

INNER PRODUCT LINEAR FUNCTIONAL REPRESENTATIONS

Why

We can identify any linear functional $F: \mathbb{R}^n \to \mathbb{R}$ with a vector $y \in \mathbb{R}^n$ so that $F(x) = \langle x, y \rangle$. We generalize this result to complete inner product spaces.

Motivating result

The following is known as the Riesz representation theorem (or Riesz-Fréchet representation theorem, or Riesz theorem, or Riesz-Fréchet theorem).

Proposition 1. Let $((V,k), \langle \cdot, \cdot \rangle)$ be a complete inner product space and let $F: V \to k$ be a continuous linear functional on V. There exists a unique $y \in V$ so that

$$F(x) = \langle x, y \rangle$$

for all $x \in V$. Moreover $||y|| = ||F||_*$.

Clearly \mathbb{R}^n is a complete inner product space, and so this this theorem says the expected. We can identify linear functionals on \mathbb{R}^n with elements (vectors) in \mathbb{R}^n .¹

¹Future editions will expand further.

