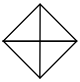


Why

Which tree is optimal for tree distribution approximation?

Definition

We want to choose a tree whose corresponding approximator for the given distribution minimizes the relative entropy with the given distribution among all tree distribution approximators. We call such a distribution an *optimal* tree approximator of the given distribution. We call a tree according to which an optimal tree approximator factors and *optimal* approximator tree.

Result

Proposition 1. *Let $A_1, \dots, A_n$ be finite nonempty sets. Define $A = \prod_{i=1}^{n} A_i$. Let $q : A \to [0, 1]$ a distribution. A tree T on $\{1, \dots, n\}$ is an optimal approximator tree if and only if it is a maximal spanning tree of the mutual information graph of q.*

Proof. First, denote the optimal tree distribution approximator of q for tree T by p^*_T. Express

$$p^*_T = q_1 \prod_{i \neq 1} q_{i|\mathrm{pa}\, i}$$

Second, express $d(q, p) = H(q, p) - H(q)$. Since $H(q)$ does not depend on T, p^*_T is a minimizer (w.r.t. T) of $d(q, p^*_T)$ if and only if it is a minimizer of $H(q, p^*_T)$.

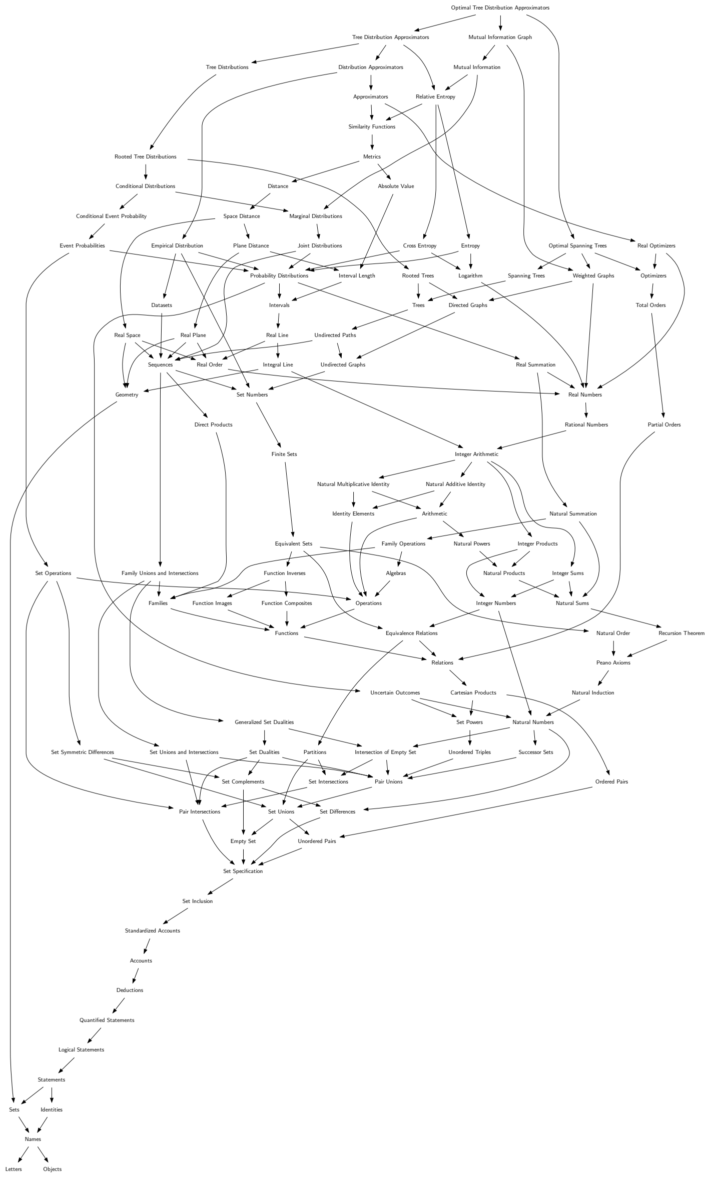
Third, express the cross entropy of p_T^* relative to q as

$$
\begin{aligned}
H(q, p_T^*) &= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) \log q_{i|pai}(a_i, a_{\mathrm{pa}\,i}) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i,\mathrm{pa}\,i}(a_i, a_{\mathrm{pa}\,i}) - \log q_{\mathrm{pa}\,i}(a_{\mathrm{pa}\,i})) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i,\mathrm{pa}\,i}(a_i, a_{\mathrm{pa}\,i}) - \log q_{\mathrm{pa}\,i}(a_{\mathrm{pa}\,i}) - \log q \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{i \neq 1} I(q_i, q_{\mathrm{pa}\,i}) \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{\{i,j\} \in T} I(q_i, q_j)
\end{aligned}
$$

where $\mathrm{pa}\,i$ denotes the parent of vertex i in T rooted at vertex 1 ($i = 2, \dots, n$). For $i = 1, \dots, n$, $H(q_i)$ does not depend on the choice of tree. Therefore selecting a tree to minimize the second term in the final expression above is equivalent to choosing a maximal spanning tree from the weighted graph with mutual information edge weights; namely, the mutual information graph of q.

□

Proposition 1 says that to we should first select a maximum spanning tree of the mutual information graph of the distribution we are approximating. Then, we should pick the best approximator to q which factors according to that three.


Optimal Tree Distribution Approximators
Tree Distribution Approximators
Mutual Information Graph
Tree Distributions
Distribution Approximators
Mutual Information
Approximators
Relative Entropy
Similarity Functions
Rooted Tree Distributions
Metrics
Conditional Distributions
Distance
Absolute Value
Conditional Event Probability
Space Distance
Marginal Distributions
Event Probabilities
Empirical Distribution
Plane Distance
Joint Distributions
Cross Entropy
Entropy
Optimal Spanning Trees
Real Optimizers
Probability Distributions
Interval Length
Rooted Trees
Logarithm
Spanning Trees
Weighted Graphs
Optimizers
Datasets
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Directed Graphs
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Real Numbers
Direct Products
Rational Numbers
Partial Orders
Finite Sets
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Identity Elements
Arithmetic
Natural Summation
Equivalent Sets
Family Operations
Natural Powers
Integer Products
Set Operations
Family Unions and Intersections
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Families
Function Images
Function Composites
Operations
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Set Symmetric Differences
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Intersection of Empty Set
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Pair Intersections
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Unordered Pairs
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Names
Letters
Objects