Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$ in the obvious way. For this reason, we call $x \in \mathbf{R}^{n\times 1}$ (meaning $x \in \mathbf{R}^n$) a *column vector*.

For the reasons that we identify $\mathbf{R}^n$ with $\mathbf{R}^{n\times 1}$, we write the vector $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.
$$

We could as easily also identify $\mathbf{R}^n$ with $\mathbf{R}^{1\times n}$. We avoid this convention. However, by analogy with the language "column vector," we refer to the *matrix* $y \in \mathbf{R}^{1\times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n\times 1}$ and $\mathbf{R}^{1\times n}$. If $a \in \mathbf{R}^{n\times 1}$, we denote $b \in \mathbf{R}^{1\times n}$ defined by $b_i = a_i$ by $a^\top$.

More generally, given a matrix $A \in \mathbf{R}^{m\times n}$, we denote the matrix $B \in \mathbf{R}^{m\times n}$ defined by $B_{ij} = A_{ji}$ by $A^\top$. Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A, and similarly call $a^\top$ the *transpose* of the vector a. Clearly, $(A^\top)^\top = A$, which includes $(a^\top)^\top = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors $\mathbf{R}^1$ (and so with the 1 by 1 matrices $\mathbf{R}^{1\times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1\times 1}$ where $A_{11} = a$, then $Ax = ax$.

Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^\top y.$$

We express the symmetry of the inner product by $x^\top y = y^\top x$.

Vectors as Matrices

Real Inner Product · Real Matrices

Linear Equations · Real Norm

Linear Functions · Space Norm · Real Vectors

Plane Inner Product · Plane Norm · N-Dimensional Space · Space Vectors · Plane Vector Sums and Differences

Right Triangle Sides Relation · Absolute Value · Distance · Plane Vectors

Real Squares · Space Distance

Square Numbers · Plane Distance

Squares

Real Arithmetic

Real Multiplication Inverse · Interval Length

Real Products · Intercept · Real Space

Real Addition Inverse

Rational Arithmetic · Rational Multiplication Inverse · Real Sums · Real Order · Real Line · Real Summation · Integral Line · Real Plane

Rational Products · Rational Order · Rational Sums · Real Numbers · Geometry

Rational Numbers

Integer Arithmetic · Arrays · Lists

Integer Products · Natural Addition Identity · Natural Multiplication Identity · Set Numbers

Arithmetic · Identity Elements · Finite Sets · Inverse Elements

Natural Powers · Natural Summation · Equivalent Sets · Direct Products

Integer Sums · Natural Products · Integer Order · Natural Equations · Comparisons · Element Functions · Family Operation · Function Inverse · Family Unions and Intersections

Integer Numbers · Natural Sums · Orders · Equation · Operation · Function Image · Function Composites · Families

Recursion Theorem · Equivalence Relations · Natural Order · Converse Relations · Functions

Peano Axiom · Relations

Natural Induction · Set Products

Natural Numbers · Set Powers · Generalized Set Dualities

Set Symmetric Differences · Partitions · Successor Sets · Intersection of Empty Set · Unordered Triples · Set Dualities · Set Unions and Intersections

Set Complements · Set Intersections · Pair Unions · Ordered Pairs

Set Differences · Set Unions · Unordered Pairs

Empty Set · Pair Intersections

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deduction

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects

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