



## Definition

A subset of  $\mathbf{R}^n$  is *affine* (or an *affine variety*, *linear variety*, or *flat*) if it contains the lines through each of its points.

Some authors use the term *affine subspace*, but we avoid this terminology because affine sets are not subspaces. In fact, they are translated subspaces.

## Examples

The empty set is trivially an affine set. The entire set of points in  $n$ -dimensional space is an affine set. Any singleton is an affine set.

Our definition is meant to capture the intuitive idea of an infinite, uncurved set, like a line or a plane in space.

## Notation

As usual, let  $L(x, y)$  denote the line between  $x, y \in \mathbf{R}^n$ . The set  $M \subset \mathbf{R}^n$  is affine if  $L(a, b) \subset M$  for all  $a, b \in M$ . For any  $x, y \in M$  and  $\lambda \in \mathbf{R}$ ,

$$(1 - \lambda)x + \lambda y \in M.$$

## Translations

The set of affine sets is closed under translation.

**Proposition 1.** *Let  $M \subset \mathbf{R}^n$  affine. Then  $M + a$  is affine for any  $a \in \mathbf{R}^n$ .*

*Proof.* Suppose  $x, y \in M + a$ . So there exists corresponding  $\tilde{x}, \tilde{y} \in M$  with  $x = \tilde{x} + a$  and  $y = \tilde{y} + a$ . For any  $\lambda \in \mathbf{R}$ ,

$$\begin{aligned} (1 - \lambda)x + \lambda y &= (1 - \lambda)(\tilde{x} + a) + \lambda(\tilde{y} + a) \\ &= ((1 - \lambda)\tilde{x} + \lambda\tilde{y}) + a. \end{aligned}$$

□

An affine set  $M$  is *parallel* to an affine set  $L$  if there exists  $a \in \mathbf{R}^n$  so that  $M = L + a$ . The relation " $M$  is parallel to  $L$ " on the set of affine sets is an equivalence relation. This notion of parallelism is more restrictive than the natural one, in which we may speak of a line being parallel to a plane. We must speak of a line which is parallel to another line within the given plane.

### **Properties**

The set of affine sets is closed under intersection.

**Proposition 2.** *If  $\mathcal{M}$  is a family of affine sets, then  $\cap \mathcal{M}$  is affine.*



