



## Why

When does the technique of row reductions prevail?

## Multivariable row reductions

Let  $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$  be a linear system with  $A_{kk} \neq 0$ . The  $k$ th row reduction of  $S$  is the linear system  $(C, d)$  with  $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$  if  $i < s \leq m$  and  $C_{st} = A_{st}$  otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable  $k$  from equations  $k + 1, \dots, m$ . We are taking the  $k$ th column of  $A$  from

$$\begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ A_{k+1,k} \\ \vdots \\ A_{mk} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We interpret the  $i$ th row reduction as *subtracting equations* of the system or *reducing rows* of the array  $A$ . If  $a^i, c^i \in \mathbf{R}^n$  denote the  $i$ th rows of  $A$  and  $C$ ,  $c^i = a^i - (A_{ik}/A_{kk})a^k$  for  $k < i \leq m$ . In other words, we obtain the  $i$ th row of matrix  $C$  by subtracting a multiple of the  $k$ th row of matrix  $A$  from the  $i$ th row of matrix  $A$ , for  $k < i \leq m$ . The following is an immediate consequence of real arithmetic.

**Proposition 1.** *Let  $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$  be a linear system which row reduces to  $(C, d)$ . Then  $x \in \mathbf{R}^n$  is a solution of  $(A, b)$  if and only if it is a solution of  $(C, d)$ .*

## Ordinary reductions

We call the system  $S$  *ordinarily reducible* if there exists a sequence of systems  $S_1, \dots, S_{m-1}$  so that  $S_1$  is the 11-reduction of  $S$  and  $S_i$  is the  $ii$ -reduction of  $S_{i-1}$  for  $i = 1, \dots, n - 1$ . In this case, we call  $S_{n-1}$  the

*final ordinary reduction* (or just *ordinary reduction*) of  $S$ . The following is an immediate consequence of Proposition 1.

**Proposition 2.** *Let  $S'$  be the (final) ordinary reduction of  $S$ . Then  $S$  and  $S'$  have equivalent solution sets.*

This process of constructing the ordinary reduction is called *Gauss elimination* or *Gaussian elimination*. We call the  $kk$ th entry of system  $S_{k-1}$  the *pivot*. In an ordinarily reducible system, the pivots are nonzero. The idea is that a system is ordinarily reducible if we can take row reductions in sequence and end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system  $(A, b)$  in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.

