



# MONOTONIC MATRIX FUNCTIONS

## Why

Since we have a partial order on the set of positive semidefinite matrices, we can study which familiar functions are have order-preserving or order-reversing properties.

## Norms

It would be nice if the matrix norm induced by the matrix scalar produce (see **Matrix Scalar Product**) was an isotonic function. In other words, if  $A, B \in \mathbf{S}^d$  satisfy  $A \geq B$ , does  $\|A\| \geq \|B\|$ ?

Since  $\|A\|^2 = \text{tr } A^2$ , we should study the trace first..

## Trace

**Proposition 1.** *Let  $f : \mathbf{S}^d \rightarrow \mathbf{R}$  defined by  $f(A) = \text{tr } A$ .*

In other words, the function  $f$  is the restriction of the trace function onto the set of symmetric matrices.

**Proposition 2.** *Let  $B \in \mathbf{S}^d$  Let  $f_B : \mathbf{S}^d \rightarrow \mathbf{R}$  defined by  $f(A) = \text{tr } AB$ .*

## Inversion

**Proposition 3.** *Let  $A \in \mathbf{S}_{++}^d$ . Then the map  $f : \mathbf{S}_{++}^d \rightarrow \mathbf{S}_{++}^d$  satisfying  $f(A) = A^{-1}$  is an isotonic function mapping the (open) positive definite cone into itself.<sup>1</sup>*

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<sup>1</sup>Future editions will include a proof.



