Set Inclusion

1 Why

We want language for all of the elements of a first set being the elements of a second set.

2 Definisions

If every element of a first set is an element of a second set we say that the first set is a *subset* of the second set. Conversely, we say that second set is a *superset* of the first set.

Every set is a subset of itself. Similarly, every set is a superset of itself. Thus, if two sets are equal, the first is a subset of the second and the second is a subset of the first. Because of our definition of set equality, the converse is also true.

The empty set is a subset of every set, since it has no elements and so satisfies our definition. Consider a set. We call the empty set and the set itself *improper subsets* of the set. All other subsets we call *proper subsets*.

Finally, the *power set* of a set is the set of all subsets of that set. It includes the set itself and the empty set.

2.1 Notation

Let A and B be sets. We denote that A is a subset of B by $A \subset B$. We read the notation $A \subset B$ aloud as "A subset B".

We can express the axiom of extension by

$$A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$$

The notation $A \subset B$ is a concise symbolism for the sentence "every element of A is an element of B." Or for the alternative notation $a \in A \implies a \in B$.

We denote the power set of A by A^*, read aloud as "powerset of A." $A \in A^*$ and $\varnothing \in A^*$. However, $A \subset A^*$ is false.

2.2 Example

Let a, b, c be distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in A^*$. As always, $\varnothing \in A^*$ and $A \in A^*$ as well. In this case, we can list the elements (which are sets) of the power set:

$$A^* = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$

2.3 Immediate Results

Proposition 1. *If $A \subset B$ and $B \subset C$ then $A \subset C$.*

Proof. Let $a \in A$. Then $a \in B$ and so then $b \in C$. Thus $a \in C$. □