



# Random Variable Sigma Algebra

## 1 Why

What does it mean for two random variables to be independent?

What are the events associated with a random variable? TODO

## 2 Definition

### 2.1 Defining Result

**Proposition 1.** *The set of inverse images the distinguished sets of a measurable space under a function from a set to that space is a sigma algebra.*

*If the first set and the function are measurable, the sigma algebra is a sub sigma algebra of the domain sigma algebra.*

*Proof.* Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Let  $f : X \rightarrow Y$  be a measurable function. Define

$$\mathcal{C} := \{f^{-1}(B) \mid B \in \mathcal{B}\}.$$

First, since  $f^{-1}(Y) = X$ ,  $X \in \mathcal{C}$ .

Second, let  $C \in \mathcal{C}$ . Then there is  $B$  such that  $C = f^{-1}(B)$ . Then  $X - C = X - f^{-1}(B) = f^{-1}(Y - B)$ . Since  $\mathcal{B}$  is a sigma algebra,  $B \in \mathcal{B}$ ,  $Y - B \in \mathcal{B}$  and so  $X - C \in \mathcal{C}$ .

Finally, let  $(C_n)_n \subset \mathcal{C}$ . Then for every  $n$  there exists a  $B_n \in \mathcal{B}$  so that  $C_n = f^{-1}(B_n)$ . Then:

$$\cup_n C_n = \cup_n f^{-1}(B_n) = f^{-1}(\cup B_n).$$

Since  $\mathcal{B}$  is a sigma algebra,  $\cup_n B_n \in \mathcal{B}$  and so  $\cup_n C_n \in \mathcal{C}$ .

Since  $f$  is measurable,  $f^{-1}(B) \in \mathcal{A}$  for every  $B \in \mathcal{B}$ , and so  $\mathcal{C} \subset \mathcal{A}$ .  $\square$

The sigma algebra *generated by a random variable* is the sigma algebra consisting of the inverse images of every measurable set of the codomain.

The sigma algebra generated by a family of random variables is the sigma algebra generated by the union of the sigma algebras generated individually by each of the random variables.

## 2.2 Notation

Let  $(X, \mathcal{A}, \mu)$  be a probability space and  $(Y, \mathcal{B})$  be a measurable space. Let  $f : X \rightarrow Y$  be a random variable. Denote by  $\sigma(f)$  the sigma algebra generated by  $f$ .

### 3 Results

**Proposition 2.** *The sigma algebra generated by a family of random variables is the smallest sigma algebra for with respect to which each random variable is measurably.*