Why

We expect measure to have the common sense properties we stated when trying to define a notion of length for the real line.

Monotonicity

An extended-real-valued function on an alebra is *monotone* if, given a first distinguished set contained in a distinguished second set, the result of the first is no greater than the result of the second.

Proposition 1. *All measures are monotone.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $A, B \in \mathcal{A}$ and $A \subset B$. Then $B = A \cup (B - A)$, a disjoint union. So

$$\mu(B) = \mu(A \cup (B - A)) = \mu(A) + \mu(B - A),$$

by the additivity of μ. Since $\mu(B-A) \geq 0$, we conclude $\mu(A) \leq \mu(B)$. $\square$

Proposition 2. *If $A \subset B$ and B finite, then $\mu(B - A) = \mu(B) - \mu(A)$.*

Subadditivity

Monotonicity along with additivity of measures give us one other convenient property: subadditivity.

An extended-real-valued function on an algebra is *subadditive* if, given a sequence of distinguished sets, the result of union of the sequence is no greater than the limit of the partial sums of the results on each element of the sequence.

Proposition 3. *All measures are subadditive.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\} \subset \mathcal{A}$. Define $\{B_n\} \subset \mathcal{A}$ with $B_n := A_n - \cup_{i=1}^{n-1} A_i$. Then $\cup_n A_n = \cup_n B_n$, $\{B_n\}$ is a disjoint

sequence, and $B_n \subset A_n$ for each n. So

$$\mu(\cup_n A_n) = \mu(\cup_n B_n) = \sum_{i=1}^{\infty} \mu(B_n) \leq \sum_{i=1}^{\infty} \mu(A_n),$$

by additivity and then montonicity of measure. $\qquad\square$

The inequality involved in subadditivity is sometimes called *Boole's inequality* or *Bonferroni's inequalities* or the *union bound*; each of these terms is most common when discussing Probability Measures.

Limits

Measures also behave well under limits.

An extended-real-valued function on an algebra *resolves under increasing limits* if the result of the union of an increasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets. An extended-real-valued function on an algebra *resolves under decreasing limits* if the result of the intersection of a decreasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets.

Proposition 4. *All measures resolve under increasing limits.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\}$ be an increasing sequence in $\mathcal{A}$. Then we want to show: $\mu(\cup_n A_n) = \lim_{n\to\infty} \mu(A_n)$. Define $\{B_n\}$ such that $B_n := A_n - \cup_{i=1}^{n-1} A_i$. Then $\{B_n\}$ is disjoint, $A_n = \cup_{i=1}^{n} B_i$ for each n, $\cup_n A_n = \cup_n B_n$, and $\mu(\cup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \mu(B_i)$, by additivity. So

$$\mu(\cup_n A_n) = \mu(\cup_n B_n)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i)$$
$$= \lim_{n\to\infty} \mu(\cup_{i=1}^{n} B_i)$$
$$= \lim_{n\to\infty} \mu(A_n).$$

$\qquad\square$

Proposition 5. *Measures resolve under decreasing limits if there is a finite set in the decreasing sequence.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\}$ be a decreasing sequence in $\mathcal{A}$ with one element finite. Then we want to show: $\mu(\cap_n A_n) = \lim_{n\to\infty} \mu(A_n)$. On one hand, let n_0 be the index of the first finite element of the sequence. Then for all $n \geq n_0$, the sequence is finite because of the monotonicity of measure. Denote this decreasing finite subsequence of sets by $\{B_n\}$. Then $\cap_n A_n = \cap_n B_n$ and $\lim_n A_n = \lim_n B_n$.

On the other hand, the sequence $\{B_1 - B_n\}$ is an increasing sequence in $\mathcal{A}$. Also $\cap_n B_n = B_1 - \cup_n(B_1 - B_n)$. So

$$
\begin{aligned}
\mu(\cap_n B_n) &= \mu(B_1 - \cup_n(B_1 - B_n)) \\
&= \mu(B_1) - \mu(\cup_n(B_1 - B_n)) \\
&= \mu(B_1) - \lim_n \mu(B_1 - B_n) \\
&= \mu(B_1) - \left(\lim_n \mu(B_1) - \mu(B_n)\right) \\
&= \lim_n B_n.
\end{aligned}
$$

$\square$

Measure Properties

Measures

Sigma Algebras

Extended Real Numbers

Real Series

Real Limits

Absolute Value

Interval Length

Intervals

Real Line

Real Order

Real Sequences

Rational Order

Real Numbers

Subset Algebras

Sequences

Integral Line

Rational Numbers

Cardinality

Lists

Integer Arithmetic

Set Numbers

Direct Products

Integer Products

Natural Multiplicative Identity

Natural Additive Identity

Finite Sets

Arithmetic

Identity Elements

Equivalent Sets

Family Unions and Intersections

Natural Powers

Integer Sums

Integer Order

Comparisons

Operations

Function Inverse

Natural Products

Integer Numbers

Orders

Function Composites

Function Images

Families

Natural Sums

Natural Order

Converse Relations

Equivalence Relations

Functions

Recursion Theorem

Relations

Peano Axioms

Ordered Pair Projections

Natural Induction

Set Products

Subset Systems

Generalized Set Dualities

Natural Numbers

Ordered Pairs

Set Powers

Set Symmetric Differences

Successor Sets

Unordered Triples

Set Dualities

Intersection of Empty Set

Partitions

Set Unions and Intersections

Set Complements

Pair Unions

Set Intersections

Set Differences

Set Unions

Pair Intersections

Empty Set

Unordered Pairs

Geometry

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects