Why

We want a complete norm on the vector space of continuous functions.

Result

Proposition 1. *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $(f_n)_n$ be an egoprox sequence in $C[a, b]$.

Candidate. $(f_n)_n$ is egoprox means $\forall \varepsilon > 0, \exists N$ so that

$$m, n > N \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon.$$

Since $\|f_n - f_m\|_{\sup} < \varepsilon \longrightarrow |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in [a, b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a, b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit $l_x \in R$ such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a, b]$. Define $f : [a, b] \to R$ by $f(x) = l_x$ for each $x \in [a, b]$.

Candidate is Limit. First, we argue that $\|f_n - f\|_{\sup} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $(f_n)_n$ is an egoprox sequence, there exists n_0 so that

$$n, m \geq n_0 \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon/2.$$

So for all $x \in [a, b]$,

$$n, m \geq n_0 \longrightarrow |f_n(x) - f_m(x)| < \varepsilon/2.$$

For all $x \in [a, b]$, and $n \geq n_0$,

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

The sequence $\{f_k(x)\}_{k=m}^{\infty}$ is a final part of $\{f_k(x)\}_{k=1}^{\infty}$, and so has the same limit, $f(x)$. Therefore, using continuity of subtraction and the absolute value,

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for $n \geq n_0$, $x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$, from which we deduce $\|f_n - f\|_{\sup} < \varepsilon$. Thus $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

Limit is Continuous. Next, we argue that f is continuous. Let $x_0 \in [a,b]$. Let $\varepsilon > 0$. Since $f_n \longrightarrow f$ there exists n_0 so that

$$\|f_{n_0} - f\|_{\sup} < \varepsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a,b]$. Using $|f(x_0) - f_{n_0}(x_0)| < \varepsilon/3$,

$$|f(x_0) - f(x)| \& < \varepsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a,b]$. Using the triangle inequality,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

for all $x \in [a,b]$. Using $|f_{n_0}(x_0) - f(x)| < \varepsilon/3$

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \varepsilon/3$$

for all $x \in [a,b]$. Since f_{n_0} is continuous, there exists $\delta > 0$ so that

$$|x_0 - x| < \delta \longrightarrow |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3,$$

for $x \in [a,b]$. In this case,

$$|f(x_0) - f(x)| \& < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since ε was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere. Some call the above the *three epsilon argument*.

□

Supremum Norm Complete

Supremum Norm

Norms

Functionals

Least Upper Bounds

Real Norm

Vectors

Space Norm

Real Vectors

N-Dimensional Space

Plane Norm

Space Vector

Plane Vector Sums and Differences

Real Limits

Distance

Plane Vector

Absolute Value

Space Distance

Plane Distance

Interval Length

Real Space

Intervals

Real Plane

Real Line

Integral Line

Real Order

Real Sequences

Geometry

Rational Order

Real Numbers

Fields

Sequences

Rational Numbers

Rings

Lists

Integer Arithmetic

Set Numbers

Integer Products

Natural Multiplicative Identity

Natural Additive Identity

Finite Sets

Arithmetic

Identity Elements

Direct Products

Equivalent Sets

Integer Order

Integer Sums

Family Unions and Intersections

Natural Powers

Comparisons

Function Inverses

Natural Products

Orders

Function Composites

Function Images

Families

Integer Numbers

Natural Sums

Operations

Converse Relations

Natural Order

Functions

Equivalence Relations

Recursion Theorem

Relations

Peano Axioms

Set Products

Natural Induction

Set Powers

Natural Numbers

Generalized Set Dualities

Partitions

Unordered Triples

Intersection of Empty Set

Successor Sets

Set Unions and Intersections

Set Dualities

Set Symmetric Differences

Ordered Pairs

Set Intersections

Pair Unions

Set Complements

Set Unions

Pair Intersections

Set Differences

Unordered Pairs

Empty Set

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects