



Why

We want to solve linear equations. Our approach is to “eliminate” variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

Two-variable example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$3x_1 + 2x_2 = 10, \text{ and}$$

$$6x_1 + 5x_2 = 20.$$

We can list the coefficients in a two-dimensional array $A = (3, 2; 6, 5)$ and $b = (10, 20)$. We can eliminate x_1 from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$3x_1 + 2x_2 = 10 \text{ and}$$

$$x_2 = 0.$$

The key insight is that this system has the *same solution set*. We call the process of moving between these two systems a *row reduction*.

Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate x_1 (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear

system $S_1 = (A^1, b^1)$ in which

$$A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and} \quad b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The trick is that, since $A'_{22} \neq 0$, we can take the same route to eliminate x_2 , to obtain the system $S_2 = (A^2, b^2)$ in which

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \text{and} \quad b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Likewise for x_3 , we obtain $S_3 = (A^3, b^3)$ in which

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one, (A^3, b^3) , is easy to solve. We solve it by *back substitution*. First, since $2x_4 = 3$, we find $x_4 = 3/2$. Second, since $2x_3 + 2x_4 = -1$, we find $x_3 = -2$. Similarly we find $x_2 = 1/2$ and $x_1 = 5/4$.

Definition

Let $S = (A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system. The lower *row reduction* of S for index i with $A_{ii} \neq 0$ (or the i -row reduction) is the linear system $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$ if $i < s \leq m$ and A_{st} otherwise. We say that the system (A, b) is *ordinarily reducible*.

Let $a^k, \tilde{a}^k \in \mathbf{R}^n$ denote the k th row of A and \tilde{A} , respectively. Then if $k \neq i$, $\tilde{a}^k = a^k - \alpha_k a^i$ where $\alpha_k = A_{kj}/A_{ij}$. In other words, a row k

of the matrix \tilde{A} is obtained by subtracting a multiple of the i th row of matrix A from row k of matrix A . We are “reducing” the rows of A .

Proposition 1. *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to (C, d) . Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d) .¹*

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = \left(\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

We then subtract three times row 2 from row 3 and four times row 2 from row 4 to obtain

$$S_2 = \left(\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

Finally, we subtract two times row 3 from row 4 to obtain S_4 , which we can write as

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ x_2 + x_3 + x_4 &= 0, \\ 2x_3 + 2x_4 &= -1, \quad \text{and} \\ 2x_4 &= 3. \end{aligned}$$

We can now back-substitute to find $x_4 = 3/2$, $x_3 = -2$, $x_2 = 1/2$ and $x_1 = 5/4$. The above proposition says that this is the only solution of S , as well.

¹Future editions will include an account.

