Measurable Function Operations

1 Why

Under which operations is the set of measurable functions closed?

2 Overview

Measurable functions are closed under composition and concatenation. Taking these facts together with the observation that continuous functions are measurable, we conclude that measurable functions are closed under addition, multiplication, and (with suitable nonzero assumptions) division.

3 Results

Proposition 1. *The composition of two measurable functions is measurable.*

Proof. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$, $(Z, \mathcal{C})$ be measurable spaces. Let $f : X \to Y$ and $g : Y \to Z$ be measurable functions. Define $h = g \circ f$.

Let $C \in \mathcal{C}$. Measurability of g implies $g^{-1}(C) \in \mathcal{B}$. This fact, together with measurability of f, implies $f^{-1}(g^{-1}(C)) \in \mathcal{A}$. Since $h^{-1} = f^{-1} \circ g^{-1}$, we conclude that h is measurable. $\square$

Proposition 2. *A continuous function between topological spaces is measurable with respect to the topological sigma algebras.*

Proposition 3. *The concatentation of two measurable functions is measurable.*

Proof. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ be measurable spaces. Let $f : X \to Y$ and $g : X \to Z$ be measurable. Define $h : X \to Y \times Z$ by $h(x) = (f(x), g(x))$. $\square$

Proposition 4. *Let $(X, \mathcal{A})$ be a measurable space and let R denote the real numbers. Let $f, g : X \to R$ be measurable.*

Then, $f + g$ and fg are measurable.