Why

What is the best linear predictor if we choose according to a particular norm.

Definition

Suppose we have a paired dataset of n records with inputs in $\mathbf{R}^d$ and outputs in $\mathbf{R}$. A *norm weighted least squares linear predictor* for a norm $g : \mathbf{R}^n \to \mathbf{R}$ is a linear transformation $f : \mathbf{R}^d \to \mathbf{R}$ (the field is $\mathbf{R}$) which minimizes

$$g(y - Ax).$$

Weight matrix

Let $\|\cdot\|_W$ be the weighted norm for some positive semidefinite weight matrix W. We want to find x to minimize

$$\|y - AX\|_W.$$

This problem is referred to by many authors as *weighted least squares* or the *weighted least squares problem*.

Diagonal weight matrix

A special case of norm weighted least squares with a weighted norm is the usual weighted least squares problem (see Weighted Least Squares Linear Predictors). Consider weighted least squares with weights $w \in \mathbf{R}^n$, $w \geq 0$. Define $W \in \mathbf{R}^{n \times n}$ so that $W_{ii} = w_i$ and $W_{ij} = 0$ when $i \neq j$. So, in particular, W is a diagonal matrix and

$$\|y - Ax\|_W = \sum_{i=1}^{n} w_i (y_i - x^\top a_i)^2.$$

Solution

Proposition 1. *There exists a unique weighted least squares linear predictor and its parameters are given by*

$$(A^\top W A)^{-1} A^\top W y.$$

Norm Weighted Least Square Linear Regression

Weighted Least Square Linear Regression

Least Square Linear Regression

Linear Prediction

Matrix Transpose — Loss Function — Input Designs — Real Function Approximation

Real Matrix Inverse

Predictor Performance Metrics — Consistent Induction — Vector Space Bases — Regression

Real Matrix-Matrix Products

Prediction — Span

Real Matrix-Vector Products — Subspace — Linearly Dependent Vectors — Norm

Real Inner Products — Vectors as Matrices — Linear Transformation — Real Subspace — Functionals

Real Inner Product — Real Matrices — Transformation — Real Vectors Span — Linear Combination — Real Function Space

Space Inner Product — Real Norm — Linear Equation — Real Linear Combination — Vectors — Approximation

Space Norm — Real Vectors — Arrays — Linear Functions — Similarly Functions

Plane Inner Product — Plane Norm — Space Vectors — Plane Vector Sums and Differences — N-Dimensional Space — Metrics

Right Triangle Solve Relation — Plane Vectors — Distance — Absolute Value

Real Squares — Space Distance — Real Functions

Real Arithmetic — Plane Distance — Internal Length

Real Multiplication Inverse — Intervals

Real Products — Real Space

Square Number — Real Addition Inverse — Real Plane — Real Line — Lists — Real Optimizers

Rational Multiplicative Inverse — Square — Integral Line — Real Sums — Real Order — Real Summation

Rational Arithmetic — Geometry — Real Numbers — Fields — Rational Order

Inverse Elements — Rational Products — Rational Sums — Rational Numbers — Rings — Optimizers

Element Functions — Set Numbers — Direct Products

Integer Arithmetic — Finite Sets — Natural Summation

Natural Multiplicative Identity — Natural Additive Identity — Integer Products

Comparison — Identity Elements — Arithmetic — Function Inverse — Integer Sums — Family Operations — Natural Equations — Family Unions and Intersections

Operation — Natural Powers — Function Image — Function Composition — Integer Order — Orders — Families — Equations

Converse Relations — Natural Products — Functions — Integer Numbers — Natural Order

Equivalence Relations — Natural Sums

Relations — Recursion Theorem

Peano Axioms

Set Products — Natural Induction

Set Powers — Natural Numbers — Generalized Set Dualities

Set Symmetric Differences — Partitions — Unordered Triples — Successor Sets — Intersection of Empty Set — Set Dualities — Set Unions and Intersections

Ordered Pairs — Set Complements — Set Intersections — Pair Unions

Set Union — Set Differences — Pair Intersections

Unordered Pairs — Empty Set

Set Specification

Set Equality

Set Inclusion

Deduction

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects

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