

EVENT PROBABILITIES

Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Suppose p is a distribution on Ω . For any event $A \subset \Omega$, we call the value $\sum_{a \in A} p(a)$ the *event probability*. We refer to the probability of A. The probability A is the sum of the probabilities of its outcomes.

We can define a function $\mathbf{P}: \mathcal{P}(\Omega) \to \mathbf{R}$ by $\mathbf{P}(A) = \sum_{a \in A} p(a)$. We call \mathbf{P} the event probability function (or the probability measure) induced by p. Since \mathbf{P} depends on the sample space Ω and the distribution p, we ocassionally denote this dependence by $\mathbf{P}_{\Omega,p}$ or \mathbf{P}_p .

Example: die

Define $p:\{1,\ldots,6\}\to \mathbf{R}$ by $p(\omega)=1/6$ for $\omega=1,\ldots,6$. Define the event $E=\{2,4,6\}$. Then

$$\mathbf{P}(E) = \sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of P

As a result of the conditions on p, \mathbf{P} satisfies

- 1. $\mathbf{P}(A) \geq 0$ for all $A \subset \Omega$;
- 2. $\mathbf{P}(\Omega) = 1 \text{ (and } \mathbf{P}(\varnothing) = 0);$
- 3. $\mathbf{P}(A) + \mathbf{P}(B)$ for all $A, B \subset \Omega$ and $A \cap B = \emptyset$. This statement follows from the more general identity

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

for $A, B \subset \Omega$, by using $P(\emptyset) = 0$ of (2) above.

Do all such **P** satisfying (1)-(3) have a corresponding underlying probability distribution? In other words, suppose $f: \mathcal{P}(\Omega) \to \mathbf{R}$ satisfies (1)-(3). These three conditions are sometimes called the *axioms of probability* for finite sets.

Define $q: \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an event probability function (or a probability measure).

Probability by cases

Let **P** be a probability event function. Suppose A_1, \ldots, A_n partition Ω . Then for any $B \subset \Omega$,

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

Some authors call this the *law of total probability*.

