Why

We extend integrability to complex functions.[1]

Definition

A *measurable* complex-valued function on a measurable space is one whose real and imaginary parts are both measurable. An *integrable* complex-valued function on a measurable space is one whose real and imaginary parts are both integrable.

The *integral* of a integrable complex-valued function on a measurable space is the complex number whose real part is the integral of the real part of the function and whose imaginary part is the integral imaginary part of the function.

Notation

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let C denote the set of complex numbers. Let $f : X \to C$ be a function. f is measurable if $\text{Re}(f)$ and $\text{Im}(f)$ are measurable. f is integrable if $\text{Re}(f)$ and $\text{Im}(f)$ is integrable. If f is integrable, we denote its integral by $\int f d\mu$. We have defined it by:

$$\int f d\mu = \int \text{Re}(f) d\mu + \int \text{Im}(f) d\mu.$$

Results

Proposition 1. *A linear combination of two integrable complex-valued functions is an integrable complex-valued function.*

Proposition 2. *The integral is a linear operator on the vector space of integrable complex-valued functions.*

Proposition 3. *The absolute value of the integral of a complex-valued function is smaller than the integral of the absolute value of the function.*

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[1]Future editions will modify this motivation.

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbf{C}$ integrable. There exists $\alpha \in C$ with $|\alpha| = 1$ such that

$$|\int f d\mu| = \alpha \int f d\mu.$$

Since the integral is homogenous,

$$|\int f d\mu| = \int \alpha f d\mu \; = \int \mathrm{Re}(\alpha f) d\mu + i \int \mathrm{Im}(\alpha f) d\mu.$$

Since $\mathbf{C} mod \int f d\mu$ is real, $\int \mathrm{Im}(\alpha f) d\mu = 0$, so

$$|\int f d\mu| = \int \mathrm{Re}(\alpha f) d\mu \; \leq \int |\alpha f| d\mu = \int |f| d\mu,$$

since $\mathrm{Re}(z) \leq |z|$ for all complex numbers z and $|\alpha| = 1$. $\quad\square$