$$\diamondsuit$$

Sigma Algebra Event Independence

1   Why

TODO

2   Definition

The sigma algebra *generated by an event* is the sigma algebra consisting of the empty set, the event, the complement (in the base set) of the event, and the base set.

A family of events events are *independent* if the sigma algebras generated by the events are independent.

2.1   Notation

Let $(X, \mathcal{A}, \mu)$ be a probability space. Let $A \in \mathcal{A}$ be an event. The sigma algebra generated by A is $\{\varnothing, A, X - A, X\}$. We denote it by $\sigma(A)$.

Let $B \in \mathcal{A}$. If A is independent of B we write $A \perp B$.

3   Equivalent Condition

Proposition 1. *Two events are independent if and only if the measure of their intersection is the product of their measures.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a probability space. Let $A, B \in \mathcal{A}$.

($\Rightarrow$) If $A \perp B$, then by definition $A \in \sigma(A)$ and $B \in \sigma(B)$ and so:

$$\mu(A \cap B) = \mu(A)\mu(B).$$

($\Leftarrow$) Conversely, let $a \in \sigma(A)$ and $b \in \sigma(B)$. If $a = \varnothing$ or $b = \varnothing$ then $a \cap b = \varnothing$. So

$$\mu(a \cap b) = \mu(\varnothing) = \mu(a)\mu(b),$$

since one of the two measures on the right hand side is zero. On the other hand, if $a = X$, then $a \cap b = b$ and so

$$\mu(a \cap b) = \mu(b) = \mu(a)\mu(b),$$

since $\mu(a) = \mu(X) = 1$. Likewise if $b = X$.

So it remains to verify $\mu(a \cap b) = \mu(a)\mu(b)$ for the cases $a \in \{A, X - A\}$ and $b \in \{B, X - B\}$. If $a = A$, and $b = B$, then the identity follows by hypothesis. Next, observe that $A \cap (X - B) = A - (A \cap B)$ and $(A \cap B) \subset A$ so $\mu(X) < \infty$ allows us to deduce:

$$
\begin{aligned}
\mu(A \cap (X - B)) &= \mu(A - (A \cap B)) \\
&= \mu(A) - \mu(A \cap B) \\
&= \mu(A)(1 - \mu(B)) \\
&= \mu(A)\mu(X - A).
\end{aligned}
$$

Similar for $X - A$ and B. Finally, recall that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$. So then,

$$
\begin{aligned}
\mu((X - A) \cap (X - B)) &= 1 - \mu(A \cup B) \\
&= 1 - \mu(A) - \mu(B) + \mu(A \cap B) \\
&= 1 - \mu(A) - \mu(B) + \mu(A)\mu(B) \\
&= (1 - \mu(A))(1 - \mu(B)) \\
&= \mu(X - A)\mu(X - B).
\end{aligned}
$$

$\square$