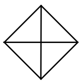


Why

We want to describe how fast a function grows or declines.[1]

Definition

Let $f : \mathsf{R} \to \mathsf{R}$. The *lower growth class* of f (*toward infinity*) is the set of all functions $g : \mathsf{R} \to \mathsf{R}$ for which there exists $C, M > 0$ so that $|g(x)| \leq C\,|f(x)|$ for all $x > M$. The intuition is that if $h : \mathsf{R} \to \mathsf{R}$ is in the lower growth class of f, h does not grow faster than f. In this case we say that *h grows at order f*.

The *lower limit class of f at x_0* is the set of all functions $g : \mathsf{R} \to \mathsf{R}$ for which there exists $C, \varepsilon > 0$ so that $|g(x)| \leq C\,|f(x)|$ for all $|x - x_0| < \varepsilon$. The intuition is that for x sufficiently close to x_0, the magnitude of f is bounded by a constant times the magnitude of g. Often x_0 is 0.

The *upper growth class* of f (*toward infinity*) is the set of all functions $g : \mathsf{R} \to \mathsf{R}$ for which there exists $C, M > 0$ so that $|g(x)| \geq C\,|f(x)|$ for all $x > M$. The intuition is that if h is in the upper growth class of f, h grows at least as fast as f. We similarly define the *upper growth class at a limit x_0*.

The *(exact) growth class* of f is the set of all functions $g : \mathsf{R} \to \mathsf{R}$ for which there exists C_1, C_2, M so that $C_1\,|f(x)| \leq |g(x)| \leq C_2\,|f(x)|$ for all $x > M$. The intuition is that if h is

[1]Future editions will expand this vague introduction.

in the growth class of f, then h and f grow at the same rate. Again, we similarly define the *growth class at limit* x_0.

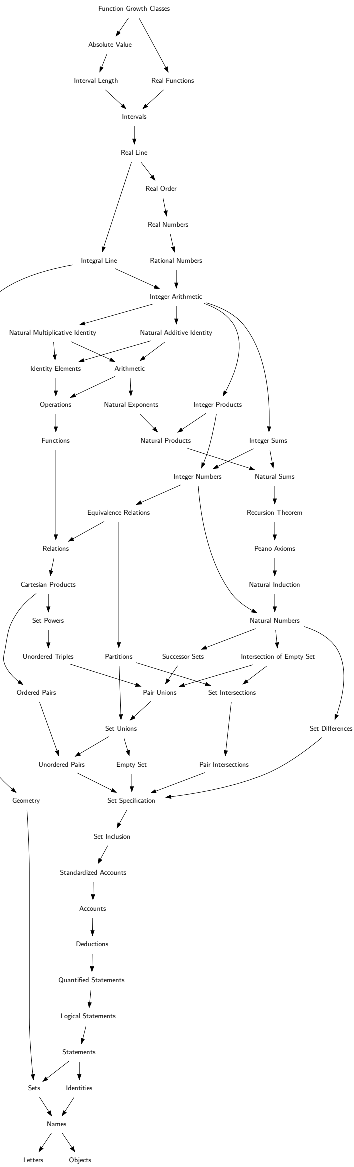
Notation

We denote the upper, lower and exact growth classes of a function $f : \mathsf{R} \to \mathsf{R}$ by of f by $O(f)$, $\Omega(f)$ and $\Theta(f)$, respectively. We read the notation $O(f)$ as "order at most f," we read $\Omega(f)$ as "order at least f," and $\Theta(f)$ as "order exactly f."

The letter O is a mnemonic for order, and Ω and Θ build on this mnemonic. The term order appears to arise from the use of growth classes when discussing Taylor approximations. In this case of small x (i.e., $|x| < 1$), $|x^p| < |x^q|$ if $q < p$ and so higher order terms are "smaller" and "negligible." This notation is sometimes called *Big O notation* or *Laundau's symbol.*

Let $\phi, \psi : \mathsf{R} \to \mathsf{R}$. Many authors use $\phi = O(\psi)$ or $\phi(t) = O(\psi(t))$ to assert that ϕ is in the upper growth class of ψ at some understood limit (e.g., 0 or ∞). In other words, the equation asserts that there exists some positive constant $C > 0$ sot that, for all t sufficiently close to the understood limit, $|\phi(t)| \leq C\,|\psi(t)|$.[2] For example, the statement $\sin^2(t) = P(t^2)$ as $t \to 0$ (or for $t \to 0$) means that there exists constants $C, \varepsilon > 0$ so that, $|t| < \varepsilon \longrightarrow \left|\sin^2(t)\right| \leq Ct^2$.

[2] Often also defined $|\phi(t)| < C\psi(t)$, with no absolute value on ψ.


Function Growth Classes
Absolute Value
Interval Length
Real Functions
Intervals
Real Line
Real Order
Real Numbers
Integral Line
Rational Numbers
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Identity Elements
Arithmetic
Operations
Natural Exponents
Integer Products
Functions
Natural Products
Integer Sums
Integer Numbers
Natural Sums
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Cartesian Products
Natural Induction
Set Powers
Natural Numbers
Unordered Triples
Partitions
Successor Sets
Intersection of Empty Set
Ordered Pairs
Pair Unions
Set Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
Geometry
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects