



## Why

Vectors can be identified with matrices of width 1.

## Canonical identification

We identify  $\mathbf{R}^n$  with  $\mathbf{R}^{n \times 1}$  in the obvious way. For this reason, we call  $x \in \mathbf{R}^{n \times 1}$  (meaning  $x \in \mathbf{R}^n$ ) a *column vector*.

For the reasons that we identify  $\mathbf{R}^n$  with  $\mathbf{R}^{n \times 1}$ , we write the vector  $a = (a_1, a_2, a_3) \in \mathbf{R}^3$  as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

We could as easily also identify  $\mathbf{R}^n$  with  $\mathbf{R}^{1 \times n}$ . We avoid this convention. However, by analogy with the language “column vector,” we refer to the *matrix*  $y \in \mathbf{R}^{1 \times n}$  as a *row vector*.

## Matrix transpose

We frequently move from  $\mathbf{R}^{n \times 1}$  and  $\mathbf{R}^{1 \times n}$ . If  $a \in \mathbf{R}^{n \times 1}$ , we denote  $b \in \mathbf{R}^{1 \times n}$  defined by  $b_i = a_i$  by  $a^\top$ .

More generally, given a matrix  $A \in \mathbf{R}^{m \times n}$ , we denote the matrix  $B \in \mathbf{R}^{m \times n}$  defined by  $B_{ij} = A_{ji}$  by  $A^\top$ . Notice that the entries of  $i$  and  $j$  have swapped. We call the matrix  $B$  the *transpose* of  $A$ , and similarly call  $a^\top$  the *transpose* of the vector  $a$ . Clearly,  $(A^\top)^\top = A$ , which includes  $(a^\top)^\top = a$ .

## Reals as vectors

There is a similar, and similarly obvious, identification of scalars  $a \in \mathbf{R}$  with the 1-vectors  $\mathbf{R}^1$  (and so with the 1 by 1 matrices  $\mathbf{R}^{1 \times 1}$ ). Given our definition of matrix-vector products, if we identify  $a \in \mathbf{R}$  with  $A \in \mathbf{R}^{1 \times 1}$  where  $A_{11} = a$ , then  $Ax = ax$ .

## Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^\top y.$$

We express the symmetry of the inner product by  $x^\top y = y^\top x$ .

