

## CHORDAL GRAPHS AND VERTEX SEPARATORS

## Why

We characterize chordal graphs using vertex separators, and vice versa.<sup>1</sup>

## Main Result

**Proposition 1** (Chordal Graphs and Vertex Separators). An undirected graph is chordal if and only if all minimal vertex separators are complete.

*Proof.* Let G = (V, E) be an undirected graph. First, suppose that all minimal vertex separators of G are complete. Let c be a cycle of length greater than 3. Let v, w be nonconsecutive vertices in c. If v and w are adjacent in G, then  $\{v, w\} \in E$  is a chord. If v and w are nonadjacent, then vw-separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any vw-separator T. Split the cycle into the path from v to w, call it  $p_1$  and the path from v to v, call it  $v_2$ . v must include an interior point of v, call it v, otherwise v and v are connected. Similarly, v must include an interior point of v, call it v, and v are not consecutive in v, since they are distinct from v and v.

Let S be a minimal vw-separator. Let  $s, t \in S$  be two non-consecutive vertices in the cycle different from v and w By assumption S is complete, so s and t are adjacent in G.

Second, let G = (V, E) be a chordal graph. Let S be a minimal vwseparator. Let  $C_v$  and  $C_w$  be the connected components containing v and w of the subgraph induced by V - S.

If  $\nu mS = 1$ , then S is complete. Otherwise, let  $x, y \in S$  by distinct. We want to show  $\{x, y\} \in E$ . The key insight is that x is adjacent to vertices in  $C_v$  and  $C_w$ . If there were no such vertex,  $S - \{x\}$  would be a vw-separator and S would not be minimal. Similarly with y. Also,  $\nu mC_v$ ,  $\nu mC_w > 1$ .

<sup>&</sup>lt;sup>1</sup>Future editions will expand and may include graphics.

With these observations, there exists a path from x to y through  $C_v$ . Let  $p_v = (x, v_1, \ldots, v_k, y)$  be a path of shortest length with at least one interior vertex (so  $k \geq 1$ ) from x to y using interior vertices in  $v_1, \ldots, v_k \in C_v$ . Let  $p_w = (y, w_1, \ldots, w_l, x)$  be a path of shortest length with at least one interior vertex (so  $l \geq 1$ ) from y to x using interior vertices  $w_1, \ldots, w_k \in C_w$ . Use  $p_v$  and  $p_w$  to define the cycle  $c = (x, v_1, \ldots, v_k, y, w_1, \ldots, w_l, x)$  which has length at least four. G is chordal, so c has a chord.

We argue that the chord of c is  $\{x,y\}$ . Since  $C_w$  and  $C_V$  are different connected components (whose vertices are not included in S), there are no edges  $\{v_i, w_j\}$  for i = 1, ..., k and j = 1, ..., l. Since  $p_v$  and  $p_w$  are paths of shortest length, they have no chords. In particular, there is no edge  $\{v_i, v_j\}$  for |i - j| > 1 or  $\{v_i, x\}$  for i = 1, ..., l. Similarly, there is no edge  $\{w_i, w_j\}$  for |i - j| > 1 or  $\{w_i, y\}$  for i = 1, ..., l. The only remaining pair is  $\{x, y\}$ , and so it must be the chord.

