Bourbaki

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Contents

1 Objects

1.1 Why

We want to talk about things.

1.2 Definition

We use the word **object** with its usual sense in the English language. An object may be tangible, in that we can hold or touch it, or an object may be abstract, in that we can do neither.

1.2.1 Notation

To aid in discussing and denoting objects, let us tend to give them short names. A single Latin letter regularly suffices: for example, a, b or c. To aid our memory, we tend to choose the letter mnemonically.

We denote that the object a and the object b are the same object by $a = b$, read aloud as "a is b." We denote that the object a and b are different by $a \neq b$, read aloud as "a is not b."

2 Sets

2.1 Why

We want to talk about none, one, or several objects considered as a whole, for which we will use the word *set*.

2.2 Definition

A **set** is an abstract object which we think of as several objects considered at once. We say that the set **contains** the objects so considered. We call these the **elements** of the set.

We call the set which contains no objects the **empty set**. We call a set which contains only a single object a **singleton**. A singleton is not the same as the object it contains. Besides these two cases, we think of sets as containing two or more objects.

The objects a set contains may be other sets. This may be subtle at first glance, but becomes familiar with experience.

2.2.1 Notation

Let us tend to denote sets by upper case Latin letters: for example, A, B, and C. To aid our memory, let us tend to use the lower case form of the letter for an element of the set. For example, let A and B be non-empty sets. Let us tend to denote by a an element of A, and likewise, by b an element of B

Let us denote that an object a is an element of a set A by $a \in A$. We read the notation $a \in A$ aloud as "a in A." The $\in$ is a stylized lower case Greek letter: ϵ. It is read aloud "ehp-sih-lawn" and is a mnemonic for "element of". We write $a \notin A$, read aloud as "a not in A," if a is not an element of A.

If we have named the elements of a set, and can list them, let us do so between braces. For example, let a, b, and c be three distinct objects. Denote by $\{a, b, c\}$ the set containing theses three objects and only these three objects. We can further compress notation, and denote this set of three objects by A: so, $A = \{a, b, c\}$. Then $a \in A$, $b \in A$, and $c \in A$. Moreover, if d is an object and $d \in A$, then $d = a$ or $d = b$ or $d = c$.

We denote the empty set by $\varnothing$. Note that $\varnothing \neq \{\varnothing\}$. The left hand side, $\varnothing$, is the empty set. The right hand side, $\{\varnothing\}$, is the singleton whose element is the empty set. We distinguish the set containing one element from the element itself.

3 Set Examples

3.1 Why

We give some examples of objects and sets.

3.2 Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, $\ldots$, x, y, z. This set has twenty-six elements.

4 Set Extension

4.1 Why

When are two sets the same?

4.2 Definition

Two sets are **equal** if and only if they have the same elements.

4.2.1 Notation

Let A and B be two sets. As with any objects, we denote that A and B are equal by $A = B$.

5 Subsets

5.1 Why

We want to speak of sets which contain all the elements of other sets.

5.2 Two Sets

A **subset** of a set A is any set B for which each element of the set B is an element of the set A. In this case, we say that B is a subset of A. Conversely, we say that A is a **superset** of B.

Every set is a subset of itself. So if the set A is the set B, then A is a subset of B and B is a subset of A. Conversely, if A is a subset of B and B is a subset of A, then A is B. To argue that A is B, we argue that membership in A implies membership in B and second, we argue that membership in B implies membership in A.

The **power set** of a set is the set of all subsets of that set. It includes the set itself and the empty set. We call these two sets **improper subsets** of the set. We call all other sets **proper subsets**.

5.2.1 Notation

Let A and B be sets. We denote that A is a subset of B by $A \subset B$. We read the notation $A \subset B$ aloud as "A subset B".

If $A \subset B$ and $B \subset A$, then $A = B$. The converse also holds.

We denote the power set of A by 2^A, read aloud as "two to the A." $A \in 2^A$ and $\varnothing \in 2^A$. However, $A \subset 2^A$ is false.

5.2.2 Examples

Let a, b, c be distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in 2^A$. As always, $\varnothing \in 2^A$ and $A \in 2^A$ as well. In this case, we can list the elements (which are sets) of the power set:

$$2^A = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$

6 Set Specification

6.1 Why

We specify a subset of a known set via a property.

6.2 Definition

TODO: add the axiom

Let A be a nonempty set. We use the notation

$$\{a \in A \mid \text{———} \}$$

to indicate a subset of A that satisfies some property specified after the $\mid$. We read the symbol $\mid$ aloud as "such that." We read the whole notation aloud as "a in A such that..."

We call the notation **set-builder notation**. Set-builder notation avoids enumerating elements.

6.3 Example

For example, let L be the set of Latin letters and V the set of Latin vowels. A first notation for V is $\{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$. A second notation for V is $\{l \in L \mid l \text{ is a vowel}\}$. We may prefer the second, in cases when it saves time. This notation is really indispensable for sets which have many members, too many to reasonably write down.

7 Ordered Pairs

7.1 Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

7.2 Definition

Let A and B be non-empty sets. Let $a \in A$ and $b \in B$. The **ordered pair** of a and b is the set $\{\{a\}, \{a, b\}\}$. The **first element** of $\{\{a\}, \{a, b\}\}$ is a and the **second element** is b.

The **cartesian product** of A and B is the set of all ordered pairs. If $A \neq B$, the ordering causes the cartesian product of A and B to differ from the cartesian product of B with A. If $A = B$, however, the symmetry holds.

7.2.1 Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the cartesian product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

8 Relations

8.1 Why

How can we relate the elements of two sets?

8.2 Definition

A **relation** between two non-empty sets A and B is a subset of $A \times B$. A relation on a single set C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The pair (a, b) may or may not be in a relation on A and B. If $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

8.2.1 Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as "a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

9 Functions

9.1 Why

We want a notion for a correspondence between two sets.

9.2 Definition

A **functional** relation on two sets relates each element of the first set with a unique element of the second set. A **function** is a functional relation.

The **domain** of the function is the first set and **codomain** of the function is the second set. The function **maps** elements **from** the domain **to** the codomain. We call the codomain element associated with the domain element the **result** of **applying** the function to the domain element.

9.2.1 Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially

f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in A \to B$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

10 Operations

10.1 Why

We want to "combine" elements of a set.

10.2 Definition

Let A be a non-empty set. An **operation** on A is a function from ordered pairs of elements in the set to the same set. We use operations to combine the elements. We operate on pairs.

10.2.1 Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\,g\,b$ instead. We call this **infix notation**.

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

11 Algebras

11.1 Why

We name a set together with an operation.

11.2 Definition

An **algebra** is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The **ground set** of the algebra is the set on which the operation is defined.

11.2.1 Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

12 Solving Equations

12.1 Why

If I am holding 3 stones in both hands, and I have two of them in my left hand, how many might I have in my right hand? If I am holding fewer than 5 pebbles, how many might I have in my right hand?

12.2 Overview

I am asserting above that I have three pebbles and that I have two in my left hand. We know that the total nu A solution of the equation is any possible number of pebbles I could be holding in my right hand such that the sum is three.

Discuss solutions in a set. Discuss existence of solution. Discuss uniqueness of solution.

12.2.1 Notation

We can denote the pebble equation as find n so that

$$2 + n = 3$$

13 Integer Numbers

13.1 Why

TODO

14 Groups

14.1 Why

We generalize the algebraic structure of addition over the integers.

14.2 Definition

A **group** is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra **group addition**. A **commutative group** is a group whose operation commutes.

14.2.1 Notation

TODO

15 Fields

15.1 Why

We generalize the algebraic structure of addition and multiplication over the rationals.

15.2 Definition

A **field** is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra **field addition**. We call the operation of the second algebra **field multiplication**.

15.2.1 Notation

TODO

16 Absolute Value

16.1 Why

We want a notion of distance between elements of the real line.

16.2 Definition

We define a function mapping a real number to its length from zero.

17 Vectors

17.1 Why

We speak of objects which we can add and scale.

17.2 Definition

A **vector space** is a tuple of three objects with properties.

The objects are: (1) a commutative group, (2) a field, and (3) a function from the product of the field ground set and group ground set to the group ground set.

We call the ground set of the commutative group the **vectors** and we call the group operation **vector addition**. We call the field the **scalars**. The field has addition and multiplication of scalars which we call **scalar addition** and **scalar multiplication**, respectively. The **scaling map** is the map on pairs of scalars and vectors.

The first property is that the result of scaling a vector by a scalar, and then scaling the result by a second scalar is the same as first multiplying the scalars and then scaling the vector. The second property is that the result of scaling a vector by the scalar sum of two scalars is the same as scaling the vector by each scalar separately and then adding the vectors. The third property is that the result of scaling the vector sum of two vectors is the same as scaling each individually and then adding the two vectors.

17.2.1 Notation

Let $(V, +)$ be a commutative group and $(F, \tilde{+}, \cdot)$ be a field. Let $s : F \times V \to V$. $((V, +), (F, \tilde{+}, \cdot), s)$ is a vector space if

1. $s(b, s(a, v)) = s(b \cdot a, v)$ for all $a, b \in F$ and $v \in V$.
2. $s(a \tilde{+} b, v) = s(a, v) + s(b, v)$ for all $a, b \in F$ and $v \in V$.
3. $s(a, u + v) = s(a, u) + s(a, b)$ for all $a \in F$ and $u, v \in V$.

We now introduce four notational simplifications.

First, we denote both vector and scalar addition by $+$. So $a + b$ denotes $a \tilde{+} b$ for all $a, b \in F$. Second, we contract $\cdot$ So ab denotes $a \cdot b$ for all $a, b \in F$. Third, we contract the scaling map. So av denotes $s(a, v)$ for $a \in F$ and $v \in V$. Fourth, we take the scaling map to have precedence so that $au + av$ means $(au) + (av)$ for all $a \in F$ and $u, v \in V$.

We denote: (1) $b(av) = (ba)v$; (2) $(a + b)v = av + bv$; and (3) $a(u + v) = au + av$ for all $a, b \in F$ and $u, v \in V$.

Finally, we say: let (V, F) be a vector space, assuming the above notational conventions.

18 Linear Combinations

19 Linear Independence

20 Vector Bases

20.1 Why

TODO

21 Vector Space Dimension

21.1 Why

We noticed that the number of vectors in any basis is the same.

21.2 Definition

Consider a vector space.

A vector space is **finite-dimensional**. if it has a finite basis; otherwise it is it is **infinite-dimensional**.

The **dimension** of a finite-dimensional vector space is the number of elements in any basis.

21.2.1 Notation

22 Norms

22.1 Why

We want to measure the size of an element in a vector space.

22.2 Definition

A **norm** is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisifies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms.

A **norm space** is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

22.2.1 Notation

Let (X, F) be a vector space where F is the field of real numbers or the field of complex numbers. Let R denote the set of real numbers. Let $f : X \to R$. The functional f is a norm if

1. $f(v) \geq 0$ for all $x \in V$
2. $f(v) = 0$ if and only if $x = 0 \in X$.
3. $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in F$, $x \in X$
4. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.

In this case, for $x \in X$, we denote $f(x)$ by $|x|$, read aloud "norm x". The notation follows the notation of absolute value as a norm. When we wish to distinguish the norm from the absolute value function, we may write $\|x\|$. In some cases, we go further, and for a norm indexed by some parameter α or set A we write $\|x\|_\alpha$ or $\|x\|_A$.

23 Norm Examples

23.1 Why

We give some standard examples of norms.

23.2 Examples

Example 1. *The absolute value function is a norm on the vector space of real numbers.*

Example 2. *The Euclidean distance is a norm on the various real spaces.*