Why

We want an estimator for the parameters of a linear function, given observations of the function with additive noise.

Definition

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Let $x : \Omega \to \mathbf{R}^d$. Define $g : \Omega \to (\mathbf{R}^d \to \mathbf{R})$ by $g(\omega)(a) = a^\top x(\omega)$, for $a \in \mathbf{R}^d$. In other words, for each outcome $\omega \in \Omega$, $g_\omega : \mathbf{R}^d \to \mathbf{R}$ is a linear function with parameters $x(\omega)$. g_ω is the function of interest.

Let $a^1, \ldots, a^n \in \mathbf{R}^d$ a dataset with data matrix $A \in \mathbf{R}^{n \times d}$. Let $e : \Omega \to \mathbf{R}^n$ independent of x, and define $y : \Omega \to \mathbf{R}^n$ by

$$y = Ax + e.$$

In other words, $y_i = x^\top a^i + e_i$.

We call (x, A, e) a *probabilistic linear model*. Other terms include *linear model*, *linear regression model*, *bayesian linear regression*, and *bayesian analysis of the linear model*.[1] We call x the parameters, A a *design*, e the *error* or *noise* vector, and y the *observation* vector.

One may want an estimator for the parameters x in terms of y or one may be modeling the function g and want to predict $g(a)$ for $a \in A$ not in the dataset.

[1] The word bayesian is in reference to treating the object of interest—x—as a random variable.

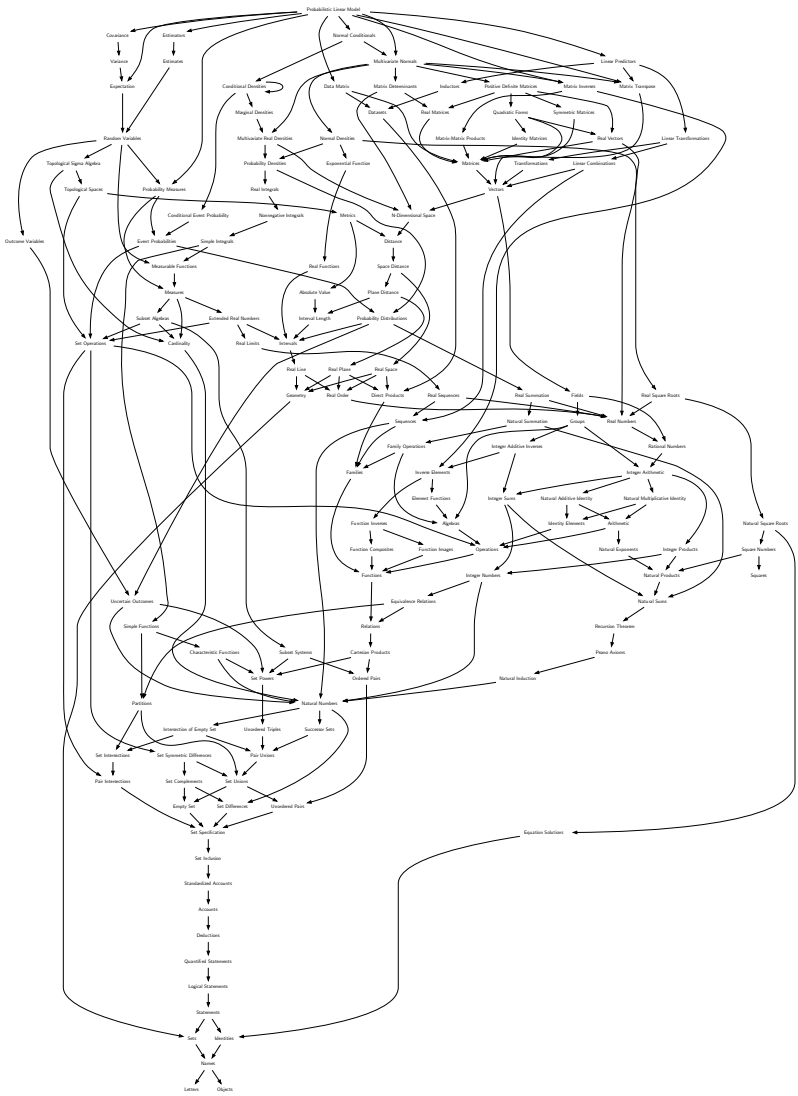
Mean and variance

Proposition 1. $\mathsf{E}(y) = A\,\mathsf{E}(x) + \mathsf{E}(w)$[2]

Proposition 2. $\mathsf{cov}((x, y)) = A\,\mathsf{cov}(x)A^\top + \mathsf{cov}\,e$[3]

[2]By linearity. Full account in future editions.

[3]Full account in future editions.


Probabilistic Linear Model
Covariance
Estimators
Normal Conditionals
Variance
Estimates
Multivariate Normals
Linear Predictors
Expectation
Data Matrix
Matrix Determinants
Induction
Positive Definite Matrices
Matrix Inverses
Matrix Transpose
Conditional Densities
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Datasets
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Multivariate Real Densities
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Real Sequences
Real Summation
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Real Square Roots
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Natural Summation
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Real Numbers
Family Operations
Integer Addition Inverses
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Inverse Elements
Integer Arithmetic
Element Functions
Integer Sums
Natural Addition Identity
Natural Multiplication Identity
Algebras
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Natural Products
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Uncertain Outcomes
Equivalence Relations
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Characteristic Functions
Subset Systems
Cartesian Products
Peano Axioms
Set Powers
Ordered Pairs
Natural Induction
Partitions
Natural Numbers
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Set Symmetric Differences
Pair Unions
Pair Intersections
Set Complements
Set Unions
Empty Set
Set Differences
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Set Specification
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