



Real Integrals

1 Why

We define the area under an extended real function.

2 Definition

The *positive part* of an extended-real-valued function is the function mapping each element to the maximum of the function's result and zero. The *negative part* of an extended-real-valued function is the function mapping each element to the maximum of the negative of function's result and zero.

We decompose an extended-real-valued function as the difference of its positive part and its negative part. Both the positive and negative parts are non-negative extended-real-valued functions.

Consider a measure space. An *integrable* function is a measurable extended real function for which the non-negative integral of the positive part and the non-negative integral of the negative part of the function are finite.

The *integral* of an integrable function is the difference of the non-negative integral of the positive part and the non-negative integral of the negative part.

If one but not both of the parts of the function are finite, we say that the integral *exists* and again define it as before. In this way we avoid arithmetic between two infinities.

2.1 Notation

Let A a non-empty set. Let $g : A \rightarrow [-\infty, \infty]$. We denote the positive part of g by g^+ and the negative part of g by g^- :

$$g^+(x) = \max\{g(x), 0\} \quad \text{and} \quad g^-(x) = \max\{-g(x), 0\}.$$

Moreover, we decompose g as $g = g^+ - g^-$. We observed that $g^+(x) \geq 0$ and $g^-(x) \geq 0$ for all $x \in X$.

Let (X, \mathcal{A}, μ) be a measure space. Let $f : X \rightarrow [-\infty, +\infty]$ measurable and one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite (if both are finite, f is integrable).

We denote the integral of f with respect to the measure μ by $\int f d\mu$. We defined:

$$\int f d\mu = \left(\int f^+ d\mu \right) - \left(\int f^- d\mu \right).$$