## Why

Many functions of interest are additive and homogenous.

## Definition

A transformation is *linear* (a *linear transformation*, *linear map*) if the result of a linear combination of the two vectors is the linear combination of the results of the vectors (using the same coefficients). The transformation is linear *with respect to* the field of the two vector spaces.

We use the term transformation (see Transformations) to emphasize and remind that the function is defined on a *vector* space. Of course, **R** is a vector space and so a function $f : \mathbf{R} \to \mathbf{R}$ may be linear. The linear maps from **R** to **R** are the the *linear functions* (see Real Linear Functions).

## Notation

Let $(V, \mathbf{F})$ and $(W, \mathbf{F})$ be two vector spaces over the same field. Suppose $T : V \to W$. $T$ is linear means

$$T(\alpha u + \beta v) = \beta T(u) + \alpha T(v) \quad \text{for all } \alpha, \beta \in F \text{ and } u, v \in V.$$

As usual, the condition that $T$ is linear condition is equivalent to the two conditions:

1. $T(u + v) = f(u) + T(v)$ for all $u, v \in V$, and

2. $f(\lambda u) = \lambda f(u)$ for all $\lambda \in \mathbf{F}$ and $u \in V$.

If $T$ satisfies (1), we call $T$ *additive* (has the property of *additivity*). If $T$ satisfies (2) we call $T$ *homogeneous* (has the property of *homogeneity*).

For linear maps, it is common to denote $T(v)$ by $Tv$; notice that we have dropped the usual parentheses.

We denote the set of all linear maps by $\mathcal{L}(V, W)$. It is understood when using this notation that $V$ and $W$ are vector spaces with respect to the same field **F**.

**Examples**

Throughout, we consider vector spaces $V$ and $W$ over some fixed field $\mathbf{F}$.

*Constant zero map.* The map $T \in \mathcal{L}(V, W)$ defined by

$$T(v) = 0 \in W \quad \text{for all } v \in V$$

is called the *zero map* (or *zero transformation*). It is common to overload the symbol 0 so that $0 \in \mathcal{L}(V, W)$ denotes the zero map. In other words, the map 0 is defined by

$$0v = 0$$

Some care is required to interpret this equation. The 0 on the left hand side refers to a function, from $V$ to $W$. The 0 on the right hand side is the additive identity in $W$. Usually context disambiguates the overloaded notation.

*The identity map.* The map $T \in \mathcal{L}(V, V)$ defined by

$$Tv = v \quad \text{for all } v \in V$$

is called the *identity map* (or *identity transformation*) It is common to denote this map by $I$.

*Differentiation of polynomials* Suppose $P$ is the set of all polynomials with coefficients in $\mathbf{R}$. (Some authors denote this set by $\mathcal{P}(\mathbf{R})$. Recall that every $p \in \mathcal{P}(\mathbf{R})$ is differentiable and $p' \in \mathcal{P}(\mathbf{R})$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ defined by

$$Tp = p'$$

is linear. To see this, recall $(f + g)' = f' + g'$ and $(\lambda f)' = \lambda f'$ whenever $f, g$ are differentiable and $\lambda \in \mathbf{R}$ (see Derivative of Sumsand Derivatives of Scalar Multiples) .

*Integration of polynomials* As in the previous paragraph, $\mathcal{P}(\mathbf{R})$ denotes the vector space of polynomials with coefficients in $\mathbf{R}$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$ defined by

$$Tp = \int_{[0,1]} p$$

is linear To see this, recall that $\int (f + g) = \int f + \int g$ and $\int \lambda f = \lambda \int f$ whenever $f, g$ are differentiable and $\lambda \in \mathbf{R}$ (see Real Integral Additivityand Real Integral Homogeneity.

*Multipliciation by a quadratic.* As in the previous paragraph, $\mathcal{P}(\mathbf{R})$ denotes the vector space of polynomials with coefficients in $\mathbf{R}$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ defined by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbf{R}, p \in \mathcal{P}(\mathbf{R})$$

is linear. (Prove this).

*Sequence backward shift.* Denote the space of infinite sequences in a field $\mathbf{F}$ by $\mathbf{F}^{\mathbf{N}}$ as usual. Define $T \in \mathcal{L}(\mathbf{F}^{\mathbf{N}}, \mathbf{F}^{\mathbf{N}})$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

$T$ is called the *backward shift operator.*

*From real space the the real plane.* Define $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

*From $\mathbf{F}^n$ to $\mathbf{F}^m$.* Generalizing the previous example, suppose $m$ and $n$ are natuarl numbers, and let $A_{i,j} \in \mathbf{F}$ for $i = 1, \dots, m$ and $j = 1, \dots, m$. Define $T \in \mathcal{L}(\mathbf{F}^3, \mathbf{F}^2)$ by

$$T(x_1, \cdots, x_n) = (A_{1,1}x + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

(It happens that every linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$ has this form.)

*A counterexample:* $\cos$[1] Notice $\cos(x + y) = \cos(x) + \cos(y)$. True, cos is not homogenous. that $\cos 2x = 2\cos(x)$ and But this does not hold for all reals: $\cos \lambda x \neq \lambda \cos(x)$.

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[1]Need to add a sheet for trigonometric functions.

Linear Transformations

Real Polynomial Derivatives · Derivatives of Scalar Multiples · Derivatives of Sums · Real Integral Additivity · Real Integral Homogeneity

Real Linear Functions · Real Differentiable Functions · Simple Integral Additivity · Simple Integral Homogeneity

Vector Space of Polynomials · Transformations · Linear Combinations · Simple Integrals

Real Polynomials · Fields · Real Functions · Vectors · Real Linear Combinations · Simple Functions · Measurable Functions

Length Measure · Measures · Real Vector

Real Arithmetic · Rings · Sigma Algebras · N-Dimensional Space · Plane Vector Sums and Differences · Space Vectors

Real Series · Extended Real Numbers · Distance · Plane Vectors

Real Multiplicative Inverse · Real Limits · Space Distance

Absolute Value · Plane Distance

Real Products · Interval Length · Real Space

Real Additive Inverse · Intervals · Real Plane

Rational Multiplicative Inverse · Rational Arithmetic · Real Sums · Real Line · Real Order · Integral Line · Real Sequences

Rational Products · Rational Order · Rational Sums · Real Numbers · Subset Algebras · Sequence · Geometry

Rational Numbers · Cardinality · Lists

Integer Arithmetic · Natural Fractions · Set Numbers · Direct Products

Integer Products · Natural Additive Identity · Natural Multiplicative Identity · Finite Sets · Inverse Elements

Natural Arithmetic · Identity Elements · Equivalent Sets · Element Functions

Integer Sums · Integer Order · Natural Powers · Operations · Function Inverses · Comparisons · Family Unions and Intersections

Integer Numbers · Natural Products · Orders · Function Images · Function Composites · Families

Natural Sums · Equivalence Relations · Natural Order · Functions · Converse Relations

Recursion Theorem · Relations

Peano Axioms · Ordered Pair Projections

Natural Induction · Characteristic Functions · Set Products · Subset Systems

Natural Numbers · Generalized Set Dualities · Set Powers · Ordered Pairs

Set Symmetric Differences · Partitions · Intersection of Empty Set · Successor Sets · Set Dualities · Unordered Triples · Set Unions and Intersections

Set Complements · Set Intersections · Pair Unions

Set Differences · Set Unions · Pair Intersections

Empty Set · Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects