



REAL MATRICES

Why

We compress the notation for linear equations.

Definition

A *real matrix* (*real-valued matrix*, *matrix of real numbers*, *matrix*) is a two-dimensional array of real numbers. Recall that we are interested in solutions of the linear equations

$$\begin{aligned}y_1 &= A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n, \\y_2 &= A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n, \\&\vdots \\y_n &= A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n.\end{aligned}$$

We have suggestively used the notation A_{ij} for the coefficients of the equations, so they are the entries of a matrix $A \in \mathbf{R}^{m \times n}$.

We call n and m the *dimensions* of the matrix. We call n the *height* and m the *width*. If the height of the matrix is the same as the width of the matrix then we call the matrix *square*. If the height is larger than the width, we call the matrix *tall*. If the width is larger than the height, we call the matrix *wide*.

Matrix-vector products

Using the notation $A \in \mathbf{R}^{m \times n}$ and $x \in \mathbf{R}^n$ we want a compressed way to write the above system of linear equations. Define the *real matrix-vector product* z of A with x by $z_i = \sum_{j=1}^n A_{ij}x_j$. We denote the matrix vector product z by Ax .

Notation

We express the above system of linear equations as

$$y = Ax,$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}, \quad \text{and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

The compact notation $y = Ax$ is sometimes called the *matrix form* of the m linear equations and A the *coefficient matrix*.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function $x \mapsto Ax$. In other words, we are interested in the existence of an inverse element of A . The geometric interpretation is that A transforms the vector x .

Note on terminology

The etymology of the word “matrix” is from the Latin “mater,” meaning mother, and has an old sense in the English language similar to the sense of the English word “womb.” The matrix is source of many determinants (discussed later).¹

¹Future editions may elide this discussion.

