

ORDINARY ROW REDUCTIONS

Why

When does the technique of row reductions prevail?

Multivariable row reductions

Let $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$. The kth row reduction of S is the linear system (C, d) with $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$ if $i < s \le m$ and $C_{st} = A_{st}$ otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable k from equations k + 1, ..., m. We are taking the kth column of A from

$$\left[egin{array}{c} A_{1k} \ dots \ A_{kk} \ A_{k+1,k} \ dots \ A_{mk} \end{array}
ight] \qquad \left[egin{array}{c} A_{1k} \ dots \ A_{kk} \ 0 \ dots \ 0 \end{array}
ight].$$

We interpret the *i*th row reduction as subtracting equations of the system or reducing rows of the array A. If $a^i, c^i \in \mathbb{R}^n$ denote the *i*th rows of A and C, $c^i = a^i - (A_{ik}/A_{kk})a^k$ for $k < i \le m$, In other words, we obtain the *i*th row of matrix C by subtracting a multiple of the kth row of matrix A from the *i*th row of matrix A, for $k < i \le m$. The following is an immediate consequence of real arithmetic.

Proposition 1. Let $(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n)$ be a linear system which row reduces to (C, d). Then $x \in \mathbb{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d).

Ordinary reductions

We call the system S ordinarily reducible if there exists a sequence of systems S_1, \ldots, S_{m-1} so that S_1 is the 11-reduction of S and S_i is the *ii*-reduction of S_{i-1} for $i = 1, \ldots, n-1$. In this case, we call S_{n-1} the

final ordinary reduction (or just ordinary reduction) of S. The following is an immediate consequence of Proposition 1.

Proposition 2. Let S' be the (final) ordinary reduction of S. Then S and S' have equivalent solution sets.

This process of constructing the ordinary reduction is called *Gauss elimination* or *Gaussian elimination*. We call the kkth entry of system S_{k-1} the *pivot*. In an ordinarily reducible system, the pivots are nonzero.

The idea is that a system is ordinarily reducible if we can take row reductions in sequence an end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system (A, b) in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.

