

## Why

We want a norm on the vector space of continuous functions.

## Definition

Consider a function from a closed real interval to the real numbers. The absolute supremum of the function is the supremum of the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

**Proposition 1.** The functional mapping  $f \in C[a, b]$  to its absolute supremum is a norm.

*Proof.* Let R denote the set of real numbers. Define  $\phi: C[a,b] \to R$  by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a,b]\}.$$

- 1.  $|f(x)| \ge 0$  for all  $x \in [a, b]$ , so  $\phi(f) \ge 0$ .
- 2. If  $\phi(f) = 0$  then  $|f(x)| \le 0$  for all x and so f(x) = 0 for all  $x \in [a, b]$ . If f = 0, then |f(x)| = 0 for all  $x \in [a, b]$
- 3. For all  $\alpha$  real,  $|\alpha f(x)| = |\alpha||f(x)|$ . So  $\phi(\alpha f) = |\alpha|\phi(f)$
- 4. For all  $f, g \in C[a, b]$ , and  $x \in [a, b]$ ,  $|f(x) + g(x)| \le |f(x)| + |g(x)|$  by the triangle inequality for absolute value. Thus,

$$\begin{split} \phi(f+g) & \leq \sup\{|f(x)| + |g(x)| \mid x \in [a,b]\} \\ & \leq \sup\{|f(x)| \mid x \in [a,b]\} + \sup\{|g(x)| \mid x \in [a,b]\} \\ & = \phi(f) + \phi(g) \end{split}$$

We call the functional  $\phi$  defined above the *supremum norm*.

## **Notation**

Let  $f \in C[a, b]$ . We denote the supremum norm of f by  $||f||_{\sup}$ .

