

Expectation Deviation Upper Bound

Why

We bound the probability that a random variance deviates from its mean using its variance.

Result

PROPOSITION 1. Let f be a square-integrable real-valued random variable on the probability space (X, \mathcal{A}, μ) . Then for t > 0,

$$\mu(|f - \mathsf{E}(f)| \ge t) \le \frac{\mathsf{var}\, f}{t^2}.$$

Proof. The set $|f - \mathbf{E}(f)| \ge t$ is $\{x \in X \mid |f(x) - \mathbf{E}(f)| \ge t\}$. This set is $\{x \in X \mid (f(x) - \mathbf{E}(f))^2 \ge t^2\}$. By using the nonnegative inequality

$$\mu(\{x \in X \mid (f(x) - \mathbf{E}(f))^2 \ge t^2\}) \le \frac{\mathbf{E}(f - \mathbf{E}(f))}{t^2}.$$

We recognize the numerator of the right hand side as the variance. \Box

The above is also called *Chebychev's Inequality*

