Why

We want to approximate a given distribution with one which factors according to a tree.

Definition

Given $q : A \to [0,1]$, we want to find a distribution p on A and tree T on $\{1, \ldots, n\}$ to

$$\text{minimize} \quad d_{kl}(q, p)$$
$$\text{subject to} \quad p \text{ factors according to } T.$$

where d_{kl} is the relative entropy as a criterion of approximation. We call such a distribution a *tree distribution approximator* (or *tree approximator*) and we call the tree the *approximator tree*.

Result

Prop. 1. *Let $A_1, \ldots, A_n$ be finite non-empty sets. Define $A = \prod_{i=1}^{n} A_i$. Let $q : A \to [0,1]$ a distribution and T a tree on $\{1, \ldots, n\}$. The distribution $p_T^* : A \to [0,1]$ defined by*

$$p_T^* = q_1 \prod_{i \neq 1} q_{i|\text{pa}\,i}$$

minimizes the relative entropy with q among all distributions on A which factor according to T.

Proof. Let $p : A \to [0,1]$ be a distribution which factors ac-

cording to T. First, express

$$p = p_1 \prod_{i \neq i} p_{i|\mathrm{pa}\,i}$$

where $\mathrm{pa}\,i$ is the parent of vertex i in T rooted at vertex 1 $(i = 2, \ldots, n)$.

Second, recall that the relative entropy of q with p is $H(q, p) - H(q)$. Since $H(q)$ does not depend on p, p is a minimizer of the relative of q with p if and only if p is a minimizer of $H(q, p)$.

Third, express

$$H(q, p) = -\sum_{a \in A} q(a) \log p(a)$$

$$= -\sum_{a \in A} q(a) \left(\log p_1(a_1) + \sum_{i \neq 1} \log p_{i|\mathrm{pa}\,i}(a_i, a_{\mathrm{pa}\,i}) \right)$$

$$= H(q_1, p_1) + \sum_{i \neq 1} \sum_{a_{\mathrm{pa}\,i} \in A_{\mathrm{pa}\,i}} q_{\mathrm{pa}\,i}(a_{\mathrm{pa}\,i}) H\left(q_{i|\mathrm{pa}\,i}(\cdot, a_{\mathrm{pa}\,i}), p_{i|\mathrm{pa}\,i}(\cdot, a_{\mathrm{pa}\,i}) \right)$$

which separates across p_1 an $p_{i|\mathrm{pa}\,i}(\cdot, a_{\mathrm{pa}\,i})$ for $i = 2, \ldots, n$ and $a_{pa_i} \in A_{\mathrm{pa}\,i}$.

Fourth, recall $H(\cdot, \cdot) \geq 0$ and is zero on repeated pairs. By this, we mean, for example, $H(p_1, p_1) = 0$. So $p_1 = q_1$ and $p_{i|\mathrm{pa}\,i} = q_{i|\mathrm{pa}\,i}$ are solutions.

$\square$

Proposition 1 states the form of an optimal approximator given a tree. A natural next question is to select the tree.

Tree Distribution Approximators

Tree Distributions

Distribution Approximators

Relative Entropy

Approximators

Similarity Functions

Metrics

Rooted Tree Distributions

Distance

Absolute Value

Conditional Distributions

Space Distance

Conditional Event Probability

Marginal Distributions

Plane Distance

Event Probabilities

Joint Distributions

Empirical Distribution

Discrete Entropy

Cross Entropy

Interval Length

Rooted Trees

Probability Distributions

Logarithm

Real Space

Intervals

Trees

Directed Graphs

Datasets

Real Plane

Real Line

Undirected Paths

Lists

Undirected Graphs

Real Summation

Real Optimizers

Real Order

Integral Line

Geometry

Set Numbers

Real Numbers

Rational Order

Finite Sets

Optimizers

Rational Numbers

Direct Products

Natural Summation

Integer Arithmetic

Family Operations

Equivalent Sets

Natural Additive Identity

Natural Multiplicative Identity

Integer Products

Set Operations

Family Unions and Intersections

Function Inverses

Algebras

Identity Elements

Comparisons

Arithmetic

Integer Order

Integer Sums

Families

Function Image

Function Composites

Operations

Orders

Natural Powers

Integer Numbers

Functions

Equivalence Relations

Converse Relations

Natural Order

Natural Products

Relations

Natural Sums

Recursion Theorem

Peano Axioms

Uncertain Outcomes

Set Products

Natural Induction

Generalized Set Dualities

Set Powers

Natural Numbers

Set Symmetric Differences

Set Unions and Intersections

Set Dualities

Unordered Triples

Intersection of Empty Set

Successor Sets

Partitions

Set Complements

Pair Unions

Set Intersections

Ordered Pairs

Pair Intersections

Set Differences

Set Unions

Empty Set

Unordered Pairs

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects

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