Why

We discuss a decomposition using eigenvalues and eigenvectors.[1]

Defining result

An *eigenvalue decomposition* of a matrix $A \in \mathbf{R}^{n \times n}$ is an ordered pair (X, Λ) in which X is invertible, Λ is diagonal, and $A = X\Lambda X^{-1}$.

In this case, $AX = X\Lambda$, in other words,

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

in which x_i is the ith column of X and λ_i is the ith diagonal element of Λ. We have $Ax_i = \lambda_i x_i$ for $i = 1, \ldots, n$. In other words, the ith column of X is an eigenvector of A and the jth entry of Λ is the corresponding eigenvalue. If X is orthonormal, so that $X^{-1} = X^\top$, then we can interpret such a decomposition as a change of basis to *eigenvector coordinates*. If $Ax = b$, and $A = X\Lambda X^{-1}$ then $(X^{-1}b) = \Lambda(X^{-1}x)$. Here, $X^{-1}x$ expands x is the basis of columns of X. So to compute Ax , we first expand into the basis of columns of X, scale by Λ, and then interpret the result as the coefficients of a linear combination of the columns of X.

In this case that $A = X\Lambda X^\top$ for an eigenvalue decomposition (X, Λ) of A, we can also write

$$A = X\Lambda X^\top = \sum_{i=1}^{n} \Lambda_{ii} x_i x_i^\top.$$

Proposition 1. *Every real symmetric matrix has an eigenvalue decomposition (X, Λ) in which X is orthonormal.*[2]

[1] Future editions will expand.

[2] In future editions, this may be the motivating result for the definition of eigenvalues.

Eigenvalue Decomposition
Orthogonal Matrices
Orthonormal Set of Vectors
Matrix Transpose
Real Inner Product Norms
Eigenvalues and Eigenvectors
Orthonormal Set of Real Vectors
Real Inner Products
Norms
Vectors as Matrices
Matrices
Matrix Determinants
Real Inner Product
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Plane Vector Sums and Differences
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Absolute Value
Plane Vectors
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Real Square Roots
Right Triangle Sides Relation
Space Distance
Marc Polynomials
Real Squares
Real Arithmetic
Plane Distance
Real Multiplication Inverses
Interval Length
Natural Square Roots
Real Products
Intervals
Real Space
Square Numbers
Real Additive Inverses
Real Line
Real Plane
Real Summation
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Peano Axioms
Natural Induction
Set Products
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Set Powers
Generalized Set Duaities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Duaities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects