

#### REAL MATRIX NULLSPACE

#### Definition

The nullspace (or kernel) of a matrix  $A \in \mathbb{R}^{m \times n}$  is the set

$$\{x \in \mathbf{R}^n \mid Ax = 0\}.$$

It is the set of vectors mapped to zero by A. Equivalently, it is the set of vectors orthogonal to the rows of A.

#### Notation

We denote the nullspace of  $A \in \mathbf{R}^{m \times n}$  by  $\text{null}(A) \subset \mathbf{R}^n$ . Some authors denote the nullspace of A by  $\mathcal{N}(A)$ .

## A subspace

The nullspace of a matrix is a subspace (this justifies the terminology nullspace!). There are a few routes to see this.

The first is direct. If  $w, z \in \text{null}(A)$ , then Aw = 0 and Az = 0. So then A(w + z) = Aw + Az = 0. So null(A) is closed under vector addition. Also  $A(\alpha w) = \alpha(Aw) = 0$  for all  $\alpha \in \mathbb{R}$ . [In particular A0 = 0, so  $0 \in \text{null}(A)$ ; i.e., null(A) contains the origin.] So null(A) is closed under scalar multiplication.

The second is by thinking about orthogonal complements. Second, we can view the  $\operatorname{null}(A)$  as the set of vectors orthogonal to all the rows of A. In other words,  $\operatorname{null}(A) = \{\tilde{a}_1, \dots, \tilde{a}_m\}^{\perp}$ . The orthogonal complement of any set is a subspace (see Orthogonal Real Subspaces).

# Ambiguity in solutions

Suppose we have a solution to the system of linear equation with data (A, y). In other words, we have a vector  $x \in \mathbb{R}^n$  so that y = Ax. If we have a vector  $z \in \text{null}(A)$ , then x + z is also a solution to the system (A, y), since

$$A(x+z) = Ax + Az = Ax + 0 = y$$

Conversely, suppose there were another solution  $\tilde{x} \in \mathbb{R}^n$  to the system (A, y). Then  $y = Ax = A\tilde{x}$ , so

$$0 = y - y = Ax - A\tilde{x} = A(x - \tilde{x}).$$

Consequently,  $(x - \tilde{x}) \in \text{null}(A)$ , and so  $\tilde{x}$  is the solution x plus some vector in the null space of A. Consequently we are interested in whether A has vectors in its nullspace.

## Zero nullspace

The origin 0 is always in the nullspace of A. However, this vector does not mean that we can find different solutions, since x + 0 = x for all  $x \in \mathbb{R}^n$ . If, on the other hand, there is a nonzero vector  $z \in \text{null}(A)$ , then  $x + z \neq x$ , and x + z is a solution for (A, y). We think about A as a function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . In the case that there is a nonzero element in the nullspace, A maps different vectors to the same vector. Here, x and x + z both map to y. In this case, the function is not invertible, because it is not one-to-one. If, however, zero is the only element of the null space, the function is one-to-one. So call A one-to-one if null(A) = 0.

## **Equivalent statements**

A matrix  $A \in \mathbb{R}^{m \times n}$  is *one-to-one* if the linear function  $f : \mathbb{R}^n \to \mathbb{R}^m$  defined by f(x) = Ax is one-to-one. In this case, if there exists  $x \in \mathbb{R}^n$  so that y = Ax, then there is only one such x. Different elements in  $\mathbb{R}^n$  map to different elements in  $\mathbb{R}^m$ .

