



## Why

Can permuting the rows or columns of a matrix be represented by matrix multiplication?

## Definition

Let  $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  be a permutation of  $n$ . The *permutation matrix* of  $\sigma$  is the matrix  $P$  defined by  $P_{ij} = 1$  if  $\sigma(i) = j$  and 0 otherwise. This is sometimes called the *column representation* (in contrast to the row representation, in which  $P_{ij} = 1$  if  $\sigma(j) = i$ ).

Let  $A \in \mathbf{R}^{n \times n}$ . Then pre-multiplying  $A$  by  $P$  permutes the rows of  $A$ . In other words  $PA$  has the same rows as  $A$  but permuted according to  $\sigma$ . Similarly, post-multiplying by  $P$  permutes the columns of  $A$ . In other words,  $AP$  has the same columns as  $A$  but permuted according to  $\sigma$ . Clearly, we can also speak of permuting the components of a vector.

## Composition

Let  $\pi, \sigma \in S_n$  with corresponding permutation matrices  $P_\sigma$  and  $P_\pi$ . Then  $P_\pi P_\sigma A$  has the same rows as  $A$  but permuted by  $\pi\sigma$ . Likewise,  $AP_\pi P_\sigma$  has the same columns as  $A$  but permuted by  $\pi\sigma$ . Clearly, the identity permutation on  $\{1, 2, \dots, n\}$  is the identity  $I \in \mathbf{R}^{n \times n}$ .

## Inverses

It is clear from the definition that  $P_{\sigma}^{-1} = P_{\sigma^{-1}}$  and so if  $P$  is a permutation matrix then  $P^{-1}$  is  $P^{\top}$ .



