Why

A simple way to obtain a sigma algebra, is to start with some sets, and then to add all the sets needed to make the starting set closed under the various operations.

Definition

The *generated sigma algebra* for a set of subsets is the smallest sigma algbera containing the set of subsets. We must prove the existence and uniqueness of this sigma algebra.

Proposition 1. *The intersection of a non-empty set of sigma algebras over the same set is a sigma algebra.*

Proof. Given a family of sigma algebras $\{(A, \mathcal{A}_\alpha\}_{\alpha \in I}$ over some set, define $\mathcal{A} = \cap_{\alpha \in I} \mathcal{A}_\alpha$.

1. For all $\alpha \in I$, $A \in \mathcal{A}_\alpha$, thus $A \in \mathcal{A}$; condition (a).
2. For all $B \in \mathcal{A}$, for all $\alpha \in I$, $B \in \mathcal{A}_\alpha$. Thus, for all $\alpha \in I$, $C_A(B) \in \mathcal{A}_\alpha$. And so $C_A(B) \in \mathcal{A}$; condition (b).
3. For all sequences $\{B_n\} \subset \mathcal{A}$, $\{B_n\} \subset \mathcal{A}_\alpha$ for all α. Thus $\cup_n B_n \in \mathcal{A}_\alpha$ for all α and so $\cup_n B_n \in \mathcal{A}$; condition (c).

$\square$

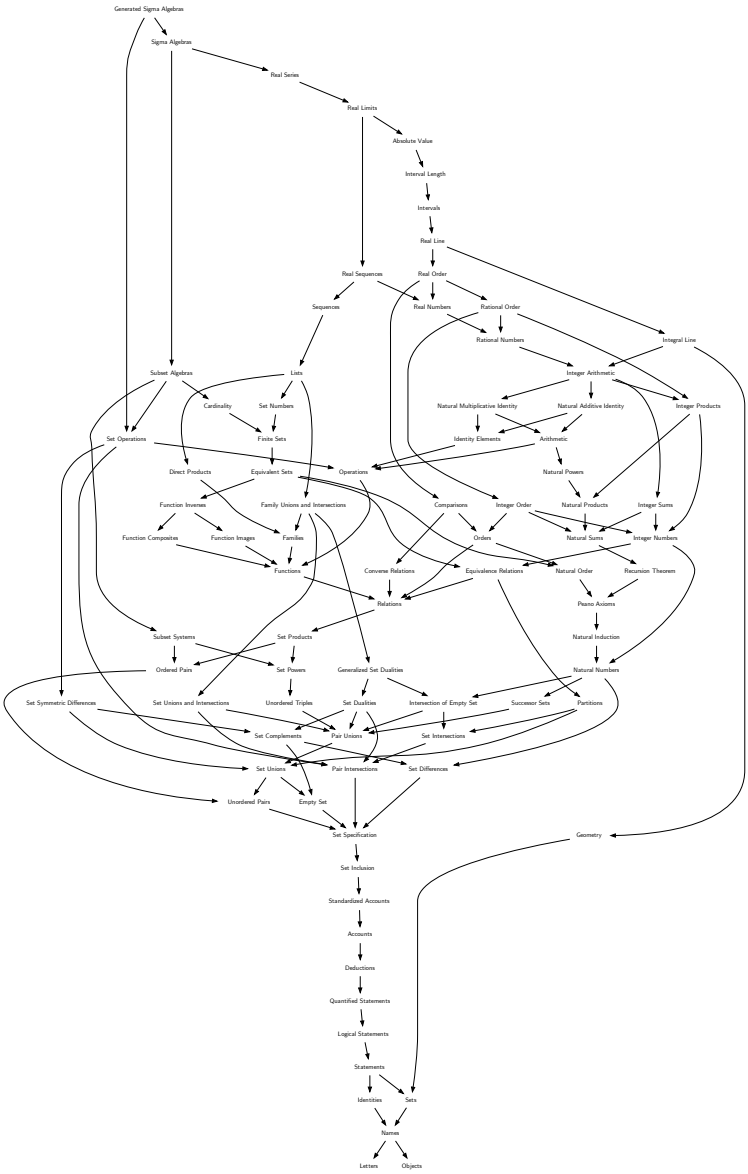
On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

Proposition 2. *If A is a set and $\mathcal{A} \subset 2^A$, then there is a unique a smallest sigma algebra containing $\mathcal{A}$.*

Proof. We know of one sigma algebra containing $\mathcal{A}$: the power set of A. Thus, the set of sigma algebras containing $\mathcal{A}$ is not empty. Proposition 1 implies the intersection of all such sigma algebras (containing $\mathcal{A}$) is a sigma algebra. The intersection contains $\mathcal{A}$, and is contained in all other sigma algebras with this property, so is a smallest sigma algebra containing $\mathcal{A}$. If $\mathcal{B}, \mathcal{C}$ were two smallest sigma algebras, then $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C} \subset \mathcal{B}$, but then $\mathcal{B} = \mathcal{C}$; thus the smallest sigma algebra is unique. $\square$

Notation

Let A be a set and $\mathcal{A} \subset \mathcal{P}(A)$. We denote the sigma algebra generated by $\mathcal{A}$ by $\sigma(\mathcal{A})$.


Generated Sigma Algebras
Sigma Algebras
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Sequences
Real Order
Sequences
Real Numbers
Rational Order
Integral Line
Rational Numbers
Integer Arithmetic
Subset Algebras
Lists
Integer Products
Cardinality
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Set Operations
Finite Sets
Identity Elements
Arithmetic
Direct Products
Equivalent Sets
Operations
Natural Powers
Function Inverses
Family Unions and Intersections
Comparisons
Integer Order
Natural Products
Integer Sums
Function Composites
Function Images
Families
Orders
Natural Sums
Integer Numbers
Functions
Converse Relations
Equivalence Relations
Natural Order
Recursion Theorem
Relations
Peano Axioms
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Set Powers
Generalized Set Dualities
Natural Numbers
Set Symmetric Differences
Set Unions and Intersections
Unordered Triples
Set Dualities
Intersection of Empty Set
Successor Sets
Partitions
Set Complements
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Set Differences
Unordered Pairs
Empty Set
Set Specification
Geometry
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects