



## Why

We want to describe how fast a function grows or declines.<sup>1</sup>

## Definition

Let  $f : \mathbf{R} \rightarrow \mathbf{R}$ . The *lower growth class* of  $f$  (*toward infinity*) is the set of all functions  $g : \mathbf{R} \rightarrow \mathbf{R}$  for which there exists  $C, M > 0$  so that  $|g(x)| \leq C|f(x)|$  for all  $x > M$ . The intuition is that if  $h : \mathbf{R} \rightarrow \mathbf{R}$  is in the lower growth class of  $f$ ,  $h$  does not grow faster than  $f$ . In this case we say that  $h$  *grows at order*  $f$ .

The *lower limit class* of  $f$  at  $x_0$  is the set of all functions  $g : \mathbf{R} \rightarrow \mathbf{R}$  for which there exists  $C, \varepsilon > 0$  so that  $|g(x)| \leq C|f(x)|$  for all  $|x - x_0| < \varepsilon$ . The intuition is that for  $x$  sufficiently close to  $x_0$ , the magnitude of  $f$  is bounded by a constant times the magnitude of  $g$ . Often  $x_0$  is 0.

The *upper growth class* of  $f$  (*toward infinity*) is the set of all functions  $g : \mathbf{R} \rightarrow \mathbf{R}$  for which there exists  $C, M > 0$  so that  $|g(x)| \geq C|f(x)|$  for all  $x > M$ . The intuition is that if  $h$  is in the upper growth class of  $f$ ,  $h$  grows at least as fast as  $f$ . We similarly define the *upper growth class at a limit*  $x_0$ .

The (*exact*) *growth class* of  $f$  is the set of all functions  $g : \mathbf{R} \rightarrow \mathbf{R}$  for which there exists  $C_1, C_2, M$  so that  $C_1|f(x)| \leq |g(x)| \leq C_2|f(x)|$  for all  $x > M$ . The intuition is that if  $h$  is

---

<sup>1</sup>Future editions will expand this vague introduction.

in the growth class of  $f$ , then  $h$  and  $f$  grow at the same rate. Again, we similarly define the *growth class at limit*  $x_0$ .

## Notation

We denote the upper, lower and exact growth classes of a function  $f : \mathbf{R} \rightarrow \mathbf{R}$  by  $O(f)$ ,  $\Omega(f)$  and  $\Theta(f)$ , respectively. We read the notation  $O(f)$  as “order at most  $f$ ,” we read  $\Omega(f)$  as “order at least  $f$ ,” and  $\Theta(f)$  as “order exactly  $f$ .”

The letter  $O$  is a mnemonic for order, and  $\Omega$  and  $\Theta$  build on this mnemonic. The term order appears to arise from the use of growth classes when discussing Taylor approximations. In this case of small  $x$  (i.e.,  $|x| < 1$ ),  $|x^p| < |x^q|$  if  $q < p$  and so higher order terms are “smaller” and “negligible.” This notation is sometimes called *Big O notation*, *Landau’s symbol*, *Landau notation* or *Landau’s notation*.

Let  $\phi, \psi : \mathbf{R} \rightarrow \mathbf{R}$ . Many authors use  $\phi = O(\psi)$  or  $\phi(t) = O(\psi(t))$  to assert that  $\phi$  is in the upper growth class of  $\psi$  at some understood limit (e.g., 0 or  $\infty$ ). In other words, the equation asserts that there exists some positive constant  $C > 0$  so that, for all  $t$  sufficiently close to the understood limit,  $|\phi(t)| \leq C|\psi(t)|$ .<sup>2</sup> For example, the statement  $\sin^2(t) = O(t^2)$  as  $t \rightarrow 0$  (or for  $t \rightarrow 0$ ) means that there exists constants  $C, \varepsilon > 0$  so that,  $|t| < \varepsilon \longrightarrow |\sin^2(t)| \leq Ct^2$ .

---

<sup>2</sup>Often also defined  $|\phi(t)| < C\psi(t)$ , with no absolute value on  $\psi$ .

