Length Common Notions

1 Why

We want to define the length of a subset of real numbers.

2 Notions

We take two common notions:

1. The length of a whole is the sum of the lengths of its parts; the *additivity principle*.
2. The length of a whole is the at least the length of any whole it contains the *containment principle*.

The task is to make precise the use of "whole,", "parts," and "contains." We start with intervals.

3 Definition

By whole we mean set. By part we mean an element of a partition. By contains we mean set containment.

The *length* of an interval is the difference of its endpoints: the larger minus the smaller.

Two intervals are *non-overlapping* if their intersection is a single point or empty. The *length* of the union of two non-overlapping intervals is the sum of their lengths.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A *countably simple* subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A *cover* of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

3.1 Cuts

If a, b are real numbers and $a < b$, then we *cut* an interval with a and b as its endpoints by selecting c such that $a < c$ and $c < b$. We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the *cut pieces*.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

4 All sets

Proposition 1. *Not all subsets of real numbers are simple.*

Exhibit: R is not finite.

Proposition 2. *Not all subsets of real numbers are countably simple.*

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

4.1 Notation