The Bourbaki Project

Edited by N. C. Landolfi

Edition 1 — Summer 2021

Printed in Menlo Park, California

Editor's Preface

This project is one of the more ambitious with which I am affiliated. Its two-fold goal is to explain mathematics to the novice and provide standardized language for the expert. The reader should note that I have cut this edition under the pressure of time, in accordance with my annual goals for the project, and not because I felt we had reached a reasonable landmark, or that the content was particularly polished.

So then, what is here? An attempt to talk about language, symbols, intangible objects and logical reasoning enough to get to a few principles having to do with intangible objects called sets and a few things you can build out of these sets. The construction of real numbers and their relation to the lines of geometry becomes quite sparse toward the end, but the outline is included. The n-dimensional real space is touched upon, and barely metric spaces, barely topological spaces.

On that last point, I should mention that the original goal for this edition was to reach topological spaces. We agreed that this topic involved sufficiently abstract concepts which could test the project's assumptions. We all agree, now, that there was much more to be said (to the novice) about topics much preliminary to topological spaces. More than we anticipated. We could, early on, define a topological space in terms of sets. But we could not say why we cared. And this idea, that we might say why we wanted a new concept before we introduced it, was an assumption we were testing with this project.

What were the other assumptions to be tested? First, that the concepts and discussion could be so ordered that we only use prior concepts and discussion. Second, that we could structure the book so that topics are treated by short, two-page sheets. Third, that such a treatment would be useful as a reference. Fourth, that we could standardize language (perhaps formally) and use it in all theorems, definitions and proofs.

These traits would undoubtedly be useful. The sheets could serve both as a beginner's guide and a reference. When reaching for a particular topic, the prerequisites would be clear, fine-grained, and each one only two-pages long. And a standardized language to facilitate understanding and communication is a centuries-old endeavor. That no such text exists, to our knowledge, must indicate that its construction is accompanied by great difficulty. But that is not to say impossible, and computers and screens may facilitate the process.

The text you hold is the first edition. And we might call it a first attempt. It is incomplete and with flaws. But that is not to say useless. There is visible in it the form of what is to come, if only you look at it properly. And, in any case, it is time that we have a first edition.

N.C.L.
16 July 2021
Menlo Park, California

The Bourbaki Project is a collection of documents describing mathematical concepts, terms, results and notation.

Sheets

We call these documents *sheets*. They are only ever two-pages long and sometimes shorter. They can be printed on a single sheet of paper, hence the name sheet. In a book, they occupy two facing pages. The decision to cap at two pages is arbitrary. But our experience suggests it is convenient.

Prerequisites

Each sheet is labeled with the names of those sheets which are its immediate prerequisites, with the names of those sheets for which it is an immediate prerequisite, and a diagram illustrating the dependencies between all its prerequisites.

For example, the sheet Relationsneeds the sheet Ordered Pairs. The reason, in this case, is that the concept of a relation is discussed using the concept of an ordered pair of objects. And since the phrase "ordered pair of objects" makes sense only if we know what is meant by object (discussed in the sheet Objects), the sheet Relationsneeds the sheet Objects also. The reader unacquainted with ordered pairs and objects must read (at least) these two sheets before the sheet on relations. In this case (and in every case) the prerequisites are naturally ordered. Objectsought to be read first, before Ordered Pairs, before Relations. Such an ordering always exists because we ensure that if a sheet X needs a sheet Y, then Y can not need X or any sheet that needs X. A sheet is an immediate prerequisite if it is not prerequisite to any other prerequisite.

Preface

The project is like a map. The landmarks are sheets, or really concepts. Walking is reading. And you must walk along the trails specified by the prerequisites.

Aims

Our primary aim is two-fold. First, to provide useful exposition to teach the concepts to an unacquainted reader (here the prerequisites help). And second, to serve as a reference for further work. It is a welcomed concomitant that we better understand and develop the mathematical concepts ourselves.

Caveats

There are two caveats. First, we give only one path to concepts. The point is that our way of structuring the concepts (and hence the prerequisites) is just one way, and there are many ways, since there are equivalent concepts, alternate proofs, and so on. The second caveat is a wink. These sheets are fiction. They contain only ideas. We have done our best to eliminate all false statements. The game for the practical cogitator is to fit these puzzle pieces to reality.

Contents

Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *symbol* is a written mark. A *script* is a collection of symbols called *letters*. In *phonetic* languages the letters correspond to sounds and rules for composing these letters into successions called written words. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into written words which are meant to denote the spoken words of the English language. The written words on this page are several letters one after the other. For example, the word "word" is composed of the letters "w", "o", "r", "d".

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. The *lower case latin letters*.

a	b	c	d	e	f	g	h	i
j	k	l	m	n	o	p	q	r
s	t	u	v	w	x	y	z	

And the *upper case latin letters*.

A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest. Read from right to left and top to bottom, we have listed these letters in the *conventional ordering* of the *Latin alphabet*.

Arabic numerals

We also use the *Arabic numerals*.

0 1 2 3 4 5 6 7 8 9

Other symbols

We also use the following symbols.

$$' \quad (\quad) \quad \{ \quad \} \quad \vee \quad \wedge \quad \neg \quad \forall \quad \exists \quad \Rightarrow \quad \longleftrightarrow \quad = \quad \in \quad \to \quad \sim$$

Letters (1) does not immediately need any sheet.

Letters (1) is immediately needed by:

Names (3)

Letters (1) gives the following terms.

language, affections, spoken word, symbol, script, letters, phonetic, lower case latin letters, upper case latin letters, conventional ordering, Latin alphabet, Arabic numerals.

Letters

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*. Intangible objects are also called *abstract*.

Examples

A pebble is a tangible object. It is an object, surely, and we can hold and touch it, of course.

The color of the pebble is *intangible*. We can call it an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible.

These sheets discuss other intangible objects and little else besides.

Objects (2) does not immediately need any sheet.

Objects (2) is immediately needed by:

Names (3)

Objects (2) gives the following terms.

object, tangible, intangible, abstract.

Objects

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an *accent* $'$ or subscripts or superscripts. When we write the symbols we say that the composite symbol formed *denotes* the object. We call it the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{\text{A}}$ or $\boxed{\text{A'}}$ or $\boxed{\text{A}_0}$. The box works well to group the symbols and clarifies that $\boxed{\text{A}}\boxed{\text{A}}$ is different from $\boxed{\text{AA}}$. But experience shows that we need not use boxes.

We indicate a name for an object with italics. Instead of $\boxed{\text{A'}}$ we use A', instead of $\boxed{\text{A}_0}$ we use A_0. Experience shows that this subtlety is enough for clarity and it agrees with traditional and modern practice. Other examples include A'', A''', A'''', B, C, D, E, F, f, f' f_a.

No repetitions

We never use the same name to refer to two different objects. Using the same name for two different objects causes confusion. We make clear when we reuse symbols to mean different objects. We tend to introduce the names used at the beginning of a paragraph or section.

Names are objects

There is an odd aspect in these considerations. The symbol A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can reference themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the marks on the page which are meant to look similar are the several uses of a name.

Names as placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is best understood by example.

Names (3) immediately needs:

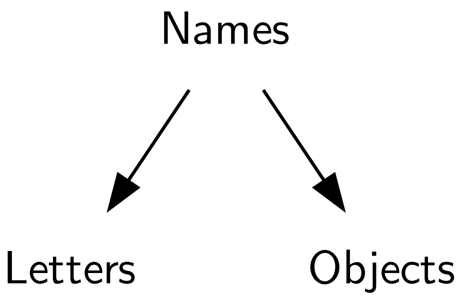
Letters (1)
Objects (2)

Names (3) is immediately needed by:

Identities (4)
Sets (5)

Names (3) gives the following terms.

accent, denotes, name, placeholder.


Names
Letters
Objects

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B." In other words, A and B are two names for the same object.

Symmetry

Let A denote an object and let B denote an object. "A equals B" means the same as "B equals A". The identity of the names is not dependent on the order in which the names are given. We call this the *symmetry of identity*. It means we can switch the spots of A and B and say the same thing. In other words, there are two ways to make the statement.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself. This fact is called the *reflexivity of identity*. A name is equal to itself because an object is itself.

Identities (4) immediately needs:

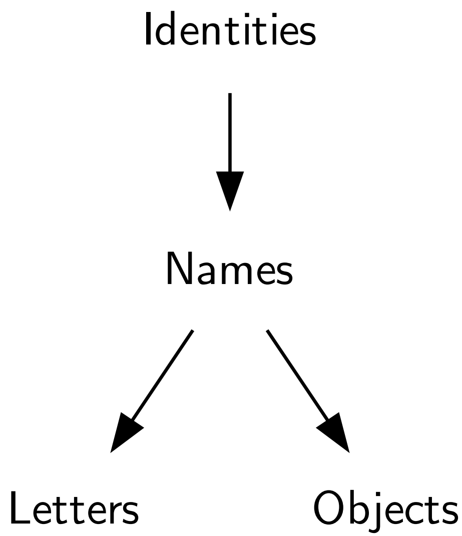
Names (3)

Identities (4) is immediately needed by:

Statements (6)

Identities (4) gives the following terms.

is, equation, indeterminate, is, equal, name, identity, symmetry of identity, reflexivity of identity, reflexive, symmetric, transitive, equals, reflexive, symmetric, transitive.


Identities
Names
Letters
Objects

Why

A pack of wolves, a bunch of grapes, or a flock of pigeons. We want to talk about none, one, or several objects considered together, as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set* (or *aggregate*[1]). We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

A set may have other sets as its members. This is subtle but becomes familiar. We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object belongs to the set. In other words, the object is a member or an element of the set. We say "The object denoted by a belongs to the set denoted by A".

Not symmetric

Notice that belonging is not symmetric. Saying "the object denoted by a belongs to the set denoted by A" does not mean the same as "the set denoted by A belongs to the object denoted by a." In fact, the latter

[1]The German word being *Menge*.

sentence is nonsensical unless the object denoted by a is also a set.

Not transitive

Let a denote an object and let A and B both denote sets. If the object denoted by a is "a part of" the set denoted by A, and the set denoted by A is "a part of" the set denoted by B, then usual English usage would suggest that a is "a part of" the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing.

The relation of belonging does not follow this familiar usage. In contrast, if an object is an element of a set, that set may be an element of another set, but this does not mean that the the first object is also an element of that other set. The upshot is that sets are nested: we can have intangible groups of intangible groups, and have them be different than the intangible group of all the members of each group.

Examples

The hairs on your head, the grains of sand on the beaches of Earth, the blades of grass in a field are all examples of sets. Although we can not readily visualize all the elements at once, we can conceive of them, and visualize the elements one by one.

Sets (5) immediately needs:

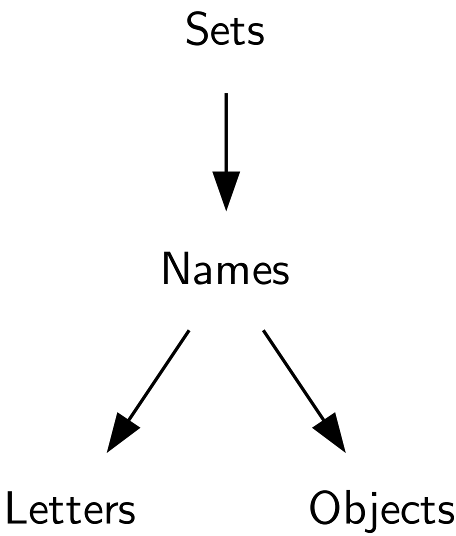
Names (3)

Sets (5) is immediately needed by:

Geometry (22)
Statements (6)

Sets (5) gives the following terms.

set, aggregate, belong, members, elements, in, empty, nonempty.


Sets
Names
Letters
Objects

Why

We want symbols to represent identity and belonging.

Definition

In the English language, nouns are words that name people, places and things. In these sheets, names (see Names) serve the role of nouns. In the English language, verbs are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings.

As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense.

A *statement* is a succession of symbols.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object (see Identities).

We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And if we read it aloud, we would read a as "the object denoted by a", then $=$ as "is", then b as "the object denoted by b". Altogether then, "the object denoted by a is the object denoted by b." We might box these three symbols $\boxed{a = b}$ to make clear that they are meant to be read together, but experience shows that (as with English sentences and words) we do not need boxes.

The symbol $=$ is (appropriately) a symmetric symbol. If we flip it left and right, it is the same symbol. This reflects the symmetry of the English sentences represented (see Identities). The symbols $a = b$ mean the same as the symbols $b = a$.

Belonging

As a second example, consider the symbol $\in$. Let a denote an object and let A denote a set.

We agree that $\in$ means "belongs to" in the sense of "is an element of" or "is a member of" (see Sets). Then we write $a \in A$. We read these symbols as "the object denoted by a belongs to the set denoted by A".[2]

The symbol $\in$ is not symmetric. If we flip it left and right it looks different. This reflects that $a \in A$ does not the mean the same as $A \in a$ (see Sets). As with english words, the order of symbols is significant. The word "word" is not the same as the word "drow".

Our symbolism for belonging reflects the concept's lack of symmetry.

[2]The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for the ancient Greek word ἐστί

which means, roughly, "belongs". Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of".

Statements (6) immediately needs:

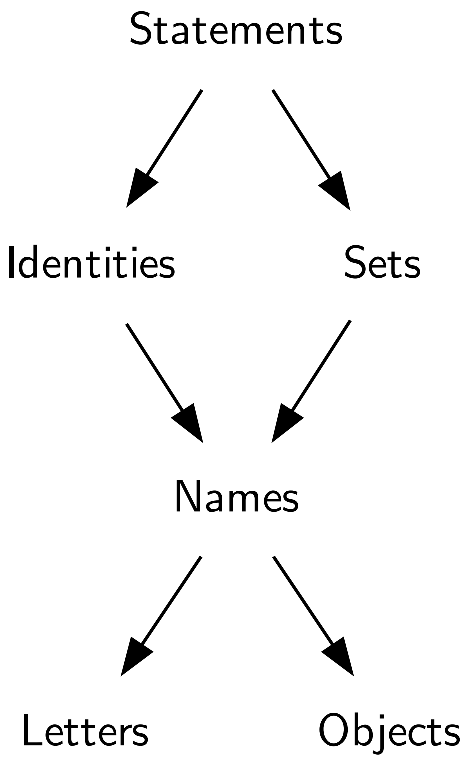
Identities (4)
Sets (5)

Statements (6) is immediately needed by:

Logical Statements (7)

Statements (6) gives the following terms.

statement, relational symbol, name symbol, relational symbol, name symbol, relational symbols, terminal, assertion, membership assertion, identity assertion, primitive sentence, logical form, sentence, belongs to, member.


Statements
Identities
Sets
Names
Letters
Objects

Why

We want symbols for “and”, “or”, “not”, and “implies”.[3]

Overview

We call $=$ and $\in$ *relational symbols.* They say how the objects denoted by a pair of placeholder names relate to each other in the sense of being or belonging. We call $\_ = \_$ and $\_ \in \_$ *simple statements.* They denote simple sentences “the object denoted by _ is the object denoted by _” and “the object denoted by _ belongs to the set denoted by _”. The symbols introduced here are *logical symbols* and statements using them are *logical statements.*

Conjunction

Consider the symbol $\land$. We will agree that it means “and”. If we want to make two simple statements like $a = b$ and $a \in A$ at once, we write $(a = b) \land (a \in A)$. The symbol $\land$ is symmetric, reflecting the fact that a statement like $(a \in A) \land (a = b)$ means the same as $(a = b) \land (a \in A)$.

Disjunction

Consider the symbol $\lor$. We will agree that it means “or” in the sense of either one, the other, or both. If we want to say that at least one of the simple statements like $a = b$ and $a \in A$, we write write $(a = b) \lor (a \in A)$. The symbol $\lor$ is also symmetric, reflecting the fact that a statement like $(a \in A) \lor (a = b)$ means the same as $(a = b) \lor (a \in A)$.

Negation

Consider the symbol $\neg$. We will agree that it means “not”. We will use it to say that one object “is not” another object and one object “does

[3] This sheet does not explain logic. In the next edition there will be several more sheets serving this function.

not belong to" another object. If we want to say the opposite of a simple statement like $a = b$ we will write $\neg(a = b)$. We read it aloud as "not a is b" or (the more desirable) "a is not b". Similarly, $\neg(a \in A)$ we read as "not, the object denoted by a belongs to the set denoted by A". Again, the more desirable pronunciation goes "the object denoted by a does not belong to the set A." For these reasons, we introduce two new symbols $\neq$ and $\notin$. $a \neq b$ means $\neg(a = b)$ and $a \notin A$ means $\neg(a \in A)$.

Implication

Consider the symbol $\longrightarrow$. We will agree that it means "implies". For example $(a \in A) \longrightarrow (a \in B)$ means "the object denoted by a belongs to the object denoted by A implies the object denoted by a belongs to the set denoted by B". It is the same as $(\neg(a \in A)) \vee (a \in B)$. In other words, if $a \in A$, then always $a \in B$. The symbol $\longrightarrow$ is not symmetric, since implication is not symmetric. The symbol $\longleftrightarrow$ means "if and only if".[4]

[4]Future editions will expand.

Logical Statements (7) immediately needs:

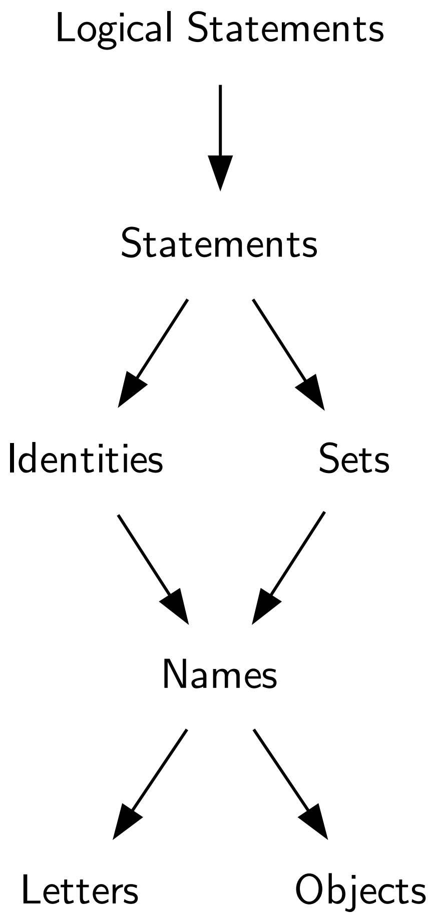
Statements (6)

Logical Statements (7) is immediately needed by:

Quantified Statements (8)

Logical Statements (7) gives the following terms.

relational symbols, simple statements, logical symbols, logical statements.


Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want symbols for talking about the existence of objects and for making statements which hold for all objects.[5]

Definition

If we say there exists an object that is blue, we mean the same as if we say that not every object is not blue. If we say that every object is blue, we mean the same as if we say there does not exist an object that is not blue. In other words, "there exists an object so that _" is the same as "not every object is not _". Or, "every object is _" is the same as "there does not exist an object that is _".

When we assert something of every object we also assert the nonexistence of the contrary of that assertion. And likewise when we assert that an object exists with some conditions, we assert that not every object exists without that condition.

The content of our assertions will be logical statements (see Logical Statements) and when we want to make them for all objects or for no object we will use the following symbols. The symbols introduced here are *quantifier symbols* and statements using them are *quantified statements.*

Existential quantifier

Consider the symbol $\exists$. We agree that it means "there exists an object". We write $(\exists x)(\_)$ and then substitute any logical statement which uses the name x for _ . For example, we write $(\exists x)(x \in A)$ to mean "there exists an object in the set denoted by A."

We call $\exists$ the *existential quantifier* symbol.

[5]This sheet does not explain quantifiers. In the next edition there will be several more sheets serving this function

Universal quantifier

Consider the symbol $\forall$. We agree that it means "for every object". We write $(\forall x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\forall x)((x \in A) \longrightarrow (x \in B)$ to mean, "every object which is in the set denoted by A is in the set denoted by B". We call $\forall$ the *universal quantifier* symbol.

Binding

When we have a name following a $\forall$ or $\exists$ we say that the name is *bound*. If a name is bound, then the statement uses it in one sense but not in another. The name is only used in that single statement. Regular names in statements we call *unbound* or *free*.

Negations

The statement $\neg(\forall x)(\_)$ is the same as $(\exists x)(\neg(\_))$ and $\neg(\exists x)(\_)$ is the same as $(\forall x)(\neg(\_))$.

Quantified Statements (8) immediately needs:

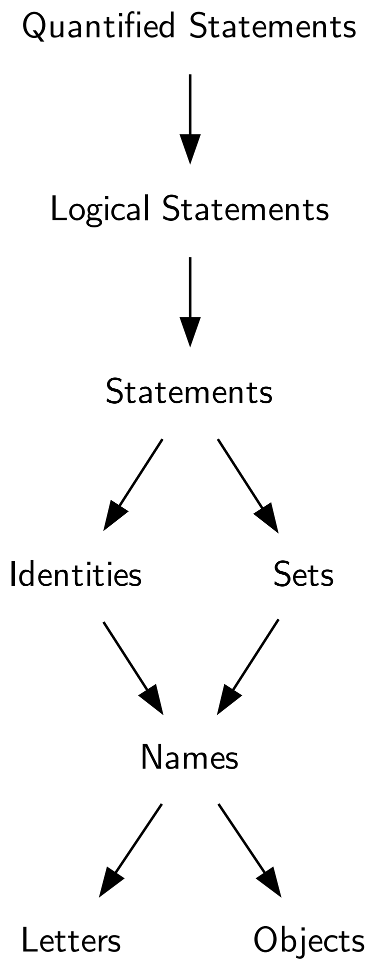
Logical Statements (7)

Quantified Statements (8) is immediately needed by:

Deductions (9)

Quantified Statements (8) gives the following terms.

quantifier symbols, quantified statements, existential quantifier, universal quantifier, bound, unbound, free.


Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to make conclusions.

Discussion

A *conclusion* is a statement that holds necessarily as a consequence of other statements. We have a list of quantified logical statements, and we call them *premisses.* We want to state which other statements hold necessarily if the premisses hold. A sequence of statements, each of which follows from the previous, ending with a *conclusion* is called a *proof* of the conclusion. The process is *deduction.* A *deduction* is a statement which follows necessarily from other premisses.

A *proposition* is another term for a statement. An unproven statement (or premiss) is also called a *principle.* We will often set apart propositions and principles from the text. We bold them and label them with Arabic numerals (see Letters) to enable us to reference them.

Examples

Since principles have no proofs, they will look like

Principle 1. *(Here is where the statement would go).*

Since propositions have proofs, but are used like principles, they will appear stated first, and followed by their proof.

Proposition 1. *(Here is where the statement would go).*

Proof. (Here is the where the account would go). □

Methods of proof

We outline a few of the methods of proof used in this text.

Forward reasoning

If we have as premisses that a statement P implies a statement Q, and we have P, then we have Q. It is common that this reasoning is done in chains. P implies Q, and Q implies R. So if we have P then we have Q and if we have Q then we have R. So in other words, we can also deduce that P implies R.

Contradiction

A contradiction occurs when we can deduce a statement and its opposite from the same premisses. If we can deduce a contradiction when we append to a list of premisses a given premiss we can conclude that the given premiss is false.

Terms

To make propositions and principles easy to state, we will often introduce new terms. Doing so is a process of *definition*. These definitions are abbreviations for more complicated to explain objects or properties of objects. In other words, all definitions are *nominal*, which means that they just name things which are already known to exist. They are made to give us language and to save space. When we are defining a term, we will put it in italics.

Deductions (9) immediately needs:

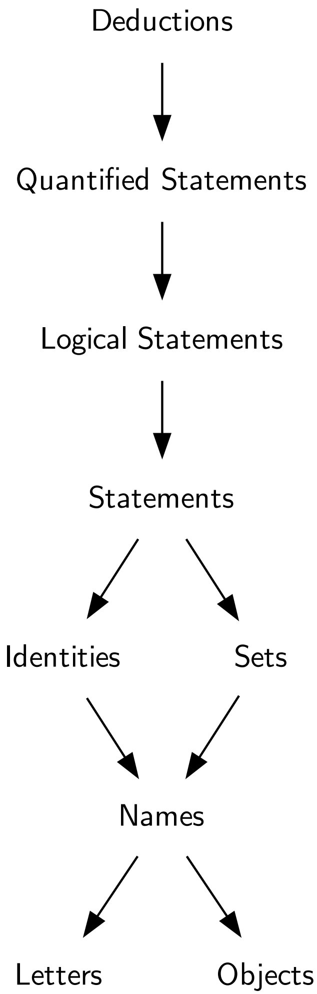
Quantified Statements (8)

Deductions (9) is immediately needed by:

Set Inclusion (10)

Deductions (9) gives the following terms.

conclusion, premisses, conclusion, proof, deduction, deduction, proposition, principle, definition, nominal.


Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to discuss when two sets are the same, and to do so we want to say when all the elements of one set are in another set.

Definition

Denote a set by A and a set by B. If every element of the set denoted by A is an element of the set denoted by B, then we say that the set denoted by A is a *subset* of the set denoted by B.

We say that the set denoted by A is *included* in the set denoted by B. We say that the set denoted by B is a *superset* of the set denoted by A or that the set denoted by B *includes* the set denoted by A.

Every set is included in and includes itself. If the set denoted by B is a subset of the set denoted by A, but B is not A, we call B a *proper subset* of A.

Notation

Let A denote a set and B denote a set. We denote that the set A is included in the set B by $A \subset B$. In other words, $A \subset B$ means $(\forall x)((x \in A) \longrightarrow (x \in B))$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A". $B \supset A$ also means $(\forall x)((x \in A) \longrightarrow (x \in B))$.

Some authors use the notation $\subseteq$ for $\subset$, and use $B \subsetneq A$ to indicate that the set denoted by B is a *proper* subset of the set denoted by A.

Properties

There are some properties that our intuition suggests inclusion should have. First, every set should include itself. We describe this fact by saying that inclusion is *reflexive*.

Proposition 2 (Reflexive). *Every set is included in itself.*

Proof. Suppose A is a set. Then we have $(\forall x)(x \in A \longrightarrow x \in A)$ In other words, $A \subset A$. $\square$

Next, we expect that if one set is included in another, This fact is described by saying that inclusion is *transitive*

Proposition 3 (Transitive)**.** *If a set is included in another, and the latter in yet another, then the first is included in the last.*

Proof. Suppose A, B, C are sets. If $A \subset B$ and $B \subset C$ Thus $A \subset C$ by modus ponens. $\square$

Equality $(=)$ shares these two properties. Let A denote an object. Then $A = A$. Let B and C also denote objects. If $A = B$ and $B = C$, then $A = C$. Of course, inclusion is not symmetric.. Belonging $(\in)$ may be, but need not be reflexive and transitive.

Set Inclusion (10) immediately needs:

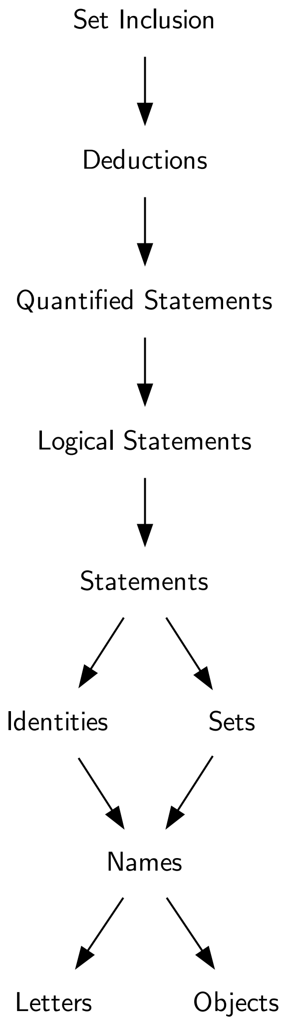
Deductions (9)

Set Inclusion (10) is immediately needed by:

Set Equality (11)

Set Inclusion (10) gives the following terms.

subset, included, superset, includes, proper subset, improper subsets, proper subsets, reflexive, transitive.


Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

When are two sets the same?

Definition

Let A and B denote sets. If $A = B$ then every element of A is an element of B and every element of B is an element of A. In other words, $(A = B) \longrightarrow ((A \subset B) \wedge (B \subset A))$.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Then $A = B$? We define it to be so. In other words, sets are *determined* by their members.

Principle 2 (Extension)**.** *Two sets are the same (or* equal*) if every member of one is a member of the other and vice versa.*

In other words, two sets are identical if and only if every element of one is an element of the other. This principle is sometimes called the *principle of extension* (or *axiom of extension*). We refer to the elements of a set as its *extension.* Roughly speaking, this principle states that we know the extension of a set, then we know the set. A set is *determined* by its extension.

Deductive principle

We can use this definition to deduce $A = B$ if we first deduce $A \subset B$ and $B \subset A$. With these two implications, we use the principle of extension to conclude that the sets are the same. In other words, $(A = B) \longleftrightarrow ((A \subset B) \wedge (B \subset A))$. We also describe this fact by saying that inclusion ($\subset$) is *antisymmetric.*

Belonging and sets compared with ancestry and humans

Compare the principle of extension for identifying sets from their elements with an analogous principle for identifying people from their ancestors.

We can consider a person's ancestors. Namely, the person's parents, grandparents, great grandparents and so on. It is clear that if we label the same human with two names A and B, then A and B have the same ancestors. In other words, same human implies same ancestors. This is the analog of "if two sets are equal they have the same members".

On the other hand, if we have two people denoted by A and B, and we know that A has the same ancestors as B, we can not conclude that A and B denote the same human. For example, siblings have the same ancestors but are different people. This direction, same ancestors implies same human, is the analogue of "if they have the same elements, two sets are the same". It is false for humans and ancestors, but we define it to be true for sets and members.

The principle of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Set Equality (11) immediately needs:

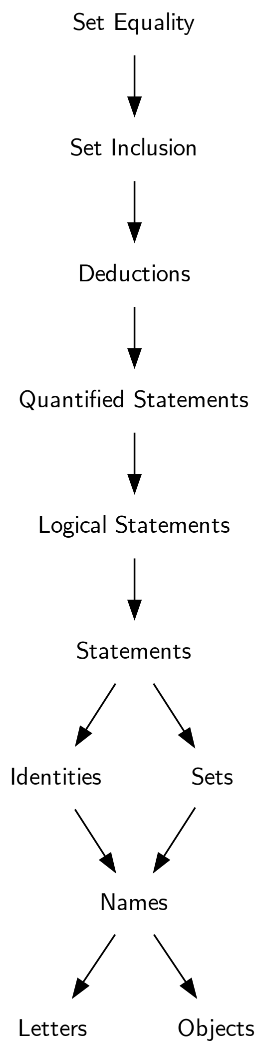
Set Inclusion (10)

Set Equality (11) is immediately needed by:

Set Specification (12)

Set Equality (11) gives the following terms.

equal, principle of extension, axiom of extension, extension, antisymmetric.


Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to construct new sets out of old ones. So, can we always construct subsets?

Definition

We will say that we can. More specifically, if we have a set and some statement which may be true or false for the elements of that set, a set exists containing all and only the elements for which the statement is true.

Roughly speaking, the principle is like this. We have a set which contains some objects. Suppose the set of playing cards in a usual deck exists. We are taking as a principle that the set of all fives exists, so does the set of all fours, as does the set of all hearts, and the set of all face cards. Roughly, the corresponding statements are "it is a five", "it is a four", "it is a heart", and "it is a face card".

Principle 3 (Specification)**.** *For any statement and any set, there is a subset whose elements satisfy the statement.*

We call this the *principle of specification.* We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence. The principle of extensionsays that this set is unique. All our basic principles about sets (other than the principle of extension) assert that we can construct new sets out of old ones in reasonable ways.

Notation

Let A denote a set. Let s denote a statement in which the symbol x and A appear unbound. We assert that there is a set, denote it by B, for which belonging is equivalent to membership in A and satisfaction of s. In other words,

$$(\forall x)((x \in B) \longleftrightarrow ((x \in A) \land s(x))).$$

We denote B by $\{x \in A \mid s(x)\}$. We read the symbol $\mid$ aloud as "such that." We read the whole notation aloud as "a in A such that..." We call it *set-builder notation.*

Nothing contains everything

As an example of the principle of specification and an important consequence, consider the statement $x \notin x$. Using this statement and the principle of specification, we can prove that there is no set which contains every other set.

Proposition 4. *No set contains all sets.*[6]

Proof. Suppose there exists a set, denote it A which contains all sets. In other words, suppose $(\exists A)(\forall x)(x \in A)$. Use the principle of specification to construct $B = \{x \in A \mid x \notin x\}$. So $(\forall x)(x \in B \longleftrightarrow (x \in A \land x \notin x))$ In particular, $(B \in B \longleftrightarrow (B \in A \land B \notin B))$. So $B \notin A$. $\square$

[6]We might call such a set, if we admitted its existence, a *universe of discourse* or *universal set*. With the principle of specification, a "principle of a universal set" would give a contradiction (called *Russell's paradox*).

Set Specification (12) immediately needs:

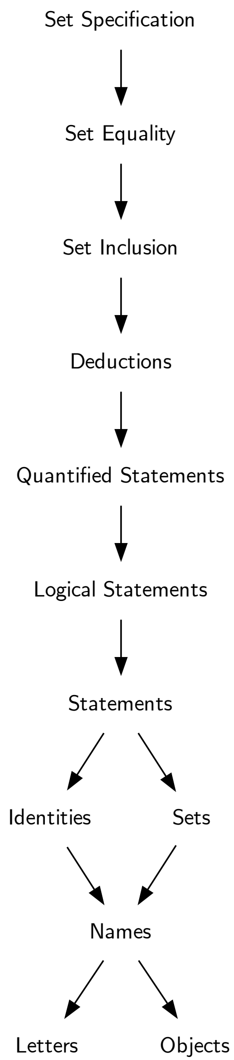
Set Equality (11)

Set Specification (12) is immediately needed by:

Empty Set (13)
Pair Intersections (18)
Set Differences (24)
Unordered Pairs (14)

Set Specification (12) gives the following terms.

principle of specification, specifying, set-builder notation, universe of discourse, universal set, Russell's paradox.


Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Can a set have no elements?

Definition

Sure. A set exists by the principle of existence (see Sets); denote it by A. Specify elements (see Set Specification) of any set that exists using the universally false statement $x \neq x$. We denote that set by $\{x \in A \mid x \neq x\}$. It has no elements. In other words, $(\forall x)(x \notin A)$. The principle of extension (see Set Equality) says that the set obtained is unique (contradiction).[7] We call the unique set with no elements the *empty set*. If a set is not the empty set, we call it *nonempty*.

Notation

We denote the empty set by $\varnothing$.

Properties

It is immediate from our definition of the empty set and of the definition of inclusion (see Set Inclusion) that the empty set is included in every set (including itself).

Proposition 5. $(\forall A)(\varnothing \subset A)$

Proof. Suppose toward contradiction that $\varnothing \not\subset A$. Then there exists $y \in \varnothing$ such that $y \notin A$. But this is impossible, since $(\forall x)(x \notin \varnothing)$. □

[7] This account will be expanded in the next edition.

Empty Set (13) immediately needs:

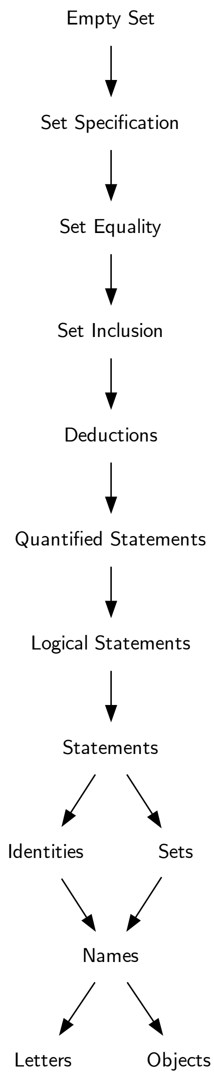
Set Specification (12)

Empty Set (13) is immediately needed by:

Set Complements (25)
Set Intersections (19)
Set Unions (15)

Empty Set (13) gives the following terms.

empty set, nonempty.


Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Can we always make a set out of two objects?

Definition

We say yes.

Principle 4 (Pairing)**.** *Given two objects, there exists a set containing them.*

We refer to this as the *principle of pairing*. Denote one object by a and the other by b. This principle gives us the existence of a set that contains the objects. The principle of specification (see Set Specification) gives use the subset for the statement "$x = a \lor x = b$". The principle of extension (see Set Equality) says this set is unique. We call this set a *pair* or an *unordered pair*.

If the object denoted by a is the object denoted by b, then we call the pair the *singleton* of the object denoted by a. Every element of the singleton of the object denoted by a is a.

In other words, the principle of pairing says that every object is an element of some set. That set may be the singleton, or it may be the pair with any other object. We can construct several sets using this principle: the singleton of the object denoted by a, the singleton of the singleton of the object denoted by a, the singleton of the singleton of the singleton of the object denoted by a, and so on.

Notation

We denote the set which contains a and b as elements and nothing else by $\{a, b\}$. The pair of a with itself is the set $\{a, a\}$ is the singleton of a. We denote it by $\{x\}$. The principle of pairing also says that $\{\{a\}\}$ exists and $\{\{\{a\}\}\}$ exists, as well as $\{a, \{a\}\}$.

Note well that $a \neq \{a\}$. a denotes the object a. $\{a\}$ denotes the set whose only element is a. In other words $(\forall x)(x \in \{a\} \longleftrightarrow x = a)$. The moral is that a sack with a potato is not the same thing as a potato.

Unordered Pairs (14) immediately needs:

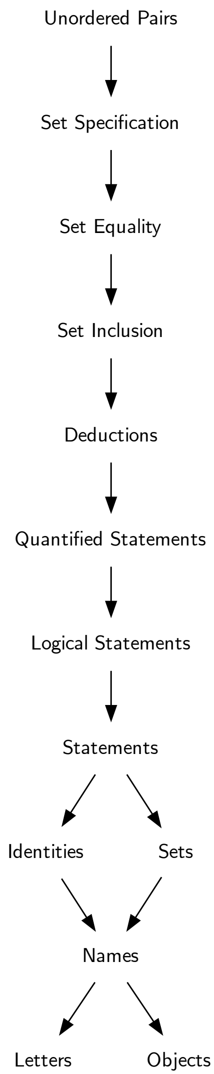
Set Specification (12)

Unordered Pairs (14) is immediately needed by:

Ordered Pairs (35)
Set Unions (15)

Unordered Pairs (14) gives the following terms.

principle of pairing, pair, unordered pair, singleton.


Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Can we combine sets?

Definition

We say yes. For example, if we have a first set denoted A and a second set denoted B, then we want a third set including all the elements of the set denoted by A and the elements of the set denoted by B. If an object appears in the set denoted by A and in the set denoted by B, it appears in the new set. If an object appears in one set but not the other, it appears in the new set. Indeed, if we have a set of sets, the same should hold.

Principle 5 (Union)**.** *Given a set of sets, there exists a set which contains all elements which belong to any of the sets.*

We call this the *principle of union.* If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the principle of union may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification (see Set Specification) to form the set which contains only those elements which are appear in at least one of any of the sets. The set is unique by the principle of extension. We call that unique set *the union* of the sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\bigcup \mathcal{A}$. So

$$(\forall x)((x \in (\cup \mathcal{A})) \longleftrightarrow (\exists A)((A \in \mathcal{A}) \land x \in A)).$$

Simple facts

It is reasonable for the union of the empty set to be empty and for the union of the singleton of a set to be itself.[8]

Proposition 6. $\cup\varnothing = \varnothing$

Proposition 7. $\cup\{A\} = A$

[8]Future editions will include the account.

Set Unions (15) immediately needs:

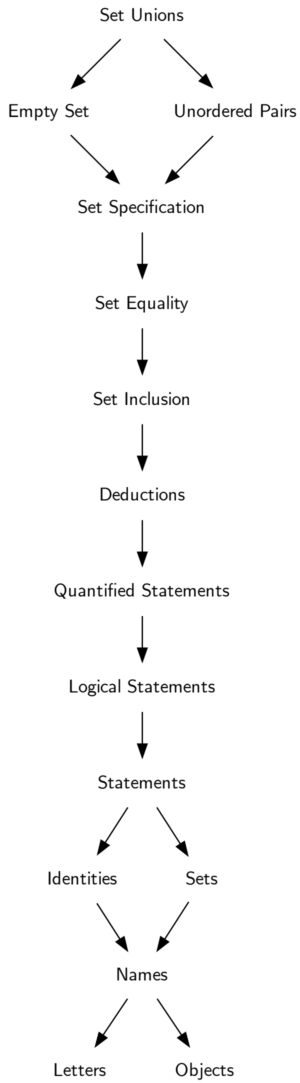
Empty Set (13)
Unordered Pairs (14)

Set Unions (15) is immediately needed by:

Ordered Pair Projections (38)
Pair Unions (16)
Partitions (26)
Set Rings (**??**)
Set Symmetric Differences (29)

Set Unions (15) gives the following terms.

principle of union, the union.


Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We often unite the elements of one set with another.

Discussion

Let A and B denote sets. We call $\cup\{A, B\}$ the *pair union* of A and B. We denote the union of the pair $\{A, B\}$ by $A \cup B$. Clearly the pair union does not depend on the order of A and B. In other words, $A \cup B = B \cup A$.

Facts

Here are some basic facts about unions of a pair of sets.[9] Let A and B denote sets.

Proposition 8 (Identity Element)**.** $A \cup \varnothing = A$

Proposition 9 (Commutativity)**.** $A \cup B = B \cup A$

Proposition 10 (Commutativity)**.** $(A \cup B) \cup C = A \cup (B \cup C)$

Proposition 11 (Idempotence)**.** $A \cup A = A.$

Proposition 12. $A \subset B \longleftrightarrow A \cup B = B$

[9]Proofs will appear in the next edition.

Pair Unions (16) immediately needs:

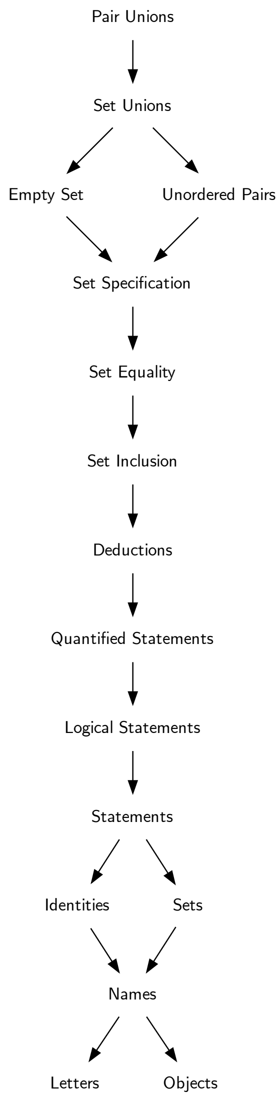
Set Unions (15)

Pair Unions (16) is immediately needed by:

Intersection of Empty Set (20)
Set Dualities (27)
Set Unions and Intersections (21)
Successor Sets (55)
Uncertain Outcomes (**??**)
Unordered Triples (17)
Venn Diagrams (23)

Pair Unions (16) gives the following terms.

pair union.


Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

$\{a\} \cup \{b\} = \{a, b\}$

Definition

Let a, b and c denote objects. From the associativity of pair unions (see Pair Unions), we have

$$(\{a\} \cup \{b\}) \cup \{b\} = \{a\} \cup (\{b\} \cup \{c\}).$$

So we will drop the parentheses, and write $\{a\} \cup \{b\} \cup \{c\}$. We call such a set the *unordered triple* of a, b and c. The unordered triple of a, b and c is the set containing these elements and no others.

Notation

Such sets are so commonplace that we denote the unordered triple of a, b and c by $\{a, b, c\}$.

Quadruples

Let d denote an object. Again, the associativity of pair unions allows us to drop the parentheses from

$$(((\{a\} \cup \{b\}) \cup \{c\}) \cup \{d\})).$$

We can therefore write $\{a\} \cup \{b\} \cup \{c\} \cup \{d\}$ without ambiguity. We call this set the *unordered quadruple.* As before, the unordered quadruple contains of a, b, c and d contains a, b, c, and d and nothing besides these.

Notation

We denote the unordered quadruple of the objected denoted by a, b, c and d, denote this set by $\{a, b, c, d\}$.

The case of several named objects

In a similar way we speak of *unordered pentuples*, *unordered sextuples*, *unordered septuples* and so on. If we have several objects named, we denote the set containing these objects be writing their names in between the left brace { and right brace}, separating the names by commas. For example, if we A, b, x and Y and z denote objects, then we denote the set containing these elements by

$$\{A, b, x, Y, z\}.$$

Unordered Triples (17) immediately needs:

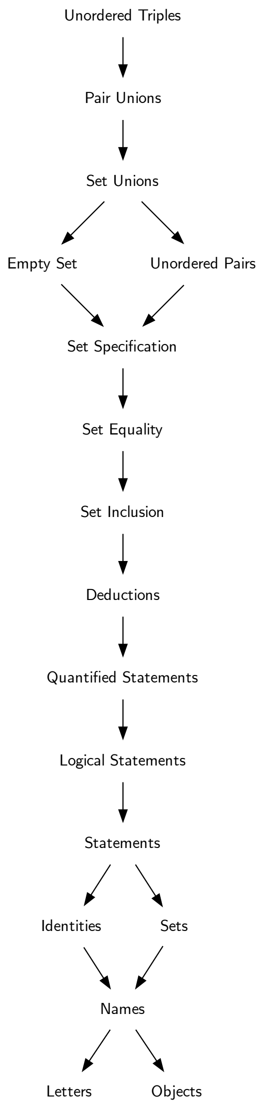
Pair Unions (16)

Unordered Triples (17) is immediately needed by:

Ordering Sets (34)
Set Powers (30)

Unordered Triples (17) gives the following terms.

unordered triple, unordered quadruple, unordered pentuples, unordered sextuples, unordered septuples.


Unordered Triples
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Does a set exist containing the elements shared between two sets? How might we construct such a set?

Definition

Let A and B denote sets. Consider the set $\{x \in A \mid x \in B\}$. This set exists by the principle of specification (see Set Specification). Moreover $(y \in \{x \in A \mid x \in B\}) \longleftrightarrow (y \in A \land y \in B)$. In other words, $\{x \in A \mid x \in B\}$ contains all the elements of A that are also elements of B.

We can also consider $\{x \in B \mid x \in A\}$, in which we have swapped the positions of A and B. Similarly, the set exists by the principle of specification (see Set Specification) and again $y \in \{x \in B \mid x \in A\} \longleftrightarrow (y \in B \land y \in B)$. Of course, $y \in A \land y \in B$ means the same as[10] $y \in B \land y \in A$ and so by the principle of extension (see Set Equality)

$$\{x \in A \mid x \in B\} = \{x \in B \mid x \in A\}.$$

We call this set the *pair intersection* of the set denoted by A with the set denoted by B.

Notation

We denote the intersection fo the set denoted by A with the set denoted by B by $A \cap B$. We read this notation aloud as "A intersect B".

Basic properties

All the following results are immediate.[11]

Proposition 13. $A \cap \varnothing = \varnothing$

Proposition 14 (Commutativity)**.** $A \cap B = B \cap A$

[10] Future editions will name and cite this rule.

[11] Proofs of these results will appear in the next edition.

Proposition 15 (Associativity)**.** $(A \cap B) \cap C = A \cap (B \cap C)$

Proposition 16. $A \cap A = A$

Proposition 17. $(A \subset B) \longleftrightarrow (A \cap B = A)$.

Pair Intersections (18) immediately needs:

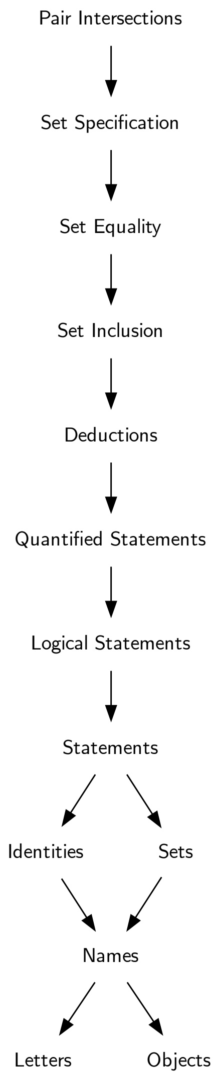
Set Specification (12)

Pair Intersections (18) is immediately needed by:

Operations (72)
Set Dualities (27)
Set Intersections (19)
Set Unions and Intersections (21)
Uncertain Outcomes (**??**)
Venn Diagrams (23)

Pair Intersections (18) gives the following terms.

pair intersection.


Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We can consider intersections of more than two sets.

Definition

Let $\mathcal{A}$ denote a set of sets. In other words, every element of $\mathcal{A}$ is a set. And suppose that $\mathcal{A}$ has at least one set (i.e., $\mathcal{A} \neq \varnothing$). Let C denote a set such that $C \in \mathcal{A}$. Then consider the set,

$$\{x \in C \mid (\forall A)(A \in \mathcal{A} \longrightarrow x \in A)\}.$$

This set exists by the principle of specification (see Set Specification). Moreover, the set does not depend on which set we picked. So the dependence on C does not matter. It is unique by the axiom of extension (see Set Equality). This set is called the *intersection* of $\mathcal{A}$.

Notation

We denote the intersection of $\mathcal{A}$ by $\bigcap \mathcal{A}$.

Equivalence with pair intersections

As desired, the the set denoted by $\mathcal{A}$ is a pair (see Unordered Pairs) of sets, the pair intersection (see Pair Intersections) coincides with intersection as we have defined it in this sheet.[12]

Proposition 18. $\bigcap\{A, B\} = A \cap B$

[12] A full account of the proof will appear in future editions.

Set Intersections (19) immediately needs:

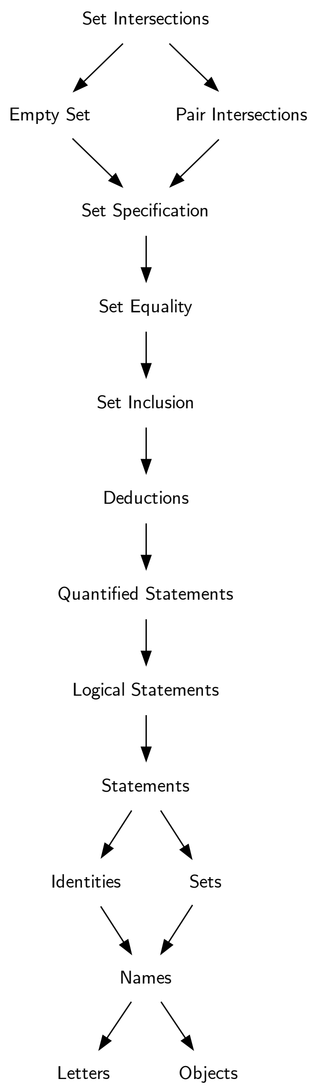
Empty Set (13)
Pair Intersections (18)

Set Intersections (19) is immediately needed by:

Intersection of Empty Set (20)
Partitions (26)
Powers and Intersections (31)

Set Intersections (19) gives the following terms.

intersection.



Set Intersections
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We only define set intersections for nonempty sets of sets. Why?

Discussion

Which objects are specified by the sentence $(\forall x \in \varnothing)(x \in X)$? Well, since no objects fail to satisfy the statement,[13] the sentence specifies all objects. So in other words, the condition we used to define set intersections (see Set Intersections) specifies the "set of everything." In order to maintain other more desirable set principles like selection, we have said that such a set does not exist (see Set Specification).

If, however, all sets under consideration are subsets of one paticular set—denote it E—then we can define intersections as follows. Let $\mathcal{C}$ be a possibly nonempty collection of sets

$$\bigcap \mathcal{C} = \{X \in E \mid (\forall X \in \mathcal{C})(x \in X)\}.$$

This definition agrees with that given in Set Intersections. In particular, it is the intersection of the set $\mathcal{C} \cup \{E\}$

Another definition

This begs the following question. Why not define intersections by selecting from the union. Let $\mathcal{A}$ be a possibly nonempty set of sets. Then define:

$$\bigcap \mathcal{A} = \{x \in \bigcup \mathcal{A} \mid (\forall A \in \mathcal{A})(x \in A)\}.$$

If $\mathcal{A}$ is empty, so is $\bigcup \mathcal{A}$ and then there are no elements in the set to select from so $\bigcap \mathcal{A}$ is empty. This does not agree with the previous definitions for the empty set, but does for all other sets of sets.

For these reasons, the intersection of the empty set is delicate.[14]

[13]Future editions will offer an account of this.

[14]Future editions will expand on the preference for the former definition.

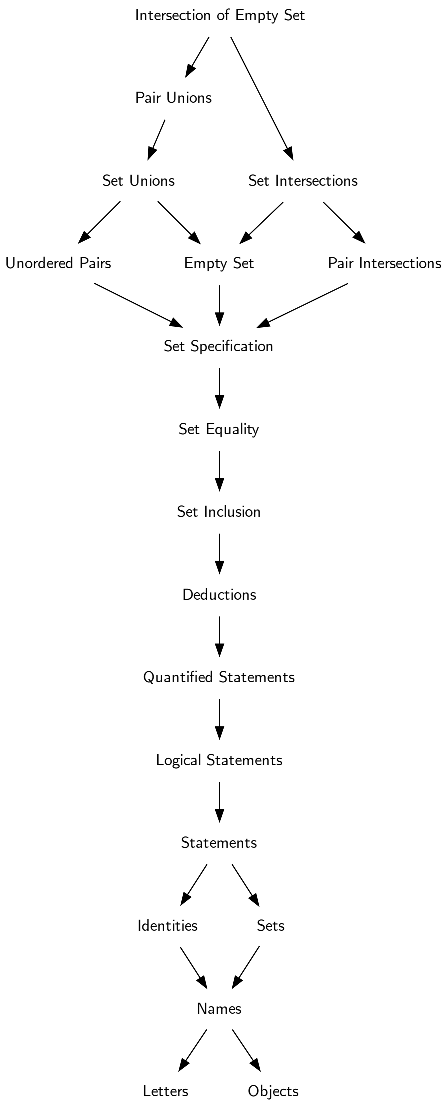
Intersection of Empty Set (20) immediately needs:

Pair Unions (16)
Set Intersections (19)

Intersection of Empty Set (20) is immediately needed by:

Generalized Set Dualities (33)
Natural Numbers (56)

Intersection of Empty Set (20) gives no terms.



Intersection of Empty Set
Pair Unions
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We study how intersection and union interact.

Results

The following are easy results.[15] They are known as the *distributive laws*.

Proposition 19. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proposition 20. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

[15] The accounts will appear in future editions.

Set Unions and Intersections (21) immediately needs:

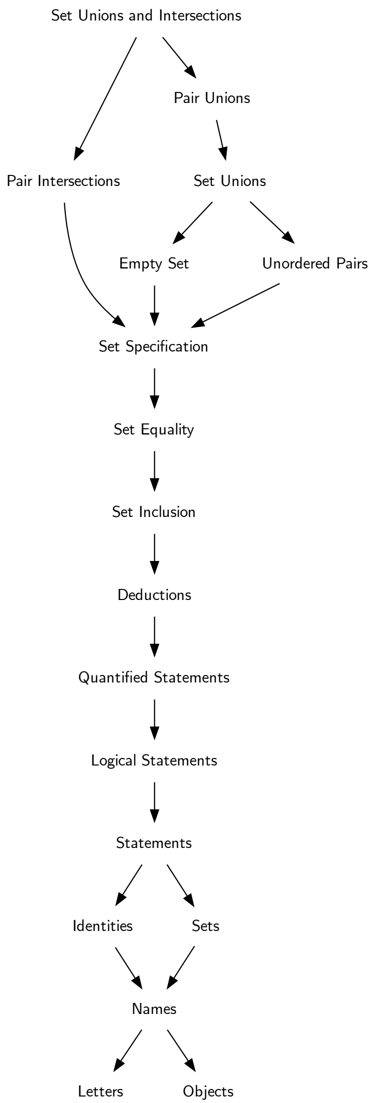
Pair Intersections (18)
Pair Unions (16)

Set Unions and Intersections (21) is immediately needed by:

Family Unions and Intersections (46)
Generalized Inclusion-Exclusion Formula (**??**)

Set Unions and Intersections (21) gives the following terms.

distributive laws.


Set Unions and Intersections
Pair Unions
Pair Intersections
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We need some basic geometric concepts.[16]

Definitions

A *point* is that which has no part.[17] A *line* is a breadthless length. The *extremities of a line*[18] are points. A *straight line* is a line which lies evenly with the points on itself. A *surface* is that which has length and breadth only. The *extremities of a surface* are lines.

A *plane surface* is a surface which lies evenly with the straight lines on itself. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line. And when teh lines containing the angle are straight, the angle is called *rectilineal.* When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other side is called a *perpendicular* to that on which it stands.

A *boundary* is that which is an extremity of any thing. A *figure* is that which is contained by any boundary or boundaries. A *circle* is a plane figure contained by one line such that allthe straight lines falling upon it from one point among those lying withing the figure are equal to one another. The point is called the *center* of the circle. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.[19]

[16]This sheet will be expanded into several in future editions.

[17]This and all that follows is taken (nearly) verbatim from Heath's translation of Book I of Euclid's Elements. In future editions, there will be a reference to the Litterae manuscript of this text.

[18]We have departed from Heath and made extremity here a term.

[19]We end here. Of course, Euclid goes on to discuss semicircles, rectilineal figures, etc.

Geometry (22) immediately needs:

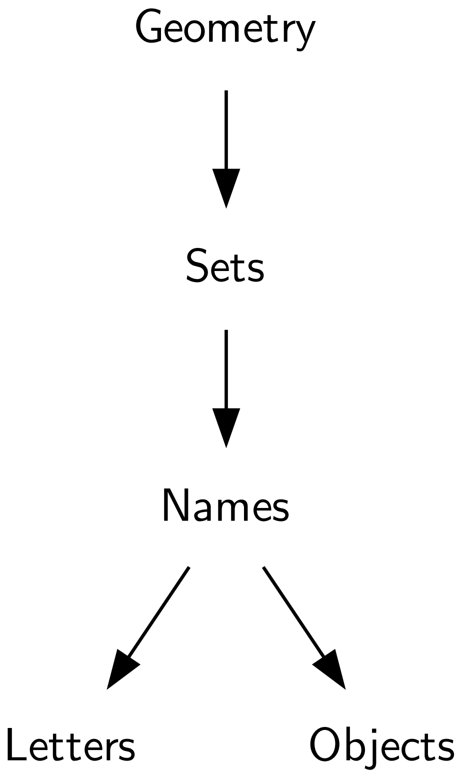
Sets (5)

Geometry (22) is immediately needed by:

Area (**??**)
Integral Line (82)
Real Convex Hulls (**??**)
Real Plane (116)
Real Space (118)
Right Triangle Sides Relation (**??**)
Squares (**??**)
Venn Diagrams (23)

Geometry (22) gives the following terms.

point, line, extremities of a line, straight line, surface, extremities of a surface, plane surface, plane angle, rectilineal, right, perpendicular, boundary, figure, circle, center, diameter.


Geometry
Sets
Names
Letters
Objects

Why

We want to visualize the operations of union and intersection.

Discussion

A Venn diagram is several (possibly overlapping) plane figures.[20]

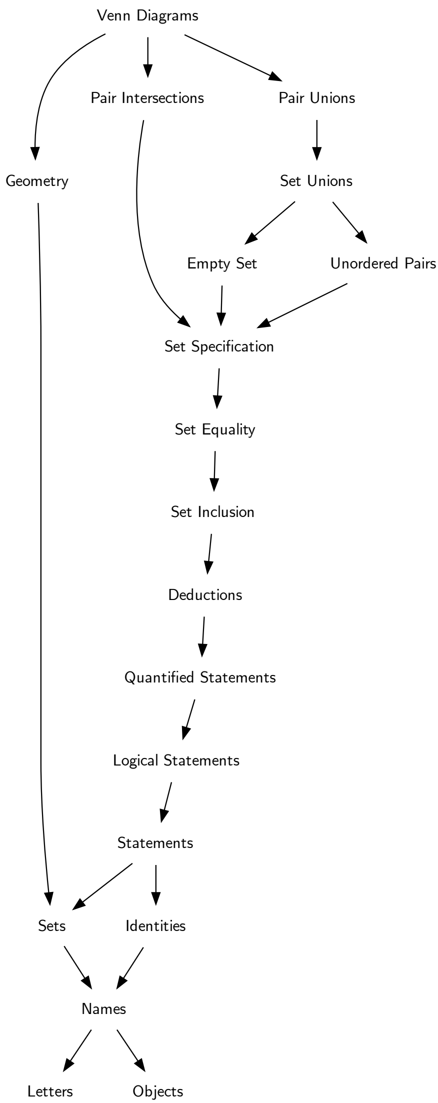
[20]Future editions will include the highly desirable illustrative figures.

Venn Diagrams (23) immediately needs:

Geometry (22)
Pair Intersections (18)
Pair Unions (16)

Venn Diagrams (23) is not immediately needed by any sheet.

Venn Diagrams (23) gives no terms.


Venn Diagrams
Pair Intersections
Pair Unions
Geometry
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We consider elements of one set which are not contained in another set.

Definition

Let A and B denote sets. The *difference* between A and B is the set $\{x \in A \mid x \notin B\}$. In other words, the difference between A and B is the set of all points of A which do not belong to B.

It is not necessary that $B \subset A$; the difference is called *proper* if $A \supset B$. This terminology is from that of proper subsets.

Notation

We denote the difference between A and B by $A - B$. Some authors use $-$ or $\sim$, but we will avoid this.

Properties

The following are straightforward.[21]

Proposition 21. $A - \varnothing = A$

Proposition 22. $A - A = \varnothing$

[21] Accounts will appear in future editions.

Set Differences (24) immediately needs:

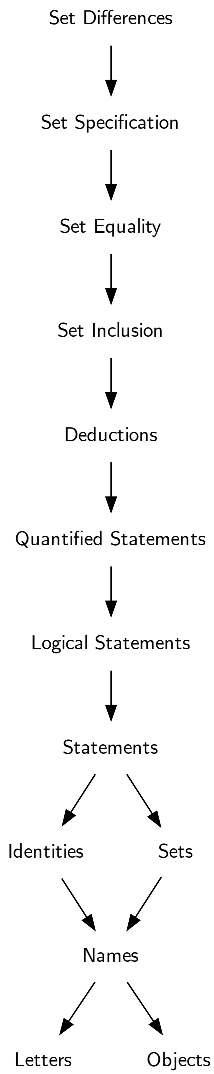
Set Specification (12)

Set Differences (24) is immediately needed by:

Natural Numbers (56)
Set Complements (25)
Set Rings (**??**)
Vertex Separators (**??**)

Set Differences (24) gives the following terms.

difference, proper.



Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

It is often the case in considering set differences that all sets considered are subsets of one set.

Definition

Let A and B denote sets. In many cases, we take the difference between a set and one contained in it. In other words, we assume that $B \subset A$. In this case, we often take complements relative to the same set A. So we do not refer to it, and instead refer to the relative complement of B in A as the *complement* of B.

Notation

Let A denote a set, and let B denote a set for which $B \subset A$. We denote the relative complement of B in A by $C_A(B)$. When we need not mention the set A, and instead speak of the complement of B without qualification, we denote this complement by $C(B)$.

Complement of a complement

One nice property of a complement when $B \subset A$ is:

Proposition 23. $(B \subset A) \longleftrightarrow (C_A(C_A(B)) = B)$

Basic facts

Let E denote a set and let A and B denote sets satisfying $A, B \subset E$. Then take all complements with respect to E. Here are some immediate consequences of the definition.[22]

Proposition 24. $C(C(A)) = A$

Proposition 25. $C(\varnothing) = E$

Proposition 26. $C(E) = \varnothing$

[22] Future editions will include accounts.

Proposition 27. $A \subset B \longleftrightarrow C(B) \subset C(A)$

Set Complements (25) immediately needs:

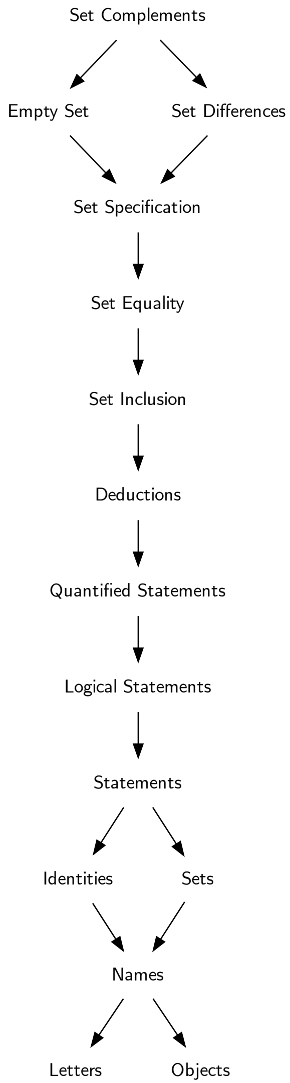
Empty Set (13)
Set Differences (24)

Set Complements (25) is immediately needed by:

Set Dualities (27)
Set Symmetric Differences (29)

Set Complements (25) gives the following terms.

complement.


Set Complements
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We divide a set into disjoint subsets whose union is the whole set. In this way we can handle each subset of the main set individually, and so handle the entire set piece by piece.

Decomposing a set

Two sets A and B *divide* a set X if $A \cup B = X$ and $A \cap B = \varnothing$. Although every element is in either A or B, no element is in both.

If $\mathcal{A}$ is a set of sets, and $A, B \in \mathcal{A}$, then $\mathcal{A}$ is *pairwise disjoint* if $A \cap B = \varnothing$ whenever $A \neq B$.

Definition

A *partition* (or *decomposition*, *set partition*) of a set X is a set of *nonempty*, *pairwise disjoint*, subsets of X whose union is X. We call the elements of a partition the *parts* (or *pieces*, *blocks*, *cells*) of the partition.

When speaking of a partition, we commonly call the set of sets *mutually exclusive* (or *non-overlapping*), by which we mean that they are pairwise disjoint, and *collectively exhaustive*, by which we mean that their union is full set.[23]

Other terminology

Occasionally, the term *unlabeled partition* is used, and the term *partition* is reserved for a separate concept. In this case, the term *allocation* is sometimes used as an abbreviation for unlabeled partition.

[23]Future editions will include diagrams.

Partitions (26) immediately needs:

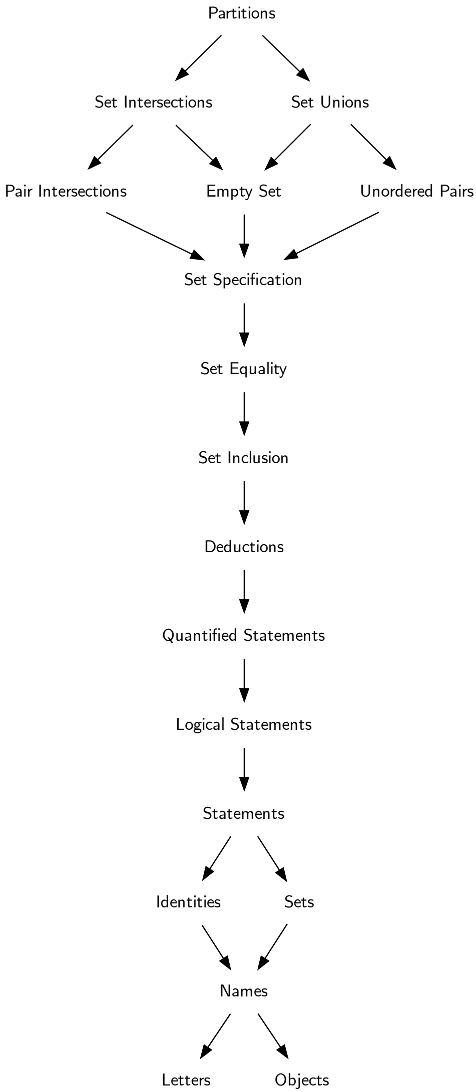
Set Intersections (19)
Set Unions (15)

Partitions (26) is immediately needed by:

Contingency Tables (**??**)
Equivalence Relations (40)
Integer Partitions (**??**)
Marked Graphs (**??**)
Number Partitions (**??**)
Numbered Partitions (**??**)
Set Exercises (28)
Split Graphs (**??**)

Partitions (26) gives the following terms.

divide, pairwise disjoint, partition, decomposition, set partition, parts, pieces, blocks, cells, mutually exclusive, non-overlapping, collectively exhaustive, unlabeled partition, allocation.


Partitions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How does taking complements relate to forming unions and intersections.

Complements of unions or intersections

Let E denote a set. Let A and B denote sets and $A, B \subset E$. All complements are taken with respect to E. The following are known as *DeMorgan's Laws.*[24]

Proposition 28. $C(A \cup B) = C(A) \cap C(B)$

Proposition 29. $C(A \cap B) = C(A) \cup C(B)$

Principle of duality

As a result of DeMorgan's Laws[25] and basic facts about complements (see Set Complements) theorems about sets often come in pairs. In other words, given an inclusion or identity relation involving complements, unions and intersections of some set (above E) if we replace all sets by their complemnets, swap unions and intersections, and flip all inclusions we obtain another, true, result. The correspondence is called the *principle of duality for sets.*

[24] Proofs will appear in a future edition.

[25] Future editions will change the name to remove the reference to DeMorgan in accordance with the project's policy on naming.

Set Dualities (27) immediately needs:

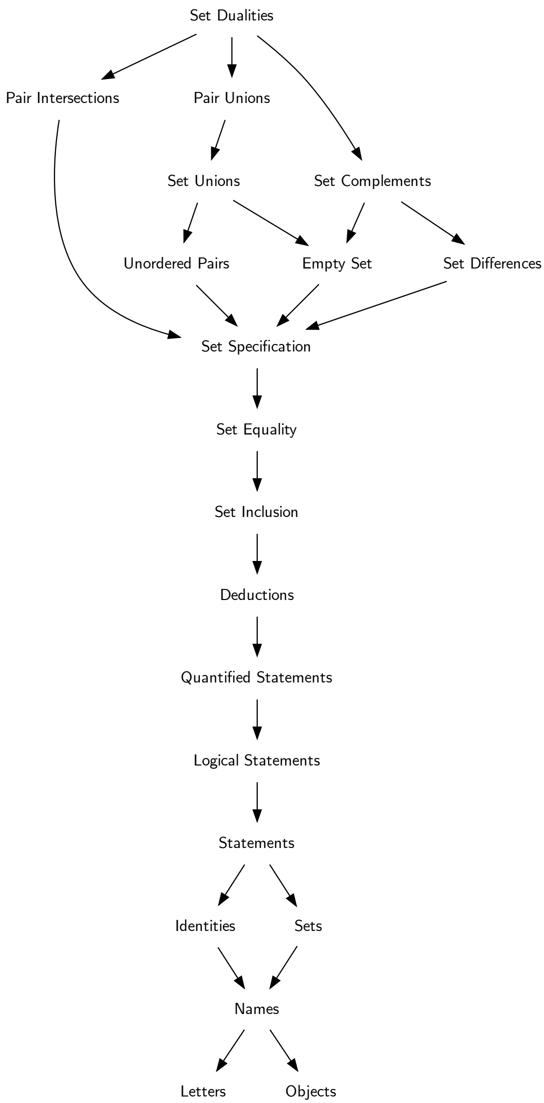
Pair Intersections (18)
Pair Unions (16)
Set Complements (25)

Set Dualities (27) is immediately needed by:

Generalized Set Dualities (33)
Set Exercises (28)

Set Dualities (27) gives the following terms.

DeMorgan's Laws, principle of duality for sets.


Set Dualities
Pair Intersections
Pair Unions
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Here are some exercises on sets.[26]

Exercise 1. *Let A, B, C denote sets. Show $((A \cap B) \cup C = A \cap (B \cup C)) \longleftrightarrow (C \subset A)$ Observe that the condition does not involve B.*

Exercise 2.

$$A - B = A \cap B'.$$

Exercise 3.

$$A \subset B \textit{ if and only if} A - B = \varnothing.$$

Exercise 4.

$$A - (A - B) = A \cap B.$$

Exercise 5.

$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

Exercise 6.

$$(A \cap B) \subset ((A \cap C) \cup (A \cap C')).$$

Exercise 7.

$$((A \cup C) \cap (B \cup C')) \subset (A \cup B).$$

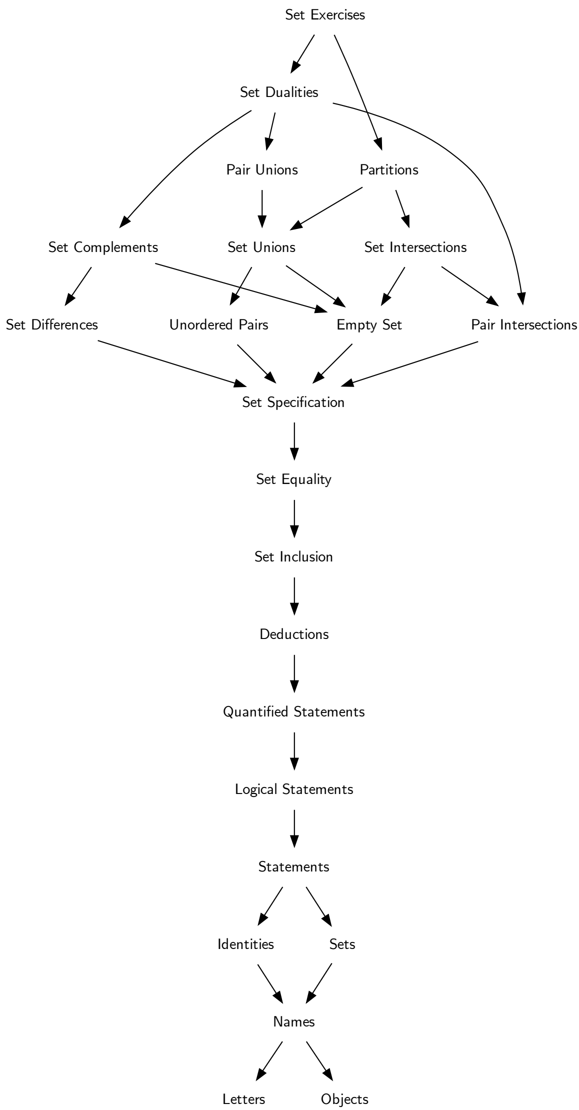
[26] Future editions will give the hypotheses more clearly.

Set Exercises (28) immediately needs:

Partitions (26)
Set Dualities (27)

Set Exercises (28) is not immediately needed by any sheet.

Set Exercises (28) gives no terms.



Set Exercises
Set Dualities
Pair Unions
Partitions
Set Complements
Set Unions
Set Intersections
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to consider the non-overlapping elements of a pair of sets.

Definition

In other words, we want to consider the set of elements which is one or the other but not in both. The *symmetric difference* (or *Boolean sum*) of a set with another set is the union of the difference between the latter set and the former set and the difference between the former and the latter.

Notation

Let A and B denote sets. We denote the symmetric difference by $A + B$, so that

$$A + B = (A - B) \cup (B - A)$$

Properties

Here are some immediate properties of symmetric differences.[27]

Proposition 30 (Commutative)**.** $A + B = B + A$.

Proposition 31 (Associative)**.** $(A + B) + C = A + (B + C)$.

Proposition 32 (Identity)**.** $(A + \varnothing) = A$

Proposition 33 (Inverse)**.** $(A + A) = \varnothing$

[27] Future editions will have more detailed (but obvious) hypotheses stated.

Set Symmetric Differences (29) immediately needs:

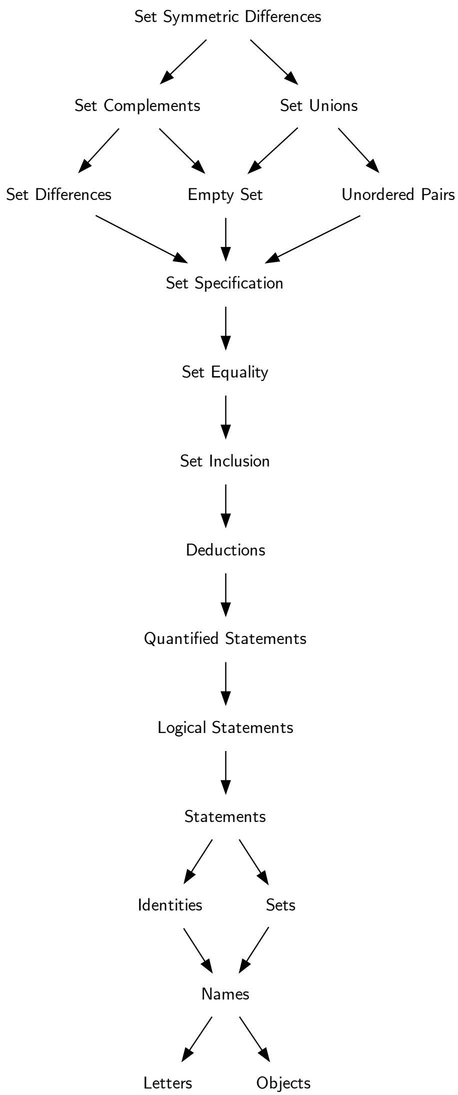
Set Complements (25)
Set Unions (15)

Set Symmetric Differences (29) is immediately needed by:

Operations (72)

Set Symmetric Differences (29) gives the following terms.

symmetric difference, Boolean sum.


Set Symmetric Differences
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to consider all the subsets of a given set.

Definition

We do not yet have a principle stating that such a set exists, but our intuition suggests that it does.

Principle 6 (powers)**.** *For every set, there exists a set of its subsets.*

We call the existence of this set the *principle of powers* and we call the set the *power set.*[28] As usual, the principle of extension gives uniqueness (see Set Equality). The power set of a set includes the set itself and the empty set.

Notation

Let A denote a set. We denote the power set of A by $\mathcal{P}(A)$, read aloud as "powerset of A." $A \in \mathcal{P}(A)$ and $\varnothing \in \mathcal{P}(A)$. However, $A \subset \mathcal{P}(A)$ is false.

Examples

Let a, b, c denote distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in \mathcal{P}(A)$. Showing each of the following is straightforward.

1. The empty set: $\mathcal{P}(\varnothing) = \{\varnothing\}$
2. Singletons: $\mathcal{P}(\{a\}) = \{\varnothing, \{a\}\}$
3. Pairs: $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
4. Triples:

$$\mathcal{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

[28] This terminology is standard, but unfortunate. Future editions may change these terms.

Properties

We can guess the following easy properties.[29]

Proposition 34. $\varnothing \in \mathcal{P}(A)$

Proposition 35. $A \in \mathcal{P}(A)$

We call A and $\varnothing$ the *improper* subsets of A. All other subsets we call *proper*.

Basic fact

Proposition 36. $E \subset F \longrightarrow \mathcal{P}(E) \subset \mathcal{P}(F)$

[29] Future editions will expand this account.

Set Powers (30) immediately needs:

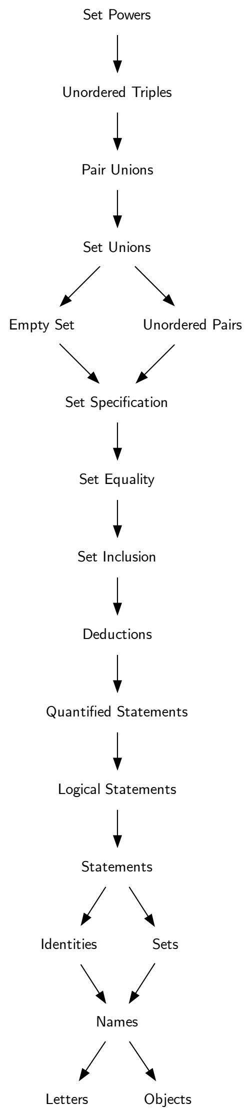
Unordered Triples (17)

Set Powers (30) is immediately needed by:

Characteristic Functions (**??**)
Lattices (**??**)
Powers and Intersections (31)
Powers and Unions (32)
Set Products (37)
Subset Systems (**??**)

Set Powers (30) gives the following terms.

principle of powers, power set, improper, proper.


Set Powers
Unordered Triples
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How does the power set relate to an intersection?

Notation preliminaries

First, if we have a set of sets—denote it $\mathcal{C}$—and all members are subsets of a fixed set—denote it E—then the set of sets is a subset of $\mathcal{P}(E)$. In this case, we can write

$$\bigcap\{X \in \mathcal{P}(E) \mid x \in \mathcal{C}\}$$

Which is a sort of justification for the notation

$$\bigcap_{X \in \mathcal{C}} X.$$

Basic properties

Here are some basic interactions between the powerset and intersections.[30]

Proposition 37. $\mathcal{P}(A) \cap \mathcal{P}(F) = \mathcal{P}((A \cap F))$

Proposition 38. $\bigcap_{X \in \mathcal{A}} \mathcal{P}(A) = \mathcal{P}((\bigcap_{X \in \mathcal{A}} A))$

Proposition 39. $\bigcap_{X \in \mathcal{P}(E)} X = \varnothing$

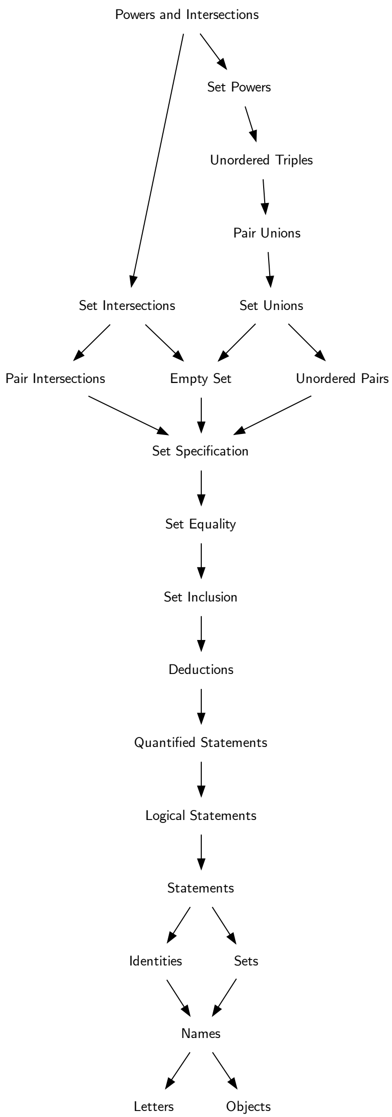
[30]Future editions will expand on these propositions and provide accounts of them.

Powers and Intersections (31) immediately needs:

Set Intersections (19)
Set Powers (30)

Powers and Intersections (31) is not immediately needed by any sheet.

Powers and Intersections (31) gives no terms.



Powers and Intersections
Set Powers
Unordered Triples
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How does the power set relate to a union?

Notation preliminaries

Let E denote a set. Let $\mathcal{A}$ denote a set of subsets of the set denoted by E. We define $\bigcup_{A \in \mathcal{A}} A$ to mean $\bigcup \mathcal{A}$.

Basic properties

Here are some basic interactions between the powerset and unions.[31]

Proposition 40. $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}((E \cup F))$

Proposition 41. $\bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}((\bigcup_{X \in \mathcal{C}} X))$

Proposition 42. $E = \bigcup \mathcal{P}(E)$

Proposition 43. $\mathcal{P}((\bigcup E)) \supset E.$

Typically $E \neq \mathcal{P}((\bigcup E))$, in which case E is a proper subset.

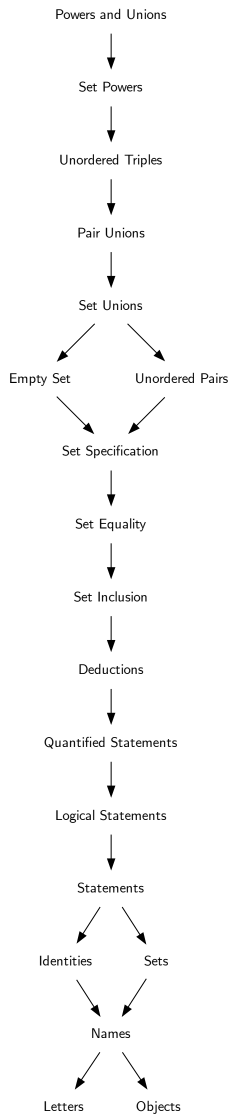
[31]Future editions will expand on these propositions and provide accounts of them.

Powers and Unions (32) immediately needs:

Set Powers (30)

Powers and Unions (32) is not immediately needed by any sheet.

Powers and Unions (32) gives no terms.


Powers and Unions
Set Powers
Unordered Triples
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

If all sets considered in a union or intersection are subsets of a fixed set, then the union and intersection of any set of sets is well defined. We can then derive generalized version of DeMorgan's laws.[32]

New notation

Let E denote a set. Let $\mathcal{A}$ denote a set of subsets of E. Then define

$$\bigcup_{A \in \mathcal{A}} A := \bigcup \mathcal{A}, \quad \bigcap_{A \in \mathcal{A}} A := \bigcap \mathcal{A}.$$

In this case we have

Proposition 44. $C(\cup_{A \in \mathcal{A}} A) = \cap_{A \in \mathcal{A}} C(A)$.

Proposition 45. $C(\cap_{A \in \mathcal{A}} A) = \cup_{A \in \mathcal{A}} C(A)$.

[32]In future editions, this sheet may not exist.

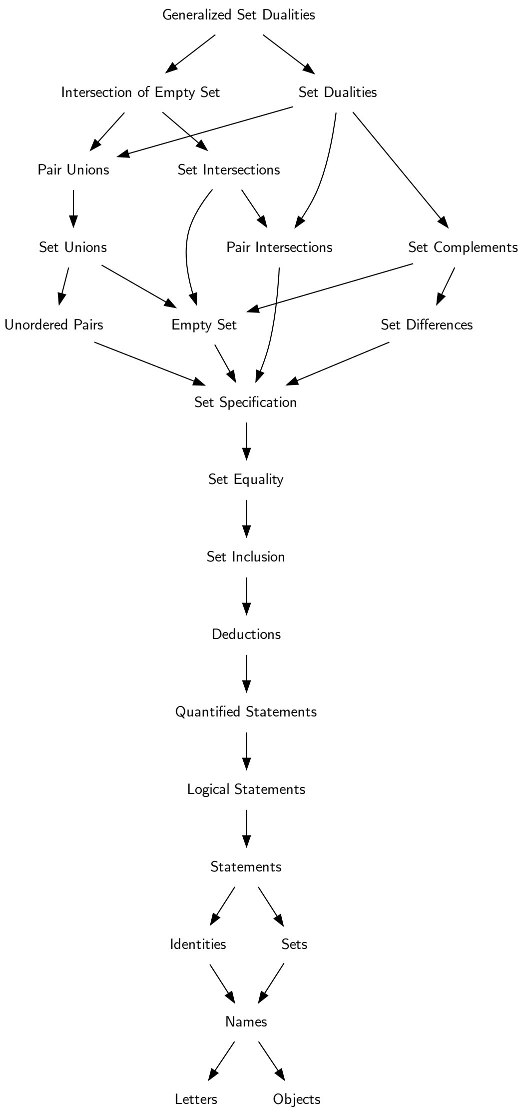
Generalized Set Dualities (33) immediately needs:

Intersection of Empty Set (20)
Set Dualities (27)

Generalized Set Dualities (33) is immediately needed by:

Family Unions and Intersections (46)

Generalized Set Dualities (33) gives no terms.


Generalized Set Dualities
Intersection of Empty Set
Set Dualities
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to arrange the elements of a set in an order using only the concept of sets.[33]

Discussion

What does this mean? Well, we often arrange objects in orders. For example, the letters of this page are arranged into words. Take two such words: 'note' and 'tone'. If letters are objects, what are words?

A first guess is that words seem like groups of letters, and sets seem like groups, and so a word is a set of letters. So, the word 'note' is the set 'n', 'o', 't', 'e', and then word 'tone' is the set 't', 'o', 'n', 'e'. The rub, of course, is that these are the same set.

The trick is that a word is not just the set of letters, it is that set in some order. Since 'tone' and 'note' have the same letters, they have the same set of letters.

The question is whether there is a way of saying what a word is in terms of letters by using sets in such a way that the set corresponding to 'tone' is distinguishable from the set corresponding to 'note'.

The way we read English offers a hint. When reading 'tone' we scan from left to right seeing 't', then 'to', then 'ton' then 'tone'. Suppose that for each spot in the ordering of the letters, we consider those letters that appear at or before the spot. In other words, we can consider the sets 't', 't', 'o', 't', 'o', 'n', 't', 'o', 'n', 'e'. Let us say that 'tone' corresponds to the set of these sets, denoted by $\mathcal{C}$,

$$\mathcal{C} = \{\{n, o, t\}, \{n, o, t, e\}, \{t\}, \{o, t\}\}.$$

Given $\mathcal{C}$, can we recover 'tone' (instead of 'note')? Sure. First, look for a set contained in all the others. The singleton 't' is the only one. So the first letter is 't'. Next look for a set distinct from 't' which is contained

[33] This sheet needs revision.

in all the rest. The pair 'o', 't' is the only one. Since we already have 't', the next letter is 'o'. We do the same twice more, getting 'n' and 'e', in that order.

There is a certain peculiarity in all these considerations. Every time we write down a set, we write the names (see Names) of the elements in some order. Indeed, whenever we speak of objects, we must say their names in some order. But of course, no matter how we denote or speak of the set, the concept of set has no concept of ordering.

Generally

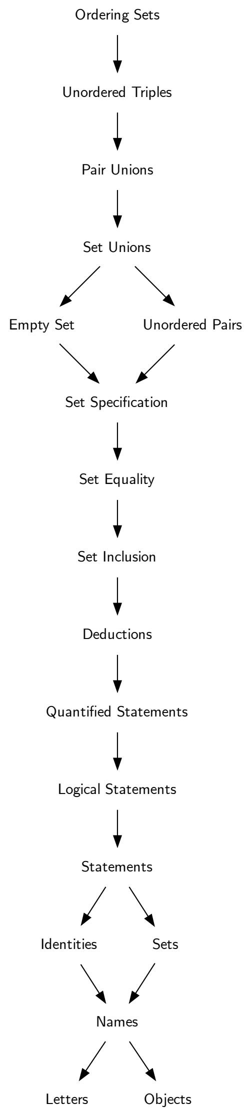
Let a, b, c and d denote objects, no two of which are the same (i.e., $a \neq b$, $b \neq c$, etc.). Suppose we want to consider the elements of the quadruple $\{a, b, c, d\}$ in the order c, b, d, a. We include in the set all objects that that occur at or before that position. For the order c, b, d, a of the objects in the set $\{a, b, c, d\}$ we use $\{c\}$, $\{c, b\}$, $\{c, b, d\}$ and $\{c, b, d, a\}$.

Ordering Sets (34) immediately needs:

Unordered Triples (17)

Ordering Sets (34) is not immediately needed by any sheet.

Ordering Sets (34) gives no terms.


Ordering Sets
Unordered Triples
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to order two objects.

Definition

Let a and b denote objects. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is the object denoted by a and the *second coordinate* is the object denoted by b.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b).

Equality

Our intuition of two objects in order dictates that if we have the same objects in the same order then we have the same ordered pair. Conversely, if we have two identical ordered pairs, they must consist of the same objects in the same location. In other words, two ordered pairs should be equal if and only if they consist of the same objects in the same order. Our definition agrees with this intuition. Indeed,

Proposition 46. $(((a, b) = (x, y)) \longleftrightarrow (a = x \land b = y))$[34]

[34]The proof of this proposition will be found in future editions.

Ordered Pairs (35) immediately needs:

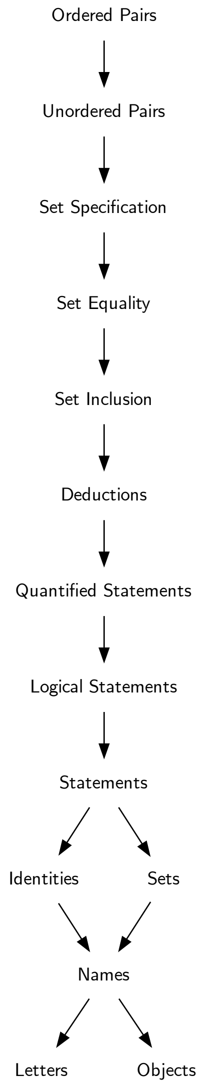
Unordered Pairs (14)

Ordered Pairs (35) is immediately needed by:

Multisets (**??**)
Ordered Pair Pathologies (36)
Product Sections (**??**)
Set Products (37)
Subset Systems (**??**)

Ordered Pairs (35) gives the following terms.

ordered pair, first coordinate, second coordinate.


Ordered Pairs
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Why define ordered pairs in terms of sets? Why not make them their own intangible object?

Pathologies

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second "coordinate" in Ordered Pairsrather than the term "element" (used in Sets). Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $(a, b) \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $(a, b) \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life.

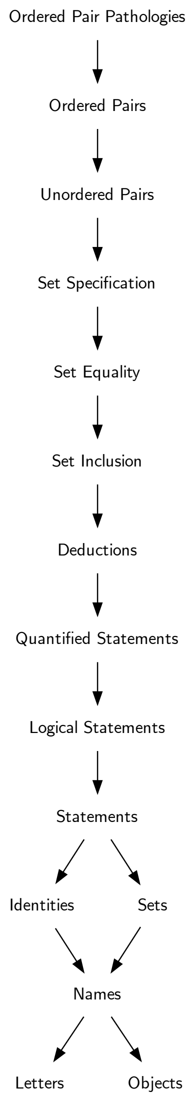
Suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction using already developed concepts.

Ordered Pair Pathologies (36) immediately needs:

Ordered Pairs (35)

Ordered Pair Pathologies (36) is not immediately needed by any sheet.

Ordered Pair Pathologies (36) gives no terms.


Ordered Pair Pathologies
Ordered Pairs
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Does a set exist which contains all ordered pairs of elements from two sets?

Discussion

The answer is easily seen to be yes. Ordered pairs are just sets, containing two sets. One set has one object, and so is a singleton. The other has two objects, and so is a pair. So to construct the set of all ordered pairs, we need only specify certain members of some set containing all singletons and pairs. The power set of the union of the two sets will suffice.

To see this, suppose A and B are two sets. If $a \in A$, then $a \in A \cup B$. Likewise if $b \in B$, then $b \in A \cup B$. Hence $\{a\} \subset A$ and $\{b\} \subset B$, so that $\{a\}, \{b\} \in \mathcal{P}(A \cup B)$. In other words, the singletons are members of the power set. Similarly, $\{a, b\} \in \mathcal{P}(A \cup B)$. In other words, the pairs are elements of the power set. Thus the set of sets containing singletons and pairs is a power set of the power set of $A \cup B$. In symbols, $\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$.

Definition

We define the set of "all ordered pairs" from A and B by specifying the appropriate pairs of this set.[35]

$$\{(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid a \in A \land b \in B\}$$

We name this set the *product* of the set denoted by A and the set denoted by B is the set of all ordered pairs. This set is also called the *set product* (or *cartesian product*[36]). If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

[35]The specific statement used here requires some translation. A discussion of this and the full statement will appear in a future edition.

[36]This second term is universal, but avoided in accordance with the project policy on naming.

Notation

We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.[37]

Empty set

It turns out the product of the empty set with any other set is always empty.

Proposition 47. *Suppose A is a set. Then $A \times \varnothing = \varnothing \times A = \varnothing$.*

Proof. This follows from the definition of the set product, since there is no element in the emepty set, and so the statement used in the specification always evaluates to false. $\square$

[37] Future editions may include a table figure visualizing the product.

Set Products (37) immediately needs:

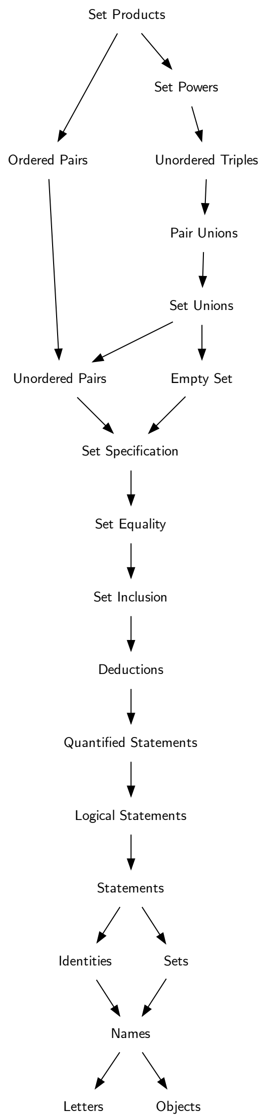
Ordered Pairs (35)
Set Powers (30)

Set Products (37) is immediately needed by:

Ordered Pair Projections (38)
Uncertain Outcomes (**??**)

Set Products (37) gives the following terms.

product, set product, cartesian product.


Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

The product of two sets is a (sub)set of ordered pairs. Is every set of ordered pairs a subset of a product of two sets?

Result

The answer is easily seen to be yes. Let R denote a set of ordered pairs. So for $x \in R$, $x = \{\{a\}, \{a, b\}\}$. First consider $\bigcup R$. Then $\{a\} \in \bigcup R$ and $\{a, b\} \in \bigcup R$. Next consder $\bigcup \bigcup R$. Then $a, b \in \bigcup \bigcup R$. So if we want two sets—denote them by A and B—so that $R \subset A \times B$, we can take both A and B to be the set $\bigcup \bigcup R$.

Projections

We often want to shrink the sets A and B to include only the *relevant* members. In other words, to include only those members which appear as either the first coordinate (for A) or second coordinate (for B) in an element of R. We can do this by specifying the elements of $\bigcup \bigcup R$ which are actually a first coordinate or second coordinate for some ordered pair in the set R.

Define

$$A' = \{a \in A \mid (\exists b)((a, b) \in R)\},$$

and likewise

$$B' = \{b \in B \mid (\exists a)((a, b) \in R)\}.$$

We call A' the *projection onto the first coordinate* and B' the *projection onto the second coordinate.*

Ordered Pair Projections (38) immediately needs:

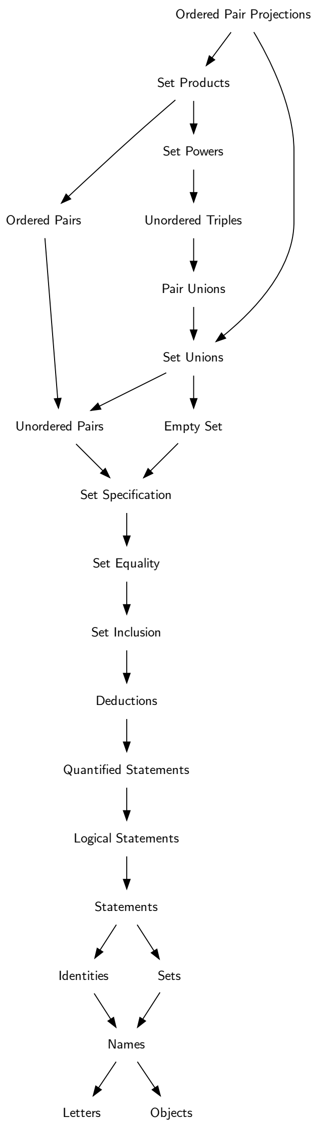
Set Products (37)
Set Unions (15)

Ordered Pair Projections (38) is immediately needed by:

Relations (39)

Ordered Pair Projections (38) gives the following terms.

projection onto the first coordinate, projection onto the second coordinate..


Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How can we relate the elements of two sets?

Definition

A *relation* is a set of ordered pairs (see Ordered Pairs). So if an object z is an element of a relation, there exist two other objects x and y so that $z = (x, y)$.

Domain and range

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation (the projection onto the first coordinate, see Ordered Pair Projections). The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation (the projection onto the second coordinate).

When the domain of a relation R is a subset of X and the range is a subset of Y, we say R is a relation *between* X and Y or (*from* X *to* Y). If $X = Y$, then we speak of R as a relation *on* (or *in*) X.

Notation

If R is a relation, we express that $(x, y) \in R$ by writing $x \, R \, y$, which we read aloud as "x is in relation R to y". We denote the domain of R by $\operatorname{dom} R$ and the range of R by $\operatorname{range} R$.

Examples

Empty relation

For an uninteresting relation, consider the empty set. We call the empty set the *empty relation*. In the empty (set) relation, no object is related to any other. Both the domain and range of $\varnothing$ are $\varnothing$.

Total relation

Next, consider the product of any two sets X and Y. In $X \times Y$, all objects are related. The domain is X and the range is Y.

Equality

For a more interesting example, define $R \subset X \times X$ by

$$R = \{(x, y) \in X \times X \mid x = y\}.$$

This relation is the *relation of equality* (see Identities) between two objects. Here $x \, R \, y \longleftrightarrow x = y$. $\operatorname{dom} R = \operatorname{range} R = X$.

Belonging

Another similar example is if we consider the set X and $\mathcal{P}(X)$, and the relation

$$R := \{(x, y) \in X \times \mathcal{P}(X) \mid x \in y\}.$$

This relation is the *relation of belonging* (see Sets). Here $x \, R \, y \longleftrightarrow x \in y$. Here $\operatorname{dom} R = X$ and $\operatorname{range} R = \mathcal{P}(X)$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

- A relation is *reflexive* if every element is related to itself.
- A relation is *symmetric* if two objects are related regardless of their order.
- A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related.

Equality is reflexive, symmetric and transitive whereas belonging is neither. Exercise: what is inclusion?

Relations (39) immediately needs:

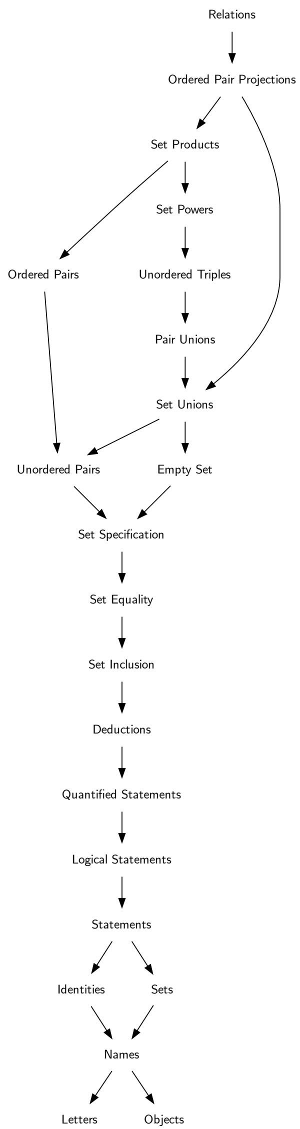
Ordered Pair Projections (38)

Relations (39) is immediately needed by:

Converse Relations (53)
Equivalence Relations (40)
Functions (41)
Orders (**??**)
Relation Composites (52)

Relations (39) gives the following terms.

relation, domain, range, between, from, to, on, in, empty relation, relation of equality, relation of belonging, reflexive, symmetric, transitive.


Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to handle at once all the objects of a set which are indistinguishable or equivalent in some aspect.

Definition

An *equivalence relation* on a set X is a reflexive, symmetric, and transitive relation on X (see Relations). The smallest equivalence relation in a set X is the relation of equality in X. The largest equivalence relation in a set X is $X \times X$.

Equivalence relations are useful because they partition (see Partitions) the set. If R is an equivalence relation on X, the *equivalence class* of an object $x \in X$ is the set $\{y \in X \mid x \; R \; y\}$. We call the set of equivalence classes the *quotient set* of the set under the relation (or the *quotient* of the set *by the relation*). An equally good name is the divided set of the set under the relation, but this terminology is not standard. The language in both cases reminds us that the relation partitions the set into equivalence classes.

If $\mathcal{C}$ is a partition of X, we can define a relation R on X for which $x \, R \, y \longleftrightarrow (\exists A \in \mathcal{C})(x \in A \land y \in A)$. In other words, if x and y are in the same piece (see Partitions) of $\mathcal{C}$.

The key result is that every equivalence relation partitions the set and every partition of the set is an equivalence relation. Moreover, if we start with an equivalence relation, look for the partition, and then get the relation defined by the partition, we end up with the relation we started with. Likewise, if we start with a partition relation, get the equivalence relation, and then get the partition defined by the relation, we end up with the partition we started with. Before stating and proving this result, we give some notation.

Notation

Let R denote an equivalence relation on a set denoted by X. We denote the equivalence class of $x \in X$ by x/R. We denote the set of equivalence classes of R by X/R, read aloud as "X modulo R" or "x mod R". We denote the equivalence class of an element $x \in X$ by $[x]$.

Main Results

The proofs of these results are straightforward.[38]

Proposition 48. *$X/\mathcal{C}$ is an equivalence relation.*

Proposition 49. *X/R is a a partition.*

Proposition 50. *If R is an equivalence relation on X, then $X/(X/R) = R$*

Proposition 51. *If $\mathcal{C}$ is a partition of X, then $X/(X/\mathcal{C}) = \mathcal{C}$.*

These last two propositions make clear the rationale for the notation. The function mapping an element to its equivalence class is onto and is sometimes called the *projection*.

[38]Nonetheless, the full accounts will appear in future editions.

Equivalence Relations (40) immediately needs:

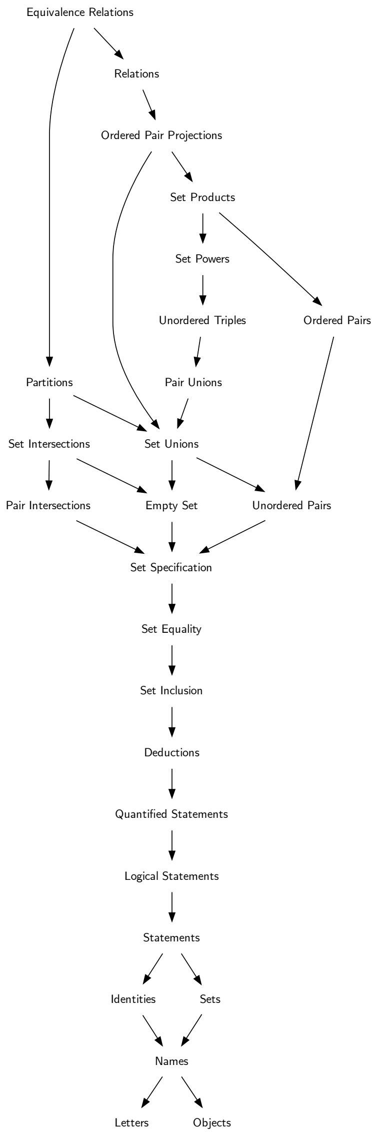
Partitions (26)
Relations (39)

Equivalence Relations (40) is immediately needed by:

Canonical Maps (44)
Equivalent Sets (65)
Integer Numbers (79)
Inverses of Composite Relations (54)
Matrix Similarity (**??**)

Equivalence Relations (40) gives the following terms.

equivalence relation, equivalence class, quotient set, quotient, by the relation, projection.


Equivalence Relations
Relations
Ordered Pair Projections
Set Products
Set Powers
Unordered Triples
Ordered Pairs
Partitions
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want a notion for a correspondence between two sets.

Definition

A *function* f (or *correspondence*, *mapping*, *map*) *from* a set X *to* a set Y is a relationwhose domain is X and whose range is a subset of Y, such that for each $x \in X$,

1. there exists $y \in Y$ so that $(x, y) \in f$
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$; where y and z are in Y

We often summarize these two conditions by saying: to every element $x \in X$ there corresponds a *unique* element $y \in Y$ so that $(x, y) \in f$.

We call this unique element $y \in Y$ the *result* of the function *at* the *argument* x. We call Y *a codomain*—notice our use of the word "a", since the codomain is not a property of the function. If the range is Y we say that f is a function from X *onto* Y (or call f *onto*, *surjective*). If distinct elements of X are mapped to distinct elements of Y, we say that the function is *one-to-one* (or *injective*).

We say that the function *maps* (or *takes*) elements from the domain to the codomain. Since the word "function" and the verb "maps" connote activity, some authors refer to the set of ordered pairs as the *graph* of a function and avoid defining the term "function" as we have, in terms of sets.

Notation

Given sets X and Y, we abbreviate the statement that the object denoted by f is a function whose domain is a X and whose codomain is a set Y by

$$f : X \to Y$$

We read the notation aloud as "f from X to Y." We emphasize again that the *range* of f need not be Y, but must necessarily be a subset.

We denote by Y^X the set of functions from X to Y. This set is contained in the power set $\mathcal{P}((X \times Y))$. A reasonable but nonstandard notation is $X \to Y$, read as "A to B." All the following three statements have the same meaning:

$$f : X \to Y, \quad f \in Y^X, \quad f \in (X \to Y).$$

We tend to denote functions by lower case latin letters; especially f, g, and h. f is a mnemonic for function and g and h are nearby in the usual ordering of the Latin letters.

Suppose $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a." Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

Examples

If $X \subset Y$, the function $\{(x, y) \in X \times Y \mid x = y\}$ is the *inclusion function* of X into Y. We often introduce such a function as "the function from X to Y defined by $f(x) = y$". We mean by this that f is a function and that we are specifying the appropraite ordered pairs using the statement, called *argument-value notation.* The inclusion function of X into X is called the *identity function* of X. If we view the identity function as a relation on X, it is the relation of equality on X.

The functions $f : (X \times Y) \to X$ defined by $f(x, y) = x$ is the *pair projection* of $X \times Y$ ono X. Similarly $g : (X \times Y) \to Y$ defined by $g(x, y) = y$ is the pair projection of $X \times Y$ onto Y.

The identity function is one-to-one and onto, the inclusion functions are one-to-one but not always onto, and the pair projections are usually not one-to-one.

Functions (41) immediately needs:

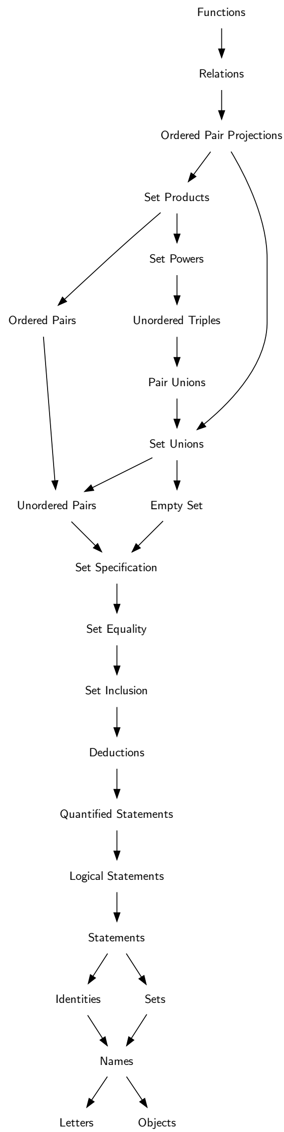
Relations (39)

Functions (41) is immediately needed by:

Canonical Maps (44)
Categories (**??**)
Constant Functions (**??**)
Equations (**??**)
Families (45)
Function Composites (49)
Function Images (43)
Function Restrictions and Extensions (42)
Operations (72)

Functions (41) gives the following terms.

function, correspondence, mapping, map, from, to, unique, result, at, argument, codomain, onto, onto, surjective, one-to-one, injective, maps, takes, graph, inclusion function, argument-value notation, identity function, pair projection.


Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

The relationship between the inclusion map and the identity map is characteristic of making small functions out of large ones.[39]

Definition

Let $X \subset Y$ and $f : Y \to Z$. There is a natural function $g : X \to Z$, namely the one defined by $g(x) = f(x)$ for all $x \in X$. We call g the *restriction* of f to X. We call f an *extension* of g to Y. Clearly, there may be more than one extension of a function

Notation

We denote the restriction of $f : Y \to Z$ to the set $X \subset Y$ by $f \mid X$ or $f_{|X}$.

Example

A simple example is the that the inclusion mapping from X to Y with $X \subset Y$ is a restriction of the identity map on X

An extension order

Here is a natural order involving set extensions and restrictions. Fix two sets A and B. Let F be the set of all functions $f : X \to Y$ with $X \subset A$ and $Y \subset B$. Define a relation R in F by $(f, g) \in R$ if $\operatorname{dom} f \subset \operatorname{dom} g$ and $f(x) = g(x)$ for all x in $\operatorname{dom} f$. In other words, $(f, g) \in R$ if f is a restriction of g (or, equivalently, g is an extension of f. We recognize that R is a special case of the inclusion partial order by recognizing the elements of F as subsets $A \times B$.

[39] Future editions will modify this language.

Function Restrictions and Extensions (42) immediately needs:

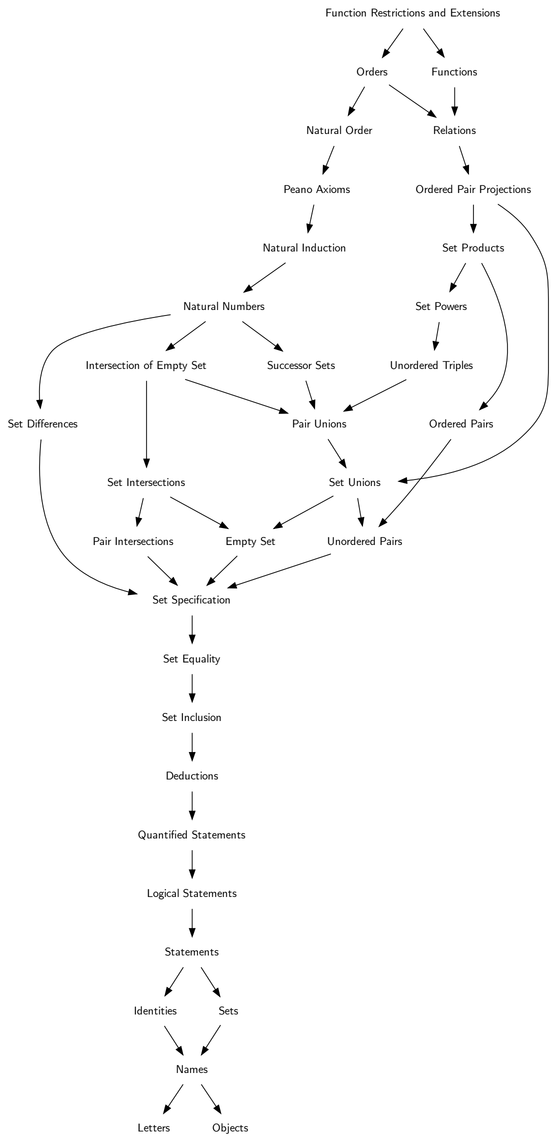
Functions (41)
Orders (**??**)

Function Restrictions and Extensions (42) is immediately needed by:

Convex Multivariate Functions (**??**)
Natural Integer Isomorphism (89)

Function Restrictions and Extensions (42) gives the following terms.

restriction, *extension*.


Function Restrictions and Extensions
Orders
Functions
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Differences
Pair Unions
Ordered Pairs
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We consider the set of results of a set of domain elements.

Definition

The *image* of a set of domain elements under a function is the set of their results. Though the set of domain elements may include several distinct elements, the image may still be a singleton, since the function may map all of elements to the same result.

Using this language, the range (see Functions) of a function is the image of its domain. The range includes all possible results of the function. If the range does not include some element of the codomain, then the function maps no domain elements to that codomain element.

Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for every $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. Following this notation for function images, we denote the range of f by $f(A)$. In this notation, we can record that f maps X onto Y by $f(X) = Y$.

Notational ambiguity

The notation $f(A)$ is can be ambiguous in the case that A is both an element and a set of elements of the domain of f. For example, consider $f : \{\{a\}, \{b\}, \{a, b\}\} \to X$. Then $f(\{a, b\})$ is ambiguous. We will avoid this ambiguity by making clear which we mean in particular cases.

Inverse images

Similarly to how we can define $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ for $A \subset X$

$$f(A) = \{y \in Y \mid (\exists x)(x \in a \land y = f(x))\},$$

we can define $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ for $B \subset X$

$$f^{-1}(B) = \{x \in X \mid (\exists y)(y \in B \wedge y = f(x))\}.$$

In other words, $f^{-1}(B)$ is the set of all elements of the domain which give the elements in B of the range. We call $f^{-1}(B)$ the *inverse image* of B. Another name less commonly used is *counter image* or *counterimage*.

Connections

Here are some connections.[40]

Proposition 52. *Let $f : X \to Y$ and $B \subset Y$. $f(f^{-1}(B)) \subset B$. If f is onto, then $f(f^{-1}(B)) \subset B$.*

Proposition 53. *Let $f : X \to Y$ and $A \subset X$. $A \subset f^{-1}(f(A))$. If f is one-to-one, then $A = f^{-1}(f(A))$.*

[40] The proofs are straightfoward, and will appear in future editions.

Function Images (43) immediately needs:

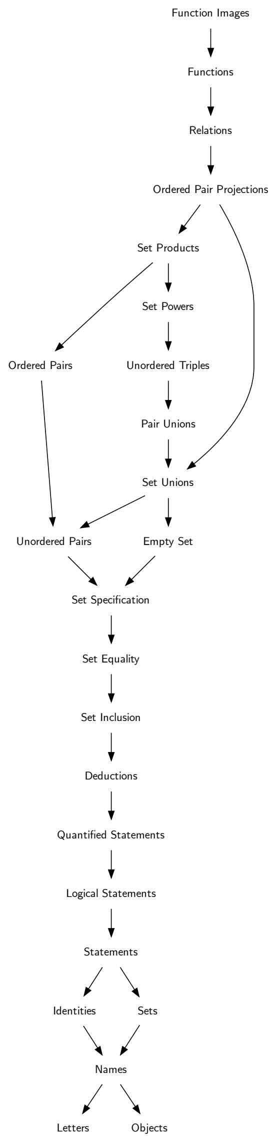
Functions (41)

Function Images (43) is immediately needed by:

Function Inverses (50)

Function Images (43) gives the following terms.

image, inverse image, counter image, counterimage.


Function Images
Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How do equivalence classes and functions relate?

Definition

We can associate to each element of a set its equivalence class under an equivalence relation. Let X denote a set and R an equivalence relation. We call the function $f : X \to X/R$ defined by $f(x) = x/R$ the *canonical map* from X to X/R.

Conversely, if f is an arbitrary function from X onto Y, we can naturally define an equivalence relation R in X so that for $a, b \in X$, $a\,R\,b \longleftrightarrow f(a) = f(b)$ f was onto, so for each $y \in Y$, there exists an $x \in X$ with $f(x) = y$. Now let $g : Y \to X/R$ be defined by $g(y) = x/R$. The values of g are the subset X which are mapped to the same value under f. Moreover, the function g is one-to-one.

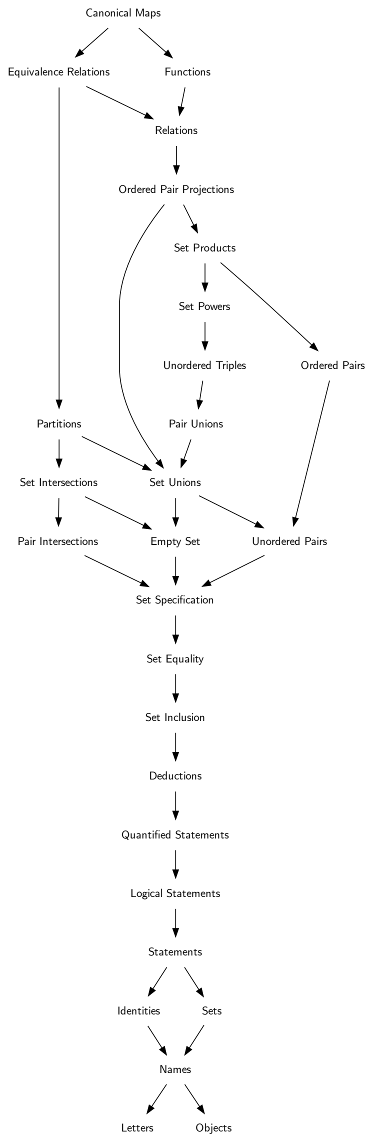
Canonical Maps (44) immediately needs:

Equivalence Relations (40)
Functions (41)

Canonical Maps (44) is not immediately needed by any sheet.

Canonical Maps (44) gives the following terms.

canonical map.


Canonical Maps
Equivalence Relations
Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Unordered Triples
Ordered Pairs
Partitions
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We often use functions to keep track of several objects by the objects of some well-known set with which they correspond. In this case, we use specific language and notation.

Definition

Let I and X denote sets. A *family* is a function from I to X. We call an element of I an *index* and we call I the *index set*. Of course, the letter I was picked here to be a mnemonic for "index". We call the range of the family the *indexed set* and we call the value of the family at an index i a *term* of the family at i or the *ith term* of the family.

Experience shows that it is useful to discuss sets using indices, especially when discussing a set of sets. If the values of the family are sets, we speak of a *family of sets*. Indeed, we often speak of a *family of* whatever object the values of the function are. So for instance, a family of subsets of X is understood to be a function from some index set into $\mathcal{P}(X)$.

Notation

Let $x : I \to X$ be a family. We denote the ith term of x by x_i. We sometimes denote the family by $\{x_i\}_{i \in I}$.

Families (45) immediately needs:

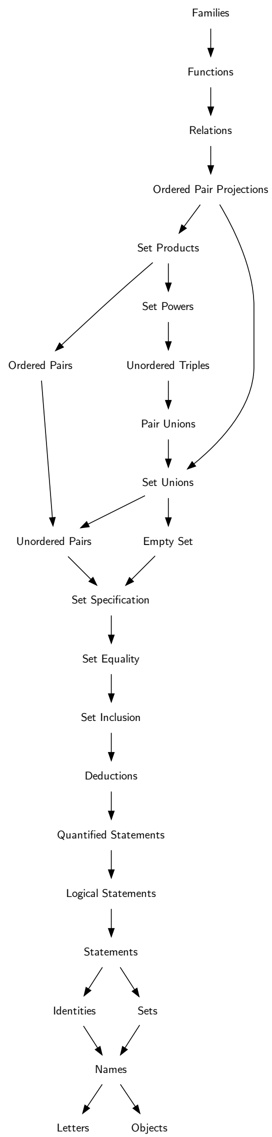
Functions (41)

Families (45) is immediately needed by:

Direct Products (47)
Family Operations (**??**)
Family Unions and Intersections (46)

Families (45) gives the following terms.

family of sets, ordered family, family, index, index set, indexed set, term, ith term, family of sets, family of.


Families
Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We can use families to think about unions and intersections.

Family unions

Let $A : I \to \mathcal{P}(X)$ be a family of subsets. We refer to the union (see Set Unions) of the range (see Relations) of the family the *family union*. We denote it $\cup_{i \in I} A_i$.

Proposition 54. $(x \in \cup_{i \in I} A_i) \longleftrightarrow (\exists i)(x \in A_i)$

If $I = \{a, b\}$ is a pair with $a \neq b$, then $\cup_{i \in I} = A_a \cup A_b$.

There is no loss of generality in considering family unions. Every set of sets is a family: consider the identity function from the set of sets to itself.

We can also show generalized associative and commutative law[41] for unions.

Proposition 55. *Let $\{I_j\}$ be a family of sets and define $K = \cup_j I_j$. Then $\cup_{k \in K} A_k = \cup_{j \in J} (\cup_{i \in I_j} A_i)$.*[42]

Family intersection

If we have a nonempty family of subsets $A : I \to \mathcal{P}(X)$, we call the intersection (see Set Intersections) of the range of the family the *family intersection*. We denote it $\cap_{i \in I} A_i$.

Proposition 56. $x \in \cap_{i \in I} A_i \longleftrightarrow (\forall i)(x \in A_i)$

Similarly we can derive associative and commutative laws for intersection.[43] They can be derived as for unions, or from the facts of unions using generalized DeMorgan's laws (see Generalized Set Dualities).

[41] The commutative law will appear in future editions.
[42] An account will appear in future editions.
[43] Statements of these will be given in future editions.

Connections

The following are easy.[44]

Let $\{A_i\}$ be a family of subsets of X and let $B \subset X$.

Proposition 57. $B \cap \bigcup_i A_i = \bigcup_i (B \cap A_i)$

Proposition 58. $B \cup \bigcap_i A_i = \bigcap_i (B \cup A_i)$

Let $\{A_i\}$ and $\{B_j\}$ be families of sets.[45]

Proposition 59. $(\bigcup_i A_i) \cap (\bigcup_j B_j) = \bigcup_{i,j} (A_i \cap B_j)$

Proposition 60. $(\bigcap_i A_u) \cup (\bigcap_j B_j) = \bigcap_{i,j} (A_i \cup B_j)$.

Proposition 61. $\cap_i X_i \subset X_j \subset \cup_i X_i$ *for each* j.

[44] Nevertheless, full accounts will appear in future editions.

[45] An account of the notation used and the proofs will appear in future editions.

Family Unions and Intersections (46) immediately needs:

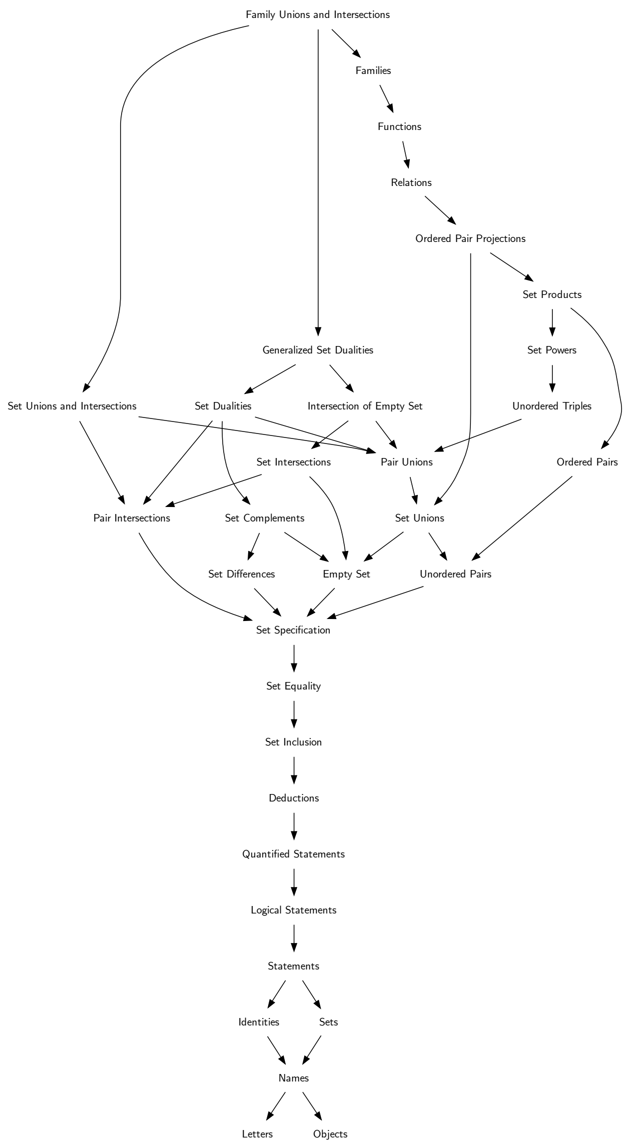
Families (45)
Generalized Set Dualities (33)
Set Unions and Intersections (21)

Family Unions and Intersections (46) is immediately needed by:

Family Products and Unions (48)
Generalized Inclusion-Exclusion Formula (**??**)
Inverses Unions Intersections and Complements (51)
Lists (**??**)

Family Unions and Intersections (46) gives the following terms.

family union, family intersection.



Family Unions and Intersections
Families
Functions
Relations
Ordered Pair Projections
Set Products
Generalized Set Dualities
Set Powers
Set Unions and Intersections
Set Dualities
Intersection of Empty Set
Unordered Triples
Set Intersections
Pair Unions
Ordered Pairs
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We generalize the product of two sets to a product of a family of sets. To do so we discuss sets of families.

Discussion for pairs

Suppose X and Y are nonempty sets. There is a natural correspondence between the product $X \times Y$ (see Set Products) and the set of families

$$Z = \{z : \{i, j\} \to (A \cup B) \mid z_i \in A \text{ and } z_j \in B\}$$

where $\{i, j\}$ is any unordered pair with $i \neq j$.

The set Z can be put in one-to-one correspondence with $X \times Y$. The family $z \in Z$ corresponds with the pair (z_i, z_j). The pair (a, b) corresponds to the family $z \in Z$ defined by $z(i) = a$ and $z(j) = b$. So, ordered pairs can be put in one-to-one correspondence with families. The generalization of Cartesian products to more than two sets generalizes the notion for families.

Definition

Suppose $\{X_i\}_{i\in I}$ is a family of sets. The *direct product* (or *Cartesian product*, *family Cartesian product*) of A is the set of all families (i.e., functions) $a : I \to X$ which satisfy $a_i \in A_i$ for every $i \in I$.

A function on a product is called a *function of several variables* and, in particular, a function on the product $X \times Y$ is called a *function of two variables.*

Notation

We denote the product of the family $\{A_i\}_{i\in I}$ by

$$\textstyle\prod_{i\in I} A_i$$

We read this notation as "product over i in I of A sub-i." Other notation in use includes $\times_{i\in I} A_i$.

Projections

The word "projection" is used in two senses with families. Let I be a set, and let $\{A_i\}_{i \in I}$ be a family of sets. Define $A = \prod_{i \in I} A_i$.

First, let $J \subset I$. There is a natural correspondence between the elements of A and those of $\prod_{j \in J} A_j$. To each element $a \in A$, we restrict a to J and this is restriction is an element of $\prod_{j \in J} A_j$. The correspondence is called the *projection* of A onto $\prod_{i \in J} A_i$. The projection in this sense is a set of families.

Second, consider the value of a family $a \in A$ at j. We call a_j the *projection of a onto index j* or the *j-coordinate* of a. This word *coordinate* is meant to follow the language used in defining ordered pairs. The projection in this sense is an element of A_j. The jth projection is a function mapping $\prod_{i \in I} X_i$ to X_j.

Direct Products (47) immediately needs:

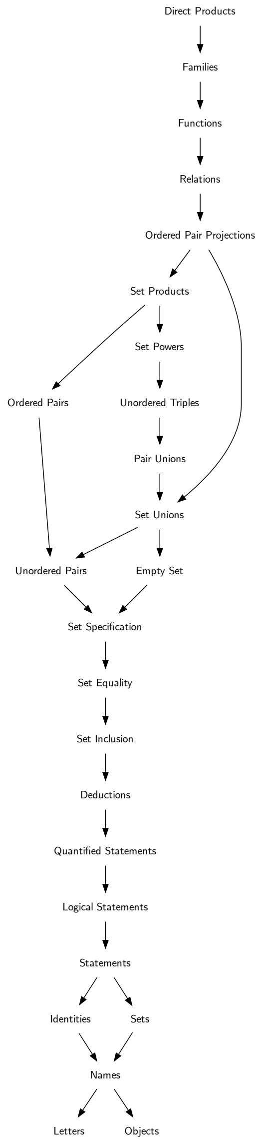
Families (45)

Direct Products (47) is immediately needed by:

Choice Functions (**??**)
Family Products and Unions (48)
Lists (**??**)
Size of Direct Product (**??**)

Direct Products (47) gives the following terms.

n-tuples, sequences, direct product, Cartesian product, family Cartesian product, function of several variables, function of two variables, consecutive, projection, projection of a onto index j, j-coordinate, coordinate.


Direct Products
Families
Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We study how family unions and direct products interact.

Result

The following is easy.[46]

Proposition 62. $(\cup_i A_i) \times (\cup_j B_j) = \cup_{i,j}(A_i \times B_j)$.

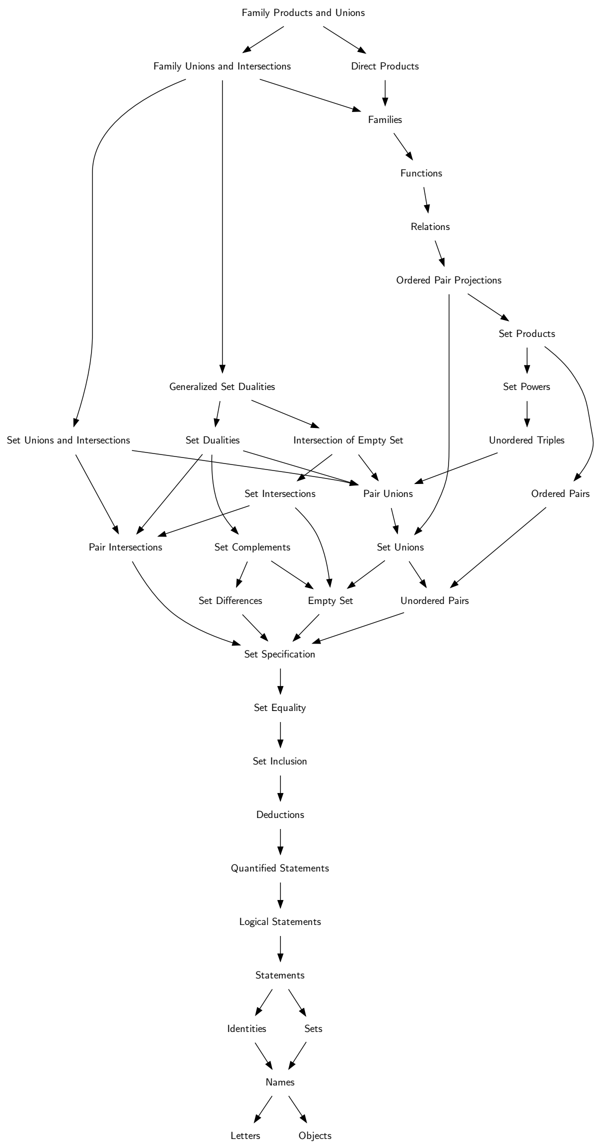
[46] An account will appear in future editions.

Family Products and Unions (48) immediately needs:

Direct Products (47)
Family Unions and Intersections (46)

Family Products and Unions (48) is not immediately needed by any sheet.

Family Products and Unions (48) gives no terms.


Family Products and Unions
Family Unions and Intersections
Direct Products
Families
Functions
Relations
Ordered Pair Projections
Set Products
Generalized Set Dualities
Set Powers
Set Unions and Intersections
Set Dualities
Intersection of Empty Set
Unordered Triples
Set Intersections
Pair Unions
Ordered Pairs
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to have language for applying two functions one after the other. We apply a first function then a second function.

Definition

Consider two functions. Suppose the range of the first is a subset of the domain of the second. In other words, every value of the first is in the domain (and so can be used as an argument) for the second. In this case we say that the second function is *composable* with the first.

The *composite* (or *composition*) of the second function *with* the first function is the function which associates to an element in the first's domain the element in the second's codomain that the second function associates with the result of the first.

In other words, we take an element in the first's domain. We apply the first function to it. We obtain an element in the first's codomain, which is also an element in the second's domain. We apply the second function to this result. We obtain an element in the second's codomain. The composition of the second function with the first is the function so constructed. Of course the order of composition is important.

Notation

Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$. We denote the composition of g with f by $g \circ f$ read aloud as "g composed with f. To make clear the domain and codomain, we denote the composition $g \circ f : A \to C$. The function $g \circ f$ is defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

Sometimes the notation gf is used for $g \circ f$.

Basic properties

Function composition is associative but not commutative.[47] Indeed, even if $f \circ g$ is defined, $g \circ f$ may not be.

Proposition 63 (Associative)**.** *Let* $f : X \to Y$*,* $g : Y \to Z$ *and* $h : Z \to U$ *Then* $(f \circ g) \circ h = f \circ (g \circ h)$[48]

[47]Future editions will include a counterexample.

[48]The proof is straightforward. Future editions will include it.

Function Composites (49) immediately needs:

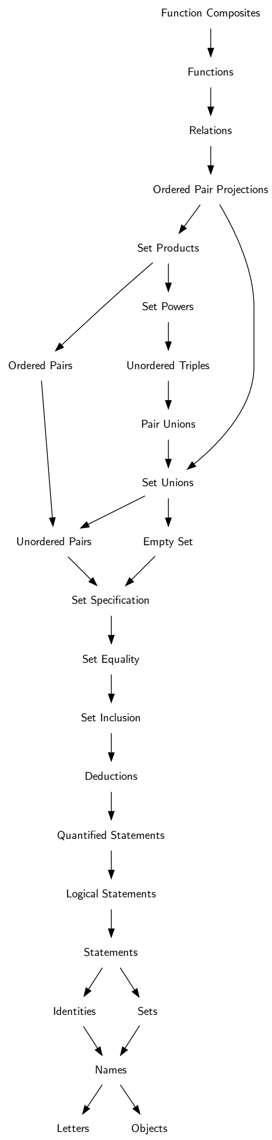
Functions (41)

Function Composites (49) is immediately needed by:

Composition Graphs (**??**)
Function Inverses (50)
Subsequences (71)
Typed Graphs (**??**)

Function Composites (49) gives the following terms.

composable, composite, composition, with.


Function Composites
Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want a notion of reversing functions.

Definition

Reversing functions does not make sense if the function is not one-to-one. Let $f : X \to Y$. If x_1 goes to y and x_2 goes to y (i.e., $f(x_1) = f(x_2) = y$), then what should y go to. One answer is that we should have a function which gives all the domain values which could lead to y. This is the inverse image (see Function Images) $f^{-1}(\{y\})$. Nor does reversing functions make sense if f is not onto. If there does not exist $x \in X$ so that $y = f(x)$, then $f^{-1}(\{y\}) = \varnothing$.

In the case, however, that the function is one-to-one and onto, then each element of the domain corresponds to one and only one element of the codomain and vice versa. In this case, for all $y \in Y$, $f^{-1}(\{y\})$ is a singleton $\{x\}$ where $f(x) = y$. In this case, we define a fucntion $g : Y \to X$ so that $g(y) = x$ if and only if $f(x) = y$.

Proposition 64 (Uniqueness)**.** *Let $f : A \to B$, $g : B \to A$, and $h : B \to A$. If g and h are both inverse functions of f, then $g = h$.*

Proposition 65 (Existence)**.** *If a function is one-to-one and onto, it has an inverse; and conversely.*[49]

Composites and inverses

Let $f : X \to Y$ and $g : Y \to Z$. Then g^{-1} maps $\mathcal{P}(Z)$ to $\mathcal{P}(Y)$ and f^{-1} maps $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Then the following is immediate

Proposition 66. $(gf)^{-1} = f^{-1}g^{-1}$

[49] A proof will appear in future editions.

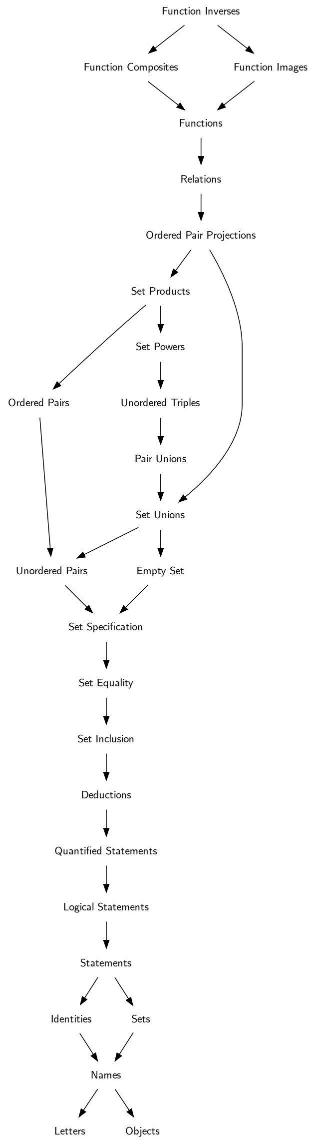
Function Inverses (50) immediately needs:

Function Composites (49)
Function Images (43)

Function Inverses (50) is immediately needed by:

Equivalent Sets (65)
Inverse Elements (78)
Invertible Linear Transformations (**??**)
Isometries (**??**)

Function Inverses (50) gives no terms.


Function Inverses
Function Composites
Function Images
Functions
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

The inverse of a function interacts nicely with family unions, family intersections and complements.

Results

Let $f : X \to Y$. Throughout this sheet, let $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$. And take $\{B_i\}$ to be a family of subsets of Y.[50]

Proposition 67. $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$

Proposition 68. $f^{-1}(\cup_i B_i) = \cap_i f^{-1}(B_i)$

Proposition 69. $f^{-1}(Y - B) = X - f^{-1}(B)$

Properties for function image

Notice that $f(\cup_i A_i) = \cup_i f(A_i)$ but not for interesctions. Nor is there a similar correspondence for complements. There are some relations, which we list below.[51]

Proposition 70. *$f(A \cap B) = f(A) \cap f(B)$ if and only if f is one-to-one.*

Proposition 71. *For all $A \subset X$, $f(X - A) = Y - f(A)$ if and only if f is one-to-one.*

Proposition 72. *For all $A \subset X$, $Y - f(A) \subset f(X - A)$ if and only if f is onto.*

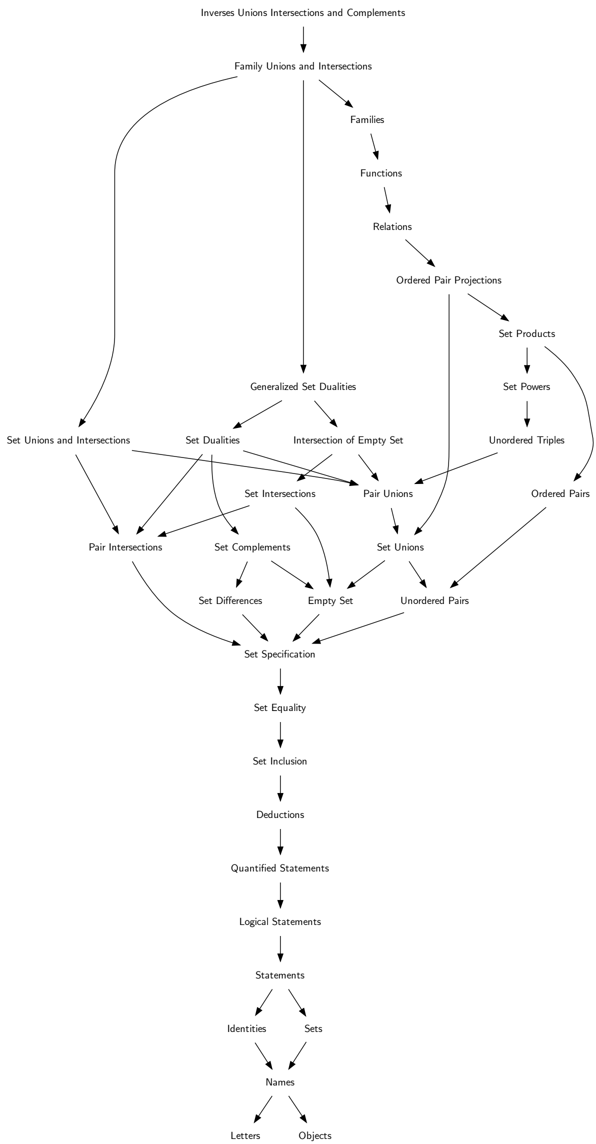
[50] The proofs of the following will appear in future editions.
[51] Accounts of these facts will appear in future editions.

Inverses Unions Intersections and Complements (51) immediately needs:

Family Unions and Intersections (46)

Inverses Unions Intersections and Complements (51) is not immediately needed by any sheet.

Inverses Unions Intersections and Complements (51) gives no terms.


Inverses Unions Intersections and Complements
Family Unions and Intersections
Families
Functions
Relations
Ordered Pair Projections
Set Products
Generalized Set Dualities
Set Powers
Set Unions and Intersections
Set Dualities
Intersection of Empty Set
Unordered Triples
Set Intersections
Pair Unions
Ordered Pairs
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

If x is related to y and y to z, then x and z are related.

Definition

Let R be a relation from X to Y and S a relation from Y to Z. The *composite relation* from X to Z contains the pair $(x, z) \in (X \times Z)$ if and only if there exists a $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S$. This composite relation is sometimes called the *relative product*.

Notation

We denote the composite relation of R and S by $R \circ S$ or RS.

Example

Let X be the set of people and let R be the relation in X "is a brother of" and S be the relation in X "is a father of". Then RS is the relation "is an uncle of".

Properties

Composition of relation is associative but not commutative.[52]

[52] A fuller account will appear in future editions.

Relation Composites (52) immediately needs:

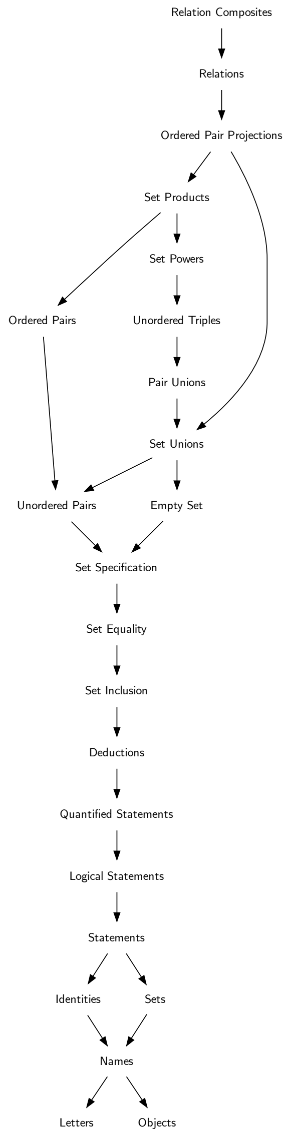
Relations (39)

Relation Composites (52) is immediately needed by:

Inverses of Composite Relations (54)

Relation Composites (52) gives the following terms.

composite relation, relative product.


Relation Composites
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

If x is related to y, the y is related to x, but how?

Definition

If R is a relation between X and Y, then the *converse* or *inverse* relation of R is a relation on Y and X relating $y \in Y$ to $x \in X$ if and only if $x\,R\,y$. If $R = R^{-1}$ then R is symmetric.

Notation

We denote the converse relation of R by R^{-1}.

Example

Let X be the set of people and let R be a relation in X. If R is "is a father of", then R^{-1} is "is a son of". If R is "is a mother of", then R^{-1} is "is a daughter of". If R is "is a brother of", then R^{-1} is "is a brother of". The relation "is a brother of" is symmetric.

Converse Relations (53) immediately needs:

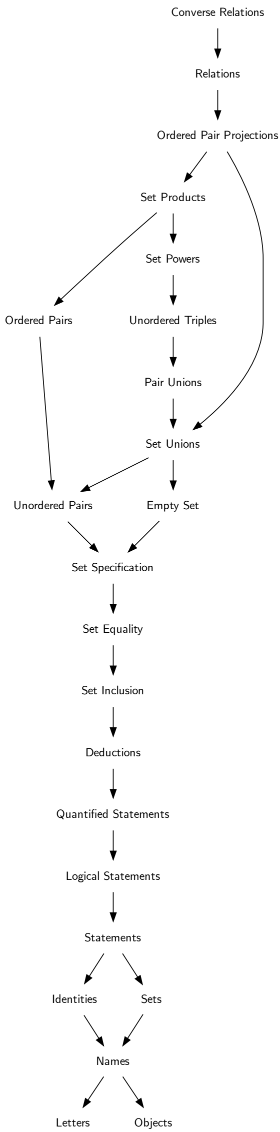
Relations (39)

Converse Relations (53) is immediately needed by:

Comparisons (**??**)
Inverses of Composite Relations (54)

Converse Relations (53) gives the following terms.

converse, inverse.



Converse Relations
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How do inverse and converse relations interact.

Results

Let R be a relation between X and Y and let S be a relation between Y and Z.

Proposition 73. $(RS)^{-1} = S^{-1}R^{-1}$

Identity relations

Recall that I is the identity relation on X if $x\,I\,y$ if and only if $x = y$.

Proposition 74. *Let R be a relation on X. Let I be the identity relation on X. Then $RI = IR = R$.*

One would like $RR^{-1} \supset I$, $R^{-1}R \supset I$. The father of the son is the father and the son of the father is the son. But the empty relation violates these claims.

Relation properties

Proposition 75. *R is symmetric if and only if $R \subset R^{-1}$*

Proposition 76. *R is reflextive if and only if $I \subset R$*

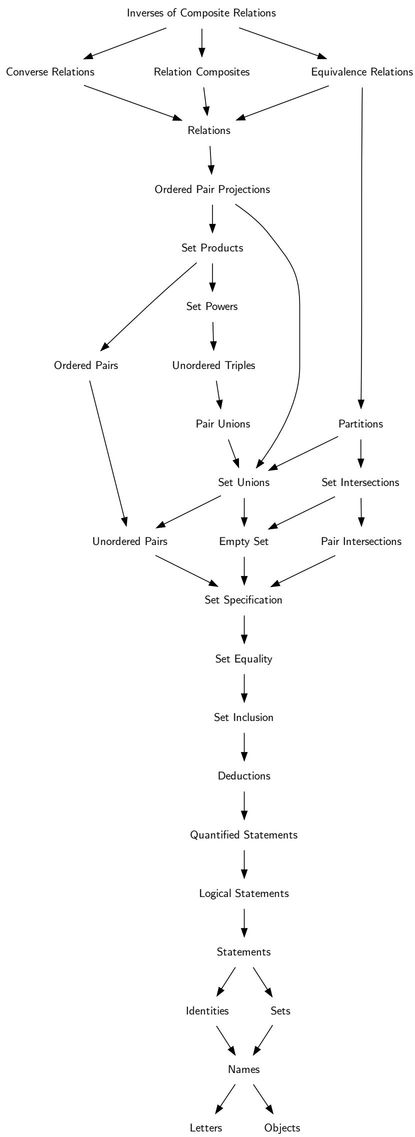
Proposition 77. *R is transitive if and only if $RR \subset R$.*

Inverses of Composite Relations (54) immediately needs:

Converse Relations (53)
Equivalence Relations (40)
Relation Composites (52)

Inverses of Composite Relations (54) is not immediately needed by any sheet.

Inverses of Composite Relations (54) gives no terms.


Inverses of Composite Relations
Converse Relations
Relation Composites
Equivalence Relations
Relations
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Partitions
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want numbers to count with.[53]

Definition

The *successor* of a set is the set which is the union of the set with the singleton of the set. In other words, the successor of a set A is $A \cup \{A\}$. This definition is primarily of interest for the particular sets introduced here.

These sets are the following (and their successors): We call the empty set *zero*.[54] We call the successor of the empty set *one*. In other words, one is $\varnothing \cup \{\varnothing\} = \{\varnothing\}$. We call the successor of one *two*. In other words, two is $\{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$. Likewise, the successor of two we call *three* and the successor of three we call *four*. And we continue as usual,[55] using the English language in the typical way.

A set is a *successor set* if it contains zero and if it contains the successor of each of its elements.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ \coloneqq x \cup \{x\}$$

We denote one by 1. We denote two by 2. We denote three by 3. We

[53]Future editions will expand on this sheet with a more justified why.
[54]In future editions, zero may be a separate sheet.
[55]Future editions will assume less in the introduction of natural numbers.

denote four by 4. So

$$\begin{aligned}
0 &= \varnothing \\
1 &= 0^+ = \{0\} \\
2 &= 1^+ = \{0, 1\} \\
3 &= 2^+ = \{0, 1, 2\} \\
4 &= 3^+ = \{0, 1, 2, 3\}
\end{aligned}$$

Successor Sets (55) immediately needs:

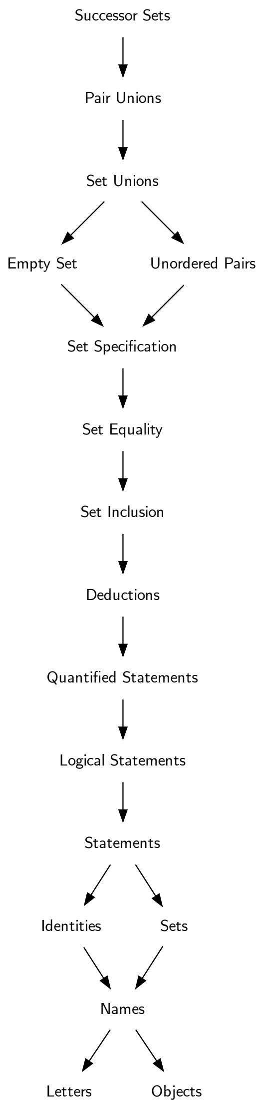
Pair Unions (16)

Successor Sets (55) is immediately needed by:

Natural Numbers (56)

Successor Sets (55) gives the following terms.

successor, zero, one, two, three, four, successor set.


Successor Sets
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What are numbers? We want to count, forever. Does a set exist which contains zero, and one, and two, and three, and all the rest?

Definition

In Successor Sets, we said "and we continue as usual using the English language..." in our definition of zero, and one and two and three. Can this really be carried on and on? We will say yes. We will say that there exists a set which contains zero and contains the successor of each of its elements.

Principle 7 (Natural Numbers)**.** *A set which contains 0 and contains the successor of each of its elements exists.*

This principle is sometimes called the *principle of infinity* (or *axiom of infinity*).

We want this set to be unique. The principle says one successor set exists, but not that it is unique. To see that it is unique, notice that the intersection of a nonempty family of successor sets is a successor set.[56] Consider the intersection of the family of all successor sets. The intersection is nonempty by the principle of infinity (see Intersection of Empty Setfor this subtlety). The principle of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique. We summarize:

Proposition 78 (Minimal Successor Set)**.** *There exists a unique smallest successor set.*

The *set of natural numbers* is the minimal successor set. A *natural number* (or *number*, *natural*) is an element of this minimal successor set.

[56] This account will be expanded in future editions.

Notation

We denote the unique smallest successor set by ω.[57] We denote the set of natural numbers without 0 by N, a mnemonic for natural. In other words $\mathsf{N} = \omega - \{0\}$. We often denote elements of ω or N by n, a mnemonic for number, or m, the letter before m in the conventional ordering of the Latin alphabet(see Letters).

We denote the natural numbers up to n by $\{1, 2, \dots, n\}$. Recall that n *is a set.* In other words, we have defined n so that $n - \{0\} = \{1, 2, \dots, n\}$.

[57]We use this notation to follow many authorities on the subject, and to meet the exigencies of time in producing this first edition. Future editions are likely to rework the treatment.

Natural Numbers (56) immediately needs:

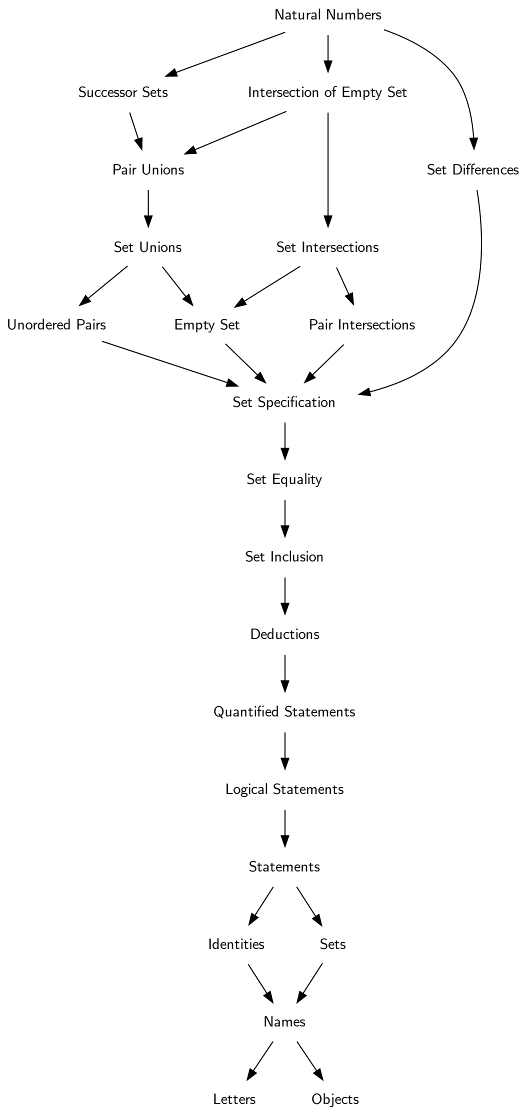
Intersection of Empty Set (20)
Set Differences (24)
Successor Sets (55)

Natural Numbers (56) is immediately needed by:

Categorical Outcome Variables (**??**)
Characteristic Functions (**??**)
Integer Numbers (79)
Natural Induction (57)
Natural Numbers Exercises (**??**)
Number Factorizations (**??**)
Prime Numbers (**??**)
Uncertain Outcomes (**??**)

Natural Numbers (56) gives the following terms.

principle of infinity, axiom of infinity, set of natural numbers, natural number, number, natural, zero, natural numbers with zero, addition.


Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to show something holds for every natural number.[58]

Definition

The most important property of the set of natural numbers is that it is the unique smallest successor set. In other words, if S is a successor set contained in ω (see Natural Numbers), then $S = \omega$. This is useful for proving that a particular property holds for the set of natural numbers.

To do so we follow standard routine. First, we define the set S to be the set of natural numbers for which the property holds. This step uses the principle of selection (see Set Selection) and ensures that $S \subset \omega$. Next we show that this set S is indeed a successor set. The first part of this step is to show that $0 \in S$. The second part is to show that $n \in S \longrightarrow n^+ \in S$. These two together mean that S is a successor set, and since $S \subset \omega$ by definition, that $S = \omega$. In other words, the set of natural numbers for which the property holds is the entire set of natural numbers. We call this the *principle of mathematical induction.*

[58]Future editions will modify this superficial why.

Natural Induction (57) immediately needs:

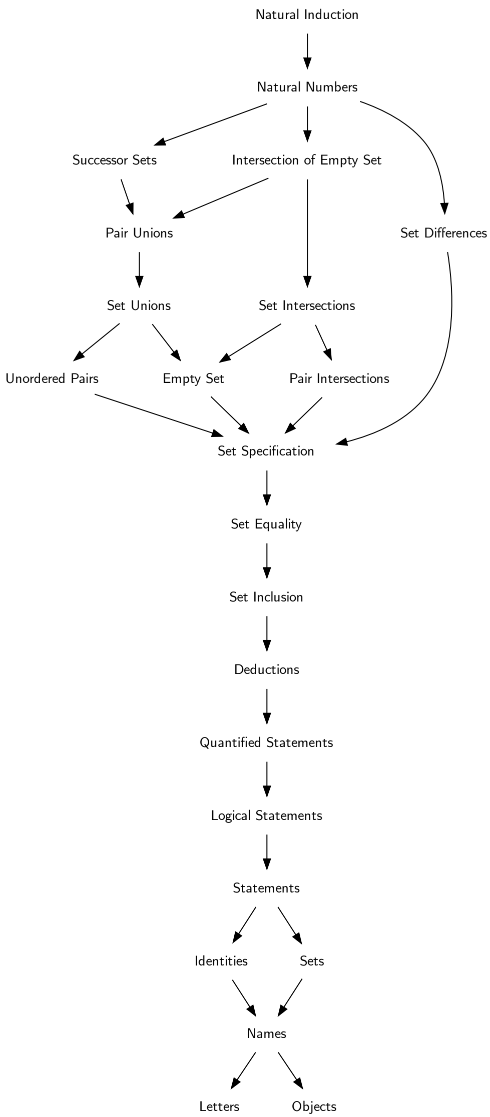
Natural Numbers (56)

Natural Induction (57) is immediately needed by:

Peano Axioms (58)

Natural Induction (57) gives the following terms.

Peano's axioms, principle of mathematical induction..


Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Historically considered a fountainhead for all of mathematics.

Discussion

So far we know that ω is the unique smallest successor set. In other words, we know that $0 \in \omega$, $n \in \omega \longrightarrow n^+ \in \omega$ and that if these two properties hold of some $S \subset \omega$, then $S = \omega$. We can add two important statements to this list. First, that 0 is the successor of no number. In other words, $n^+ \neq 0$ for all $n \in \omega$. Second, that if two numbers have the same successor, then they are the same number In other words, $n^+ = m^+ \longrightarrow n = m$

These five properties were historically considered the fountainhead of all of mathematics. One by the name of Peano used them to show the elementary properties of arithmetic. They are:

1. $0 \in \omega$.
2. $n \in \omega \longrightarrow n^+ \in \omega$ for all $n \in \omega$.
3. If S is a successor set contained in ω, then $S = \omega$.
4. $n^+ \neq 0$ for all $n \in \omega$
5. $n^+ = m^+ \longrightarrow n = m$ for all $n, m \in \omega$.

These are collectively known as the *Peano axioms.* Recall that the third statement in this list is the *principle of mathematical induction.*

Statements

Here are the statements.

Proposition 79 (Peano's First Axiom)**.** *$0 \in \omega$.*

Proposition 80 (Peano's Second Axiom)**.** *$n \in \omega \longrightarrow n^+ \in \omega$.*

Proposition 81 (Peano's Third Axiom)**.** *Suppose* $S \subset \omega$, $0 \in S$, *and* $(n \in S \longrightarrow n^+ \in S$. *Then* $S = \omega$.

Proposition 82 (Peano's Fourth Axiom)**.** $n^+ \neq 0$ *for all* $n \in \omega$.

The last one uses the following two useful facts.

Proposition 83. $x \in n \longrightarrow n \not\subset x$.

Proposition 84. $(x \in y \wedge y \in n) \longrightarrow x \in n$

This latter proposition is sometimes described by saying that n is a *transitive set.* This notion of transitivity is not the same as that described in Relations. Using these one can show:

Proposition 85 (Peano's Fifth Axiom)**.** *Suppose* $n, m \in \omega$ *with* $n^+ = m^+$. *Then* $n = m$.

Peano Axioms (58) immediately needs:

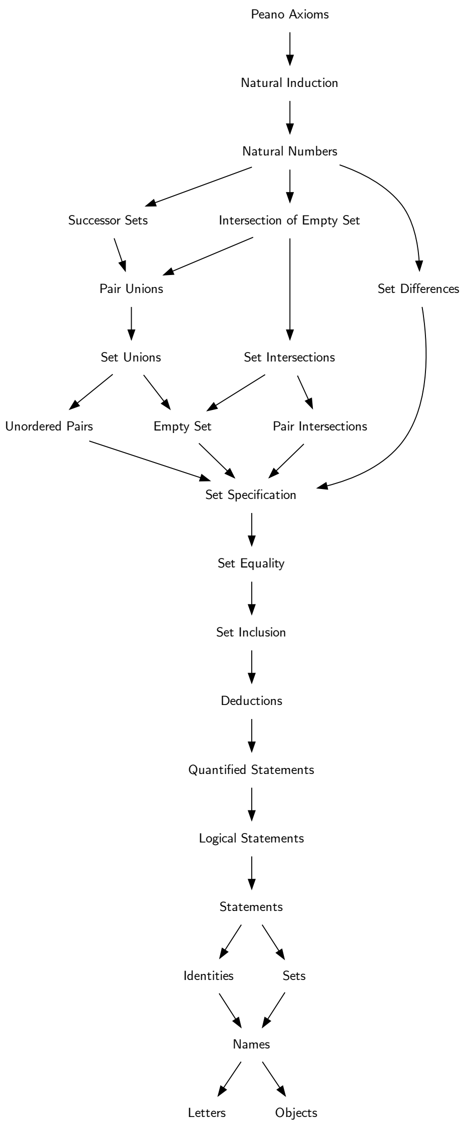
Natural Induction (57)

Peano Axioms (58) is immediately needed by:

Natural Order (63)
Recursion Theorem (59)

Peano Axioms (58) gives the following terms.

Peano axioms, principle of mathematical induction, transitive set.


Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

It is natural to want to define a sequence by giving its first term and then giving its later terms as functions of its earlier ones. In other words, we want to define sequences inductively.[59]

Proposition 86 (Recursion theorem)**.** *Let X be a set, let $a \in X$ and let $f : X \to X$. There exists a unique function u so that $u(0) = a$ and $u(n^+) = f(u(n))$.*[60]

When one uses the recursion theorem to assert the existence of a function with the desired properties, it is called *definition by induction.*

[59] Future editions will expand on this. We are really headed toward natural addition, multiplication and exponentiation.

[60] The account is somewhat straightforward, given a good understanding of the results of Peano Axioms. The full account will appear in future editions.

Recursion Theorem (59) immediately needs:

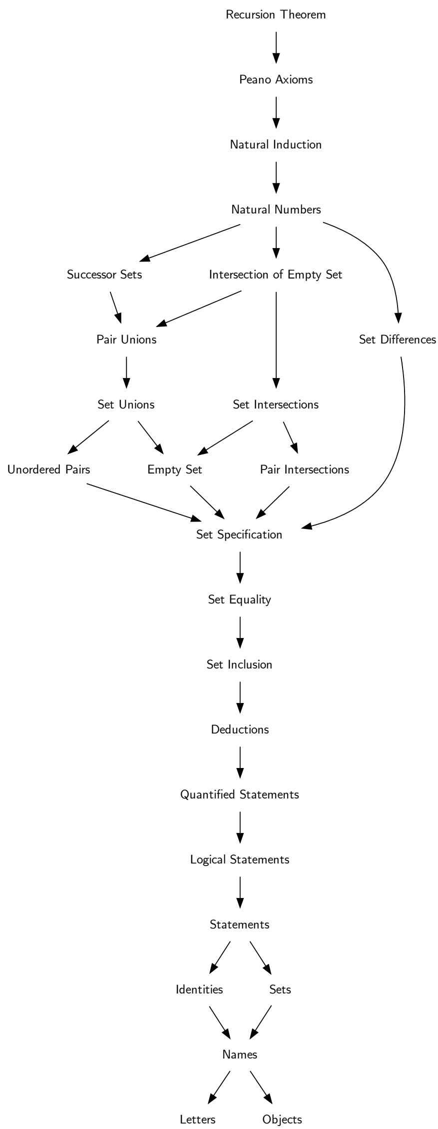
Peano Axioms (58)

Recursion Theorem (59) is immediately needed by:

Natural Sums (60)

Recursion Theorem (59) gives the following terms.

definition by induction.


Recursion Theorem
Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to combine two groups.[61]

Defining result

Proposition 87. *For each natural number m, there exists a function $s_m : \omega \to \omega$ which satisfies*

$$s_m(0) = m \quad \textit{and} \quad s_m(n^+) = (s_m(n))^+$$

for every natural number n.

Proof. The proof uses the recursion theorem (see Recursion Theorem).[62] □

Let m and n be natural numbers. The value $s_m(n)$ is the *sum* of m with n.

Notation

We denote the sum $s_m(n)$ by $m + n$.

Properties

The properties of sums are direct applications of the principle of mathematical induction (see Natural Induction).[63]

Proposition 88 (Associative)**.** *Let k, m, and n be natural numbers. Then*

$$(k + m) + n = k + (m + n).$$

Proposition 89 (Commutative)**.** *Let m and n be natural numbers. Then*

$$m + n = n + m.$$

[61] Future editions will change this section.
[62] Future editions will give the entire account.
[63] Future editions will include the accounts.

Relation to addition

Proposition 90 (Distributive)**.** *Let k, m, and n be natural numbers. Then*

$$k \cdot (m + n) = (k \cdot m) + (k \cdot n).$$

Natural Sums (60) immediately needs:

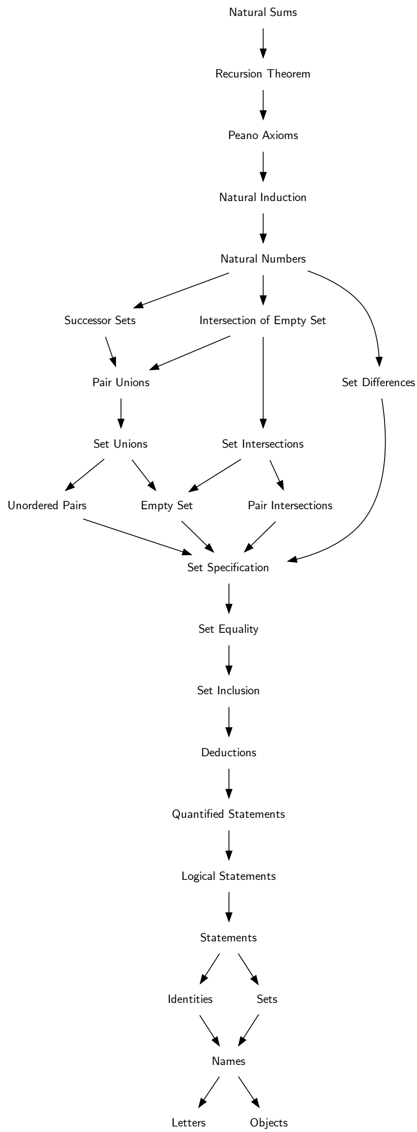
Recursion Theorem (59)

Natural Sums (60) is immediately needed by:

Integer Order (83)
Integer Partitions (**??**)
Integer Sums (80)
Natural Equations (**??**)
Natural Products (61)
Natural Summation (**??**)
Number of Disjoint Unions (**??**)
Number Partitions (**??**)

Natural Sums (60) gives the following terms.

sum.



Natural Sums
Recursion Theorem
Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to add repeatedly.

Definitiong result

Proposition 91. *For each natural number m, there exists a function $p_m : \omega \to \omega$ which satisfies*

$$p_m(0) = 0 \quad \textit{and} \quad p_m(n^+) = (p_m(n))^+ + m$$

for every natural number n.

Proof. The proof uses the recursion theorem (see Recursion Theorem).[64] □

Let m and n be natural numbers. The value $p_m(n)$ is the *product* of m with n.

Notation

We denote the product $p_m(n)$ by $m \cdot n$. We often drop the $\cdot$ and write $m \cdot n$ as mn.

Properties

The properties of products are direct applications of the principle of mathematical induction (see Natural Induction).[65]

Proposition 92 (Associativity)**.** *Let k, m, and n be natural numbers. Then*

$$(k \cdot m) \cdot n = k \cdot (m \cdot n).$$

Proposition 93. *Let m and n be natural numbers. Then*

$$m \cdot n = n \cdot m.$$

[64]Future editions will give the entire account.
[65]Future editions will include the accounts.

Natural Products (61) immediately needs:

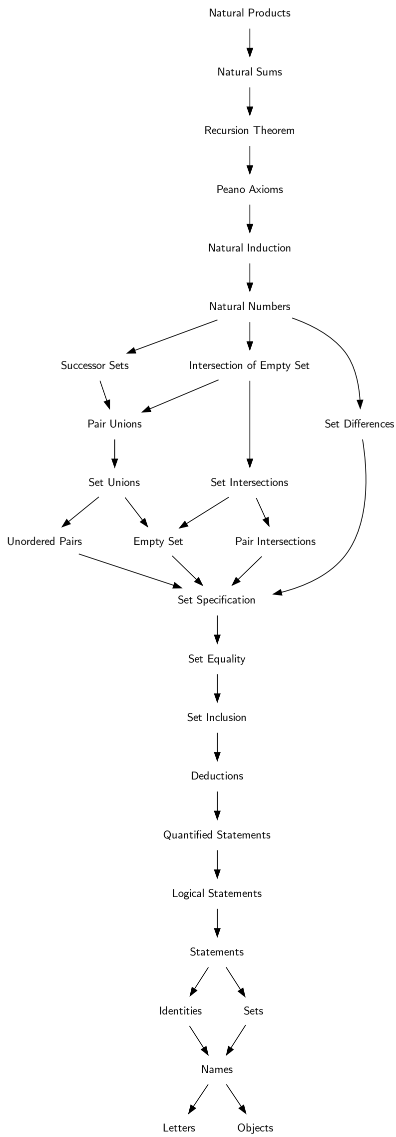
Natural Sums (60)

Natural Products (61) is immediately needed by:

Factorials (**??**)
Integer Products (81)
Natural Powers (62)
Number of Set Products (**??**)
Order and Arithmetic (64)
Prime Numbers (**??**)
Square Numbers (**??**)

Natural Products (61) gives the following terms.

product, sum, add, addition, product, multiply, multiplication.


Natural Products
Natural Sums
Recursion Theorem
Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to repeatedly multiply.

Defining result

Proposition 94. *For each natural number m, there exists a function $e_m : \omega \to \omega$ which satisfies*

$$e_m(0) = 1 \quad \textit{and} \quad e_m(n^+) = (e_m(n))^+ \cdot m$$

for every natural number n.

Proof. The proof uses the recursion theorem (see Recursion Theorem).[66] □

Let m and n be natural numbers. The value $p_m(n)$ is the *power* of m with n. Or the *nth power of m*

Notation

We denote the nth power of m by m^n.

Properties

Here are some basic properties of powers.

Proposition 95. *Let k, m, and n be natural numbers. Then*

$$m^n m^k = m^{k+k}.$$

Proposition 96. *Let k, m, and n be natural numbers. Then*

$$(m^n)^k = m^{nk}.$$

[66] Future editions will give the entire account.

Natural Powers (62) immediately needs:

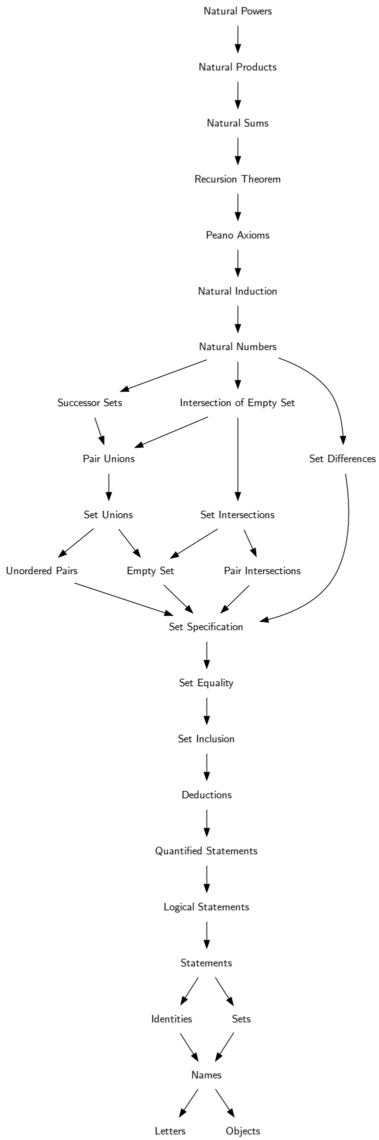
Natural Products (61)

Natural Powers (62) is immediately needed by:

Bit Strings (**??**)
Natural Arithmetic (73)
Natural Number Notation (**??**)

Natural Powers (62) gives the following terms.

power, *nth power of m*.


Natural Powers
Natural Products
Natural Sums
Recursion Theorem
Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We count in order.[67]

Defining result

We say that two natural numbers m and n are *comparable* if $m \in n$ or $m = n$ or $n \in m$.

Proposition 97. *Any two natural numbers are comparable.*[68]

In fact, more is true.

Proposition 98. *For any two natural numbers, exactly one of $m \in n$, $m = n$ and $n \in m$ is true.*[69]

Proposition 99. $m \in n \longleftrightarrow m \subset n$.

If $m \in n$, then we say that m is *less than* n. We also say in this case that m is *smaller than* n. If we know that $m = n$ or m is less than n, we say that m is *less than or equal to* n.

Notation

If m is less than n we write $m < n$, read aloud "m less than n." If m is less than or equal to n, we write $m \leqq n$, read alout "m less than or equal to n."

Properties

Notice that $<$ and $\leqq$ are relations on ω (see Relations).[70]

Proposition 100 (Reflexivity). *$\leqq$ is reflexive, but $<$ is not.*

---

[67] Future editions will expand.

[68] Future editions will include an account.

[69] Use the fact that no natural number is a subset of itself. Future editions will expand this account. See Peano Axioms).

[70] Proofs of the following propositions will appear in future editions.

Proposition 101 (Symmetry)**.** *Both* $\leqq$ *and* $<$ *are not symmetric.*

Proposition 102 (Transitivity)**.** *Both* $\leqq$ *and* $<$ *are transitive.*

Proposition 103 (Antisymmetry)**.** *If* $m \leqq n$ *and* $n \leqq n$*, then* $m = n$*.*

Natural Order (63) immediately needs:

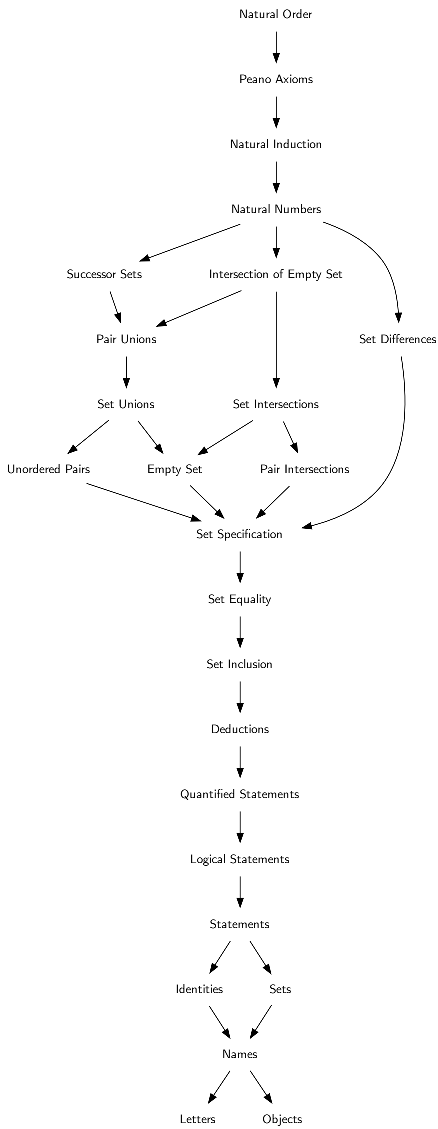
Peano Axioms (58)

Natural Order (63) is immediately needed by:

Equivalent Sets (65)
Natural Equations (**??**)
Order and Arithmetic (64)
Orders (**??**)

Natural Order (63) gives the following terms.

Peano's axioms, comparable, less than, smaller than, less than or equal to.


Natural Order
Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How does arithmetic preserve order?

Results

The following are standard useful results.[71]

Proposition 104. *If $m < n$, then $m + k < n + k$ for all k.*

Proposition 105. *If $m < n$ and $k \neq 0$, then $m \cdot k < n \cdot k$.*

Proposition 106 (Least Element)**.** *If E is a nonempty set of natural numbers, there exists $k \in E$ such that $k \leq m$ for all $m \in E$.*

Proposition 107 (Greatest Element)**.** *If E is a nonempty set of natural numbers, there exists $k \in E$ such that $m \leq k$ for all $m \in E$.*

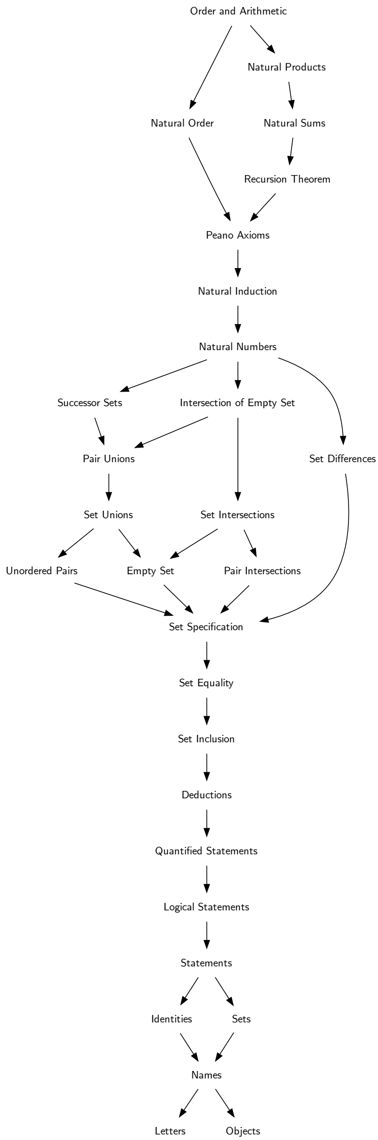
[71]The accounts of which will appear in future editions.

Order and Arithmetic (64) immediately needs:

Natural Order (63)
Natural Products (61)

Order and Arithmetic (64) is not immediately needed by any sheet.

Order and Arithmetic (64) gives no terms.


Order and Arithmetic
Natural Products
Natural Order
Natural Sums
Recursion Theorem
Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to talk about the size of a set.

Definition

Two sets are *equivalent* if there exists a bijection between them. Let X be a set. Then set equivalence as a relation in $\mathcal{P}(X)$ is an equivalence relation (see Equivalence Relations).

Notation

If A and B are sets and they are equivalent, then we write $A \sim B$, read aloud as "A is equivalent to B."

Basic result

Every set is equivalent to itself, whether two sets are equivalent does not depend on the order in which we consider them, and if two sets are equivalent to the same set then they are equivalent to each other. These facts can be summarized by the following proposition.

Proposition 108. *Let X a set. Then $\sim$ is an equivalence relation on $\mathcal{P}(X)$.*[72]

For natural numbers

Proposition 109. *Every proper subset of a natural number is equivalent to some smaller natural number.*[73]

Equivalence to subsets

It is unusual that a set can be equivalent to a proper subset of itself.

Proposition 110. *A set may be equivalent to a proper subset of itself.*

[72] The proof is direct and will appear in future editions.
[73] The proof, which uses induction, will appear in future editions.

Proof. The example is the set of natural numbers and the function $f(n) = n^+$. It is a bijection from ω onto $\mathbf{N}$.[74] $\square$

However, this never holds for natural numbers.

Proposition 111. *If $n \in \omega$ then $n \not\sim x$ for any $x \subset n$ and $x \neq n$.*

[74] The account will be expanded in future editions.

Equivalent Sets (65) immediately needs:

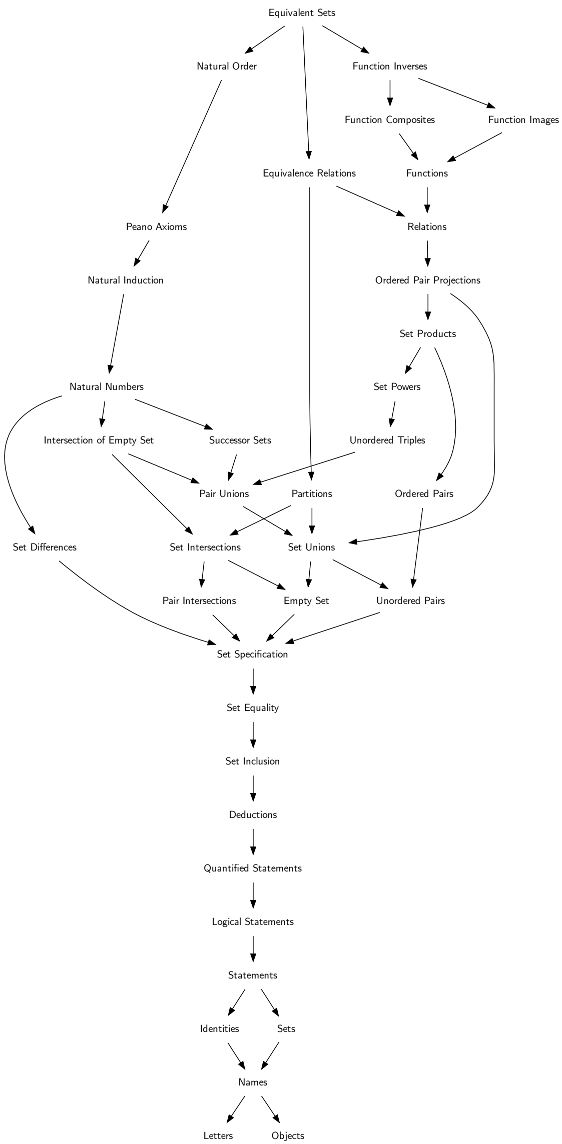
Equivalence Relations (40)
Function Inverses (50)
Natural Order (63)

Equivalent Sets (65) is immediately needed by:

Finite Sets (66)

Equivalent Sets (65) gives the following terms.

equivalent.


Equivalent Sets
Natural Order
Function Inverses
Function Composites
Function Images
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Ordered Pair Projections
Set Products
Natural Numbers
Set Powers
Intersection of Empty Set
Successor Sets
Unordered Triples
Pair Unions
Partitions
Ordered Pairs
Set Differences
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

As with introducing Equivalent Sets, we want to talk about the size of a set.[75]

Definition

A *finite* set is one that is equivalent to some natural number; an infinite set is one which is not finite. From this we can show that ω is infinite. This justifies the language "principle of infinity" with Natural Numbers. The principle of infinity asserts the existence of a particular infinite set; namely ω.

Motivation for set number

It happens that if a set is equivalent to a natural number, it is equivalent to only one natural number.

Proposition 112. *A set can be equivalent to at most one natural number.*[76]

A consequence is that a finite set is never equivalent to a proper subset of itself. So long as we are considering finite sets, a piece (subset) is always less than than the whole (original set).

Proposition 113. *A finite set is never equivalent to a proper subset of itself.*

Subsets of finite sets

Every subset of a natural number is equivalent to a natural number.[77] A consequence is:

[75] Will be expanded in future editions.

[76] Future edition will include proof, which uses comparability of numbers and the results of Equivalent Sets.

[77] This requires proof, and may become a proposition in future editions.

Proposition 114. *Every subset of a finite set is finite.*[78]

Unions of finite sets

Proposition 115. *If A and B are finite, then $A \cup B$ is finite.*

Products of finite sets

Proposition 116. *If A and B are finite, then $A \times B$ is finite.*

Powers of finite sets

Proposition 117. *If A is finite then $\mathcal{P}(A)$ is finite.*

Functions between finite sets

Proposition 118. *If A and B are finite, then A^B is finite.*

[78] An account will appear in future editions.

Finite Sets (66) immediately needs:

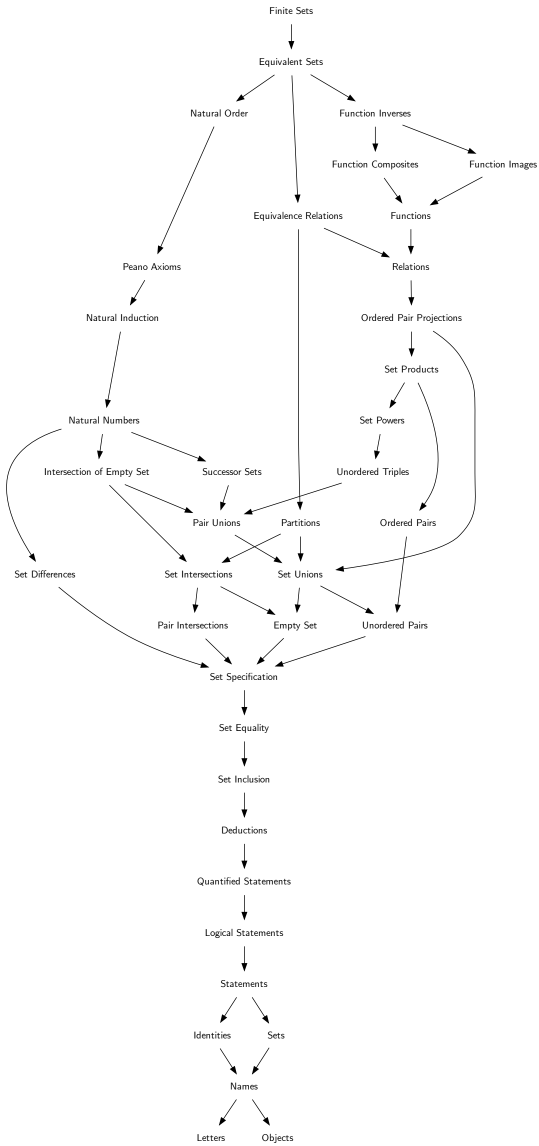
Equivalent Sets (65)

Finite Sets (66) is immediately needed by:

Groups (87)
Set Numbers (67)
Submodular Functions (**??**)

Finite Sets (66) gives the following terms.

finite.


Finite Sets
Equivalent Sets
Natural Order
Function Inverses
Function Composites
Function Images
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Ordered Pair Projections
Set Products
Natural Numbers
Set Powers
Intersection of Empty Set
Successor Sets
Unordered Triples
Pair Unions
Partitions
Ordered Pairs
Set Differences
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to count the number of elements in a set.

Defining result

Proposition 119. *A set can be equivalent to at most one natural number.*[79]

The *number* (or *size*) of a finite set is the unique natural number equivalent to it.

Notation

We denote the number of a set by $|A|$. Equally good notation, which we will not use in these sheets, is $\#(A)$.

Restriction to a finite set

If we restrict $E \mapsto |E|$ to the domain $\mathcal{P}(X)$ of some set X then $|\cdot| : \mathcal{P}(X) \to \omega$ is a function.[80]

Properties

Proposition 120. $A \subset B \longrightarrow |A| \leq |B|$

[79] A proof will appear in future editions.
[80] Future editions will clarify this point.

Set Numbers (67) immediately needs:

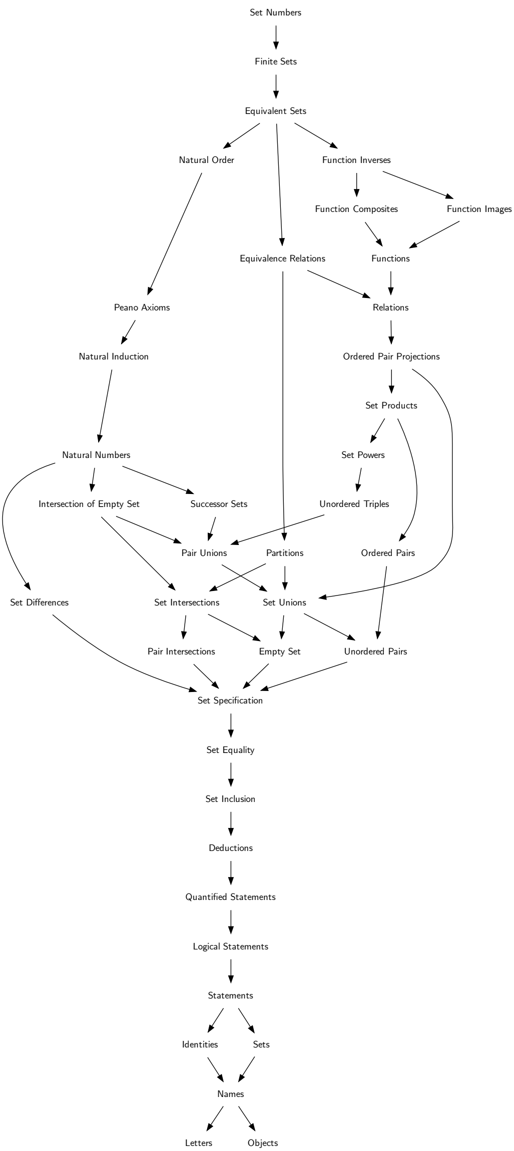
Finite Sets (66)

Set Numbers (67) is immediately needed by:

Cardinality (??)
Categorical Outcome Variables (??)
Decision Processes (??)
Decisions (??)
Directed Graphs (??)
Empirical Distribution (??)
Finite Set Examples (68)
Games (??)
Lists (??)
Number of Disjoint Unions (??)
Outcome Probabilities (??)
Permutations (??)
Undirected Graphs (??)

Set Numbers (67) gives the following terms.

number, size.


Set Numbers
Finite Sets
Equivalent Sets
Natural Order
Function Inverses
Function Composites
Function Images
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Ordered Pair Projections
Set Products
Natural Numbers
Set Powers
Intersection of Empty Set
Successor Sets
Unordered Triples
Pair Unions
Partitions
Ordered Pairs
Set Differences
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of lower case latin letters (introduced in Letters): a, b, c, …, x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Finite Set Examples (68) immediately needs:

Set Numbers (67)

Finite Set Examples (68) is immediately needed by:

Size of Direct Product (**??**)

Finite Set Examples (68) gives no terms.


Finite Set Examples
Set Numbers
Finite Sets
Equivalent Sets
Natural Order
Function Inverses
Function Composites
Function Images
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Ordered Pair Projections
Set Products
Natural Numbers
Set Powers
Intersection of Empty Set
Successor Sets
Unordered Triples
Pair Unions
Partitions
Ordered Pairs
Set Differences
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How does the number of elements change with unions, and products.

Results

There are a few nice relations.[81] Recall from Finite Setsthat the union and product of finite sets is finite. Also, the power of a finite set is finite.

Proposition 121. *Let A and B be finite sets with $A \cap B = \varnothing$. Then* $|A \cup B| = |A| + |B|$.

Proposition 122. *Let A and B be a finite sets Then* $|A \times B| = |A| \cdot |B|$.

Proposition 123. *Let A and B be a finite sets Then* $\left|A^B\right| = |A|^{|B|}$.

Proposition 124. *Let A be a finite set. Then* $|\mathcal{P}(A)| = 2^{\nu m A}$.

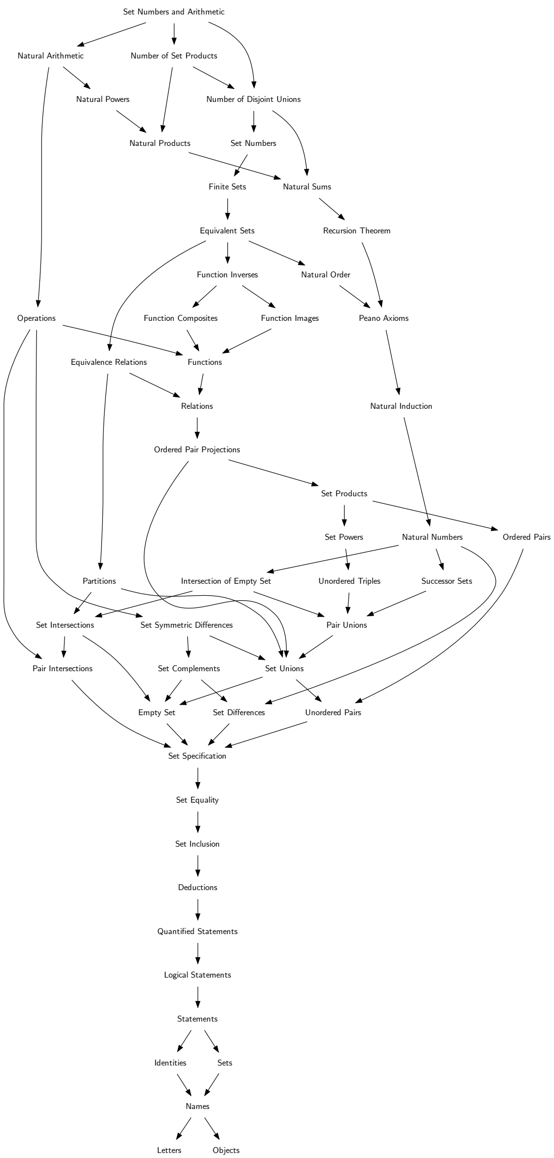
[81] Proofs will appear in future editions.

Set Numbers and Arithmetic (69) immediately needs:

Natural Arithmetic (73)
Number of Disjoint Unions (??)
Number of Set Products (??)

Set Numbers and Arithmetic (69) is not immediately needed by any sheet.

Set Numbers and Arithmetic (69) gives no terms.


Set Numbers and Arithmetic
Natural Arithmetic
Number of Set Products
Natural Powers
Number of Disjoint Unions
Natural Products
Set Numbers
Finite Sets
Natural Sums
Equivalent Sets
Recursion Theorem
Function Inverses
Natural Order
Operations
Function Composites
Function Images
Peano Axioms
Equivalence Relations
Functions
Relations
Natural Induction
Ordered Pair Projections
Set Products
Set Powers
Natural Numbers
Ordered Pairs
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to speak of infinite processes, and to do so we define sequences indexed by $\mathbf{N}$. In other words, important families are those indexed by the natural numbers.

Definition

A *sequence* (or *infinite sequence*) is a family whose index set is $\mathbf{N}$ (the set of natural numbers without zero). The *nth term* or *coordinate* of a sequence is the result of the nth natural number, $n \in \mathbf{N}$.[82]

Notation

Let A be a non-empty set and $a : \mathbf{N} \to A$. Then a is a (infinite) sequence in A. $a(n)$ is the nth term. We also denote a by $(a_n)_n$ and $a(n)$ by a_n. If $\{A_n\}_{n \in \mathbf{N}}$ is an infinite sequence of sets, then we denote the direct product of the sequence by $\prod_{i=1}^{\infty} A_i$.

Natural unions and intersections

We denote the family of the infinite sequence of sets $(A_n)_n$ by $\cup_{i=1}^{\infty} A_i$. Similarly, we denote the intersection of an infinite sequence of sets by $\cap_{i=1}^{\infty} A_i$, respectively.

[82]Future editions may also comment that we are introducing language for the steps of an infinite process.

Sequences (70) immediately needs:

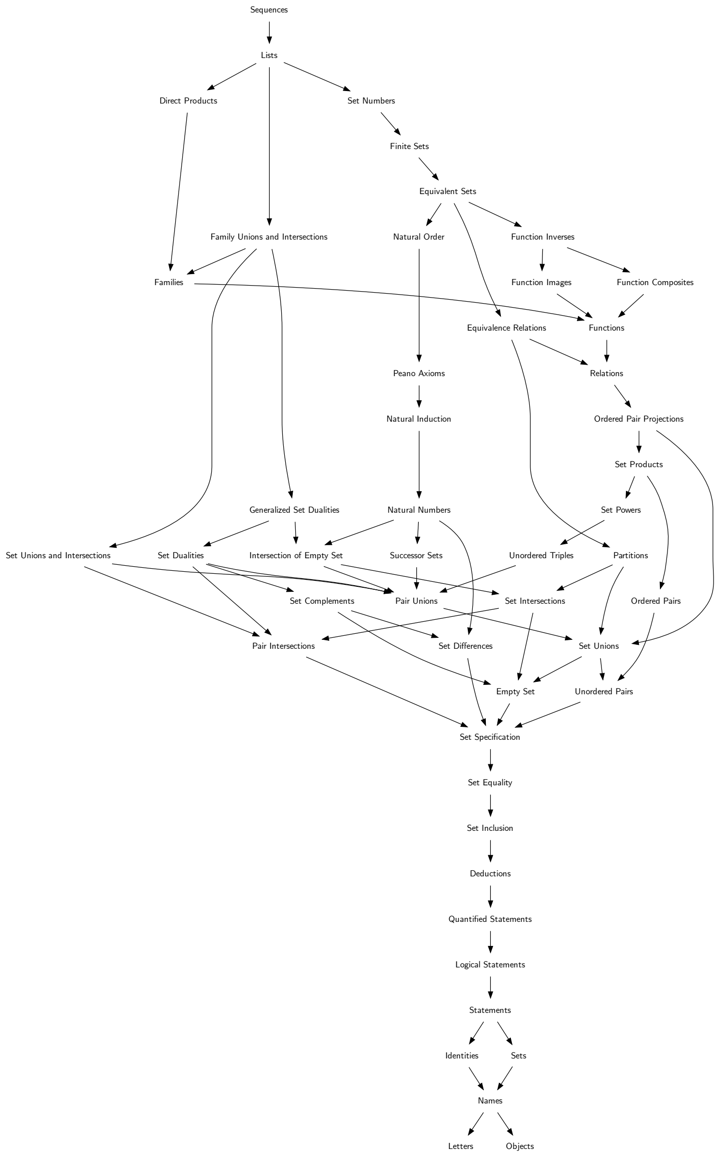
Lists (??)

Sequences (70) is immediately needed by:

Factorials (??)
Monotone Classes (??)
Monotone Sequences (??)
Nets (??)
Real Sequences (??)
Sequential Decisions (??)
Subsequences (71)

Sequences (70) gives the following terms.

sequence, infinite sequence, nth term, coordinate.


Sequences
Lists
Direct Products
Set Numbers
Finite Sets
Equivalent Sets
Family Unions and Intersections
Natural Order
Function Inverses
Families
Function Images
Function Composites
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Ordered Pair Projections
Set Products
Generalized Set Dualities
Natural Numbers
Set Powers
Set Unions and Intersections
Set Dualities
Intersection of Empty Set
Successor Sets
Unordered Triples
Partitions
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Pair Intersections
Set Differences
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to select particular terms of sequence.

Definition

A *subindex* is a monotonically increasing function from and to the natural numbers. Roughly, it selects some ordered infinite subset of natural numbers. A *subsequence* of a first sequence is any second sequence which is the composition of the first sequence with a subindex.

Notation

Let $i : \mathbf{N} \to \mathbf{N}$ such that $n < m \Rightarrow i(n) < i(m)$. Then i is a subindex. Let $b = a \circ i$. Then b is a subsequence of a. We denote it by $\{b_{i(n)}\}_n$ and the nth term by $b_{i(n)}$.

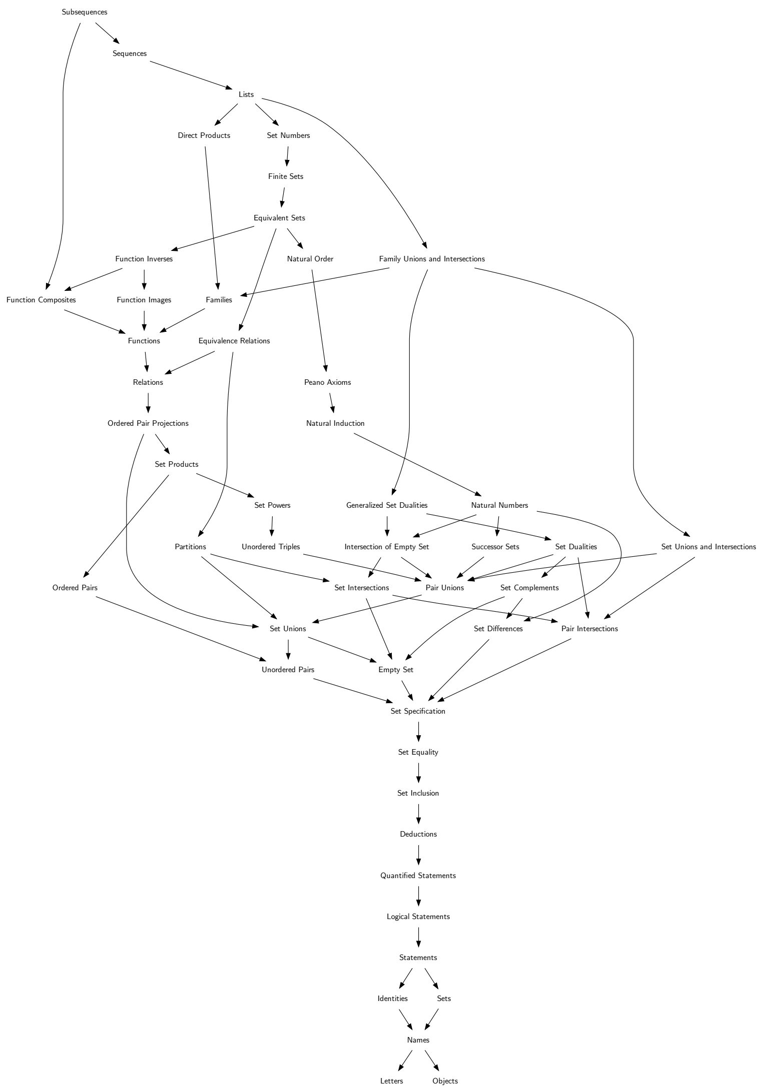
Subsequences (71) immediately needs:

Function Composites (49)
Sequences (70)

Subsequences (71) is not immediately needed by any sheet.

Subsequences (71) gives the following terms.

subindex, subsequence.


Subsequences
Sequences
Lists
Direct Products
Set Numbers
Finite Sets
Equivalent Sets
Function Inverses
Natural Order
Family Unions and Intersections
Function Composites
Function Images
Families
Functions
Equivalence Relations
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Set Powers
Generalized Set Dualities
Natural Numbers
Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Dualities
Set Unions and Intersections
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We have seen several concepts that consist of associating a pair of sets with a third set. For example, set unions and set intersections

Definition

An *operation* (or *binary operation*, *law of composition*) on a set A is a function from $A \times A$ to A.

Roughly speaking, operations *combine* (or *compose*) elements. We *operate* on ordered pairs.

Example: set operations

Let X be a set. Define $g : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ by $g(A, B) = A \cup B$. Then g, the function which associates with two sets their unionis an operation on $\mathcal{P}(X)$. Likewise, define $h : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ by $h(A, B) = A \cap B$.

Naming their properties

$\cup$ has several nice properties. For one $A \cup B = B \cup A$ and $(A \cup B) \cup C = A \cup (B \cup C)$.

An operation with the first property, that the ordered pair (A, B) and (B, A) have the same result is called *commutative.* An operation with the second property, that when given three objects the order in which we operate does not matter is called *associative.* $\cap$ shares these properties with $\cup$.

We call the operation of *forming unions* the function $(A, B) \mapsto A \cup B$. We call the operation of *forming intersections* the function $(A, B) \mapsto A \cap B$. We call the operation of *forming symmetric differences* the function $(A, B) \mapsto A + B$. Since forming unions commutes and is associative and likewise with forming intersections, forming symmetric differences also commutes.

Algebras

Of course, any operation is defined on some set. For this reason, we define an *algebra* (or *algebraic structure*) as an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* (or *underlying set*, *carrier set*, *domain*) of the algebra is the set on which the operation is defined.

Operations (72) immediately needs:

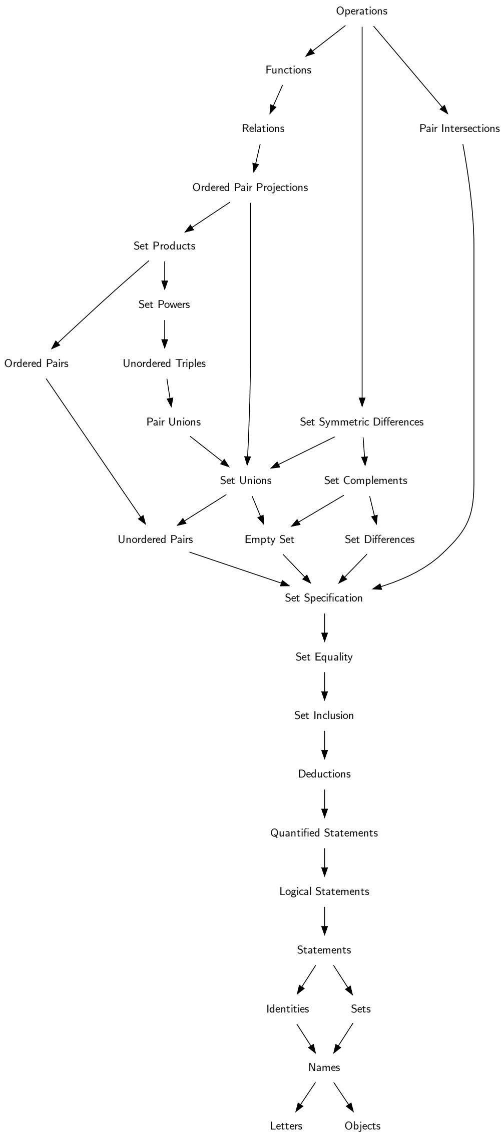
Functions (41)
Pair Intersections (18)
Set Symmetric Differences (29)

Operations (72) is immediately needed by:

Commutative Operations (**??**)
Element Functions (74)
Extended Real Numbers (**??**)
Family Operations (**??**)
Identity Elements (75)
Isomorphisms (86)
Natural Arithmetic (73)
Pointwise and Measure Limits (**??**)
Subset Algebras (**??**)
Uncertain Events (**??**)

Operations (72) gives the following terms.

operation, binary operation, law of composition, combine, compose, operate, commutative, associative, forming unions, forming intersections, forming symmetric differences, algebra, algebraic structure, ground set, underlying set, carrier set, domain.


Operations
Functions
Relations
Pair Intersections
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We name the operations which produce natural sums, products and powers.

Definition

Consider the set of natural numbers. The we can define three functions corresponding to sums, products and powers which are operations (see Operations) on this set.

We call *addition* the function $+ : \omega \times \omega \to \omega$, which maps two natural numbers m and n to their sum $m + n$. We call *multiplication* the function $\cdot : \omega \times \omega \to \omega$, which maps two natural numbers m and n to their product $m \cdot n$. We call *exponentiation* the function $(m, n) \mapsto m^n$.

In other words, we can think of sums, products, and powers as obtainable by applying a function to pairs of natural numbers. This function gives another natural numbers We call these three operations the operations of *arithmetic*.

Natural Arithmetic (73) immediately needs:

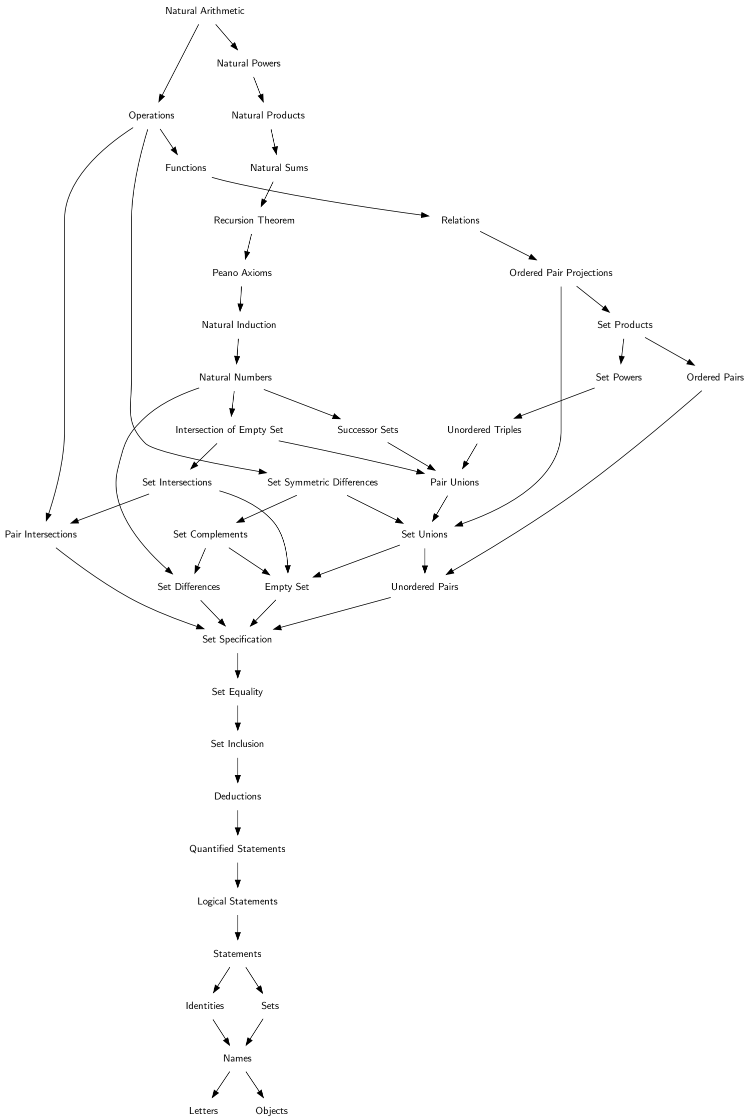
Natural Powers (62)
Operations (72)

Natural Arithmetic (73) is immediately needed by:

Natural Additive Identity (76)
Natural Fractions (**??**)
Natural Multiplicative Identity (77)
Set Numbers and Arithmetic (69)

Natural Arithmetic (73) gives the following terms.

addition, multiplication, exponentiation, arithmetic.


Natural Arithmetic
Natural Powers
Operations
Natural Products
Functions
Natural Sums
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Take an element of an algebra, and consider the function defined on the ground set which maps elements to the result of the operation applied to the fixed element and the given element.

Definition

Let $(A, +)$ be an algebra. For each $a \in A$, denote by $+_a : A \to A$ the function defined by

$$+_a(b) = a + b.$$

We call $+_a$ the *left element function* of a.

Similarly, denote by $+^a : A \to A$ the function defined by

$$+^a(b) = b + a.$$

We call $+^a$ the *right element function* of a

The idea is that elements of an algebra can always be associated with functions.

Element Functions (74) immediately needs:

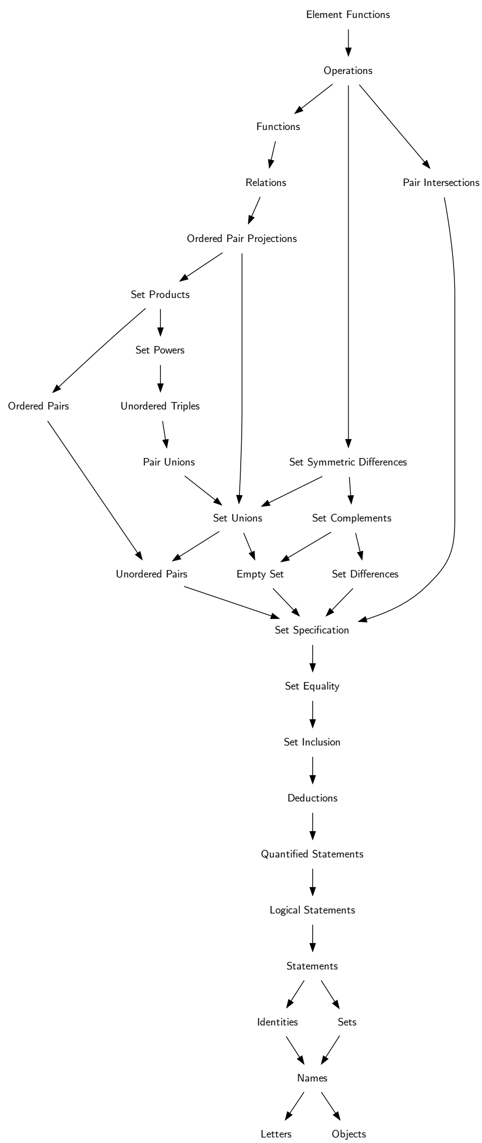
Operations (72)

Element Functions (74) is immediately needed by:

Inverse Elements (78)

Element Functions (74) gives the following terms.

left element function, right element function.


Element Functions
Operations
Functions
Relations
Pair Intersections
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We can construct functions on the ground set of an algebra by fixing an element in the ground set and defining a function which maps elements to the result of the operation applied to the fixed element and the given element.

Definition

Let $(A, +)$ be an algebra. For each $a \in A$, denote by $+_a : A \to A$ the function defined by

$$+_a(b) = a + b.$$

If $+_a$ is the identity function on A then we call a a *left identity element* of the algebra.

Similarly, denote by $+^a : A \to A$ the function defined by

$$+^a(b) = b + a.$$

If $+^a$ is the identity function on A then we call a a *right identity element* of the algebra.

An *identity element* of the algebra is an element which is both a left and right identity. If the operation commutes, then a left identity and right identities are the same.

Identity Elements (75) immediately needs:

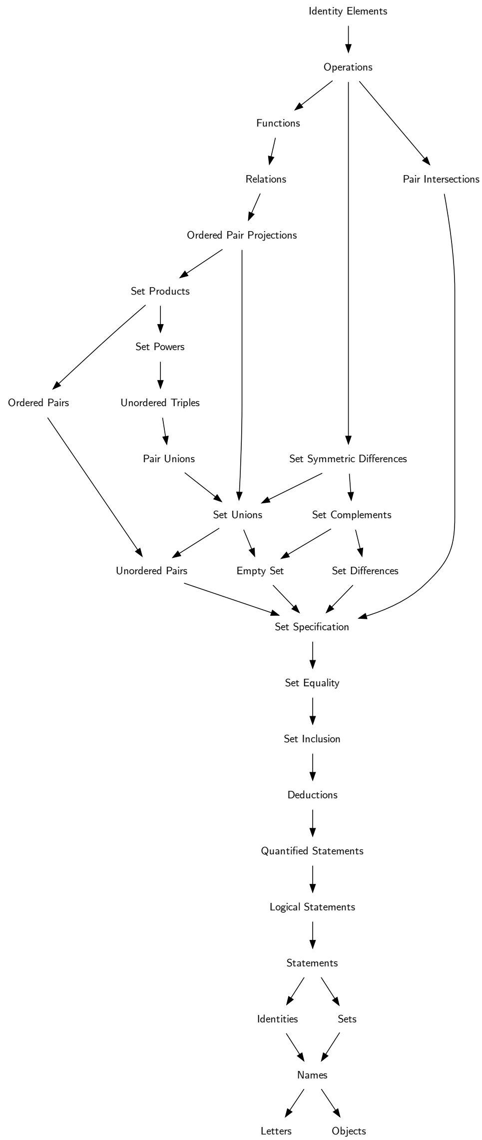
Operations (72)

Identity Elements (75) is immediately needed by:

Natural Additive Identity (76)
Natural Multiplicative Identity (77)

Identity Elements (75) gives the following terms.

left identity element, right identity element, identity element.


Identity Elements
Operations
Functions
Relations
Pair Intersections
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the identity element of addition of the natural numbers.

Result

Proposition 125. *0 is the identity element of ω under $+$.*

Proof. By definition $0 + n = n$ (see Natural Sums). □

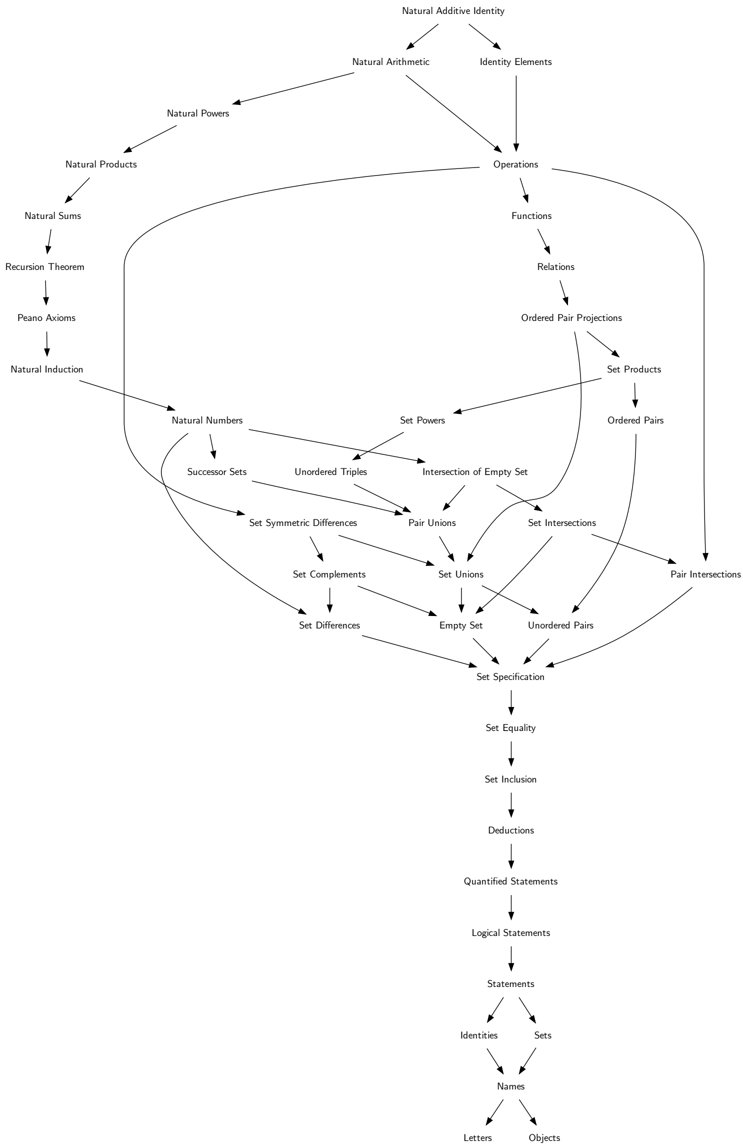
Natural Additive Identity (76) immediately needs:

Identity Elements (75)
Natural Arithmetic (73)

Natural Additive Identity (76) is immediately needed by:

Integer Arithmetic (84)

Natural Additive Identity (76) gives no terms.


Natural Additive Identity
Natural Arithmetic
Identity Elements
Natural Powers
Natural Products
Operations
Natural Sums
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Successor Sets
Unordered Triples
Intersection of Empty Set
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Set Unions
Pair Intersections
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the identity element of natural multiplication?

Proposition 126. *1 is the identity element of ω under $\cdot$.*

Proof. By definition $1 \cdot n = n$ (see Natural Products). □

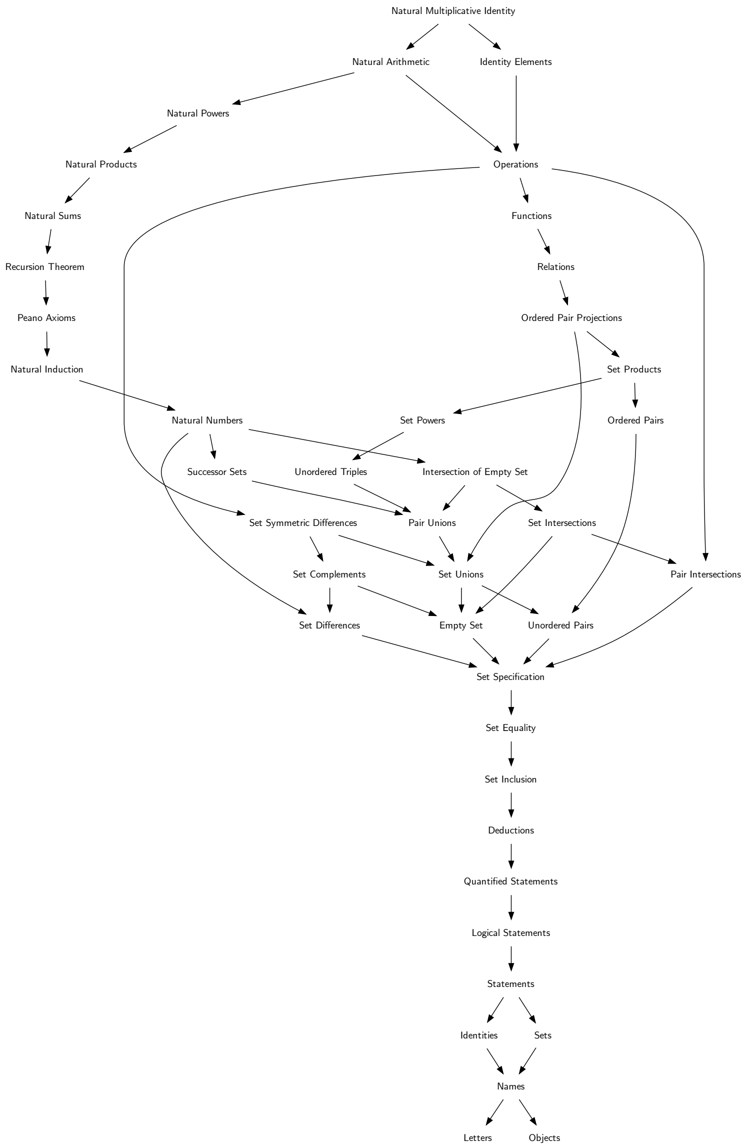
Natural Multiplicative Identity (77) immediately needs:

Identity Elements (75)
Natural Arithmetic (73)

Natural Multiplicative Identity (77) is immediately needed by:

Integer Arithmetic (84)

Natural Multiplicative Identity (77) gives no terms.


Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Natural Products
Operations
Natural Sums
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Successor Sets
Unordered Triples
Intersection of Empty Set
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Set Unions
Pair Intersections
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Is the inverse of an element function the element function of a different element?

Definition

The *inverse* of an element of an algebra (also called the *inverse element*) is the element (if it exists) whose corresponding element function under the operation is the inverse of the first element's function.

Notation

Let $(A, +)$ be an algebra. Let $a \in A$. If the inverse element for a exists and is unique we denote it by a^{-1}. In other words $+^{a^{-1}} \circ +^{a} = \mathrm{id}_A$

Inverse Elements (78) immediately needs:

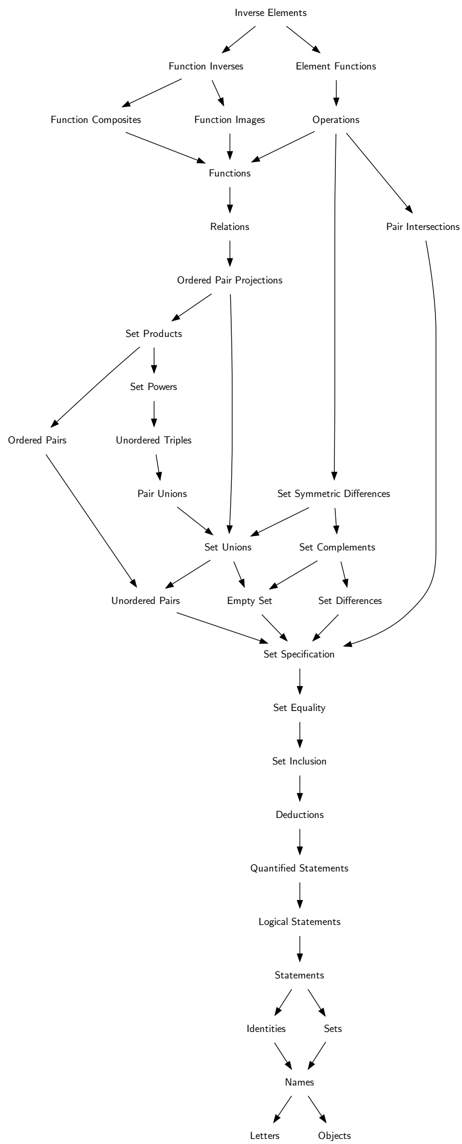
Element Functions (74)
Function Inverses (50)

Inverse Elements (78) is immediately needed by:

Integer Additive Inverses (90)
Rational Multiplicative Inverses (96)
Real Matrix Inverses (**??**)

Inverse Elements (78) gives the following terms.

inverse, inverse element.


Inverse Elements
Function Inverses
Element Functions
Function Composites
Function Images
Operations
Functions
Relations
Pair Intersections
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to subtract numbers.[83]

Definition

Consider the set $\omega \times \omega$. This set is the set of ordered pairs of ω. In other words, the ordered pairs of natural numbers.

We call two such pairs (a, b) and (c, d) of $\omega \times \omega$ *integer equivalent* if

$$a + d = b + c$$

Briefly, the intuition is that (a, b) represents a less b, or in the usual notation "$a - b$".[84] So this equivalence relation says these two are the same if $a - b = c - d$. Rearranging gives $a + d = b + c$.

Proposition 127. *Integer equivalence is an equivalence relation.*[85]

The *set of integer numbers* is the set of equivalence classes (see Equivalence Relations) under integer equivalence on $\omega \times \omega$. We call an element an *integer number* (or *integer*).

Notation

We denote the set of integers by $\mathbf{Z}$. If we denote integer equivalence by $\sim$ then $\mathbf{Z} = (\omega \times \omega)/\sim$.

[83] Future editions will change this why. In particular, by referencing Inverse Elementsand the lack thereof in ω.
[84] This account will be expanded in future editions.
[85] The proof is straightforward. It will be included in future editions.

Integer Numbers (79) immediately needs:

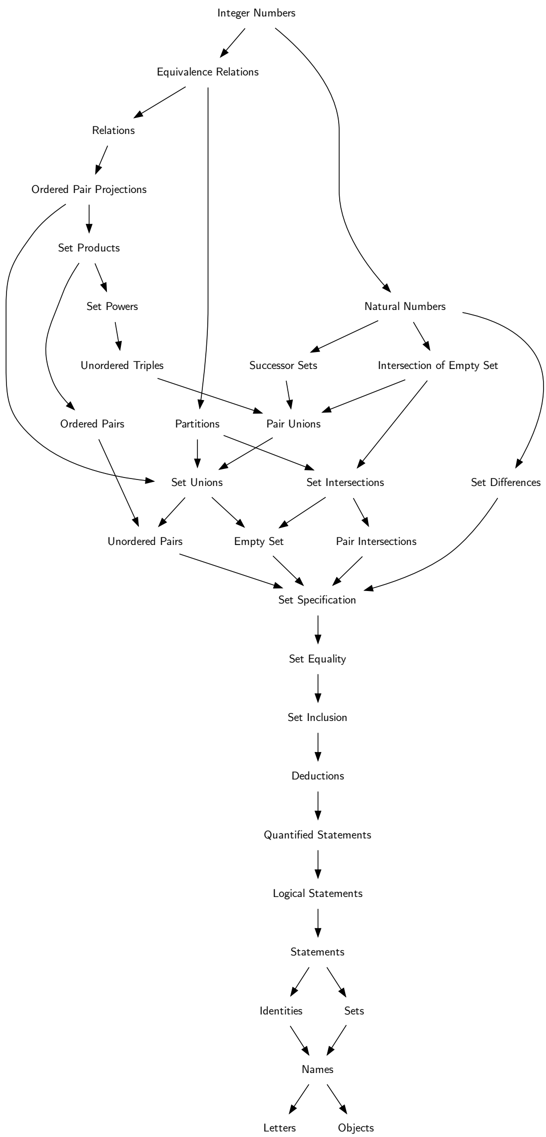
Equivalence Relations (40)
Natural Numbers (56)

Integer Numbers (79) is immediately needed by:

Decision Problems (**??**)
Digital Integers (**??**)
Integer Order (83)
Integer Products (81)
Integer Sums (80)

Integer Numbers (79) gives the following terms.

integer equivalent, set of integer numbers, integer number, integer.


Integer Numbers
Equivalence Relations
Relations
Ordered Pair Projections
Set Products
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Ordered Pairs
Partitions
Pair Unions
Set Unions
Set Intersections
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want sums to follow those of natural numbers.[86]

Definition

Consider $[(a, b)], [(c, d)] \in \mathbf{Z}$. We define the *integer sum* of $[(a, b)]$ with $[(c, d)]$ as $[(a + c, b + d)]$.[87]

Notation

We denote the sum of $[(a, b)]$ and $[(c, d)]$ by $[(a, b)] + [(b, c)]$ So if $x, y \in \mathbf{Z}$ then the sum of x and y is $x + y$.

[86] Future editions will modify this.

[87] One needs to show that this is well-defined. The account will appear in future editions.

Integer Sums (80) immediately needs:

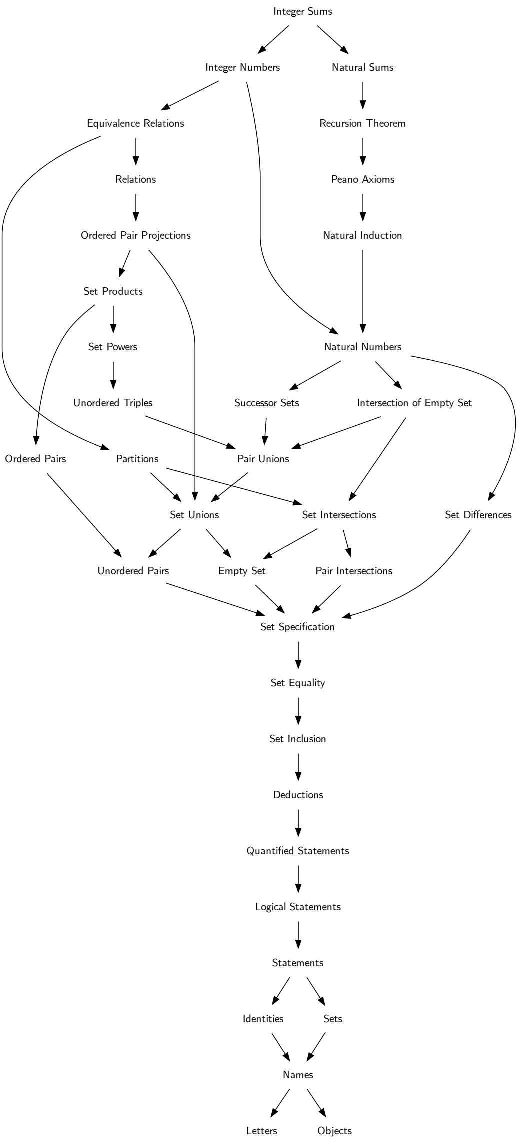
Integer Numbers (79)
Natural Sums (60)

Integer Sums (80) is immediately needed by:

Integer Additive Inverses (90)
Integer Arithmetic (84)

Integer Sums (80) gives the following terms.

integer sum.


Integer Sums
Integer Numbers
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Ordered Pairs
Partitions
Pair Unions
Set Unions
Set Intersections
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want sums to follow those of natural numbers.[88]

Definition

Consider $[(a, b)], [(b, c)] \in \mathbf{Z}$. The *integer product* of $[(a, b)]$ with $[(b, c)]$ is $[(ac + bd, ad + bc)]$.[89]

Notation

We denote the product of $[(a, b)]$ and $[(c, d)]$ by $[(a, b)] \cdot [(b, c)]$ So if $x, y \in \mathbf{Z}$ then the sum of x and y is $x \cdot y$. As with natural products, we often drop the $\cdot$ and write xy for $x \cdot y$.

[88] Future editions will modify this.

[89] One needs to show that this is well-defined. The account will appear in future editions.

Integer Products (81) immediately needs:

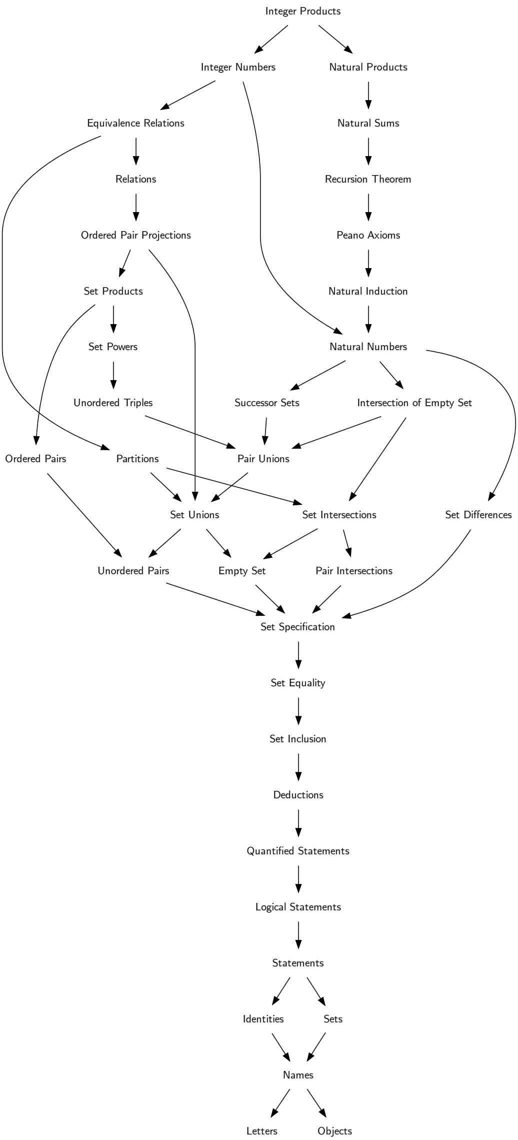
Integer Numbers (79)
Natural Products (61)

Integer Products (81) is immediately needed by:

Integer Arithmetic (84)
Integer Powers (**??**)
Rational Order (97)
Rational Products (93)

Integer Products (81) gives the following terms.

integer product.

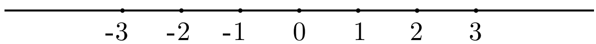

Integer Products
Integer Numbers
Natural Products
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Ordered Pairs
Partitions
Pair Unions
Set Unions
Set Intersections
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We are constantly thinking of the integers as the endpoints of equal length segments of a line.

Discussion

We commonly associate elements of the integers with the endpoints of equal-length segments of a real line. Take segment S_0 of L with endpoints p and q. Associate the point p with 0. Associate the point q with 1. Take a segment S_1 of equal length, non-overlapping with S_0, who shares the endpoint q. Associate the second endpoint of this segment 2. Continue with the rest. We call the line so formed the *integral line* of unit S_0.



Integral Distance

Let $f : \mathbf{Z} \to \mathbf{Z}$ be defined by $f(a, b) = a - b$ if $a > b$ and $f(a, b) = b - a$ if $b > a$. Notice that f is symmetric: $f(a, b) = f(b, a)$. The (geometric) interpretation of f is the distance between the points associated with the two integers $a, b \in \mathbf{Z}$ in some integral line. We call f the *integral distance*. Notice that $f(a, b) > 0$ for all $a, b \in \mathbf{Z}$.

Notation

We denote the distance between $a, b \in \mathbf{Z}$ by $|a - b|$.

Integral Line (82) immediately needs:

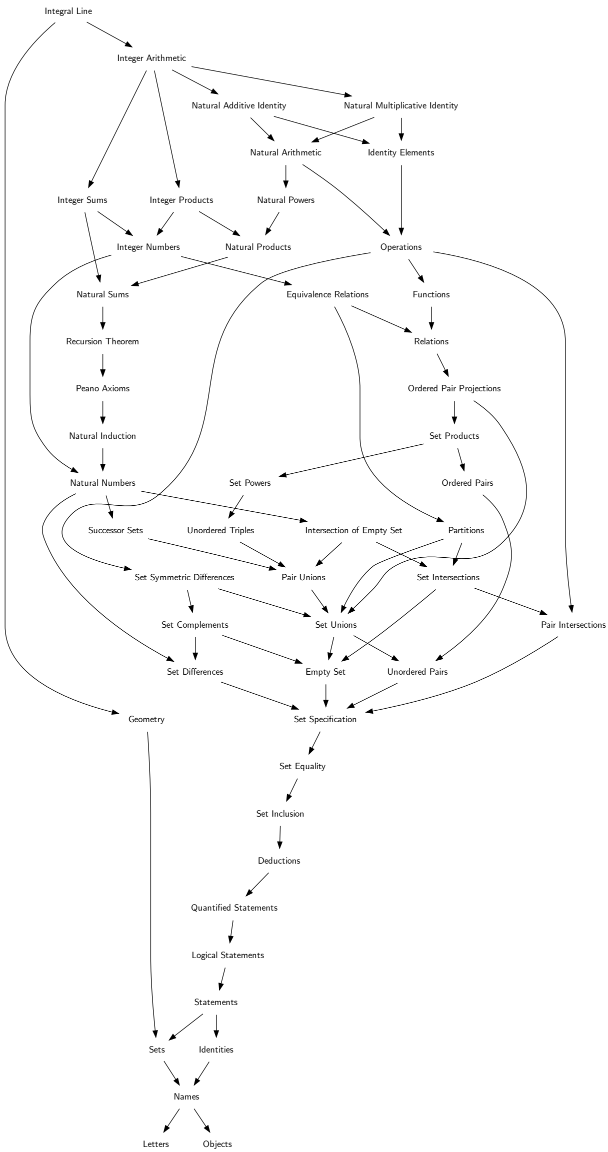
Geometry (22)
Integer Arithmetic (84)

Integral Line (82) is immediately needed by:

Chordal Graphs (**??**)
Real Line (112)

Integral Line (82) gives the following terms.

integral line, *integral distance*.


Integral Line
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Integer Sums
Integer Products
Natural Powers
Integer Numbers
Natural Products
Operations
Natural Sums
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Set Unions
Pair Intersections
Set Differences
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We want to order the integers.

Definition

Consider $[(a,b)], [(b,c)] \in \mathbf{Z}$. If $a + d < b + c$, then we say that $[(a,b)]$ is *less than* $[(b,c)]$.[90] If $[(a,b)]$ is less than $[(b,c)]$ or equal, then we say that $[(a,b)]$ is *less than or equal to* $[(b,c)]$.

Notation

If $x, y \in \mathbf{Z}$ and x is less than y, then we write $x < y$. If x is less than or equal to y, we write $x \leq y$.

Positive and negative integers

We call an integer z *positive* if $z > 0$ and we call z *negative* if $z < 0$.[91] We call an integer z *nonnegative* if $z > 0$ or $z = 0$ and *nonpositive* if $z < 0$ or $z = 0$.

Notation

We denote the set $\{z \in \mathbf{Z} \mid z \geq 0_Z\}$ by $\mathbf{Z}_{++}$.

---

[90] One needs to show that this is well-defined. The account will appear in future editions.

[91] Some authors use the term positive for the case when $z > 0$ or $z = 0$. We use the term nonnegative in this case.

Integer Order (83) immediately needs:

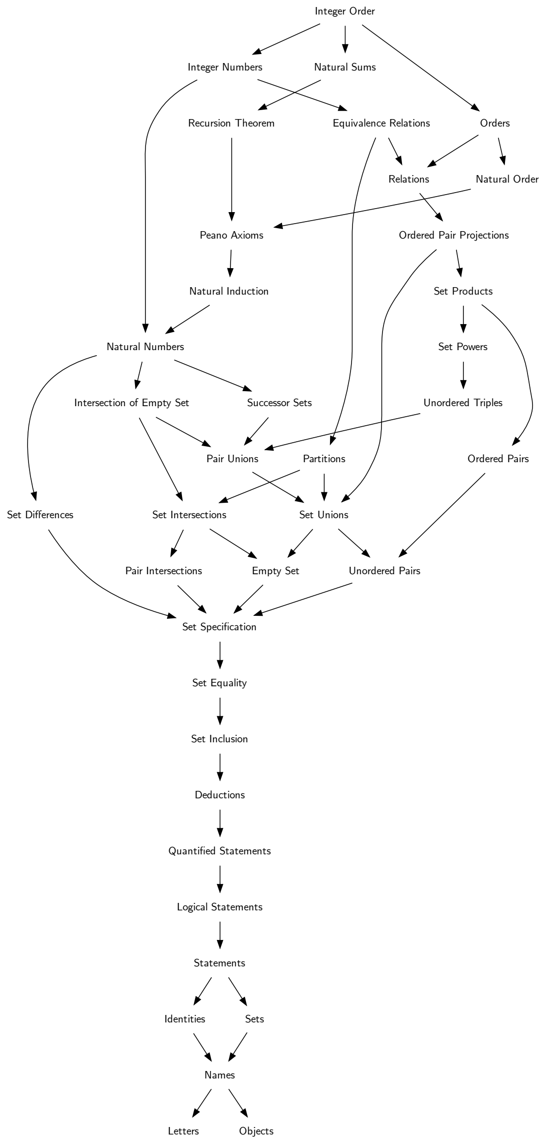
Integer Numbers (79)
Natural Sums (60)
Orders (**??**)

Integer Order (83) is immediately needed by:

Integer Arithmetic and Order (85)
Natural Integer Isomorphism (89)
Rational Order (97)

Integer Order (83) gives the following terms.

less than, less than or equal to, positive, negative, nonnegative, nonpositive.


Integer Order
Integer Numbers
Natural Sums
Recursion Theorem
Equivalence Relations
Orders
Relations
Natural Order
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Intersection of Empty Set
Successor Sets
Unordered Triples
Pair Unions
Partitions
Ordered Pairs
Set Differences
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What are addition and multiplication for integers? What are the identity elements?

Definition

We call the operation of forming integer sums *integer addition.* We call the operation of forming integer products *integer multiplication.*

Results

It is easy to see the following.[92]

Proposition 128. *The additive identity for* $\mathbf{Z}$ *is* $[(0, 0)]$.

Proposition 129. *The multiplicative identity for* $\mathbf{Z}$ *is* $[(1, 0)]$.

Notation

We denote the additive identity of $\mathbf{Z}$ by $0_{\mathbf{Z}}$ and the multiplicative identity by $1_{\mathbf{Z}}$. When it is clear from context, we call $0_{\mathbf{Z}}$ "zero" and we call $1_{\mathbf{Z}}$ "one".

Distributive

Proposition 130. *For integers* $x, y, z \in \mathbf{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.[93]

---

[92] Nonetheless, the full accounts will appear in future editions.
[93] An account will appear in future editions.

Integer Arithmetic (84) immediately needs:

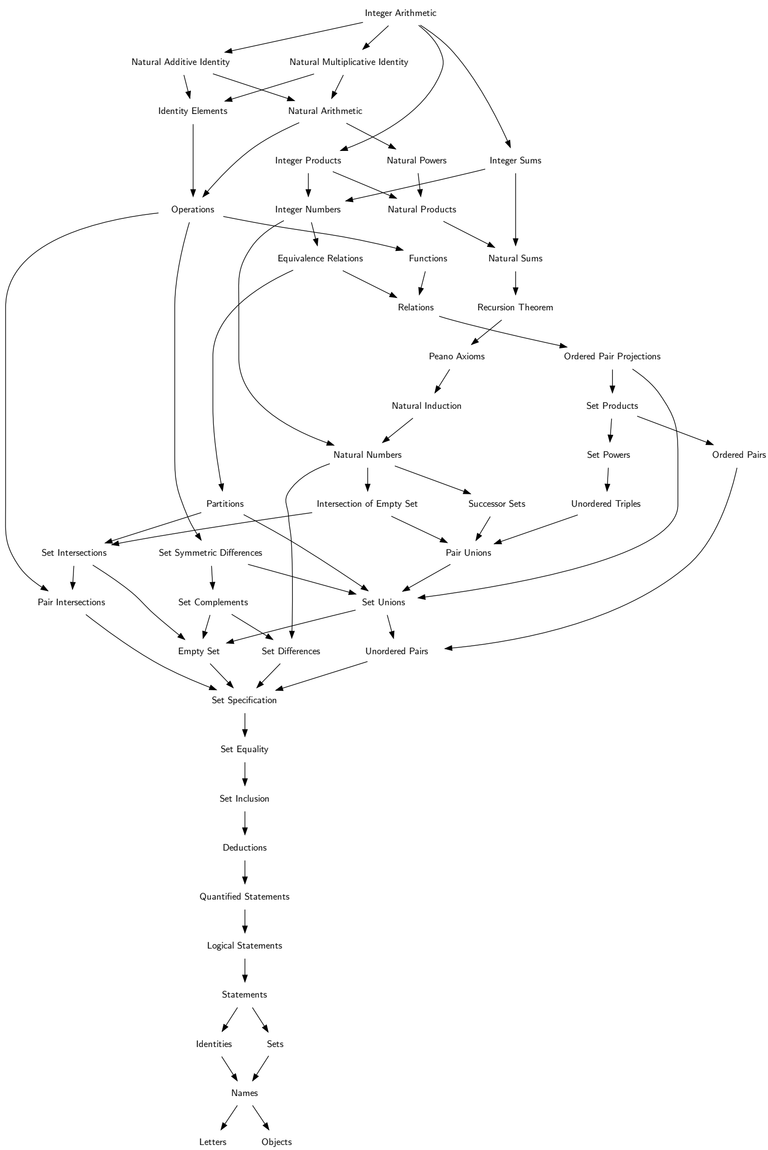
Integer Products (81)
Integer Sums (80)
Natural Additive Identity (76)
Natural Multiplicative Identity (77)

Integer Arithmetic (84) is immediately needed by:

Integer Arithmetic and Order (85)
Integer Divisors (**??**)
Integral Line (82)
Modular Arithmetic (**??**)
Rational Arithmetic (94)
Rational Multiplicative Inverses (96)
Rational Numbers (91)
Rings (88)

Integer Arithmetic (84) gives the following terms.

integer addition, integer multiplication.


Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Identity Elements
Natural Arithmetic
Integer Products
Natural Powers
Integer Sums
Operations
Integer Numbers
Natural Products
Equivalence Relations
Functions
Natural Sums
Relations
Recursion Theorem
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

How does arithmetic interact with integers.

Results

We can show the following.[94]

Proposition 131. *Let* $a, b, c, d \in \mathbf{Z}$*. If* $a \leq b$ *and* $c \leq d$*, then* $a+b \leq c+d$*.*

Proposition 132. *Let* $a, b, c, d \in \mathbf{Z}$ *with* $a, b \geq 0_{\mathbf{Z}}$*. If* $a \leq b$ *and* $c \leq d$*, then* $a \cdot c \leq a \cdot d$*.*

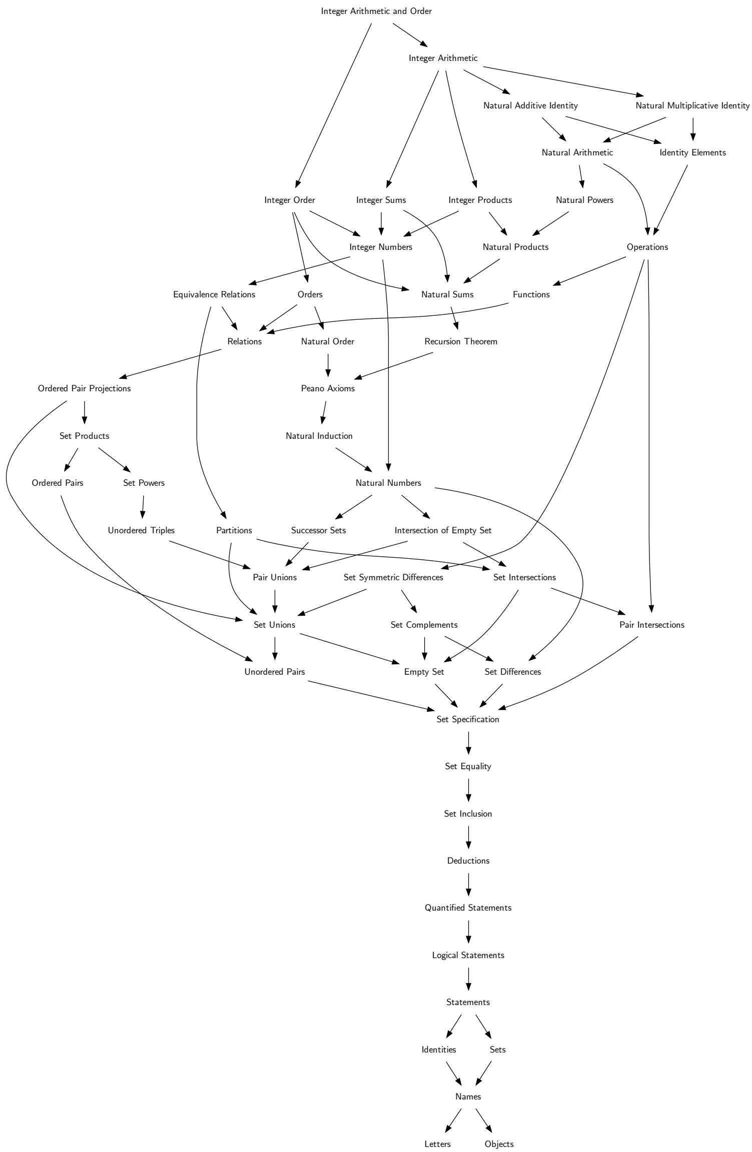
[94] Accounts will appear in future editions.

Integer Arithmetic and Order (85) immediately needs:

Integer Arithmetic (84)
Integer Order (83)

Integer Arithmetic and Order (85) is not immediately needed by any sheet.

Integer Arithmetic and Order (85) gives no terms.


Integer Arithmetic and Order
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Integer Order
Integer Sums
Integer Products
Natural Powers
Integer Numbers
Natural Products
Operations
Equivalence Relations
Orders
Natural Sums
Functions
Relations
Natural Order
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Partitions
Successor Sets
Intersection of Empty Set
Pair Unions
Set Symmetric Differences
Set Intersections
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We often have two algebras for which we can identify elements of the ground set.

Definition

Let $(A, +_A)$ and $(B, +_B)$ be two algebras.[95]

An *isomorphism* between these two algebras is a bijection $f : A \to B$ satisfying:

$$f(a +_A a') = f(a) +_B f(a')$$

and

$$f^{-1}(b +_B b') = f^{-1}(b) +_A f^{-1}(b').$$

If there exists an isomorphism between two algebras we say that the algebras are *isomorphic*.

[95] Future editions will change this notation to avoid clashes with right and left identity elements (see Identity Elements).

Isomorphisms (86) immediately needs:

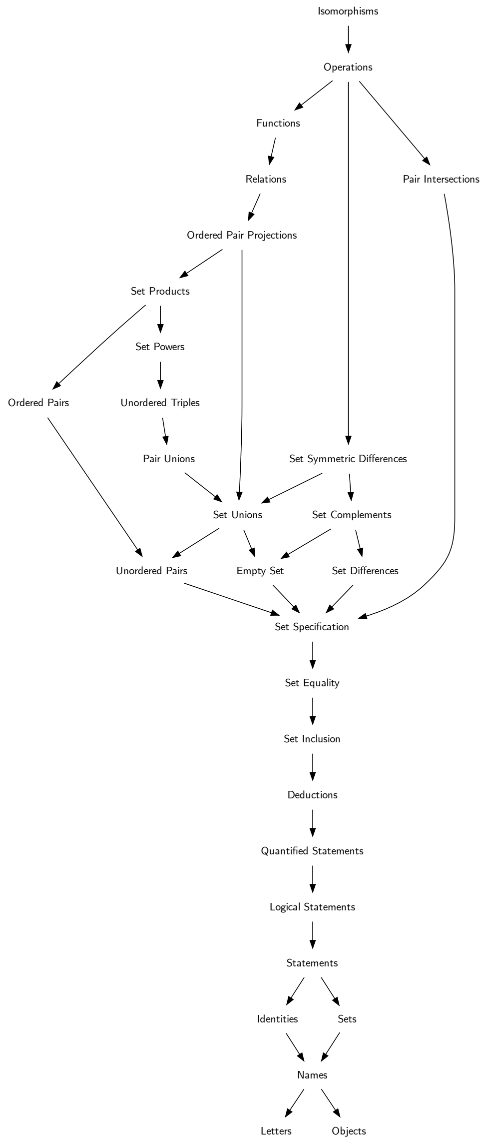
Operations (72)

Isomorphisms (86) is immediately needed by:

Natural Integer Isomorphism (89)

Isomorphisms (86) gives the following terms.

isomorphism, isomorphic.


Isomorphisms
Operations
Functions
Relations
Pair Intersections
Ordered Pair Projections
Set Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We further drop conditions on the structure of the binary operations, and study only the algebraic structure of addition over the integers.

Definition

A *group* is an algebra$(G, \circ)$ for which $\circ : G \times G \to G$ is associative, has an identity element in G, and has inverse elements. A group is a *commutative group* (or *abelian group*) if $\circ$ is commutative. A group is a *finite group* if G is a finite set.

Additive groups

Suppose that $(R, +, \cdot)$ is ring. Then $(R, +)$ is a commutative group. Conversely, suppose $(G, +)$ is a commutative group. Define multiplication on S by $a \cdot b = 0$ for all $a, b \in R$. Then $(S, +, \cdot)$ is a ring, called the *zero ring* of $(G, +)$. For this reason, it is customary to write $+$ for the operation $\circ$ when handling commutative groups.

Group Operations

Along with the group operation, we call the function which maps an element to its inverse element the *group operations*.

Groups (87) immediately needs:

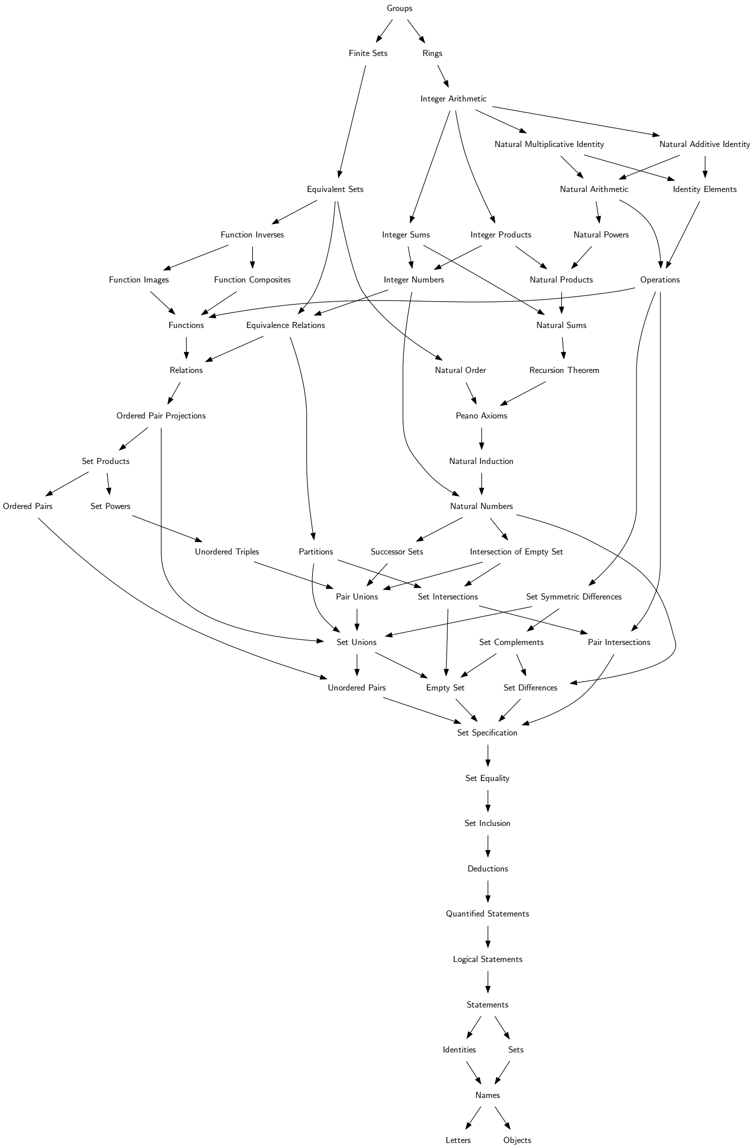
Finite Sets (66)
Rings (88)

Groups (87) is immediately needed by:

Homomorphisms (99)
Linear Representations of Groups (**??**)
Monoids (**??**)
Permutations (**??**)
Subgroups (**??**)
Topological Groups (**??**)

Groups (87) gives the following terms.

group, commutative group, abelian group, finite group, zero ring, group operations.


Groups
Finite Sets
Rings
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Equivalent Sets
Natural Arithmetic
Identity Elements
Function Inverses
Integer Sums
Integer Products
Natural Powers
Function Images
Function Composites
Integer Numbers
Natural Products
Operations
Functions
Equivalence Relations
Natural Sums
Relations
Natural Order
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Partitions
Successor Sets
Intersection of Empty Set
Pair Unions
Set Intersections
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We generalize the algebraic structure of *addition* and *multiplication* over the integers.[96]

Definition

A *ring* (or *ring with identity*) (R, f, g) is a set A and two binary operations on R satisfying the following set of conditions.

(A) (i) f is *associative.* (ii) f is *commutative*, (iii) A has an *identity element* for f (i.e., there is $e \in R$ with $f(r, e) = f(e, r) = r$ for all $r \in R$ (iv) R has *inverse elements* for f (i.e., for any $r \in R$, there is $\tilde{r}$ satisfying $f(r, \tilde{r}) = f(\tilde{r}, r) = e$)

(B) (i) g is *associative*; (ii) R has an *identity element* for g (i.e., for any $r \in R$, there is $\tilde{e} \in A$ satisfying $g(r, \tilde{e}) = g(\tilde{e}, r) = r$)

(C) (i) g *left distributes*:

$$g(f(x, y), \alpha) = f(g(\alpha, x), g(\alpha, y)) \quad \text{for all } x, y, \alpha \in R$$

(ii) g *right distributes*:

$$g(\alpha, f(x, y)) = f(g(\alpha, x), g(\alpha, y)) \quad \text{for all } x, y, \alpha \in R$$

Conditions (A) concern f, conditions (B) concern g, and conditions (C) relate the two.

Clearly, $\mathbf{Z}$ with addition and multiplication is a ring. The element referred to in (A.2) is $0 \in \mathbf{Z}$, so we refer to this element in any ring as the *additive identity.* That referred to (A.3) is $1 \in \mathbf{Z}$, so we refer to this element in any ring as the *multiplicative identity.* We refer to the elements mentioned in (A.4) as *additive inverses.* We call to f *ring addition* and g *ring multiplication.*

---

[96] Future editions will likely modify this sheet, and give a genetic treatment involving the solution of polynomial equations by Galois.

A ring which for which multiplication is commutative is called a *commutative ring*. Note that a ring is *always* commutative with respect to addition, here the term commutative refers to multiplication. A ring for which there are inverse elements, excepting 0, is called a *division ring*).

Of course, the integers form a ring with the usual notion of addition and multiplication. For another trivial example, consider $\{0\}$ with $0+0 = 0$ and $0 \cdot 0 = 0$; this is called the *zero ring* (any ring isomorphic to this one is called a *trivial ring* or *zero ring*).

Notation

The notation commonly adopted in discussing rings relies on analogy with the set of integers $\mathbf{Z}$. We denote the ring addition by $+$ and ring multiplication by $\cdot$. Moreover, we denote the ring's additive identity by 0 and the ring's multiplicative identity by 1. Finally, we denote the additive inverse of $a \in A$ by $-a$.

Rewriting the conditions (A), (B), (C) in this notation gives familiar-looking relations, from when the objects involved were integers. (A) (1) $a + (b + c) = (a + b) + c$; (2) $a + b = b + a$; (3) $a + 0 = 0 + a = a$; (4) $a + (-a) = 0$. (B) (1) $a(bc) = (ab)c$; (2) $1a = a1 = a$. (C) (1) $(a + b)c = ac + bc$; (2) $c(a + b) = ca + cb$.

Immediate consequences

We need not require that $0x = 0$, because we can deduce it:

$$0x + x = (0 + 1)x = 1x = x.$$

Similarly, $(-a)b = -(ab)$ since

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

Other familiar relations among the integers, e.g. $(-a)(-b) = ab$, may be deduced.

Rings (88) immediately needs:

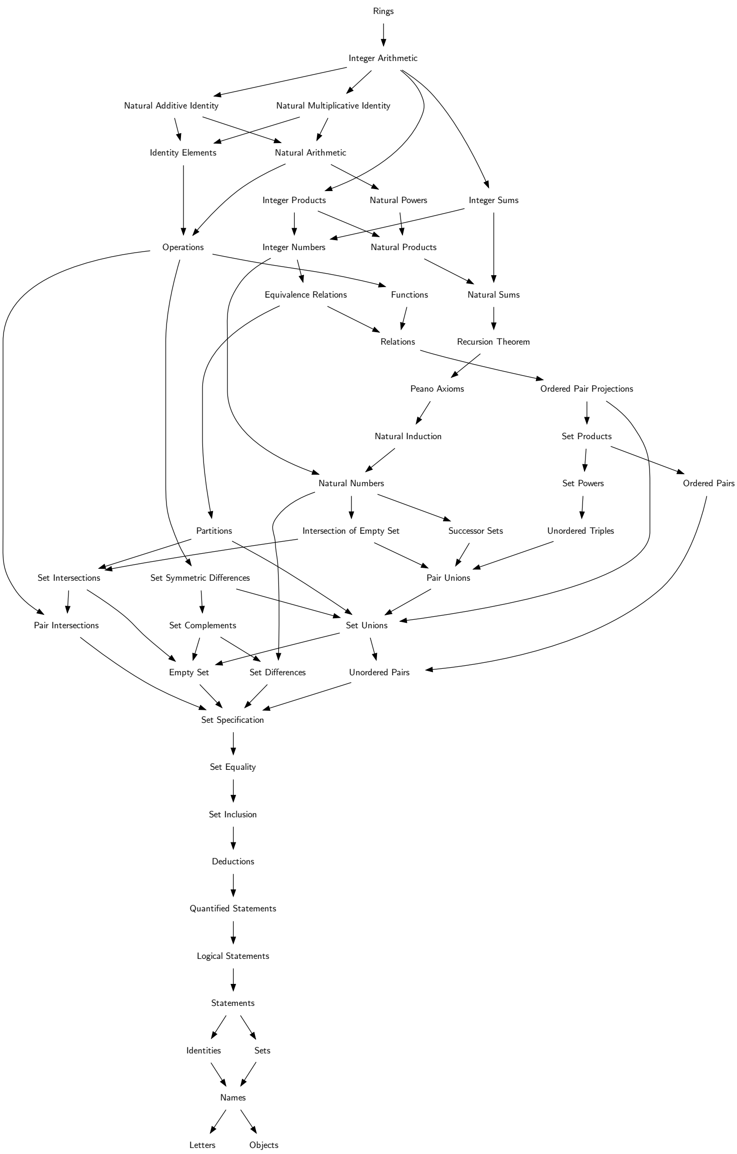
Integer Arithmetic (84)

Rings (88) is immediately needed by:

Fields (98)
Groups (87)
Homomorphisms (99)
Modules (**??**)
Polynomials (**??**)
Ring Ideals (**??**)
Semirings (**??**)
Subrings (**??**)

Rings (88) gives the following terms.

ring, ring with identity, additive identity, multiplicative identity, additive inverses, ring addition, ring multiplication, commutative ring, division ring, zero ring, trivial ring, zero ring.


Rings
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Identity Elements
Natural Arithmetic
Integer Products
Natural Powers
Integer Sums
Operations
Integer Numbers
Natural Products
Equivalence Relations
Functions
Natural Sums
Relations
Recursion Theorem
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Do the natural numbers correspond (in the sense of Isomorphisms) to elements of integers.

Main result

Indeed, the natural numbers correspond to the Z_+.

Proposition 133. *$(\mathbf{Z}_{++}, + \mid \mathbf{Z}_{++})$ and $(\omega, +)$ are isomorphic.*

Proof. The function is $f(n) = [(n, 0)]$.[97] □

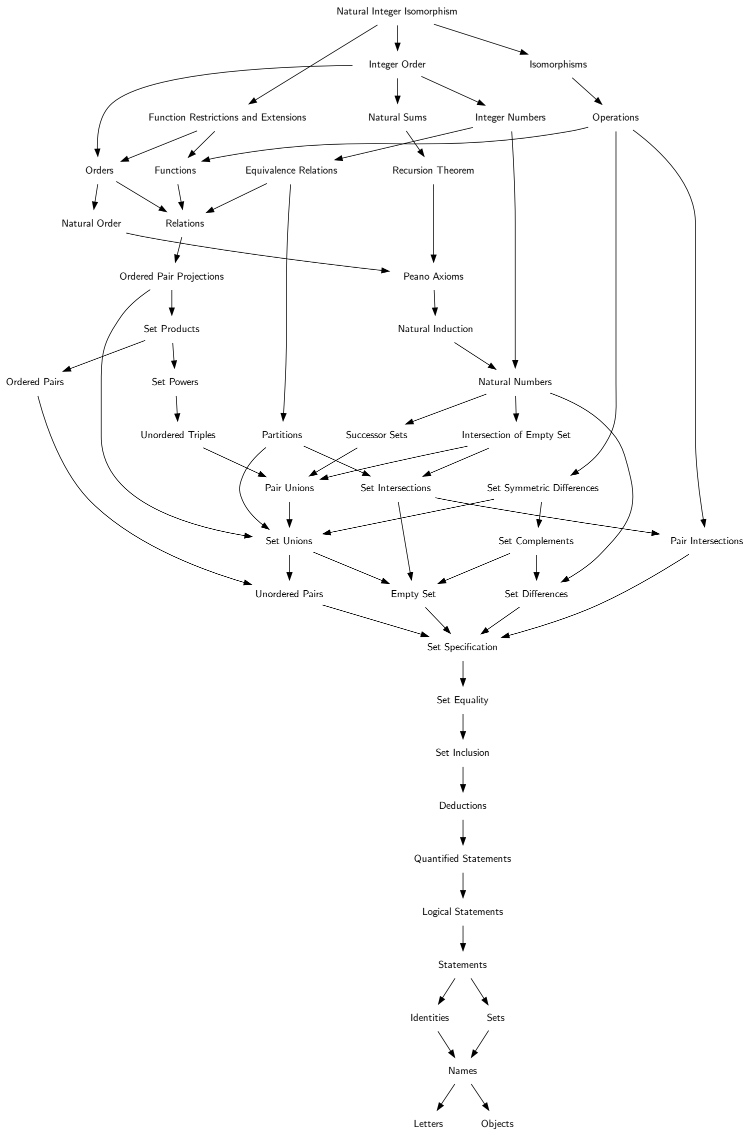
[97] The full account will appear in future editions.

Natural Integer Isomorphism (89) immediately needs:

Function Restrictions and Extensions (42)
Integer Order (83)
Isomorphisms (86)

Natural Integer Isomorphism (89) is not immediately needed by any sheet.

Natural Integer Isomorphism (89) gives no terms.



Natural Integer Isomorphism
Integer Order
Isomorphisms
Function Restrictions and Extensions
Natural Sums
Integer Numbers
Operations
Orders
Functions
Equivalence Relations
Recursion Theorem
Natural Order
Relations
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Partitions
Successor Sets
Intersection of Empty Set
Pair Unions
Set Intersections
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the additive inverse of $[(a, b)]$ in the integers?

Result

Proposition 134. *The additive inverse of* $[(a, b)] \in \mathbf{Z}$ *is* $[(b, a)]$.

Notation

We denote the additive inverse of $z \in \mathbf{Z}$ by $-z$. We denote $a + (-b)$ by $a - b$.

Subtraction

We call the operation $(a, b) \mapsto a - b$ *subtraction.*

Integer Additive Inverses (90) immediately needs:

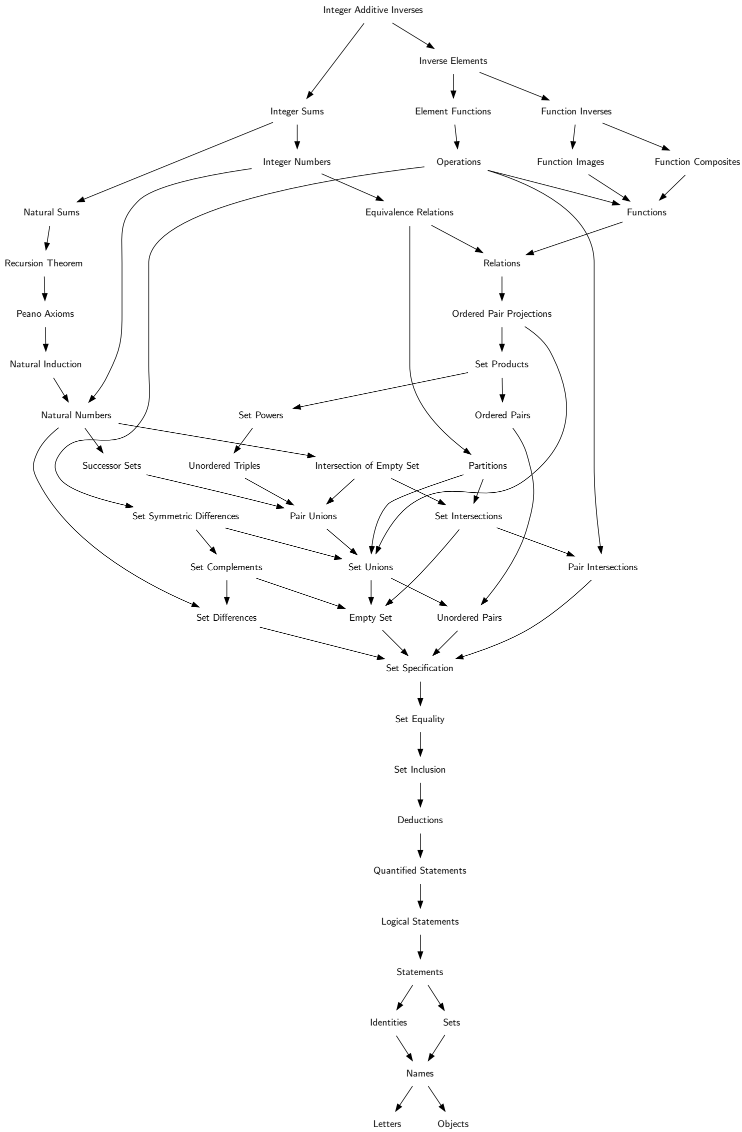
Integer Sums (80)
Inverse Elements (78)

Integer Additive Inverses (90) is immediately needed by:

Rational Additive Inverses (95)

Integer Additive Inverses (90) gives the following terms.

subtraction.


Integer Additive Inverses
Inverse Elements
Integer Sums
Element Functions
Function Inverses
Integer Numbers
Operations
Function Images
Function Composites
Natural Sums
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Set Unions
Pair Intersections
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Rational equivalence

Consider $\mathbf{Z} \times (\mathbf{Z} - \{0_{\mathbf{Z}}\})$. We say that the elements (a, b) and (c, d) of this set are *rational equivalent* if $ad = bc$. Briefly, the intuition is that (a, b) represents a over b In the usual notation, (a, b) represents "a/b". So this equivalence relation says these two are the same if $a/b = c/d$ or else $ad = bc$.

Proposition 135. *Rational equivalence is an equivalence relation on* $\mathbf{Z} \times (\mathbf{Z} - \{0_{\mathbf{Z}}\})$.[98]

Definition

The *set of rational numbers* is the set of equivalence classes (see Equivalence Classes) of $\mathbf{Z} \times (\mathbf{Z} - \{0_{\mathbf{Z}}\})$ under ratioanl equivalence. We call an element of the set of rational numbers a *rational number* or *rational.* We call the set of rational numbers the *set of rationals* or *rationals* for short.

Notation

We denote the set of rationals by $\mathbf{Q}$.[99] If we denote rational equivalence by $\sim$ then $\mathbf{Q} = (\mathbf{Z} \times (\mathbf{Z} - \{0_{\mathbf{Z}}\}))/\sim$.

[98] Future editions will include an account.

[99] From what we can tell, $\mathbf{Q}$ is a mnemonic for "quantity", from the latin "quantitas." It may also be a mnemonic for quotient.

Rational Numbers (91) immediately needs:

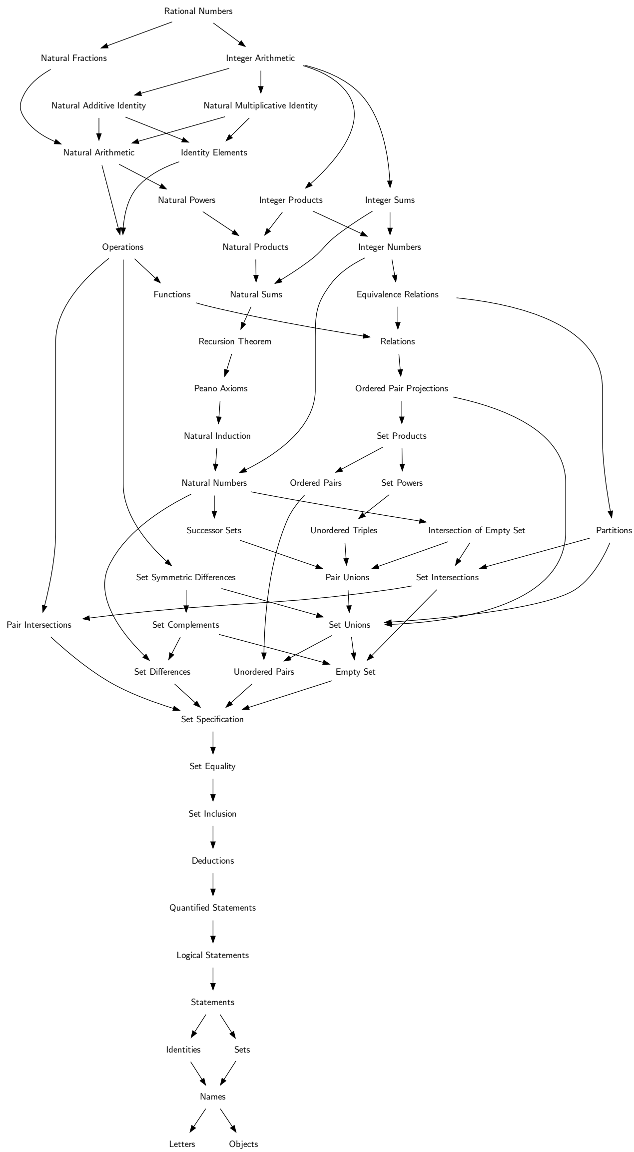
Integer Arithmetic (84)
Natural Fractions (**??**)

Rational Numbers (91) is immediately needed by:

Fields (98)
Rational Order (97)
Rational Products (93)
Rational Sums (92)
Real Numbers (101)

Rational Numbers (91) gives the following terms.

rational equivalent, set of rational numbers, rational number, rational, set of rationals, rationals.


Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Integer Products
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to add rationals.[100]

Definition

Let $[(a, b)], [(b, c)] \in \mathbf{Q}$. The *rational sum* of $[(a, b)]$ with $[(b, c)]$ is $[(ad + bc, bd)]$.[101]

Notation

We denote the rational sum of $q, r \in \mathbf{Q}$ by $q + r$.

[100] Future editions will expand on this why.
[101] An account that this is well-defined will appear in future editions.

Rational Sums (92) immediately needs:

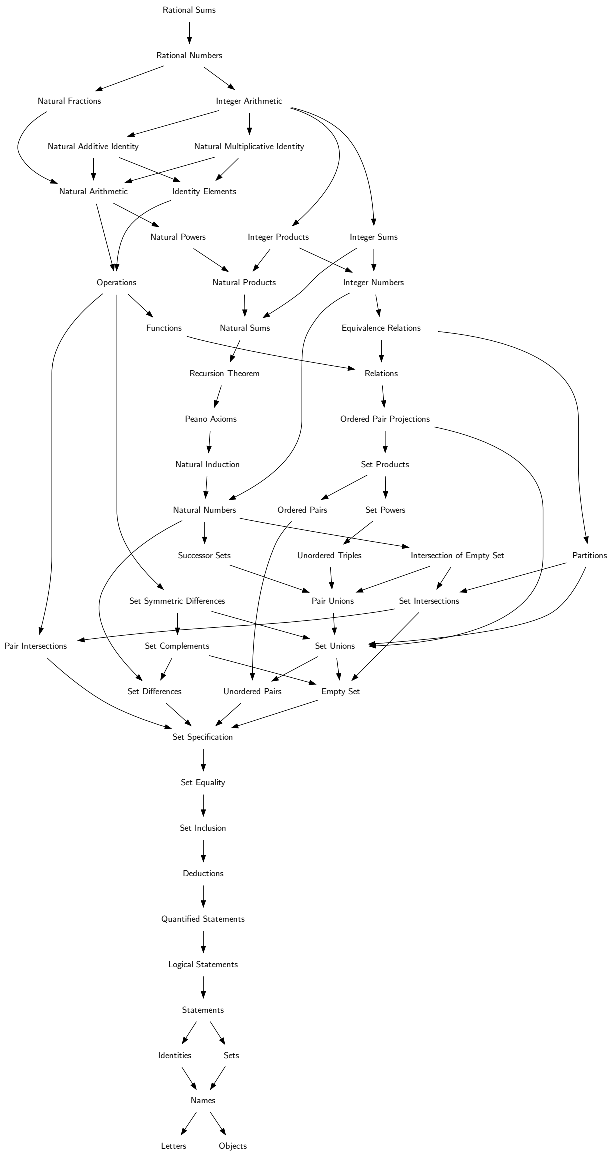
Rational Numbers (91)

Rational Sums (92) is immediately needed by:

Rational Additive Inverses (95)
Rational Arithmetic (94)

Rational Sums (92) gives the following terms.

rational sum.


Rational Sums
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Integer Products
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to multiply rationals.[102]

Definition

Let $[(a, b)], [(b, c)] \in \mathbf{Q}$. The *rational product* of $[(a, b)]$ with $[(b, c)]$ is $[(ac, bd)]$.[103]

Notation

We denote the rational product of $q, r \in \mathbf{Q}$ by $q \cdot r$.

[102] Future editions will expand on this why.
[103] An account that this is well-defined will appear in future editions.

Rational Products (93) immediately needs:

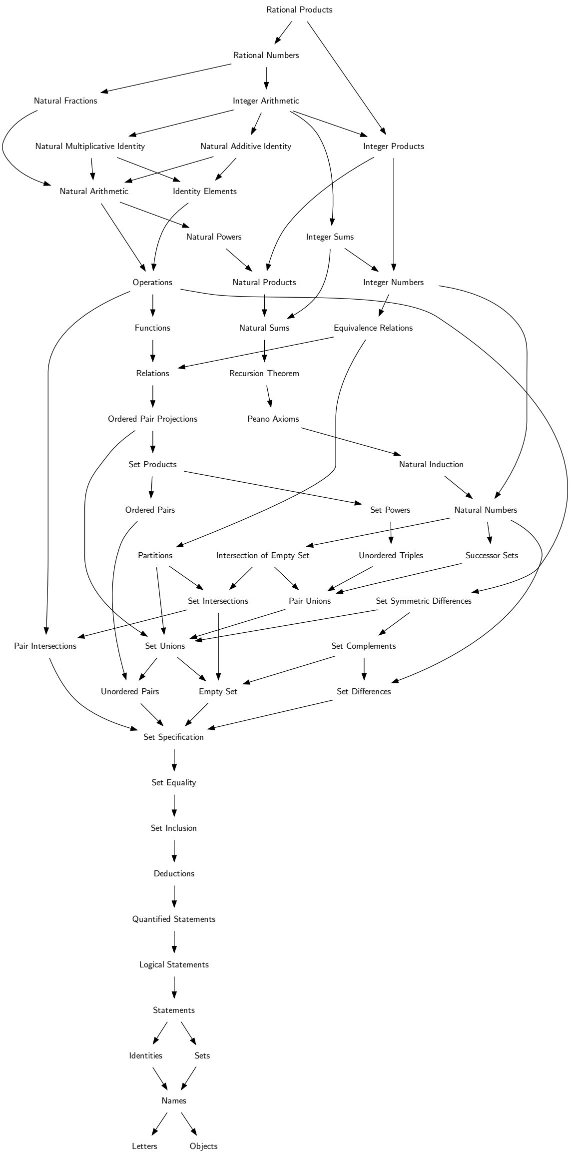
Integer Products (81)
Rational Numbers (91)

Rational Products (93) is immediately needed by:

Rational Arithmetic (94)
Rational Multiplicative Inverses (96)

Rational Products (93) gives the following terms.

rational product.


Rational Products
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Symmetric Differences
Pair Intersections
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What are addition and multiplication for rationals? What are the identity elements?

Definition

We call the operation of forming rationals sums *rational addition*. We call the operation of forming rational products *rational multiplication*.

Results

It is easy to see the following.[104]

Proposition 136. *The additive identity for* $\mathbf{Q}$ *is* $[(0_{\mathbf{Z}}, 1_{\mathbf{Z}})]$.

Proposition 137. *The multiplicative identity for* $\mathbf{Z}$ *is* $[(1_{\mathbf{Z}}, 1_{\mathbf{Z}})]$.

Notation

We denote the additive identity of $\mathbf{Q}$ by $0_{\mathbf{Q}}$ and the multiplicative identity by $1_{\mathbf{Q}}$. We denote the set $\{q \in \mathbf{Q} \mid q \geq 0_Q\}$ by $\mathbf{Q}_+$.

Distributive

Proposition 138. *For rationals* $x, y, z \in \mathbf{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.[105]

[104] Nonetheless, the full accounts will appear in future editions.
[105] An account will appear in future editions.

Rational Arithmetic (94) immediately needs:

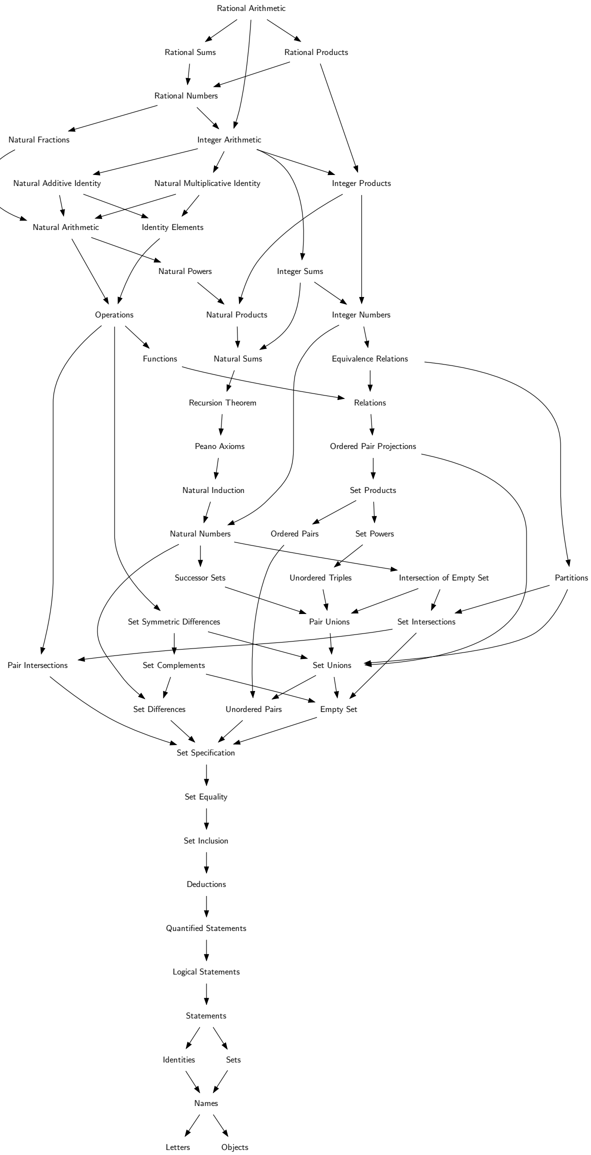
Integer Arithmetic (84)
Rational Products (93)
Rational Sums (92)

Rational Arithmetic (94) is immediately needed by:

Integer Rational Homomorphism (100)
Real Products (105)

Rational Arithmetic (94) gives the following terms.

rational addition, rational multiplication.


Rational Arithmetic
Rational Sums
Rational Products
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the additive inverse of $[(a, b)]$ in the rationals?

Result

Proposition 139. *The additive inverse of* $[(a, b)] \in \mathbf{Q}$ *is* $[(-a, b)]$.

Notation

We denote the additive inverse of $q \in \mathbf{Q}$ by $-q$. We denote $a + (-b)$ by $a - b$.

Subtraction

We call the operation $(a, b) \mapsto a - b$ *subtraction.*

Rational Additive Inverses (95) immediately needs:

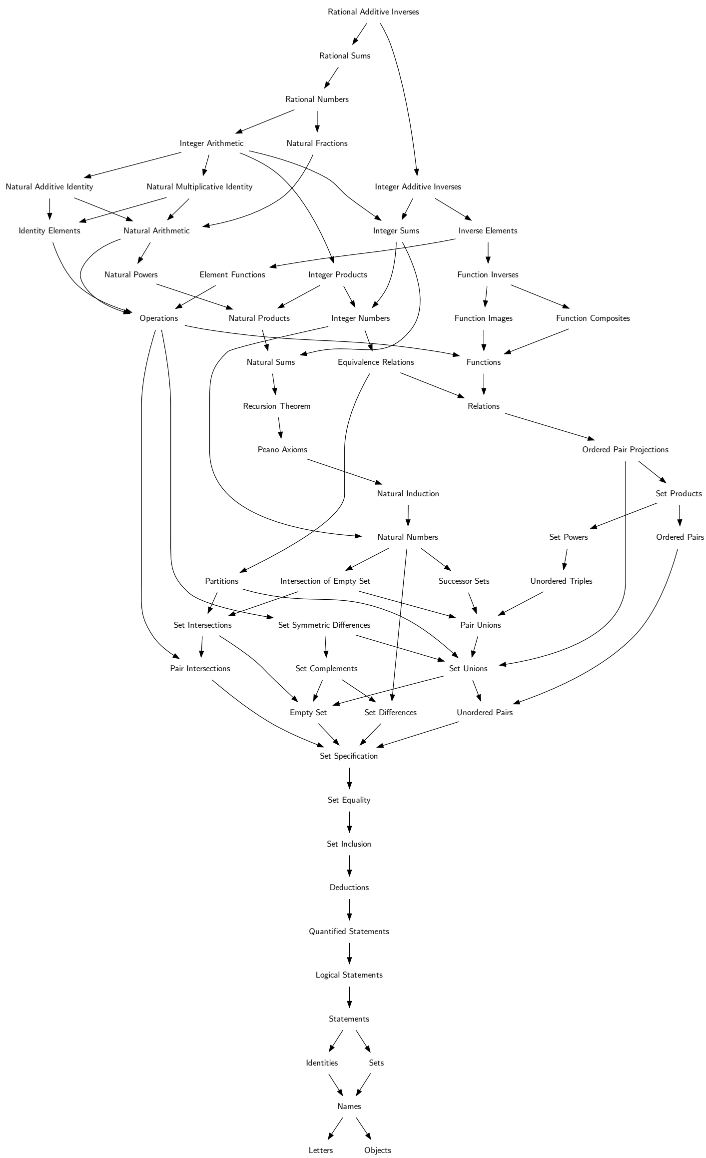
Integer Additive Inverses (90)
Rational Sums (92)

Rational Additive Inverses (95) is immediately needed by:

Integer Rational Homomorphism (100)

Rational Additive Inverses (95) gives the following terms.

subtraction.


Rational Additive Inverses
Rational Sums
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Additive Identity
Natural Multiplicative Identity
Integer Additive Inverses
Identity Elements
Natural Arithmetic
Integer Sums
Inverse Elements
Natural Powers
Element Functions
Integer Products
Function Inverses
Operations
Natural Products
Integer Numbers
Function Images
Function Composites
Natural Sums
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the multiplicative inverse of $[(a, b)]$ in the rationals?

Result

Proposition 140. *The multiplicative inverse of* $[(a, b)] \in \mathbf{Q}$ *if* $b \neq 0_{\mathbf{Z}}$ *is* $[(b, a)]$.

Notation

We denote the multiplicative inverse of $q \in \mathbf{Q}$ by q^{-1}. We denote $q \cdot (r^{-1})$ by q/r.

Division

We call the operation $(a, b) \mapsto a/b$ *rational division.*

Rational Multiplicative Inverses (96) immediately needs:

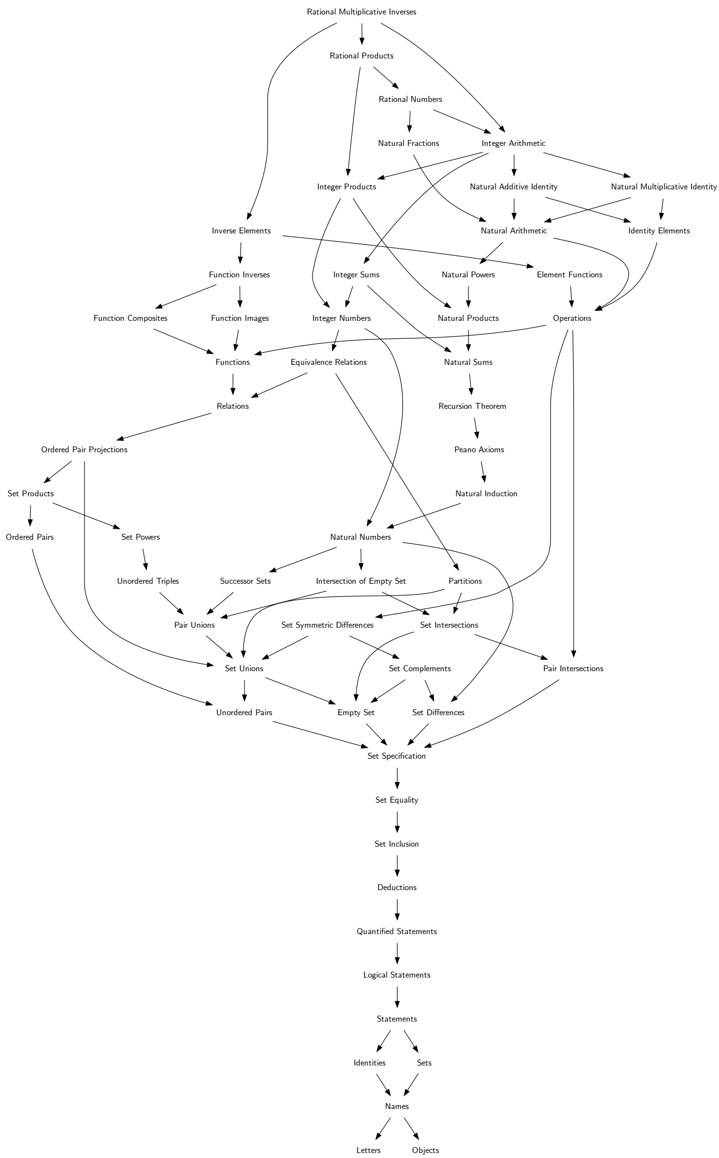
Integer Arithmetic (84)
Inverse Elements (78)
Rational Products (93)

Rational Multiplicative Inverses (96) is immediately needed by:

Integer Rational Homomorphism (100)
Real Multiplicative Inverses (106)

Rational Multiplicative Inverses (96) gives the following terms.

rational division.



Rational Multiplicative Inverses
Rational Products
Rational Numbers
Natural Fractions
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Inverse Elements
Natural Arithmetic
Identity Elements
Function Inverses
Integer Sums
Natural Powers
Element Functions
Function Composites
Function Images
Integer Numbers
Natural Products
Operations
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Pair Unions
Set Symmetric Differences
Set Intersections
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to order the rationals.

Definition

Consider $[(a, b)], [(b, c)] \in \mathbf{Q}$ with $0_{\mathbf{Z}} < b, d$ If $ad < bc$, then we say that $[(a, b)]$ is *less than* $[(b, c)]$.[106] If $[(a, b)]$ is less than $[(b, c)]$ or equal, then we say that $[(a, b)]$ is *less than or equal to* $[(b, c)]$.

Notation

If $x, y \in \mathbf{Q}$ and x is less than y, then we write $x < y$. If x is less than or equal to y, we write $x \leq y$.

[106] One needs to show that this is well-defined. The account will appear in future editions.

Rational Order (97) immediately needs:

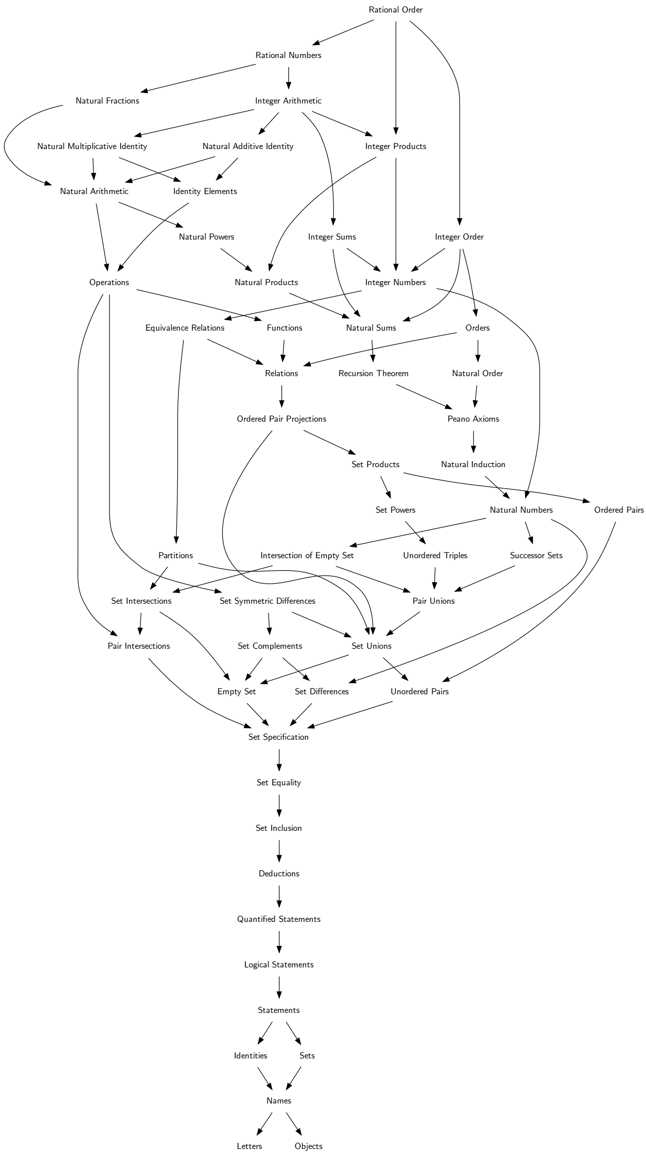
Integer Order (83)
Integer Products (81)
Rational Numbers (91)

Rational Order (97) is immediately needed by:

Complete Fields (109)
Real Order (104)

Rational Order (97) gives the following terms.

less than, less than or equal to.



Rational Order
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Equivalence Relations
Functions
Natural Sums
Orders
Relations
Recursion Theorem
Natural Order
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Set Powers
Natural Numbers
Ordered Pairs
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is a ring $(R, +, \cdot)$ for which $\cdot$ is commutative (i.e., $ab = ba$ for all $a, b \in R$) and $\cdot$ has inverses for all elements except 0. In this case, we refer to *field addition* and *field multiplication*.

Notation

Since our guiding example is the set of rationals $\mathbf{Q}$ with addition and multiplication defined in the usual manner, and we use a bold font for $\mathbf{Q}$, we tend to denote an arbitrary field by $\mathbf{F}$, a mnemonic for "field."

Field operations

Along with field addition and field multiplication, we call the function which takes an element of a field to its additive inverse and the function which takes an element of a field to its multiplicative inverse the *field operations*.

Fields (98) immediately needs:

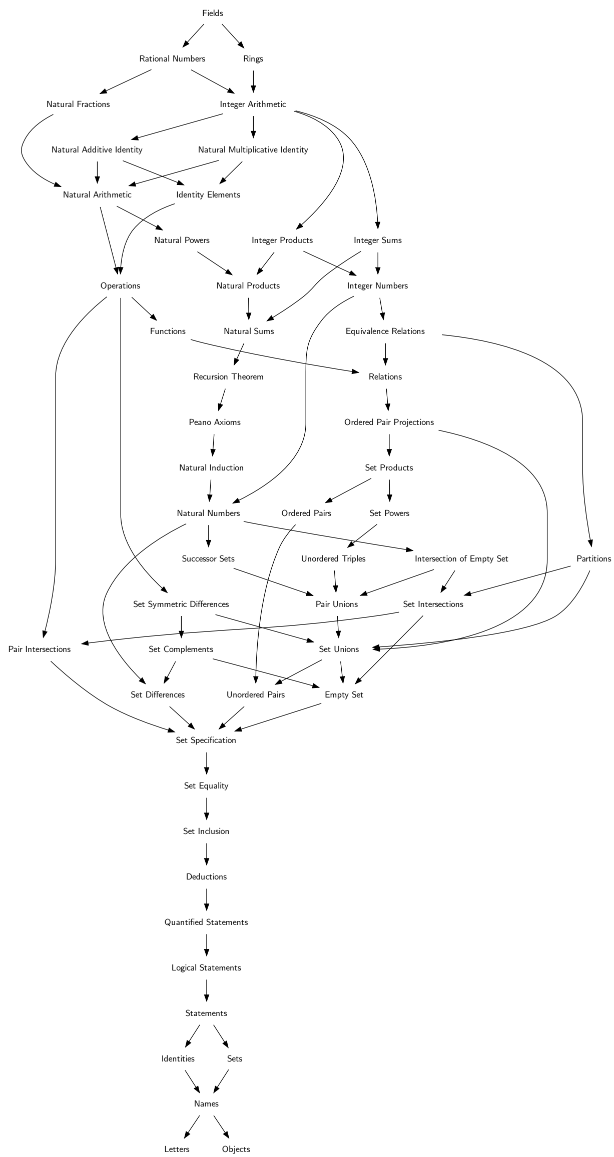
Rational Numbers (91)
Rings (88)

Fields (98) is immediately needed by:

Homomorphisms (99)
Topological Fields (**??**)
Vector Space of Polynomials (**??**)
Vectors (**??**)

Fields (98) gives the following terms.

field, field addition, field multiplication, field operations.


Fields
Rational Numbers
Rings
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Integer Products
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We name a function which preserves algebraic structure.

Definition

A *group homomorphism* between two groups $(A, +)$ and $(B, \tilde{+})$ is a bijection $f : A \to B$ such that $f(1_A) = 1_B$ for $1_A \in A$ and $1_B \in B$ and $f(a + a') = f(a) \tilde{+} f(a')$ for all $a, a' \in A$. We define a *ring homomorphism* and *field homomorphism* similarly.

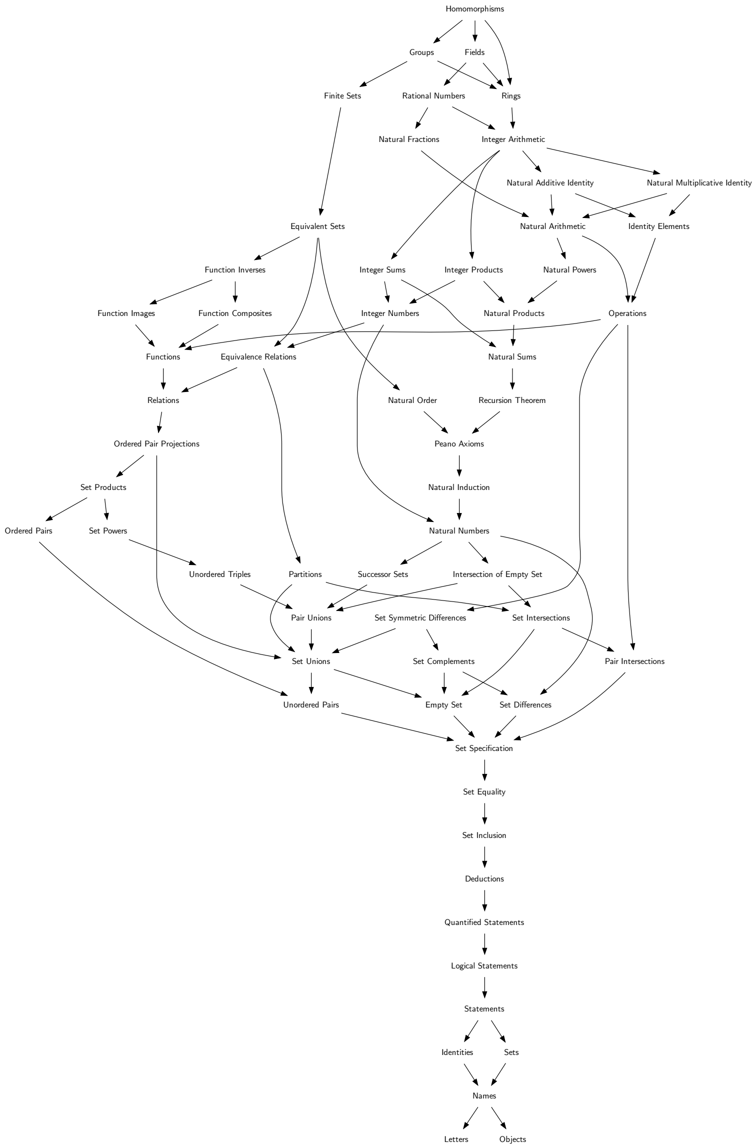
Homomorphisms (99) immediately needs:

Fields (98)
Groups (87)
Rings (88)

Homomorphisms (99) is not immediately needed by any sheet.

Homomorphisms (99) gives the following terms.

group homomorphism, ring homomorphism, field homomorphism.


Homomorphisms
Groups
Fields
Finite Sets
Rational Numbers
Rings
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Equivalent Sets
Natural Arithmetic
Identity Elements
Function Inverses
Integer Sums
Integer Products
Natural Powers
Function Images
Function Composites
Integer Numbers
Natural Products
Operations
Functions
Equivalence Relations
Natural Sums
Relations
Natural Order
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Partitions
Successor Sets
Intersection of Empty Set
Pair Unions
Set Symmetric Differences
Set Intersections
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Do the integer numbers correspond (in the sense of Homomorphisms) to elements of the rationals.

Main result

Indeed, roughly speaking the integers correspond to rationals whose denominator is 1. Define

$$\tilde{Q} := \{[(a,b)] \in \mathbf{Q} \mid b = 1_{\mathbf{Z}}\}.$$

Proposition 141. *The rings* $(\tilde{\mathbf{Q}}, +_{\mathbf{Q}} \mid \tilde{\mathbf{Q}}, \cdot_{\mathbf{Q}} \mid \tilde{\mathbf{Q}})$ *and* $(Z, +_{\mathbf{Z}}, \cdot_{\mathbf{Z}})$ *are homomorphic.*[107]

Proof. The function is $f : \mathbf{Z} \to \mathbf{Q}$ with $f(z) = [(z, 1)]$.[108] □

[107]Indeed, more is true and will be included in future editions. There is an *order perserving* ring homomorphism.

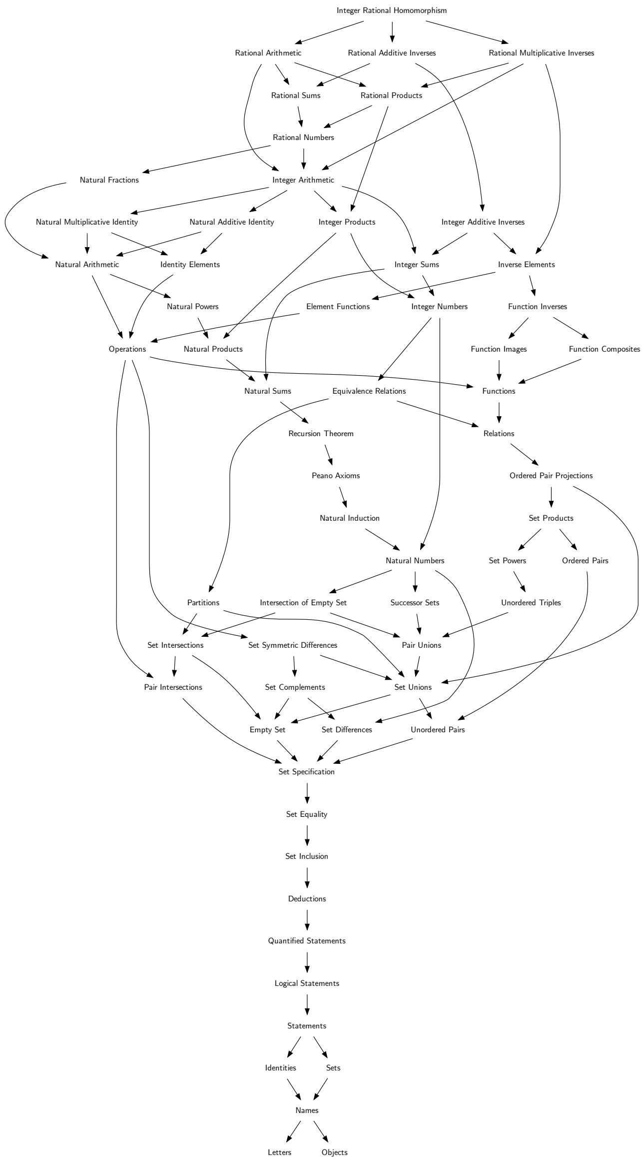
[108]The full account will appear in future editions.

Integer Rational Homomorphism (100) immediately needs:

Rational Additive Inverses (95)
Rational Arithmetic (94)
Rational Multiplicative Inverses (96)

Integer Rational Homomorphism (100) is not immediately needed by any sheet.

Integer Rational Homomorphism (100) gives no terms.


Integer Rational Homomorphism
Rational Arithmetic
Rational Additive Inverses
Rational Multiplicative Inverses
Rational Sums
Rational Products
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Integer Additive Inverses
Natural Arithmetic
Identity Elements
Integer Sums
Inverse Elements
Natural Powers
Element Functions
Integer Numbers
Function Inverses
Operations
Natural Products
Function Images
Function Composites
Natural Sums
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want a set which corresponds to our notion of points on a line.[109]

Rational cuts

We call a subset R of $\mathbf{Q}$ a *rational cut* if (a) $R \neq \varnothing$, (b) $R \neq \mathbf{Q}$, (c) for all $q \in R$, $r \leq q \longrightarrow r \in R$, and (d) R has no greatest element. Briefly, the intuition is that the point is the set of all rationals to less than (or, potentially, equal to) some particular rational number.[110]

Definition

The *set of real numbers* is the set of all rational cuts. This set exists by an application of the principle of selection (see Set Selection) to the power set (see Set Powers) of $\mathbf{Q}$. We call an element of the set of real numbers a *real number* or a *real*. We call the set of real numbers the *set of reals* or *reals* for short.

Notation

We follow tradition and denote the set of real numbers by $\mathbf{R}$, likely a mnemonic for "real."

Other terminology

Some authors call a real number a *quantity* or a *continuous quantity*. The real numbers, then, are said to be *continuous*. When contrasting (using this terminology) a finite set with the real numbers, one refers to the finite set as *discrete*.[111]

---

[109] Future editions will modify and expand this justification.
[110] This brief intuition will be expanded upon in future sheets.
[111] Future editions may move this discussion later, to the discussion of the cardinality of the reals.

Real Numbers (101) immediately needs:

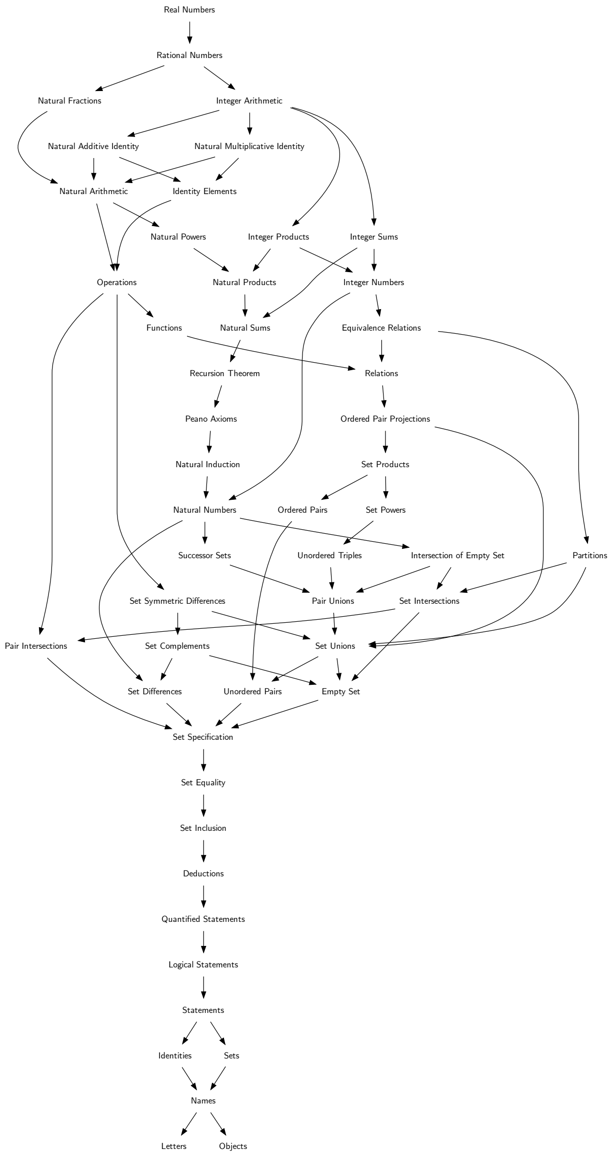
Rational Numbers (91)

Real Numbers (101) is immediately needed by:

Dynamical Systems (**??**)
Logarithm (**??**)
Observation Sequences (**??**)
Quantizations (**??**)
Real Continuity (124)
Real Optimizers (**??**)
Real Order (104)
Real Sequences (**??**)
Real Set Closures (**??**)
Real Summation (**??**)
Real Sums (102)
Regressors (**??**)
Semirings (**??**)
Unbiased Estimators (**??**)

Real Numbers (101) gives the following terms.

rational cut, set of real numbers, real number, real, set of reals, reals, quantity, continuous quantity, continuous, discrete.


Real Numbers
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Integer Products
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to add real numbers.[112]

Definition

The *real sum* of two real numbers R and S is the set

$$\{t \in \mathbf{Q} \mid \exists r \in R, s \in S \text{ with } t = r + s\}.$$

Notation

We denote the sum of two real numbers x and y by $x + y$.

Properties

We can show the following.[113]

Proposition 142 (Associative)**.** $x + (y + z) = (x + y) + z$

Proposition 143 (Commutative)**.** $x + y = y + x$

Proposition 144 (Identity)**.** *The set of negative rational numbers is the additive identity.*

We denote the additive identity of $\mathbf{R}$ under $+$ by $0_{\mathbf{R}}$. When it is clear from context, we call $0_{\mathbf{R}}$ "zero".

[112] Future editions will expand.
[113] Accounts will appear in future editions.

Real Sums (102) immediately needs:

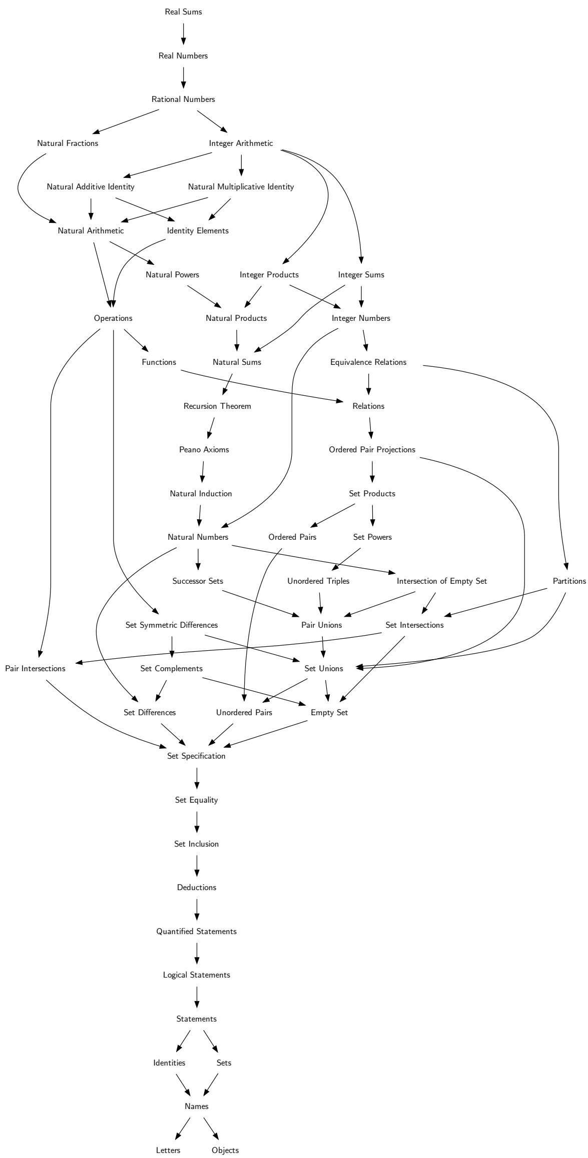
Real Numbers (101)

Real Sums (102) is immediately needed by:

Real Additive Inverses (103)

Real Sums (102) gives the following terms.

real sum.


Real Sums
Real Numbers
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Integer Products
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the additive inverse for reals.[114]

Main result

Proposition 145. *Let $R \in \mathsf{R}$. The set $\{-r \mid r \in R \text{ and } s \notin R\}$ is an additive inverse of R in R.*

Notation

We denote the additive inverse of $R \in \mathsf{R}$ by $-R$.

[114] Future editions will expand.

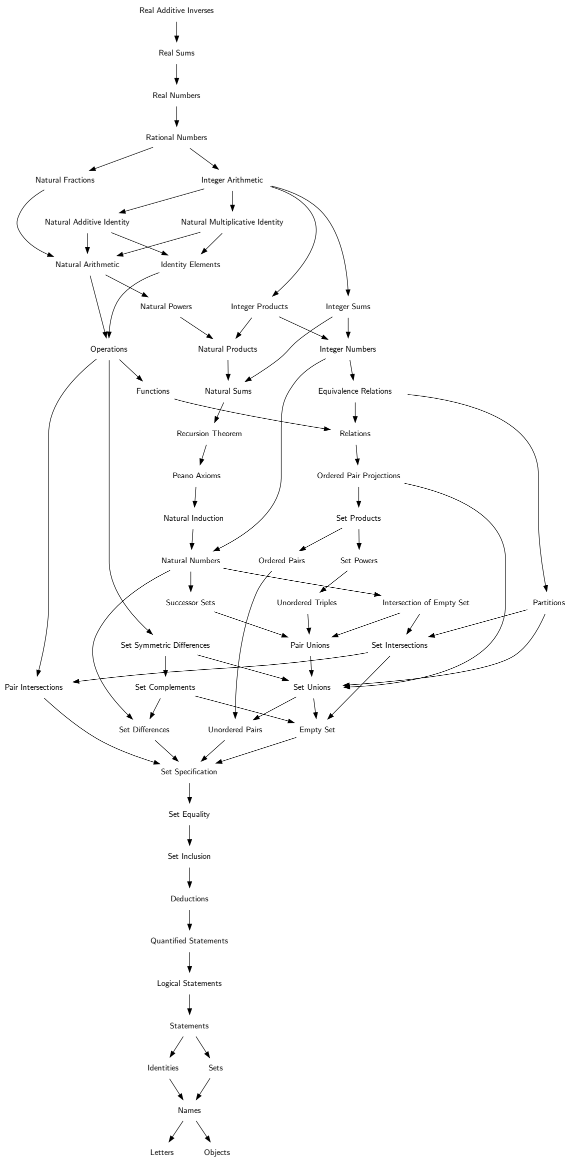
Real Additive Inverses (103) immediately needs:

Real Sums (102)

Real Additive Inverses (103) is immediately needed by:

Real Products (105)

Real Additive Inverses (103) gives no terms.


Real Additive Inverses
Real Sums
Real Numbers
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Natural Powers
Integer Products
Integer Sums
Operations
Natural Products
Integer Numbers
Functions
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Ordered Pairs
Set Powers
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Pair Intersections
Set Complements
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to order the real numbers.[115]

Definition

For $R, S \in \mathsf{R}$ define the total order $\succeq$ by $R \succeq S$ if and only if $R \subset S$. As is usual with comparisons, we use the terms *less than* and *less than or equal to*.

Notation

If R is less than S we write $R < S$. If R is less than or equal to S we write $R \leq S$.

[115] Future editions will expand

Real Order (104) immediately needs:

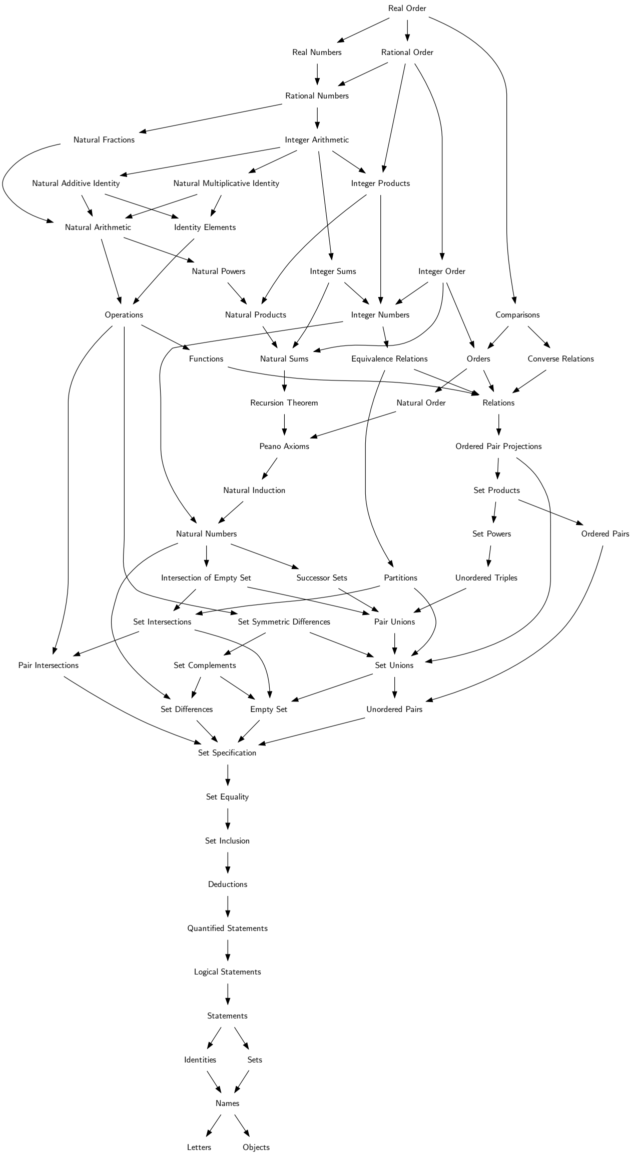
Comparisons (??)
Rational Order (97)
Real Numbers (101)

Real Order (104) is immediately needed by:

Complete Fields (109)
Floating Point Representations (??)
Greatest Lower Bounds (??)
Least Upper Bounds (108)
Monotone Real Functions (??)
Real Line (112)
Real Plane (116)
Real Products (105)

Real Order (104) gives the following terms.

less than, less than or equal to, less than, less than or equal to.


Real Order
Real Numbers
Rational Order
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Intersection of Empty Set
Successor Sets
Partitions
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to multiply real numbers.[116]

Definition

The *real product* of two real numbers R and S is defined

1. if R or S is $\{q \in \mathbf{Q} \mid q < 0_{\mathbf{Q}}\}$, then the $\{q \in \mathbf{Q} \mid q < 0_{\mathbf{Q}}\}$
2. otherwise,
   (a) if R or S is $0_{\mathbf{R}}$, then $0_{\mathbf{R}}$.
   (b) if $R, S \neq 0_{\mathbf{R}}$ and $0_{\mathbf{R}} \in R, S$, let T be
   $$\{t \in \mathbf{Q} \mid r \in R, s \in S, r, s \geq 0_{\mathbf{Q}}, t = r \cdot s\}$$
   then $T \cup \{q \in \mathbf{Q} \mid q \leq 0_{\mathbf{Q}}\}$[117]
   (c) If $R, S \neq 0_{\mathbf{R}}$, $0_{\mathbf{R}} \in R$ and $0_{\mathbf{R}} \notin S$, then the additive inverse of the product of $-R$ with S.
   (d) If $R, S \neq 0_{\mathbf{R}}$, $0_{\mathbf{R}} \notin R$ and $0_{\mathbf{R}} \in S$, then the additive inverse of the product of R with $-S$.
   (e) If $R, S \neq 0_{\mathbf{R}}$, and $0_{\mathbf{R}} \notin R, S$, then the product of $-R$ with $-S$.

Notation

We denote the product of two real numbers x and y by $x \cdot y$.

Properties

Proposition 146 (Associative). $x + (y + z) = (x + y) + z$

Proposition 147 (Commutative). $x + y = y + x$

Proposition 148 (Identity). *The set of all rationals less than $1_{\mathbf{Q}}$ is the multiplicative identity.*

[116] Future editions will expand.
[117] We use $\geq$ in the usual way, it will be defined earlier in future editions.

We denote the the multiplicative identity by 1_{R}. When it is clear from context, we call 1_{R} "one".

Real Products (105) immediately needs:

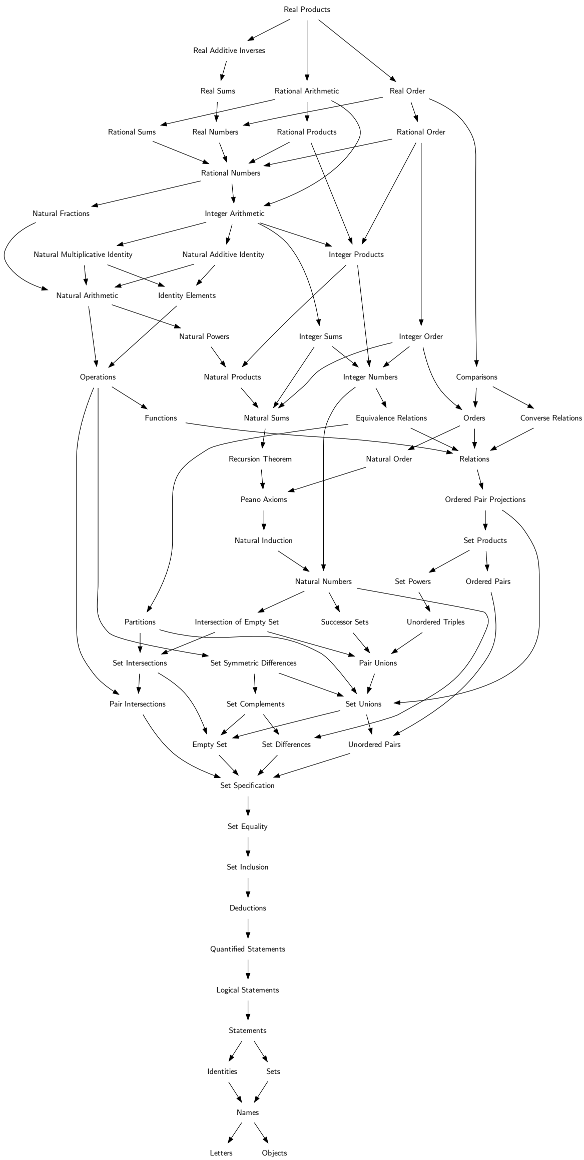
Rational Arithmetic (94)
Real Additive Inverses (103)
Real Order (104)

Real Products (105) is immediately needed by:

Real Multiplicative Inverses (106)

Real Products (105) gives the following terms.

real product.


Real Products
Real Additive Inverses
Real Sums
Rational Arithmetic
Real Order
Rational Sums
Real Numbers
Rational Products
Rational Order
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What is the multiplicative inverse in the reals?

Result

We can show the following.[118]

Proposition 149. *The multiplicative inverse of $R \in \mathbf{R}$, $R \neq 0_{\mathbf{R}}$,*

1. *if $0_{\mathbf{Q}} \in R$, then*

$$S = \{q \in \mathbf{Q} \mid q \leq 0_{\mathbf{Q}}\} \cup \left\{r^{-1} \;\middle|\; \exists s < r, (r \notin R)\right\}$$

   is a multiplicative inverse of R.

2. *if $0_{\mathbf{Q}} \notin R$, then case (1) applies to $-R$. Let S be the multiplicative inverse of $-R$. Then the additive inverse of S, i.e., $-S$ is a multiplicative inverse of R.*

Notation

We denote the multiplicative inverse of $r \in \mathbf{R}$ by r^{-1}. We denote $q \cdot (r^{-1})$ by q/r.

Division

We call the operation $(a, b) \mapsto a/b$ *real division.* We call the product of a and the multiplicative inverse of b the *(real) quotient* of a and b.

[118] The account will appear in future editions.

Real Multiplicative Inverses (106) immediately needs:

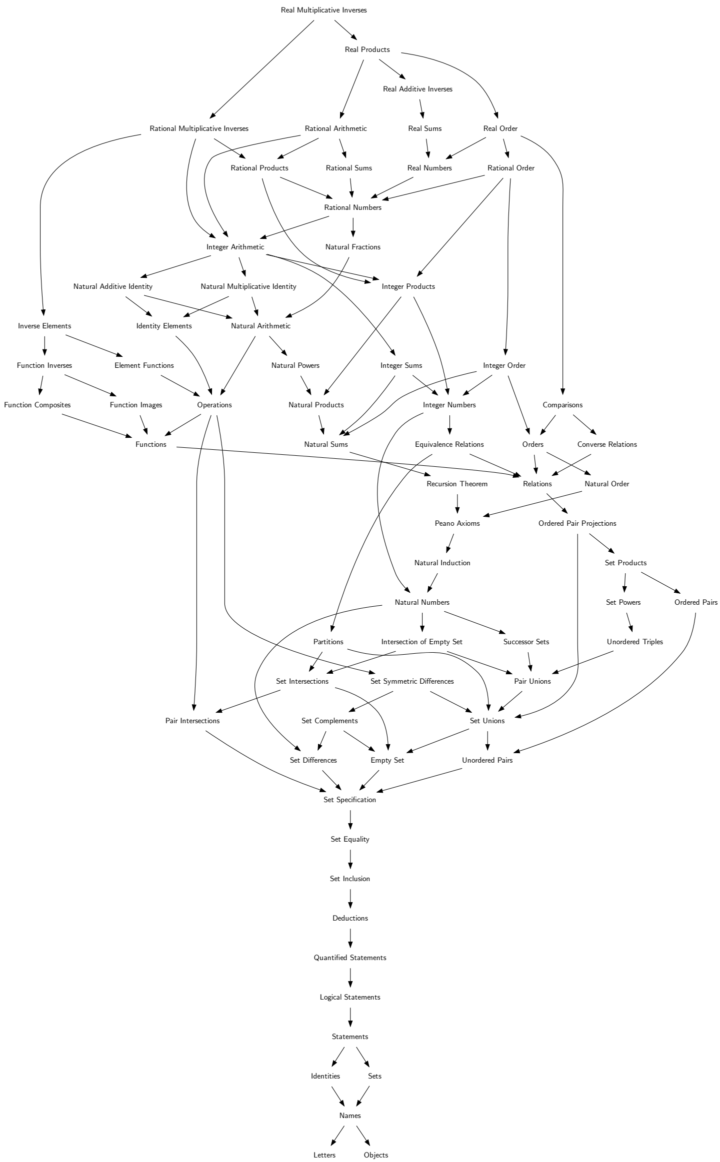
Rational Multiplicative Inverses (96)
Real Products (105)

Real Multiplicative Inverses (106) is immediately needed by:

Real Arithmetic (107)

Real Multiplicative Inverses (106) gives the following terms.

real division, (real) quotient.


Real Multiplicative Inverses
Real Products
Real Additive Inverses
Rational Multiplicative Inverses
Rational Arithmetic
Real Sums
Real Order
Rational Products
Rational Sums
Real Numbers
Rational Order
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Inverse Elements
Identity Elements
Natural Arithmetic
Function Inverses
Element Functions
Natural Powers
Integer Sums
Integer Order
Function Composites
Function Images
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Relations
Natural Order
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

What are addition and multiplication for reals? What are the identity elements?

Definition

We call the operation of forming real sums *real addition*. We call the operation of forming real products *real multiplication*.

Results

It is easy to see the following.[119]

Distributive

Proposition 150. *For reals* $x, y, z \in \mathsf{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

[119] Nonetheless, the full accounts will appear in future editions.

Real Arithmetic (107) immediately needs:

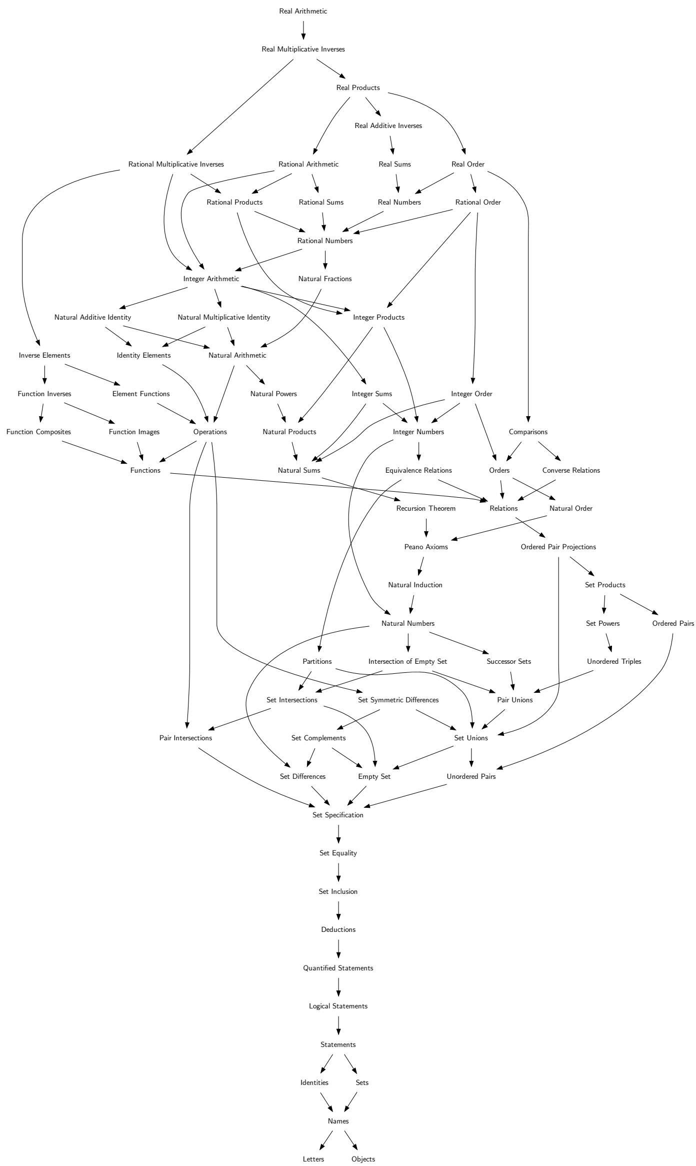
Real Multiplicative Inverses (106)

Real Arithmetic (107) is immediately needed by:

Complex Numbers (??)
Periodic Functions (??)
Rational Real Homomorphism (111)
Real Binomial Expansions (??)
Real Modular Arithmetic (??)
Real Polynomials (??)
Real Squares (??)

Real Arithmetic (107) gives the following terms.

real addition, real multiplication.


Real Arithmetic
Real Multiplicative Inverses
Real Products
Real Additive Inverses
Rational Multiplicative Inverses
Rational Arithmetic
Real Sums
Real Order
Rational Products
Rational Sums
Real Numbers
Rational Order
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Inverse Elements
Identity Elements
Natural Arithmetic
Function Inverses
Element Functions
Natural Powers
Integer Sums
Integer Order
Function Composites
Function Images
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Relations
Natural Order
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Definition

Suppose $(A, \leq)$ is a partially ordered set.

An *upper bound* for $B \subset A$ is an element $a \in A$ so that $b \leq a$ for all $b \in B$. A set is *bounded from above* if it has a least upper bound. A *least upper bound* for B is an element $c \in A$ so that c is an upper bound and $c < a$ for all other upper bounds a.

Proposition 151. *If there is a least upper bound it is unique.*[120]

We call the unique least upper bound of a set (if it exists) the *supremum.*

Notation

We denote the supremum of a set $B \subset A$ by $\sup A$.

[120] Proof in future editions.

Least Upper Bounds (108) immediately needs:

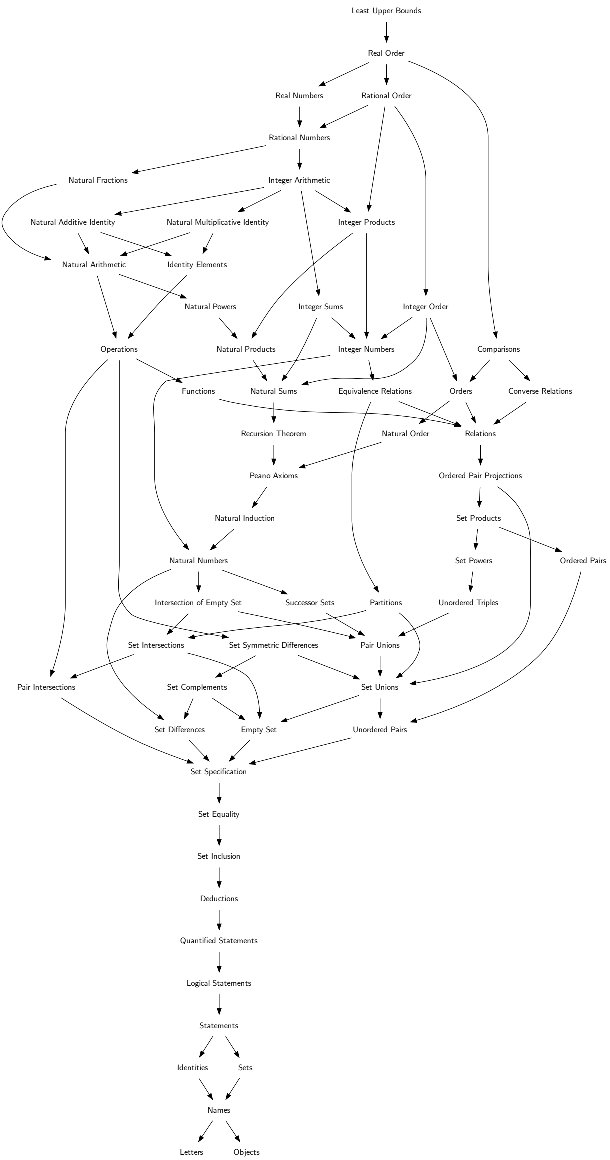
Real Order (104)

Least Upper Bounds (108) is immediately needed by:

Complete Fields (109)
Lattices (**??**)
Supremum Norm (**??**)

Least Upper Bounds (108) gives the following terms.

upper bound, bounded from above, least upper bound, supremum.


Least Upper Bounds
Real Order
Real Numbers
Rational Order
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Intersection of Empty Set
Successor Sets
Partitions
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want the a field which corresponds to points on the real line.[121]

Definition

An ordered field[122] is *complete* if every nonempty subset bounded from above has a least upper bound.

[121]Future editions are likely to modify this why.

[122]To be defined in future editions, but we take the usual definition of a field with an order. See, for example Rational Orderor Real Order).

Complete Fields (109) immediately needs:

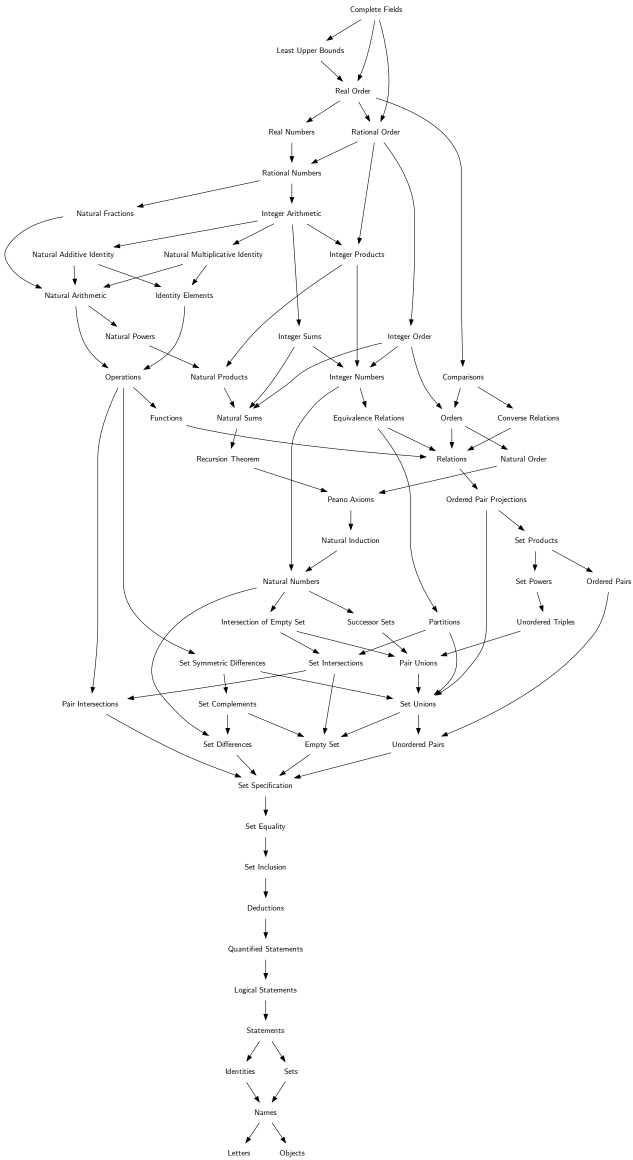
Least Upper Bounds (108)
Rational Order (97)
Real Order (104)

Complete Fields (109) is immediately needed by:

Real Completeness (110)

Complete Fields (109) gives the following terms.

complete.


Complete Fields
Least Upper Bounds
Real Order
Real Numbers
Rational Order
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Relations
Natural Order
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Intersection of Empty Set
Successor Sets
Partitions
Unordered Triples
Set Symmetric Differences
Set Intersections
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Is the set of real numbers a complete ordered field (in the sense of Complete Fields?

Main result

Proposition 152. $(\mathbf{R}, +, \cdot, <)$ *is a complete ordered field.*[123]

Proof. The supremum of a set of nonempty real numbers bounded from above R is $\cup R$. $\square$

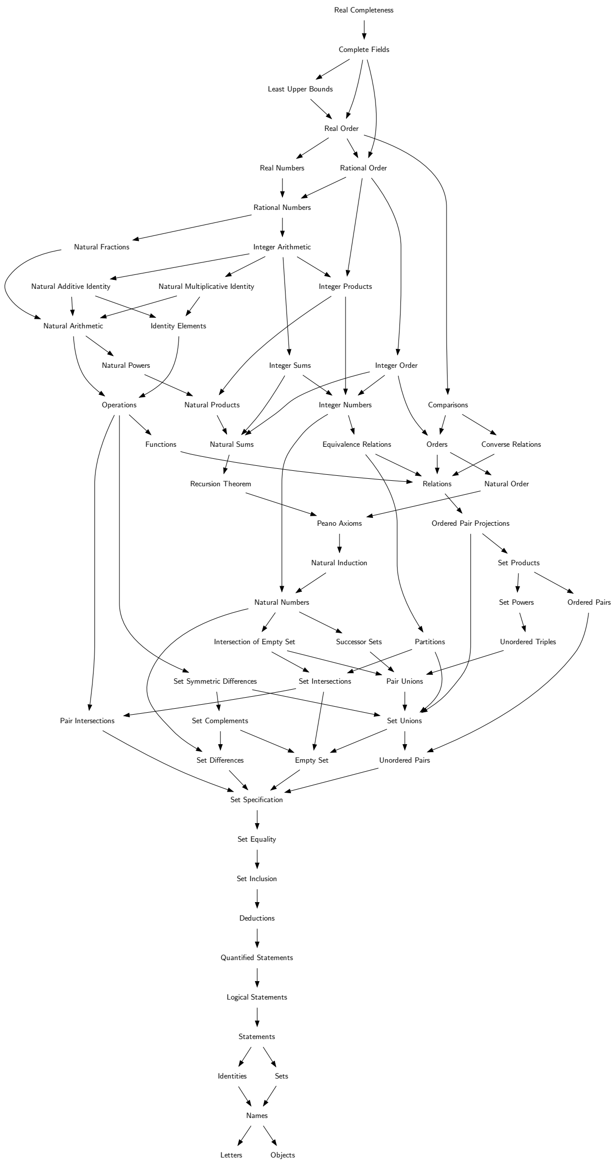
[123]The account will appear in future editions.

Real Completeness (110) immediately needs:

Complete Fields (109)

Real Completeness (110) is not immediately needed by any sheet.

Real Completeness (110) gives no terms.


Real Completeness
Complete Fields
Least Upper Bounds
Real Order
Real Numbers
Rational Order
Rational Numbers
Natural Fractions
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Natural Arithmetic
Identity Elements
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Relations
Natural Order
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Intersection of Empty Set
Successor Sets
Partitions
Unordered Triples
Set Symmetric Differences
Set Intersections
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Do the rational numbers correspond (in the sense of Homomorphisms) to elements of the reals.

Main result

Indeed, roughly speaking the rationals correspond to elements of the reals which are bounded above by that rational. Denote by $\tilde{\mathbf{R}}$ the set $\{q \in \mathbf{R} \mid \exists s \in \mathbf{Q}, q = \{t \in \mathbf{Q} \mid t < s\}\}$.

Proposition 153. *The fields* $(\tilde{\mathbf{R}}, +_{\mathbf{R}} \mid \tilde{\mathbf{R}}, \cdot_{\mathbf{R}} \mid \tilde{\mathbf{R}})$ *and* $(Q, +_{\mathbf{Q}}, \cdot_{Q})$ *are homomorphic.*[124]

Proof. The function is $f : \mathbf{Q} \to \mathbf{R}$ with $f(q) = \{(r \in \mathbf{Q} \mid r < q\}$ □

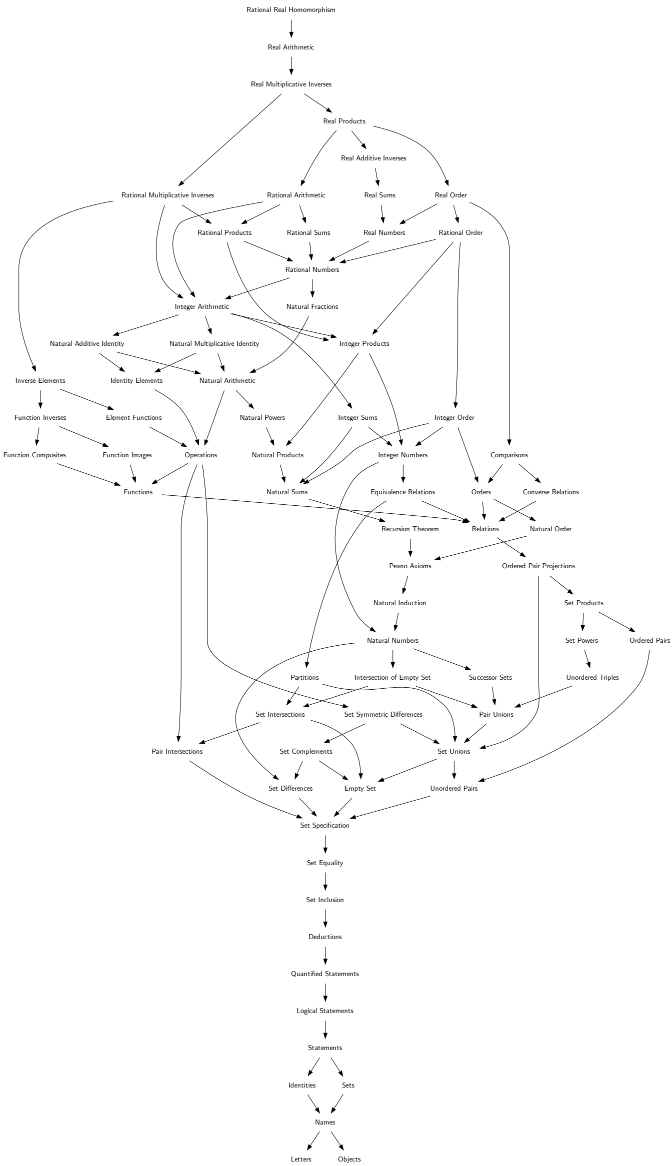
[124] Indeed, more is true and will be included in future editions. There is an *order perserving* field homomorphism.

Rational Real Homomorphism (111) immediately needs:

Real Arithmetic (107)

Rational Real Homomorphism (111) is not immediately needed by any sheet.

Rational Real Homomorphism (111) gives no terms.


Rational Real Homomorphism
Real Arithmetic
Real Multiplicative Inverses
Real Products
Real Additive Inverses
Rational Multiplicative Inverses
Rational Arithmetic
Real Sums
Real Order
Rational Products
Rational Sums
Real Numbers
Rational Order
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Inverse Elements
Identity Elements
Natural Arithmetic
Function Inverses
Element Functions
Natural Powers
Integer Sums
Integer Order
Function Composites
Function Images
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Relations
Natural Order
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We are constantly thinking of the real numbers as the points of a line.[125]

Discussion

We commonly associate elements of the real numbers (see Real Numbers) with points on a line (see Geometry).

Principle 8 (Point Sets)**.** *Given a line, there exists a set of its (infinite) points.*

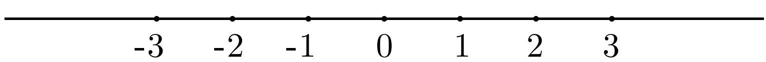
Principle 9 (Real Line Correspondence)**.** *Let P be the set of points for a line. There exists a one-to-one correspondence mapping elements of P onto elements of $\mathbf{R}$.*

For this reason, we sometimes call elements of the real numbers *points*. We call the point associated with 0 the *origin*.

Visualization

To visualize the correspondence we draw a line. We then associate a point of the line with the $0 \in \mathbf{R}$. We can label it so. We then pick a unit length. We associate the points a unit length away from zero with $1 \in \mathbf{R}$ (on the right) and $-1 \in \mathbf{R}$ (on the left). We do the same for two and 2 and -2, 3 and -3, and then we say that we could continue the process indefinitely.

We can visualize the image below



[125]Future editions will modify this sheet.

Real Line (112) immediately needs:

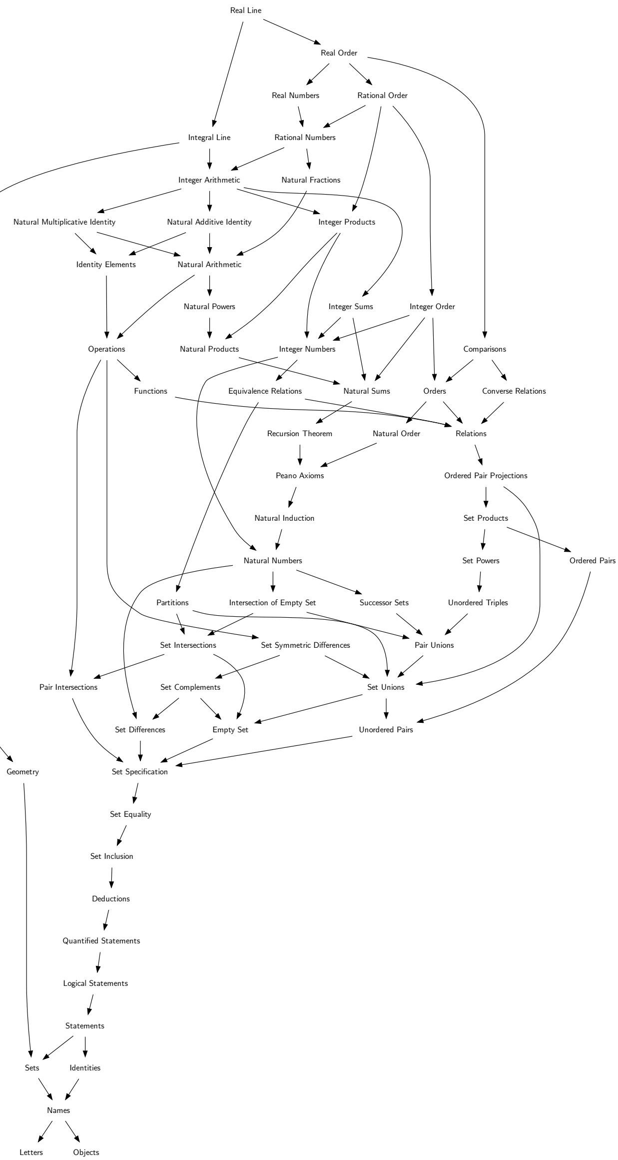
Integral Line (82)
Real Order (104)

Real Line (112) is immediately needed by:

Intervals (113)
Length Measure (**??**)

Real Line (112) gives the following terms.

points, origin.


Real Line
Real Order
Real Numbers
Rational Order
Integral Line
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Identity Elements
Natural Arithmetic
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Equivalence Relations
Natural Sums
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We name and denote subsets of the set of real numbers which correspond to segments of a line.

Definition

Take two real numbers, with the first less than the second.

An *interval* is one of four sets:

1. the set of real numbers larger than the first number and smaller than the second; we call the interval *open*.
2. the set of real numbers larger than or equal to the first number and smaller than or equal to the second number; we call the interval *closed*.
3. the set of real numbers larger than the first number and smaller than or equal to the second; we call the interval *open on the left* and *closed on the right*
4. the set of real numbers larger than or equal to the first number and smaller than the second; we call the interval *closed on the left* and *open on the right*.

If an interval is neither open nor closed we call it *half-open* or *half-closed*

We call the two numbers the *endpoints* of the interval. An open interval does not contain its endpoints. A closed interval contains its endpoints. A half-open/half-closed interval contains only one of its endpoints. We say that the endpoints *delimit* the interval.

Notation

Let a, b be two real numbers which satisfy the relation $a < b$.

We denote the open interval from a to b by (a, b). This notation, although standard, is the same as that for ordered pairs; no confusion arises with adequate context.[126]

We denote the closed interval from a to b by $[a, b]$. We record the fact $(a, b) \subset [a, b]$ in our new notation.

We denote the half-open interval from a to b, closed on the right, by $(a, b]$ and the half-open interval from a to b, closed on the left, by $[a, b)$.[127]

The *unit interval* is the set $[0_{\mathbf{R}}, 1_{\mathbf{R}}]$ and we sometimes denote it by $\mathbf{I}$.

[126] In future editions, we may use $\rlap{(}|a, b\rlap{)}|$ or even $\rlap{[}|a, b\rlap{]}|$.

[127] Some authors use]a, b], [a, b[and]a, b[.

Intervals (113) immediately needs:

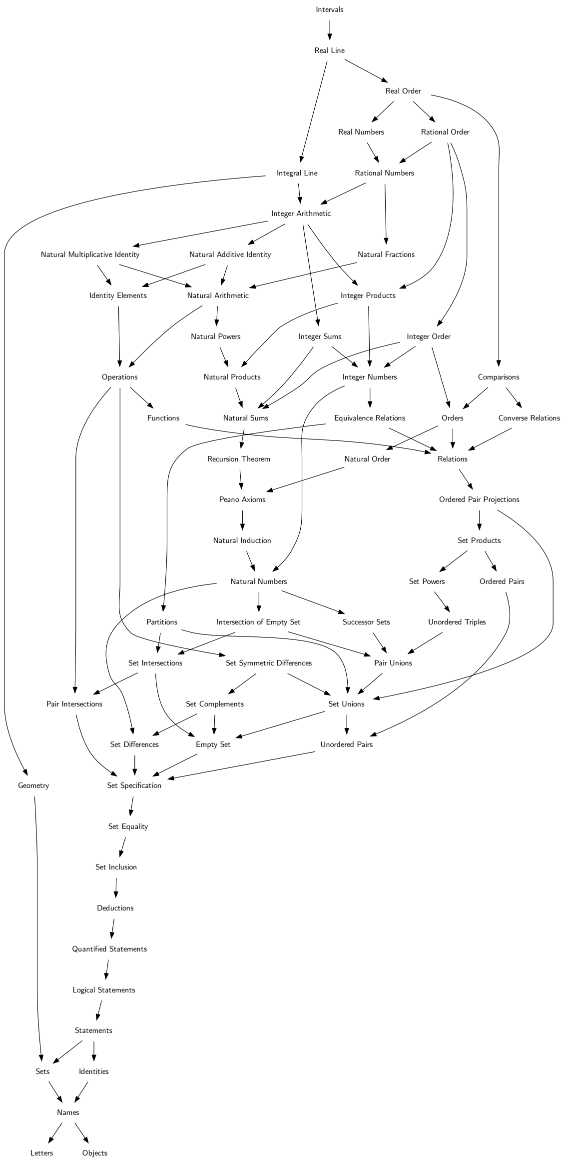
Real Line (112)

Intervals (113) is immediately needed by:

Interval Graphs (**??**)
Interval Length (114)
Interval Partitions (**??**)
N-Dimensional Line Segments (**??**)
Product Sections (**??**)
Real Functions (123)
Rectangles (**??**)

Intervals (113) gives the following terms.

interval, open, closed, open on the left, closed on the right, closed on the left, open on the right, half-open, half-closed, endpoints, delimit, unit interval.


Intervals
Real Line
Real Order
Real Numbers
Rational Order
Integral Line
Rational Numbers
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Natural Fractions
Identity Elements
Natural Arithmetic
Integer Products
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Natural Sums
Equivalence Relations
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

Toward defining the length of a subset of real numbers, we start by defining the length of an interval.

Definition

The *length* of an interval is the difference of its endpoints: the larger less the smaller.

Notation

Let a, b be real numbers which satisfy the relation $a < b$. The length of (a, b), $[a, b]$ $[a, b)$ and $(a, b]$ is, in each case, $b - a$.

For example, the length of the interval $(0, 1)$ is 1.

Interval Length (114) immediately needs:

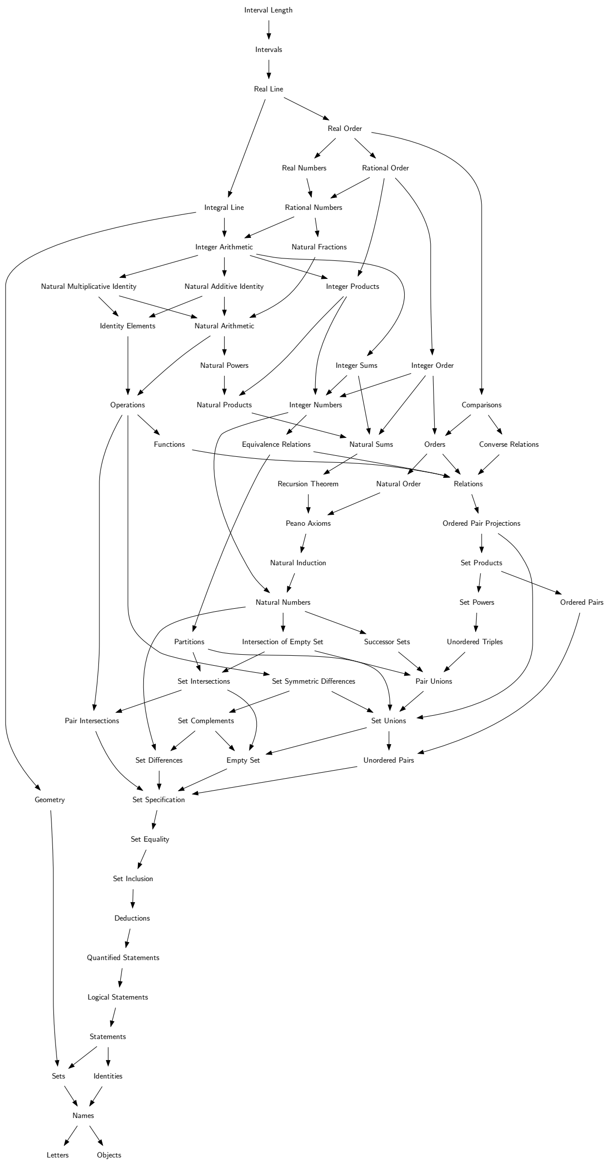
Intervals (113)

Interval Length (114) is immediately needed by:

Absolute Value (115)
Length Common Notions (**??**)
Plane Distance (117)

Interval Length (114) gives the following terms.

length.


Interval Length
Intervals
Real Line
Real Order
Real Numbers
Rational Order
Integral Line
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Identity Elements
Natural Arithmetic
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Equivalence Relations
Natural Sums
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We want a notion of distance between elements of the real line.

Definition

The *absolute value* of a real number is the greater of itself and its additive inverse. In other words, if x is positive, then the absolute value of x is x. If x is negative, then the absolute value of x is $-x$ (a positive real number).

Notation

We denote the absolute value of a real number $x \in \mathbf{R}$ by $|x|$.

Distance

The absolute value can be interpreted as the distance between the point corresponding to the real number and the point corresponding to 0. We can generalize this idea. Consider $x, y \in \mathbf{R}$. If $x > y$, then $x - y > 0$ and so the distance between the corresponding points is $x - y$. If $x < y$ then $y - x > 0$, and so the distance is $y - x$.

The observation is that $|-x| = |x|$. So

$$|y - x| = |-(x - y)| = |x - y|.$$

So if we just care about the distance between the points corresponding to y and x, we can consider $|x - y|$, without regard for their order. In other words, the function $(x, y) \mapsto |x - y|$ is symmetric in x and y.

Absolute Value (115) immediately needs:

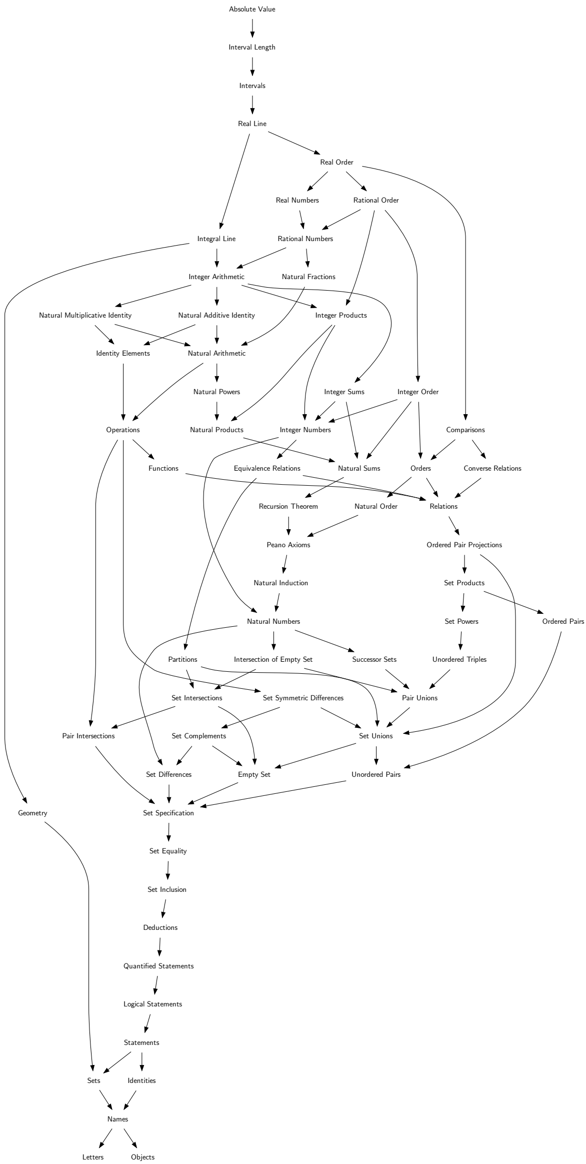
Interval Length (114)

Absolute Value (115) is immediately needed by:

Complex Numbers (**??**)
Convergence In Measure (**??**)
Convergence In Probability (**??**)
Function Growth Classes (**??**)
Functionals (**??**)
Integrable Function Spaces (**??**)
Metric Space Examples (**??**)
Metrics (125)
Plane Norm (**??**)
Pointwise and Measure Limits (**??**)
Real Continuity (124)
Real Egoprox Sequences (**??**)
Real Integral Monotone Convergence (**??**)
Real Limits (**??**)
Real Norm (**??**)
Supremum Norm (**??**)

Absolute Value (115) gives the following terms.

absolute value.


Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Numbers
Rational Order
Integral Line
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Identity Elements
Natural Arithmetic
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Equivalence Relations
Natural Sums
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We are constantly thinking of the elements of $\mathbf{R}^2$ as points of a plane.[128]

Discussion

We commonly associate elements of $\mathbf{R}^2$ with points on a plane. (see Geometry).

Principle 10 (Line Sets)**.** *Given a plane, there exists a set of its (infinite) lines.*

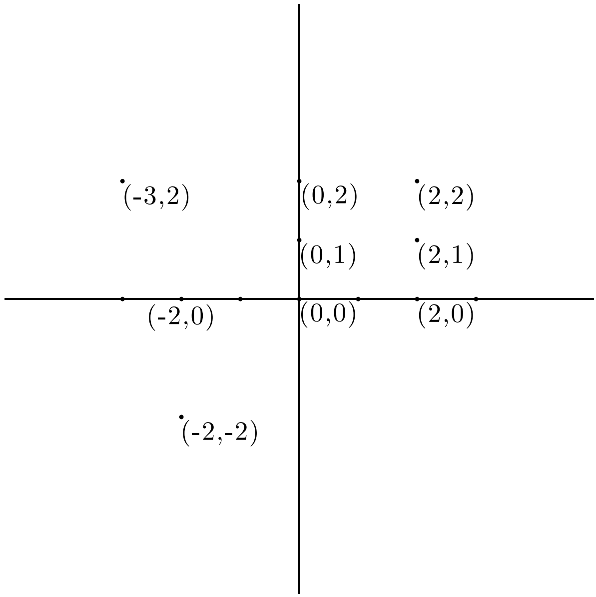
Principle 11 (Real Plane Correspondence)**.** *Let L be the set of lines of a plane. Then $\cup L$ is the set of points of the plane. There exists a one-to-one correspondence mapping elements of $\cup L$ onto elements of $\mathbf{R}^2$.*

For this reason, we sometimes call elements of $\mathbf{R}^2$ *points*. We call the point associated with $(0,0)$ the *origin*. We call the element of $\mathbf{R}^2$ which corresponds to a point the *coordinates* of the point.

Visualization

To visualize the correspondence we draw two perpendicular lines. We then associate a point of the line with $(0,0) \in \mathbf{R}^2$. We can label it so. We then pick a unit length. And proceed as usual.[129]

[128] Future editions will modify this sheet.
[129] Future editions will expand this.



Given that we have identified a plane with $\mathbf{R}^2$ in this way, we call $(x, y) \in \mathbf{R}^2$ the *coordinates* of the point it corresponds to. Many authors refer to this identification as a *Cartesian coordinate system* (or *Rectangular coordinate system*, x-y *coordinate system*).

Real Plane (116) immediately needs:

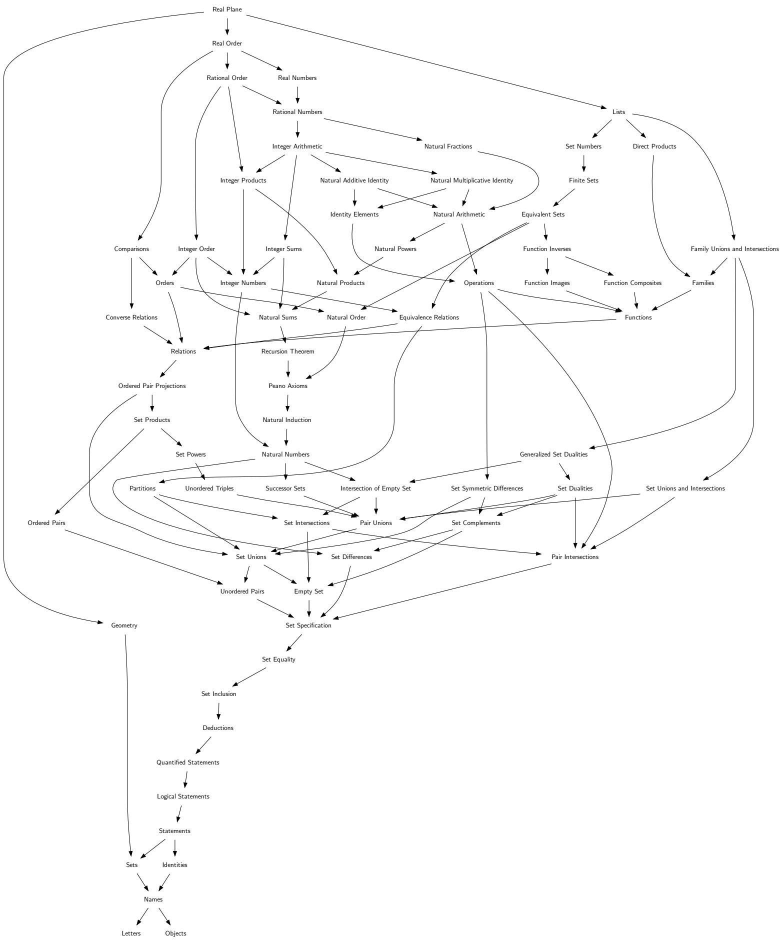
Geometry (22)
Lists (**??**)
Real Order (104)

Real Plane (116) is immediately needed by:

Area Measure (**??**)
Circular Coordinates (**??**)
Complex Plane (**??**)
Plane Distance (117)
Plane Vectors (**??**)
Real Space (118)
Rectangles (**??**)

Real Plane (116) gives the following terms.

points, origin, coordinates, coordinates, Cartesian coordinate system, Rectangular coordinate system, x-y coordinate system.


Real Plane
Real Order
Rational Order
Real Numbers
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Comparisons
Integer Order
Integer Sums
Natural Powers
Function Inverses
Family Unions and Intersections
Orders
Integer Numbers
Natural Products
Operations
Function Images
Function Composites
Families
Converse Relations
Natural Sums
Natural Order
Equivalence Relations
Functions
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Partitions
Unordered Triples
Successor Sets
Intersection of Empty Set
Set Symmetric Differences
Set Dualities
Set Unions and Intersections
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

What is the distance between two points in a plane?

Definition

We define the distance between two points in the plane as the length of the line segment connecting them.[130] In terms of their coordinates $(x_1, x_2), (y_1, y_2) \in \mathbf{R}^2$, the *plane distance* of two points is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This is sometimes referred to as the *Euclidean distance.* We have thus defined a function mapping $\mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}$.

[130] This intuition will be expanded in future editions.

Plane Distance (117) immediately needs:

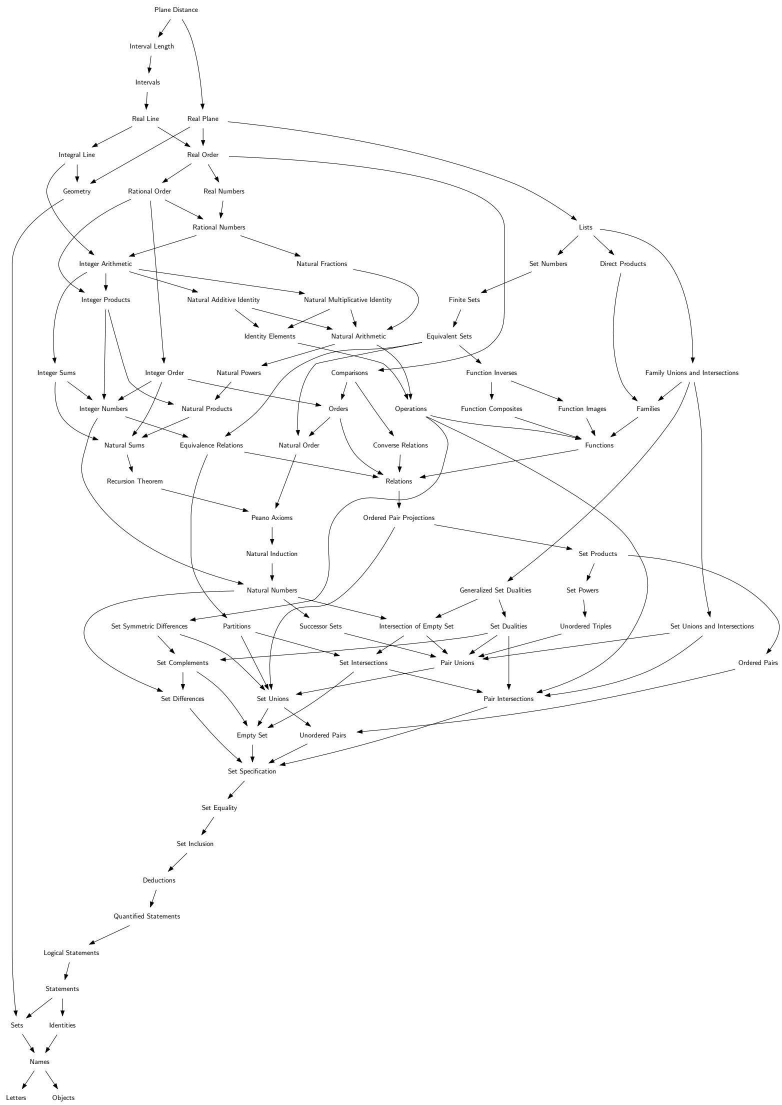
Interval Length (114)
Real Plane (116)

Plane Distance (117) is immediately needed by:

Complex Distance (**??**)
Plane Inner Product (**??**)
Plane Norm (**??**)
Space Distance (119)

Plane Distance (117) gives the following terms.

plane distance, Euclidean distance.


Plane Distance
Interval Length
Intervals
Real Line
Real Plane
Integral Line
Real Order
Geometry
Rational Order
Real Numbers
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Integer Sums
Integer Order
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We are constantly thinking of $\mathbf{R}^3$ as points of space.[131]

Definition

We commonly associate elements of $\mathbf{R}^3$ with points in space. (see Geometry).

Principle 12 (Plane Sets)**.** *There exists a set of all planes.*

Principle 13 (Real Space Correspondence)**.** *Let P be the set of all planes of space. Then $\cup P$ is the set of all lines and $\cup\cup P$ is the set of all points. There exists a one-to-one correspondence mapping elements of $\cup\cup P$ onto elements of $\mathbf{R}^3$.*

For this reason, we sometimes call elements of $\mathbf{R}^3$ *points*. We call the point associated with $(0, 0, 0)$ the *origin*. We call the element of $\mathbf{R}^3$ which corresponds to a point the *coordinates* of the point.

Visualization

To visualize the correspondence we draw three perpendicular lines. We call these *axes*. We then associate a point of the line with $(0, 0, 0) \in \mathbf{R}^3$. We can label it so. We then pick a unit length. And proceed as usual.[132]

[131] Future editions will modify this sheet.
[132] Future editions will expand this.

Real Space (118) immediately needs:

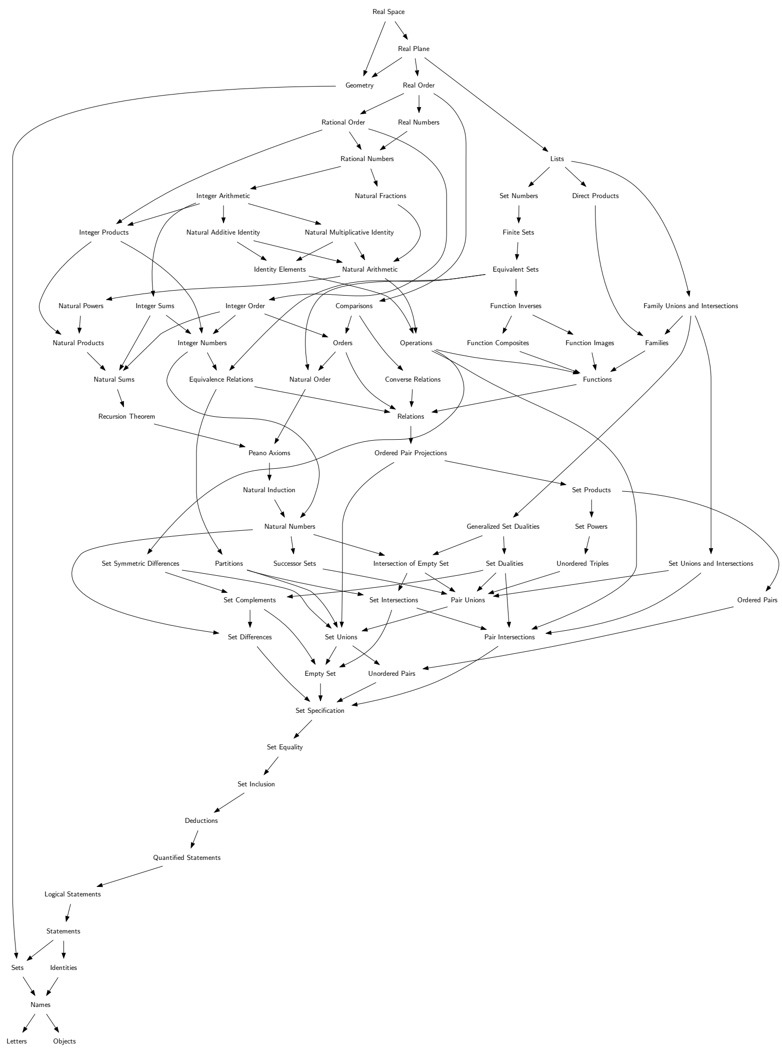
Geometry (22)
Real Plane (116)

Real Space (118) is immediately needed by:

Cubes (**??**)
Space Distance (119)
Volume Measure (**??**)

Real Space (118) gives the following terms.

points, origin, coordinates, axes.


Real Space
Real Plane
Geometry
Real Order
Rational Order
Real Numbers
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Natural Products
Integer Numbers
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

What is the distance between two points in space?

Definition

We define the distance between two points in space as the length of the line segment connecting them. In terms of their coordinates $(x_1, x_2, x_3), (y_1, y_2, x_3) \in \mathbf{R}^3$, the *space distance* of two points is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

This is sometimes referred to as the *Euclidean distance*. We have thus defined a function mapping $\mathbf{R}^3 \times \mathbf{R}^3$ into $\mathbf{R}$.

Space Distance (119) immediately needs:

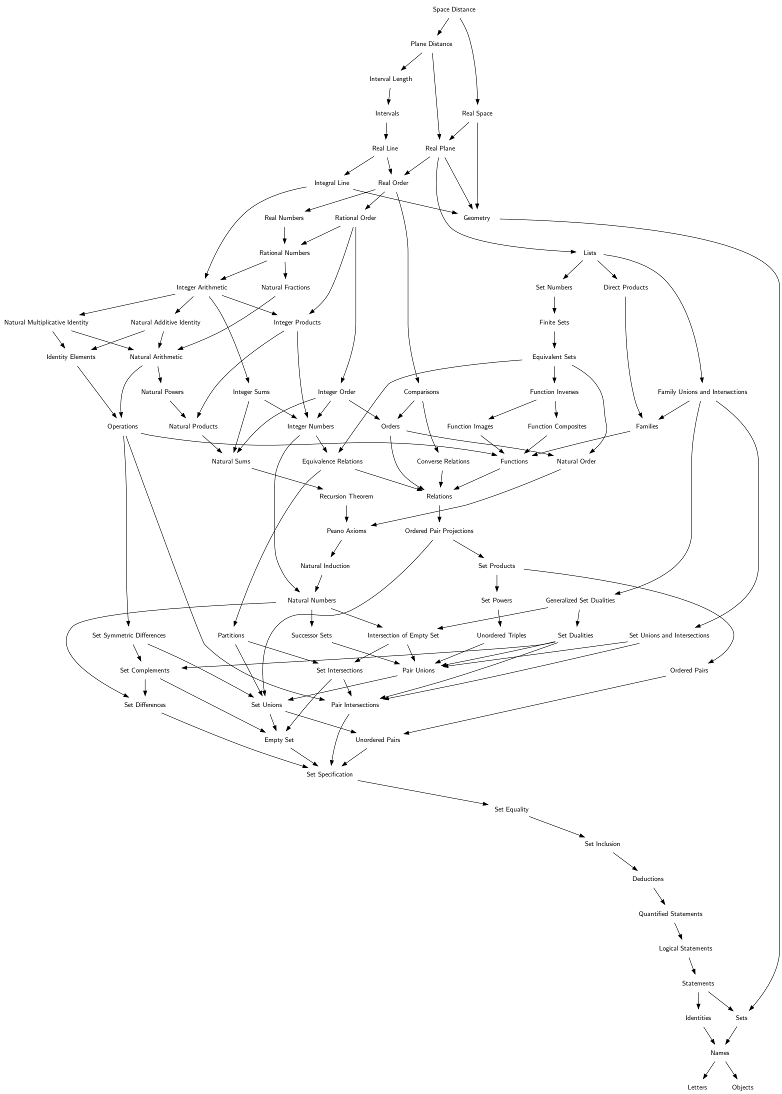
Plane Distance (117)
Real Space (118)

Space Distance (119) is immediately needed by:

Distance (120)
Space Norm (**??**)

Space Distance (119) gives the following terms.

space distance, Euclidean distance.


Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Integral Line
Real Order
Real Numbers
Rational Order
Geometry
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Operations
Natural Products
Integer Numbers
Orders
Function Images
Function Composites
Families
Natural Sums
Equivalence Relations
Converse Relations
Functions
Natural Order
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to talk about the "distance" between objects in a set.

Common notions

Our inspiration is the notion of distance in the plane (see Plane Distance) or in space (see Space Distance). The objects are points and the distance between them is the length of the line segment joining them. We note a few properties of this notion of distance:

1. The distance between any two distinct objects is not zero.
2. The distance between any two objects does not depend on the order in which we consider them.
3. The distance between two objects is no larger than the sum of the distances of each with any third object

The first observation is natural: if two points are not the same, then they are some distance apart. In other words, the line segment between them has length.

The second observation is natural: the line segment connecting two points does not depend on the order specifying the points. This observation justifies the word "between." If it were not the case, then we should use different words, and be careful to speak of the distance "from" a first point "to" a second point.

The third property is a non-obvious property of distance in the plane. It says, in other words, that the length of any side of a triangle is no larger than the sum of the lengths of the two other sides. With experience in geometry, the observation may become natural. But it does not seem to be superficially so.

A more muddled but superficially natural justification for our concern with third observation is that it says something about the transitivity of closeness. Two objects are close if their distance is small. Small is a relative concept, and needs some standard of comparison. Let us fix two points, take the distance between them, and call it a unit. We call two objects close with respect to our unit if their distance is less than a unit.

In this language, the third observation says that if we know two objects are each half of a unit distance from a third object, then the two objects are close (their distance is less than a unit). We might call this third object the reference object. Here, then, is the usefulness of the third property: we can infer closeness of two objects if we know their distance to a reference object.

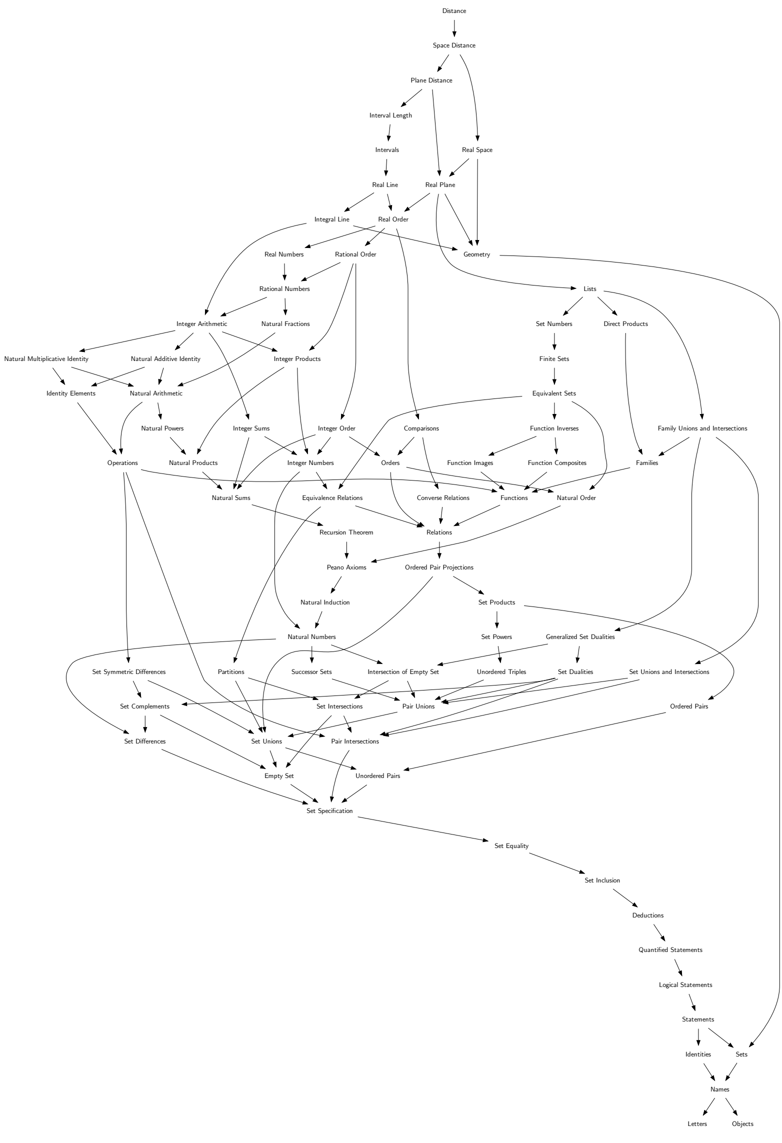
Distance (120) immediately needs:

Space Distance (119)

Distance (120) is immediately needed by:

Distance Asymmetry (121)
Metrics (125)
N-Dimensional Space (122)

Distance (120) gives no terms.


Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Integral Line
Real Order
Real Numbers
Rational Order
Geometry
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Operations
Natural Products
Integer Numbers
Orders
Function Images
Function Composites
Families
Natural Sums
Equivalence Relations
Converse Relations
Functions
Natural Order
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

Sometimes "distance" as used in the English language refers to an asymmetric concept. This apparent paradox further illuminates the symmetry property.

Apparent paradox

Distance in the plane is symmetric: the distance from one point to another does not depend on the order of the points so considered. We took this observation as a definiting property of our abstract notion of distance. The meaning, strength, and limitation of this property is clarified by considering an asymmetric case.

Contrast walking up a hill with walking down it. The "distance" between these two points, the top of the hill and a point on its base, may not be symmetric with respect to the time taken or the effort involved. Experience suggests that it will take longer to walk up the hill than to walk down it. A superficial justification may include reference to the some notion of uphill walking requiring more effort.

If we were going to model the top and base of the hill as points in space, however, the distance between them is the same: it is symmetric. It is even the same if we take into account that some specific path, a trail say, must be followed.

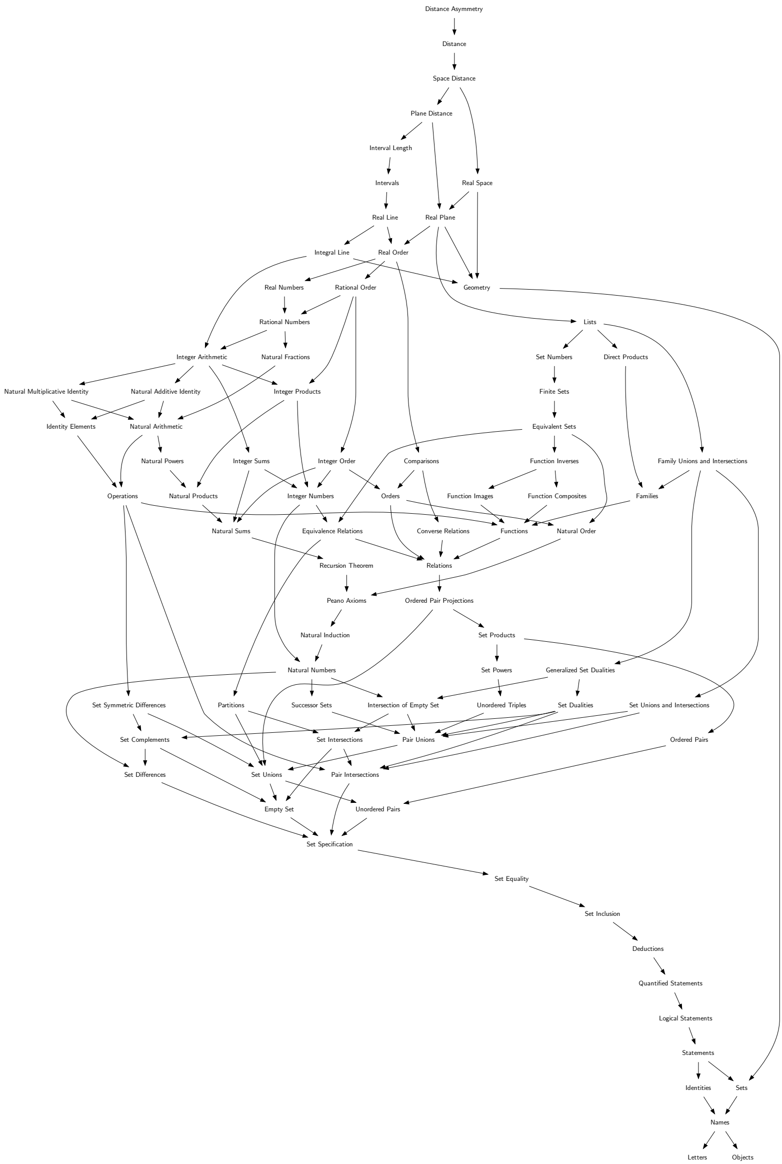
If planning a backpacking trip, such symmetry appears foolish. The distance between two locations must not be considered symmetric. Going up the mountain takes longer than going down. It may justify, in the English phrase, "going around, rather than going over."

Distance Asymmetry (121) immediately needs:

Distance (120)

Distance Asymmetry (121) is not immediately needed by any sheet.

Distance Asymmetry (121) gives no terms.



Distance Asymmetry
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Integral Line
Real Order
Real Numbers
Rational Order
Geometry
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Operations
Natural Products
Integer Numbers
Orders
Function Images
Function Composites
Families
Natural Sums
Equivalence Relations
Converse Relations
Functions
Natural Order
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

If $\mathbf{R}$ corresponds to a line, and $\mathbf{R}^2$ to a plane, and $\mathbf{R}^3$ to space, does $\mathbf{R}^4$ correspond to anything? What of $\mathbf{R}^5$?

Definition

Let n be a natural number. We call the set $\mathbf{R}^n$ *n-dimensional space* (or *Euclidean n-space*, *real coordinate space*, *real Euclidean space*). We call elements of $\mathbf{R}^n$ *points*. We identify $\mathbf{R}^1$ with $\mathbf{R}$ in the obvious way.

We call the point associated with $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ with $x_i = 0$ for $1 \leqq i \leqq n$ the *origin*. We denote the origin by 0. Similarly, we denote the point x with $x_i = 1$ for all $i = 1, \dots, n$ by 1.

Visualization

We can not visualize n-dimensional space. Thus, our intuition for it comes from real space (see Real Space).

Distance

A natural notion of distance for $\mathbf{R}^n$ generalizes that in $\mathbf{R}^2$ and $\mathbf{R}^3$. We define the *distance* (or *Euclidean distance*) between $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbf{R}^n$ as

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

Does this have the properties that distance has in the plane and in space? We discussed these properties It does. Denote the function which associates to $x, y \in \mathbf{R}^n$ their distance $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. So $d(x, y)$ is the distance between the points corresponding to x and y.

Proposition 154. *d is non-negative, symmetric, and the distance between two points is no larger than the sum of the distances with any third object.*[133]

[133] Future editions will include an account.

Order

Let $x, y \in \mathbf{R}^n$. If $x_i < y_i$ for all $i = 1, \ldots, n$ then we say x is *less than* y. Likewise, if $x_i \leq y_i$ for all $i = 1, \ldots, n$ then we say $x \leq y$. Likewise for $>$ and $\geq$.

Notation

If $x \in \mathbf{R}^n$ is less than $y \in \mathbf{R}^n$ then we write $x < y$. Similarly for $x \leq y$, $x > y$ and $x \geq y$. Other notation in the literature for $\mathbf{R}^n$ includes E^n, which is a mnemonic for "euclidean."

N-Dimensional Space (122) immediately needs:

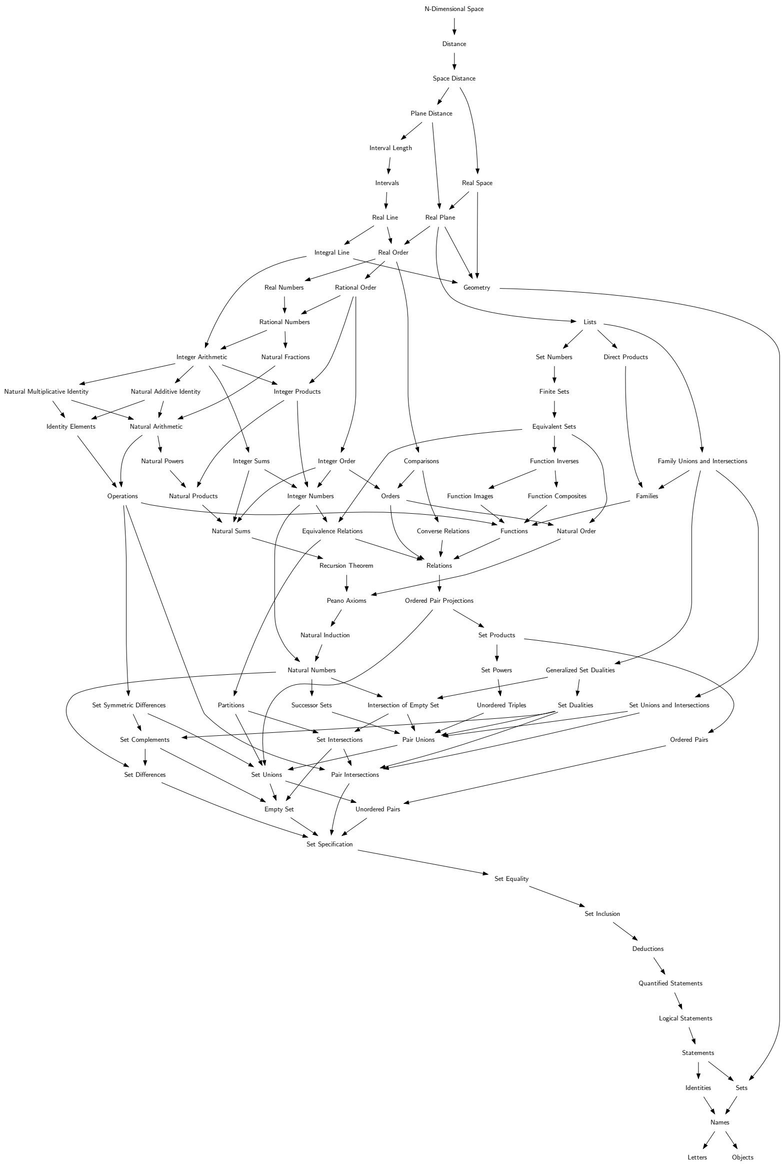
Distance (120)

N-Dimensional Space (122) is immediately needed by:

Borel Sigma Algebra (??)
Convex Multivariate Functions (??)
Data Fitting (??)
Hyperrectangles (??)
Linear Functions (??)
Multivariate Functions (??)
Multivariate Real Densities (??)
N-Dimensional Lines (??)
N-Dimensional Volume Measure (??)
Optimization Problems (??)
Real Affine Transformations (??)
Real Balls (??)
Real Cones (??)
Real Inner Product (??)
Real Linear Transformations (??)
Real Open Sets (??)
Real Translates (??)
Real Vectors (??)

N-Dimensional Space (122) gives the following terms.

n-dimensional space, Euclidean n-space, real coordinate space, real Euclidean space, points, origin, distance, Euclidean distance, less than.


N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Integral Line
Real Order
Real Numbers
Rational Order
Geometry
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Operations
Natural Products
Integer Numbers
Orders
Function Images
Function Composites
Families
Natural Sums
Equivalence Relations
Converse Relations
Functions
Natural Order
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We name those functions—and important set—whose range is contained in the real numbers.

Definition

A *real function* is a real-valued function. The domain is often an interval of real numbers, but may be any non-empty set.

Notation

Given *any* set A, $f : A \to \mathbf{R}$ is a real function. If $A = \mathbf{R}$, then $f \in \mathbf{R} \to \mathbf{R}$.

We often speak of functions defined on intervals. Given $a, b \in \mathbf{R}$, then $g : [a, b] \to \mathbf{R}$ is a real function defined on a closed interval. The function $h : (a, b) \to \mathbf{R}$ is a real function defined on an open interval.

We regularly declare the interval and the function at once. For example, "let $f : [a, b] \to \mathbf{R}$" is understood to mean "let a and b be real numbers with $a < b$, let $[a, b]$ be the closed interval with them as endpoints, and let f be a real-valued function whose domain is this interval". We read the notation $f : [a, b] \to \mathbf{R}$ aloud as "f from closed a b to $\mathbf{R}$." We use $f : (a, b) \to \mathbf{R}$ similarly (read aloud "f from open a b to $\mathbf{R}$").

Examples

Example 1. *Given $c \in \mathbf{R}$, define $f : \mathbf{R} \to \mathbf{R}$ by*

$$f(x) = c \quad \text{for all } x \in \mathbf{R}$$

Example 2. *Define $f : \mathbf{R} \to \mathbf{R}$ by*

$$f(x) = 2x^2 + 1 \quad \text{for all } x \in \mathbf{R}$$

Example 3. *Define $f : \mathbf{R} \to \mathbf{R}$ by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Real Functions (123) immediately needs:

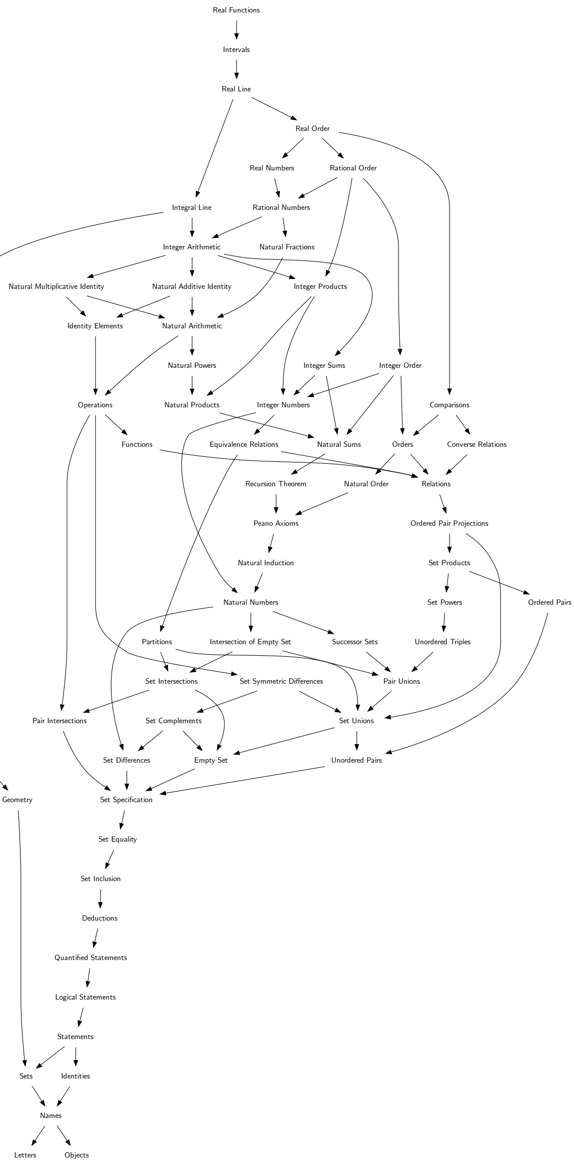
Intervals (113)

Real Functions (123) is immediately needed by:

Analytic Functions (??)
Complex Functions (??)
Dimension Reducers (??)
Exponential Function (??)
Function Growth Classes (??)
Monotone Real Functions (??)
Optimization Problems (??)
Outcome Probabilities (??)
Real Convex Functions (??)
Real Differentiable Functions (??)
Real Function Graphs (??)
Real Function Space (??)
Real Linear Functions (??)
Real Rational Functions (??)
Rectangular Functions (??)
Simple Functions (??)
Submodular Functions (??)
Threshold Graphs (??)
Weighted Graphs (??)

Real Functions (123) gives the following terms.

real function.


Real Functions
Intervals
Real Line
Real Order
Real Numbers
Rational Order
Integral Line
Rational Numbers
Integer Arithmetic
Natural Fractions
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Identity Elements
Natural Arithmetic
Natural Powers
Integer Sums
Integer Order
Operations
Natural Products
Integer Numbers
Comparisons
Functions
Equivalence Relations
Natural Sums
Orders
Converse Relations
Recursion Theorem
Natural Order
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

What does it mean for a function to continuous, or uninterrupted.

Definition

Consider a function from the real numbers to the real numbers.

The function is *continuous at a point* in its domain if for every positive real number, there is a positive real number such that every pointin the domain which is the second positive number close to the first element has result which is the first positive number close to the second.

A function is *continuous* if it is continuous at every point of its domain.

Notation

Let $f : \mathbf{R} \to \mathbf{R}$. Then f is continuous at $x \in \mathbf{R}$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(|x - y| < \delta \longrightarrow |f(x) - f(y)| < \varepsilon)$$

for all $y \in \mathbf{R}$.

Then f is continuous.

$$(\forall x \in R)(\forall \varepsilon > 0)(\exists \delta > 0)(|x - y| < \delta \longrightarrow |f(x) - f(y)| < \varepsilon)$$

for all $y \in \mathbf{R}$.

Real Continuity (124) immediately needs:

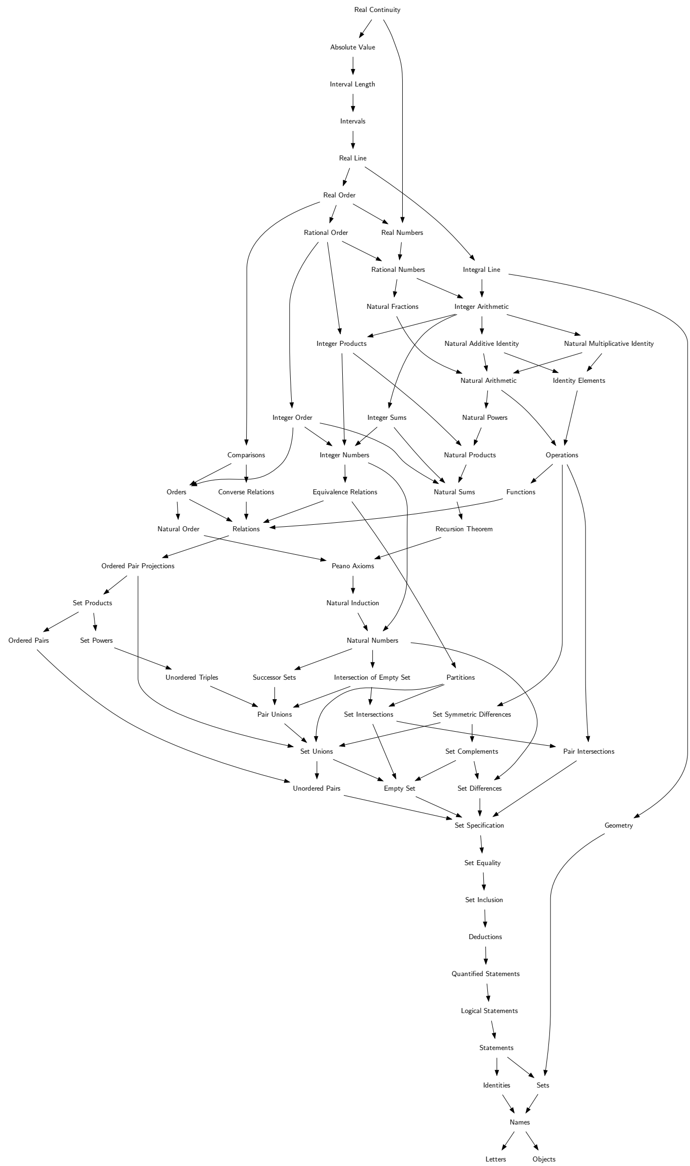
Absolute Value (115)
Real Numbers (101)

Real Continuity (124) is immediately needed by:

Metric Continuity (127)
Real Uniform Continuity (**??**)

Real Continuity (124) gives the following terms.

continuous at a point, continuous.


Real Continuity
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Rational Order
Real Numbers
Rational Numbers
Integral Line
Natural Fractions
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Natural Arithmetic
Identity Elements
Integer Order
Integer Sums
Natural Powers
Comparisons
Integer Numbers
Natural Products
Operations
Orders
Converse Relations
Equivalence Relations
Natural Sums
Functions
Natural Order
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Pair Unions
Set Intersections
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Set Specification
Geometry
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to talk about a set with a prescribed quantitative degree of closeness (or distance) between its elements.

Definition

The correspondences which serve as a degree of closeness, or measure of distance, must satisfy our previously developed (see Distance) notion of distance.

A function on ordered pairs which does not depend on the order of the elements so considered is *symmetric*. A function into the real numbers which takes only nonnegative values is *nonnegative*. A repeated pair is an ordered pair of the same element twice. A function which satisfies a triangle inequality for any three elements is *triangularly transitive*.

A *metric* (or *distance function*) is a function on ordered pairs of elements of a set which is symmetric, non-negative, zero only on repeated pairs, and triangularly transitive. A *metric space* is an ordered pair whose first coordinate is a nonempty set and whose second coordinate is a metric.

In a metric space, we say that one pair of objects is *closer* together if the metric of the first pair is smaller than the metric value of the second pair.

Notice that a set can be made into different metric spaces by using different metrics.

Notation

Let A be a set. We commonly denote a metric by the letter d, as a mnemonic for "distance." Let $d : A \times A \to \mathbf{R}$. Then d is a metric if:

1. it is non-negative, which we tend to denote by

$$d(a, b) \geq 0 \quad \forall a, b \in A.$$

2. it is 0 only on repeated pairs, which we tend to denote by

$$d(a, b) = 0 \longleftrightarrow a = b, \quad \forall a, b \in A.$$

3. it is symmetric, which we tend to denote by:

$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$

4. it is triangularly transitive, which we tend to denote by

$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of A with d by (A, d). Another common choice of letter for a metric is ρ.

Examples

R with the absolute value distance is a metric space. As is $\mathbf{R}^2$ and $\mathbf{R}^3$ with the Euclidean distance. $\mathbf{R}^n$ with Euclidean metric is an example of a metric space for which the objects (n-dimensional tuples of real numbers) are impossible to visualize.

Metrics (125) immediately needs:

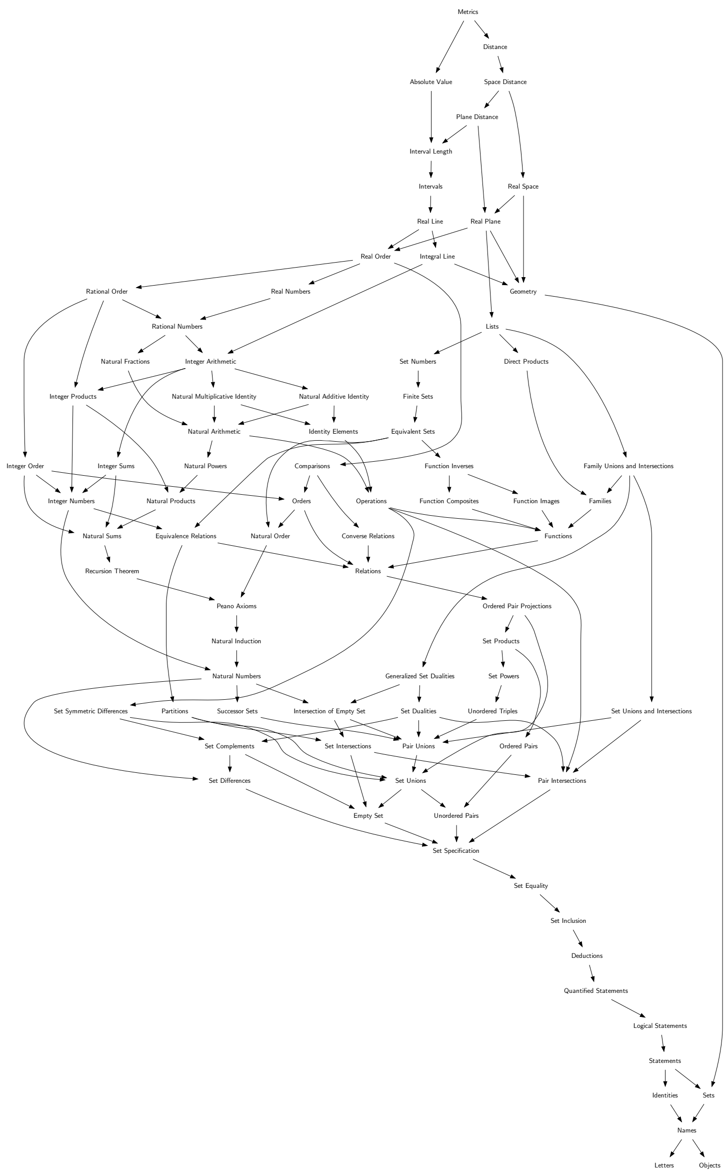
Absolute Value (115)
Distance (120)

Metrics (125) is immediately needed by:

Discrete Metric (??)
Egoprox Sequences (??)
Isometries (??)
Metric Balls (??)
Metric Continuity (127)
Metric Limits (??)
Metric Space Examples (??)
Metric Space Functions (126)
Nearest Neighbor Predictors (??)
Norm Metrics (??)
Product Metrics (??)
Similarity Functions (??)
Topologies (128)

Metrics (125) gives the following terms.

symmetric, nonnegative, triangularly transitive, metric, distance function, metric space, closer.


Metrics
Distance
Absolute Value
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Rational Order
Real Numbers
Geometry
Rational Numbers
Lists
Natural Fractions
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Natural Arithmetic
Identity Elements
Equivalent Sets
Integer Order
Integer Sums
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to talk about functions from one set with a metric into another set with a metric.

Definition

A *function* from a first metric space to a second metric space is a function from the first set to the second set.

Notation

Suppose (A, d) and (B, d') be metric spaces. We denote that f is a function from the first metric space to the second metric space by 0 $f : (A, d) \to (B, d')$.

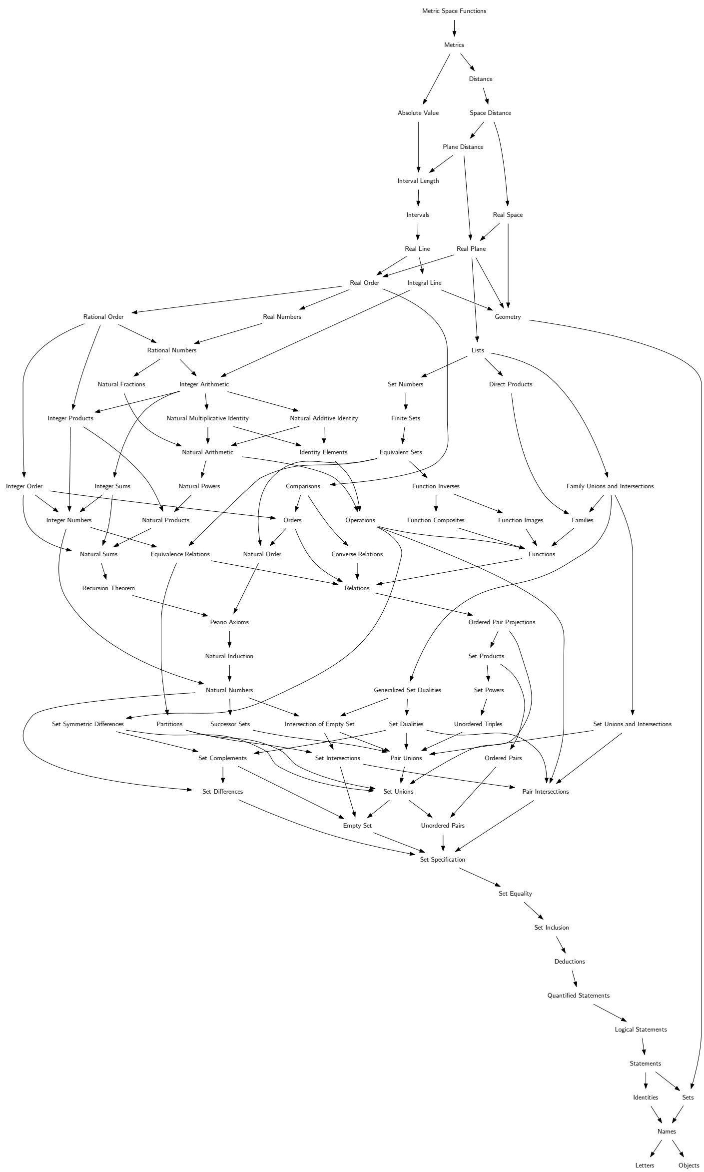
Metric Space Functions (126) immediately needs:

Metrics (125)

Metric Space Functions (126) is not immediately needed by any sheet.

Metric Space Functions (126) gives the following terms.

function.


Metric Space Functions
Metrics
Distance
Absolute Value
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Rational Order
Real Numbers
Geometry
Rational Numbers
Lists
Natural Fractions
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Natural Arithmetic
Identity Elements
Equivalent Sets
Integer Order
Integer Sums
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We define continuity for functions between metric spaces.

Definition

Our inspiration is continuity of functions from the set of real numbers to the set of real numbers. There we decided on a definition which codified our intuition that numbers which are sufficiently close to each other are mapped to numbers that are close to each other.

A function from a first metric space to a second metric space is *continuous at* an object of its domain if, for every positive real number (no matter how small), there is a second positive real number (possibly, though not necessarily, smaller) so that every element in the domain whose distance to the fixed object is less than the second positive number has a result under the function whose distance to the result of the fixed object is less than the first positive number.

A function between metric spaces is continuous if it is *continuous* at every object of its domain.

Notation

Let (A, d) and (B, d') be metric spaces. Let $f : (A, d) \to (B, d')$. Then f is continuous at $\bar{a} \in A$, if for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all $a \in A$,

$$d(\bar{a}, a) < \delta \longrightarrow d'(f(\bar{a}), f(a)) < \varepsilon.$$

Metric Continuity (127) immediately needs:

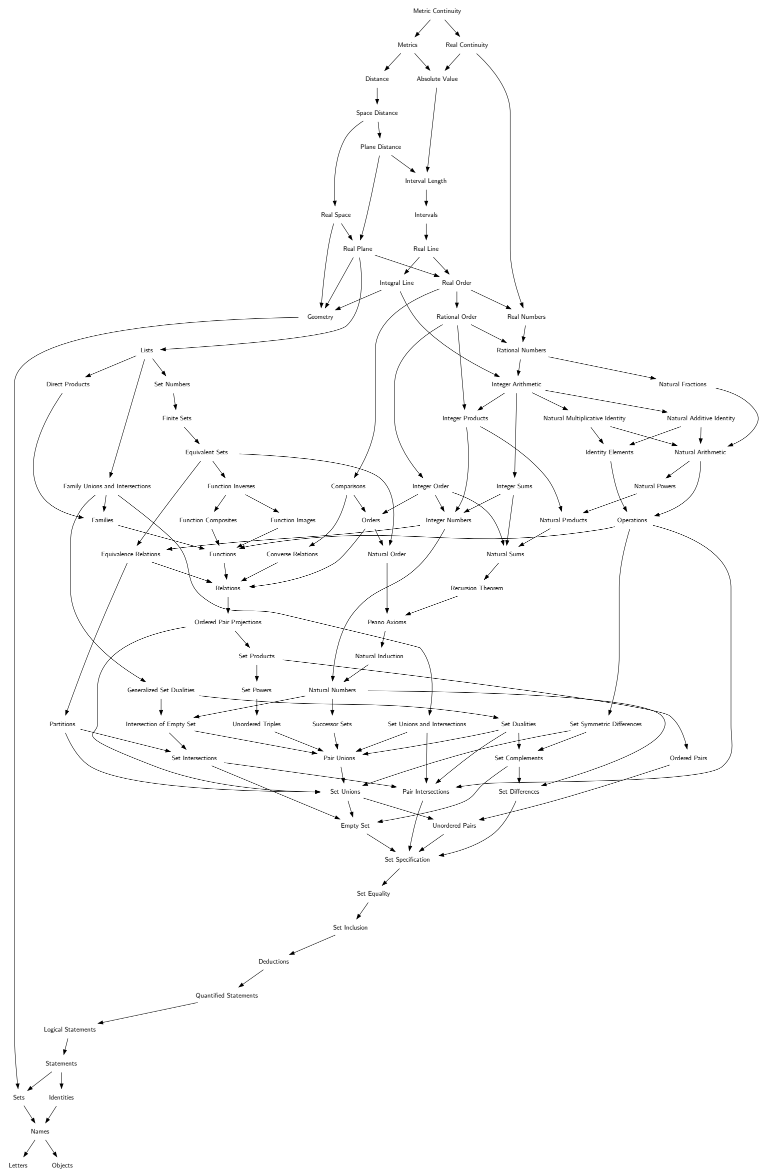
Metrics (125)
Real Continuity (124)

Metric Continuity (127) is immediately needed by:

Continuous Linear Transformations (**??**)

Metric Continuity (127) gives the following terms.

continuous at, continuous.


Metric Continuity
Metrics
Real Continuity
Distance
Absolute Value
Space Distance
Plane Distance
Interval Length
Real Space
Intervals
Real Plane
Real Line
Integral Line
Real Order
Geometry
Rational Order
Real Numbers
Lists
Rational Numbers
Direct Products
Set Numbers
Integer Arithmetic
Natural Fractions
Finite Sets
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Equivalent Sets
Identity Elements
Natural Arithmetic
Family Unions and Intersections
Function Inverses
Comparisons
Integer Order
Integer Sums
Natural Powers
Families
Function Composites
Function Images
Orders
Integer Numbers
Natural Products
Operations
Equivalence Relations
Functions
Converse Relations
Natural Order
Natural Sums
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Unions and Intersections
Set Dualities
Set Symmetric Differences
Set Intersections
Pair Unions
Set Complements
Ordered Pairs
Set Unions
Pair Intersections
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects

Why

We want to generalize the notion of continuity.

Definition

Given a set X, a *topology* on X is a set of subsets of X for which (1) the empty set base set are distinguished (2) the intersection of a *finite* family of distinguished subsets is distinguished, and (3) the union of a family of distinguished subsets is distinguished. We call the elements of the topology the *open sets*.

A *topological space* is an ordered pair: a base set and a set distinguished subsets of the base set which are a topology.

Notation

Let X be a non-empty set. For the set of distinguished sets, we tend to use $\mathcal{T}$, a mnemonic for topology, read aloud as "script T". We tend to denote elements of $\mathcal{T}$ by O, a mnemonic for open. We denote the topological space with base set X and topology $\mathcal{T}$ by $(X, \mathcal{T})$. We denote the properties satisfied by elements of $\mathcal{T}$:

1. $X, \varnothing \in \mathcal{T}$
2. if $O_1, \dots, O_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} O_i \in \mathcal{T}$
3. if $O_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} \in \mathcal{T}$

Examples

R with the open intervals as the open sets is a topological space.

Topologies (128) immediately needs:

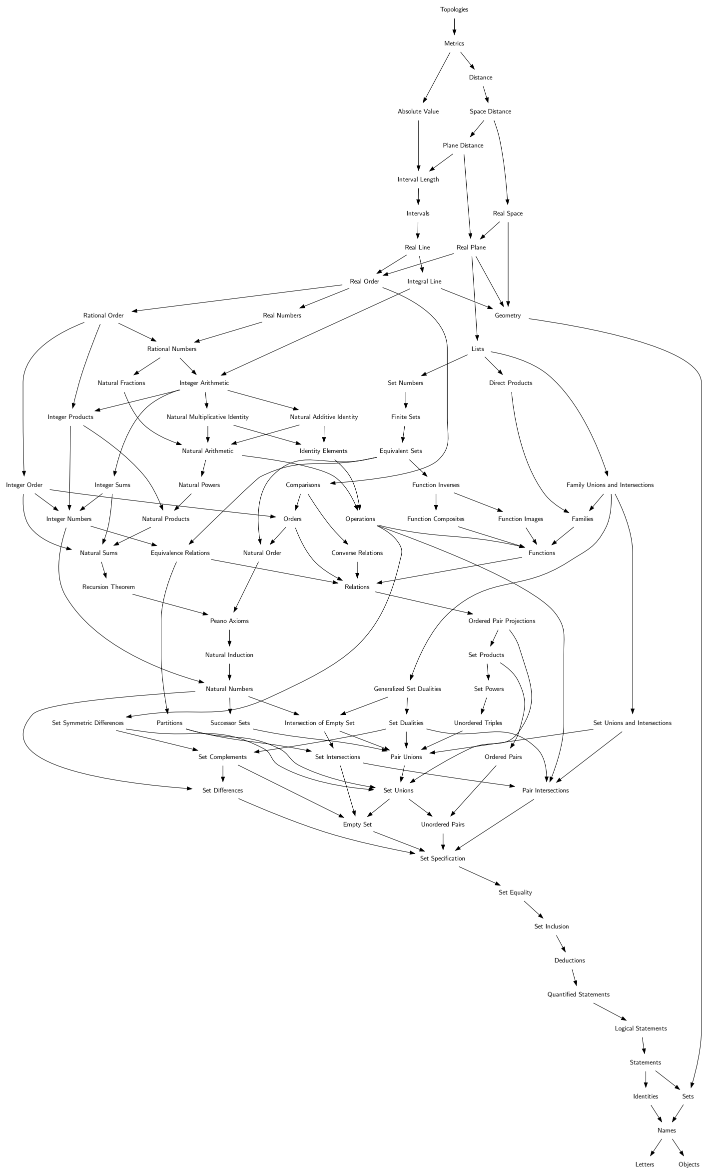
Metrics (125)

Topologies (128) is immediately needed by:

Generated Topologies (**??**)
Topological Groups (**??**)
Topological Neighborhoods (**??**)
Topological Sigma Algebra (**??**)
Topology Bases (**??**)

Topologies (128) gives the following terms.

topology, open sets, topological space.


Topologies
Metrics
Distance
Absolute Value
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Rational Order
Real Numbers
Geometry
Rational Numbers
Lists
Natural Fractions
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Natural Arithmetic
Identity Elements
Equivalent Sets
Integer Order
Integer Sums
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Note on Printing

The font is *Computer Modern.* The document was typeset using LaTeX. This pamphlet was printed, folded, and stitched in Menlo Park, California.