



# Tree-Structured Distributions

## 1 Why

## 2 Definition

### 2.1 Rooted Definition

Consider a set of outcomes which is a direct product of  $n$  sets. Consider a distribution on this set of outcomes. The distribution has  $n$  marginals distributions. The  $n(n-1)$  conditionals of each component given another. Consider a rooted tree over  $n$  vertices.

The distribution *factors according to the rooted tree* if the probability for every outcome can be written as a product of the marginal probability of the outcome at the root with the produce of the probabilities of the conditionals correponding to the edges in the tree.

#### 2.1.1 Notation

Let  $p$  be a distribution on a  $d$ -dimensional space.  $p$  *factors according to the rooted tree* on  $\{1, \dots, d\}$  rooted at a vertex  $k$  if it can be written as a product of  $p_k$  and the conditionals of  $p_{i|j}$

for  $i, j = 1, \dots, d$  and  $i \neq j$  and  $i \neq k$  where  $j$  is the parent of  $i$  in the rooted tree.

## 2.2 Defining Result

**Proposition 1.** *If a distribution factors according to a tree rooted at a vertex it factors according to that tree rooted at any vertex.*

## 2.3 Undirected Definition

A distribution  $p$  factors according to the tree  $T$  if it factors according to the  $T$  rooted at any vertex.

# 3 Existence and Uniqueness

Trees are not a property of distributions, since there is no one-to-one correspondence, as demonstrated by the following propositions.

## 3.1 Existence

A distribution  $p$  need not factor according to a tree.

## 3.2 Uniqueness

A distribution  $p$  may factor according to multiple trees.