Why

We can approximate a density with a tree density similar to how we can approximate a distribution with a tree distribution.

Definition

We use the differential relative entropy as a criterion of approximation. An *optimal tree approximator* of density for a tree is a density which factors according to a tree and minimizes its differential relative entropy with the given density.

Notation

Let $g : \mathbf{R}^n \to \mathbf{R}$ be a density and T be a tree on $\{1, \ldots, n\}$. An optimal tree approximator of g for T is a density f that factors according to T and minimizes $d(g, f)$. In other words, given g and T we want to find f to

$$\begin{array}{rl} \text{minimize} & d(g, f) \\ \text{subject to} & f \text{ factors according to } T. \end{array}$$

Result

Proposition 1. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a density and T be a tree on $\{1, \ldots, n\}$. The density $f_T^* : \mathbf{R}^d \to \mathbf{R}$ defined by*

$$f^* _T = g _1 \prod _i \neq 1 g _i \mid \operatorname{pa} i$$

minimizes the differential relative entropy with g among all densities on $\mathbf{R}^n$ which factor according to T ($\operatorname{pa} i$ is the parent of i in T rooted at vertex 1, $i = 2, \ldots, n$).

Proof. Let $f : \mathbf{R}^d \to \mathbf{R}$ be a density factoring according to T. First, express

$$f = f_1 \prod_{i=1} f_{i|\operatorname{pa} i}.$$

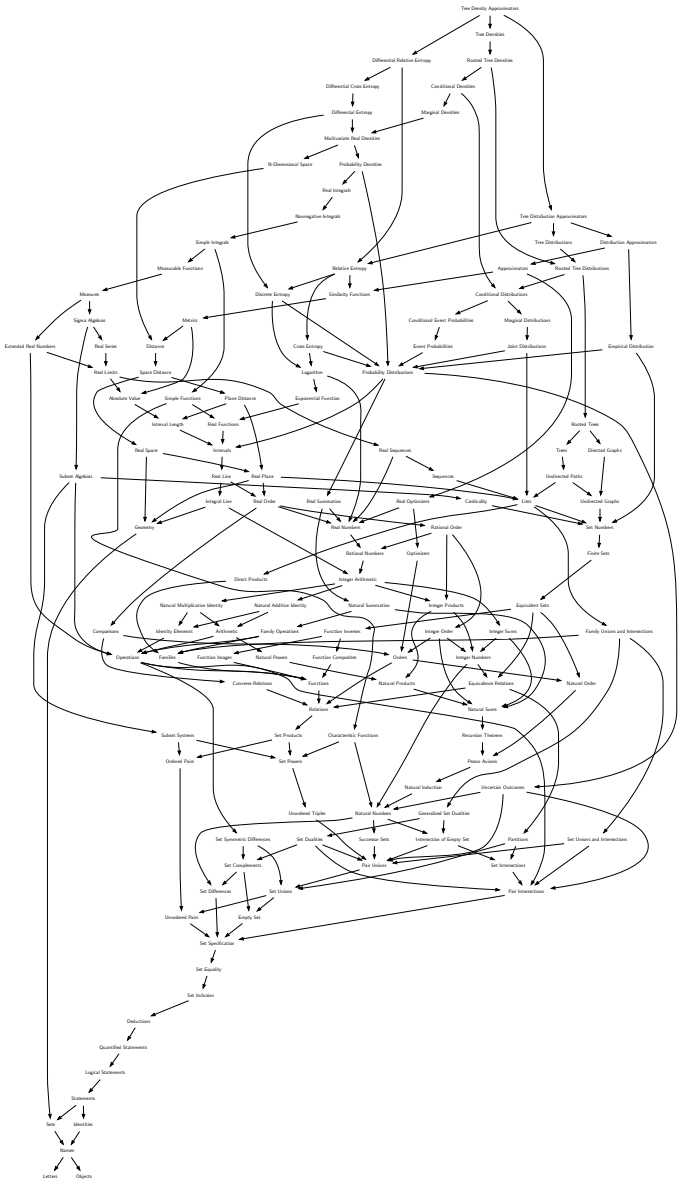
Second, recall that $d(g, f) = h(g, f) - h(g)$. Since $h(g)$ does not depend on f, f is a minimizer of $d(g, f)$ if and only if f is a minimizer of $h(g, f)$.

Third, express

$$\begin{aligned} h(g, f) \& &= -\int_{\mathbf{R}^d} g \log f \\ \& &= -\int_{\mathbf{R}^d} g(x) \left(\log f_i(x_i) + \sum_{i \neq 1} \log f_{i|\mathrm{pa}\, i}(x_i, x_{\mathrm{pa}\, i}) \right) dx \\ \& &= h(g_1, f_1) + \sum_{i \neq 1} \left(\int_{\mathbf{R}} g_{\mathrm{pa}\, i}(\xi) h\big(g_{i|\mathrm{pa}\, i}(\cdot, \xi), f_{i|\mathrm{pa}\, i}(\cdot, \xi)\big) d\xi \right) \end{aligned}$$

which separates across f_1 and $f_{i|\mathrm{pa}\, i}(\cdot, \xi)$ for $i = 1, \ldots, n$ and $\xi \in \mathbf{R}$. In particular, since $g_{pai} \geq 0$, we can minimize the integrand pointwise.

Fourth, recall $h(\phi, \psi) \geq 0$ for densities ϕ, ψ of any dimension, and zero if $\phi = \psi$. So $f_1 = g_1$ and $f_{i|\mathrm{pa}\, i} = g_{i|\mathrm{pa}\, i}$ are solutions. $\square$


Tree Density Approximators
Tree Densities
Differential Relative Entropy
Rooted Tree Densities
Differential Cross Entropy
Conditional Densities
Differential Entropy
Marginal Densities
Multivariate Real Densities
N-Dimensional Space
Probability Densities
Real Integrals
Nonnegative Integrals
Tree Distribution Approximators
Simple Integrals
Tree Distributions
Distribution Approximators
Measurable Functions
Relative Entropy
Approximation
Rooted Tree Distributions
Measures
Discrete Entropy
Similarity Functions
Conditional Distributions
Sigma Algebras
Metrics
Conditional Event Probabilities
Marginal Distributions
Extended Real Numbers
Real Series
Distances
Cross Entropy
Event Probabilities
Joint Distributions
Empirical Distribution
Real Limits
Space Distances
Logarithm
Probability Distributions
Absolute Value
Simple Functions
Plane Distances
Exponential Function
Interval Length
Real Functions
Rooted Trees
Real Spaces
Intervals
Real Sequences
Trees
Directed Graphs
Subset Algebras
Real Line
Real Plane
Sequences
Undirected Paths
Integral Line
Real Order
Real Summation
Real Optimizers
Cardinality
Lists
Undirected Graphs
Geometry
Real Numbers
Rational Order
Set Numbers
Rational Numbers
Optimizers
Finite Sets
Direct Products
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Natural Summation
Integer Products
Equivalent Sets
Comparators
Identity Elements
Arithmetic
Family Operations
Function Inverses
Integer Order
Integer Series
Family Unions and Intersections
Operations
Function Image
Natural Powers
Function Composition
Orders
Integer Numbers
Converse Relations
Functions
Natural Products
Equivalence Relations
Natural Order
Relations
Natural Sums
Subset Systems
Set Products
Characteristic Functions
Recursion Theorem
Ordered Pairs
Set Powers
Peano Axioms
Natural Induction
Uncertain Outcomes
Unordered Triples
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Set Dualities
Successor Sets
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Terms
Letters
Objects