Why

We want to discuss real-valued random functions whose family of random variables have simple densities.[1]

Definition

A *normal random function* is a real-valued random function whose family of real-valued random variables has the property that any subfamily is jointly normal.

For this reason, we call the family of random variables (or stochastic process) corresponding to the random function a *gaussian process* or *normal process*.

Notation

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and A a set. Let $x : \Omega \to (A \to \mathbf{R})$ be a random function with family $y : A \to (\Omega \to \mathbf{R})$.

The random function x is a normal if, for all $a^1, \ldots, a^m \in A$, $(y(a^1), \ldots, y(a^m))$ is jointly normal.

Mean and covariance function

Proposition 1. *Let $x : \Omega \to (A \to \mathbf{R})$ be a normal random function with family $X : A \to (\Omega \to \mathbf{R})$. There exists unique functions $m : A \to \mathbf{R}$ and $k : A \times A \to \mathbf{R}$ so that the mean of the random variable X_a is $m(a)$ for all A and the covariance of the random variables X_a and $X_{a'}$ is $k(a, a')$ for all $a, a' \in A$.*[2]

For this reason, we call m the *mean function* and k the *covariance function* of the random function.

Conversely, let $m : A \to \mathbf{R}$ and $k : A \times A \to \mathbf{R}$. Then if k satisfies the

[1] Future editions will expand.
[2] Future editions may include an account.

property that for all $a^1, \ldots, a^m$, the $m \times m$ matrix

$$\begin{pmatrix} k(a^1, a^1) & \cdots & k(a^1, a^m) \\ \vdots & \ddots & \vdots \\ k(a^m, a^1) & \cdots & k(a^m, a^m) \end{pmatrix}$$

is positive semidefinite, then we can construct a Gaussian process with mean function m and covariance function k.[3] For this reason, we call k with such a property *positive semidefinite* or a *covariance function.* Notice, of course, that k is symmetric. The matrix above is sometimes called the *Gram matrix* for k and $a^1, \ldots, a^m$.

Example

Let $A = \{1, \ldots, n\}$ and let $K \in \mathbf{R}^{n \times n}$ be symmetric positive semidefinite. Define $m : A \to \mathbf{R}$ to be $m \equiv 0$ (the constant zero function) and $k(i, j) = K_{ij}$. Then the normal random function $x : \Omega \to (A \to \mathbf{R})$ with mean m and covariance k is in one to one correspondence with the gaussian random vectors with mean zero.

[3] Some authors belabor this point because of the natural inclination to want to specify an *inverse* covariance function, which need not satisfy the consistency property. The consistency property ensures that any marginal of a subfamilys density is the density of that further subfamily. Future editions may expand.

Normal Random Functions
Random Functions
Multivariate Normals
Positive-Definite Matrices
Quadratic Forms
Symmetric Matrices
Real Matrix Inverses
Multivariate Real Distribution
Matrix Trace
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Real Matrix-Matrix Products
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Natural Square Roots
Absolute Value
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Real Arithmetic
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Simple Functions
Probability Distributions
Interval Length
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Real Functions
Real Sequences
Real Accumulation
Intervals
Real Space
Real Products
Sequences
Real Line
Real Plane
Real Addition Inverses
Real Lim
Real Order
Real Sums
Rational Multiplication Inverses
Rational Arithmetic
Permutations
Integral Lim
Lim
Geometry
Real Numbers
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Element Functions
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Natural Accumulation
Equivalent Sets
Natural Multiplication Identity
Natural Addition Identity
Integer Products
Family Operations
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Identity Elements
Commutation
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Natural Order
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Relations
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Recursive Theorems
Characteristic Functions
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Natural Induction
Generalized Set Unions
Unordered Triples
Natural Numbers
Set Symmetric Differences
Set Unions
Set Unions and Intersections
Successor Sets
Intersection of Empty Set
Partitions
Set Complements
Pair Unions
Set Intersections
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Arguments
Axioms
Deductions
Quantified Statements
Logical Statements
Statements
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