



Definition

A *hyperplane* is a $(n - 1)$ -dimensional affine set in \mathbf{R}^n .

Discussion

Since the $n - 1$ -dimensional subspaces are the orthogonal complements of the one-dimensional subspaces, they are the sets which can be specified by

$$\{x \in \mathbf{R}^n \mid x \perp b\} + a$$

for $a, b \in \mathbf{R}^n$. The hyperplanes are translates of these,

$$\begin{aligned} \{x \in \mathbf{R}^n \mid x \perp b\} + a &= \{x + a \mid \langle x, b \rangle = 0\} \\ &= \{y \mid \langle y - a, b \rangle = 0\} = \{y \mid \langle y, b \rangle = \beta\}, \end{aligned}$$

where $\beta = \langle a, b \rangle$.

Characterization

Proposition 1. *$H \subset \mathbf{R}^n$ is a hyperplane if and only if there exists $\beta \in \mathbf{R}$ and nonzero $b \in \mathbf{R}^n$ so that*

$$H = \{x \in \mathbf{R}^n \mid \langle x, b \rangle = \beta\}.$$

Remark 1. *b and β are unique up to a common nonzero multiple. For example, b, β and $2b, 2\beta$ give the same hyperplane.*

Remark 2. *Any such vector b is called a normal (or normal vector) to the hyperplane.*

Remark 3. *If H is a hyperplane not containing the origin, then there is a unique unit vector u and $\alpha > 0$ so that*

$$H = \{x \in \mathbf{R}^n \mid \langle x, u \rangle = \alpha\}$$

