



## Why

In ordinary reduction, we obtain a sequence of row reducers.

### Factorization of $A$ from a sequence of reducers

Let  $(A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$  be an ordinarily reducible linear system. The *ordinary reducer sequence* is a sequence of reducer matrices  $L_1, \dots, L_{m-1}$  with  $A_1 = L_1 A$  and  $A_i = L_i A_{i-1}$  for  $2 \leq i \leq m-1$ . In other words,  $U \in \mathbf{R}^{m \times m}$  defined by

$$U = L_{m-1} \cdots L_2 L_1 A \quad (1)$$

is the ordinary row reduction of  $A$ .  $U$  is upper triangular.

If  $L_{m-1} \cdots L_2 L_1$  in Equation (1) is invertible, then we have

$$A = (L_{m-1} \cdots L_2 L_1)^{-1} U,$$

which is a factorization of  $A$ . Each  $L_i$  is invertible, so

$$(L_{m-1} \cdots L_2 L_1)^{-1} = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}.$$

So we are interested in the inverse of  $L_i$  for  $i \leq m-1$ . Recall that if  $x_1$  is the first column of  $A$ , and  $x_2$  is the second column of  $L_1 A$  and  $x_k$  is the  $k$ th column of  $L_{k-1} \cdots L_1 A$  for  $k = 2, \dots, m-1$ , then

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -\ell_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -\ell_{mk} & & & 1 \end{bmatrix}$$

where  $\ell_{jk} = x_{jk}/x_{kk}$  for  $k < j \leq m$ .

## Properties

The two important properties of the  $L_i$  is that they have simple inverses and a simple product. Define

$$\ell_k = (0, \dots, 0, \ell_{k+1,k}, \dots, \ell_{m,k})$$

so that  $L_k = L_k - \ell_k e_k^\top$  where  $(e_k)_i$  is 1 if  $k = i$  and 0 otherwise.

**Prop:**  $L_i^{-1}$  is  $L_i$  with the subdiagonal entries negated. *Proof.* From the sparsity pattern of  $\ell_k$ , we have  $e_k^\top \ell_k = 0$ . So

$$(I - \ell_k e_k^\top)(I + \ell_k e_k^\top) = I - \ell_k e_k^\top \ell_k e_k^\top = I.$$

**Prop:**  $L_k^{-1} L_{k+1}^{-1}$  is the unit lower-triangular matrix with the entries of both  $L_k^{-1}$  and  $L_{k+1}^{-1}$  in their usual places. *Proof.* From the sparsity pattern of  $\ell_{k+1}$  we have  $e_k^\top \ell_{k+1} = 0$  so that

$$L_k^{-1} L_{k+1}^{-1} = (I + \ell_k e_k^\top)(I + \ell_{k+1} e_{k+1}^\top) = I + \ell_k e_k^* + \ell_{k+1} e_{k+1}^*.$$

Combining these two results, we deduce that

$$L_1^{-1} L_2^{-1} \dots L_{m-1}^{-1} = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{m1} & \ell_{m2} & \dots & \ell_{m,m-1} & 1 \end{bmatrix}$$

If we define  $L = L_1^{-1} \dots L_{m-1}^{-1}$  we obtain  $A = LU$ . In other words, we have a factorization (the *ordinary reducer factorization*) of  $A$  in terms of two matrices. The first,  $L$  is unit lower triangular. The second,  $U$ , is upper triangular.

