Iterated Rectangular Integrals

1 Why

Toward a theorem for iterated integrals, we show a result for rectangular functions. TODO

2 Result

Proposition 1. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measurable spaces. Let $E \in \mathcal{A} \times \mathcal{B}$. Let $\mu = \mu_1 \times \mu_2$ be the product measure on the product sigma algebra $\mathcal{A}_1 \times \mathcal{A}_2$. Let $f : X_1 \times X_2 \to [-\infty, \infty]$ be an integrable function.*

The **product measure** of the measures of two sigma finite measure spaces is the unique measure which assigns to every rectangle with measurable sides the product of the measures of the sides. We prove that such a measure exists, and is unique.

2.1 Defining Result

Proposition 2. *Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *be sigma-finite measurable spaces. There is a unique measure* π *on* $\mathcal{A} \times \mathcal{B}$ *such that*

$$\pi(A \times B) = \mu(A) \times \nu(B)$$

for all $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$. *Furthermore, for any* $E \in \mathcal{A} \times \mathcal{B}$.

$$\pi(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

Proof. TODO □

2.2 Notation

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be sigma-finite measurable spaces. We denote the product measure by $\mu \times \nu$. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$(\mu \times \nu)(A \times B) = \mu(A) \times \nu(B).$$