

## MATRIX SCALAR PRODUCT

# Why

We have seen that the matrices are a vector space. Are they an inner product space?

#### Definition

The matrix scalar product of  $A \in \mathbb{R}^{n \times k}$  and  $B \in \mathbb{R}^{n \times k}$  is the following product

$$\sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} b_{ij}.$$

Using the matrix trace, we can denote this as  $\operatorname{tr} A^{\top} B$ . Some authors call this the *Euclidean matrix scalar product*, matrix inner product or Frobenius inner product.

**Proposition 1.** The matrix scalar product is an inner product.

For example, symmetry of the product is a consequence of the fact that a square matrix and its transpose have identical traces. Commutativity of the trace yields  $\operatorname{tr} A^\top B = \operatorname{tr} B A^\top$ , where the LHS is the scalar product of  $B^\top$  and  $A^\top$ . In other words, transposition "preserves" the matrix scalar product.

With this inner product,  $\mathbf{R}^{n \times k}$  is a Euclidean vector space (see Inner procuts) of dimension nk. For the case of k = 1, we recover a model<sup>1</sup> for the usual space  $\mathbf{R}^n$ .

### Notation

We commonly denote the matrix inner product by  $\langle A, B \rangle$ .

## Induced norm

The matrix inner product induces a norm in the usual way. This norm is sometimes called the *matrix-vector norm* (or *Frobenius norm*) and is

<sup>&</sup>lt;sup>1</sup>Future editions will define this term.

often denoted for a matrix  $A \in \mathbf{R}^{m \times n}$  by  $||A||_F$ .

