

## PROBABILISTIC FUNCTIONAL INDUCTORS

## Why

Let  $X = \{a, b\}$  and  $Y = \{0, 1\}$ . The dataset ((a, 0)) is consistent, but it is not functionally complete. On the other hand, the dataset ((a, 0), (b, 0), (a, 0), (a, 0), (a, 0), (a, 1)) is complete but it is not functionally consistent.

In general, if  $y_i \neq y_j$  for some i and j where  $x_i = x_j$ , then the dataset is not functionally consistent. In the preceding example, both (a, 0) and (a, 1) appear.

If we emphasize the "predictive" aspect of a functional inductor, we interpret the input as an object we "see before" the output. And so treat  $y \in Y$  as an uncertain outcome which is the element associated to  $x \in X$ .

In this case, we may use the language of probability to discuss this uncertain outcome. If, for example, Y is finite, we can associate a distribution with each input  $x \in X$ .

## Definition

Let  $(X, \mathcal{X})$  and  $(Y, \mathcal{Y})$  be measurable spaces.

A probabilistic functional inductor (for a dataset of size n in  $X \times Y$ ) is a function mapping a dataset in  $(X \times Y)^n$  to a family of measures on  $(Y, \mathcal{Y})$ , indexed by X. We call a function from inputs to output measures a probabilistic predictor. We call the distribution a probabilistic prediction.

## Notation

Let  $\mathcal{M}(Y, \mathcal{Y})$  be the set of measures on Y. Let D be a dataset in  $(X \times Y)^n$ . Let  $g: X \to \mathcal{M}(Y, \mathcal{Y})$  a probabilistic predictor. Let  $G_n(X \times Y)^n \to (X \to \mathcal{M}(Y, \mathcal{Y}))$  be a predictive probabilistic inductor. Then  $G_n(D)$  is a family of measures  $\{g_x: \mathcal{Y} \to [0, 1]\}_{x \in X}$ .

