Direct Products

Why

We generalize the product of two sets to a product of a family of sets. To do so we discuss sets of families.

Discussion for pairs

Let A and B be sets. There is a natural correspondence between the product set $A \times B$ (see Cartesian Products) and the set of families

$$Z = \{z : \{i, j\} \to (A \cup B) \mid z_i \in A \text{ and } z_b \in B\}.$$

The family $z \in Z$ corresponds with the pair (z_i, z_i). The pair (a, b) corresponds to the family $z \in Z$ defined by $z(i) = a$ and $z(j) = b$. So, ordered pairs can be put in one-to-one correspondence with families. The generalization of Cartesian products to more than two sets generalizes the notion for families.

Definition

Let X be a set. Let $A : I \to X$ be a family of subsets of X. The *direct product* or *family Cartesian product* of A is the set of all families $a : I \to X$ which satisfy $a_i \in A_i$ for every $i \in I$.

A function on a product is called a *function of several variables* and, in particular, a function on the product $X \times Y$ is called a *function of two variables.*

Notation

We denote the product of the family $\{A_i\}$ by

$$\prod_{i \in I} A_i$$

We read this notation as "product over i in I of A sub-i."

Projections

The word "projection" is used in two senses with families. Let I be a set, and let $\{A_i\}$ be a family of sets. Define $A = \prod_{i \in I} A_i$.

First, let $J \subset I$. There is a natural correspondence between the elements of A and those of $\prod_{j \in J} A_j$. To each element $a \in A$, we restrict a to J and this is restriction is an element of $\prod_{j \in J} A_j$. The correspondence is called the *projection* of A onto $\prod_{i \in J} A_i$. The projection in this sense is a set of families.

Second, consider the value of a family $a \in A$ at j. We call a_j the *projection of a onto index j* or the *j-coordinate* of a. This word *coordinate* is meant to follow the language used in defining ordered pairs. The projection in this sense is an element of A_j.

Direct Products
↓
Families
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Functions
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Relations
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Cartesian Products
↓ ↘
Set Powers
↙ ↘
Ordered Pairs   Unordered Triples
↓   ↓
Pair Unions
↓
Set Unions
↙   ↓
Unordered Pairs   Empty Set
↘   ↙
Set Specification
↓
Set Inclusion
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Standardized Accounts
↓
Accounts
↓
Deductions
↓
Quantified Statements
↓
Logical Statements
↓
Statements
↙ ↘
Identities   Sets
↘ ↙
Names
↙ ↘
Letters   Objects