Why

We generalize our notion of norm on real vectors to abstract vector spaces.

Definition

A *norm* is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisifies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms on those vectors.

A *normed space* (or *norm space*) is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

Notation

Let $(X, \mathbf{F})$ be a vector space where $\mathbf{F}$ is the field of real numbers or the field of complex numbers. Let $f : X \to \mathbf{R}$. The functional f is a norm if

1. $f(v) \geq 0$ for all $x \in V$
2. $f(v) = 0$ if and only if $x = 0 \in X$.
3. $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in \mathbf{F}$, $x \in X$
4. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.

In this case, for $x \in X$, we denote $f(x)$ by $\|x\|$, read aloud "norm x". The notation follows the notation for the absolute value function is a norm on the vector space of real numbers. In some cases, we go further, and for a norm indexed by some parameter α or set A we write $\|x\|_\alpha$ or $\|x\|_A$.

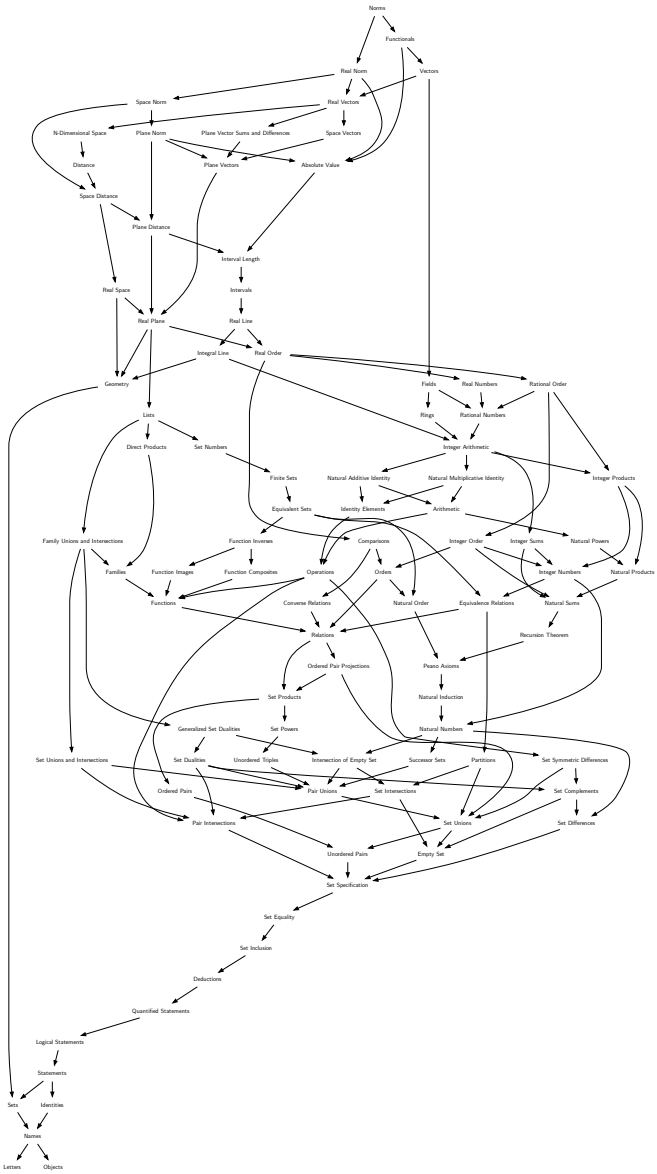
When the field is assumed or clear from context, it is succinct to say let $(V, \|\cdot\|)$ be a normed space.

Examples

The absolute value function is a norm on the vector space of real numbers. In addition, the (Euclidean norm) is a norm on the vector space $\mathbf{R}^n$.

Other terminology

The descriptive but slight more verbose *norm vector space* and *normed vector space* are also in usage.


Norms
Functionals
Real Norm
Vectors
Space Norm
Real Vectors
N-Dimensional Space
Plane Norm
Plane Vector Sums and Differences
Space Vectors
Distance
Plane Vectors
Absolute Value
Space Distance
Plane Distance
Interval Length
Real Space
Intervals
Real Plane
Real Line
Integral Line
Real Order
Geometry
Fields
Real Numbers
Rational Order
Rings
Rational Numbers
Lists
Direct Products
Set Numbers
Integer Arithmetic
Finite Sets
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Equivalent Sets
Identity Elements
Arithmetic
Family Unions and Intersections
Function Inverses
Comparisons
Integer Order
Integer Sums
Natural Powers
Families
Function Images
Function Composites
Operations
Orders
Integer Numbers
Natural Products
Functions
Converse Relations
Natural Order
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Intersection of Empty Set
Successor Sets
Partitions
Set Symmetric Differences
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects