

DISTANCE COVARIANCE FUNCTIONS

Why

It is common to consider random functions whose domain is time, space, or a n-dimensional space.

Definition

Let (X, d) be a metric space.

A distance covariance function $k: X \times X \to \mathbf{R}$ is a covariance function satisfying

$$k(x,y) > k(x,y) \longleftrightarrow d(x,y) < d(x,y).$$

In other words, the covariance decreases as the points get further apart.

Example: Squared Exponential

Let $k: X \times X \to \mathbf{R}$ be defined by

$$k(x,y) = \exp(-d(x,y)).$$

Then k is a distance covariance function. It is often called the squared exponential covariance function.

Let $\alpha, \sigma \in \mathbf{R}$. Define $k': X \times X \to \mathbf{R}$ by

$$k'(x,y) = \alpha \exp(-d(x,y)/\sigma^2)$$

then k' is still a covariance function. In this context σ is often referred to as the *characteristic length-scale* of the process.

The scalar α is sometimes called a "prefactor" that "controls" the "overall variance."

Suppose $(X, d) = (\mathbb{R}^n, ||\cdot||)$. Then the squared exponential covariance function

$$\alpha \exp(-\|x-y\|/(2\sigma^2))$$

is sometimes called the radial basis function or gaussian covariance function.¹ Also called an exponentiated quadratic kernel.

¹For reasons that will be included in future editions.

