

#### REAL MATRIX-VECTOR PRODUCTS

# Why

We explore matrix-vector multiplication.

## **Definition**

Given a matrix  $A \in \mathbb{R}^{m \times n}$  and a vector  $x \in \mathbb{R}^n$ , the product of A with x is the vector  $y \in \mathbb{R}^m$  defined by

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \quad i = 1, \dots, m.$$

#### Notation

We denote the product of A with x by Ax. With which we concisely write the system of linear equations (A, b) as b = Ax.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function  $x \mapsto Ax$ . In other words, we are interested in the existence of an inverse element of A. The geometric interpretation is that A transforms the vector x.

Conversely, we can view x as transforming (acting on) A. Let  $a^j \in \mathbb{R}^m$  denote the jth column of A, then

$$Ax = \sum_{j=1}^{n} x_j a^j.$$

In other words, y is linear combination of the columns of A.

### **Properties**

We call the function  $f: \mathbf{R}^n \to \mathbf{R}^m$  defined by f(x) = Ax the matrix multiplication function (or matrix-vector product function) associated with A. f is satisfies the following two important properties:

$$1. \ A(x+y) = Ax + Ay$$

2.  $A(\alpha x) = \alpha Ax$ .

We call such a function f linear. In other words, the matrix multiplication function is linear. Conversely, if  $g: \mathbb{R}^n \to \mathbb{R}^m$  is linear, there exists a matrix inducing g.

**Proposition 1.** Let  $f : \mathbb{R}^n \to \mathbb{R}^m$  be linear. Then there exists a unique  $A \in \mathbb{R}^{m \times n}$  satisfying f(x) = Ax for all  $x \in \mathbb{R}^n$ .

*Proof.* Evaluate f at the standard unit vectors  $e_i$ . The ith component of  $e_i$  is 1 and all other components are 0.

Moreover, this matrix representation of f is unique.

**Proposition 2.** If  $A, B \in \mathbb{R}^{m \times n}$  are two matrices so that f(x) = Ax = Bx, then A = B.

Proof. We have Ax - Bx = 0 so (A - B)x = 0 for every x. In particular  $y^{\top}(A - B)x = 0$  for every  $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ . In particular,  $e_i^{\top}(A - b)e_j = 0$ . Conclusion:  $A_{ij} - B_{ij} = 0$ , and conclude that  $A_{ij} = B_{ij}$ . Thus, A = B. Evaluate f at the standard unit vectors  $e_i$ . The ith component of  $e_i$  is 1 and all other components are 0.

