Why

We characterize chordal graphs using vertex separators, and vice versa.[1]

Main Result

Proposition 1 (Chordal Graphs and Vertex Separators)**.** *An undirected graph is chordal if and only if all minimal vertex separators are complete.*

Proof. Let $G = (V, E)$ be an undirected graph. First, suppose that all minimal vertex separators of G are complete. Let c be a cycle of length greater than 3. Let v, w be nonconsecutive vertices in c. If v and w are adjacent in G, then $\{v, w\} \in E$ is a chord. If v and w are nonadjacent, then vw-separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any vw-separator T. Split the cycle into the path from v to w, call it p_1 and the path from w to v, call it p_2. T must include an interior point of p_1, call it u_1, otherwise v and w are connected. Similarly, T must include an interior point of p_2, call it u_2. u_1 and u_2 are not consecutive in c, since they are distinct from x and y.

Let S be a minimal vw-separator. Let $s, t \in S$ be two non-consecutive vertices in the cycle different from v and w By assumption S is complete, so s and t are adjacent in G.

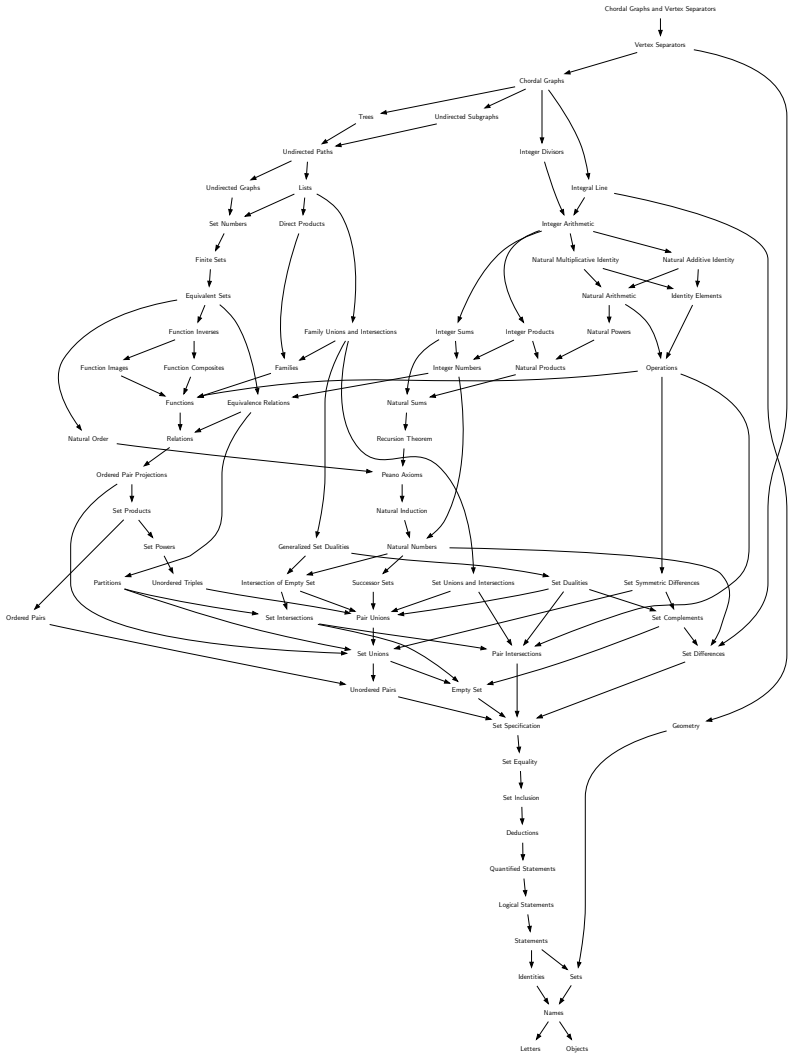
Second, let $G = (V, E)$ be a chordal graph. Let S be a minimal vw-separator. Let C_v and C_w be the connected components containing v and w of the subgraph induced by $V - S$.

If $\nu m S = 1$, then S is complete. Otherwise, let $x, y \in S$ by distinct. We want to show $\{x, y\} \in E$. The key insight is that x is adjacent to vertices in C_v and C_w. If there were no such vertex, $S - \{x\}$ would be a vw-separator and S would not be minimal. Similarly with y. Also, $\nu m C_v, \nu m C_w \geq 1$.

[1] Future editions will expand and may include graphics.

With these observations, there exists a path from x to y through C_v. Let $p_v = (x, v_1, \ldots, v_k, y)$ be a path of shortest length with at least one interior vertex (so $k \geq 1$) from x to y using interior vertices in $v_1, \ldots, v_k \in C_v$. Let $p_w = (y, w_1, \ldots, w_l, x)$ be a path of shortest length with at least one interior vertex (so $l \geq 1$) from y to x using interior vertices $w_1, \ldots, w_k \in C_w$. Use p_v and p_w to define the cycle $c = (x, v_1, \ldots, v_k, y, w_1, \ldots, w_l, x)$ which has length at least four. G is chordal, so c has a chord.

We argue that the chord of c is $\{x, y\}$. Since C_w and C_V are different connected components (whose vertices are not included in S), there are no edges $\{v_i, w_j\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Since p_v and p_w are paths of shortest length, they have no chords. In particular, there is no edge $\{v_i, v_j\}$ for $|i - j| > 1$ or $\{v_i, x\}$ for $i = 1, \ldots, l$. Similarly, there is no edge $\{w_i, w_j\}$ for $|i - j| > 1$ or $\{w_i, y\}$ for $i = 1, \ldots, l$. The only remaining pair is $\{x, y\}$, and so it must be the chord. $\square$


Chordal Graphs and Vertex Separators
Vertex Separators
Chordal Graphs
Trees
Undirected Subgraphs
Undirected Paths
Integer Divisors
Undirected Graphs
Lists
Integral Line
Set Numbers
Direct Products
Integer Arithmetic
Finite Sets
Natural Multiplicative Identity
Natural Additive Identity
Equivalent Sets
Natural Arithmetic
Identity Elements
Function Inverses
Family Unions and Intersections
Integer Sums
Integer Products
Natural Powers
Function Images
Function Composites
Families
Integer Numbers
Natural Products
Operations
Functions
Equivalence Relations
Natural Sums
Natural Order
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Set Powers
Generalized Set Dualities
Natural Numbers
Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Unions and Intersections
Set Dualities
Set Symmetric Differences
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Set Unions
Pair Intersections
Set Differences
Unordered Pairs
Empty Set
Set Specification
Geometry
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects