



# Supremum Norm Complete

## 1 Why

We want a complete norm on the vector space of continuous functions.

## 2 Result

**Proposition 1.** *The supremum norm is complete.*

*Proof.* Let  $R$  denote the real numbers. Let  $(f_n)_n$  be an egoprox sequence in  $C[a, b]$ .

**Candidate.**  $(f_n)_n$  is egoprox means  $\forall \varepsilon > 0, \exists N$  so that

$$m, n > N \implies |f_n - f_m|_{\sup} < \varepsilon.$$

Since  $|f_n - f_m|_{\sup} < \varepsilon \implies |f_n(x) - f_m(x)| < \varepsilon$  for all  $x \in [a, b]$ , the sequence of real numbers  $\{f_n(x)\}_n$  is egoprox for each  $x \in [a, b]$ . Since the metric space  $(R, |\cdot|)$  is complete, there is a limit  $l_x \in R$  such that  $f_n(x) \longrightarrow l_x$  as  $n \longrightarrow \infty$ , for each  $x \in [a, b]$ . Define  $f : [a, b] \rightarrow R$  by  $f(x) = l_x$  for each  $x \in [a, b]$ .

**Candidate is Limit.** First, we argue that  $|f_n - f|_{\sup} \longrightarrow 0$  as  $n \longrightarrow \infty$ . Since  $(f_n)_n$  is an egoprox sequence, there exists  $n_0$  so that

$$n, m \geq n_0 \implies |f_n - f_m|_{\sup} < \varepsilon/2.$$

So for all  $x \in [a, b]$ ,

$$n, m \geq n_0 \implies |f_n(x) - f_m(x)| < \varepsilon/2.$$

For all  $x \in [a, b]$ , and  $n \geq n_0$ ,

$$\lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

The sequence  $\{f_k(x)\}_{k=m}^{\infty}$  is a final part of  $\{f_k(x)\}_{k=1}^{\infty}$ , and so has the same limit,  $f(x)$ . Therefore, using continuity of subtraction and the absolute value,

$$\lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for  $n \geq n_0$ ,  $x \in [a, b]$ ,  $|f_n(x) - f(x)| < \varepsilon$ , from which we deduce  $|f_n - f|_{\sup} < \varepsilon$ . Thus  $f_n \rightarrow f$  as  $n \rightarrow \infty$ .

**Limit is Continuous.** Next, we argue that  $f$  is continuous. Let  $x_0 \in [a, b]$ . Let  $\varepsilon > 0$ . Since  $f_n \rightarrow f$  there exists  $n_0$  so that

$$|f_{n_0} - f|_{\sup} < \varepsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all  $x \in [a, b]$ . Using  $|f(x_0) - f_{n_0}(x_0)| < \varepsilon/3$ ,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all  $x \in [a, b]$ . Using the triangle inequality,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

for all  $x \in [a, b]$ . Using  $|f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3$

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \varepsilon/3$$

for all  $x \in [a, b]$ . Since  $f_{n_0}$  is continuous, there exists  $\delta > 0$  so that

$$|x_0 - x| < \delta \implies |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3,$$

for  $x \in [a, b]$ . In this case,

$$|f(x_0) - f(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since  $\varepsilon$  was arbitrary,  $f$  is continuous at  $x_0$ . Since  $x_0$  was arbitrary,  $f$  is continuous everywhere. Some call the above the *three epsilon argument*.  $\square$