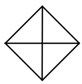


Result

Proposition 1. *Suppose P is a* probability measure*on a finite set of* out-comes*Ω. For any two* events*A, B with $P(A), P(B) > 0$, we have*

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

Proof. By definition, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

And also symmetrically,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

From this second equation we have $P(A \cap B) = P(B \mid A)P(A)$, which we can substitute into the numerator of the first expression to obtain the result. □

This result is known by many names including *Bayes' rule*, *Bayes rule* (no possessive), *Bayes' formula*, and *Bayes' theorem.*

It is a *basic* consequence of the *definition* of conditional probability, but it is *useful* in the case that we are given problem data in terms of the probabilities on the right hand side of the above equation.

Examples

Diagnostic test

Suppose we want to model the situation in which a rare disease afflicts 0.5% of a population and we have a diagnostic test that is 99% accurate.

We consider the population Ω of people. We agree to partition the population into D and H so that

$$D \cup H = \Omega \quad \text{and} \quad D \cap H = \varnothing$$

D is the set of people with the disease, and H is the set of *healthy* people without the disease. Similarly, we agree to partition the population into R and N so that

$$R \cup N = \Omega \quad \text{and} \quad P \cap N = \varnothing$$

R is the set of people who test positive, and N is the set of people who test negative.

We agree that 0.5% of the the population being afflicted means, $P(D) = 0.005$. We agree that having a 99% accurate test means *means*

$$P(D \mid R) = 0.99 \quad \text{and } P(H \mid N) = 0.99$$

Now, what is the conditional probability of having the disease given that the test is positive? Using Bayes rule,

$$P(D \mid R) = \frac{P(R \mid D)P(D)}{P(R \mid D)P(D) + P(R \mid H)P(H)}$$

Using our supposed values,

$$P(D \mid R) = \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995} \approx 0.33$$

This may be viewed as suprising, since the test is perceived to be accurate.

The frequentist interpretation is clear: if have many outcomes, say a thousand individuals, we may expect that about five of these thousand have the disease. The test is likely to diagnose these correctly. However, of the other 995 people, about 1% of them—say 10 people—will be misdiagnosed. Thus we may expect to see about 15 positive test results, but only five of which correspond to individuals with the disease.

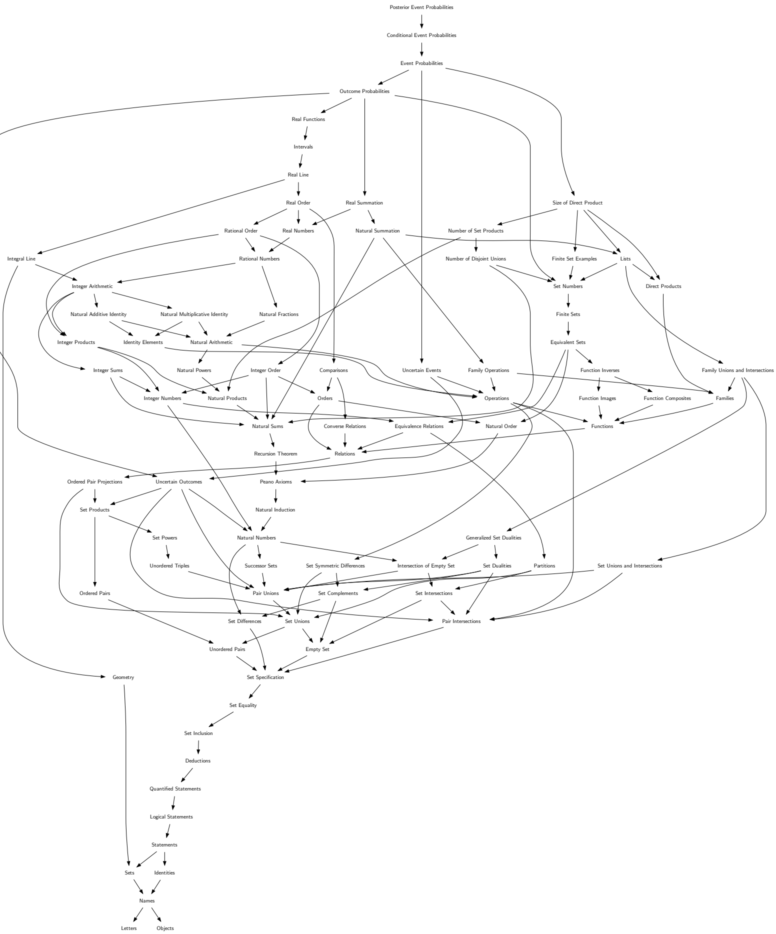
Compound form

More is true.

Proposition 2. *Suppose P is a* finite probability measure*on a set of* outcomes*Ω. For any three* events*A, B, C with $P(A), P(B), P(C) > 0$, we have*

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C)P(A \mid C)}{P(B \mid C)}.$$

Proof. Future editions will include, the strategy is the same as above. $\square$


Posterior Event Probabilities
Conditional Event Probabilities
Event Probabilities
Outcome Probabilities
Real Functions
Intervals
Real Line
Real Order
Real Summation
Size of Direct Product
Rational Order
Real Numbers
Natural Summation
Number of Set Products
Integral Line
Rational Numbers
Number of Disjoint Unions
Finite Set Examples
Lists
Integer Arithmetic
Set Numbers
Direct Products
Natural Additive Identity
Natural Multiplicative Identity
Natural Fractions
Finite Sets
Integer Products
Identity Elements
Natural Arithmetic
Equivalent Sets
Integer Sums
Natural Powers
Integer Order
Comparisons
Uncertain Events
Family Operations
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Images
Function Composites
Families
Natural Sums
Converse Relations
Equivalence Relations
Natural Order
Functions
Recursion Theorem
Relations
Ordered Pair Projections
Uncertain Outcomes
Peano Axioms
Set Products
Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Unordered Triples
Successor Sets
Set Symmetric Differences
Intersection of Empty Set
Set Dualities
Partitions
Set Unions and Intersections
Ordered Pairs
Pair Unions
Set Complements
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects