



## Pointwise vs Measure Limits

### 1 Why

How does convergence pointwise (or almost everywhere pointwise) relate to convergence in measure?

### 2 Results

**Proposition 1.** *There exists a measure space and a sequence of measurable real-valued functions on that space converging everywhere (and so almost everywhere) but not converging in measure.*

**Proposition 2.** *There exists a measure space and a sequence of measurable real-valued functions on that space converging in measure but not converging almost everywhere (nor everywhere).*

**Proposition 3.** *On finite measure spaces, all sequences of measurable real-valued functions converging almost everywhere converge in measure.*

*Proof.* Let  $(X, \mathcal{A}, \mu)$  be a measure space. Let  $(f_n)_n$  be a sequence of measurable functions on  $X$  such that  $f_n \rightarrow f$  almost everywhere. Let  $\varepsilon > 0$ .

For each  $x \in X$ , if  $|f_n(x) - f(x)| > \varepsilon$  for infinitely many  $n$ , then  $f_n(x) \not\rightarrow f(x)$ . Let  $A$  be the set of such  $x$ . and let  $B = \{x \in X \mid f_n(x) \not\rightarrow f(x)\}$ .  $A$  is a subset of  $B$ . The measure of  $B$  is

zero since  $f_n \rightarrow f$ . Use the the monotonicity of measure to conclude.  $\mu(A) \leq \mu(B) = 0$ . Since  $\mu(A) \geq 0$ ,  $\mu(A) = 0$ .

For natural  $k$ , let  $E_k$  be the  $\{x \in X \mid |f_k(x) - f(x)| > \varepsilon\}$ . Then  $x \in A$  means that for every natural  $n$ , there exists a  $k \geq n$  such that  $x \in E_k$ . In particular, for every  $n$ ,  $x$  is in  $\cup_{k=n}^{\infty} E_k$ ; denote this set by  $B_n$ . If  $x$  is in  $B_n$  for every  $n$ , then  $x \in \cap_{n=1}^{\infty} B_n$ . So we can write

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} B_n.$$

The sequence of sets  $(B_n)_n$  is decreasing. So since  $\mu$  is finite,

$$\lim_{n \rightarrow \infty} \mu(B_n) = \mu(A) = 0.$$

For every  $n$ , the set  $B_n$  contains  $\{x \in X \mid |f_n(x) - f(x)| > \varepsilon\}$ , namely  $E_n$ , the first set in the union. So then  $\mu(E_n) \leq \mu(B_n)$  by monotonicity and so

$$0 \leq \lim_{n \rightarrow \infty} \mu(E_n) \leq \lim_{n \rightarrow \infty} \mu(B_n) = 0,$$

and we conclude  $\lim_n \mu(E_n) = 0$ . Since  $\varepsilon$  was arbitrary, we conclude  $f_n \rightarrow f$  in measure.

□

**Proposition 4.** *On any measure space, for a sequence of measurable real-valued functions converging in measure to a measurable real-valued limit function, there exists a subsequence converging to the limit function almost everywhere.*

*Proof.* Let  $(X, \mathcal{A}, \mu)$  be a measure space. Let  $(f_n)_n$  be a sequence of measurable functions on  $X$  such that  $f_n \rightarrow f$  in measure.

There exists  $n_1$  so that

$$\mu(\{x \in X \mid |f_{n_1}(x) - f(x)| > 1\}) < \frac{1}{2}.$$

Can find  $n_2 > n_1$  so that

$$\mu\left(\left\{x \in X \mid |f_{n_2}(x) - f(x)| > \frac{1}{2}\right\}\right) < \frac{1}{4}.$$

We can inductively find a sequence  $\{n_k\}_k$  so that:

$$\mu\left(\left\{x \in X \mid |f_{n_k}(x) - f(x)| > \frac{1}{k}\right\}\right) \leq \frac{1}{2^k}.$$

□