Definition

The *nullspace* (or *kernel*) of a matrix $A \in \mathbf{R}^{m \times n}$ is the set

$$\{x \in \mathbf{R}^n \mid Ax = 0\}.$$

It is the set of vectors mapped to zero by A. Equivalently, it is the set of vectors orthogonal to the rows of A.

Notation

We denote the nullspace of $A \in \mathbf{R}^{m \times n}$ by $\mathrm{null}(A) \subset \mathbf{R}^n$. Some authors denote the nullspace of A by $\mathcal{N}(A)$.

A subspace

The nullspace of a matrix is a subspace (this justifies the terminology null*space*!). There are a few routes to see this.

The first is direct. If $w, z \in \mathrm{null}(A)$, then $Aw = 0$ and $Az = 0$. So then $A(w + z) = Aw + Az = 0$. So $\mathrm{null}(A)$ is closed under vector addition. Also $A(\alpha w) = \alpha(Aw) = 0$ for all $\alpha \in \mathbf{R}$. [In particular $A0 = 0$, so $0 \in \mathrm{null}(A)$; i.e., $\mathrm{null}(A)$ contains the origin.] So $\mathrm{null}(A)$ is closed under scalar multiplication.

The second is by thinking about orthogonal complements. Second, we can view the $\mathrm{null}(A)$ as the set of vectors orthgonal to all the rows of A. In other words, $\mathrm{null}(A) = \{\tilde{a}_1, \ldots, \tilde{a}_m\}^\perp$. The orthogonal complement of any set is a subspace (see Orthogonal Real Subspaces).

Ambiguity in solutions

Suppose we have a solution to the system of linear equation with data (A, y). In other words, we have a vector $x \in \mathbf{R}^n$ so that $y = Ax$. If we have a vector $z \in \mathrm{null}(A)$, then $x + z$ is also a solution to the system (A, y), since

$$A(x + z) = Ax + Az = Ax + 0 = y$$

Conversely, suppose there were another solution $\tilde{x} \in \mathbf{R}^n$ to the system (A, y). Then $y = Ax = A\tilde{x}$, so

$$0 = y - y = Ax - A\tilde{x} = A(x - \tilde{x}).$$

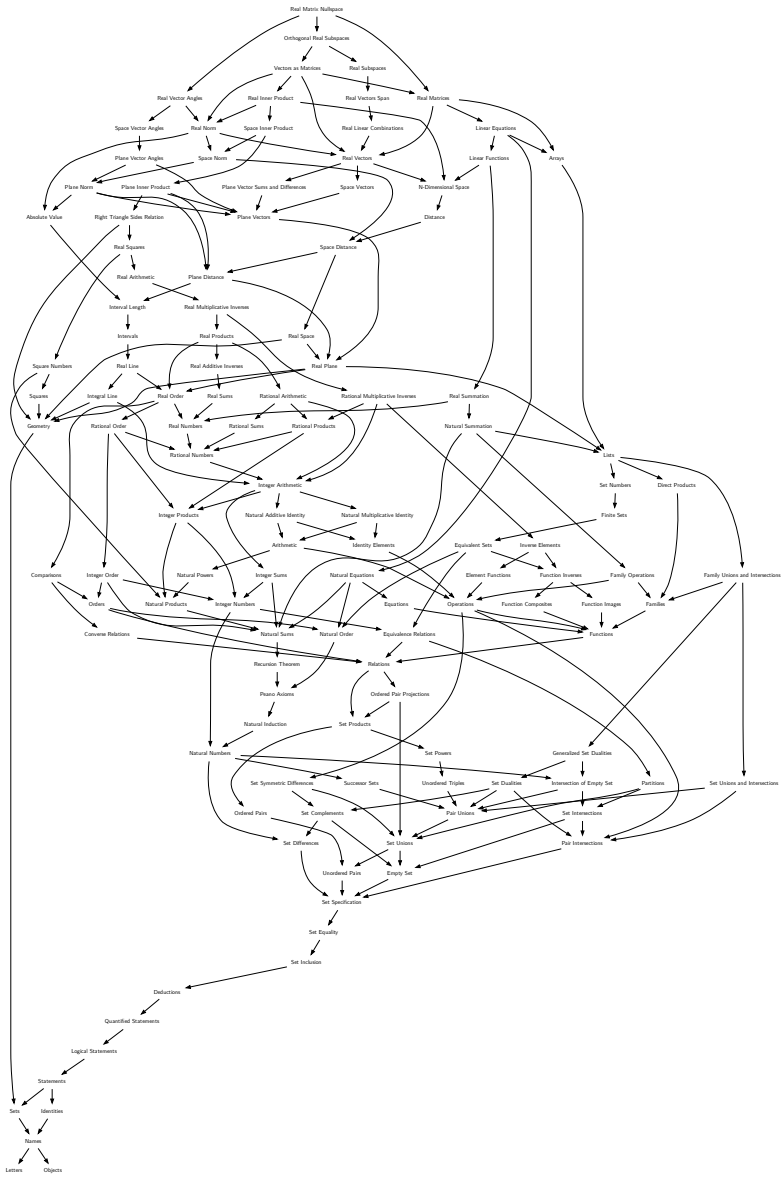
Consequently, $(x - \tilde{x}) \in \text{null}(A)$, and so $\tilde{x}$ is the solution x plus some vector in the null space of A. Consequently we are interested in whether A has vectors in its nullspace.

Zero nullspace

The origin 0 is always in the nullspace of A. However, this vector does not mean that we can find different solutions, since $x + 0 = x$ for all $x \in \mathbf{R}^n$. If, on the other hand, there is a nonzero vector $z \in \text{null}(A)$, then $x + z \neq x$, and $x + z$ is a solution for (A, y). We think about A as a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. In the case that there is a nonzero element in the nullspace, A maps different vectors to the same vector. Here, x and $x+z$ both map to y. In this case, the function is *not invertible*, because it is not one-to-one. If, however, zero is the only element of the null space, the function is one-to-one. So call A *one-to-one* if $\text{null}(A) = 0$.

Equivalent statements

A matrix $A \in \mathbf{R}^{m \times n}$ is *one-to-one* if the linear function $f : \mathbf{R}^n \to \mathbf{R}^m$ defined by $f(x) = Ax$ is one-to-one. In this case, if there exists $x \in \mathbf{R}^n$ so that $y = Ax$, then there is only one such x. Different elements in $\mathbf{R}^n$ map to different elements in $\mathbf{R}^m$.


Real Matrix Nullspace
Orthogonal Real Subspaces
Vectors as Matrices
Real Subspaces
Real Vector Angles
Real Inner Product
Real Vectors Span
Real Matrices
Space Vector Angles
Real Norm
Space Inner Product
Real Linear Combinations
Linear Equations
Arrays
Plane Vector Angles
Space Norm
Real Vectors
Linear Functions
Plane Norm
Plane Inner Product
Plane Vector Sums and Differences
Space Vectors
N-Dimensional Space
Absolute Value
Right Triangle Sides Relation
Plane Vectors
Distance
Real Squares
Space Distance
Real Arithmetic
Plane Distance
Interval Length
Real Multiplicative Inverses
Intervals
Real Products
Real Space
Square Numbers
Real Line
Real Additive Inverses
Real Plane
Squares
Integral Line
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Real Summation
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Natural Summation
Lists
Rational Numbers
Set Numbers
Direct Products
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Inverse Elements
Comparisons
Integer Order
Natural Powers
Integer Sums
Natural Equations
Element Functions
Function Inverses
Family Operations
Family Unions and Intersections
Orders
Natural Products
Integer Numbers
Equations
Operations
Function Composition
Function Images
Families
Converse Relations
Natural Sums
Natural Order
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects