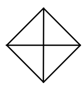


Why

If we model some measured value as a random variable with induced distribution $p : V \to \mathbf{R}$, then one interpretation of $p(v)$ for $v \in V$ is the *proportion* of times in a large number of trials that we *expect* to measure the value v.

Definition

Given a distribution $p : \Omega \to \mathbf{R}$ and a *real-valued* outcome variable $x : \Omega \to \mathbf{R}$, the *expectation* (or *mean*) of x under p is $\sum_{\omega \in \Omega} p(\omega)x(\omega)$.

Notation

We denote the expectation of x under p by $\mathbf{E}(x)$. When there is no chance of ambiguity, we write $\mathbf{E}(x)$.

Properties

Let $x, y : \Omega \to \mathbf{R}$ be two outcome variables and $p : \Omega \to \mathbf{R}$ a distribution. Let $\alpha, \beta \in \mathbf{R}$. Define $z = \alpha x + \beta y$ by $z(\omega) = \alpha x(\omega) + \beta y(\omega)$. Then $\mathbf{E}(z) = \alpha\mathbf{E}(x) + \beta\mathbf{E}(z)$.

Example: expectation

Suppose $\Omega = \{1, 2, 3, 4, 5\}$ with $p(1) = 0.1$, $p(2) = 0.15$, $p(3) = 0.1$, $p(5) = 0.25$ and $p(5) = 0.4$. Define $x : \Omega \to \mathbf{R}$ by

$$x(a) = \begin{cases} -1 & \text{if } a = 1 \text{ or } a = 2, \\ 1 & \text{if } a = 3 \text{ or } a = 4, \\ 2 & \text{if } a = 5. \end{cases}$$

The expectation of x under p is

$$\mathsf{E}x = -1 - 0.15 + 0.1 + 0.25 + 2(0.4) = 0.9.$$

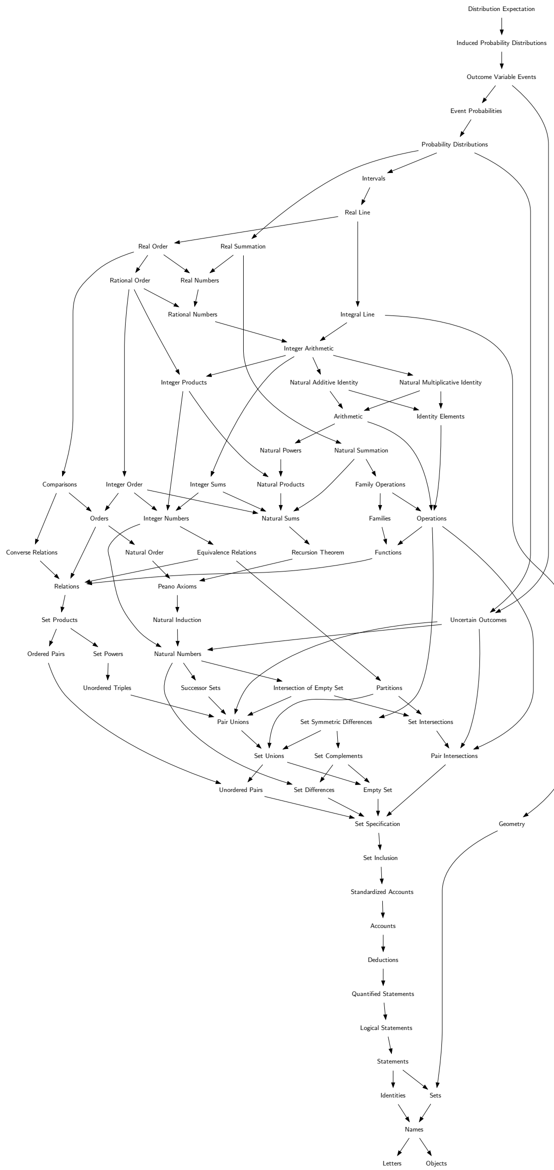
Two routes for computation

Denote by $p_x : V \to \mathsf{R}$ the induced distribution of $x : \Omega \to V$ (where $V \subset \mathsf{R}$). Then $\mathsf{E}(x) = \sum_{v \in V} p_x(v)v$ since

$$\begin{aligned}
\textstyle\sum_{\omega \in \Omega} p(\omega)x(\omega) &= \sum_{v \in V} \sum_{\omega \in x^{-1}(v)} x(\omega)p(\omega) \\
&= \sum_{v \in V} v \sum_{\omega \in x^{-1}(v)} p(\omega) \\
&= \sum_{v \in V} x(v)p_x(v).
\end{aligned}$$

Interpretations

We interpret the mean as the center of mass of the induced distribution.


Distribution Expectation
Induced Probability Distributions
Outcome Variable Events
Event Probabilities
Probability Distributions
Intervals
Real Line
Real Order
Real Summation
Rational Order
Real Numbers
Rational Numbers
Integral Line
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Arithmetic
Identity Elements
Natural Powers
Natural Summation
Comparisons
Integer Order
Integer Sums
Natural Products
Family Operations
Orders
Integer Numbers
Natural Sums
Families
Operations
Converse Relations
Natural Order
Equivalence Relations
Recursion Theorem
Functions
Relations
Peano Axioms
Set Products
Natural Induction
Uncertain Outcomes
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Pair Unions
Set Symmetric Differences
Set Intersections
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Set Differences
Empty Set
Set Specification
Geometry
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects