Supremum Norm Complete

1 Why

We want a complete norm on the vector space of continuous functions.

2 Result

Proposition 1. *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $\{f_n\}_n$ be an egoprox sequence in $C[a, b]$.

Candidate. $\{f_n\}_n$ is egoprox means $\forall \epsilon > 0, \exists N$ so that

$$m, n > N \implies |f_n - f_m|_{\sup} < \epsilon.$$

Since $|f_n - f_m|_{\sup} < \epsilon \implies |f_n(x) - f_m(x)| < \epsilon$ for all $x \in [a, b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a, b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit $l_x \in R$ such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a, b]$. Define $f : [a, b] \to R$ by $f(x) = l_x$ for each $x \in [a, b]$.

Candidate is Limit. First, we argue that $|f_n - f|_{\sup} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\{f_n\}_n$ is an egoprox sequence, there exists n_0

so that
$$n, m \geq n_0 \implies |f_n - f_m|_{\text{sup}} < \epsilon/2.$$

So for all $x \in [a, b]$,
$$n, m \geq n_0 \implies |f_n(x) - f_m(x)| < \epsilon/2.$$

For all $x \in [a, b]$, and $n \geq n_0$,
$$\lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \epsilon/2 < \epsilon.$$

The sequence $\{f_k(x)\}_{k=m}^{\infty}$ is a final part of $\{f_k(x)\}_{k=1}^{\infty}$, and so has the same limit, $f(x)$. Therefore, using continuity of subtraction and the absolute value,
$$\lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for $n \geq n_0$, $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$, from which we deduce $|f_n - f|_{\text{sup}} < \epsilon$. Thus $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

Limit is Continuous. Next, we argue that f is continuous. Let $x_0 \in [a, b]$. Let $\epsilon > 0$. Since $f_n \longrightarrow f$ there exists n_0 so that
$$|f_{n_0} - f|_{\text{sup}} < \epsilon/3.$$

By the triangle inequality,
$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a, b]$. Using $|f(x_0) - f_{n_0}(x_0)| < \epsilon/3$,
$$|f(x_0) - f(x)| < \epsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a, b]$. Using the triangle inequality,
$$|f(x_0) - f(x)| < \epsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

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for all $x \in [a, b]$. Using $|f_{n_0}(x_0) - f(x)| < \epsilon/3$

$$|f(x_0) - f(x)| < \epsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \epsilon/3$$

for all $x \in [a, b]$. Since f_{n_0} is continuous, there exists $\delta > 0$ so that

$$|x_0 - x| < \delta \implies |f_{n_0}(x_0) - f_{n_0}(x)| < \epsilon/3,$$

for $x \in [a, b]$. In this case,

$$|f(x_0) - f(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Since ϵ was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere. We call the above the **three epsilon argument**. $\qquad\square$