

VECTORS AS MATRICES

Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify \mathbf{R}^n with $\mathbf{R}^{n\times 1}$ in the obvious way. For this reason, we call $x\in \mathbf{R}^{n\times 1}$ (meaning $x\in \mathbf{R}^n$) a column vector.

For the reasons that we identify \mathbb{R}^n with $\mathbb{R}^{n\times 1}$, we write the vector $a=(a_1,a_2,a_3)\in \mathbb{R}^3$ as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

We could as easily also identify \mathbb{R}^n with $\mathbb{R}^{1 \times n}$. We avoid this convention. However, by analogy with the language "column vector," we refer to the *matrix* $y \in \mathbb{R}^{1 \times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n\times 1}$ and $\mathbf{R}^{1\times n}$. If $a\in\mathbf{R}^{n\times 1}$, we denote $b\in\mathbf{R}^{1\times n}$ defined by $b_i=a_i$ by a^{\top} .

More generally, given a matrix $A \in \mathbf{R}^{m \times n}$, we denote the matrix $B \in \mathbf{R}^{m \times n}$ defined by $B_{ij} = A_{ji}$ by A^{\top} . Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A, and similarly call a^{\top} the *transpose* of the vector a. Clearly, $(A^{\top})^{\top} = A$, which includes $(a^{\top})^{\top} = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors \mathbf{R}^1 (and so with the 1 by 1 matrices $\mathbf{R}^{1\times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1\times 1}$ where $A_{11} = a$, then Ax = ax.

Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$||x|| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n = x^{\top}y.$$

We express the symmetry of the inner product by $x^{\top}y = y^{\top}x$.

