Defining result

Proposition 1. *Let $(X, \mathcal{A})$ be a measurable space. Let μ and ν be finite measures with $\nu \ll \mu$. There exists $g : X \to [0, \infty)$ such that*

$$\nu(A) = \int_A g \, d\mu$$

for all $A \in \mathcal{A}$. The function g is μ-almost everywhere unique.

Proof. Define

$$\mathcal{F} = \left\{ f : X \to [0, \infty) \,\middle|\, f \text{ measurable and } \int_A f d\mu \leq \nu(A) \right\}.$$

The function $f \equiv 0$ is in $\mathcal{F}$, since it is a measurable simple function whose integral over every measureable set is zero. If f_1 and f_2 are in $\mathcal{F}$, then $f_1 \vee f_2$ is in $\mathcal{F}$. To check, let $A \in \mathcal{A}$, and define the sets $A_1 = \{x \in A \mid f_1(x) > f_2(x)\}$ and $A_2 = \{x \in A \mid f_1(x) \leq f_2(x)\}$. A_1 and A_2 partition A, so

$$\int_A f_1 \vee f_2 = \int_{A_1} f_1 \vee f_2 + \int_{A_2} f_1 \vee f_2$$

$$= \int_{A_1} f_1 + \int_{A_2} f_2$$

$$\leq \nu(A_1) + \nu(A_2)$$

Since A_1 and A_2 partition A,

$$\nu(A_1) + \nu(A_2) = \nu(A_1 \cup A_2) = \nu(A).$$

Select a sequence of functions $(f_n)_n$ in $\mathcal{F}$ so that

$$\lim_n \int f_n = \sup\left\{ \int f \,\middle|\, f \in \mathcal{F} \right\}.$$

Toward ensuring the sequence is increasing, define $g_1 = f_1$, $g_2 = g_1 \vee f_2$, and $g_n = g_{n-1} \vee f_n$ for $n \geq 3$. Using the observation in the previous paragraph, $g_n \in \mathcal{F}$ for each n.

Let g be the pointwise limit of the $(g_n)_n$. The monotone convergence of integrals shows

$$\int_A g = \lim_n \int_A g_n.$$

for each $A \in \mathcal{A}$. Since $\int_A g_n \leq \nu(A)$, so too is the limit and thus so too is $\int_A g$. Thus, $g \in \mathcal{F}$. By construction, for $A = X$, $\int g = \sup\{\int f \mid f \in \mathcal{F}\}$. We have constructed an element of $\mathcal{F}$ attaining the supremum.

We know that the integral of g on A with respect to μ is bounded above by $\nu(A)$. We want the gap to be zero. Regardless of the gap, the function $\nu_0 : \mathcal{A} \to [0, \infty)$ defined by

$$\nu_0(A) = \nu(A) - \int (g, A, \mu),$$

for each $A \in \mathcal{A}$ is a positive measure. If ν_0 is identically zero, then there is no gap.

Suppose there is a gap: then there exists a measurable set with strictly positive measure under ν_0. Since the base set contains this set, and measures are monotone, the base set must have stricty positive measure. Since μ is finite, there exists a natural number n so that

$$\nu_0(X) > \frac{1}{n}\mu(X).$$

Define a new measure $\nu_1 = \nu_0 - \frac{1}{n}\mu$. Denote a signed-set decomposition of ν_1 by (P, N). Then $\nu_1(A \cap P) \geq 0$, or equivalently,

$$\nu_0(A \cap P) - \frac{1}{n}\mu(A \cap P) \geq 0,$$

for all A, and so

$$\nu(A) = \nu_0(A) + \int (g, A, \mu)$$
$$\geq \nu_0(A \cap P) + \int (g, A, \mu).$$

$\square$

Many authorities refer to this result as the *Radon-Nikodym theorem*.

Notation

Measure Derivatives

Measures

Sigma Algebras

Real Series

Extended Real Numbers

Real Limits

Absolute Value

Interval Length

Intervals

Real Line

Real Order

Real Sequences

Rational Order

Real Numbers

Sequences

Subset Algebras

Integral Line

Rational Numbers

Lists

Cardinality

Integer Arithmetic

Direct Products

Set Numbers

Natural Additive Identity

Natural Multiplicative Identity

Natural Fractions

Finite Sets

Integer Products

Identity Elements

Natural Arithmetic

Equivalent Sets

Integer Order

Integer Sums

Natural Powers

Comparisons

Operations

Family Unions and Intersections

Function Inverses

Integer Numbers

Natural Products

Orders

Families

Function Images

Function Composites

Natural Sums

Converse Relations

Equivalence Relations

Natural Order

Functions

Recursion Theorem

Relations

Ordered Pair Projections

Peano Axioms

Natural Induction

Set Products

Subset Systems

Natural Numbers

Set Powers

Generalized Set Dualities

Ordered Pairs

Set Symmetric Differences

Partitions

Successor Sets

Intersection of Empty Set

Unordered Triples

Set Dualities

Set Unions and Intersections

Set Complements

Set Intersections

Pair Unions

Set Differences

Set Unions

Pair Intersections

Empty Set

Unordered Pairs

Geometry

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects