Product Measures

1 Why

We want to generalize the construction of cover area as generated as a product of two cover lengths, and more generally for arbitrary measure spaces. TODO

2 Definition

The *product measure* of the measures of two sigma finite measure spaces is the unique measure which assigns to every rectangle with measurable sides the product of the measures of the sides. We prove that such a measure exists, and is unique.

2.1 Defining Result

Proposition 1. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be sigma-finite measurable spaces. There is a unique measure π on $\mathcal{A} \times \mathcal{B}$ such that*

$$\pi(A \times B) = \mu(A) \times \nu(B)$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, for any $E \in \mathcal{A} \times \mathcal{B}$.

$$\pi(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

Proof. TODO $\qquad\square$

2.2   Notation

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be sigma-finite measurable spaces. We denote the product measure by $\mu \times \nu$. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$(\mu \times \nu)(A \times B) = \mu(A) \times \nu(B).$$