Definition

Two subspaces $S, T \subset \mathbf{R}^n$ are *orthogonal* if

$$x^\top y = 0 \text{ for all } x \in S, y \in T.$$

For any set $S \subset \mathbf{R}^n$ (not necessarily a subspace), the *orthogonal complement* of S is the set

$$S^\perp = \{x \in \mathbf{R}^n \mid x^\top y = 0 \text{ for all } y \in S\}.$$

$S^\perp$ is the set of all vectors which are orthogonal to every vector in S.

Orthgonal complement is a subspace

Notice that $S^\perp$ is always a subspace. If $x \in S^\perp$, then $x^\top y = 0$ for all $y \in S$. So then $(\alpha x)^\top y = \alpha(x^\top y) = 0$ for all $\alpha \in \mathbf{R}$ and $y \in S$. We conclude $\alpha x \in S^\perp$ for all $\alpha \in \mathbf{R}$. In other words, $S^\perp$ is closed under scalar multiplication. If $x, z \in S^\perp$, then $(x + z)^\top y = x^\top y + z^\top y = 0 + 0 = 0$. We conclude that $x + z \in S^\perp$ for all $x, z \in S^\perp$. In other words, $S^\perp$ is closed under vector addition. Consequently, $S^\perp$ is a subspace.

Orthogonal Real Subspaces
Vectors as Matrices
Real Subspaces
Real Inner Product
Real Vector Span
Real Matrices
Real Norm
Space Inner Product
Real Linear Combinations
Real Linear Equations
Space Norm
Real Vectors
Linear Functions
Arrays
Plane Norm
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Plane Inner Product
Absolute Value
Distance
Plane Vectors
Right Triangle Sides Relation
Space Distance
Real Squares
Plane Distance
Real Arithmetic
Interval Length
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Plane
Square Numbers
Real Additive Inverses
Real Line
Integral Line
Square
Real Order
Rational Arithmetic
Real Sums
Rational Multiplicative Inverses
Real Summation
Geometry
Rational Order
Rational Sums
Real Numbers
Rational Products
Natural Summation
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Inverse Elements
Comparisons
Integer Order
Integer Sums
Natural Powers
Natural Equations
Element Functions
Function Inverses
Family Operations
Family Unions and Intersections
Orders
Integer Numbers
Natural Products
Equations
Operations
Function Image
Function Composites
Families
Converse Relations
Natural Sums
Natural Order
Equivalence Relations
Functions
Recursion Theorem
Relations
Power Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects