Tail Measure Lower Bound

1 Why

2 Result

We bound below the measure that a non-negative measurable real-valued function exceeds some value by its integral.

Proposition 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $g : X \to [0, \infty]$ be measurable and square-integrable. Then for all t such that $\int t d\mu \in [0, \int g d\mu)$,*

$$\mu(\{x \in X \mid g(x) > t\}) \geq \frac{(\int (g-t) d\mu)^2}{\int g^2 d\mu}.$$

Proof. Let t such that $\int t d\mu \in [0, \int g)$. We have selected t so that $\int (g-t) d\mu \geq 0$. Define $h = (g-t)^+$ and $A = \{x \in X \mid h(x) > 0\}$. Then

$$\int (g-t) d\mu \leq \int h d\mu = \int h \chi_A d\mu \leq \sqrt{\int h^2 d\mu \int \chi_A^2 \, d\mu}$$

□

Now $g^2 > h^2$, so $\int g^2 d\mu \geq \int h^2 d\mu$. Also $\chi_A^2 = \chi_A$ so $\int \chi_A^2 = \mu(A)$. $h(x) > 0$ if and only if $g(x) \geq t$ for all x. So $A = \{x \in X \mid g(x) \geq t\}$. Combining we have:

$$\int (g-t) d\mu \leq \sqrt{\left(\int g^2 d\mu \right) \mu(A)}.$$

Proposition 2. *Let X be a random variable with $\mathbf{E}(X^2) \leq \infty$. Then for all $t \in [0, \mathbf{E}(X))$, we have*

$$P(X > t) \geq \frac{(\mathbf{E}(X) - t)^2}{\mathbf{E}\, X^2}.$$

The above is also called the *Paley-Zygmund Inequality*.