Set Equality

1 Why

When are two sets the same?

2 Definition

Consider the sets A and B. If A is B, then every element of A is an element of B and every element of B is an element of A.

What of the converse? If every element of A is an element of B and vice versa is A the same as B? We declare the affirmative.

Thus: Two sets are the same if and only if they have the same elements. We call the elements of a set its *extension* and this above statement is sometimes called the *axiom of extension*.

The importance is that we have given ourselves a way to argue two sets are equivalent. Argue the consequence of the first paragraph, and the use the axiom of extension to conclude that the sets are the same.

2.1 Notation

Let A and B be two sets. As with any objects, we denote that A and B are equal by $A = B$. The axiom of extension is

$$A = B \Leftrightarrow (a \in A \implies a \in B) \wedge (b \in B \implies b \in A).$$

3 A Contrast

It is useful to compare the axiom of extension for sets with elements to a an analogue for human beings with ancesors (parents, grandparents, and so on). If two human beings are equal, then they have the same set of ancestors. (this is analogue of the the "only if" part of the axiom of extension). However, if two human beings have the same set of ancestors, they need not be the same (this is the analogue of the "if" part of the axiom of extension). Siblings have the same ancestors, but are different people.

So we conclude that the axiom of extension is not just a logically necessary consequence of equality. In fact, it contains aspects of our notion of belonging.