Why

If we model some measured value as a random variable with induced distribution $p : V \to \mathbf{R}$, then one interpretation of $p(v)$ for $v \in V$ is the *proportion* of times in a large number of trials that we *expect* to measure the value v.[1]

Definition

Given a distribution $p : \Omega \to \mathbf{R}$ and a *real-valued* outcome variable $x : \Omega \to \mathbf{R}$, the *expectation* (or *mean*) of x under p is $\sum_{\omega \in \Omega} p(\omega)x(\omega)$.

Notation

We denote the expectation of x under p by $\mathbf{E}(x)$. When there is no chance of ambiguity, we write $\mathbf{E}(x)$.

Properties

Let $x, y : \Omega \to \mathbf{R}$ be two outcome variables and $p : \Omega \to \mathbf{R}$ a distribution. Let $\alpha, \beta \in \mathbf{R}$. Define $z = \alpha x + \beta y$ by $z(\omega) = \alpha x(\omega) + \beta y(\omega)$. Then $\mathbf{E}(z) = \alpha \mathbf{E}(x) + \beta \mathbf{E}(z)$.

Example: expectation

Suppose $\Omega = \{1, 2, 3, 4, 5\}$ with $p(1) = 0.1$, $p(2) = 0.15$, $p(3) = 0.1$, $p(5) = 0.25$ and $p(5) = 0.4$. Define $x : \Omega \to \mathbf{R}$ by

$$x(a) = \begin{cases} -1 & \text{if } a = 1 \text{ or } a = 2, \\ 1 & \text{if } a = 3 \text{ or } a = 4, \\ 2 & \text{if } a = 5. \end{cases}$$

The expectation of x under p is

$$\mathbf{E}x = -1 - 0.15 + 0.1 + 0.25 + 2(0.4) = 0.9.$$

[1]Future editions may modify this explanation, and take a genetic approach via summary statistics.

Two routes for computation

Denote by $p_x : V \to \mathbf{R}$ the induced distribution of $x : \Omega \to V$ (where $V \subset \mathbf{R}$). Then $\mathbf{E}(x) = \sum_{v \in V} p_x(v)v$ since

$$
\begin{aligned}
\sum_{\omega \in \Omega} p(\omega)x(\omega) &= \sum_{v \in V} \sum_{\omega \in x^{-1}(v)} x(\omega)p(\omega) \\
&= \sum_{v \in V} v \sum_{\omega \in x^{-1}(v)} p(\omega) \\
&= \sum_{v \in V} x(v)p_x(v).
\end{aligned}
$$

Interpretations

We interpret the mean as the center of mass of the induced distribution.

Outcome Variable Expectation

Induced Probability Distributions

Outcome Variable Events

Event Probabilities

Probability Distributions

Intervals

Real Line

Real Order   Real Summation

Rational Order   Real Numbers

Rational Numbers

Integral Line

Integer Arithmetic

Integer Products   Natural Additive Identity   Natural Multiplicative Identity

Arithmetic   Identity Elements

Natural Powers   Natural Summation

Comparisons   Integer Order   Integer Sums   Natural Products   Family Operations

Orders   Integer Numbers   Natural Sums   Families   Operations

Converse Relations   Natural Order   Equivalence Relations   Recursion Theorem   Functions

Relations   Peano Axioms

Set Products   Natural Induction   Uncertain Outcome

Ordered Pairs   Set Powers   Natural Numbers

Unordered Triples   Successor Sets   Intersection of Empty Set   Partitions

Pair Unions   Set Symmetric Differences   Set Intersections

Set Unions   Set Complements   Pair Intersections

Unordered Pairs   Set Differences   Empty Set

Set Specification   Geometry

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects