Why

We want to talk about sequences of real numbers which, as we go further and further in the sequence, get closer and closer to some fixed real number.

Definition

We need to talk about "where" a sequence is "approaching."

A *limit* of a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers is a real number x_0 with the property that, for any interval centered at x_0, we can find a final part of the sequence wholly contained in that interval; no matter how small the interval. By making the interval small, we capture to intuition that the sequence is "close" to its limit.

In other words, you propose a limit for a sequence. To test this proposal, I specify some small positive real number. Then we look for a final part of the sequence which is wholly contained in the interval whose width is twice the small positive number. If we can always find the final part, no matter how small the positive number I specified, then the proposed limit is true.

Existence

Some sequences have no limit. Consider the sequence which alternates between $+1$ and -1. To show that the limit does not exist, we argue indirectly. We take any real number and test it with the interval length one. No matter which real number we have selected, $+1$ and -1 are a distance two apart, and so can not both be contained in an interval of width one.

In the case that there exists a limit for a sequence, we say that the sequence *converges to* its limit (or we say that it *converges*, or call it *convergent* or *converging*).

Uniqueness

If a sequence has a limit, it has only one limit. So, from here on, we will speak of *the limit* of the sequence.

To see this uniqueness, suppose that two real numbers satisfy the limiting property. We now argue indirectly: suppose also that they are not equal. Denote the distance between them by x. Then ask for final parts in intervals of width $x/2$ for both limits.

Approximation

We use limits to speak about the terminating behavior of infinite processes. We think about the sequence as approximating the limit. The sequence may never actually take the value of its limit, so the limit need not be in the set of terms of the sequence. But the idea, of course, is that the set of terms, especially those "far out" in the sequence, are close to the limit.

The definition, moreover, ensures that the sequence will get arbitrarily close. We can operationalize this property, by taking the first element of that final part after which all elements are close to the limit. This element of the sequence approximates the limit value well.

Notation

Let $(a_n)_n$ be a sequence of real numbers. Let a be a real number. We denote that a is the limit of $(a_n)_n$ by

$$a = \lim_{n \to \infty} a_n.$$

The abbreviation "lim" is from the Latin word *limes*, which means boundary, or limit.

We read this statement aloud as "a is the limit of a sub n." The above statement asserts two facts: (1) the sequence $(a_n)_n$ has a limit and (2) the limit is the real number a. We sometimes abbreviate this by writing $a = \lim_n a_n$.

Real Limits

Absolute Value

Interval Length

Intervals

Real Line

Real Order — Real Sequences

Rational Order — Real Numbers — Sequences

Integral Line — Rational Numbers — Lists

Integer Arithmetic — Natural Fractions — Set Numbers — Direct Products

Integer Products — Natural Multiplicative Identity — Natural Additive Identity — Finite Sets

Identity Elements — Natural Arithmetic — Equivalent Sets — Family Unions and Intersections

Integer Sums — Natural Powers — Integer Order — Comparisons — Function Inverses

Integer Numbers — Natural Products — Operations — Orders — Function Composites — Function Images — Families

Natural Sums — Equivalence Relations — Converse Relations — Natural Order — Functions

Recursion Theorem — Relations

Peano Axioms — Ordered Pair Projections

Natural Induction — Set Products

Natural Numbers — Set Powers — Generalized Set Dualities

Set Symmetric Differences — Set Partitions — Successor Sets — Intersection of Empty Set — Unordered Triples — Set Dualities — Set Unions and Intersections

Set Complements — Set Intersections — Pair Unions — Ordered Pairs

Set Differences — Set Unions — Pair Intersections

Empty Set — Unordered Pairs

Geometry — Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects