¹ □

We call $(V^*, \|\cdot\|_*)$ the *dual space* (or *Banach dual* of V). Notice that $(V^*, \|\cdot\|_*)$ is complete regardless of whether the original normed space $(V, \|\cdot\|)$ is complete.

Basic dual norm property

Notice that the dual norm satisfies a familiar property.

Proposition 2. *For any vector x in a normed space $(V, \|\cdot\|)$ and any continuous linear functional F on E ,*

$$|F(x)| \leq \|F\|_* \|x\|.$$

¹Future editions will include an account.

Proof. If $x = 0$, then $\|x\| = 0$ and $F(x) = 0$ (F is linear). Otherwise, $x/\|x\|$ is a unit vector and so

$$\|F\|_* \geq |F(x/\|x\|)| = \frac{|F(x)|}{\|x\|}.$$

where the inequality is from the definition of $\|\cdot\|_*$ (as a supremum) and the equality follows from the linearity of F . \square

