



## FUNCTIONS

### Why

We want a notion for a correspondence between two sets.

### Definition

A *function* (or *mapping* or *map*)  $f$  from a set  $X$  to a set  $Y$  is a relation (see **Relations**) whose domain is  $X$  and whose range is a subset of  $Y$  such that for each  $x \in X$ , there exists a unique  $y \in Y$  so that  $(x, y) \in f$ .

We call the unique  $y \in Y$  the *result* of the function *at* the *argument*  $x$ . We call  $Y$  the *codomain*. If the range is  $Y$  we say that  $f$  is a function from  $X$  *onto*  $Y$  (or  $f$  is *surjective*). If distinct elements of  $X$  are mapped to distinct elements of  $Y$ , we say that the function is *one-to-one* (or  $f$  is *injective*).

We say that the function *maps* elements from the domain to the codomain. Since the word “function” and the verb “maps” connote activity, some authors refer to the set of ordered pairs as the *graph* of a function and avoid defining “function” in terms of sets.

### Notation

Let  $X$  and  $Y$  denote sets. We denote a function named  $f$  whose domain is  $X$  and whose codomain is  $Y$  by  $f : X \rightarrow Y$ . We read the notation aloud as “ $f$  from  $X$  to  $Y$ ”. We denote the set of all functions from  $X$  to  $Y$  (which is a subset of  $(X \times Y)^*$ )

by  $Y^X$ . A less standard but equally good notation is  $X \rightarrow Y$ , read aloud as “ $A$  to  $B$ ”. Using the notations introduced so far, we denote that  $f \in (A \rightarrow B)$  by  $f : A \rightarrow B$ . We tend to denote function by lower case latin letters, especially  $f$ ,  $g$ , and  $h$ .  $f$  is a mnemonic for function and  $g$  and  $h$  are nearby.

Let  $f : A \rightarrow B$ . For each element  $a \in A$ , we denote the result of applying  $f$  to  $a$  by  $f(a)$ , read aloud “ $f$  of  $a$ .” We sometimes drop the parentheses, and write the result as  $f_a$ , read aloud as “ $f$  sub  $a$ .” Let  $g : A \times B \rightarrow C$ . We often write  $g(a, b)$  or  $g_{ab}$  instead of  $g((a, b))$ . We read  $g(a, b)$  aloud as “ $g$  of  $a$  and  $b$ ”. We read  $g_{ab}$  aloud as “ $g$  sub  $a$   $b$ .”

## Examples

If  $X \subset Y$ , the function  $\{(x, y) \in X \times Y \mid x = y\}$  is the *inclusion function* of  $X$  into  $Y$ . We often introduce such a function as “the function from  $X$  to  $Y$  defined by  $f(x) = x$ ”. We mean by this that  $f$  is a function and that we are specifying the appropriate ordered pairs using the statement, called *argument-value notation*. The inclusion function of  $X$  into  $X$  is called the *identity function* of  $X$ . If we view the identity function as a relation on  $X$ , it is the relation of equality on  $X$ .

The functions  $f : (X \times Y) \rightarrow X$  defined by  $f(x, y) = x$  is the *pair projection* of  $X \times Y$  onto  $X$ . Similarly  $g : (X \times Y) \rightarrow Y$  defined by  $g(x, y) = y$  is the pair projection of  $X \times Y$  onto  $Y$ . The identity function is one-to-one and onto, the inclusion functions are one-to-one but not always onto, and the pair

projections are usually not one-to-one.

