Why

We characterize chordal graphs using vertex separators, and vice versa.[1]

Main Result

Proposition 1 (Chordal Graphs and Vertex Separators)**.** *An undirected graph is chordal if and only if all minimal vertex separators are complete.*

Proof. Let $G = (V, E)$ be an undirected graph.

First, suppose that all minimal vertex separators of G are complete. Let c be a cycle of length greater than 3. Let v, w be nonconsecutive vertices in c. If v and w are adjacent in G, then $\{v, w\} \in E$ is a chord. If v and w are nonadjacent, then vw-separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any vw-separator T. Split the cycle into the path from v to w, call it p_1 and the path from w to v, call it p_2. T must include an interior point of p_1, call it u_1, otherwise v and w are connected. Similarly, T must include an interior point of p_2, call it u_2. u_1 and u_2 are not consecutive in c, since they are distinct from x and y.

Let S be a minimal vw-separator. Let $s, t \in S$ be two non-consecutive vertices in the cycle different from v and w

[1] Future editions will expand and may include graphics.

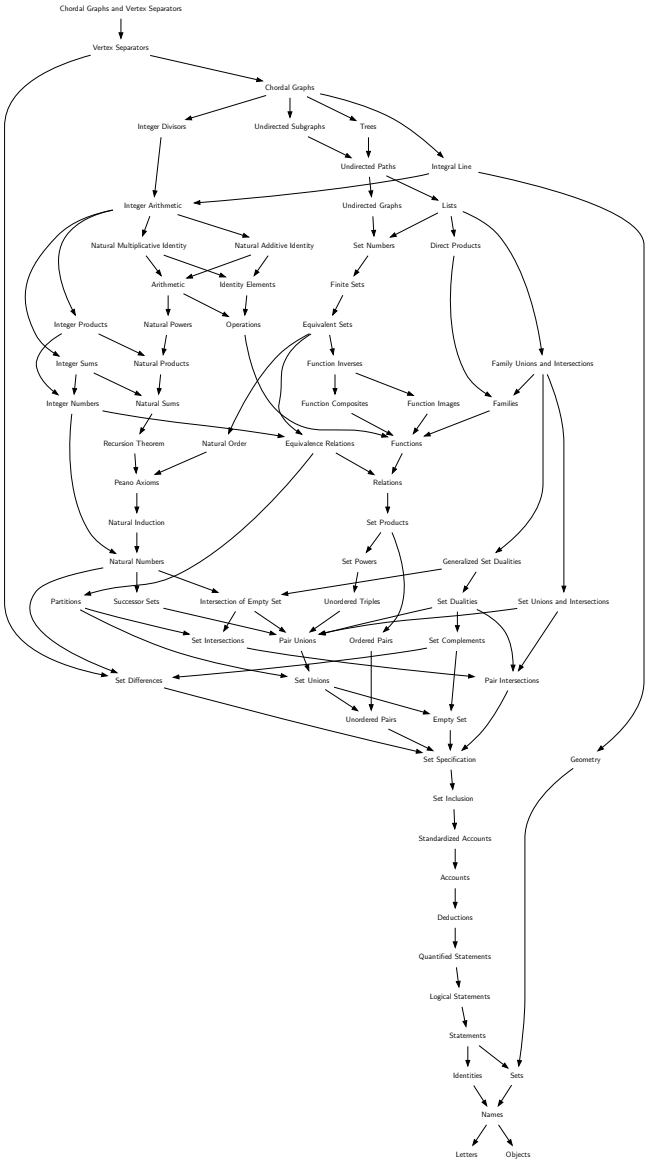
By assumption S is complete, so s and t are adjacent in G.

Second, let $G = (V, E)$ be a chordal graph. Let S be a minimal vw-separator. Let C_v and C_w be the connected components containing v and w of the subgraph induced by $V - S$.

If $|S| = 1$, then S is complete. Otherwise, let $x, y \in S$ by distinct. We want to show $\{x, y\} \in E$. The key insight is that x is adjacent to vertices in C_v and C_w. If there were no such vertex, $S - \{x\}$ would be a vw-separator and S would not be minimal. Similarly with y. Also, $|C_v|, |C_w| \geq 1$.

With these observations, there exists a path from x to y through C_v. Let $p_v = (x, v_1, \ldots, v_k, y)$ be a path of shortest length with at least one interior vertex (so $k \geq 1$) from x to y using interior vertices in $v_1, \ldots, v_k \in C_v$. Let $p_w = (y, w_1, \ldots, w_l, x)$ be a path of shortest length with at least one interior vertex (so $l \geq 1$) from y to x using interior vertices $w_1, \ldots, w_k \in C_w$. Use p_v and p_w to define the cycle $c = (x, v_1, \ldots, v_k, y, w_1, \ldots, w_l, x)$ which has length at least four. G is chordal, so c has a chord.

We argue that the chord of c is $\{x, y\}$. Since C_w and C_V are different connected components (whose vertices are not included in S), there are no edges $\{v_i, w_j\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Since p_v and p_w are paths of shortest length, they have no chords. In particular, there is no edge $\{v_i, v_j\}$ for $|i - j| > 1$ or $\{v_i, x\}$ for $i = 1, \ldots, l$. Similarly, there is no edge $\{w_i, w_j\}$ for $|i - j| > 1$ or $\{w_i, y\}$ for $i = 1, \ldots, l$. The only remaining pair is $\{x, y\}$, and so it must be the chord. $\square$


Chordal Graphs and Vertex Separators
Vertex Separators
Chordal Graphs
Integer Divisors
Undirected Subgraphs
Trees
Undirected Paths
Integral Line
Integer Arithmetic
Undirected Graphs
Lists
Natural Multiplicative Identity
Natural Additive Identity
Set Numbers
Direct Products
Arithmetic
Identity Elements
Finite Sets
Integer Products
Natural Powers
Operations
Equivalent Sets
Integer Sums
Natural Products
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Sums
Function Composites
Function Images
Families
Recursion Theorem
Natural Order
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Intersections
Pair Unions
Ordered Pairs
Set Complements
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Geometry
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects