Expectation Deviation Upper Bound

1 Why

We bound the probability that a random variance deviates from its mean using its variance.

2 Result

Proposition 1. *Let f be a square-integrable real-valued random variable on the probability space $(X, \mathcal{A}, \mu)$. Then for $t > 0$,*

$$\mu(|f - \mathsf{E}(f)| \geq t) \leq \frac{\mathsf{var}\, f}{t^2}.$$

Proof. The set $|f - \mathsf{E}(f)| \geq t$ is $\{x \in X \mid |f(x) - \mathsf{E}(f)| \geq t\}$. This set is $\{x \in X \mid (f(x) - \mathsf{E}(f))^2 \geq t^2\}$. By using the non-negative inequality

$$\mu(\{x \in X \mid (f(x) - \mathsf{E}(f))^2 \geq t^2\}) \leq \frac{\mathsf{E}(f - \mathsf{E}(f))}{t^2}.$$

We recognize the numerator of the right hand side as the variance. □

The above is also called *Chebychev's Inequality*