



## Why

It is believable that  $1/2, 1/4, 1/8, \dots$  has a convergent series. And likewise with  $1/3, 1/9, 1/27, \dots$ . What of  $a_k = x^k$  for  $x \in \mathbf{R}$ .

## Definition

Let  $x \in \mathbf{R}$ . The *geometric series* of  $x$  is the series of the sequence  $(a_k)$  defined by  $a_k = x^k$ .

## Characterization of convergence

Does the geometric series of  $x$  converge? In other words, does  $(s_n)$  defined by  $s_n = \sum_{k=1}^n x^k$  have a limit.

For  $x = 1$  and  $x = -1$ , we have seen (see *Real Series*) that the series diverges. However for the cases  $x = 1/2$  and  $x = 1/3$  the geometric series converges.

**Proposition 1.** *If  $|x| < 1$ , then the geometric series of  $x$  converges and*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n x^k = \frac{x}{1-x}$$

*If  $|x| \geq 1$  then the geometric series of  $x$  diverges.*

*Proof.* Define  $s_n = \sum_{k=1}^n x^k$ . Then

$$\begin{aligned} x \cdot s_n &= x \cdot (x^1 + x^2 + \dots + x^n) \\ &= x^2 + x^3 + \dots + x^{n+1} \\ &= s_n - x + x^{n+1}. \end{aligned}$$

From which we deduce,  $s_n(1-x) = x(1-x^n)$ . If  $x \neq 1$ , then

$$s_n = \frac{1}{1-x}(1-x^n)$$

If  $|x| < 1$ , then using the algebra of limits (see *Real Limit Algebra*) we deduce

$$\lim_{n \rightarrow \infty} \frac{1}{1-x}(1-x^n) = \frac{1}{1-x}(1-0) = \frac{1}{1-x},$$

since  $\lim_{k \rightarrow \infty} x^k = 0$  for  $|x| < 1$ .

If  $x = 1$  or  $x = -1$ , then we have seen that  $(s_n)$  diverges.<sup>1</sup>

□

---

<sup>1</sup>Future editions will include the trivial account about the case  $|x| > 1$ .

