



Why

We generalize our notion of norm on real vectors to abstract vector spaces.

Definition

A *norm* is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisfies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms on those vectors.

A *normed space* (or *norm space*) is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

Notation

Let (X, \mathbf{F}) be a vector space where \mathbf{F} is the field of real numbers or the field of complex numbers. Let R denote the set of real numbers. Let $f : X \rightarrow R$. The functional f is a norm if

1. $f(v) \geq 0$ for all $x \in V$
2. $f(v) = 0$ if and only if $x = 0 \in X$.
3. $f(\alpha x) = |\alpha|f(x)$ for all $\alpha \in \mathbf{F}$, $x \in X$
4. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.

In this case, for $x \in X$, we denote $f(x)$ by $\|x\|$, read aloud “norm x”. The notation follows the notation of absolute value as a norm. In some cases, we go further, and for a norm indexed by some parameter α or set A we write $\|x\|_\alpha$ or $\|x\|_A$.

Examples

The absolute value function is a norm on the vector space of real numbers. In addition, the (Euclidean norm) is a norm on the vector space \mathbf{R}^n .

