

REAL SUBSPACES

Definition

A nonempty set $S \subset \mathbb{R}^n$ is called a *subspace* (*linear subspace*) if

- 1. $x + y \in S$ for all $x, y \in S$, and
- 2. $\alpha x \in S$ for all $\alpha \in \mathbb{R}$, $x \in S$.

We say that that S is (1) closed under vector addition and (2) closed under scalar multiplication.

Examples

The set $S_1 = \mathbb{R}^n$ is a subspace. In other words, the entire set is a subspace of itself. The set $S_2 = \{0\}$, consisting of a single point, the origin, is a subspace. S_1 is the biggest subspace. In other words, if S' is another subspace of \mathbb{R}^n , then $S' \subset S_1$. If S is a subspace, it is nonempty, so there is $x \in S$, and it is closed under scalar multiplication, so $0 \cdot x = 0 \in S$. In other words, every subspace contains the origin. So S_2 is the smallest subspace, in the sense that if S' is another subspace $S_2 \subset S'$.

The span (see Real Vectors Span) of a set of vectors v_1, \ldots, v_k is a subspace. For two subspaces $S, T \subset \mathbf{R}^n$, their sum

$$S+T=\{x+y\mid x\in S,y\in T\}$$

is a subspace.

Geometric intuition

Roughly speaking, a subspace S is a flat set which passes through the origin. In \mathbb{R}^2 , the subspaces are the lines. In \mathbb{R}^3 , the lines and the planes.

