



## Why

We expect measure to have the common sense properties we stated when trying to define a notion of length for the real line.

## Monotonicity

An extended-real-valued function on an algebra is *monotone* if, given a first distinguished set contained in a distinguished second set, the result of the first is no greater than the result of the second.

**Proposition 1.** *All measures are monotone.*

*Proof.* Let  $(A, \mathcal{A}, \mu)$  be a measure space. Let  $A, B \in \mathcal{A}$  and  $A \subset B$ . Then  $B = A \cup (B - A)$ , a disjoint union. So

$$\mu(B) = \mu(A \cup (B - A)) = \mu(A) + \mu(B - A),$$

by the additivity of  $\mu$ . Since  $\mu(B - A) \geq 0$ , we conclude  $\mu(A) \leq \mu(B)$ .  $\square$

**Proposition 2.** *If  $A \subset B$  and  $B$  finite, then  $\mu(B - A) = \mu(B) - \mu(A)$ .*<sup>1</sup>

## Subadditivity

Monotonicity along with additivity of measures give us one other convenient property: subadditivity.

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<sup>1</sup>Future editions will contain a proof.

An extended-real-valued function on an algebra is *subadditive* if, given a sequence of distinguished sets, the result of union of the sequence is no greater than the limit of the partial sums of the results on each element of the sequence.

**Proposition 3.** *All measures are subadditive.*

*Proof.* Let  $(A, \mathcal{A}, \mu)$  be a measure space.

Let  $\{A_n\} \subset \mathcal{A}$ . Define  $\{B_n\} \subset \mathcal{A}$  with  $B_n := A_n - \cup_{i=1}^{n-1} A_i$ . Then  $\cup_n A_n = \cup_n B_n$ ,  $\{B_n\}$  is a disjoint sequence, and  $B_n \subset A_n$  for each  $n$ . So

$$\mu(\cup_n A_n) = \mu(\cup_n B_n) = \sum_{i=1}^{\infty} \mu(B_n) \leq \sum_{i=1}^{\infty} \mu(A_n),$$

by additivity and then monotonicity of measure. □

*The inequality involved in subadditivity is sometimes called Boole's inequality or Bonferroni's inequalities or the union bound; each of these terms is most common with discussion Probability Measures.*

## Limits

Measures also behave well under limits.

An extended-real-valued function on an algebra *resolves under increasing limits* if the result of the union of an increasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets. An extended-real-valued function on an algebra *resolves under decreasing*

*limits* if the result of the intersection of a decreasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets.

**Proposition 4.** *All measures resolve under increasing limits.*

*Proof.* Let  $(A, \mathcal{A}, \mu)$  be a measure space. Let  $\{A_n\}$  be an increasing sequence in  $\mathcal{A}$ . Then we want to show:  $\mu(\cup_n A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$ .

Define  $\{B_n\}$  such that  $B_n := A_n - \cup_{i=1}^{n-1} A_i$ . Then  $\{B_n\}$  is disjoint,  $A_n = \cup_{i=1}^n B_i$  for each  $n$ ,  $\cup_n A_n = \cup_n B_n$ , and  $\mu(\cup_{i=1}^n B_i) = \sum_{i=1}^n \mu(B_i)$ , by additivity. So

$$\begin{aligned} \mu(\cup_n A_n) &= \mu(\cup_n B_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(B_i) \\ &= \lim_{n \rightarrow \infty} \mu(\cup_{i=1}^n B_i) \\ &= \lim_{n \rightarrow \infty} \mu(A_n). \end{aligned}$$

□

**Proposition 5.** *Measures resolve under decreasing limits if there is a finite set in the decreasing sequence.*

*Proof.* Let  $(A, \mathcal{A}, \mu)$  be a measure space. Let  $\{A_n\}$  be a decreasing sequence in  $\mathcal{A}$  with one element finite. Then we want to show:  $\mu(\cap_n A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$ .

On one hand, let  $n_0$  be the index of the first finite element of the sequence. Then for all  $n \geq n_0$ , the sequence is finite because of the monotonicity of measure. Denote this decreasing finite subsequence of sets by  $\{B_n\}$ . Then  $\cap_n A_n = \cap_n B_n$  and  $\lim_n A_n = \lim_n B_n$ .

On the other hand, the sequence  $\{B_1 - B_n\}$  is an increasing sequence in  $\mathcal{A}$ . Also  $\cap_n B_n = B_1 - \cup_n (B_1 - B_n)$ . So

$$\begin{aligned}
 \mu(\cap_n B_n) &= \mu(B_1 - \cup_n (B_1 - B_n)) \\
 &= \mu(B_1) - \mu(\cup_n (B_1 - B_n)) \\
 &= \mu(B_1) - \lim_n \mu(B_1 - B_n) \\
 &= \mu(B_1) - \left( \lim_n \mu(B_1) - \mu(B_n) \right) \\
 &= \lim_n \mu(B_n).
 \end{aligned}$$

□

