Why

What are the affine sets in terms of subspaces?

Affine sets which are subspaces

The subspaces of $\mathbf{R}^n$ are the affine sets which contain the origin.

Proposition 1. $M \subset \mathbf{R}^n$ *is a subspace if and only if M is affine and* $0 \in M$.

Proof. ($\Rightarrow$) Suppose M is a subspace. Then $0 \in M$. Also $\alpha x + \beta y \in M$ for all $\alpha, \beta \in \mathbf{R}$ and $x, y \in \mathbf{R}^n$ In particular, $(1 - \lambda)x + \lambda y \in M$ for all $\lambda \in \mathbf{R}$, $x, y \in \mathbf{R}^n$. In other words, M contains the line through x and y.

($\Leftarrow$) Suppose M is affine and $0 \in M$. M is closed under scalar multiplication since

$$\alpha x = (1 - \alpha)0 + \alpha x$$

is in the line through 0 and x. M is closed under vector addition since

$$(1/2)(x + y) = (1 - 1/2)x + (1/2)y$$

is in the line through x and y. Thus, $x + y = 2(1/2)(x + y) \in M$. $\qquad\square$

Affine sets as translated subspaces

Proposition 2. *Suppose $M \neq \varnothing$ is affine. Then there exists a unique subspace L and vector $a \in \mathbf{R}^n$ for which $M = L + a$. Moreover,*

$$L = M - M = \{x - y \mid x, y \in M\}.$$

Proof. First, uniqueness. Suppose L_1 and L_2 are subspaces parallel to M. We will show that $L_2 \supset L_1$ (and similarly, $L_1 \supset L_2$).

Since L_1 and L_2 are both parallel to M, they are also parallel to each other. Consequently, there exists $a \in \mathbf{R}^n$ with $L_2 = L_1 + a$. Since $0 \in L_2$ (it is a subspace, after all), $-a \in L_1$. Since L_1 is a subspace, $a \in L_1$.

So $x + a \in L_1$ for every $a \in L_1$, and so $L_2 = L_1 + a \subset L_1$. A similar argument gives $L_1 \supset L_2$.

If $y \in M$, then $M + (-y) = M - y$ is a translate of M containing zero (since $y - y = 0$). In other words, the affine set $M - y$ is a subspace. This, then, is the unique subspace parallel to M. Since y was arbitrary, the subspace parallel to M is $L = \cup_{y \in M} M - y = \{x - y \mid x, y \in M\}$. $\quad\square$

Real Affine Sets and Subspaces

Real Affine Sets

Real Subspaces

Real Affine Combinations

Real Vectors Span

N-Dimensional Lines

Real Linear Combinations

Real Translates

Real Vectors

N-Dimensional Space

Plane Vector Sums and Differences

Space Vectors

Distance

Plane Vectors

Space Distance

Plane Distance

Interval Length

Intervals

Real Space

Real Line

Real Plane

Integral Line

Real Order

Geometry

Rational Order

Real Numbers

Rational Numbers

Natural Fractions

Lists

Integer Arithmetic

Set Numbers

Direct Products

Integer Products

Natural Additive Identity

Natural Multiplicative Identity

Finite Sets

Identity Elements

Natural Arithmetic

Equivalent Sets

Natural Powers

Integer Sums

Integer Order

Comparisons

Function Inverse

Family Unions and Intersections

Natural Products

Integer Numbers

Orders

Operation

Function Image

Function Composites

Families

Natural Sums

Natural Order

Converse Relations

Equivalence Relations

Functions

Recursion Theorem

Relations

Peano Axioms

Ordered Pair Projections

Natural Induction

Set Products

Natural Numbers

Set Powers

Generalized Set Dualities

Set Symmetric Differences

Successor Sets

Unordered Triples

Set Dualities

Intersection of Empty Set

Set Partitions

Set Unions and Intersections

Set Complements

Pair Unions

Set Intersections

Ordered Pairs

Set Differences

Set Unions

Pair Intersections

Empty Set

Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects