



Why

Can we represent the function associating a linear system with its row reduction by matrix multiplication?

Main observation

The following proposition affirmatively answers the question.

Proposition 1. *Let $(A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$ and (C, d) the k th reduction of (A, b) . Then there exists a matrix $L \in \mathbf{R}^{m \times m}$ so that $C = LA$ and $d = Lb$.*

Proof. Define $L \in \mathbf{R}^{m \times m}$ by $L_{st} = 1$ if $s = t$, $-A_{sj}/A_{ik}$ if $k < s \leq m$ and zero otherwise. \square

For this reason, we call L in Proposition 1 a *row reducer matrix* or *row reducing matrix* or *row reducer*. The row reducer matrix for the k th reduction of (A, b) has the form

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & A_{ik}/A_{kk} & 1 & & \\ & & \vdots & & \ddots & \\ & & A_{mk}/A_{kk} & & & 1 \end{bmatrix}$$

So the following is immediate

Prop. 2. *Row reducing matrices are unit lower triangular.*

Example

For example, the $(1,1)$ -reduction of (A, b) in which

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 6 & 7 & 9 & 5 \\ 8 & 7 & 9 & 8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

is the linear system

$$A' = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and} \quad b' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The row reducer is $L \in R^{4 \times 4}$ defined by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that $A' = LA$ and $b' = Lb$, and clearly L is unit lower triangular.

