



Why

We want to describe how fast a function grows or declines.¹

Definition

Let $f : \mathbf{R} \rightarrow \mathbf{R}$. The *lower growth class* of f (*toward infinity*) is the set of all functions $g : \mathbf{R} \rightarrow \mathbf{R}$ for which there exists $C, M > 0$ so that $|g(x)| \leq C|f(x)|$ for all $x > M$. The intuition is that if $h : \mathbf{R} \rightarrow \mathbf{R}$ is in the lower growth class of f , h does not grow faster than f . In this case we say that h *grows at order* f .

The *lower limit class* of f at x_0 is the set of all functions $g : \mathbf{R} \rightarrow \mathbf{R}$ for which there exists $C, \varepsilon > 0$ so that $|g(x)| \leq C|f(x)|$ for all $|x - x_0| < \varepsilon$. The intuition is that for x sufficiently close to x_0 , the magnitude of f is bounded by a constant times the magnitude of g . Often x_0 is 0.

The *upper growth class* of f (*toward infinity*) is the set of all functions $g : \mathbf{R} \rightarrow \mathbf{R}$ for which there exists $C, M > 0$ so that $|g(x)| \geq C|f(x)|$ for all $x > M$. The intuition is that if h is in the upper growth class of f , h grows at least as fast as f . We similarly define the *upper growth class* at a limit x_0 .

The (*exact*) *growth class* of f is the set of all functions $g : \mathbf{R} \rightarrow \mathbf{R}$ for which there exists C_1, C_2, M so that $C_1|f(x)| \leq |g(x)| \leq C_2|f(x)|$ for all $x > M$. The intuition is that if h is in the growth class of f , then h and f grow at the same rate. Again, we similarly define the *growth class* at limit x_0 .

Notation

We denote the upper, lower and exact growth classes of a function $f : \mathbf{R} \rightarrow \mathbf{R}$ by $O(f)$, $\Omega(f)$ and $\Theta(f)$, respectively. We read the notation $O(f)$ as “order at most f ,” we read $\Omega(f)$ as “order at least f ,” and $\Theta(f)$ as “order exactly f .”

¹Future editions will expand this vague introduction.

The letter O is a mnemonic for order, and Ω and Θ build on this mnemonic. The term order appears to arise from the use of growth classes when discussing Taylor approximations. In this case of small x (i.e., $|x| < 1$), $|x^p| < |x^q|$ if $q < p$ and so higher order terms are “smaller” and “negligible.” This notation is sometimes called *Big O notation*, *Landau’s symbol*, *Landau notation* or *Landau’s notation*.

Let $\phi, \psi : \mathbf{R} \rightarrow \mathbf{R}$. Many authors use $\phi = O(\psi)$ or $\phi(t) = O(\psi(t))$ to assert that ϕ is in the upper growth class of ψ at some understood limit (e.g., 0 or ∞). In other words, the equation asserts that there exists some positive constant $C > 0$ so that, for all t sufficiently close to the understood limit, $|\phi(t)| \leq C|\psi(t)|$.² For example, the statement $\sin^2(t) = P(t^2)$ as $t \rightarrow 0$ (or for $t \rightarrow 0$) means that there exists constants $C, \varepsilon > 0$ so that, $|t| < \varepsilon \rightarrow |\sin^2(t)| \leq Ct^2$.

²Often also defined $|\phi(t)| < C\psi(t)$, with no absolute value on ψ .

