



Why

What is the projection of a vector onto an affine set?

Result

Proposition 1. *Suppose $a \in \mathbf{R}^n$ and $U \in \mathbf{R}^{n \times k}$ with $U^\top U = I$. Define the affine set $W(a, U) = a + \text{range}(U)$. Then*

$$\text{proj}_{W(a, U)}(x) = UU^\top x + (I - UU^\top)a.$$

Proof. The minimizer of $J : \mathbf{R}^n \rightarrow \mathbf{R}$ defined by

$$J(z) = \|a + Uz - x\| = \|Uz - (x - a)\|,$$

is $z^\star = U^\top(x - a)$. So the projection of x onto $W(a, U)$ is

$$a + Uz^\star = a + UU^\top(x - a) = UU^\top x + (I - UU^\top)a.$$

□

