Why

We want language for all of the elements of a first set being the elements of a second set.

Definisions

Given two sets A and B, if every element of A is an element of B then we call that A is a *subset* of the B. We say that A is *included* in B. We say that B is a *superset* of A or that B *includes* A. A set A includes and is included in itself.

If $A = B$, then A includes B and B includes A. The axiom of extension asserts the converse also holds. If A includes B and B includes A, then $A = B$. In other words, if A is a subset of B and B a subset of A, then $A = B$.

The empty set is a subset of every other set. Suppose toward contradiction that A were a set which did not include the empty set. Then there would exist an element in the empty set which is not in A. But then the empty set would not be empty. We call the empty set and A *improper subsets* of A. All other subsets we call *proper subsets.* In other words, B is an improper subset of A if and only if A includes B, $B \neq A$ and $B \neq \varnothing$.

Notation

Given two sets A and B, we denote that A is included in B by $A \subset B$. We read the notation $A \subset B$ aloud as "A is included in B" or “A subset B". Or we write $B \supset A$, and read it aloud “B includes A" or “B superset A".

In this notation, we express the axiom of extension

$$A = B \Leftrightarrow (A \supset B) \wedge (A \subset B).$$

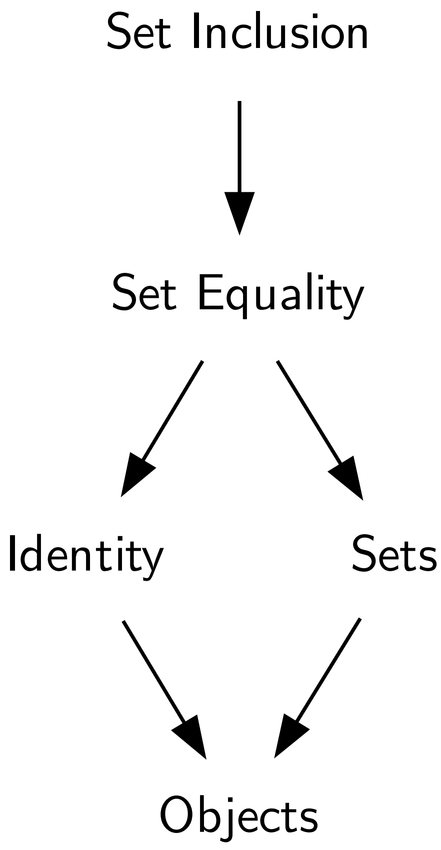
The notation $A \subset B$ is a concise symbolism for the sentence “every element of A is an element of B." Or for the alternative notation $a \in A \implies a \in B$.

Properties

Given a set A, $A \subset A$. Like equality, we say that inclusion is *reflexive*. Given sets A and B, if $A \subset B$ and $B \subset C$ then $A \subset C$. Like equality, we say that inclusion is *transitive*. If $A \subset B$ and $B \subset A$, then $A = B$ (by the axiom of extension). Unlike equality, which is symmetric, we say that inclusion is *antisymmetric*.

Comparison with belonging

Given a set A inclusion is reflexive. $A \subset A$ is always true. Is $A \in A$ ever true? Also, inclusion is transitive. Whereas belonging is not.


Set Inclusion
Set Equality
Identity
Sets
Objects