Why

We characterize chordal graphs using vertex separators, and vice versa.[1]

Main Result

Proposition 1 (Chordal Graphs and Vertex Separators)**.** *An undirected graph is chordal if and only if all minimal vertex separators are complete.*

Proof. Let $G = (V, E)$ be an undirected graph. First, suppose that all minimal vertex separators of G are complete. Let c be a cycle of length greater than 3. Let v, w be nonconsecutive vertices in c. If v and w are adjacent in G, then $\{v, w\} \in E$ is a chord. If v and w are nonadjacent, then vw-separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any vw-separator T. Split the cycle into the path from v to w, call it p_1 and the path from w to v, call it p_2. T must include an interior point of p_1, call it u_1, otherwise v and w are connected. Similarly, T must include an interior point of p_2, call it u_2. u_1 and u_2 are not consecutive in c, since they are distinct from x and y.

Let S be a minimal vw-separator. Let $s, t \in S$ be two non-consecutive vertices in the cycle different from v and w By assumption S is complete, so s and t are adjacent in G.

Second, let $G = (V, E)$ be a chordal graph. Let S be a minimal vw-separator. Let C_v and C_w be the connected components containing v and w of the subgraph induced by $V - S$.

If $\nu m S = 1$, then S is complete. Otherwise, let $x, y \in S$ by distinct. We want to show $\{x, y\} \in E$. The key insight is that x is adjacent to vertices in C_v and C_w. If there were no such vertex, $S - \{x\}$ would be a vw-separator and S would not be minimal. Similarly with y. Also, $\nu m C_v, \nu m C_w \geq 1$.

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[1]Future editions will expand and may include graphics.

With these observations, there exists a path from x to y through C_v. Let $p_v = (x, v_1, \ldots, v_k, y)$ be a path of shortest length with at least one interior vertex (so $k \geq 1$) from x to y using interior vertices in $v_1, \ldots, v_k \in C_v$. Let $p_w = (y, w_1, \ldots, w_l, x)$ be a path of shortest length with at least one interior vertex (so $l \geq 1$) from y to x using interior vertices $w_1, \ldots, w_k \in C_w$. Use p_v and p_w to define the cycle $c = (x, v_1, \ldots, v_k, y, w_1, \ldots, w_l, x)$ which has length at least four. G is chordal, so c has a chord.

We argue that the chord of c is $\{x, y\}$. Since C_w and C_V are different connected components (whose vertices are not included in S), there are no edges $\{v_i, w_j\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Since p_v and p_w are paths of shortest length, they have no chords. In particular, there is no edge $\{v_i, v_j\}$ for $|i - j| > 1$ or $\{v_i, x\}$ for $i = 1, \ldots, l$. Similarly, there is no edge $\{w_i, w_j\}$ for $|i - j| > 1$ or $\{w_i, y\}$ for $i = 1, \ldots, l$. The only remaining pair is $\{x, y\}$, and so it must be the chord. $\qquad\square$

Chordal Graphs and Vertex Separators

Vertex Separators

Chordal Graphs

Trees   Undirected Subgraphs

Undirected Paths   Integer Divisors

Lists   Undirected Graphs   Integral Line

Direct Products   Set Numbers   Integer Arithmetic

Finite Sets   Natural Multiplicative Identity   Natural Additive Identity

Equivalent Sets   Identity Elements   Arithmetic

Family Unions and Intersections   Function Inverses   Natural Powers   Integer Products   Integer Sums

Families   Function Composites   Function Images   Operations   Natural Products   Integer Numbers

Functions   Equivalence Relations   Natural Sums

Relations   Natural Order   Recursion Theorem

Ordered Pair Projections   Peano Axioms

Set Products   Natural Induction

Generalized Set Dualities   Set Powers   Natural Numbers

Set Unions and Intersections   Set Dualities   Unordered Triples   Successor Sets   Intersection of Empty Set   Partitions   Set Symmetric Differences

Ordered Pairs   Pair Unions   Set Intersections   Set Complements

Pair Intersections   Set Unions   Set Differences

Unordered Pairs   Empty Set

Set Specification   Geometry

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects

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