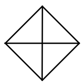


Why

We extend our notion of length, area, and volume beyond the Lebesgue measure on the product spaces of real numbers.

Definition

An extended-real-valued non-negative function on an *algebra* is *finitely additive* if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a *sigma algebra* is *countably additive* if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A *finitely additive measure* is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A *countably additive measure* is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call countably additive measures *measures*, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countably additive.

In the context of measure, we call a countably unitable subset algebra a *measurable space.* We call the distinguished sets *measurable* sets. A *measure space* is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of the measurable space.

Notation

Suppose A a nonempty set and $\mathcal{A}$ is a sigma algebra on A so that the pair $(A, \mathcal{A})$ is a measurable space.

Let $\mu : \mathcal{A} \to [0, \infty]$ a measure; thus: (a) $\mu(\varnothing) = 0$ and (b) for disjoint $\{A_n\} \subset \mathcal{A}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ The triple $(A, \mathcal{A}, \mu)$ is a measure space.

We use μ since it is a mnemonic for "measure". We often also us ν to denote measures, since it is after μ in the Greek alphabet, and λ, since it is before μ in the Greek alphabet.

Examples

Example 1. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure. We call μ the* counting measure.

Example 2. *Let $(A, \mathcal{A})$ measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise. Then μ is a measure. We call μ the* point mass *concentrated at a.*

Example 3. *The Lebesgue measure on the measurable space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is a measure.*

Example 4. *Let $\mathcal{A}$ the co-finite algebra on N. Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.*

Example 5. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ be 0 if $A = \varnothing$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.*

Example 6. *Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = \mathcal{P}(A)$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \varnothing$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.*

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \varnothing$ then using finite additivity we obtain a contradiction $1 = (B \cup C) = (B) + (C) = 2$. $\square$

Why

We want to generalize the notions of length, area, and volume.

Definition

A *measurable space* is a sigma algebra. We call the distinguished subsets the *measurable sets*. A *measure* on a measurable space is a function from the sigma algebra to the positive extended reals. A *measure space* is a measurable space and a measure.

Notation

Properties

Prop. 1. *Let $(A, \mathcal{A})$ be a measurable space and $m : \mathcal{A} \to [0, \infty]$ be a measure.*

If $B \subset C \subset A$, then $m(B) \leq m(C)$. We call this property the of measures monotonicity of measure.

Prop. 2. *For a measure space $(A, \mathcal{A}, m)$.*

If $B \subset C \subset A$, then $m(B) \leq m(C)$.

We call this property the monotonicity of measure.

Prop. 3. *For a measure space $(A, \mathcal{A}, m)$.*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

We this property the sub-additivty of measure.

Prop. 4. *For a measure space $(A, \mathcal{A}, m)$.*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

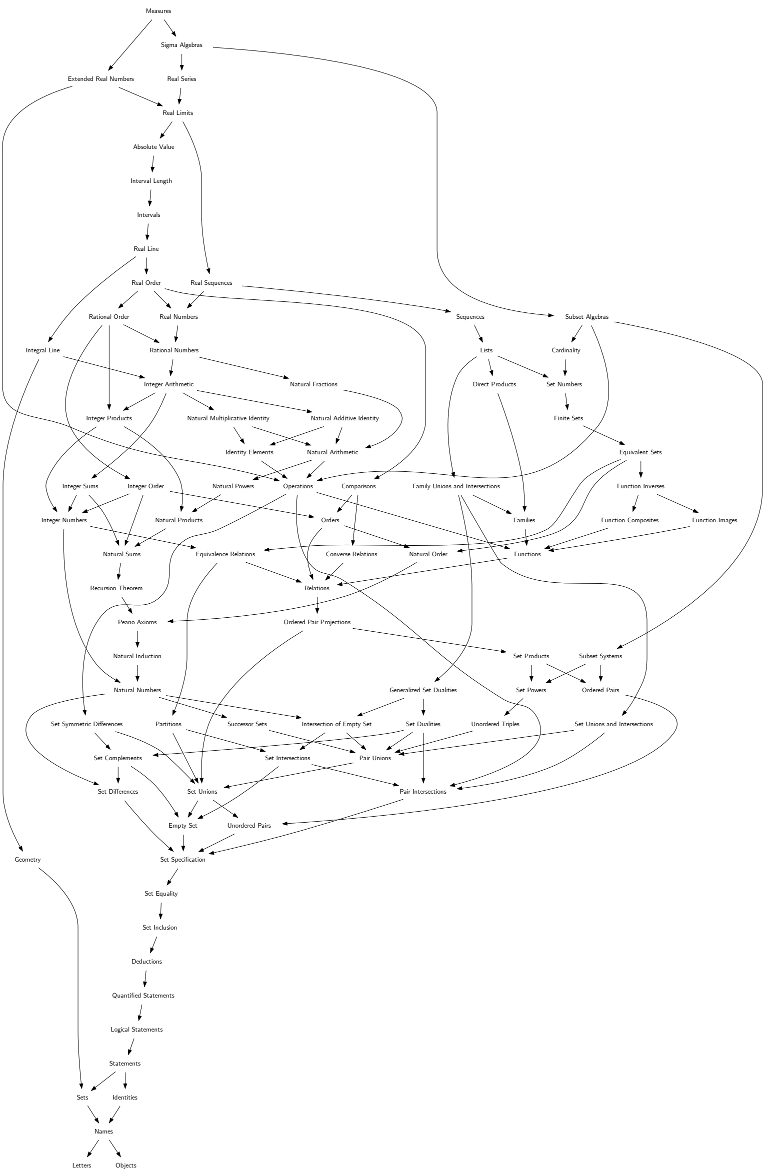
We this property the sub-additivty of measure.

Prop. 5. *For a measure space* $(A, \mathcal{A}, m)$.

$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

Prop. 6. *For a measure space* $(A, \mathcal{A}, m)$.

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$


Measures
Sigma Algebras
Extended Real Numbers
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Rational Order
Real Numbers
Sequences
Subset Algebras
Integral Line
Rational Numbers
Lists
Cardinality
Integer Arithmetic
Natural Fractions
Direct Products
Set Numbers
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Integer Sums
Integer Order
Natural Powers
Operations
Comparisons
Family Unions and Intersections
Function Inverses
Integer Numbers
Natural Products
Orders
Families
Function Composites
Function Images
Natural Sums
Equivalence Relations
Converse Relations
Natural Order
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Subset Systems
Natural Numbers
Generalized Set Dualities
Set Powers
Ordered Pairs
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects