Why

If we model some measured value as a random variable with induced distribution $p : V \to \mathbf{R}$, then one interpretation of $p(v)$ for $v \in V$ is the *proportion* of times in a large number of trials that we *expect* to measure the value v.[1]

Definition

Given a distribution $p : \Omega \to \mathbf{R}$ and a *real-valued* outcome variable $x : \Omega \to \mathbf{R}$, the *expectation* (or *mean*) of x under p is $\sum_{\omega \in \Omega} p(\omega)x(\omega)$.

Notation

We denote the expectation of x under p by $\mathbf{E}_p(x)$. When there is no chance of ambiguity, we write $\mathbf{E}(x)$.

Properties

Let $x, y : \Omega \to \mathbf{R}$ be two outcome variables and $p : \Omega \to \mathbf{R}$ a distribution. Let $\alpha, \beta \in \mathbf{R}$. Define $z = \alpha x + \beta y$ by $z(\omega) = \alpha x(\omega) + \beta y(\omega)$. Then $\mathbf{E}(z) = \alpha\mathbf{E}(x) + \beta\mathbf{E}(z)$. Many authors refer to this property as the *linearity of expecation*.

Example: expectation

Suppose $\Omega = \{1, 2, 3, 4, 5\}$ with $p(1) = 0.1$, $p(2) = 0.15$, $p(3) = 0.1$, $p(5) = 0.25$ and $p(5) = 0.4$. Define $x : \Omega \to \mathbf{R}$ by

$$x(a) = \begin{cases} -1 & \text{if } a = 1 \text{ or } a = 2, \\ 1 & \text{if } a = 3 \text{ or } a = 4, \\ 2 & \text{if } a = 5. \end{cases}$$

[1] Future editions may modify this explanation, and take a genetic approach via summary statistics.

The expectation of x under p is

$$\mathbf{E}x = -1 - 0.15 + 0.1 + 0.25 + 2(0.4) = 0.9.$$

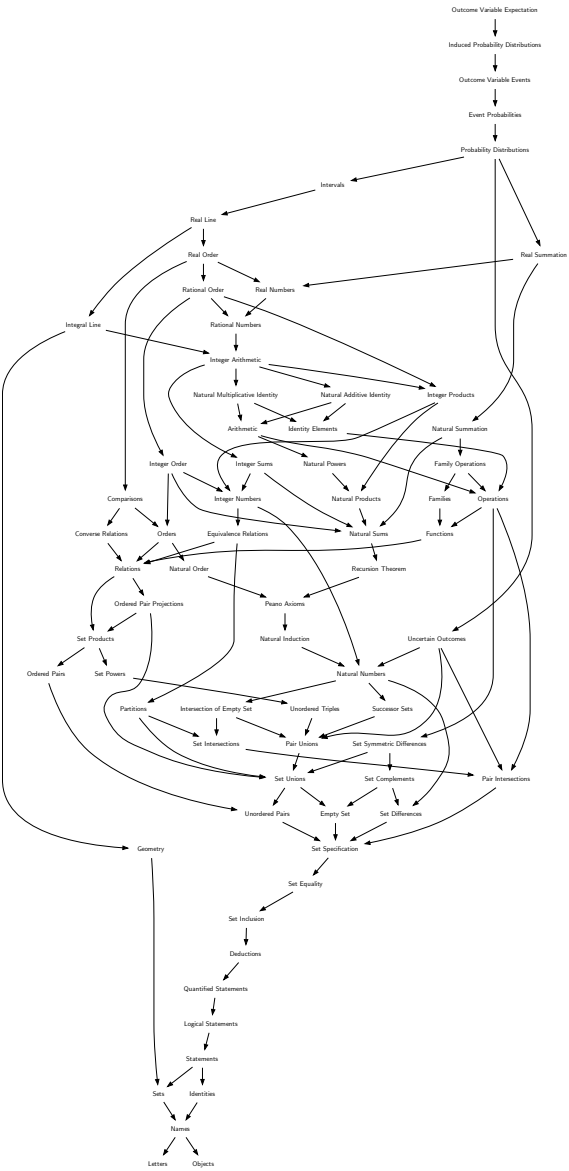
Two routes for computation

Denote by $p_x : V \to \mathbf{R}$ the induced distribution of $x : \Omega \to V$ (where $V \subset \mathbf{R}$). Then $\mathbf{E}(x) = \sum_{v \in V} p_x(v)v$ since

$$\begin{aligned}
\textstyle\sum_{\omega \in \Omega} p(\omega)x(\omega) &= \sum_{v \in V} \sum_{\omega \in x^{-1}(v)} x(\omega)p(\omega) \\
&= \sum_{v \in V} v \sum_{\omega \in x^{-1}(v)} p(\omega) \\
&= \sum_{v \in V} x(v)p_x(v).
\end{aligned}$$

Interpretations

We interpret the mean as the center of mass of the induced distribution.


Outcome Variable Expectation
Induced Probability Distributions
Outcome Variable Events
Event Probabilities
Probability Distributions
Intervals
Real Line
Real Summation
Real Order
Rational Order
Real Numbers
Integral Line
Rational Numbers
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Arithmetic
Identity Elements
Natural Summation
Integer Order
Integer Sums
Natural Powers
Family Operations
Comparisons
Integer Numbers
Natural Products
Families
Operations
Converse Relations
Orders
Equivalence Relations
Natural Sums
Functions
Relations
Natural Order
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Uncertain Outcomes
Ordered Pairs
Set Powers
Natural Numbers
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects