Why

How big can a quadratic form be? How small?

Result

Proposition 1. *Suppose $A \in \mathbf{S}^n$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_n x^\top x \leq x^\top A x \leq \lambda_1 x^\top x,$$

for all $x \in \mathbf{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ with $A = Q \Lambda Q^\top$. Express

$$\begin{aligned} x^\top A x = x^\top Q \Lambda Q^\top x &= (Q^\top x)^\top \Lambda (Q^\top x) \\ &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &= \lambda_1 \sum_{i=1}^n (q_i^\top x) = \lambda_1 \|Q^\top x\|^2 = \lambda_1 \|x\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} x^\top A x &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &\geq \lambda_n \sum_{i=1}^n (q_i^\top x) = \lambda_n \|Q^\top x\|^2 = \lambda_n \|x\|^2. \end{aligned}$$

□

Notation

For this reason, it is common to order the eigenvalues of $A \in \mathbf{S}^n$ by magnitude with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. λ_1 is sometimes denoted $\lambda_{\max}$ and λ_n is sometimes denoted $\lambda_{\min}$.

Optimization implication

If $z = \alpha x$, then $z^\top A z = \alpha^2 x^\top A x$. Consider finding $x \in \mathbf{R}^n$ to maximize

$$\begin{array}{ll} \text{maximize} & x^\top A x \\ \text{subject to} & \|x\| = 1. \end{array}$$

Since the objective is $x^\top A x \leq \lambda_1$ for all $x \in \mathbf{R}^n$ with $\|x\| = 1$, a solution of this problem is the eigenvector $q_1 \in \mathbf{R}^n$ corresponding to λ_1. In other words, these inequalities are tight.

Quadratic Form Inequalities
Quadratic Forms
Matrix Trace
Identity Matrices
Real Matrix-Matrix Products
Symmetric Real Matrix Eigenvalues
Real Matrix-Vector Products
Matrix Transpose
Symmetric Real Matrices
Eigenvalues and Eigenvectors
Vectors as Matrices
Matrices
Norms
Matrix Determinants
Real Inner Products
Real Linear Combinations
Real Matrices
Functionals
Multivariate Polynomials
Real Norm
Vectors
Linear Equations
Multivariate Functions
Space Norm
Real Vectors
Linear Functions
Plane Inner Product
N-Dimensional Space
Plane Norm
Plane Vector Sums and Differences
Space Vectors
Right Triangle Sides Relation
Distance
Plane Vectors
Absolute Value
Real Polynomials
Real Squares
Space Distance
Real Arithmetic
Plane Distance
Arrays
Real Multiplicative Inverses
Interval Length
Music Polynomials
Real Products
Real Space
Intervals
Permutations
Square Numbers
Real Additive Inverses
Real Plane
Real Line
Fields
Polynomials
Groups
Rational Multiplicative Inverses
Rational Arithmetic
Squares
Integral Line
Real Sums
Real Summations
Line
Rings
Rational Products
Rational Sums
Integers
Real Numbers
Rational Order
Set Numbers
Rational Numbers
Finite Sets
Integer Arithmetic
Direct Products
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Identity Elements
Arithmetic
Inverse Elements
Natural Powers
Natural Summation
Equivalent Sets
Element Functions
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Natural Products
Integer Sums
Integer Order
Comparisons
Family Operations
Natural Equations
Family Unions and Intersections
Operations
Function Compositions
Function Images
Integer Numbers
Natural Sums
Orders
Equations
Families
Functions
Recursion Theorems
Natural Order
Converse Relations
Equivalence Relations
Peano Axioms
Relations
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Ordered Pairs
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Notes
Letters
Objects