Measures

1 Why

We want to generalize the notion of length, area, volume beyond the Lebesgue measure on the product spaces of real numbers.

2 Definition

An extended-real-valued non-negative function on an algebra is *finitely additive* if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a sigma algebra is *countably additive* if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A *finitely additive measure* is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A *countably additive measure* is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call

countably additive measures *measures*, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countable additive.

In the context of measure, we call a countably unitable subset algebra a *measurable space*. We call the distinguished sets *measurable* sets. A *measure space* is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of teh measurable space.

3 Notation

Let A a set. Let $\mathcal{A}$ a sigma algebra on A. The pair $(A, \mathcal{A})$ is a measurable space.

Let $\mu : \mathcal{A} \to [0, \infty]$ a measure; thus: (a) $\mu(\varnothing) = 0$ and (b) for disjoint $\{A_n\} \subset \mathcal{A}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ The triple $(A, \mathcal{A}, \mu)$ is a measure space.

We use μ since it is a mnemonic for “measure". We often also us ν to denote measures, since it is after μ in the Greek alphabet, and λ, since it is before μ in the Greek alphabet.

4 Examples

Example 1. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$*

otherwise. Then μ is a measure. We call μ the counting measure.

Example 2. *Let $(A, \mathcal{A})$ measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise. Then μ is a measure. We call μ the point mass concentrated at a.*

Example 3. *Let R denote the real numbers. The Lebesgue measure on the measurable space $(R, \mathcal{B}(R))$ is a measure.*

Example 4. *Let N be the natural numbers. Let $\mathcal{A}$ the finite co-finite algebra on N. Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.*

Example 5. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ be 0 if $A = \varnothing$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.*

Example 6. *Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = 2^A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \varnothing$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.*

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \varnothing$ then using finite additivity we obtain a contradiction $1 = \mu(B \cup C) = \mu(B) + \mu(C) = 2$. $\square$