



## Why

What of lower upper triangular factorizations when the system is symmetric?

## Definition

A *symmetric lower upper triangular factorization* of  $A$  is a pair of matrices ( $L \in \mathbf{R}^{m \times m}, U \in \mathbf{R}^{m \times m}$ ) where  $L$  is unit lower triangular,  $U$  is upper triangular and  $A = LU$ .

$$A = LL^\top.$$

Other terminology includes *lower upper triangular factorization*, *LU decomposition*, *LU factorization*. Define  $R = L^\top$ , then

$$A = R^\top R.$$

Let  $A \in \mathbf{R}^{n \times n}$ . Then the ordinary row reduction of  $A$  is a matrix  $U$  which is upper triangular. A *lower upper triangular decomposition*  $A$  is a pair of matrices ( $L, L^\top$ ) where  $L \in \mathbf{R}^{n \times n}$  is lower triangular, has nonnegative real diagonal entries, and satisfies

$$A = LL^\top.$$

Other terminology includes *lower upper triangular factorization*, *LU decomposition*, *LU factorization*. Define  $R = L^\top$ , then

$$A = R^\top R.$$

## Basic properties

**Proposition 1.** *Let  $A \in \mathbf{R}^{m \times m}$  be positive definite. Then there exists unique lower triangular matrix  $L \in \mathbf{R}^{n \times n}$  so that*

$$A = LL^\top.$$

So, in the case that  $A$  is positive definite, a lower upper triangular decomposition exists and is unique. Therefore we refer to it as *the upper lower triangular decomposition* of  $A$ . It is also known (universally) as the *Cholesky decomposition* or *Cholesky factorization* of  $A$ .

**Proposition 2.** *If  $A$  is positive semidefinite, there exists a permutation matrix  $P$  for which there is a unique  $L$  so that*

$$P^\top AP = LL^\top.$$

This second proposition says that there is a unique Cholesky decomposition for a particular permutation (or pivoting) of  $A$ .

## Unitriangular form

A *lower diagonal upper decomposition* (or *lower diagonal upper factorization*) of a matrix  $A$  a sequence  $(L, D, L^\top)$  where  $L \in \mathbf{R}^{n \times n}$  is unit lower triangular,  $D \in \mathbf{R}^{n \times n}$  is diagonal with real nonnegative entries and

$$A = LDL^\top.$$

Other terminology includes *LDL decomposition*, *LDL factorization*, *LDU factorization*, *LDU decomposition*.

If  $(L \in \mathbf{R}^{n \times n}, D \in \mathbf{R}^{n \times n}, L^\top)$  is a LDU decomposition of  $A \in \mathbf{R}^{n \times n}$ , then  $(LD^{1/2}A = LDL^\top$  then  $(\tilde{L}D^{1/2}, D^{1/2}L^\top)$  is a LU decomposition. Conversely, if  $(B, B^\top)$  is a LU decomposition and  $S$  is the diagonal matrix satisfying  $S_{ii} = B_{ii}$  for  $i = 1, \dots, n$ , then  $(BS^{-1}, S^2, S^{-1}B^\top)$  is a LDU decomposition of  $A$ .



