Measure Derivatives

1 Why

TODO

2 Definition

2.1 Defining Result

Proposition 1. *Let $(X, \mathcal{A})$ be a measurable space. Let μ and ν be finite measures with $\nu \ll \mu$.*

There exists $g : X \to [0, \infty)$ such that

$$\nu(A) = \int_A g \, d\mu$$

for all $A \in \mathcal{A}$. The function g is μ-almost everywhere unique.

Proof. Define

$$\mathcal{F} = \left\{ f : X \to [0, \infty) \;\middle|\; f \text{ measurable and } \int_A f d\mu \leq \nu(A) \right\}.$$

The function $f \equiv 0$ is in $\mathcal{F}$, since it is a measurable simple function whose integral over every measureable set is zero.

If f_1 and f_2 are in $\mathcal{F}$, then $f_1 \vee f_2$ is in $\mathcal{F}$. To check, let $A \in \mathcal{A}$, and define the sets $A_1 = \{x \in A \mid f_1(x) > f_2(x)\}$ and $A_2 = \{x \in A \mid f_1(x) \leq f_2(x)\}$. A_1 and A_2 partition A, so

$$\begin{aligned} \int_A f_1 \vee f_2 &= \int_{A_1} f_1 \vee f_2 + \int_{A_2} f_1 \vee f_2 \\ &= \int_{A_1} f_1 + \int_{A_2} f_2 \\ &\leq \nu(A_1) + \nu(A_2) \end{aligned}$$

Since A_1 and A_2 partition A,

$$\nu(A_1) + \nu(A_2) = \nu(A_1 \cup A_2) = \nu(A).$$

Select a sequence of functions $(f_n)_n$ in $\mathcal{F}$ so that

$$\lim_n \int f_n = \sup \left\{ \int f \;\middle|\; f \in \mathcal{F} \right\}.$$

Toward ensuring the sequence is increasing, define $g_1 = f_1$, $g_2 = g_1 \vee f_2$, and $g_n = g_{n-1} \vee f_n$ for $n \geq 3$. Using the observation in the previous paragraph, $g_n \in \mathcal{F}$ for each n.

Let g be the pointwise limit of the $(g_n)_n$. The monotone convergence of integrals shows

$$\int_A g = \lim_n \int_A g_n.$$

for each $A \in \mathcal{A}$. Since $\int_A g_n \leq \nu(A)$, so too is the limit and thus so too is $\int_A g$. Thus, $g \in \mathcal{F}$. By construction, for $A = X$, $\int g = \sup\{\int f \mid f \in \mathcal{F}\}$. We have constructed an element of $\mathcal{F}$ attaining the supremum.

We know that the integral of g on A with respect to μ is bounded above by $\nu(A)$. We want the gap to be zero. Regardless of the gap, the function $\nu_0 : \mathcal{A} \to [0, \infty)$ defined by

$$\nu_0(A) = \nu(A) - \int (g, A, \mu).$$

for each $A \in \mathcal{A}$ is a positive measure If ν_0 is identically zero, then there is no gap.

Suppose there is a gap: then there exists a measurable set with strictly positive measure under ν_0. Since the base set contains this set, and measures are monotone, the base set must have stricty positive measure. Since μ is finite, there exists a natural number n so that

$$\nu_0(X) > \frac{1}{n}\mu(X).$$

Define a new measure $\nu_1 = \nu_0 - \frac{1}{n}\mu$. Denote a signed-set decomposition of ν_1 by (P, N). Then $\nu_1(A \cap P) \geq 0$, or equivalently,

$$\nu_0(A \cap P) - \frac{1}{n}\mu(A \cap P) \geq 0,$$

for all A so. and so

$$\begin{aligned}\nu(A) &= \nu_0(A) + \int (g, A, \mu) \\ &\geq \nu_0(A \cap P) + \int (g, A, \mu).\end{aligned}$$

$\square$

Some call the above the *Radon-Nikodym theorem* .

2.2 Notation