Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n \times 1}$ in the obvious way. For this reason, we call $x \in \mathbf{R}^{n \times 1}$ (meaning $x \in \mathbf{R}^n$) a *column vector*.

For the reasons that we identify $\mathbf{R}^n$ with $\mathbf{R}^{n \times 1}$, we write the vector $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.
$$

We could as easily also identify $\mathbf{R}^n$ with $\mathbf{R}^{1 \times n}$. We avoid this convention. However, by analogy with the language "column vector," we refer to the *matrix* $y \in \mathbf{R}^{1 \times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n \times 1}$ and $\mathbf{R}^{1 \times n}$. If $a \in \mathbf{R}^{n \times 1}$, we denote $b \in \mathbf{R}^{1 \times n}$ defined by $b_i = a_i$ by $a^\top$.

More generally, given a matrix $A \in \mathbf{R}^{m \times n}$, we denote the matrix $B \in \mathbf{R}^{m \times n}$ defined by $B_{ij} = A_{ji}$ by $A^\top$. Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A, and similarly call $a^\top$ the *transpose* of the vector a. Clearly, $(A^\top)^\top = A$, which includes $(a^\top)^\top = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors $\mathbf{R}^1$ (and so with the 1 by 1 matrices $\mathbf{R}^{1 \times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1 \times 1}$ where $A_{11} = a$, then $Ax = ax$.

Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^\top y.$$

We express the symmetry of the inner product by $x^\top y = y^\top x$.

Vectors as Matrices

Real Inner Product · Real Matrices

Space Inner Product · Real Norm · Real Linear Equations

Space Norm · Real Vectors · Linear Functions · Arrays

Plane Inner Product · Plane Norm · N-Dimensional Space · Plane Vector Sums and Differences · Space Vectors

Right Triangle Sides Relation · Absolute Value · Distance · Plane Vectors

Real Squares · Space Distance

Real Arithmetic · Plane Distance

Real Multiplicative Inverse · Interval Length

Real Products · Real Space · Intervals

Square Numbers · Real Additive Inverse · Real Line · Real Plane

Square · Integral Line · Rational Multiplicative Inverse · Rational Arithmetic · Real Sums · Real Order · Real Summation

Geometry · Rational Products · Rational Sum · Rational Order · Real Numbers · Natural Summation

Rational Numbers · Lists

Natural Fractions · Set Numbers

Integer Arithmetic · Direct Products

Integer Products · Natural Multiplicative Identity · Natural Additive Identity · Finite Sets

Natural Powers · Identity Elements · Natural Arithmetic · Inverse Elements · Equivalent Sets · Family Unions and Intersections

Natural Equations · Integer Sums · Integer Order · Element Functions · Family Operations · Comparisons · Function Inverse

Natural Products · Equations · Integer Numbers · Operations · Orders · Function Composites · Function Image · Families

Natural Sums · Converse Relations · Functions · Equivalence Relations

Natural Order · Relations

Recursion Theorem

Peano Axioms · Ordered Pair Projections

Natural Induction · Set Products

Natural Numbers · Set Power · Generalized Set Dualities

Set Symmetric Differences · Successor Sets · Unordered Triples · Set Dualities · Intersection of Empty Set · Set Partitions · Set Unions and Intersections

Set Complements · Pair Unions · Set Intersections · Ordered Pairs

Set Differences · Pair Intersections

Empty Set · Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deduction

Quantified Statements

Logical Statements

Statements

Sets · Identities

Names

Letters · Objects