Definition

Let X be a (nonempty) set and k a field. Let $F \subset (X \to k)$ and let $\langle \cdot, \cdot \rangle : F \times F \to k$ be an inner product so that $(F, \langle \cdot, \cdot \rangle)$ is a complete inner product space.

A *reproducing kernel* of $(F, \langle \cdot, \cdot \rangle)$ is a map $R : X \times X \to k$ satisfying (1) for every $y \in X$ the function $R(\cdot, y) : X \to k$ is an element of F and (2) for every $f \in F$, at every $y \in X$, $f(y) = \langle f, R(\cdot, y) \rangle$ (the *reproducing property*).

R is called a "reproducing" kernel because of the following implication of the reproducing property. Notice that $R(\cdot, y) \in F$. For this reason,

Properties

If a reproducing kernel exists, it is unique.

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Let X be nonempty (index) set. For example, X may be $\{1, 2, \ldots, N\}$, $\mathbf{Z}$, $[0, 1]$, $\mathbf{R}^d$, $\{x \in \mathbf{R}^3 \mid \|x\| \leq 1\}$ (the unit sphere), or $\{x \in \mathbf{R}^3 \mid \alpha \leq \|x\| \leq \beta\}$ (the atmosphere, or volume between two concentric spheres).

A symmetric, real-valued function $k : X \times X \to \mathbf{R}$ of two variables is said to be *positive semidefinite* if for any $n \in \mathbf{N}$, for any real $a_1, \ldots, a_n \in \mathbf{R}$ and $x_1, \ldots, x_n \in X$, we have

$$\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) \geq 0,$$

and *positive definite* if the above holds with "$>$".[1]

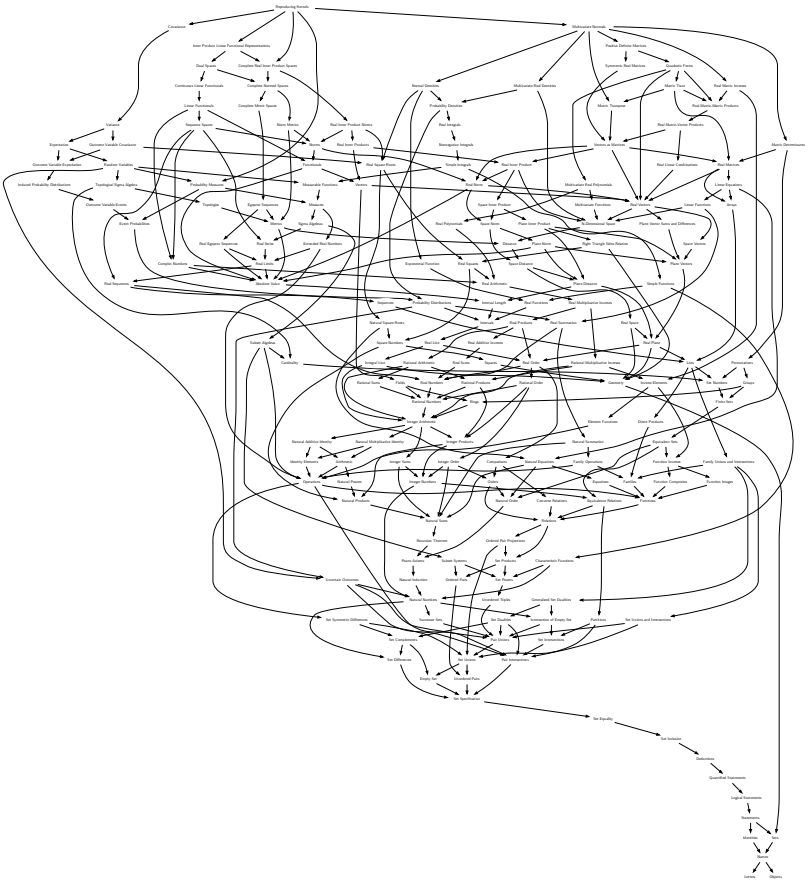
Positive semidefinite kernels are useful for the following constructive reason:

[1] Some authors use the term "positive definite" for our term positive semidefinite and the term "*strictly positive definite*" for our term "positive definite."

Proposition 1. *Let $X \neq \varnothing$ be a set. If $k : X \times X \to \mathbf{R}$ is positive semidefinite, then there exists a probability space $(\Omega, \mathbf{C}A, \mathbf{P})$ and a family of zero-mean normal real-valued random variables $\{f_x : \Omega \to \mathbf{R}\}_{x \in X}$ with covariance function k, that is,*

$$\mathbf{E} f(a) f(b) = k(a, b), \qquad \textit{for all } a, b \in X.$$

This result is known by the names *Kolmogorov extension theorem*, *Kolmogorov existence theorem*, *Kolmogorov consistency theorem* and *Daniell-Kolmogorov theorem.*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Inner Product Group Functions Representations
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Normal Distribution
Multivariate Real Matrices
Matrix Trace
Real Matrix Inverses
Linear Functionals
Complex Metric Spaces
Probability Densities
Matrix Transpose
Real Matrix Products
Variance
Sesquinorm Spaces
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Real graph
Real Matrix Inner Products
Matrix Determinants
Covariance
Random Variable Covariance
Real Inner Products
Stochastic graph
Matrix as Matrices
Real Linear Combination
Real Matrices
Covariance Matrix Estimation
Random Variables
Addition
Real Inner Norm
Simple Integrals
Real Inner Product
Multivariate Polynomials
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Unbiased Probability Distributions
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Multivariate Functions
Real Vectors
Inner Functions
Sum
Covariance Variance Errors
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Norm
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Real Eigenspace Eigenvalues
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Sum Sequence
Real Addition
Real Multiplication
Simple Functions
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