Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$ in the obvious way. For this reason, we call $x \in \mathbf{R}^{n\times 1}$ (meaning $x \in \mathbf{R}^n$) a *column vector*.

For the reasons that we identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$, we write the vector $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

We could as easily also identify $\mathbf{R}^n$ with $\mathbf{R}^{1\times n}$. We avoid this convention. However, by analogy with the language "column vector," we refer to the *matrix* $y \in \mathbf{R}^{1\times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n\times 1}$ and $\mathbf{R}^{1\times n}$. If $a \in \mathbf{R}^{n\times 1}$, we denote $b \in \mathbf{R}^{1\times n}$ defined by $b_i = a_i$ by $a^\top$.

More generally, given a matrix $A \in \mathbf{R}^{m\times n}$, we denote the matrix $B \in \mathbf{R}^{m\times n}$ defined by $B_{ij} = A_{ji}$ by $A^\top$. Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A, and similarly call $a^\top$ the *transpose* of the vector a. Clearly, $(A^\top)^\top = A$, which includes $(a^\top)^\top = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors $\mathbf{R}^1$ (and so with the 1 by 1 matrices $\mathbf{R}^{1\times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1\times 1}$ where $A_{11} = a$, then $Ax = ax$.

Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^\top y.$$

We express the symmetry of the inner product by $x^\top y = y^\top x$.

Vectors as Matrices
Real Inner Product
Real Matrices
Real Norm
Space Inner Product
Linear Equations
Space Norm
Real Vectors
Linear Functions
Arrays
Plane Norm
Plane Inner Product
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Absolute Value
Right Triangle Sides Relation
Distance
Plane Vectors
Real Squares
Space Distance
Real Arithmetic
Plane Distance
Interval Length
Real Multiplicative Inverses
Intervals
Real Space
Real Products
Square Numbers
Real Line
Real Plane
Real Additive Inverses
Squares
Integral Line
Rational Multiplicative Inverses
Rational Arithmetic
Real Order
Real Sums
Real Summation
Geometry
Rational Products
Rational Sums
Rational Order
Real Numbers
Natural Summation
Rational Numbers
Lists
Set Numbers
Direct Products
Integer Arithmetic
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Inverse Elements
Natural Powers
Integer Sums
Natural Equations
Integer Order
Comparisons
Element Functions
Function Inverses
Family Operations
Family Unions and Intersections
Natural Products
Integer Numbers
Equations
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Natural Order
Converse Relations
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects