Why

If we stack two rectangles, with equal base lengths but different heights, on top of each other, the additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property, as it ought to.

Result

Prop. 1. *The simple non-negative integral operator is additive.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathrm{SF}_+(X)$ denote the non-negative real-valued simple functions on X. Define $s : \mathrm{SF}_+(X) \to [0, \infty]$ by $s(f) = \int f d\mu$ for $f \in \mathrm{SF}_+ X$.

In this notation, we want to show that $s(f+g) = s(f)+s(g)$ for all $f, g \in \mathrm{SF}_+(X)$. Toward this end, let $f, g \in \mathrm{SF}_+(X)$ with the simple partitions:

$$\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n \subset \mathcal{A} \quad \text{and} \quad \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset [0, \infty].$$

We consider the refinement of the two partitions. TODO: this is why you don't do the unique maximal partition business. $\{A_i \cap B_j\}_{i,j=1}^{i=m, j=n}$.

First, let $\alpha \in (0, \infty)$. Then $\alpha f \in \mathrm{SF}_+(X)$, with the simple partition $\{A_n\} \subset \mathcal{A}$ and $\{\alpha a_n\} \subset [0, \infty]$.

$$s(\alpha f) = \sum_{i=1}^n \alpha a_n \mu(A_i) \ = \alpha \sum_{i=1}^n a_n \mu(A_i) = \alpha s(f).$$

If $\alpha = 0$, then αf is uniformly zero; it is the non-negative simple with partition $\{X\}$ and $\{0\}$. Regardless of the measure of X, this non-negative simple function is zero Recall that we define $0 \cdot \infty = \infty \cdot 0 = 0$. $\qquad\square$

Simple Integral Additivity

Simple Integrals

Measurable Functions

Measures

Extended Real Numbers — Sigma Algebras

Real Limits

Absolute Value

Simple Functions

Interval Length

Real Functions

Intervals

Real Line

Real Sequences — Real Order

Real Numbers

Integral Line — Rational Numbers

Sequences — Integer Arithmetic

Direct Products — Set Numbers

Natural Additive Identity — Natural Multiplicative Identity

Finite Sets

Identity Elements — Arithmetic

Equivalent Sets — Subset Algebras — Natural Powers — Integer Products

Family Unions and Intersections — Function Inverses — Set Operations

Operations — Natural Products — Integer Sums

Families — Function Images — Function Composites

Functions — Equivalence Relations — Natural Order — Recursion Theorem

Natural Sums — Integer Numbers — Cardinality

Relations — Peano Axioms

Cartesian Products — Subset Systems — Natural Induction — Characteristic Functions

Generalized Set Dualities — Ordered Pairs — Set Powers — Natural Numbers

Set Unions and Intersections — Partitions — Intersection of Empty Set — Set Dualities — Unordered Triples — Successor Sets — Set Symmetric Differences

Set Intersections — Pair Unions — Set Complements

Pair Intersections — Set Unions — Set Differences

Unordered Pairs — Empty Set

Geometry — Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects

4