Bourbaki

Contents

1 Sets

1.1 Why

We speak of a collection of objects, pre-specified or possessing a similar property.

1.2 Definition

A **set** is a collection of objects. We use **object** as usual in the English language. So a set is an object with the property that it contains other objects.

In thinking of a set, then, we regularly consider the objects it contains. We call the objects contained in a set the **members** or **elements** of the set. So we say that an object contained in a set is a **member of** or an **element of** the set.

1.2.1 Notation

We denote sets by upper case latin letters: for example, A, B, and C. We denote elements of sets by lower case latin letters: for example, a, b, and c. We denote that an object a is an element of a set A by $a \in A$. We read the notation $a \in A$ aloud as "a in A.” The $\in$ is a stylized ϵ, a mnemonic for “element of”. We write $a \notin A$, read aloud as “a not in A,” if a is not an element of A.

If we can write down the elements of A, we do so using brace notation. For example, if the set A is such that it contains only the elements a, b, c, we denote A by $\{a, b, c\}$. If the elements of a set are well-known, then we introduce the set in English and name it; often we select the name mnemonically. For example, let L be the set of latin letters.

If the elements of a set satisfy some common condition, then we use the braces and include the condition. For example, let V be the set of vowels. We can denote V by $\{l \in L \mid l \text{ is a vowel}\}$. We read the symbol $\mid$ aloud as “such that.” We read the whole notation aloud as “l in L such that l is a vowel.” We call the notation **set-builder notation**. Set-builder notation is indispensable for sets defined implicitly by some condition. Here we could have alternatively denoted V by $\{\text{“}a\text{”}, \text{“}e\text{”}, \text{“}i\text{”}, \text{“}o\text{”}, \text{“}u\text{”}\}$. We prefer the former, slighly more concise notation.

1.3 Two Sets

Let A a set. A **subset** of A is a set whose elements are also contained in A. A **superset** if A is a set which contains all the elements of A. Two sets are **equal** if they contain the same elements; equivalently if they are each subsets of each other.

The **power set** of A is the set of subsets of A. The **empty set** is the set containing no elements. The empty set is subset of every set.

1.3.1 Notation

Let A and B be sets. We denote that A is a subset of B by $A \subset B$. We read the notation $A \subset B$ aloud as "A subset B". We denote that A is equal to B by $A = B$. We read the notation $A = B$ aloud as "A equals B". We denote the empty set by $\varnothing$, read aloud as "empty." We denote the power set of A by 2^A, read aloud as "two to the A."

2 Ordered Pairs

2.1 Why

We speak of objects composed of elements from different sets.

2.2 Definition

Let A and B be non-empty sets. Let $a \in A$ and $b \in B$. An **ordered pair** is the set $\{\{a\}, \{a, b\}\}$. The **cartesian product** of A and B is the set of all ordered pairs.

We observe that two pairs are equal if they have equal elements in the same order. The ordered causes If $A \neq B$, the ordering causes the cartesian product of A and B to differ from the cartesian product of B with A. If $A = B$, however, the symmetry holds.

2.2.1 Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the caresian product of A with B by $A \times B$, read aloud as "A cross B." We record succinctly: if $A \neq B$, then $A \times B \neq B \times A$.

3 Relations

3.1 Why

We want to relate elements of two sets.

3.2 Definition

A **relation** between two non-empty sets A and B is a subset of $A \times B$. A relation on a single set C is a subset of $C \times C$.

3.2.1 Notation

We denote relations with upper case capital latin letters because they are sets. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as "a in relation R to b."

Often, instead of latin letters we use other symbols; these symbols suggest the nature of the relation. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

3.3 Properties

Let R be a relation on a non-empty set A. R is **reflexive** if $(a, a) \in R$ for all $a \in A$. R is **transitive** if $(a, b) \in R \wedge (b, c) \in$

$R \implies (a,c) \in R$ for all $a,b,c \in A$. R is **symmetric** if $(a,b) \in R \implies (b,a) \in R$ for all $a,b \in A$. R is **anti-symmetric** if $(a,b) \in R \implies (b,a) \notin R$ for all $a,b \in A$.

4 Graphs

4.1 Why

We want to visualize relations.

4.2 Definition

A **graph** is a set and a relation on the set. The graph is **undirected** if the relation is symmetric; otherwise the graph is **directed**.

A **vertex** of the graph is an element of the set. The set is called the **vertex set**. An **edge** of the graph is an element of the relation. The relation is called the **edge set**.

4.2.1 Notation

We denote the vertex set by V, a mnemonic for vertex. We denote the edge set by E, a mnemonic for edge. We denote a graph by (V, E). If the vertex set is assumed we can unambiguously refer to the graph by E.

4.2.2 Visualization

We visualize the graph by drawing a point for each vertex. If two vertices u and v are in relation, we draw a line from the point corresponding to u to the point corresponding to v with an arrow at the point corresponding to v. If the graph is undirected, we omit arrows.

4.3 Paths

A path in a relation is a sequence of elements in which consecutive elements are related. A path **cycles** if an element appears more than once. A path is **finite** if the sequence is finite. A finite path is a **loop** if it cycles once.

5 Functions

5.1 Why

We want a notion for a correspondence between two sets.

5.2 Definition

A **functional** relation on two sets relates each element of the first set with a unique element of the second set. A **function** is a functional relation.

The **domain** of the function is the first set and **codomain** of the function is the second set. The function **maps** elements **from** the domain **to** the codomain. We call the codomain element associated with the domain element the **result** of **applying** the function to the domain element.

5.2.1 Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as “A to B”.

A function is an element of th set $A \to B$, so we denote them by lower case latin letters, especially f, g, and h. Of course, f is a mnemonic for function; g and h follow f in the alphabet. We denote that $f \in A \to B$, by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

5.3 Properties

Let $f : A \to B$. The **image** of a set $C \subset A$ is the set $\{f(c) \in B \mid c \in C\}$. The **range** of f is the image of the domain. The **inverse image** of a set $D \subset B$ is the set $\{a \in A \mid f(a) \in B\}$.

The range need not equal the codomain; though it, like every other image, is a subset of the codomain. The function maps to domain **on** to the codomain if the range and codomain are equal; in this case we call the function **onto**. This language suggests that every element of the codomain is used by f. It means that for each element b of the codomain, we can find an element a of the domain so that $f(a) = b$.

An element of the codomain may be the result of several elements of the domain. This overlapping, using an element of the codomain more than once, is a regular occurrence. If a function is a unique correspondence in that every domain element has a different result, we call it **one-to-one**. This language is meant to suggest that each element of the domain corresponds to one and exactly one element of the codomain, and vice versa.

5.3.1 Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. The property that f is onto can be written succintly as $f(A) = B$. We denote the inverse image of $D \subset B$ by $f^{-1}(D)$, read aloud as "f inverse D."

6 Order Relations

6.1 Why

We want to handle elements of a set in a particular order.

6.2 Definition

Let R be a relation on a non-empty set A. R is a **partial order** if it is reflexive, transitive, and anti-symmetric.

A **partially ordered set** is a set and a partial order. The language partial is meant to suggest that two elements need not be comparable. For example, suppose R is $\{(a, a) \mid a \in A\}$; we may justifiably call this no order at all and call A totally unordered, but it is a partial order by our definition.

Often we want all elements of the set A to be comparable. We call R **connexive** if for all $a, b \in A$, $(a, b) \in R$ or $(b, a) \in R$. If R is a partial order and connexive, we call it a **total order**.

A **totally ordered set** is a set together with a total order. The language is a faithful guide: we can compare any two elements. Still, we prefer one word to three, and so we will use the shorter term **chain** for a totally ordered set; other terms include **simply ordered set** and **linearly ordered set**.

6.2.1 Notation

We denote total and partial orders on a set A by $\preceq$. We read $\preceq$ aloud as "precedes or equal to" and so read $a \preceq b$ aloud as "a precedes or is equal to b." If $a \preceq b$ but $a \neq b$, we write $a \prec b$, read aloud as "a precedes b."

7 Natural Numbers

7.1 Why

We want to count.

7.2 Definition

We define the set of **natural numbers** implicitly. There is an element of the set which we call **one**. Then we say that for each element n of the set, there is a unique corresponding element called the **successor** of n which is also in the set. The **successor function** is the implicitly defined a function from the set into itself associating elements with their successors. We call the elements **numbers** and the refer to the set itself as the **naturals**.

To recap, we start by knowing that one is in the set, and the successor of one is in the set. We call the successor of one **two**. We call the successor of two **three**. And so on using the English language in the usual manner. We are saying, in the language of sets, that the essence of counting is starting with one and adding one repeatedly.

7.2.1 Notation

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n. We denote the element called one by 1.

7.3 Induction

We assert two additonal self-evident and indispensable properties of these natural numbers. First, one is the successor of no other element. Second, if we have a subset of the naturals containing one with the property that it contains successors of its elements, then that set is equal to the natural numbers. We call this second property the **principle of mathematical induction.**

These two properties, along with the existence and uniqueness of successors are together called **Peano's axioms** for the natural numbers. When in familiar company, we freely assume Peano's axioms.

7.4 Notation

As an exercise in the notation assumed so far, we can write Peano's axioms: N is a set along with a function $s : N \to N$ such that

1. $s(n)$ is the successor of n for all $n \in N$.

2. s is one-to-one; $s(n) = s(m) \implies m = n$ for all $m, n \in N$.

3. There does not exist $n \in N$ such that $s(n) = 1$.

4. If $T \subset N$, $1 \in T$, and $s(n) \in T$ for all $n \in T$, then $T = N$.

7.5 Order

Let $\preceq$ be a relation on N where $a \preceq b$, $a, b \in N$ if we can obtain b by applying the successor function to a finitely many times. It happens that $\preceq$ is a total order, so $(N, \preceq)$ is a lattice.

8 Algebra

8.1 Why

We want to combine set elements to get other set elements.

8.2 Basics

An **operation** on a set is a function from ordered pairs of elements in the set to the same set. We use operations to combine the elements. We operate on pairs. An **algebra** is a set and an operation. We call the set the **ground set**.

8.2.1 Notation

Let A a set and $g : A \times A \to A$. We commonly forego the notation $g(a, b)$ and instead write $a\, g\, b$. We call this style **infix notation**.

Using lower case latin letters for every the elements and for the operation is confusing, but we often have special symbols for particular operations. Examples of such symbols include $+$, $-$, $\cdot$, $\circ$, and $\star$.

If we had a set A and an operation $+ : A \times A \to A$, we would write $a+b$ for the result of applying $+$ to (a,b). In denoting the algebra, we would say let $(A,+)$ be an algebra.

8.3 Operation Properties

An operation **commutes** if the result of two elements is the same regardless of their order; we call the operation **commutative**

An operation **associates** if given any three elements in order it doesn't matter whether we first operate on the first two and then with the result of the first two the third, or the second two and with the result of the second two the first.

A first operation over a set **distributes** over a second operation over the same set if the result of applying the first operation to an element and a result of the second operation is the same as applying the second operation to the results of the first operation with the arguments of the second operation.

8.3.1 Notation

Let $(A,+)$ an algebra.

We denote that $+$ commutes by asserting

$$a+b=b+a$$

for all $a,b \in A$. We denote that $+$ associates by asserting

$$(a+b)+c=(a+b)+c$$

for all $a, b, c \in A$. Let $(A, \cdot)$ a second algebra over the same set. We denote that $\cdot$ distributes over $+$ by

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

for all $a, b, c \in A$.

8.4 Identity Elements

We call $e \in A$ an **identity element** if (1) $e + a = e$ and (2) $a + e = e$ for all $a \in A$. If only (1) holds, we call e a **left identity**. If only (2) holds, we call e a **right identity**.

8.5 Inverse Elements

We call $b \in A$ an **inverse element** of $a \in A$ if (1) $b + a = e$ and (2) $a + b = e$. If only (1) holds, we call e a **left inverse**. If only (2) holds, we call e a **right inverse**.

9 Set Operations

9.1 Why

We want to consider the elements of two sets together at once, and other sets created from two sets.

9.2 Definitions

Let A and B be two sets.

The **union** of A with B is the set whose elements are in either A *or* B *or* both. The key word in the definition is *or*.

The **intersection** of A with B is the set whose elements are in both A *and* B. The keyword in the definition is *and*.

Viewed as operations, both union and intersection commute; this property justifies the language "with." The intersection is a subset of A, of B, and of the union of A with B.

The **symmetric difference** of A and B is the set whose elements are in the union but not in the intersection. The symmetric difference commutes because both union and intersection commute; this property justifies the language "and." The symmetric difference is a subset of the union.

Let C be a set containing A. The **complement** of A in C is the symmetric difference of A and C. Since $A \subset C$, the union is C and the intersection is A. So the complement is the "left-over" elements of B after removing the elements of A.

We call these four operations **set-algebraic operations**.

9.2.1 Notation

Let A, B be sets. We denote the union of A with B by $A \cup B$, read aloud as "A union B." $\cup$ is a stylized U. We denote the intersection of A with B by $A \cap B$, read aloud as "A intersect B." We denote the symmetric difference of A and B by $A \Delta B$, read aloud as "A symdiff B." "Delta" is a mnemonic for difference.

Let C be a set containing A. We denote the complement of A in C by $C - A$, read aloud as "C minus A."

9.2.2 Results

Proposition 1. *For all sets A and B the operations $\cup$, $\cap$, and Δ commute.*

Proposition 2. *Let S a set. For all sets $A, B \subset S$,*

$$\text{(1)} \quad S - (A \cup B) = (S - A) \cap (S - B)$$

$$\text{(2)} \quad S - (A \cap B) = (S - A) \cup (S - B).$$

Proposition 3. *Let S a set. For all sets $A, B \subset S$,*

$$A \Delta B = (A \cup B) \cap C_S(A \cap B)$$

$\boxed{TODO : notation}$

10 Arithmetic

10.1 Why

Counting one by one is slow so we define an algebra on the naturals.

10.2 Sums and Addition

Let m and n be two natural numbers. If we apply the successor function to m n times we obtain a number. If we apply the successor function to n m times we obtain a number. Indeed, we obtain the same number in both cases. We call this number the **sum** of m and n. We say we **add** m to n, or vice versa. We call this symmetric operation mapping (m, n) to their sum **addition**.

10.2.1 Notation

We denote the operation of addition by $+$ and so denote the sum of the naturals m and n by $m + n$.

10.3 Products and Multiplication

Let m and n naturals. If we add n copies of m we obtain a number. If we add m copies of n we obtain a number. Indeed, we obtain the same number in both cases. We call this number the **product** of m and n. We say we **multiply** m to n, or vice versa. We call this symmetric operation mapping (m, n) to their product **multiplication**.

10.3.1 Notation

We denote the operation of multiplication by $\cdot$ and so denote the product of the naturals m and n by $m \cdot n$.

11 Equivalence Relations

11.1 Why

We want to handle at once all elements which are indistinguishable or equivalent in some aspect.

11.2 Definition

Let R be a relation on A R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

For an element $a \in A$, we call the set of elements in relation R to a the **equivalence class** of a. The key observation, recorded and proven below, is that the equivalence classes partition the set A. A frequent technique is to define an appropriate equivalence relation on a large set A and then to work with the set of equivalence classes of A.

We call the set of equivalence classes the **quotient set** of A under R. An equally good name is the divided set of A under R, but this terminology is not standard. The language in both cases reminds us that $\sim$ partitions the set A into equivalence classes.

11.2.1 Notation

If R is an equivalence relation on a set A, we use the symbol $\sim$. When alone, $\sim$ is read aloud as "sim," but we still read $a \sim b$ aloud as "a equivalent to b." We denote the quotient set of A under $\sim$ by $A/\sim$, read aloud as "A quotient sim".

11.2.2 Results

$\boxed{\textit{TODO}}$

12 Families

12.1 Why

We want to generalize operations beyond two objects.

12.2 Definition

Let A, B be non-empty sets. A **family** of elements of a first set **indexed** by elements of a second set is the range of a function from the second set to the first set. We call second set the **index set**.

If the index set is a finite set, we call the family a **finite family**. If the index set a countable set, we call the family a **countable family**. If the index set is an uncountable set, we call the family a **uncountable family**.

If the codomain is a set of sets, we call the family a **family of sets**. We often use a subset of the whole natural numbers as the index set. In this case, and for other indexed sets with orders, we call the family an **ordered family**

12.2.1 Notation

Let A be a non-empty set. We denote the index set by I, a mnemonic for index. For $i \in I$, let we denote the result of applying the function to i by a_i; the notation evokes evokes function notation but avoids naming the function.

We denote the family of a_α indexed with I by $\{a_\alpha\}_{\alpha \in I}$, which is short-hand for set-builder notation. We read this notation "a sub-alpha, alpha in I."

12.3 Operations

The **pairwise extension** of a commutative operation is the function from finite families of the ground set to the ground set obtained by applying the operation pairwise to elements.

The **ordered pairwise extension** of an operation is the function from finite families ground set to the ground set obtained by applying the operation pairwise to elements in order.

12.3.1 Notation

Let $(A, +)$ be an algebra and $\{A_i\}_{i=1}^n$ a finite family of elements of A. We denote the pairwise extension by

$$\mathop{+}_{i=1}^{n} A_i$$

12.4 Family Set Algebra

We define the set whose elements are the objects which are contained in at least one family member the **family union**. We define the set whose elements are the objects which are contained in all of the family members the **family intersection**.

12.4.1 Notation

We denote the family union by $\cup_{\alpha \in I} A_\alpha$. We read this notation as "union over alpha in I of A sub-alpha." We denote family intersection by $\cap_{\alpha \in I} A_\alpha$. We read this notation as "intersection over alpha in I of A sub-alpha."

12.4.2 Results

Proposition 4. *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \{i, j\}$ then*

$$\cup_{\alpha \in I} A_\alpha = A_i \cup A_j$$

and

$$\cap_{\alpha \in I} A_\alpha = A_i \cap A_j.$$

Proposition 5. *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \varnothing$, then*

$$\cup_{\alpha \in I} A_\alpha = \varnothing$$

and

$$\cap_{\alpha \in I} A_\alpha = S.$$

Proposition 6. *For an indexed family* $\{A_\alpha\}_{\alpha\in I}$ *in* S*.*

$$C_S(\cup_{\alpha\in I}A_\alpha) = \cap_{\alpha\in I}C_S(A_\alpha)$$

and

$$C_S(\cap_{\alpha\in I}A_\alpha) = \cup_{\alpha\in I}C_S(A_\alpha).$$

13 Direct Products

13.1 Why

We can profitably generalize the notion of cartesian product to families of sets indexed by the natural numbers.

13.2 Direct Products

The **direct product** of family indexed by a subset of the naturals is the set whose elements are ordered sequences of elements from each set in the family. The ordering on the sequences comes from the natural ordering on N. If the index set is finite, we call the elements of the direct product n**-tuples**. If the index set is the natural numbers, and every set in the family is the same set A, we call the elements of the direct product the **sequences** in A.

13.2.1 Notation

For a family $\{A_\alpha\}_{\alpha \in I}$ of S with $I = \{1, \dots, n\}$, we denote the direct product by

$$\prod_{i=1}^{n} A_i.$$

We read this notation as "product over alpha in I of A sub-alpha." We denote an element of $\prod_{i=1}^{n} A_i$ by $(a_1, a_2, \ldots, a_n)$ with the understanding that $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

If I is the set of natural numbers we denote the direct product by

$$\prod_{i=1}^{\infty} A_i.$$

We denote an element of $\prod_{i=1}^{\infty} A_i$ by (a_i) with the understanding that $a_i \in A_i$ for all $i = 1, 2, 3, \ldots$. If $A_i = A$ for all $i = 1, 2, 3, \ldots$, then (a_i) is a sequence in A.

14 Sequences

14.1 Why

We introduce language for the steps of an infinite process.

14.2 Definition

A **sequence** is a function from the natural numbers to a set.

Equivalently, a sequence is an element of a direct product of a family of sets for which each set in the family is identical and the index set is the natural numbers.

14.2.1 Notation

15 Monotone Sequences

15.1 Why

If the base set of a sequence has a partial order, then we can discuss its relation to the order of sequence.

15.2 Definition

A sequence on a partially ordered set is **non-decreasing** if whenever a first index precedes a second index the element associated with the first index precedes the element associated with the second element. A sequence on a partially ordered set is **increasing** if it is non-decreasing and no two elements are the same. An increasing sequence is non-decreasing.

A sequence on a partially ordered set is **non-increasing** if whenever a first index precedes a second index the element associated with the first index succedes the element associated with the second element. A sequence on a partially ordered set is **decreasing** if it is non-increasing and no two elements are the same. A decreasing sequences is non-increasing.

A sequence on a partially ordered set is **monotone** if it is non-decreasing, or non-increasing. A sequence on a partially ordered set is **strictly monotone** if it is decreasing, or increasing.

15.2.1 Notation

Let A a non-empty set with partial order $\preceq$. Let $\{a_n\}_n$ a sequence in A.

The sequence is non-decreasing if $n \leq m \implies a_n \preceq a_m$, and increasing if $n < m \implies a_n \prec a_m$. The sequence is non-increasing if $n \leq m \implies a_n \succeq a_m$, and decreasing if $n < m \implies a_n > a_m$.

15.3 Examples

Example 7. *Let A a non-empty set and $\{A_n\}_n$ a sequence of sets in 2^A. Partially order elements of 2^A by the relation contained in.*

16 Nets

16.1 Why

We generalize the notion of sequence to index sets beyond the naturals.

16.2 Definition

A sequence is a function on the natural numbers; this set has two important properties: (a) we can order the natural numbers and (b) we can always go "further out."

To elaborate on property (b): if handed two natural numbers m and n, we can always find another, for example $\max\{m, n\}+1$, larger than m and n. We might think of larger as "further out" from the first natural number: 1.

Combining these to observations, we define a directed set:

Definition 8. *A* ***directed set*** *is a set D with a partial order $\preceq$ satisfying one additional property: for all $a, b \in D$, there exists $c \in D$ such that $a \preceq c$ and $b \preceq c$.*

Definition 9. *A* ***net*** *is a function on a directed set.*

A sequence, then, is a net. The directed set is the set of natural numbers and the partial order is $m \preceq n$ if $m \leq n$.

16.2.1 Notation

Directed sets involve a set and a partial order. We commonly assume the partial order, and just denote the set. We use the letter D as a mnemonic for directed.

For nets, we use function notation and generalize sequence notation. We denote the net $x : D \to A$ by $\{a_\alpha\}$, emulating notation for sequences. The use of α rather than n reminds us that D need not be the set of natural numbers.

17 Categories

17.1 Why

We generalize the notion of sets and functions.

17.2 Definition

A **category** is a collection of objects together with a set of **category maps** for each ordered pair of objects. The set of maps has a binary operation called **category composition**, whose induced algebra is associative and contains identities.

As the fundamental example, consider the category whose objects are sets and whose maps are functions. The sets are the objects of the category. The functions are the maps. The rule of composition is ordinary function composition. The map identities are the identity functions. We call this category the **category of sets**.

17.2.1 Notation

Our notation for categories is guided by our generalizing the notions of set and functions.

We denote categories with upper-case latin letters in script; for example, $\mathcal{C}$. We read $\mathcal{C}$ aloud as "script C." Upper case latin letters remind that the category is a set of objects. The script form reminds that these objects may themselves be sets.

We denote the objects of a category by upper-case latin letters, for example A, B, C; an allusion to the idea that these generalize sets. We denote the set of maps for an ordered pair of objects (A, B) by $A \to B$; an allusion to the function notation. We denote members of $A \to B$ using lower case latin letters, for example f, g, h; an allusion to our function notation.

18 Groups

18.1 Why

We generalize the algebraic structure of addition over the integers.

18.2 Definition

A **group** is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra **group addition**. A **commutative group** is a group whose operation commutes.

18.2.1 Notation

TODO

19 Rings

19.1 Why

We generalize the algebraic structure of addition and multiplication over the integers.

19.2 Definition

A **ring** is two algebras over the same ground set with: (1) the first algebra a commutative group (2) an identity element in the second algebra, and (3) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra **ring addition**. We call the operation of the second algebra **ring multiplication**.

19.2.1 Notation

TODO

20 Fields

20.1 Why

We generalize the algebraic structure of addition and multiplication over the rationals.

20.2 Definition

A **field** is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra **field addition**. We call the operation of the second algebra **field multiplication**.

20.2.1 Notation

$\boxed{\mathit{TODO}}$

21 Homomorphism

21.1 Why

We name a function which preserves group structure.

21.2 Definition

A **homomorphism** from group $(A, +)$ to group $(B, \tilde{+})$ is a function $f : A \to B$ such that $f(e_A) = f(e_B)$ for identities $e_A \in A$ and $e_B \in B$ and $f(a + a') = f(a) \tilde{+} f(a')$ for all $a, a' \in A$.

21.2.1 Notation

$\boxed{TODO}$

22 Cardinality

22.1 Why

We want to speak of the number of elements of a set. Subtetly arises when we can not finish counting the set's elements.

22.2 Finite Definition

If a set A is contained in a set B and not equal to B, we say that B is a **larger set** than A. Conversely, we say that A is a **smaller set** than B. We reason that we could pair the elements of B with themselves in A and still have some elements of B left over.

A **finite set** is one whose elements we can count and the process terminates. For example, $\{1, 2, 3\}$ or $\{a, b, c, d\}$. The **cardinality** of a finite set is the number of elements it contains. The cardinality of $\{1, 2, 3\}$ is 3 and the cardinality of $\{a, b, c, d\}$ is 4.

22.2.1 Notation

Let A be a non-empty set. We denote the cardinality of A by $|A|$.

22.3 Infinite Definition

Suppose we know that the counting process could never terminate. This situation superficially seems bizarre, but is in fact built in to some of our fundamental notions: namely, the natural numbers. We defined the natural numbers in a manner which made them not finite.

If we had a bag of natural numbers, we could use the total order to find the largest, and then use the existence of a successor to add a new largest number. Therefore, bizarrely, the process of counting the natural numbers can not terminate.

An **infinite set** is a non-empty set which is not finite. So the natural numbers are an infinite set. Alternatively we say that there are **infinitely many** natural numbers. The negating prefix "in" emphasizes that we have defined the nature of the size of the naturals indirectly: their size is not something we understand from the simple intuition of counting, but in contrast to the simple intuition of counting.

Still, we imagine that if we could go on forever, we could count the natural numbers; so in an infinite sense, they are countable. A **countable** set is one which is either (a) finite or (b) one for which there exists a one-to-one function mapping the natural numbers onto the set.

The natural numbers are countable: we exhibit the identity function. Less obviously the integer numbers and rational numbers are countable. Even more bizarre, the real numbers are not countable. An **uncountable** set is one which is not countable.

22.3.1 Notation

We denote the cardinality of the natural numbers by $\aleph_0$.

23 Subset Space

23.1 Why

We speak of a set and a set of its subsets satisfying properties. The utility of this abstract concept is proved by its examples, in future sheets.

23.2 Definition

A **subset space** is a a pair of sets: the second set contains subsets of the first.

We call the first set the **base set**. If the base set is finite, we call the subset space a **finite subset space**. A **distinguished subset** is an element of the second set. An **undistingished subset** is a subset of the first set which is not distinguished.

23.2.1 Notation

Let A be a set and $\mathcal{A} \subset 2^A$. We denote the subset space of A and $\mathcal{A}$ by $(A, \mathcal{A})$, read aloud as “A, script A.”

23.3 Example

Example 10. *Let A be a nonempty set. Let $\mathcal{A}$ be 2^A. Then $(A, \mathcal{A})$ is a subset space.*

24 Monotone Classes

24.1 Why

24.2 Definition

The **limit** of an increasing sequence of sets is the family union of the sequence. The **limit** of a decreasing sequence of sets is the family intersection of the sequence.

A **monotone limit** of an sequence of sets is the limit of a strictly monotone sequence.

A **monotone space** is a subset space in which monotone limits of strictly monotone sequences of distinguished sets are distinguished. We call the distinguished sets a **monotone class**.

24.2.1 Notation

Let A a non-empty set with partial order $\preceq$. Let $(A, \mathcal{A})$ be a subset space on A.

Let $\{A_n\}_n$ be an increasing or decreasing sequence in $\mathcal{A}$. We denote the limit of $\{A_n\}_n$ by $\lim_n A_n$.

If $\{A_n\}_n$ is increasing, $\lim_n A_n = \cup_n A_n$. If $\{A_n\}_n$ is decreasing, $\lim_n A_n = \cap_n A_n$.

If $(A, \mathcal{A})$ is a monotone space, then for all strictly monotone $\{A_n\}_n$ in $\mathcal{A}$, $\lim_n A_n \in \mathcal{A}$. In this case, $\mathcal{A}$ is a montone class.

25 Subset Algebra

25.1 Why

We speak of a subset space with set-algebraic properties.

25.2 Definition

A **subset algebra** is a subset space for which (1) the base set is distinguished (2) the complement of a distinguished set is distinguished (3) the union of two distinguished sets is distinguished.

We call the set of distinguished sets an **algebra** on the the base set. We justify this language by showing that the standard set operations applied to distinguished sets result in distinguished sets.

If a set of subsets is closed under complements it contains the base set if and only if it contains the empty set. So we can replace condition (1) by insisting that the algebra contain the empty set. Similarly, if a non-empty set of subsets is closed under complements and unions then it contains the base set: the union of a distinguished set and its complement. Thus we can replace condition (1) by insisting that the algebra be non-empty.

25.2.1 Notation

The notation follows that of a subset space. Let $(A, \mathcal{A})$ be a subset algebra. We also say "let $\mathcal{A}$ be an algebra on A." Moreover, since the largest element of the algebra is the base set, we can say without ambiguity: "let $\mathcal{A}$ be an algebra."

25.3 Properties

Proposition 11. *For any subset algebra, $\varnothing$ is distinguished.*

Proposition 12. *For any subset algebra, for any distinguished sets, (a) the intersection is distinguished and (b) their symmetric difference is distinguished. So, if one contains the other, the complement of the smaller in the larger is distinguished.*

Proposition 13. *For any subset algebra, for any finite family of distinguished sets, (a) the finite family union and (b) the finite family intersection are both distinguished.*

So we could have defined an algebra by insisting it be closed under finite intersections.

25.4 Examples

Example 14. *For any set A, $(A, 2^A)$ is a subset algebra.*

Example 15. *For any set A, $(A, \{A, \varnothing\})$ is a subset algebra.*

Example 16. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| < \aleph_0 \vee |C_A(B)| < \aleph_0\}.$$

$\mathcal{A}$ is an algebra; the ***finite/co-finite algebra****.*

Example 17. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| \leq \aleph_0 \vee |C_A(B)| \leq \aleph_0\}.$$

$\mathcal{A}$ is an algebra; the ***countable/co-countable algebra****.*

Example 18. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| \leq \aleph_0\}.$$

$\mathcal{A}$ is not an algebra.

Example 19. *Let A be an uncountable set. Let $\mathcal{A}$ be the collection of all countable subsets of A. $\mathcal{A}$ is not a sigma algebra.*

26 Rational Numbers

26.1 Why

26.2 Definition

27 Real Numbers

27.1 Why

27.2 Definition

28 Real Intervals

28.1 Why

We use frequently subsets of the real numbers which correspond to segments of the line.

28.2 Definition

Take two real numbers, with the first less than the second.

An **interval** is one of four sets:

1. the set of real numbers larger than the first number and smaller than the second; we call the interval **open**.

2. the set of real numbers larger than or equal to the first number and smaller than or equal to the second number; we call the interval **closed**.

3. the set of real numbers larger than the first number and smaller than or equal to the second; we call the interval **open on the left** and **closed on the right**.

4. the set of real numbers larger than or equal to the first number and smaller than the second; we call the interval **closed on the left** and **open on the right**.

If an interval is neither open nor closed we call it **half-open** or **half-closed**

We call the two numbers the **endpoints** of the interval. An open interval does not contain its endpoints. A closed interval contains its endpoints. A half-open/half-closed interval contains only one of its endpoints.

28.2.1 Notation

Denote the set of real numbers by R. Let $a, b \in R$ with $a < b$.

We denote the open interval from a to b by (a, b). This notation, although standard, is the same as that for ordered pairs; no confusion arises with adequate context.

We denote the closed interval from a to b by $[a, b]$. We record the fact $(a, b) \subset [a, b]$ in our new notation.

We denote the half-open interval from a to b, closed on the right, by $(a, b]$ and the half-open interval from a to b, closed on the left, by $[a, b)$.

29 Length

29.1 Why

We want to define the length of a subset of real numbers.

29.2 Common Notions

We take two common notions:

1. The length of the whole is the sum of the length of the parts; the **additivity principle**.
2. If one whole contains another, the first's length at least as large as the second's length; the **containment principle**.

The task is to make precise the use of "whole,", "parts," and "contains." We start with intervals.

29.3 Definition

The **length** of an interval is the difference of its endpoints: the larger minus the smaller.

Two intervals are **non-overlapping** if their intersection is a single point or empty. The **length** of the union of two non-overlapping intervals is the sum of their lengths.

A **simple** subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A **countably simple** subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A **cover** of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

29.3.1 Cuts

If a, b are real numbers and $a < b$, then we **cut** an interval with a and b as its endpoints by selecting c such that $a < c$ and $c < b$. We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the **cut pieces**.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

29.4 All sets

Proposition 20. *Not all subsets of real numbers are simple.*

Exhibit: R is not finite.

Proposition 21. *Not all subsets of real numbers are countably simple.*

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

29.4.1 Notation

30 Real Limits

30.1 Why

We want to speak of the infinite process

30.2 Definition

Take a sequence of real numbers.

We say that the sequence **converges** to a **limit point** if

31 Extended Real Numbers

31.1 Why

That some limits grow without bound leads us to add two elements to the set of real numbers.

31.2 Definition

The set of **extended real numbers** is the union of the set of real numbers with a set containing two distinct elements: one which we call **positive infinity** and one which we call **negative infinitty**.

31.3 Intervals

We

32 Real Length Impossible

32.1 Why

Given a subset of the real line, what is its length?

32.2 Background

Let $a, b \in R$ with $a \leq b$. The **length** of the closed interval of the real numbers $[a, b]$ is $b - a$. The length is non-negative.

A family $\{A_\alpha\}_{\alpha \in I}$ is **disjoint** if for $\alpha, \beta \in I$, $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \varnothing$. A set A can be **partioned** into a family if there exists a disjoint family whose union is A. A set $A \subset R$ is **simple** if it can be partitioned into a countable family whose members are closed intervals. The above discussion suggests that we should define the length of a simple set as the sum of the lengths of sets which parition it.

The above discussion suggests that if we wish to define a function length : $2^R \to R \cup \{-\infty, \infty\}$, we should ask that (1) length$(A) \geq 0$, (2) length$([a, b]) = b - a$, (3) for disjoint closed intervals $\{A_n\}_{n \in N}$, length$(A_i) = \sum_i$ length(A_i), and (4) for all $A \subset R$ and $a \in R$, length$(A + x) =$ length(A).

32.3 Converse

Define the equivalence relation $\sim$ on R by by $x \sim y$ if $x \sim y \in Q$

32.3.1 Notation

Let A be a set and $\mathcal{A} \subset 2^A$. We denote the subset algebra of A and $\mathcal{A}$ by $(A, \mathcal{A})$, read aloud as "A, script A."

32.4 Properties

Proposition 22. *For any set A, 2^A is a sigma algebra.*

Proposition 23. *The intersection of a family of sigma algebras is a sigma algebra.*

32.5 Generation

Proposition 24. *Let A a set and $\mathcal{B}$ a set of subsets. There is a unique smallest sigma algebra $(A, \mathcal{A})$ with $\mathcal{B} \subset \mathcal{A}$.*

We call the unique smallest sigma algebra containing B the **generated sigma algebra** of B.

33 Sigma Algebra

33.1 Why

For general measure theory, we need an algebra of sets closed under countable unions; we define such an object (TODO).

33.2 Definition

A **countably summable subset algebra** is a subset space for which (1) the base set is distinguished (2) the complement of a distinguished set is distinguished (3) the union of a sequence of distinguished sets is distinguished.

The name is justified, as each countably summable subset algebra is a subset algebra, because the union of $A_1, \dots, A_n$ coincides with the union of $A_1, \dots, A_n, A_n, A_n \dots$.

We say that the set of distinguished sets a **sigma algebra** on the base set; we justify this langauge, as for an algebra, by the closure properties under standard set operations.

33.2.1 Notation

The notation follows that of a subset space. Let $(A, \mathcal{A})$ be a countably summable subset algebra. We also say "let $\mathcal{A}$ be an sigma algebra on A." Moreover, since the largest element of the sigma algebra is the base set, we can say without ambiguity: "let $\mathcal{A}$ be a sigma algebra."

33.3 Examples

Example 25. *For any set A, 2^A is a sigma algebra.*

Example 26. *For any set A, $\{A, \varnothing\}$ is a sigma algebra.*

Example 27. *Let A be an infinite set. Let $\mathcal{A}$ the collection of finite subsets of A. $\mathcal{A}$ is not a sigma algebra.*

Example 28. *Let A be an infinite set. Let $\mathcal{A}$ be the collection subsets of A such that the set or its complement is finite. $\mathcal{A}$ is not a sigma algebra.*

Proposition 29. *The intersection of a family of sigma algebras is a sigma algebra.*

Example 30. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| \leq \aleph_0 \lor |C_A(B)| \leq \aleph_0\}.$$

*$\mathcal{A}$ is an algebra; the **countable/co-countable algebra**.*

$\boxed{TOOD : cleanupexamples}$

34 Generated Sigma Algebra

34.1 Why

A simple way to obtain a sigma algebra, is to ask it to obtain some sets, and then to ask it to contain all the sets it needs to fulfill the properties.

34.2 Definition

The **generated sigma algebra** for a set of subsets is the smallest sigma algbera containing the set of subsets. We must prove the existence and uniqueness of this sigma algebra.

Proposition 31. *The intersection of a non-empty set of sigma algebras on the same base set is a sigma algebra.*

Proof. Let $\{(A, \mathcal{A}_\alpha\}_{\alpha \in I}$ a familiy of sigma algebras on the same base set. Define $\mathcal{A}$ as $\cap_{\alpha \in I} \mathcal{A}_\alpha$.

1. For all $\alpha \in I$, $A \in \mathcal{A}_\alpha$, thus $A \in \mathcal{A}$; condition (a).

2. For all $B \in \mathcal{A}$, for all $\alpha \in I$, $B \in \mathcal{A}_\alpha$. Thus, for all $\alpha \in I$, $C_A(B) \in \mathcal{A}_\alpha$. And so $C_A(B) \in \mathcal{A}$; condition (b).

3. For all sequences $\{B_n\} \subset \mathcal{A}$, $\{B_n\} \subset \mathcal{A}_\alpha$ for all α. Thus $\cup_n B_n \in \mathcal{A}_\alpha$ for all α and so $\cup_n B_n \in \mathcal{A}$; condition (c).

□

On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

Proposition 32. *If A is a set and $\mathcal{A} \subset 2^A$, then there is a unique a smallest sigma algebra containing $\mathcal{A}$.*

Proof. We know of one sigma algebra containing $\mathcal{A}$: the power set of A. Thus, the set of sigma algebras containing $\mathcal{A}$ is not empty. Proposition 31 implies the intersection of all such sigma algebras (containing $\mathcal{A}$) is a sigma algebra. The intersection contains $\mathcal{A}$, and is contained in all other sigma algebras with this property, so is a smallest sigma algebra containing $\mathcal{A}$. If $\mathcal{B}, \mathcal{C}$ were two smallest sigma algebras, then $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C} \subset \mathcal{B}$, but then $\mathcal{B} = \mathcal{C}$; thus the smallest sigma algebra is unique. □

34.3 Notation

Let A be a set and $\mathcal{A} \subset 2^A$. We denote the sigma algebra generated by $\mathcal{A}$ by $\sigma(\mathcal{A})$.

35 Topological Sigma Algebra

35.1 Why

We often take the a the topology of a topological space as the generating set for the sigma algebra.

35.2 Definition

Given a topological space, the **topological sigma algebra** is the sigma algebra generated by the topology.

35.3 Notation

Let $(A, \mathcal{T})$ be a topological space. We denote the topological sigma algebra by $\sigma(\mathcal{T})$.

36 Borel Sigma Algebra

36.1 Why

We name and discuss the topological sigma algebra on the real numbers; the language and results generalize to finite direct products of the real numbers.

36.2 Definition

The **Borel sigma algebra** is the topological sigma algebra for the real numbers with the usual topology; we call its members the **Borel sets**.

36.3 Notation

Throughout this sheet we denote the real numbers by R. As usual, then, we denote the d-dimensional direct product of R by R^d. We denote the Borel sigma algebra on R^d by $\mathcal{B}(R^d)$. We denote $\mathcal{B}(R^1)$ by $\mathcal{B}(R)$.

36.4 Alternate Generations

The Borel sigma algebra is useful because it contains interesting sets besides the open sets. To make precise this statement, we show that the Borel sigma algebra is generated by, and therefore contains, other common subsets of R.

Proposition 33. *If a sigma algebra $\mathcal{A}$ includes a particular set of subsets $\mathcal{B}$, then $\mathcal{A}$ includes $\sigma(\mathcal{B})$.*

Proposition 34. *Each of*

(a) the collection of all closed subsets of R,

(b) the collection of all subintervals of R of the form $(-\infty, b]$,

(c) the collection of all subintervals of R of the form $(a, b]$,

generate $\mathcal{B}(R)$.

Proof. Denote the sigma algebra which corresponds to (a) by $\mathcal{B}_1$, that which corresponds to (b) by $\mathcal{B}_2$, and that which corresponds to (c) by $\mathcal{B}_3$. It suffices to establish $\mathcal{B}(R) \subset \mathcal{B}_3 \subset \mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{B}(R)$.

Start with $\mathcal{B}_1$. Closed sets are the complement of open sets. Thus $\mathcal{B}(R)$ contains all closed sets and so contains the sigma algebra generated by all closed sets, namely $\mathcal{B}_1$.

Next, $\mathcal{B}_2$. The intervals $(-\infty, b])$ are closed. Thus $\mathcal{B}_1$ contains all such intervals, and so contains the sigma algebra generated by such intervals, namely $\mathcal{B}_2$.

Next, $\mathcal{B}_3$. An interval $(a, b]$ is $(-\infty, b) \cap C_R((-\infty, a])$. Thus, all such intervals are contained in $\mathcal{B}_2$, and so $\mathcal{B}_2$ contains the sigma algebra generated by all such intervals, namely, $\mathcal{B}_3$.

Each open interval of R is the union of a sequence of sets $(a, b]$; namely $(a, b - 1/n]$. So $\mathcal{B}_3$ contains all open intervals (a, b). Each open set of R can be written as a countable union of open intervals (proof: $\boxed{TODO}$). Thus, $\mathcal{B}_3$ contains all open sets, and therefore contains the sigma algebra generated by the open subsets, namely $\mathcal{B}(R)$. $\square$

Proposition 35. *Each of:*

(a) the collection of all closed subsets of R^d,

(b) the collection of all closed half-spaces of R^d of the form

$$\{(x_1, \ldots, x_d) \mid x_i \leq b_i\}$$

for some index i and some b in R, and

(c) the collection of all rectangles of R^d of the form

$$\{(x_1, \ldots, x_d) \mid a_i < x_i \leq b_i\}$$

generate $B(R^d)$.

Proof. Follow the proof of Proposition 34.

The complement of open sets are closed. Closed half spaces are closed. A strip of the form $\{(x_1, \ldots, x_d) \mid a_i < x_i \leq b_i\}$ is the intersection of two half-spaces in (b). Each rectangle in (c) is the union of d such strips.

$\boxed{TODO}$: two step last piece, open rectangles are unions of rectangles in (c) and open sets are union of open rectangles.

$\square$

37 Measures

37.1 Why

We want to generalize the notion of length, area, volume beyond the Lebesgue measure on the product spaces of real numbers.

37.2 Definition

An extended-real-valued non-negative function on an algebra is **finitely additive** if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a sigma algebra is **countably additive** if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A **finitely additive measure** is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A **countably additive measure** is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call countably additive measures **measures**, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countable additive.

In the contex of measure, we call a countably unitable subset algebra a **measurable space**. We call the distinguished sets **measurable** sets. A **measure space** is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of teh measurable space.

37.3 Notation

Let A a set. Let $\mathcal{A}$ a sigma algebra on A. The pair $(A, \mathcal{A})$ is a measurable space.

Let $\mu : \mathcal{A} \to [0, \infty]$ a measure; thus: (a) $\mu(\varnothing) = 0$ and (b) for disjoint $\{A_n\} \subset \mathcal{A}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ The triple $(A, \mathcal{A}, \mu)$ is a measure space.

37.4 Examples

Example 36. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure. We call μ the* ***counting measure****.*

Example 37. *Let $(A, \mathcal{A})$ measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise.*

Then μ is a measure. We call μ the ***point mass*** *concentrated at a.*

Example 38. *Let R denote the real numbers. The Lebesgue measure on the measurable space $(R, \mathcal{B}(R))$ is a measure.*

Example 39. *Let N be the natural numbers. Let $\mathcal{A}$ the finite co-finite algebra on N. Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.*

Example 40. *Let $(A, \mathcal{A})$ a measurable space. Let $\mu : \mathcal{A} \to [0, +\infty]$ be 0 if $A = \varnothing$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.*

Example 41. *Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = 2^A$. Let $\mu : \mathcal{A} \to [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \varnothing$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.*

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \varnothing$ then using finite additivity we obtain a contradiction $1 = \mu(B \cup C) = \mu(B) + \mu(C) = 2$. $\square$

38 Finite Measures

38.1 Why

Sometimes we want finite measures.

38.2 Definition

A measurable set is **finite** if its measure is a real number. The measure space itself is **finite** if the base set is finite.

A measurable set is **sigma-finite** if there exists a sequence of finite measurable sets whose union is the set. The measure space itself is **sigma-finite** if the base set is sigma finite.

38.2.1 Notation

We denote that a measure space is finite by saying "Let $(A, \mathcal{A}, \mu)$ and $\mu(A) < +\infty$."

Example 42. *Let* $(A, \mathcal{A})$ *be a measurable space.*

The counting measure on $(A, \mathcal{A})$ *is finite if and only if the base set is finite. It is sigma finite if and only if the base set is a union of a sequence of finite sets.*

If $\mathcal{A} = 2^A$, then the counting measure is sigma finite if and only if A is countable.

Example 43. *A point mass measure is finite.*

Example 44. *Let R be the set of real numbers. The Lebesgue measure on $(R, \mathcal{B}(R))$ is sigma finite.*

39 Measure Properties

39.1 Why

We expect measure to have the common sense properties we stated when trying to define a notion of length for the real line.

39.2 Monotonicity

An extended-real-valued function on an alebra is **monotone** if, given a first distinguished set contained in a distinguished second set, the result of the first is no greater than the result of the second.

Proposition 45. *All measures are monotone.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $A, B \in \mathcal{A}$ and $A \subset B$. Then $B = A \cup (B - A)$, a disjoint union. So

$$\mu(B) = \mu(A \cup (B - A)) = \mu(A) + \mu(B - A),$$

by the additivity of μ. Since $\mu(B - A) \geq 0$, we conclude $\mu(A) \leq \mu(B)$. $\square$

Proposition 46. *$A \subset B$ and B finite means $\mu(B - A) = \mu(B) - \mu(A)$.* $\boxed{TODO}$

39.3 Subadditivity

Monotonicity along with additivity of measures give us one other convenient property: subadditivity.

An extended-real-valued function on an algebra is **subadditive** if, given a sequence of distinguished sets, the result of union of the sequence is no greater than the limit of the partial sums of the results on each element of the sequence.

Proposition 47. *All measures are subadditive.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space.

Let $\{A_n\} \subset \mathcal{A}$. Define $\{B_n\} \subset \mathcal{A}$ with $B_n := A_n - \cup_{i=1}^{n-1} A_i$. Then $\cup_n A_n = \cup_n B_n$, $\{B_n\}$ is a disjoint sequence, and $B_n \subset A_n$ for each n. So

$$\mu(\cup_n A_n) = \mu(\cup_n B_n) = \sum_{i=1}^{\infty} \mu(B_n) \leq \sum_{i=1}^{\infty} \mu(A_n),$$

by additivity and then montonicity of measure. $\square$

39.4 Limits

Measures also behave well under limits.

An extended-real-valued function on an algebra **resolves under increasing limits** if the result of the union of an increasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets. An extended-real-valued function on an algebra **resolves under decreasing**

limits if the result of the intersection of a decreasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets.

Proposition 48. *Measures resolve under increasing limits.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\}$ be an increasing sequence in $\mathcal{A}$. Then we want to show: $\mu(\cup_n A_n) = \lim_{n\to\infty} \mu(A_n)$.

Define $\{B_n\}$ such that $B_n := A_n - \cup_{i=1}^{n-1} A_i$. Then $\{B_n\}$ is disjoint, $A_n = \cup_{i=1}^{n} B_i$ for each n, $\cup_n A_n = \cup_n B_n$, and $\mu(\cup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \mu(B_i)$, by additivity. So

$$\begin{aligned}
\mu(\cup_n A_n) &= \mu(\cup_n B_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) \\
&= \lim_{n\to\infty} \mu(\cup_{i=1}^{n} B_i) \\
&= \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

□

Proposition 49. *Measures resolve under decreasing limits if there is a finite set in the decreasing sequence.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\}$ be a decreasing sequence in $\mathcal{A}$ with one element finite. Then we want to show: $\mu(\cap_n A_n) = \lim_{n\to\infty} \mu(A_n)$.

On one hand, let n_0 be the index of the first finite element of the sequence. Then for all $n \geq n_0$, the sequence is finite because of the monotonicity of measure. Denote this decreasing finite subsequence of sets by $\{B_n\}$. Then $\cap_n A_n = \cap_n B_n$ and $\lim_n A_n = \lim_n B_n$.

On the other hand, the sequence $\{B_1 - B_n\}$ is an increasing sequence in $\mathcal{A}$. Also $\cap_n B_n = B_1 - \cup_n (B_1 - B_n)$. So

$$\begin{aligned}
\mu(\cap_n B_n) &= \mu(B_1 - \cup_n (B_1 - B_n)) \\
&= \mu(B_1) - \mu(\cup_n (B_1 - B_n)) \\
&= \mu(B_1) - \lim_n \mu(B_1 - B_n) \\
&= \mu(B_1) - \left(\lim_n \mu(B_1) - \mu(B_n)\right) \\
&= \lim_n B_n.
\end{aligned}$$

□

40 Measure Space

40.1 Why

We want to generalize the notions of length, area, and volume.

40.2 Definition

A **measurable space** is a sigma algebra. We call the distinguished subsets the **measurable sets**.

A **measure** on a measurable space is a function from the sigma algebra to the positive extended reals. A **measure space** is a measurable space and a measure.

40.2.1 Notation

40.2.2 Properties

Proposition 50. *Let $(A, \mathcal{A})$ be a measurable space and $m : \mathcal{A} \to [0, \infty]$ be a measure.*

*If $B \subset C \subset A$, then $m(B) \leq m(C)$. We call this property of measures **monotonicity of measure**.*

Proposition 51. *For a measure space $(A, \mathcal{A}, m)$.*

If $B \subset C \subset A$, then $m(B) \leq m(C)$.

*We call this property the **monotonicity of measure**.*

Proposition 52. *For a measure space $(A, \mathcal{A}, m)$.*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

*We this property the **sub-additivty of measure**.*

Proposition 53. *For a measure space $(A, \mathcal{A}, m)$.*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

*We this property the **sub-additivty of measure**.*

Proposition 54. *For a measure space $(A, \mathcal{A}, m)$.*

$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

Proposition 55. *For a measure space $(A, \mathcal{A}, m)$.*

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n \to \infty} m(A_i)$$

40.2.3 Examples

Example 56. ***counting measure***

41 Topological Space

41.1 Why

We want to generalize the notion of continuity.

41.2 Definition

A **topological space** is a subset algebra for which: (1) the empty set and the base set are distinguished, (2) the intersection of a finite family of distinguished subsets is distinguished, and (3) the union of a family of distinguished subsets is distinguished. We call the set of distinguished subsets the **topology**. We call the distinguished subsets the **open sets**.

41.2.1 Notation

Let A a non-empty set. For the set of distinguished sets, we use $\mathcal{T}$, a mnemonic for topology, read aloud as "script T". We denote elements of $\mathcal{T}$ by O, a mnemonic for open. We denote the topological space with base set A and topology $\mathcal{T}$ by $(A, \mathcal{T})$. We denote the properties satisfied by elements of $\mathcal{T}$:

1. $X, \varnothing \in \mathcal{T}$

2. $\{O_i\}_{i=1}^n \subset \mathcal{T} \implies \cap_{i=1}^n O_i \in \mathcal{T}$

3. $\{O_\alpha\}_{\alpha\in I} \subset \mathcal{T} \implies \cup_{\alpha\in I} \in \mathcal{T}$

42 Matroids

42.1 Why

We generalize the notion of linear dependence.

42.2 Definition

A **matroid** is a finite subset algebra satisfying:

1. The subset of a distinguished set is distinguished.
2. For two distinguished subsets of nonequal cardinality, there is an element of the base set in the complement of the smaller set in the bigger set whose singleton union with the smaller set is a distinguished set.

An **independent subset** of a matroid is a distinguished subset. A **depenedent subset** of a matroid is an undistinguished subset.

42.2.1 Notation

We follow the notation of subset algebras, but use M for the base set, a mnemonic for matroid, and $\mathcal{I}$ for the distinguished sets, a menomic for independent.

Let $(M, \mathcal{I})$ a matroid. We denote the properties by

1. $A \in \mathcal{I} \wedge B \subset A \implies B \in \mathcal{I}$.
2. $A, B \in \mathcal{I} \wedge |A| < |B| \implies \exists x \in M : (A \cup \{x\}) \in \mathcal{I}$