

## Why

We discuss a decomposition using eigenvalues and eigenvectors. <sup>1</sup>

## Defining result

An eigenvalue decomposition of a matrix  $A \in \mathbb{R}^{n \times n}$  is an ordered pair  $(X, \Lambda)$  in which X is invertible,  $\Lambda$  is diagonal, and  $A = X\Lambda X^{-1}$ .

In this case,  $AX = X\Lambda$ , in other words,

$$\left[\begin{array}{ccc} A & \end{array}\right] \left[\begin{array}{cccc} x_1 & \cdots & x_m\end{array}\right] = \left[\begin{array}{cccc} x_1 & \cdots & x_m\end{array}\right] \left[\begin{array}{cccc} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n\end{array}\right].$$

in which  $x_i$  is the *i*th column of X and  $\lambda_i$  is the *i*th diagonal element of  $\Lambda$ . We have  $Ax_i = \lambda_i x_i$  for i = 1, ..., n. In other words, the *i*th column of X is an eigenvector of A and the *j*th entry of  $\Lambda$  is the corresponding eigenvalue.

If X is orthonormal, so that  $X^{-1} = X^{\top}$ , then we can interpret such a decomposition as a change of basis to eigenvector coordinates. If Ax = b, and  $A = X\Lambda X^{-1}$  then  $(X^{-1}b) = \Lambda(X^{-1}x)$ . Here,  $X^{-1}x$  expands x is the basis of columns of X. So to compute Ax, we first expand into the basis of columns of X, scale by  $\Lambda$ , and then interpret the result as the coefficients of a linear combination of the columns of X.

In this case that  $A = X\Lambda X^{\top}$  for an eigenvalue decomposition  $(X, \Lambda)$  of A, we can also write

$$A = X\Lambda X^{\top} = \sum_{i=1}^{n} \Lambda_{ii} x_i x_i^{\top}.$$

**Proposition 1.** Every real symmetric matrix has an eigenvalue decomposition  $(X, \Lambda)$  in which X is orthonormal.<sup>2</sup>

<sup>&</sup>lt;sup>1</sup>Future editions will expand.

<sup>&</sup>lt;sup>2</sup>In future editions, this may be the motivating result for the definition of eigenvalues.

