Why

Can we characterize positive (semi-)definite matrices in terms of their eigenvalues?

Main Result

Using eigenvalue decompositions, we can answer in the affirmative.

Proposition 1. *Suppose $A \in \mathbf{S}^d$ has smallest eigenvalue $\lambda_{\min}(A)$. Then*

$$\begin{aligned} A \in \mathbf{S}^d_+ \quad &\longleftrightarrow \quad \lambda_{\min}(A) \geq 0 \\ &\longleftrightarrow \quad \operatorname{tr} AB \geq 0 \textit{ for all } B \in \mathbf{S}^d_+. \end{aligned}$$

and

$$\begin{aligned} A \in \mathbf{S}^d_{++} \quad &\longleftrightarrow \quad \lambda_{\min}(A) > 0 \\ &\longleftrightarrow \quad \operatorname{tr} AB > 0 \textit{ for all nonzero } B \in \mathbf{S}^d_{++}. \end{aligned}$$

Eigenvalues and Definiteness
Real Positive Semidefinite Matrices
Quadratic Forms
Eigenvalue Decomposition
Symmetric Real Matrices
Matrix Trace
Orthogonal Matrices
Real Matrix-Matrix Products
Matrix Transpose
Orthonormal Set of Vectors
Eigenvalues and Eigenvectors
Real Matrix-Vector Products
Real Inner Product Norms
Monic Polynomials
Matrix Determinants
Vectors as Matrices
Real Inner Products
Orthonormal Set of Real Vectors
Norms
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Real Matrix Determinants
Real Inner Product
Independent Set of Real Vectors
Real Vector Angles
Functionals
Real Matrices
Permutations
Multivariate Real Polynomials
Space Inner Product
Real Linear Combinations
Real Norm
Space Vector Angles
Vectors
Real Linear Equations
Multivariate Functions
Space Norm
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Plane Vector Angles
Linear Functions
Arrays
Plane Inner Product
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Plane Norm
Real Square Roots
Right Triangle Solve Relation
Distance
Plane Vectors
Absolute Value
Fields
Polynomials
Groups
Real Squares
Real Polynomials
Space Distance
Rings
Real Arithmetic
Plane Distance
Real Multiplicative Inverses
Interval Length
Natural Square Roots
Real Products
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Rational Order
Rational Sums
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Direct Products
Set Numbers
Rational Numbers
Finite Sets
Integer Arithmetic
Natural Fractions
Integer Products
Inverse Elements
Natural Additive Identity
Natural Multiplicative Identity
Equivalent Sets
Integer Order
Integer Sums
Natural Equations
Element Functions
Identity Elements
Natural Arithmetic
Comparisons
Function Inverses
Family Operations
Family Unions and Intersections
Integer Numbers
Natural Powers
Orders
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Operations
Function Composites
Function Images
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Natural Products
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Equivalence Relations
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Converse Relations
Natural Sums
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Recursive Theorem
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Set Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Lattices
Objects