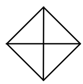


## Why

When does the technique of row reductions prevail?

## Multivariable row reductions

Let $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$. The *$k$th row reduction* of $S$ is the linear system $(C, d)$ with $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$ if $i < s \leq m$ and $C_{st} = A_{st}$ otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable $k$ from equations $k+1, \dots, m$. We are taking the $k$th column of $A$ from

$$\begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ A_{k+1,k} \\ \vdots \\ A_{mk} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We interpret the $i$th row reduction as *subtracting equations* of the system or *reducing rows* of the array $A$. If $a^i, c^i \in \mathbf{R}^n$ denote the $i$th rows of $A$ and $C$, $c^i = a^i - (A_{ik}/A_{kk})a^k$ for $k < i \leq m$, In other words, we obtain the $i$th row of matrix $C$ by subtracting a multiple of the $k$th row of matrix $A$ from the $i$th row of matrix $A$, for $k < i \leq m$. The following is an immediate consequence of real arithmetic.

**Proposition 1.** *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to $(C, d)$. Then $x \in \mathbf{R}^n$ is a solution of $(A, b)$ if and only if it is a solution of $(C, d)$.*

## Ordinary reductions

We call the system $S$ *ordinarily reducible* if there exists a sequence of systems $S_1, \dots, S_{m-1}$ so that $S_1$ is the 11-reduction of $S$ and $S_i$ is the $ii$-reduction of $S_{i-1}$ for $i = 1, \dots, n-1$. In this case, we call $S_{n-1}$ the *final ordinary reduction* (or just *ordinary reduction*) of $S$. The following is an immediate consequence of Proposition 1.

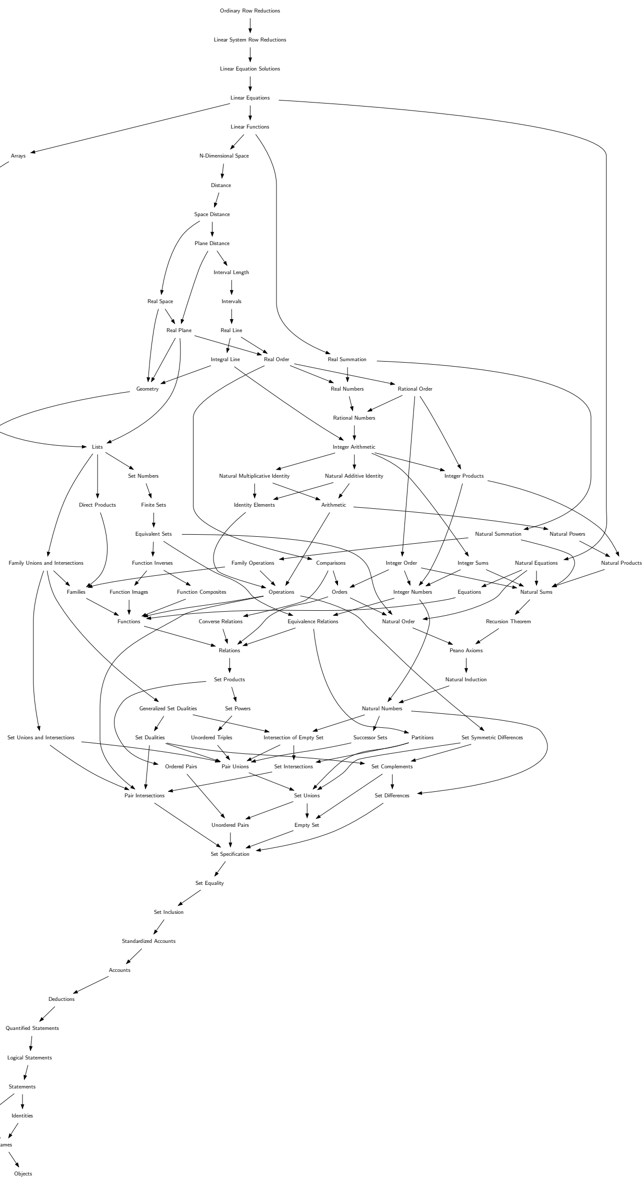
**Proposition 2.** *Let $S'$ be the (final) ordinary reduction of $S$. Then $S$ and $S'$ have equivalent solution sets.*

This process of constructing the ordinary reduction is called *Gauss elimination* or *Gaussian elimination*. We call the $kk$th entry of system $S_{k-1}$ the *pivot*. In an ordinarily reducible system, the pivots are nonzero.

The idea is that a system is ordinarily reducible if we can take row reductions in sequence an end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system $(A, b)$ in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.


Ordinary Row Reductions
Linear System Row Reductions
Linear Equation Solutions
Linear Equations
Linear Functions
Arrays
N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Real Space
Intervals
Real Plane
Real Line
Integral Line
Real Order
Real Summation
Geometry
Real Numbers
Rational Order
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Direct Products
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Natural Summation
Natural Powers
Family Unions and Intersections
Function Inverses
Family Operations
Comparisons
Integer Order
Integer Sums
Natural Equations
Natural Products
Families
Function Images
Function Composites
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Logical Statements
Statements
Sets
Identities
Names
Letters
Objects