CONTROLLED DYNAMICAL SYSTEMS

Why

We want to talk about influencing natural phenomena. [1].

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{T-1}$ be sets. For $t = 0, \ldots,$ $T - 1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \to \mathcal{X}_{t+1}$. We call the sequence

$$\left((\mathcal{X}_t)_{t=0}^{T}, (\mathcal{U}_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1} \right)$$

a *controlled kdeterministic discrete-time dynamical system*. We call the index t the *epoch*, the *stage* or the *period*.

Let $x_0 \in \mathcal{X}_0$. Let $u_0 \in \mathcal{U}_0, \ldots, u_{T-1} \in \mathcal{U}_{T-1}$. Define a state sequence $x_1 \in \mathcal{X}_1, \ldots, x_T \in \mathcal{X}_T$ by

$$x_{t+1} = f_t(x_t, u_t).$$

In this case we call x_0 the *initial state*. We call the x_t the *states*. We call the u_t a sequence of *inputs* (or *actions*, *decisions*, *choices*, or *controls*). We call f_t the *transition function* or *dynamics function*.

We call T the *horizon*. In the case that we have an infinite sequence of state sets, input sets, and dynamics, then we refer to a *infinite-horizon* dynamical system. To use language in

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[1]Future editions will modify, and may restore former editions language: "We want to talk about making decisions over time." Though this language may also be used in a sheet on finite controlled dynamical systems

contrast with this case, we refer to the dynamical system when T is finite as a *finite-horizon* dynamical system.

State

The current action u_t affects future states x_s for $s > t$, but not the current or past states. The current state x_t depends on the initial state x_0 and the sequence of past actions $u_0, \ldots, u_{t-1}$. So the state is a "link" between the past and the future. Given x_t and $u_t, \ldots, u_{s-1}$, for $s > t$, we can compute x_s. In other words, the prior actions $u_0, \ldots, u_{t-1}$ are not relevant.

This nonrelevancy of prior actions and prior states simplifies the sequential decision problem (see Sequential Decisions).

Variations

The dynamical system is called *time-invariant* if $\mathcal{X}_t, \mathcal{U}_t$ and f_t do not depend on t. A simple variation is that $\mathcal{U}_t$ depends on x_t.[2]

Examples

Finite dynamical system

A dynamical system is finite if the state and action sets are finite. For example, $\mathcal{X} = \{1, \ldots, n\}$ and $\mathcal{U} = \{1, \ldots, m\}$. Then $f_t : \mathcal{X} \times \mathcal{U} \to \mathcal{U}$ is called a *transition map*.

Or else, let (V, E) be a directed graph, then $\mathcal{X} = V, \mathcal{U}_{x_t} =$

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[2]Future editions will say more here.

$\{(u, v) \in E \mid u = x_t\}$ and $f_t(x_t, u_t) = v$ when $u_t = (x_t, v)$ is a dynamical system. Roughly this system models "moving" on the graph.

Discrete-time linear dynamical system

Let $\mathcal{X} = \mathbf{R}^n$ and $\mathcal{U} = \mathbf{R}^m$. Define $f_t : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ by

$$x_{t+1} = f_t(x_t, u_t) = A_t x_t + B_t u_t + c_t$$

for $y = 1, \ldots, T - 1$.[3]

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[3]This very special form of dynamics arises in many applications. Future editions will say more.

Controlled Dynamical Systems

N-Dimensional Space

Directed Graphs

Distance

State Representations

Dynamical Systems

Space Distance

Plane Distance

Undirected Graphs

Interval Length

Intervals

Real Line

Real Plane

Real Space

Real Order

Integral Line

Real Numbers

Geometry

Rational Numbers

Sequential Decisions

Integer Arithmetic

Sequences

Decisions

Natural Multiplicative Identity

Natural Additive Identity

Set Numbers

Total Orders

Arithmetic

Identity Elements

Direct Products

Finite Sets

Integer Products

Natural Powers

Operations

Equivalent Sets

Integer Sums

Natural Products

Family Unions and Intersections

Function Inverses

Integer Numbers

Natural Sums

Families

Function Images

Function Composites

Recursion Theorem

Natural Order

Equivalence Relations

Functions

Partial Orders

Peano Axioms

Natural Induction

Uncertain Outcomes

Cartesian Products

Relations

Natural Numbers

Generalized Set Dualities

Set Powers

Set Dualities

Set Unions and Intersections

Successor Sets

Intersection of Empty Set

Unordered Triples

Partitions

Set Complements

Pair Unions

Set Intersections

Ordered Pairs

Set Differences

Pair Intersections

Set Unions

Empty Set

Unordered Pairs

Set Specification

Set Inclusion

Standardised Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects