

## MONOTONE NEIGHBORHOODS

# Why

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#### Definition

The higher adjacency set or higher neighborhood of a vertex v in an ordered undirected graph is all vertices in the neighborhood of v whose index is greater the v. Similarly, the lower adjacency set or lower neighborhood of v is all vertices in the neighborhood of v whose index is less the v. We call these monotone neighborhoods.

The *higher degree* of a vertex is the size of the higher adjacency set and the *lower degree* of a vertex is the size of its lower adjacency set.

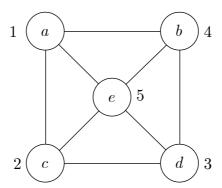
The closed monotone neighborhoods are the closed higher adjacency set, the higher adjacency set of v union with the singleton  $\{v\}$  and the closed lower adjacency set, the lower adjacency set of v union with the singleton  $\{v\}$ .

## **Notation**

We denote the higher neighborhood of v by  $\operatorname{adj}^+(v)$  and the lower neighborhood by  $\operatorname{adj}^-(v)$ .

 $<sup>^1{</sup>m Future}$  editions will include. This sheet is needed, for example, in discussing perfect elimination orderings.

# Visualization



To help think about the monotone neighborhoods of the graph we visualize ordered graphs as triangular arrays with vertices along the diagonal and a bullet in row i and column j of the array if i > j and the vertices  $\sigma(i)$  and  $\sigma(j)$  are adjacent.

An example is shown below for the ordered undirected graph in the figure (to understand this visualization, see Ordered Undirected Graphs) we use the

$$\begin{bmatrix} a & & & & \\ \bullet & c & & & \\ & \bullet & d & & \\ \bullet & & \bullet & b & \\ \bullet & \bullet & \bullet & e \end{bmatrix}$$

In this array representation the higher and lower neighborhoods are easily identified. The indices of the elements of  $\operatorname{adj}^+(v)$  are the column indices of the entries in row  $\sigma^{-1}(v)$  of the array. For example,  $\sigma^{-1}(d) = 3$ , and the only bullet entry in row three is c so  $\operatorname{adj}^-(d) = \{c\}$ . Likewise,  $\operatorname{adj}^-(c) = \{a\}$ . And so on. Similarly, the indices of  $\operatorname{adj}^+(v)$  are the row indices of the entries in column  $\sigma^{-1}(v)$ . For example,  $\sigma^{-1}(d)$  is 3, and

there are indices 4 and 5 corresponding to b and e so  $\mathrm{adj}^+(d) = \{b, e\}$ . Likewise,  $\mathrm{adj}^+(c) = \{d, e\}$ .

For this reason, we use the notation  $\operatorname{col}(v)$  and  $\operatorname{row}(v)$  for the closed upper and lower neighborhoods. So  $\operatorname{col}(v) = \operatorname{adj}^+(v) \cup \{v\}$  and  $\operatorname{row}(v) = \operatorname{adj}^-(v) \cup \{v\}$ .

