



Why

How big can a quadratic form be? How small?

Result

Proposition 1. *Suppose $A \in \mathbf{S}^n$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_n x^\top x \leq x^\top A x \leq \lambda_1 x^\top x,$$

for all $x \in \mathbf{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ with $A = Q\Lambda Q^\top$. Express

$$\begin{aligned} x^\top A x &= x^\top Q\Lambda Q^\top x = (Q^\top x)^\top \Lambda (Q^\top x) \\ &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &= \lambda_1 \sum_{i=1}^n (q_i^\top x)^2 = \lambda_1 \|Q^\top x\|^2 = \lambda_1 \|x\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} x^\top A x &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &\geq \lambda_n \sum_{i=1}^n (q_i^\top x)^2 = \lambda_n \|Q^\top x\|^2 = \lambda_n \|x\|^2. \end{aligned}$$

□

Notation

For this reason, it is common to order the eigenvalues of $A \in \mathbf{S}^n$ by magnitude with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. λ_1 is sometimes denoted λ_{\max} and λ_n is sometimes denoted λ_{\min} .

Optimization implication

If $z = \alpha x$, then $z^\top A z = \alpha^2 x^\top A x$. Consider finding $x \in \mathbf{R}^n$ to maximize

$$\begin{aligned} & \text{maximize} && x^\top A x \\ & \text{subject to} && \|x\| = 1. \end{aligned}$$

Since the objective is $x^\top A x \leq \lambda_1$ for all $x \in \mathbf{R}^n$ with $\|x\| = 1$, a solution of this problem is the eigenvector $q_1 \in \mathbf{R}^n$ corresponding to λ_1 . In other words, these inequalities are tight.

