Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$ in the obvious way. For this reason, we call $x \in \mathbf{R}^{n\times 1}$ (meaning $x \in \mathbf{R}^n$) a *column vector*.

For the reasons that we identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$, we write the vector $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

We could as easily also identify $\mathbf{R}^n$ with $\mathbf{R}^{1\times n}$. We avoid this convention. However, by analogy with the language "column vector," we refer to the *matrix* $y \in \mathbf{R}^{1\times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n\times 1}$ and $\mathbf{R}^{1\times n}$. If $a \in \mathbf{R}^{n\times 1}$, we denote $b \in \mathbf{R}^{1\times n}$ defined by $b_i = a_i$ by $a^\top$.

More generally, given a matrix $A \in \mathbf{R}^{m\times n}$, we denote the matrix $B \in \mathbf{R}^{m\times n}$ defined by $B_{ij} = A_{ji}$ by $A^\top$. Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A, and similarly call $a^\top$ the *transpose* of the vector a. Clearly, $(A^\top)^\top = A$, which includes $(a^\top)^\top = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors $\mathbf{R}^1$ (and so with the 1 by 1 matrices $\mathbf{R}^{1\times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1\times 1}$ where $A_{11} = a$, then $Ax = ax$.

Familiar concepts, new notation

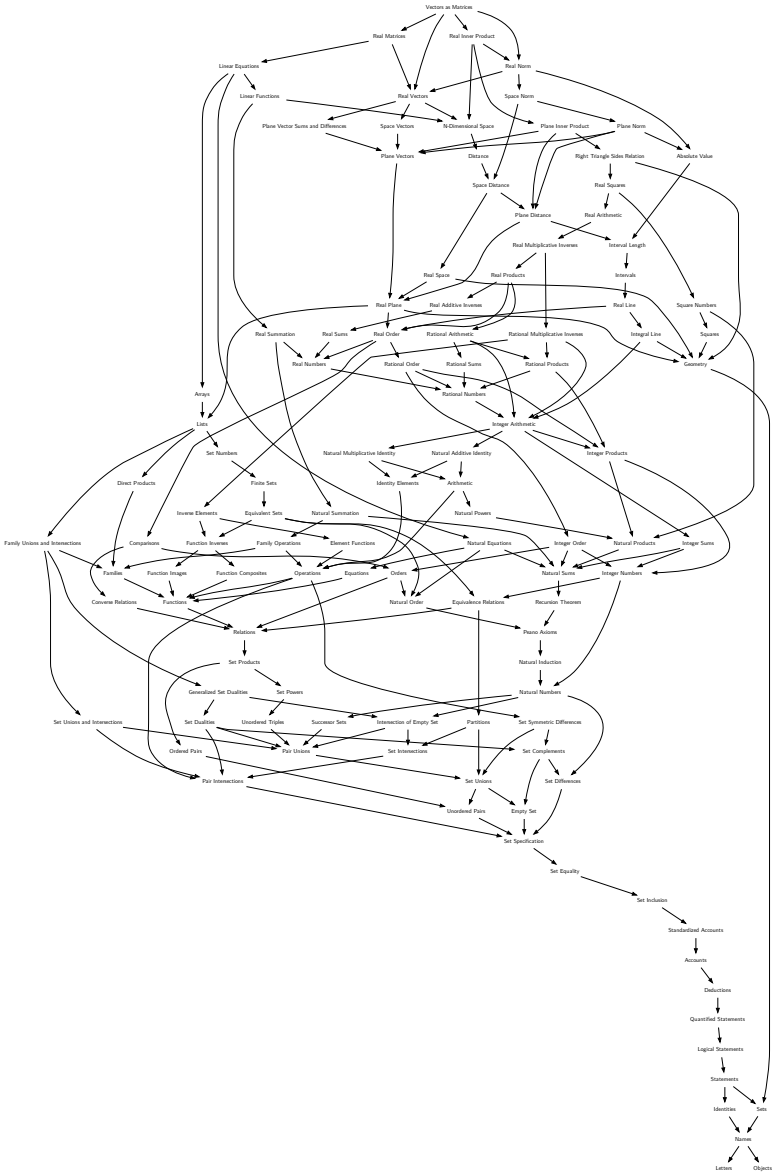
These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^\top y.$$

We express the symmetry of the inner product by $x^\top y = y^\top x$.


Vectors as Matrices
Real Matrices
Real Inner Product
Linear Equations
Real Norm
Linear Functions
Real Vectors
Space Norm
Plane Vector Sums and Differences
Space Vectors
N-Dimensional Space
Plane Inner Product
Plane Norm
Plane Vectors
Distance
Right Triangle Sides Relation
Absolute Value
Space Distance
Real Squares
Plane Distance
Real Arithmetic
Real Multiplicative Inverses
Interval Length
Real Space
Real Products
Intervals
Real Plane
Real Additive Inverses
Real Line
Square Numbers
Real Summation
Real Sums
Real Order
Rational Arithmetic
Rational Multiplicative Inverses
Integral Line
Squares
Real Numbers
Rational Order
Rational Sums
Rational Products
Geometry
Arrays
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Direct Products
Finite Sets
Identity Elements
Arithmetic
Inverse Elements
Equivalent Sets
Natural Summation
Natural Powers
Family Unions and Intersections
Comparisons
Function Inverses
Family Operations
Element Functions
Natural Equations
Integer Order
Natural Products
Integer Sums
Families
Function Images
Function Composites
Operations
Equations
Orders
Natural Sums
Integer Numbers
Convert Relations
Functions
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Set Symmetric Differences
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects