Defining result

Proposition 1. *Let* $(X, \mathcal{A})$ *be a measurable space. Let* μ *and* ν *be finite measures with* $\nu \ll \mu$.

There exists $g : X \to [0, \infty)$ *such that*

$$\nu(A) = \int_A g \, d\mu$$

for all $A \in \mathcal{A}$. *The function* g *is* μ*-almost everywhere unique.*

Proof. Define

$$\mathcal{F} = \left\{ f : X \to [0, \infty) \;\middle|\; f \text{ measurable and } \int_A f d\mu \le \nu(A) \right\}.$$

The function $f \equiv 0$ is in $\mathcal{F}$, since it is a measurable simple function whose integral over every measureable set is zero.

If f_1 and f_2 are in $\mathcal{F}$, then $f_1 \vee f_2$ is in $\mathcal{F}$. To check, let $A \in \mathcal{A}$, and define the sets $A_1 = \{x \in A \mid f_1(x) > f_2(x)\}$ and $A_2 = \{x \in A \mid f_1(x) \le f_2(x)\}$. A_1 and A_2 partition A, so

$$\begin{aligned} \int_A f_1 \vee f_2 &= \int_{A_1} f_1 \vee f_2 + \int_{A_2} f_1 \vee f_2 \\ &= \int_{A_1} f_1 + \int_{A_2} f_2 \\ &\le \nu(A_1) + \nu(A_2) \end{aligned}$$

Since A_1 and A_2 partition A,

$$\nu(A_1) + \nu(A_2) = \nu(A_1 \cup A_2) = \nu(A).$$

Select a sequence of functions $(f_n)_n$ in $\mathcal{F}$ so that

$$\lim_n \int f_n = \sup \left\{ \int f \;\middle|\; f \in \mathcal{F} \right\}.$$

Toward ensuring the sequence is increasing, define $g_1 = f_1$, $g_2 = g_1 \vee f_2$, and $g_n = g_{n-1} \vee f_n$ for $n \geq 3$. Using the observation in the previous paragraph, $g_n \in \mathcal{F}$ for each n.

Let g be the pointwise limit of the $(g_n)_n$. The monotone convergence of integrals shows

$$\int_A g = \lim_n \int_A g_n.$$

for each $A \in \mathcal{A}$. Since $\int_A g_n \leq \nu(A)$, so too is the limit and thus so too is $\int_A g$. Thus, $g \in \mathcal{F}$. By construction, for $A = X$, $\int g = \sup\left\{\int f \;\middle|\; f \in \mathcal{F}\right\}$. We have constructed an element of $\mathcal{F}$ attaining the supremum.

We know that the integral of g on A with respect to μ is bounded above by $\nu(A)$. We want the gap to be zero. Regardless of the gap, the function $\nu_0 : \mathcal{A} \to [0, \infty)$ defined by

$$\nu_0(A) = \nu(A) - \int(g, A, \mu),$$

for each $A \in \mathcal{A}$ is a positive measure. If ν_0 is identically zero, then there is no gap.

Suppose there is a gap: then there exists a measurable set with strictly positive measure under ν_0. Since the base set contains this set, and measures are monotone, the base set must have stricty positive measure. Since μ is finite, there exists a natural number n so that

$$\nu_0(X) > \frac{1}{n}\mu(X).$$

Define a new measure $\nu_1 = \nu_0 - \frac{1}{n}\mu$. Denote a signed-set decomposition of ν_1 by (P, N). Then $\nu_1(A \cap P) \geq 0$, or equivalently,

$$\nu_0(A \cap P) - \frac{1}{n}\mu(A \cap P) \geq 0,$$

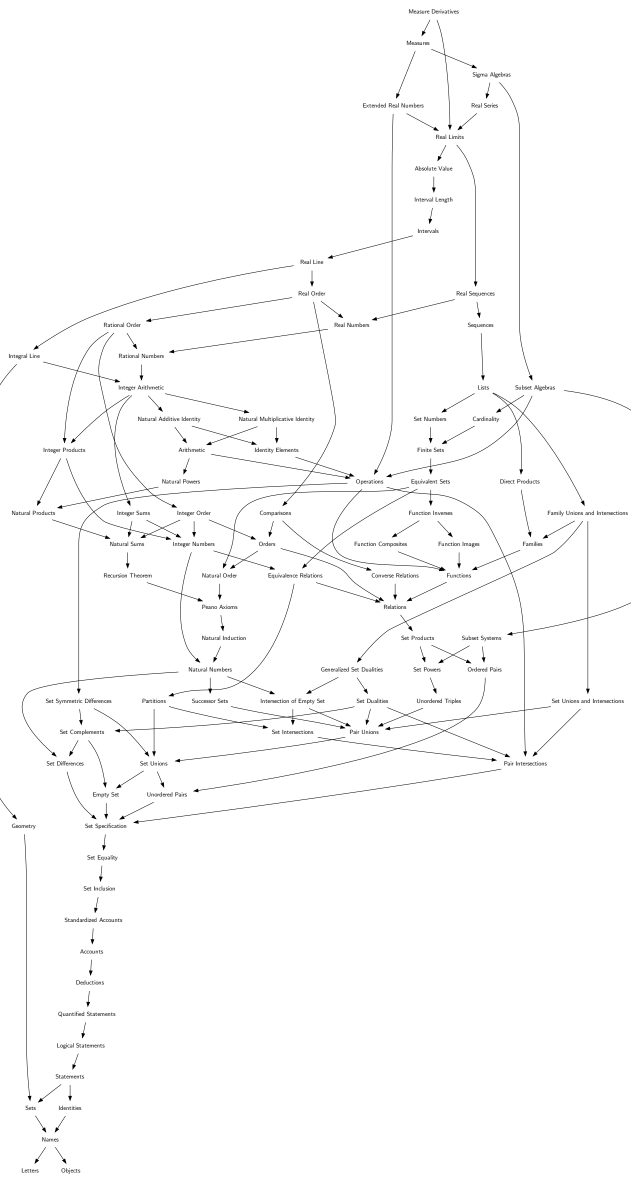
for all A, and so

$$\begin{aligned}\nu(A) &= \nu_0(A) + \int(g, A, \mu) \\ &\geq \nu_0(A \cap P) + \int(g, A, \mu).\end{aligned}$$

$\square$

Many authorities refer to this result as the *Radon-Nikodym theorem.*

Notation


Measure Derivatives
Measures
Sigma Algebras
Extended Real Numbers
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Rational Order
Real Numbers
Sequences
Integral Line
Rational Numbers
Integer Arithmetic
Lists
Subset Algebras
Natural Additive Identity
Natural Multiplicative Identity
Set Numbers
Cardinality
Integer Products
Arithmetic
Identity Elements
Finite Sets
Natural Powers
Operations
Equivalent Sets
Direct Products
Natural Products
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Natural Sums
Integer Numbers
Orders
Function Composites
Function Images
Families
Recursion Theorem
Natural Order
Equivalence Relations
Converse Relations
Functions
Peano Axioms
Relations
Natural Induction
Set Products
Subset Systems
Natural Numbers
Generalized Set Dualities
Set Powers
Ordered Pairs
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects