Why

For general measure theory, we need an algebra of sets closed under countable unions; we define such an object.[1]

Definition

A *countably summable subset algebra* is a subset algebra for which (1) the base set is distinguished (2) the complement of a distinguished set is distinguished (3) the union of a sequence of distinguished sets is distinguished.

The name is justified, as each countably summable subset algebra is a subset algebra, because the union of $A_1, \ldots, A_n$ coincides with the union of $A_1, \cdots, A_n, A_n, A_n \cdots$.

We call the set of distinguished sets a *sigma algebra* (or *sigma field*) *on* the base set. This language is justified (as for a regular subset algebra) by the closure properties of the sigma algebra under the usual set operations. We sometimes write are σ-algebra and σ-field.

A *sub-σ-algebra* (*sub-sigma-algebra*) is a subset of a sigma algebra which is itself a sigma algebra.

Notation

We often denote a sigma algebra by $\mathcal{A}$ or $\mathcal{F}$; the former is a mnemonic for "algebra" and the second is a mnemonic for "field". The calligraphic typeface is meant as a reminder that the object so denoted is a set *of sets*.

A common pattern is also to use the calligraphic font of whichever letter is being used for the base set. Thus, if A is a set, then we might choose to denote a sigma algebra on A by $\mathcal{A}$.

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[1]Future editions will make no reference to measure theory. The entire development will follow the genetic approach, and so roughly follow the historical development for handling integration.

Often, instead of saying "let $(A, \mathcal{A})$ be a countably summable subset algebra" we say instead "let $\mathcal{A}$ be a sigma algebra on A." Since the largest element of the sigma algebra is the base set, we can also say (without ambiguity): "let $\mathcal{A}$ be a sigma algebra." The base set is $\cup \mathcal{A}$.

Examples

Example 1. *For any set A, 2^A is a sigma algebra.*

Example 2. *For any set A, $\{A, \varnothing\}$ is a sigma algebra.*

Example 3. *Let A be an infinite set. Let $\mathcal{A}$ the collection of finite subsets of A. $\mathcal{A}$ is not a sigma algebra.*

Example 4. *Let A be an infinite set. Let $\mathcal{A}$ be the collection subsets of A such that the set or its complement is finite. $\mathcal{A}$ is not a sigma algebra.*

Proposition 1. *The intersection of a family of sigma algebras is a sigma algebra.*

Example 5. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| \leq \aleph_0 \vee |C_A(B)| \leq \aleph_0\}.$$

$\mathcal{A}$ *is an algebra; the* countable/co-countable algebra.[2]

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[2]Future editions will clean up and modify these examples.

Sigma Algebras

Real Series

Real Limits

Absolute Value

Interval Length

Intervals

Real Line

Real Order   Real Sequences

Rational Order   Real Numbers   Sequences   Subset Algebras

Integral Line   Rational Numbers   Lists   Cardinality

Integer Arithmetic   Natural Fractions   Direct Products   Set Numbers

Integer Products   Natural Multiplicative Identity   Natural Additive Identity   Finite Sets

Identity Elements   Natural Arithmetic   Equivalent Sets   Family Unions and Intersections

Natural Powers   Integer Sums   Integer Order   Comparisons   Operations   Function Inverses

Natural Products   Integer Numbers   Orders   Function Composites   Function Images   Families

Natural Sums   Natural Order   Converse Relations   Equivalence Relations   Functions

Recursion Theorem   Relations

Peano Axioms   Ordered Pair Projections   Set Products   Subset Systems

Natural Induction   Set Powers   Ordered Pairs

Natural Numbers   Generalized Set Dualities   Unordered Triples   Set Partitions   Set Unions and Intersections

Set Symmetric Differences   Successor Sets   Set Dualities   Intersection of Empty Set

Set Complements   Pair Unions   Set Intersections

Set Differences   Set Unions   Pair Intersections

Empty Set   Unordered Pairs

Geometry   Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets   Identities

Names

Letters   Objects