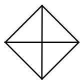


Affine MMSE Estimators

Why

[1]

Definition

We want to estimate a random variable $x : \Omega \to \mathbf{R}^n$ from a random variable $y : \Omega \to \mathbf{R}^n$ using an estimator $\phi : \mathbf{R}^m \to \mathbf{R}^n$ which is affine.[2] In other words, $\phi(\xi) = A\xi + b$ for some $A \in \mathbf{R}^{n \times m}$ and $b \in \mathbf{R}^n$. We will use the mean squared error cost.

We want to find A and b to minimize

$$\mathbf{E}\,|Ax + b - y|^2.$$

Proof. Express $\mathbf{E}(|Ax+b-y|^2)$ as $\mathbf{E}((Ax+b-y)^\top(Ax+b-y))$

$$\begin{array}{cccccccc}
+ & \mathbf{tr}(A\,\mathbf{E}(xx^\top)A^\top) & + & \mathbf{E}(x)^\top A^\top b & - & \mathbf{tr}(A^\top\,\mathbf{E}(yx^\top)) \\
+ & b^\top A\,\mathbf{E}(x) & + & b^\top b & - & b^\top\,\mathbf{E}(y) \\
- & \mathbf{tr}(A\,\mathbf{E}(xy^\top)) & - & \mathbf{E}(y)^\top b & + & \mathbf{E}(yy^\top)
\end{array}$$

The gradients with respect to b are

$$\begin{array}{cccccc}
+ & 0 & + & A\,\mathbf{E}(x) & - & 0 \\
+ & A\,\mathbf{E}(x) & + & 2b & - & \mathbf{E}(y) \\
- & 0 & - & \mathbf{E}(y) & + & 0
\end{array}$$

so $2A\,\mathbf{E}(x) + 2b - 2\,\mathbf{E}(y)$. The gradients with respect to A are

$$\begin{array}{cccccc}
+ & \mathbf{E}(xx^\top)A^\top + \mathbf{E}(xx^\top)^\top A^\top & + & \mathbf{E}(x)b^\top & - & \mathbf{E}(yx^\top)^\top \\
+ & \mathbf{E}(x)b^\top & + & 0 & - & 0 \\
- & \mathbf{E}(xy^\top) & - & 0 & + & 0
\end{array}$$

[1] Future editions will include an account.

[2] Actually, the development flips this. Future editions will correct.

so $2\,\mathbf{E}(xx^\top)A^\top + 2\,\mathbf{E}(x)b^\top - 2\,\mathbf{E}(xy^\top)$. We want A and b solutions to

$$A\,\mathbf{E}(x) + b - \mathbf{E}(y) = 0$$
$$\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)b^\top - \mathbf{E}(xy^\top) = 0$$

so first get $b = \mathbf{E}(y) - A\,\mathbf{E}(x)$. Then express

$$\begin{aligned}
\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)(\mathbf{E}(y) - A\,\mathbf{E}(x))^\top - \mathbf{E}(xy^\top) &= 0. \\
\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)\,\mathbf{E}(y)^\top - \mathbf{E}(x)\,\mathbf{E}(x)^\top A^\top - \mathbf{E}(xy^\top) &= 0. \\
(\mathbf{E}(xx^\top) - \mathbf{E}(x)\,\mathbf{E}(x)^\top)A^\top &= \mathbf{E}(xy^\top) - \mathbf{E}(x)\,\mathbf{E}(y)^\top. \\
\mathbf{cov}(x,x)A^\top &= \mathbf{cov}(x,y).
\end{aligned}$$

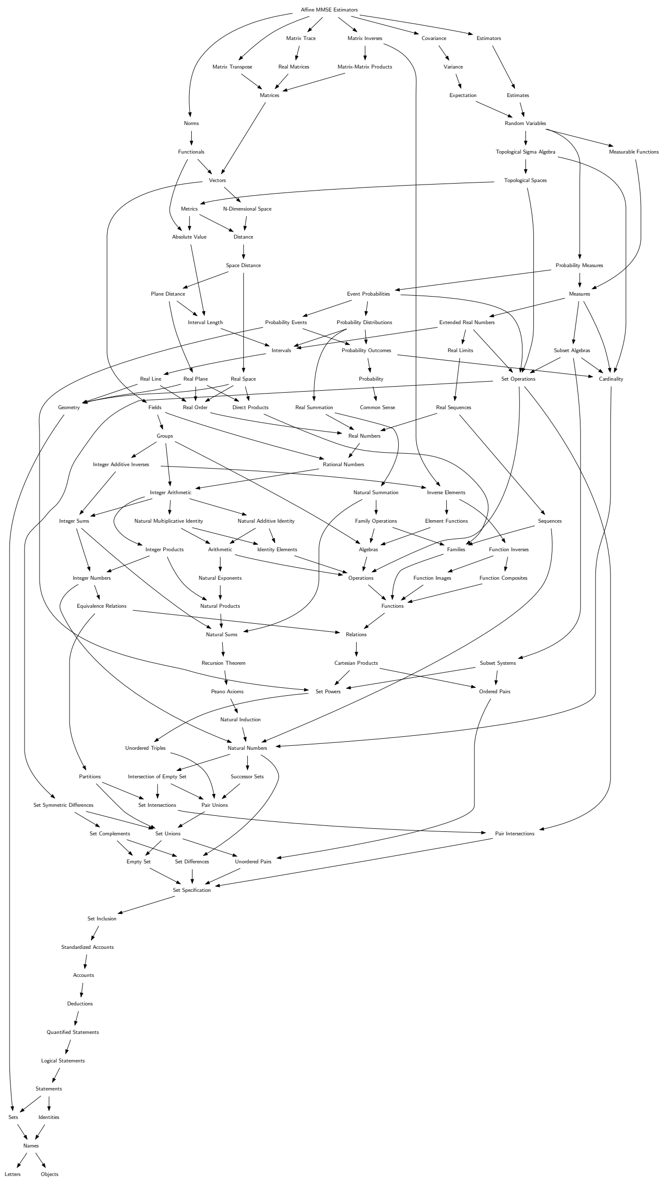
So $A^\top = \mathbf{cov}(x,x)^{-1}\,\mathbf{cov}(x,y)$ means $A = \mathbf{cov}(y,x)\mathbf{cov}(x,x)^{-1}$ is a solution. Then $b = \mathbf{E}(y) - \mathbf{cov}(y,x)\mathbf{cov}(x,x)^{-1}\,\mathbf{E}(x)$. So to summarize, the estimator $\phi(x) = Ax + b$ is

$$\mathbf{cov}(y,x)\,(\mathbf{cov}\,x,x)^{-1}\,x + \mathbf{E}(y) - \mathbf{cov}(y,x)\mathbf{cov}(x,x)^{-1}\,\mathbf{E}(x)$$

or

$$\mathbf{E}(y) + \mathbf{cov}(y,x)\,(\mathbf{cov}\,x,x)^{-1}\,(x - \mathbf{E}(x))$$

□


Affine MMSE Estimators
Matrix Trace
Matrix Inverses
Covariance
Estimators
Matrix Transpose
Real Matrices
Matrix-Matrix Products
Variance
Matrices
Expectation
Estimates
Norms
Random Variables
Functionals
Topological Sigma Algebra
Measurable Functions
Vectors
Topological Spaces
Metrics
N-Dimensional Space
Absolute Value
Distance
Space Distance
Probability Measures
Plane Distance
Event Probabilities
Measures
Interval Length
Probability Events
Probability Distributions
Extended Real Numbers
Intervals
Probability Outcomes
Real Limits
Subset Algebras
Real Line
Real Plane
Real Space
Probability
Set Operations
Cardinality
Geometry
Fields
Real Order
Direct Products
Real Summation
Common Sense
Real Sequences
Groups
Real Numbers
Integer Additive Inverses
Rational Numbers
Integer Arithmetic
Natural Summation
Inverse Elements
Integer Sums
Natural Multiplicative Identity
Natural Additive Identity
Family Operations
Element Functions
Sequences
Integer Products
Arithmetic
Identity Elements
Algebras
Families
Function Inverses
Integer Numbers
Natural Exponents
Operations
Function Images
Function Composites
Equivalence Relations
Natural Products
Functions
Natural Sums
Relations
Recursion Theorem
Cartesian Products
Subset Systems
Peano Axioms
Set Powers
Ordered Pairs
Natural Induction
Unordered Triples
Natural Numbers
Partitions
Intersection of Empty Set
Successor Sets
Set Symmetric Differences
Set Intersections
Pair Unions
Set Complements
Set Unions
Pair Intersections
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects