



## Why

We generalize our notion of norm on real vectors to abstract vector spaces.

## Definition

A *norm* is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisfies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms on those vectors.

A *normed space* (or *norm space*) is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

## Notation

Let  $(X, \mathbf{F})$  be a vector space where  $\mathbf{F}$  is the field of real numbers or the field of complex numbers. Let  $R$  denote the set of real numbers. Let  $f : X \rightarrow R$ . The functional  $f$  is a norm if

1.  $f(v) \geq 0$  for all  $x \in V$
2.  $f(v) = 0$  if and only if  $x = 0 \in X$ .
3.  $f(\alpha x) = |\alpha|f(x)$  for all  $\alpha \in \mathbf{F}$ ,  $x \in X$
4.  $f(x + y) \leq f(x) + f(y)$  for all  $x, y \in X$ .

In this case, for  $x \in X$ , we denote  $f(x)$  by  $\|x\|$ , read aloud “norm x”. The notation follows the notation of absolute value as a norm. In some cases, we go further, and for a norm indexed by some parameter  $\alpha$  or set  $A$  we write  $\|x\|_\alpha$  or  $\|x\|_A$ .

## Examples

The absolute value function is a norm on the vector space of real numbers. In addition, the (Euclidean norm) is a norm on the vector space  $\mathbf{R}^n$ .

