Why

We extend our notion of length, area, volume beyond the Lebesgue measure on the product spaces of real numbers.

Definition

An extended-real-valued non-negative function on an algebra is *finitely additive* if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a sigma algebra is *countably additive* if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A *finitely additive measure* is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A *countably additive measure* is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call countably additive measures *measures*, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countable additive.

In the context of measure, we call a countably unitable subset algebra a *measurable space*. We call the distinguished sets *measurable* sets. A *measure space* is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of the measurable space.

Notation

Let A a set. Let $\mathcal{A}$ a sigma algebra on A. The pair $(A, \mathcal{A})$ is a measurable space.

Let $\mu : \mathcal{A} \to [0, \infty]$ a measure; thus: (a) $\mu(\varnothing) = 0$ and (b) for disjoint $\{A_n\} \subset \mathcal{A}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ The triple $(A, \mathcal{A}, \mu)$ is a measure space.

We use μ since it is a mnemonic for "measure". We often also us ν to denote measures, since it is after μ in the Greek alphabet, and λ, since it is before μ in the Greek alphabet.

Examples

Example 1. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure. We call μ the counting measure.*

Example 2. *Let $(A, \mathcal{A})$ measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise. Then μ is a measure. We call μ the point mass concentrated at a.*

Example 3. *The Lebesgue measure on the measurable space $(\mathsf{R}, \mathcal{B}(\mathsf{R}))$ is a measure.*

Example 4. *Let N be the natural numbers. Let $\mathcal{A}$ the finite co-finite algebra on N. Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.*

Example 5. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ be 0 if $A = \varnothing$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.*

Example 6. *Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = \mathcal{P}(A)$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \varnothing$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.*

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \varnothing$ then using finite additivity we obtain a contradiction $1 = \mu(B \cup C) = \mu(B) + \mu(C) = 2$. □

Why

We want to generalize the notions of length, area, and volume.

Definition

A *measurable space* is a sigma algebra. We call the distinguished subsets the *measurable sets*.

A *measure* on a measurable space is a function from the sigma algebra to the positive extended reals. A *measure space* is a measurable space and a measure.

Notation

Properties

Prop. 1. *Let $(A, \mathcal{A})$ be a measurable space and $m : \mathcal{A} \to [0, \infty]$ be a measure.*

If $B \subset C \subset A$, then $m(B) \leq m(C)$. We call this property the of measures monotonicity of measure.

Prop. 2. *For a measure space $(A, \mathcal{A}, m)$.*

If $B \subset C \subset A$, then $m(B) \leq m(C)$.

We call this property the monotonicity of measure.

Prop. 3. *For a measure space $(A, \mathcal{A}, m)$.*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

We this property the sub-additivty of measure.

Prop. 4. *For a measure space $(A, \mathcal{A}, m)$.*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

We this property the sub-additivty of measure.

Prop. 5. *For a measure space $(A, \mathcal{A}, m)$.*

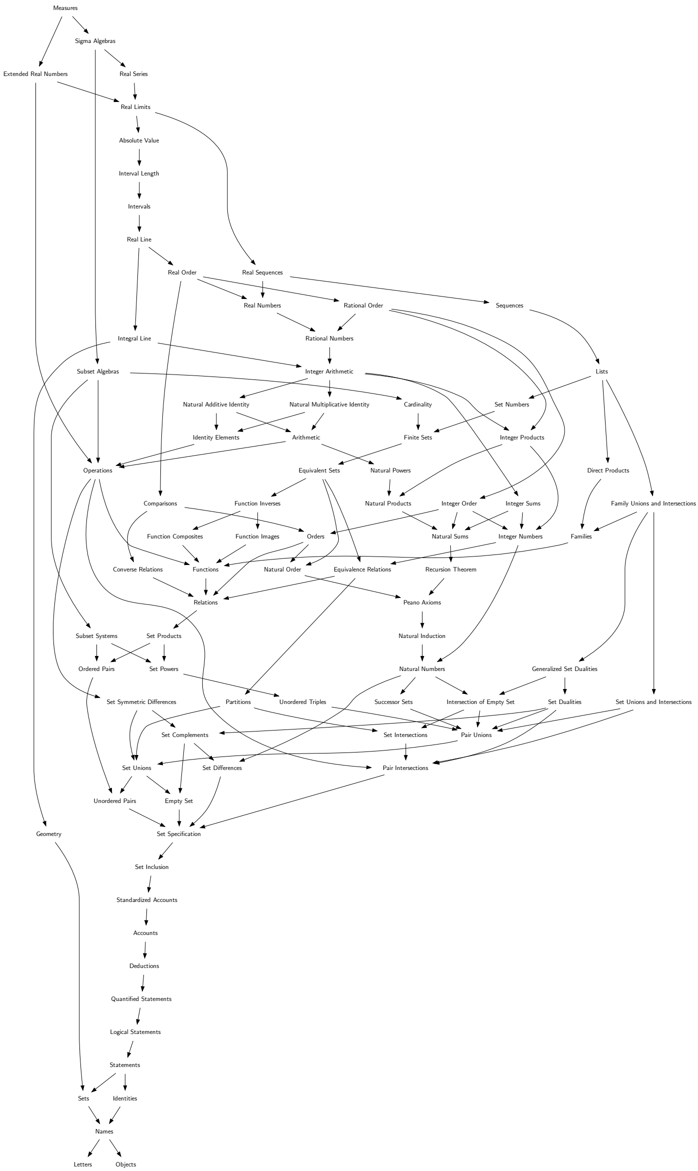
$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

Prop. 6. *For a measure space $(A, \mathcal{A}, m)$.*

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

Examples

Example 7. *counting measure*


Measures
Sigma Algebras
Extended Real Numbers
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Real Numbers
Rational Order
Sequences
Integral Line
Rational Numbers
Subset Algebras
Integer Arithmetic
Lists
Natural Additive Identity
Natural Multiplicative Identity
Cardinality
Set Numbers
Identity Elements
Arithmetic
Finite Sets
Integer Products
Operations
Equivalent Sets
Natural Powers
Direct Products
Comparisons
Function Inverses
Natural Products
Integer Order
Integer Sums
Family Unions and Intersections
Function Composites
Function Images
Orders
Natural Sums
Integer Numbers
Families
Converse Relations
Functions
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Unordered Triples
Successor Sets
Intersection of Empty Set
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects