Set Equality

Why

When are two sets the same?

Definition

Given sets denoted A and B, if $A = B$ then every element of A is an element of B and every element of B is an element of A. In other words, $(A = B) \longrightarrow ((A \subset B) \wedge (B \subset A))$.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Then $A = B$? We define it to be so.

Principle 1 (Determined by Members). *Sets are the same if every member of one is a member of the other and vice versa.*

In other words, two sets are identical if and only if every element of one is an element of the other. This principle is sometimes called the *principle of extension*. Roughly speaking, if we refer to the elements of a set as its *extension*, then we have declared that if we know the extension then we know the set. A set is determined by its extension.

ßDeductive principle

This definition allows us to deduce $A = B$ if we first deduce that each element of A is an element of B and then then that each element of B is an element of A. With these two implications, we use the principle of extension to conclude that the

sets are the same. In our notation, with the quantifiers for all sets denoted here by A and B,

$$(((A \subset B) \wedge (B \subset A)) \longleftrightarrow (A = B)).$$

Belonging and sets compared with ancestry and humans

Compare the principle of extension for identifying sets from their elements with an analogous principle for identifying people from their ancestors.

We can consider a person's ancestors. Namely, the person's parents, grandparents, great grandparents and so on. It is clear that if we label the same human with two names A and B, then A and B have the same ancestors. In other words, same human implies same ancestors. This is the analog of "if two sets are equal they have the same members".

On the other hand, if we have two people denoted by A and B, and we know that A has the same ancestors as B, we can not conclude that A and B denote the same human. For example, siblings have the same ancestors but are different people. This direction, same ancestors implies same human, is the analogue of "if they have the same elements, two sets are the same". It is false for humans and ancestors, but we define it to be true for sets and members.

The principle of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Set Equality

↓

Set Inclusion

Standardized Accounts

Definitions   Accounts

Deductions   Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Objects   Letters