



Why

We want a complete norm on the vector space of continuous functions.

Result

Prop. 1. *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $(f_n)_n$ be an egoprox sequence in $C[a, b]$.

Candidate. $(f_n)_n$ is egoprox means $\forall \varepsilon > 0, \exists N$ so that

$$m, n > N \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon.$$

Since $\|f_n - f_m\|_{\sup} < \varepsilon \longrightarrow |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in [a, b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a, b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit $l_x \in R$ such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a, b]$. Define $f : [a, b] \rightarrow R$ by $f(x) = l_x$ for each $x \in [a, b]$.

Candidate is Limit. First, we argue that $\|f_n - f\|_{\sup} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $(f_n)_n$ is an egoprox sequence, there exists n_0 so that

$$n, m \geq n_0 \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon/2.$$

So for all $x \in [a, b]$,

$$n, m \geq n_0 \longrightarrow |f_n(x) - f_m(x)| < \varepsilon/2.$$

For all $x \in [a, b]$, and $n \geq n_0$,

$$\lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

The sequence $\{f_k(x)\}_{k=m}^{\infty}$ is a final part of $\{f_k(x)\}_{k=1}^{\infty}$, and so has the same limit, $f(x)$. Therefore, using continuity of subtraction and the absolute value,

$$\lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for $n \geq n_0$, $x \in [a, b]$, $|f_n(x) - f(x)| < \varepsilon$, from which we deduce $\|f_n - f\|_{\sup} < \varepsilon$. Thus $f_n \rightarrow f$ as $n \rightarrow \infty$.

Limit is Continuous. Next, we argue that f is continuous. Let $x_0 \in [a, b]$. Let $\varepsilon > 0$. Since $f_n \rightarrow f$ there exists n_0 so that

$$\|f_{n_0} - f\|_{\sup} < \varepsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a, b]$. Using $|f(x_0) - f_{n_0}(x_0)| < \varepsilon/3$,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a, b]$. Using the triangle inequality,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

for all $x \in [a, b]$. Using $|f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3$

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \varepsilon/3$$

for all $x \in [a, b]$. Since f_{n_0} is continuous, there exists $\delta > 0$ so that

$$|x_0 - x| < \delta \rightarrow |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3,$$

for $x \in [a, b]$. In this case,

$$|f(x_0) - f(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since ε was arbitrary, f is continuous at x_0 . Since x_0 was arbitrary, f is continuous everywhere. Some call the above the *three epsilon argument*. □

