Why

When are two sets the same?

Definition

Let A and B denote sets. If $A = B$ then every element of A is an element of B and every element of B is an element of A. In other words, $(A = B) \longrightarrow ((A \subset B) \wedge (B \subset A))$.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Then $A = B$? We define it to be so. In other words, sets are *determined* by their members.

Principle 1 (Extension)**.** *Two sets are the same (or* equal*) if every member of one is a member of the other and vice versa.*

In other words, two sets are identical if and only if every element of one is an element of the other. This principle is sometimes called the *principle of extension.* We refer to the elements of a set as its *extension.* Roughly speaking, this principle states that we know the extension of a set, then we know the set. A set is *determined* by its extension.

Deductive principle

We can use this definition to deduce $A = B$ if we first deduce $A \subset B$ and $B \subset A$. With these two implications, we use the principle of extension to conclude that the sets are the same. In other words, $(A = B) \longleftrightarrow ((A \subset B) \wedge (B \subset A))$. We also describe this fact by saying that inclusion ($\subset$) is *antisymmetric.*

Belonging and sets compared with ancestry and humans

Compare the principle of extension for identifying sets from their elements with an analogous principle for identifying people from their ancestors.

We can consider a person's ancestors. Namely, the person's parents,

grandparents, great grandparents and so on. It is clear that if we label the same human with two names A and B, then A and B have the same ancestors. In other words, same human implies same ancestors. This is the analog of "if two sets are equal they have the same members".

On the other hand, if we have two people denoted by A and B, and we know that A has the same ancestors as B, we can not conclude that A and B denote the same human. For example, siblings have the same ancestors but are different people. This direction, same ancestors implies same human, is the analogue of "if they have the same elements, two sets are the same". It is false for humans and ancestors, but we define it to be true for sets and members.

The principle of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

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\begin{tikzcd}
\text{Set Equality} \arrow[d] \\
\text{Set Inclusion} \arrow[d] \\
\text{Standardized Accounts} \arrow[d] \\
\text{Accounts} \arrow[d] \\
\text{Deductions} \arrow[d] \\
\text{Quantified Statements} \arrow[d] \\
\text{Logical Statements} \arrow[d] \\
\text{Statements} \arrow[dl] \arrow[dr] & \\
\text{Identities} \arrow[dr] & \text{Sets} \arrow[dl] \\
& \text{Names} \arrow[dl] \arrow[dr] & \\
\text{Letters} & & \text{Objects}
\end{tikzcd}
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