Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering $\geq$ on S^d defined by

$$A \geq B \quad \longleftrightarrow \quad A - B \geq 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_+.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \quad \longleftrightarrow \quad A - B > 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_{++}.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$
\begin{aligned}
\alpha A &\geq 0 && \text{for all } \delta > 0, A \geq 0, \\
A + B &\geq 0 && \text{for all } A, B \geq 0, \\
A \geq B \text{ and } B \geq A &\longrightarrow A = B && \text{for all } A, B \in \mathsf{S}^d, \\
\lim_{n \to \infty} A_n = A &\longrightarrow A \geq 0 && \text{for all sequences } (A_n)_n \text{ in } \mathsf{S}^d_+.
\end{aligned}
$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that $\geq$ is antisymmetric. Moreover,

$$A \geq A \quad \text{for all } A \in \mathsf{S}^d, \text{ and}$$

$$A \geq B \text{ and } B \geq C \longrightarrow A \geq C \text{ for all } A, B, C \in \mathsf{S}^d.$$

In other words, $\geq$ is also reflexive and transitive. In other words, $\geq$ is a partial order (see Partial Orders).[1]

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[1]Future editions will include more formal accounts.

For $d = 1$, $\geq$ reduces to the familiar total order of the real line(see Real Order). The converse perspective is to see the positive semidefinite order as an extension of the order on $\mathbf{R}$ to the space $\mathbf{S}^d$. Of course, the key difference is that two matrices may not be comparable. The order is *partial*.

For example, the matrices $A, B \in \mathbf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.

Positive Semidefinite Matrix Order

Positive Semidefinite Cones

Real Convex Cones

Real Convex Sets

Cones

Real Halfspaces

Real Hyperplanes

Subspace Orthogonal Complements

Orthogonal Complements

Subspace Dimensions

Positive Definite Matrices

Symmetric Real Matrices

Quadratic Forms

Matrix Trace

Vector Space Dimension

Affine Set Dimensions

Vector Space Bases

Matrix Transpose

Real Matrix-Matrix Products

Affinely Independent Vectors

Span

Linearly Dependent Vectors

Real Matrix-Vector Products

Real Affine Sets and Subspaces

Real Affine Hulls

Real Subspace Dimensions

Subspaces

Real Inner Products

Linear Combinations

Vectors as Matrices

Real Affine Sets

Real Vector Sums

Real Subspaces

Real Inner Product

Independent Set of Real Vectors

Real Vector Span

Vectors

Real Matrices

N-Dimensional Line Segments

Real Affine Combinations

Real Translates

N-Dimensional Lines

Space Inner Product

Real Norm

Real Linear Combinations

Multivariate Real Polynomials

Linear Equations

Real Vectors

Multivariate Functions

Linear Functions

Arrays

N-Dimensional Space

Plane Inner Product

Plane Norm

Plane Vector Sums and Differences

Space Vectors

Right Triangle Sides Relation

Distance

Absolute Value

Plane Vectors

Real Squares

Space Distance

Real Polynomials

Plane Distance

Real Arithmetic

Interval Length

Real Multiplicative Inverses

Intervals

Real Space

Real Products

Square Numbers

Real Line

Real Plane

Real Addition Inverses

Squares

Integral Lines

Real Order

Real Sums

Rational Arithmetic

Rational Multiplicative Inverses

Real Summation

Geometry

Rational Order

Real Numbers

Rational Products

Rational Sums

Fields

Natural Summation

Natural Fractions

Integer Arithmetic

Rational Numbers

Rings

Lists

Set Numbers

Direct Products

Integer Order

Comparisons

Natural Addition Identity

Natural Multiplicative Identity

Finite Sets

Equivalent Sets

Integer Products

Natural Arithmetic

Identity Elements

Inverse Elements

Family Operations

Family Unions and Intersections

Integer Order

Integer Sums

Natural Equations

Element Functions

Function Inverses

Families

Natural Products

Integer Numbers

Natural Order

Equations

Operations

Function Images

Function Composites

Converse Relation

Natural Sums

Equivalence Relation

Functions

Recursion Theorem

Relations

Peano Axioms

Ordered Pair Projections

Natural Induction

Set Products

Natural Numbers

Set Powers

Generalized Set Dualities

Partitions

Set Unions and Intersections

Ordered Pairs

Set Symmetric Differences

Successor Sets

Unordered Triples

Set Dualities

Intersection of Empty Set

Set Complements

Pair Unions

Set Intersections

Pair Intersections

Set Differences

Unordered Pairs

Empty Set

Set Specification

Set Equality

Set Inclusion

Deduction

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects

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