

### NORMAL LINEAR MODEL REGRESSORS

# Why

There is a natural predictor corresponding to a normal linear model.

### Definition

Let  $(x: \Omega \to \mathbf{R}^d, A \in \mathbf{R}^{n \times d}, e: \Omega \to \mathbf{R}^n)$  be a normal linear model over the probability space  $(\Omega, \mathcal{A}, \mathbf{P})$ .

## Predictive density

We are modeling  $h_{\omega}: \mathbf{R}^d \to \mathbf{R}$  by  $h_w(a) = x(\omega)^{\top} a$ . The predictive density for a dataset  $c^1, \ldots, c^m \in \mathbf{R}^d$  is the conditional density of the random vector  $(h_{(\cdot)}(c^1), \ldots, h_{(\cdot)}(c^m))$  given y.

**Proposition 1.** The predictive density for  $c^1, \ldots, c^m \in \mathbb{R}^d$  (with data matrix  $C \in \mathbb{R}^{m \times d}$ ) is normal with mean

$$g(a) = (C\Sigma_x A^{\top}) \left( A\Sigma_x A^{\top} + \Sigma_e \right)^{-1} \gamma.$$

and covariance

$$C\Sigma_x C^{\top} - C\Sigma_x A^{\top} \left( A\Sigma_x A^{\top} + \Sigma_e \right)^{-1} A\Sigma_x C^{\top}.$$

*Proof.* Define (as usual)  $y: \Omega \to \mathbb{R}^n$  and  $z: \Omega \to \mathbb{R}^m$  by

$$y = Ax + e$$
$$z = Cx.$$

Recognize (x, y, z) as jointly normal, and use Normal Conditionals).  $\square$ 

#### Predictor

The normal linear model predictor or normal linear model regressor for the normal linear model (x, A, e) is the predictor which assigns to a new point  $a \in \mathbf{R}^d$  the mean of the predictive density at a. That is, the predictor  $g: \mathbf{R}^d \to \mathbf{R}$  defined by

$$g(a) = a^{\top} \Sigma_x A^{\top} \left( A \Sigma_x A^{\top} + \Sigma_e \right)^{-1} \gamma.$$

In the above we have substituted  $a^{\top}$  for C. In the case of normal random vectors this corresponds with the MAP estimate and the MMSE estimate.<sup>1</sup> Notice that g is *linear* in its argument, a.

The use of a normal linear model predictor is often called Bayesian linear regression. The word Bayesian is used in reference to treating the parameters of the function, x, as a random variable.

<sup>&</sup>lt;sup>1</sup>Future editions will have discussed this and include a reference to the sheet.

