Finite Signed Measures

1 Why

For the difference of two (signed) measures to be well-defined, we need one of the two to be finite. Otherwise, the measure of the difference on the base set involves subtracting ∞ from ∞.

2 Definition

A **finite** signed measure is one for which the measure of every set is finite. This condition is equivalent to the base set having finite measure (see below).

3 Result

Proposition 1. *A signed measure is finite if and only if it is finite on the base set.*

Proof. Let $(X, \mathcal{A})$ be a measurable space. Let $\mu : \mathcal{A} \to [-\infty, \infty]$ be a signed measure. ($\Rightarrow$) If μ is finite, then $\mu(X)$ is finite since $X \in \mathcal{A}$. ($\Leftarrow$) Next, suppose $\mu(X)$ is finite. Let $A \in \mathcal{A}$. Then $X = A \cup (X - A)$, with these sets disjoint, so by countable additivity of μ, $\mu(X) = \mu(A) + \mu(X - A)$. Since $\mu(X)$ finite, $\mu(A)$ and $\mu(X - A)$ are both finite. $\square$

3.1 Vector Space of Measures

If both signed measures are finite, then the difference is always well-defined. Is the difference always a signed measure? Yes.

Proposition 2. *A linear combination of finite signed measures is a finite signed measure.*

Proof. Let $(X, \mathcal{A})$ be a measurable space. Let μ and ν be finite signed measures. Let R denote the real numbers. Then $(\alpha\mu)(\varnothing) = \alpha \cdot \mu(\varnothing) = \alpha \cdot 0 = 0$. Also for $\{A_n\}_n \subset \mathcal{A}$ disjoint,

$$\begin{aligned}(\alpha\mu)(\cup A_n) = \alpha\mu(\cup A_n) &= \alpha \sum_{n=1}^{\infty} \mu(A_n) \\ &= \sum_{n=1}^{\infty} \alpha\mu(A_n) = (\alpha\mu)(A_n)\end{aligned}$$

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Similarly, $(\mu + \nu)(\varnothing) = \mu(\varnothing) + \nu(\varnothing) = 0$. And, for $\{A_n\}_n \subset \mathcal{A}$ disjoint,

$$\begin{aligned}(\mu + \nu)(\cup A_n) = \mu(\cup A_n) + \nu(\cup A_n) &= \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \nu(A_n) \\ &= \sum_{n=1}^{\infty} \mu(A_n) + \nu(A_n) = \sum_{n=1}^{\infty} (\mu + \nu)(A_n)\end{aligned}$$

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$\square$