Why

We want a complete norm on the vector space of continuous functions.

Result

Proposition 1. *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $(f_n)_n$ be an egoprox sequence in $C[a,b]$.

Candidate. $(f_n)_n$ is egoprox means $\forall \varepsilon > 0, \exists N$ so that

$$m, n > N \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon.$$

Since $\|f_n - f_m\|_{\sup} < \varepsilon \longrightarrow |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in [a,b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a,b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit $l_x \in R$ such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a,b]$. Define $f : [a,b] \to R$ by $f(x) = l_x$ for each $x \in [a,b]$.

Candidate is Limit. First, we argue that $\|f_n - f\|_{\sup} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $(f_n)_n$ is an egoprox sequence, there exists n_0 so that

$$n, m \geq n_0 \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon/2.$$

So for all $x \in [a,b]$,

$$n, m \geq n_0 \longrightarrow |f_n(x) - f_m(x)| < \varepsilon/2.$$

For all $x \in [a,b]$, and $n \geq n_0$,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

The sequence $\{f_k(x)\}_{k=m}^{\infty}$ is a final part of $\{f_k(x)\}_{k=1}^{\infty}$, and so has the same limit, $f(x)$. Therefore, using continuity of subtraction and the absolute value,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for $n \geq n_0$, $x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$, from which we deduce $\|f_n - f\|_{\sup} < \varepsilon$. Thus $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

Limit is Continuous. Next, we argue that f is continuous. Let $x_0 \in [a,b]$. Let $\varepsilon > 0$. Since $f_n \longrightarrow f$ there exists n_0 so that

$$\|f_{n_0} - f\|_{\sup} < \varepsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a,b]$. Using $|f(x_0) - f_{n_0}(x_0)| < \varepsilon/3$,

$$|f(x_0) - f(x)|\& < \varepsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a,b]$. Using the triangle inequality,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

for all $x \in [a,b]$. Using $|f_{n_0}(x_0) - f(x)| < \varepsilon/3$

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \varepsilon/3$$

for all $x \in [a,b]$. Since f_{n_0} is continuous, there exists $\delta > 0$ so that

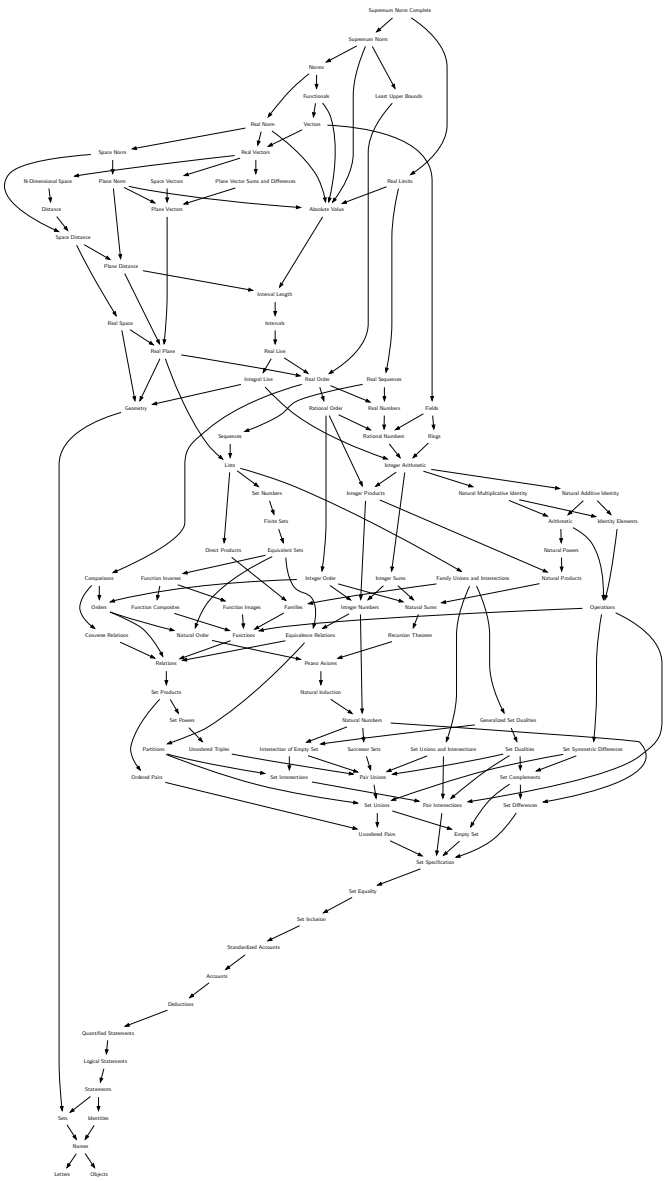
$$|x_0 - x| < \delta \longrightarrow |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3,$$

for $x \in [a,b]$. In this case,

$$|f(x_0) - f(x)|\& < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since ε was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere. Some call the above the *three epsilon argument.*

$\square$


Supremum Norm Complete
Supremum Norm
Norms
Functionals
Least Upper Bounds
Real Norm
Vectors
Space Norm
Real Vectors
N-Dimensional Space
Plane Norm
Space Vectors
Plane Vector Sums and Differences
Real Limits
Distance
Plane Vectors
Absolute Value
Space Distance
Plane Distance
Interval Length
Real Space
Intervals
Real Plane
Real Line
Integral Line
Real Order
Real Sequences
Geometry
Rational Order
Real Numbers
Fields
Sequences
Rational Numbers
Rings
Lists
Integer Arithmetic
Set Numbers
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Arithmetic
Identity Elements
Finite Sets
Natural Powers
Direct Products
Equivalent Sets
Natural Products
Comparisons
Function Inverses
Integer Order
Integer Sums
Family Unions and Intersections
Orders
Function Compositions
Function Images
Families
Integer Numbers
Natural Sums
Operations
Converse Relations
Natural Order
Functions
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
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Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Unions and Intersections
Set Dualities
Set Symmetric Differences
Set Intersections
Pair Unions
Set Complements
Ordered Pairs
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Set Differences
Unordered Pairs
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Standardized Accounts
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Statements
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Names
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