Why

We want to model uncertain outcomes in dynamical systems.[1]

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{T-1}$ be sets. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Let $\mathcal{W}_0, \ldots, \mathcal{W}_T$. Let $w_t : \Omega \to \mathcal{W}_t$ for $t = 0, \ldots, T$ be random variables. For $t = 0, \ldots, T - 1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathcal{X}_{t+1}$.

We call the sequence

$$\mathcal{D} = ((\mathcal{X}_t)_{t=0}^T), (\mathcal{U}_t)_{t=0}^{T-1}, (w_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1})$$

a *stochastic discrete-time dynamical system*. We call w_t the *noise* variables.

Problem

Let $x_0 : \Omega \to \mathcal{X}_0$ be a random variable. Define $x_1 : \Omega \to \mathcal{X}_1$, ..., $x_T : \Omega \to \mathcal{X}_t$ by

$$x_{t+1} = f_t(x_t, u_t, w_t)$$

for $t = 0, \ldots, T - 1$. Roughly speaking, the state transition functions are nondeterministic. In other words, it is uncertain which state we will arrive in given our current state and action. The choice u_t only determines the distribution of x_{t+1}. Here x_0 is (still) called the *initial state* and is a random variable, usually assumed independent of the w_t.

Let $g_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathbf{R} \cup \{\infty\}$ for $t = 0, \ldots, T - 1$ and $g_T : \mathcal{X}_T \times \mathcal{W}_T \to \mathbf{R} \cup \{\infty\}$. We call $(x_0, \mathcal{D}, (g_t)_{t=0}^T)$ a *stochastic dynamic optimization problem*. As with dynamic optimization problems, we call g_t the *stage cost function* and g_T the *terminal cost function*. It is common for these to not depend on w_T (in other words, to be deterministic). It is also common for these to take infinite values to encode constraints.

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[1] Future editions will expand.

As before, a stochastic dynamic optimization problem is just an optimization problem. Define $U = \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_{T-1}$ and let $u \in U$. Define $C : \Omega \to \mathbf{R}$ by

$$C = \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T).$$

We call C the *total cost* for actions u. It is a random variable.

Define $J : U \to \mathbf{R} \cup \{\infty\}$ by

$$J = \mathbf{E}\left(\sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T) \right).$$

$J(u)$ is the *expected total cost* for inputs u.

The optimization problem is (U, J). In other words, the objective is the mean total stage cost plus the terminal cost.

Other terminology

Stochastic dynamic optimization problems are frequently called *stochastic control problems*.