Why

We want to approximate a given distribution with one which factors according to a tree.

Definition

Given $q : A \to [0,1]$, we want to find a distribution p on A and tree T on $\{1,\dots,n\}$ to

$$\begin{aligned} \text{minimize} \quad & d_{kl}(q,p) \\ \text{subject to} \quad & p \text{ factors according to } T. \end{aligned}$$

where d_{kl} is the relative entropy as a criterion of approximation. We call such a distribution a *tree distribution approximator* (or *tree approximator*) and we call the tree the *approximator tree.*

Result

Proposition 1. *Let $A_1,\dots,A_n$ be finite non-empty sets. Define $A = \prod_{i=1}^n A_i$. Let $q : A \to [0,1]$ a distribution and T a tree on $\{1,\dots,n\}$. The distribution $p_T^* : A \to [0,1]$ defined by*

$$p_T^* = q_1 \prod_{i \neq 1} q_{i|\mathrm{pa}\, i}$$

minimizes the relative entropy with q among all distributions on A which factor according to T.

Proof. Let $p : A \to [0,1]$ be a distribution which factors according to T. First, express

$$p = p_1 \prod_{i \neq i} p_{i|\mathrm{pa}_i}$$

where pa_i is the parent of vertex i in T rooted at vertex 1 ($i = 2,\dots,n$).

Second, recall that the relative entropy of q with p is $H(q,p) - H(q)$. Since $H(q)$ does not depend on p, p is a minimizer of the relative of q with p if and only if p is a minimizer of $H(q,p)$.

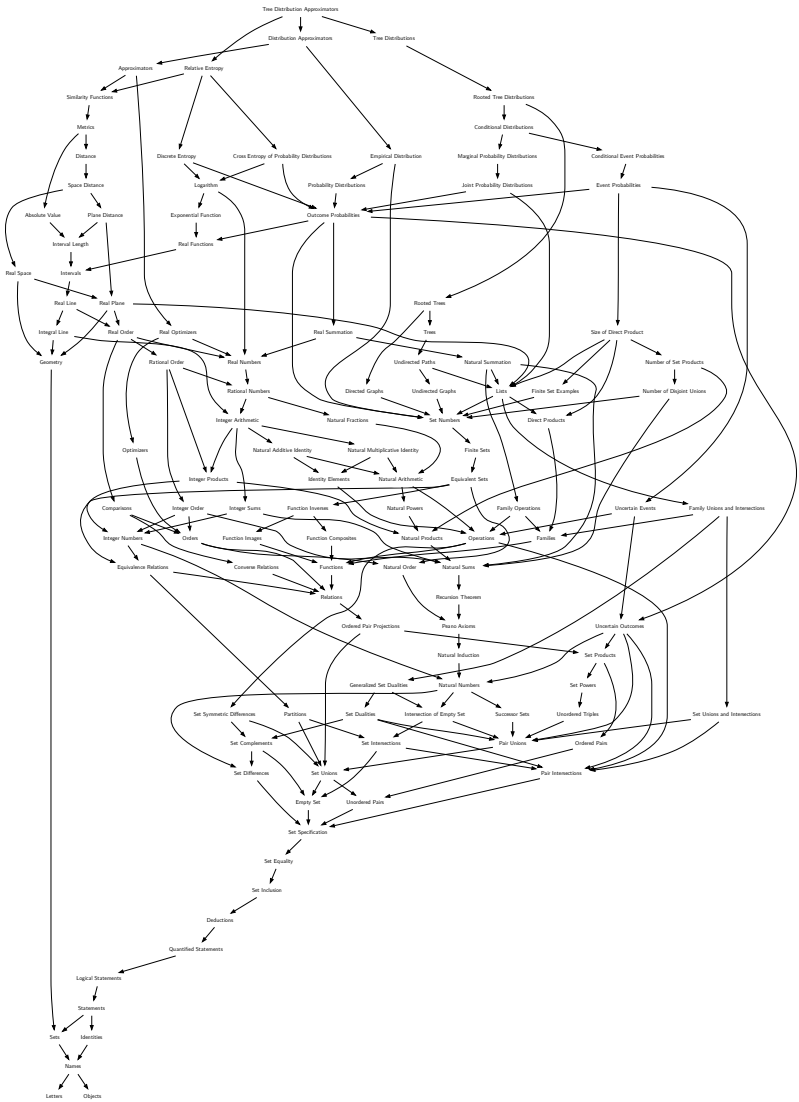
Third, express

$$
\begin{aligned}
H(q,p) &= -\sum_{a\in A} q(a)\log p(a) \\
&= -\sum_{a\in A} q(a)(\log p_1(a_1) + \sum_{i\neq 1} \log p_{i|\mathrm{pa}_i}(a_i, a_{\mathrm{pa}_i})) \\
&= H(q_1,p_1) + \sum_{i\neq 1} \sum_{a_{\mathrm{pa}_i}\in A_{\mathrm{pa}_i}} q_{\mathrm{pa}_i}(a_{\mathrm{pa}_i}) H(q_{i|\mathrm{pa}_i}(\cdot, a_{\mathrm{pa}_i}), p_{i|\mathrm{pa}_i}(\cdot, a_{\mathrm{pa}_i}))
\end{aligned}
$$

which separates across p_1 an $p_{i|\mathrm{pa}_i}(\cdot, a_{\mathrm{pa}_i})$ for $i = 2, \ldots, n$ and $a_{pa_i} \in A_{\mathrm{pa}_i}$.

Fourth, recall $H(\cdot,\cdot) \geq 0$ and is zero on repeated pairs. By this, we mean, for example, $H(p_1,p_1) = 0$. So $p_1 = q_1$ and $p_{i|\mathrm{pa}_i} = q_{i|\mathrm{pa}_i}$ are solutions. $\square$

The foregoing proposition states the form of an optimal approximator given a tree. A natural next question is to select the tree.


Tree Distribution Approximators
Distribution Approximation
Tree Distributions
Approximation
Relative Entropy
Similarity Functions
Rooted Tree Distributions
Metrics
Conditional Distributions
Distance
Discrete Entropy
Cross Entropy of Probability Distributions
Empirical Distribution
Marginal Probability Distributions
Conditional Event Probabilities
Space Distance
Logarithm
Probability Distributions
Joint Probability Distributions
Event Probabilities
Absolute Value
Plane Distance
Exponential Function
Outcome Probabilities
Interval Length
Real Functions
Real Space
Intervals
Real Line
Real Plane
Rooted Trees
Integral Line
Real Order
Real Optimizers
Real Summation
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Number of Set Products
Rational Numbers
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Lists
Finite Set Examples
Number of Disjoint Unions
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Direct Products
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Natural Multiplicative Identity
Finite Sets
Integer Products
Identity Elements
Natural Arithmetic
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Comparisons
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Natural Powers
Family Operations
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Family Unions and Intersections
Integer Numbers
Orders
Function Images
Function Composites
Natural Products
Operations
Families
Equivalence Relations
Converse Relations
Functions
Natural Order
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Recursion Theorem
Ordered Pair Projections
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Uncertain Outcomes
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Set Dualities
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects