Why

We want to record natural numbers. Recall that we denote $\varnothing$ by 0, the successor of 0 by 1 and the sucessor of 1 by 2. 1 is the success of 0 and 2 is the successor of 1. We agree to continue 3, 4, 5, 6, 7, 9, each the successor if its predecessor in the list. Given these numbers, how should we denote the successor to 9?

Discussion

We could of course have a new symbol, and denote the successor of 9 by this symbol. But then we would need as many symbols are there are natural numbers. And the natural numbers are *infinite* (not finite!). The following is a trick around this, re-using our ten symbols. Why ten?... How many fingers do you have?

Definition

The *natural notation* (or *base-ten notation*, *base-10 notation*) for a natural number is a list of elements of the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

To a notation η corresponds a natural number, n as follows. Suppose η has length k, so that $\eta = (\eta_1, \ldots, \eta_k)$. The natural corresponding to η is

$$\eta_1 + \eta_2 \cdot 9^+ + \eta_3 \cdot (9^+)^2 + \cdots + \eta_k \cdot (9^+)^{k-1}$$

We can use summation notation and the fact that $(9^+)^0 = 1$ to write

$$\sum_{i=1}^{k} \eta_i \cdot (9^+)^{i-1}$$

Notation

In a slightly different but universally standard way, we denote the natural notation $\eta = (\eta_1, \ldots, \eta_k)$ by

$$\eta_k \eta_{k-1} \cdots \eta_2 \eta_1$$

Examples

Some examples will help, and illustrate the notation. Here is a natural notation for some number: $(0, 1)$. Which number? Well, we have agreed that it is the number

$$0 + 1 \cdot 9^+ = 9^+$$

We denote the natural notation for 10. This is the natural notation for the sucessor of 9. The upshot is clear, we have denoted the sucessor to 9 using only ten symbols.

Here is a natural notation for some number: $(3, 2, 1)$. Which number? Well, we have agreed that it is the number

$$1 + 2 \cdot 9^+ + 3 \cdot (9^+)^2$$

Recall that we have notation for the successor to 9 now. We can write this number as

$$1 + 2 \cdot 10 + 3 \cdot (10)^2$$

We see that powers of 10 show up in the sum for the number corresponding to this notation. We name these powers. 100 is the number *hundred*. 1000 is the number *thousand*

Natural Number Notation

Natural Powers   Natural Summation

Natural Products

Natural Sums   Lists

Set Numbers

Direct Products   Finite Sets

Recursion Theorem   Equivalent Sets

Family Operations   Family Unions and Intersections   Natural Order   Function Inverses

Operations   Families   Peano Axioms   Function Composites   Function Images

Equivalence Relations   Functions

Natural Induction   Relations

Ordered Pair Projections

Set Products

Natural Numbers   Generalized Set Dualities   Set Powers

Set Symmetric Differences   Set Dualities   Set Unions and Intersections   Successor Sets   Intersection of Empty Set   Unordered Triples   Partitions

Set Complements   Pair Unions   Set Intersections   Ordered Pairs

Set Differences   Pair Intersections   Set Unions

Empty Set   Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects