



Why

We want to define area under a real function. We begin with functions whose area under the curve is self-evident.

Definition

Consider a measure space. The characteristic function of any measurable set is measurable. A simple function is measurable if and only if each element of its simple partition is measurable.

The *integral* of a measurable non-negative simple function is the sum of the products of the measure of each piece with the value of the function on that piece. For example, the integral of a measurable characteristic function of a subset is the measure of that subset.

The *integral operator* is the real-valued function which associates each measurable non-negative simple function with its integral. The simple integral is non-negative, so the integral operator is a non-negative function.

Notation

Let (X, \mathcal{A}, μ) be a measure space.

Let $f : X \rightarrow \mathbf{R}$ be a measurable simple function. So there exist $A_1, \dots, A_n \in \mathcal{A}$ and $a_1, \dots, a_n \in \mathbf{R}$ with:

$$f = \sum_{i=1}^n a_i \chi_{A_i}.$$

We denote the integral of f with respect to measure μ by $\int f d\mu$. We defined:

$$\int f d\mu = \sum_{i=1}^n a_i \mu(A_i).$$

