Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$ in the obvious way. For this reason, we call $x \in \mathbf{R}^{n\times 1}$ (meaning $x \in \mathbf{R}^n$) a *column vector*.

For the reasons that we identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$, we write the vector $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as

$$\begin{bmatrix} a\_1 \\ a\_2 \\ a\_3 \end{bmatrix} \text{ or } \begin{pmatrix} a\_1 \\ a\_2 \\ a\_3 \end{pmatrix}.$$

We could as easily also identify $\mathbf{R}^n$ with $\mathbf{R}^{1\times n}$. We avoid this convention. However, by analogy with the language "column vector," we refer to the *matrix* $y \in \mathbf{R}^{1\times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n\times 1}$ and $\mathbf{R}^{1\times n}$. If $a \in \mathbf{R}^{n\times 1}$, we denote $b \in \mathbf{R}^{1\times n}$ defined by $b_i = a_i$ by $a^\top$.

More generally, given a matrix $A \in \mathbf{R}^{m\times n}$, we denote the matrix $B \in \mathbf{R}^{m\times n}$ defined by $B_{ij} = A_{ji}$ by $A^\top$. Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A, and similarly call $a^\top$ the *transpose* of the vector a. Clearly, $(A^\top)^\top = A$, which includes $(a^\top)^\top = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors $\mathbf{R}^1$ (and so with the 1 by 1 matrices $\mathbf{R}^{1\times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1\times 1}$ where $A_{11} = a$, then $Ax = ax$.

Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$\|x\| = \sqrt{x\_1^2 + x\_2^2 + \cdots + x\_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x\_1 y\_1 + x\_2 y\_2 + \cdots + x\_n y\_n = x^\top y.$$

We express the symmetry of the inner product by $x^\top y = y^\top x$.

Vectors as Matrices

Real Matrices · Real Inner Product

Linear Equations · Space Inner Product · Real Norm

Arrays · Linear Functions · Real Vector · Space Norm

Plane Vector Sums and Differences · Space Vectors · N-Dimensional Space · Plane Inner Product · Plane Norm

Plane Vector · Distance · Right Triangle Sides Relation · Absolute Value

Space Distance · Real Squares

Plane Distance · Real Arithmetic

Real Multiplicative Inverse · Interval Length

Real Space · Real Products · Intervals · Square Numbers

Real Plane · Real Additive Inverse · Real Line · Square

Real Summation · Real Sums · Real Order · Rational Arithmetic · Rational Multiplicative Inverse · Integral Line · Geometry

Real Numbers · Rational Order · Rational Sums · Rational Products

Lists · Rational Numbers

Set Numbers · Finite Sets · Integer Arithmetic

Direct Products · Natural Summation · Equivalent Sets · Integer Products · Natural Additive Identity · Natural Multiplicative Identity

Family Unions and Intersections · Family Operations · Arithmetic · Identity Elements

Families · Comparisons · Natural Equations · Natural Products · Integer Sums · Function Inverses · Inverse Elements

Natural Powers · Element Functions

Natural Order · Orders · Equations · Natural Numbers · Integer Order · Integer Numbers · Function Composites · Function Images · Operations

Recursion Theorem · Equivalence Relations · Converse Relations · Functions

Peano Axioms · Relations

Set Products · Natural Induction

Generalized Set Dualities · Set Powers · Natural Numbers

Set Unions and Intersections · Partitions · Intersection of Empty Set · Set Dualities · Unordered Triples · Successor Sets · Set Symmetric Differences

Ordered Pairs · Set Intersections · Pair Unions · Set Complements

Pair Intersections · Set Unions · Set Differences

Unordered Pairs · Empty Set

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects