



# Measure Vector Space

## 1 Why

If both signed measures are finite, then their difference is always well-defined. Is the difference a finite signed measure?

## 2 Preliminary Result

**Proposition 1.** *A linear combination of finite signed measures is a finite signed measure.*

*Proof.* Let  $(X, \mathcal{A})$  be a measurable space. Let  $\mu$  and  $\nu$  be finite signed measures. Let  $R$  denote the real numbers. Then  $(\alpha\mu)(\emptyset) = \alpha \cdot \mu(\emptyset) = \alpha \cdot 0 = 0$ . Also for  $(A_n)_n \subset \mathcal{A}$  disjoint,

$$\begin{aligned} (\alpha\mu)(\cup A_n) &= \alpha\mu(\cup A_n) = \alpha \sum_{n=1}^{\infty} \mu(A_n) \\ &= \sum_{n=1}^{\infty} \alpha\mu(A_n) = (\alpha\mu)(A_n) \end{aligned}$$

.

Similarly,  $(\mu+\nu)(\emptyset) = \mu(\emptyset)+\nu(\emptyset) = 0$ . And, for  $(A_n)_n \subset \mathcal{A}$

disjoint,

$$\begin{aligned}
(\mu + \nu)(\cup A_n) &= \mu(\cup A_n) + \nu(\cup A_n) = \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \nu(A_n) \\
&= \sum_{n=1}^{\infty} \mu(A_n) + \nu(A_n) = \sum_{n=1}^{\infty} (\mu + \nu)(A_n)
\end{aligned}$$

. □

### 3 Main Result

**Proposition 2.** *The set of finite signed measures is a vector space.*

*Proof.* Use the previous proposition. Observe that the function  $\mu \equiv 0$  is a measure. and  $\nu + \mu = \nu$  for all measures  $\nu$ . □

#### 3.1 Notation

Denote the set of real numbers by  $R$ . We denote the vector space of signed measures on measurable space  $(X, \mathcal{A})$  by  $M(X, \mathcal{A}, R)$ .