

NORM WEIGHTED LEAST SQUARES LINEAR REGRESSORS

Why

What is the best linear predictor if we choose according to a particular norm.

Definition

Suppose we have a paired dataset of n records with inputs in \mathbf{R}^d and outputs in \mathbf{R} . A norm weighted least squares linear predictor for a norm $g: \mathbf{R}^n \to \mathbf{R}$ is a linear transformation $f: \mathbf{R}^d \to \mathbf{R}$ (the field is \mathbf{R}) which minimizes

$$g(y - Ax)$$
.

Weight matrix

Let $\|\cdot\|_W$ be the weighted norm for some positive semidefinite weight matrix W. We want to find x to minimize

$$||y - AX||_W$$
.

This problem is referred to by many authors as weighted least squares or the weighted least squares problem.

Diagonal weight matrix

A special case of norm weighted least squares with a weighted norm is the usual weighted least squares problem (see Weighted Least Squares Linear Predictors). Consider weighted least squares with weights $w \in \mathbb{R}^n$, $w \geq 0$. Define $W \in \mathbb{R}^{n \times n}$ so that $W_{ii} = w_i$ and $W_{ij} = 0$ when $i \neq j$. So, in particular, W is a diagonal matrix and

$$||y - Ax||_W = \sum_{i=1}^n w_i (y_i - x^\top a_i)^2.$$

Solution

Proposition 1. There exists a unique weighted least squares linear predictor and its parameters are given by

$$(A^\top W A)^{-1} A^\top W y.$$

