



# Subsets

## 1 Why

We want to speak of sets which contain all the elements of other sets.

## 2 Two Sets

If every element of a first set is an element of a second set we say that the first set is a *subset* of the second set. Conversely, we say that second set is a *superset* of the first set.

Every set is a subset of itself. Similarly, every set is a superset of itself. Thus, if two sets are equal, the first is a subset of the second and the second is a subset of the first. Because of our definition of set equality, the converse is also true. If a first set is a subset of a second set and a second set is a subset of a first, then every element of each is an element of the other and the sets are the same. This conclusion follows from the axiom of extension.

The empty set is a subset of every set, since it has no elements and so satisfies our definition. Consider a set. We call the empty set and the set itself *improper subsets* of the set. All other

subsets we call *proper subsets*.

Finally, the *power set* of a set is the set of all subsets of that set. It includes the set itself and the empty set.

## 2.1 Notation

Let  $A$  and  $B$  be sets. We denote that  $A$  is a subset of  $B$  by  $A \subset B$ . We read the notation  $A \subset B$  aloud as "A subset B".

We can state the axiom of extension in this case by

$$A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$$

The notation  $A \subset B$  is a concise symbolism for the sentence "every element of  $A$  is an element of  $B$ ." Or for the alternative notation  $a \in A \Rightarrow a \in B$ .

We denote the power set of  $A$  by  $A^*$ , read aloud as "powerset of A."  $A \in A^*$  and  $\emptyset \in A^*$ . However,  $A \subset A^*$  is false.

## 2.2 Examples

Let  $a, b, c$  be distinct objects. Let  $A = \{a, b, c\}$  and  $B = \{a, b\}$ . Then  $B \subset A$ . In other notation,  $B \in A^*$ . As always,  $\emptyset \in A^*$  and  $A \in A^*$  as well. In this case, we can list the elements (which are sets) of the power set:

$$A^* = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$