1 Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

2 Definition

Given a distribution $p : \Omega \to \mathbf{R}$, the *probability of an event $A \subset \Omega$* is $\sum_{a \in A} p(a)$, the sum of probabilities of its outcomes.

3 Notation

Define define $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ by

$$\mathbf{P}(A) = \sum_{a \in A} p(a).$$

We call $\mathbf{P}$ the *event probability function* (or *probability measure*) of (or induced by) p.

4 Example: die

Define $p : \{1, \ldots, 6\} \to \mathbf{R}$ by $p(\omega) = \text{\textonehalf}$... $p(\omega) = 1/6$ for $\omega = 1, \ldots, 6$. Define the event $E = \{2, 4, 6\}$. Then

$$\mathbf{P}(E) = \sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

5 Probability measures

Notice that for all $A \subset \Omega$, (i) $\mathbf{P}(A) \geq 0$. In particular, (ii) $\mathbf{P}(\Omega) = 1$ (and $\mathbf{P}(\varnothing) = 0$). For all $A, B \subset \Omega$, $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. In particular, if $A \cap B = \varnothing$, (iii) $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

Conversely, suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satifies (i), (ii), (iii). These three conditions are sometimes called the *axioms of probability* (for finite sets). Define $p : \Omega \to \mathbf{R}$ by

$$p(\omega) = f(\{\omega\}).$$

In case f satisfies the axioms, p is a probability distribution (nonnegative and sums to one). For this reason we call f satisfying (i)-(iii) an *event probability function* (or *probability measure*). In the case that we think of a probability event function $\mathbf{P}$ as induced by a distribution p, we write $\mathbf{P}_p$.

We conclude that p and $\mathbf{P}$ are two perspectives. We can think of elementary events (outcomes) and define their probabilities individually in a way that they sum to one and are nonnegative. Or we can think of the compound events, and define their probabilities in a way consistent with (i)-(iii).

6   Probability by cases

Let $\mathbf{P}$ be a probability event function. Suppose $A_1, \ldots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

Some authors call this the *law of total probability*.

Event Probabilities

Probability Distributions

Intervals

Real Line

Real Order

Real Summation

Rational Order

Real Numbers

Integral Line

Rational Numbers

Integer Arithmetic

Integer Products

Natural Additive Identity

Natural Multiplicative Identity

Arithmetic

Identity Elements

Natural Powers

Natural Summation

Comparisons

Integer Order

Integer Sums

Natural Products

Family Operations

Orders

Integer Numbers

Natural Sums

Families

Operations

Converse Relations

Natural Order

Equivalence Relations

Recursion Theorem

Functions

Relations

Peano Axioms

Set Products

Natural Induction

Uncertain Outcomes

Ordered Pairs

Set Powers

Natural Numbers

Unordered Triples

Successor Sets

Intersection of Empty Set

Partitions

Pair Unions

Set Intersections

Set Symmetric Differences

Set Unions

Set Complements

Pair Intersections

Unordered Pairs

Set Differences

Empty Set

Geometry

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects