Why

Which tree is optimal for tree distribution approximation?

Definition

We want to choose a tree whose corresponding approximator for the given distribution minimizes the relative entropy with the given distribution among all tree distribution approximators. We call such a distribution an *optimal* tree approximator of the given distribution. We call a tree according to which an optimal tree approximator factors and *optimal* approximator tree.

Result

Proposition 1. *Let $A_1, \ldots, A_n$ be finite nonempty sets. Define $A = \prod_{i=1}^{n} A_i$. Let $q : A \to [0, 1]$ a distribution. A tree T on $\{1, \ldots, n\}$ is an optimal approximator tree if and only if it is a maximal spanning tree of the mutual information graph of q.*

Proof. First, denote the optimal tree distribution approximator of q for tree T by p_T^*. Express

$$p_T^* = q_1 \prod_{i \neq 1} q_{i|\mathbf{pa}\, i}$$

Second, express $d(q, p) = H(q, p) - H(q)$. Since $H(q)$ does not depend on T, p_T^* is a minimizer (w.r.t. T) of $d(q, p_T^*)$ if and only if it is a minimizer of $H(q, p_T^*)$.

Third, express the cross entropy of p_T^* relative to q as

$$\begin{aligned}
H(q, p_T^*) &= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) \log q_{i|pai}(a_i, a_{\mathsf{pa}\, i}) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i,\mathsf{pa}\, i}(a_i, a_{\mathsf{pa}\, i}) - \log q_{\mathsf{pa}\, i}(a_{\mathsf{pa}\, i})) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i,\mathsf{pa}\, i}(a_i, a_{\mathsf{pa}\, i}) - \log q_{\mathsf{pa}\, i}(a_{\mathsf{pa}\, i}) - \log q \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{i \neq 1} I(q_i, q_{\mathsf{pa}\, i}) \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{\{i,j\} \in T} I(q_i, q_j)
\end{aligned}$$

where $\mathsf{pa}\, i$ denotes the parent of vertex i in T rooted at vertex 1 ($i = 2, \ldots, n$). For $i = 1, \ldots, n$, $H(q_i)$ does not depend on the choice of tree. Therefore selecting a tree to minimize the second term in the final expression above is equivalent to choosing a maximal spanning tree from the weighted graph with mutual information edge weights; namely, the mutual information graph of q.

□

Proposition 1 says that to we should first select a maximum spanning tree of the mutual information graph of the distribution we are approximating. Then, we should pick the best approximator to q which factors according to that three.


Optimal Tree Distribution Approximators
Tree Distribution Approximators
Mutual Information Graph
Tree Distributions
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Approximation
Relative Entropy
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Datasets
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Directed Graphs
Total Orders
Real Space
Real Plane
Real Line
Undirected Paths
Sequences
Real Order
Integral Line
Undirected Graphs
Real Summation
Geometry
Set Numbers
Real Numbers
Direct Products
Rational Numbers
Partial Orders
Finite Sets
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Identity Elements
Arithmetic
Natural Summation
Equivalent Sets
Family Operations
Natural Exponents
Integer Products
Set Operations
Family Unions and Intersections
Function Inverses
Algebras
Natural Products
Integer Sums
Families
Function Images
Function Composites
Operations
Integer Numbers
Natural Sums
Functions
Equivalence Relations
Natural Order
Recursion Theorem
Relations
Peano Axioms
Uncertain Outcomes
Cartesian Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Symmetric Differences
Set Unions and Intersections
Set Dualities
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Complements
Set Interactions
Pair Unions
Ordered Pairs
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects