



# Tree-Structured Distributions

## 1 Why

We consider distributions over products of several finite sets. These distributions have many outcomes, and so tabulating the probabilities associated with each outcome is usually laborious, and most often infeasible. So instead of working with all possible distributions over a product of finite sets, we work with those which can be characterized by tabulating the probabilities of fewer outcomes.

## 2 Definition

A distribution over a product of  $n$  finite sets has  $n$  one-variable marginals (one for each component) and  $n(n - 1)$  two-variable conditionals (one for each component given each other component). We will be interested in those distributions over this product which can be written as a product of these marginals and conditionals.

### 2.1 Rooted Definition

A distribution over a product of  $n$  finite sets *factors according to a rooted tree* on  $\{1, \dots, n\}$  if the probability of each outcome can be expressed as the product of the marginal probability of the component corresponding to the root of the tree and the conditionals corresponding to the edges in the tree.

### 2.1.1 Notation

Let  $A_1, \dots, A_n$  be finite sets and define  $A = \prod_{i=1}^n A_i$ . Let  $p : A \rightarrow [0, 1]$  be a distribution. If for every  $a \in A$ ,

$$p(a) = p_k(a_k) \prod_{(i,j) \in E} p_{i|j}(a_i | a_j)$$

where  $E \subset \{1, \dots, n\}^2$  and  $(\{1, \dots, n\}, E)$  is a tree, then  $p$  factors according to that same tree rooted at vertex  $k$ .

## 2.2 Defining Result

**Proposition 1.** *If a distribution factors according to a tree rooted at a vertex  $k$ , then it factors according to that tree rooted at any vertex.*

## 2.3 Undirected Definition

A distribution  $p$  factors according to the tree  $T$  if it factors according to the  $T$  rooted at any vertex.

# 3 Existence and Uniqueness

Trees are not a property of distributions, since there is no one-to-one correspondence, as demonstrated by the following propositions.

## 3.1 Existence

A distribution  $p$  need not factor according to a tree.

## 3.2 Uniqueness

A distribution  $p$  may factor according to multiple trees.