



## Why

We want to associate the natural numbers with bit strings for use on digital computers.<sup>1</sup>

## Definition

A *digital natural* is a bit string. The set of *d-bit digital natural numbers* is the set of length- $d$  bit strings  $\{0, 1\}^d$ . For example, the set of 8-bit digital naturals is the set  $\{0, 1\}^8$ .

## Correspondence with $\mathbf{N} \cup \{0\}$

We associate  $x \in \{0, 1\}^d$  corresponds to the number  $\sum_{i=1}^d x_i 2^i$ . For example, the bit string  $(0, 0, 0) \in \{0, 1\}^3$  corresponds to the natural number  $0 \in \omega$ . Likewise,  $(1, 0, 0)$  corresponds to  $1 \in \mathbf{N}$ ,  $(0, 1, 0)$  corresponds to 2,  $(1, 1, 0)$  corresponds to 3, etc.

Call the function so defined the *digital natural decoder*, and denote it by  $f : \{0, 1\}^d \rightarrow \mathbf{N} \cup \{0\}$ . In other words  $f((0, 0, 0)) = 0$ ,  $f((0, 1, 0)) = 2$ , etc. Call the set  $f(\{0, 1\}^d)$  the set of naturals *representable* by length- $d$  bit strings.

Specifically, if, for  $n \in \mathbf{N} \cup \{0, 1\}$ , there exists  $x \in \{0, 1\}^d$  so that  $f(x) = n$ , we say that  $x$  is *representable in  $d$  bits*.

## Correspondence between $d$ and $k > d$ bit naturals

Let  $x \in \{0, 1\}^d$  and  $y \in \{0, 1\}^k$  with  $k > d$ . Although  $\{0, 1\}^d \not\subset \{0, 1\}^k$ ,  $f(\{0, 1\}^d) \subset f(\{0, 1\}^k)$ . We can identify  $x \in \{0, 1\}^d$  with  $x' \in \{0, 1\}^k$  where  $x' = (x_1, \dots, x_d, 0, \dots, 0)$  so that  $f(x) = f(x')$ . Clearly then, if  $x$  is representable in  $d$  bits, it is representable in  $k > d$  bits.

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<sup>1</sup>Future editions will expand.

## Addition

We want to define addition  $\oplus : \{0, 1\}^d \times \{0, 1\}^d \rightarrow \{0, 1\}^d$  so that  $f(x \oplus x') = f(x) + f(x')$ . In general, we are stuck, because  $x + x'$  may not be representable in  $d$  bits. Suppose, however and for the time being, that it is.<sup>2</sup>

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<sup>2</sup>Future editions will complete.

