Why

Can we characterize positive (semi-)definite matrices in terms of their eigenvalues?

Main Result

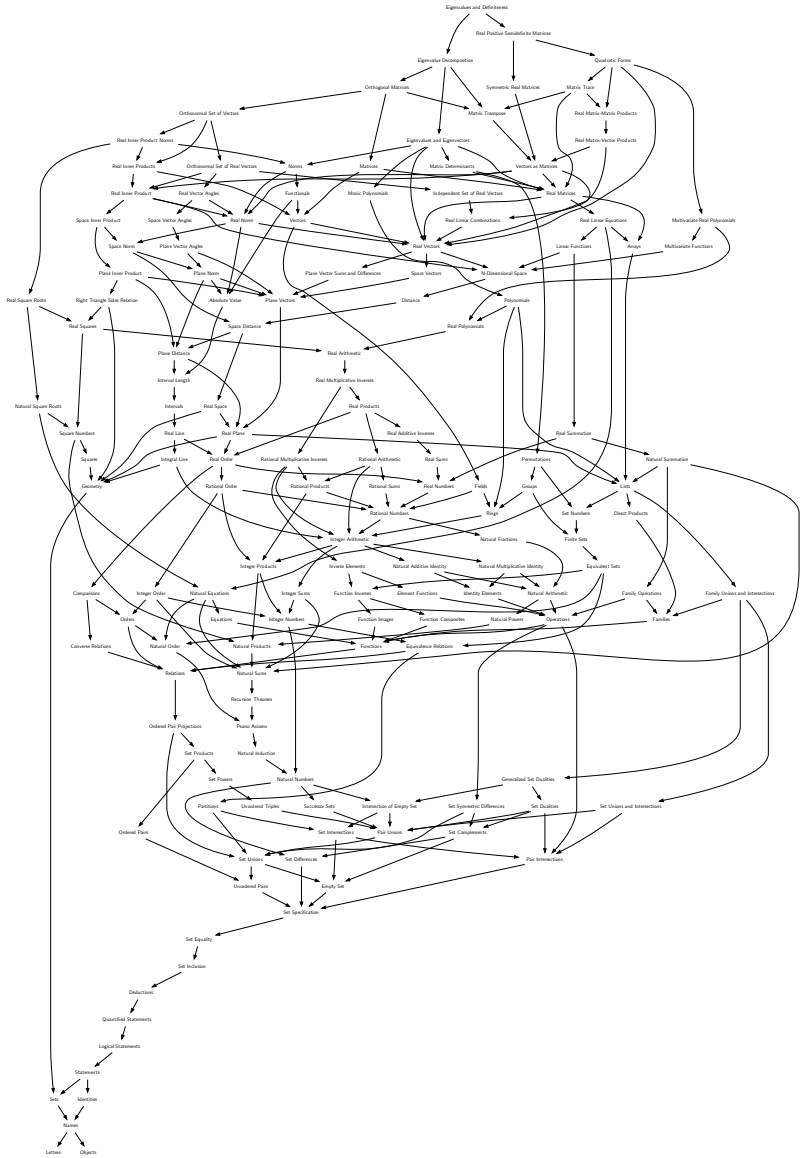
Using eigenvalue decompositions, we can answer in the affirmative.

Proposition 1. *Suppose $A \in \mathbf{S}^d$ has smallest eigenvalue $\lambda_{\min}(A)$. Then*

$$\begin{aligned} A \in \mathbf{S}^d_+ \quad &\longleftrightarrow \quad \lambda_{\min}(A) \geq 0 \\ &\longleftrightarrow \quad \operatorname{tr} AB \geq 0 \textit{ for all } B \in \mathbf{S}^d_+. \end{aligned}$$

and

$$\begin{aligned} A \in \mathbf{S}^d_{++} \quad &\longleftrightarrow \quad \lambda_{\min}(A) > 0 \\ &\longleftrightarrow \quad \operatorname{tr} AB > 0 \textit{ for all nonzero } B \in \mathbf{S}^d_{++}. \end{aligned}$$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