



Why

We want to talk about influencing natural phenomena.¹

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_{T-1}$ be sets. For $t = 0, \dots, T-1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \rightarrow \mathcal{X}_{t+1}$. We call the triple

$$((\mathcal{X}_t)_{t=0}^T, (\mathcal{U}_t)_{t=0}^{T-1}, (f_t)_{t=0}^{T-1})$$

a *controlled deterministic discrete-time dynamical system*. We call the index t the *epoch* (or *stage*, *period*).

Let $x_0 \in \mathcal{X}_0$. Let $u_0 \in \mathcal{U}_0, \dots, u_{T-1} \in \mathcal{U}_{T-1}$. Define a state sequence $x_1 \in \mathcal{X}_1, \dots, x_T \in \mathcal{X}_T$ by

$$x_{t+1} = f_t(x_t, u_t).$$

In this case we call x_0 the *initial state*. We call the x_t the *states*. We call the u_t a sequence of *inputs* (or *actions*, *decisions*, *choices*, or *controls*). We call f_t the *transition function* (or *dynamics function*).

We call T the *horizon*. In the case that we have an infinite sequence of state sets, input sets, and dynamics, then we refer to a *infinite-horizon* dynamical system. To use language in contrast with this case, we refer to the dynamical system when T is finite as a *finite-horizon* dynamical system.

State

The current action u_t affects future states x_s for $s > t$, but not the current or past states. The current state x_t depends on the initial state x_0 and the sequence of past actions u_0, \dots, u_{t-1} . So the state is a “link”

¹Future editions will modify, and may restore former editions language: “We want to talk about making decisions over time.” Though this language may also be used in a sheet on *finite* controlled dynamical systems.

between the past and the future. Given x_t and u_t, \dots, u_{s-1} , for $s > t$, we can compute x_s . In other words, the prior actions u_0, \dots, u_{t-1} are not relevant.

This nonrelevancy of prior actions and prior states simplifies the sequential decision problem (see Sequential Decisions).

Variations

The dynamical system is called *time-invariant* if \mathcal{X}_t , \mathcal{U}_t and f_t do not depend on t . A simple variation is that \mathcal{U}_t depends on x_t .²

Examples

Finite dynamical system

A dynamical system is finite if the state and action sets are finite. For example, $\mathcal{X} = \{1, \dots, n\}$ and $\mathcal{U} = \{1, \dots, m\}$. Then $f_t : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{U}$ is called a *transition map*.

Or else, let (V, E) be a directed graph, then $\mathcal{X} = V$, $\mathcal{U}_{x_t} = \{(u, v) \in E \mid u = x_t\}$ and $f_t(x_t, u_t) = v$ when $u_t = (x_t, v)$ is a dynamical system. Roughly this system models “moving” on the graph.

Discrete-time linear dynamical system

Let $\mathcal{X} = \mathbf{R}^n$ and $\mathcal{U} = \mathbf{R}^m$. Define $f_t : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$ by

$$x_{t+1} = f_t(x_t, u_t) = A_t x_t + B_t u_t + c_t$$

for $y = 1, \dots, T - 1$.³

²Future editions will say more here.

³This very special form of dynamics arises in many applications. Future editions will say more.

