Why

In ordinary reduction, we obtain a sequence of row reducers.

Factorization of A from a sequence of reducers

Let $(A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be an ordinarily reducible linear system. The *ordinary reducer sequence* is a sequence of reducer matrices $L_1, \ldots L_{m-1}$ with $A_1 = L_1 A$ and $A_i = L_i A_{i-1}$ for $2 \leq i \leq m-1$. In other words, $U \in \mathbf{R}^{m \times m}$ defined by

$$U = L_{m-1} \cdots L_2 L_1 A \tag{1}$$

is the ordinary row reduction of A. U is upper triangular.

If $L_{m-1} \cdots L_2 L_1$ in Equation (1) is invertible, then we have

$$A = (L_{m-1} \cdots L_2 L_1)^{-1} U,$$

which is a factorization of A. Each L_i is invertible, so

$$(L_{m-1} \cdots L_2 L_1)^{-1} = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}.$$

So we are interested in the inverse of L_i for $i \leq m-1$. Recall that if x_1 is the first column of A, and x_2 is the second column of $L_1 A$ and x_k is the kth column of $L_{k-1} \cdots L_1 A$ for $k = 2, \ldots, m-1$, then

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -\ell_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -\ell_{mk} & & & 1 \end{bmatrix}$$

where $\ell_{jk} = {}^{x_{jk}}\!/\!_{x_{kk}}$ for $k < j \leq m$.

Properties

The two important properties of the L_i is that they have simple inverses and a simple product. Define

$$\ell_k = (0, \cdots, 0, \ell_{k+1,k}, \cdots, \ell_{m,k})$$

so that $L_k = L_k - \ell_k e_k^\top$ where $(e_k)_i$ is 1 if $k = i$ and 0 otherwise.

Prop: L_i^{-1} *is L_i with the subdiagonal entries negated. Proof.* From the sparsity pattern of ℓ_k, we have $e_k^\top \ell_k = 0$. So

$$(I - \ell_k e_k^\top)(I + \ell_k e_k^\top) = I - \ell_k e_k^\top \ell_k e_k^\top = I.$$

Prop: $L_k^{-1} L_{k+1}^{-1}$ *is the unit lower-triangular matrix with the entries of both L_k^{-1} and L_{k+1}^{-1} in their usual places. Proof.* From the sparsity pattern of ℓ_{k+1} we have $e_k^\top \ell_{k+1} = 0$ so that

$$L_k^{-1} L_{k+1}^{-1} = (I + \ell_k e_k^\top)(I + \ell_{k+1} e_{k+1}^\top) = I + \ell_k e_k^* + \ell_{k+1} e_{k+1}^*.$$

Combining these two results, we deduce that

$$L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1} = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{m1} & \ell_{m2} & \cdots & \ell_{m,m-1} & 1 \end{bmatrix}$$

If we define $L = L_1^{-1} \cdots L_{m-1}^{-1}$ we obtain $A = LU$. In other words, we have a factorization (the *ordinary reducer factorization*) of A in terms of two matrices. The first, L is unit lower triangular. The second, U, is upper triangular.

Ordinary Reducer Factorization

Ordinary Row Reductions • Row Reducer Matrices

Linear System Row Reductions • Triangular Matrices • Matrix Transpose

Linear Equation Solutions • Matrices and Linear Equations • Diagonal Matrices

Linear Equations • Matrix-Vector Products

Linear Combinations • Matrices

Arrays • Vectors

N-Dimensional Space

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Space • Real Plane • Real Line

Real Summation • Sequences • Real Order • Integral Line

Fields • Real Numbers • Direct Products • Geometry

Groups • Rational Numbers • Set Numbers

Integer Additive Inverses • Integer Arithmetic • Finite Sets

Natural Summation • Integer Sums • Inverse Elements • Natural Additive Identity • Natural Multiplicative Identity

Arithmetic • Equivalent Sets • Family Operations • Element Functions • Identity Elements

Integer Products • Natural Powers • Natural Equations • Function Inverses • Algebras • Family Unions and Intersections

Natural Products • Integer Numbers • Equations • Natural Order • Function Composites • Function Images • Operations • Families

Natural Sums • Equivalence Relations • Functions

Recursion Theorem • Relations

Peano Axioms

Natural Induction • Cartesian Products

Natural Numbers • Generalized Set Dualities

Successor Sets • Unordered Triples • Set Powers • Intersection of Empty Set • Partitions • Set Unions and Intersections

Ordered Pairs • Set Complements • Set Dualities • Pair Unions • Set Intersections

Set Differences • Set Unions • Pair Intersections

Unordered Pairs • Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities • Sets

Names

Letters • Objects

4