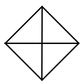


Why

We want to generalize the notion of length, area, volume beyond the Lebesgue measure on the product spaces of real numbers.

Definition

An extended-real-valued non-negative function on an algebra is *finitely additive* if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a sigma algebra is *countably additive* if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A *finitely additive measure* is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A *countably additive measure* is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call countably additive measures *measures*, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countable additive.

In the context of measure, we call a countably unitable subset algebra a *measurable space*. We call the distinguished sets *measurable* sets. A *measure space* is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of the measurable space.

Notation

Let A a set. Let $\mathcal{A}$ a sigma algebra on A. The pair $(A, \mathcal{A})$ is a measurable space.

Let $\mu : \mathcal{A} \to [0, \infty]$ a measure; thus: (a) $\mu(\varnothing) = 0$ and (b) for disjoint $\{A_n\} \subset \mathcal{A}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ The triple $(A, \mathcal{A}, \mu)$ is a measure space.

We use μ since it is a mnemonic for "measure". We often also us ν to denote measures, since it is after μ in the Greek alphabet, and λ, since it is before μ in the Greek alphabet.

Examples

Example 1. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure. We call μ the counting measure.*

Example 2. *Let $(A, \mathcal{A})$ measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise. Then μ is a measure. We call μ the point mass concentrated at a.*

Example 3. *Let R denote the real numbers. The Lebesgue* *measure on the measurable space $(R, \mathcal{B}(R))$ is a measure.*

Example 4. *Let N be the natural numbers. Let $\mathcal{A}$ the finite co-finite algebra on N. Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.*

Example 5. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ be 0 if $A = \varnothing$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.*

Example 6. *Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = A^*$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \varnothing$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.*

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \varnothing$ then using finite additivity we obtain a contradiction $1 = \mu(B \cup C) = \mu(B) + \mu(C) = 2$. $\square$

Why

We want to generalize the notions of length, area, and volume.

Definition

A *measurable space* is a sigma algebra. We call the distinguished subsets the *measurable sets*.

A *measure* on a measurable space is a function from the sigma algebra to the positive extended reals. A *measure space* is a measurable space and a measure.

Notation

Properties

PROPOSITION **7.** *Let* $(A, \mathcal{A})$ *be a measurable space and* $m : \mathcal{A} \to [0, \infty]$ *be a measure.*

If $B \subset C \subset A$*, then* $m(B) \leq m(C)$*. We call this property the of measures monotonicity of measure.*

PROPOSITION **8.** *For a measure space* $(A, \mathcal{A}, m)$*.*

If $B \subset C \subset A$*, then* $m(B) \leq m(C)$*.*

We call this property the monotonicity of measure.

PROPOSITION **9.** *For a measure space* $(A, \mathcal{A}, m)$*.*

If $\{A_n\} \subset \mathcal{A}$ *a countable family, then* $m(\cup A_n) \leq \sum_i m(A_i)$*.*

We this property the sub-additivty of measure.

PROPOSITION **10.** *For a measure space* $(A, \mathcal{A}, m)$*.*

If $\{A_n\} \subset \mathcal{A}$ *a countable family, then* $m(\cup A_n) \leq \sum_i m(A_i)$*.*

We this property the sub-additivty of measure.

PROPOSITION **11.** *For a measure space* $(A, \mathcal{A}, m)$*.*

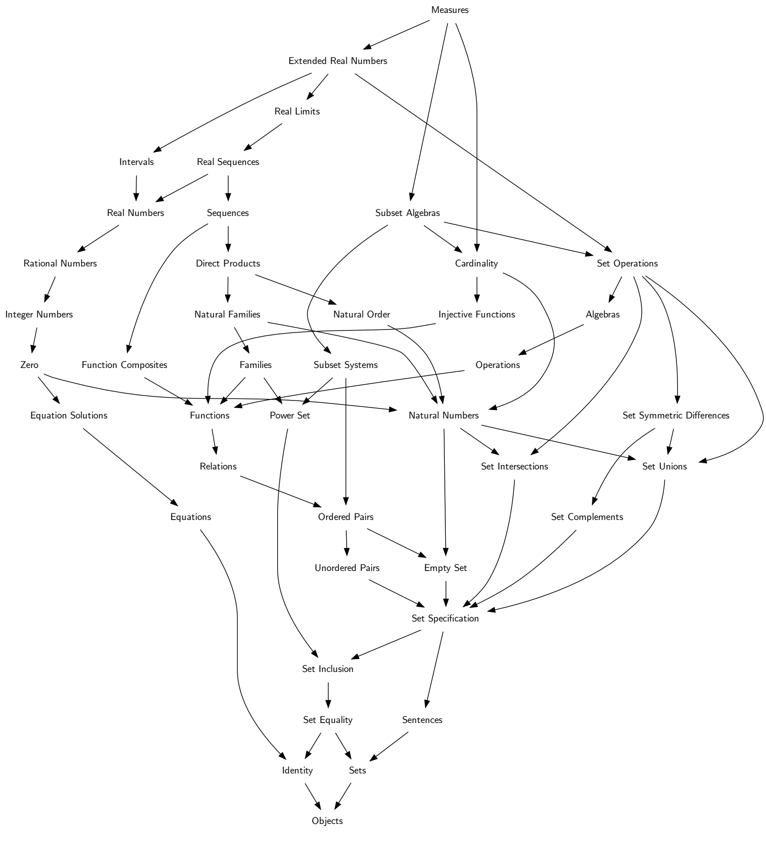
$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

PROPOSITION **12.** *For a measure space* $(A, \mathcal{A}, m)$*.*

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n\to\infty} m(A_i)$$

Examples

Example 13. *counting measure*


Measures
Extended Real Numbers
Real Limits
Intervals
Real Sequences
Real Numbers
Sequences
Subset Algebras
Rational Numbers
Direct Products
Cardinality
Set Operations
Integer Numbers
Natural Families
Natural Order
Injective Functions
Algebras
Zero
Function Composites
Families
Subset Systems
Operations
Equation Solutions
Functions
Power Set
Natural Numbers
Set Symmetric Differences
Relations
Set Intersections
Set Unions
Equations
Ordered Pairs
Set Complements
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Set Equality
Sentences
Identity
Sets
Objects