



## Why

Historically considered a fountainhead for all of mathematics.

## Discussion

So far we know that  $\omega$  is the unique smallest successor set. In other words, we know that  $0 \in \omega$ ,  $n \in \omega \longrightarrow n^+ \in \omega$  and that if these two properties hold of some  $S \subset \omega$ , then  $S = \omega$ . We can add two important statements to this list. First, that 0 is the successor of no number. In other words,  $n^+ \neq 0$  for all  $n \in \omega$ . Second, that if two numbers have the same successor, then they are the same number. In other words,  $n^+ = m^+ \longrightarrow n = m$ .

These five properties were historically considered the fountainhead of all of mathematics. One by the name of Peano used them to show the elementary properties of arithmetic. They are:

1.  $0 \in \omega$ .
2.  $n \in \omega \longrightarrow n^+ \in \omega$  for all  $n \in \omega$ .
3. If  $S$  is a successor set contained in  $\omega$ , then  $S = \omega$ .
4.  $n^+ \neq 0$  for all  $n \in \omega$ .
5.  $n^+ = m^+ \longrightarrow n = m$  for all  $n, m \in \omega$ .

These are collectively known as the *Peano axioms*. Recall that the third statement in this list is the *principle of mathematical induction*.

## Statements

Here are the statements.

**Proposition 1** (Peano's First Axiom).  $0 \in \omega$ .

**Proposition 2** (Peano's Second Axiom).  $n \in \omega \longrightarrow n^+ \in \omega$ .

**Proposition 3** (Peano's Third Axiom). *Suppose  $S \subset \omega$ ,  $0 \in S$ , and  $(n \in S \longrightarrow n^+ \in S)$ . Then  $S = \omega$ .*

**Proposition 4** (Peano's Fourth Axiom).  *$n^+ \neq 0$  for all  $n \in \omega$ .*

The last one uses the following two useful facts.

**Proposition 5.**  $x \in n \longrightarrow n \not\in x$ .

**Proposition 6.**  $(x \in y \wedge y \in n) \longrightarrow x \in n$

This latter proposition is sometimes described by saying that  $n$  is a *transitive set*. This notion of transitivity is not the same as that described in Relations. Using these one can show:

**Proposition 7** (Peano's Fifth Axiom). *Suppose  $n, m \in \omega$  with  $n^+ = m^+$ . Then  $n = m$ .*

