Generated Sigma Algebra

1 Why

A simple way to obtain a sigma algebra, is to ask it to obtain some sets, and then to ask it to contain all the sets it needs to fulfill the properties.

2 Definition

The **generated sigma algebra** for a set of subsets is the smallest sigma algbera containing the set of subsets.

Proposition 1. *The intersection of a non-empty set of sigma algebras is a sigma algebra.*

On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

Proposition 2. *Let A be and $\mathcal{A}$ a set of subsets.*

There is a smallest sigma algebra containing $\mathcal{A}$.

2.1 Notation

Let A be a set and $\mathcal{A} \subset 2^A$. We denote the subset algebra of A and $\mathcal{A}$ by $(A, \mathcal{A})$, read aloud as "A, script A."

3 Examples

Example 3. *For any set A, 2^A is a sigma algebra.*

Example 4. *For any set A, $\{A, \varnothing\}$ is a sigma algebra.*

Example 5. *Let A be an infinite set. Let $\mathcal{A}$ the collection of finite subsets of A. $\mathcal{A}$ is not a sigma algebra.*

Example 6. *Let A be an infinite set. Let $\mathcal{A}$ be the collection subsets of A such that the set or its complement is finite. $\mathcal{A}$ is not a sigma algebra.*

Proposition 7. *The intersection of a family of sigma algebras is a sigma algebra.*

Example 8. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| \leq \aleph_0 \vee |C_A(B)| \leq \aleph_0\}.$$

*$\mathcal{A}$ is an algebra; the **countable/co-countable algebra**.*

4 Generation

Proposition 9. *Let A a set and $\mathcal{B}$ a set of subsets. There is a unique smallest sigma algebra $(A, \mathcal{A})$ with $\mathcal{B} \subset \mathcal{A}$.*

We call the unique smallest sigma algebra containing B the **generated sigma algebra** of B.