



GEOMETRIC SERIES

Why

It is believable that $1/2, 1/4, 1/8, \dots$ has a convergent series. And likewise with $1/3, 1/9, 1/27, \dots$. What of $a_k = x^k$ for $x \in \mathbf{R}$.

Definition

Let $x \in \mathbf{R}$. The *geometric series* of x is the series of the sequence (a_k) defined by $a_k = x^k$.

Characterization of convergence

Does the geometric series of x converge? In other words, does (s_n) defined by $s_n = \sum_{k=1}^n x^k$ have a limit.

For $x = 1$ and $x = -1$, we have see (see **Real Series**) that the series diverges. However for the cases $x = 1/2$ and $x = 1/3$ the geometric series converges.

Proposition 1. *If $|x| < 1$, then the geometric series of x converges and*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n x^k = \frac{x}{1-x}$$

If $|x| \geq 1$ then the geometric series of x diverges.

Proof. Define $s_n = \sum_{k=1}^n x^k$. Then

$$\begin{aligned} x \cdot s_n &= x \cdot (x^1 + x^2 + \dots + x^n) \\ &= x^2 + x^3 + \dots + x^{n+1} \\ &= s_n - x + x^{n+1}. \end{aligned}$$

From which we deduce, $s_n(1 - x) = x(1 - x^n)$. If $x \neq 1$, then

$$s_n = \frac{1}{1 - x}(1 - x^n)$$

If $|x| < 1$, then using the algebra of limits (see Real Limit Algebra) we deduce

$$\lim_{n \rightarrow \infty} \frac{1}{1 - x}(1 - x^n) = \frac{1}{1 - x}(1 - 0) = \frac{1}{1 - x},$$

since $\lim_{k \rightarrow \infty} x^k = 0$ for $|x| < 1$.

If $x = 1$ or $x = -1$, then we have seen that (s_n) diverges.¹

□

¹Future editions will include the trivial account about the case $|x| > 1$.

