



# Tail Measure Lower Bound

## 1 Why

## 2 Result

We bound below the measure that a non-negative measurable real-valued function exceeds some value by its integral.

**Proposition 1.** *Let  $(X, \mathcal{A}, \mu)$  be a measure space. Let  $g : X \rightarrow [0, \infty]$  be measurable and square-integrable. Then for all  $t$  such that  $\int t d\mu \in [0, \int g d\mu)$ ,*

$$\mu(\{x \in X \mid g(x) > t\}) \geq \frac{(\int (g - t) d\mu)^2}{\int g^2 d\mu}.$$

*Proof.* Let  $t$  such that  $\int t d\mu \in [0, \int g)$ . We have selected  $t$  so that  $\int (g - t) d\mu \geq 0$ . Define  $h = (g - t)^+$  and  $A = \{x \in X \mid h(x) > 0\}$ . Then

$$\int (g - t) d\mu \leq \int h d\mu = \int h \chi_A d\mu \leq \sqrt{\int h^2 d\mu \int \chi_A^2 d\mu}$$

□

Now  $g^2 > h^2$ , so  $\int g^2 d\mu \geq \int h^2 d\mu$ . Also  $\chi_A^2 = \chi_A$  so  $\int \chi_A^2 = \mu(A)$ .  $h(x) > 0$  if and only if  $g(x) \geq t$  for all  $x$ . So  $A = \{x \in X \mid g(x) \geq t\}$ . Combining we have:

$$\int (g - t) d\mu \leq \sqrt{\left(\int g^2 d\mu\right) \mu(A)}.$$

**Proposition 2.** *Let  $X$  be a random variable with  $\mathbf{E}(X^2) \leq \infty$ . Then for all  $t \in [0, \mathbf{E}(X))$ , we have*

$$P(X > t) \geq \frac{(\mathbf{E}(X) - t)^2}{\mathbf{E} X^2}.$$

The above is also called the *Paley-Zygmund Inequality*.