Why

Result

We bound below the measure that a non-negative measurable real-valued function exceeds some value by its integral.

Prop. 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $g : X \to [0, \infty]$ be measurable and square-integrable. Then for all t such that $\int t d\mu \in [0, \int g d\mu)$,*

$$\mu(\{x \in X \mid g(x) > t\}) \geq \frac{(\int (g-t) d\mu)^2}{\int g^2 d\mu}.$$

Proof. Let t such that $\int t d\mu \in [0, \int g)$. We have selected t so that $\int (g - t) d\mu \geq 0$. Define $h = (g-t)^+$ and $A = \{x \in X \mid h(x) > 0\}$. Then

$$\int (g-t) d\mu \leq \int h d\mu = \int h \chi_A d\mu \leq \sqrt{\int h^2 d\mu \int \chi_A^2 \, d\mu}$$

□

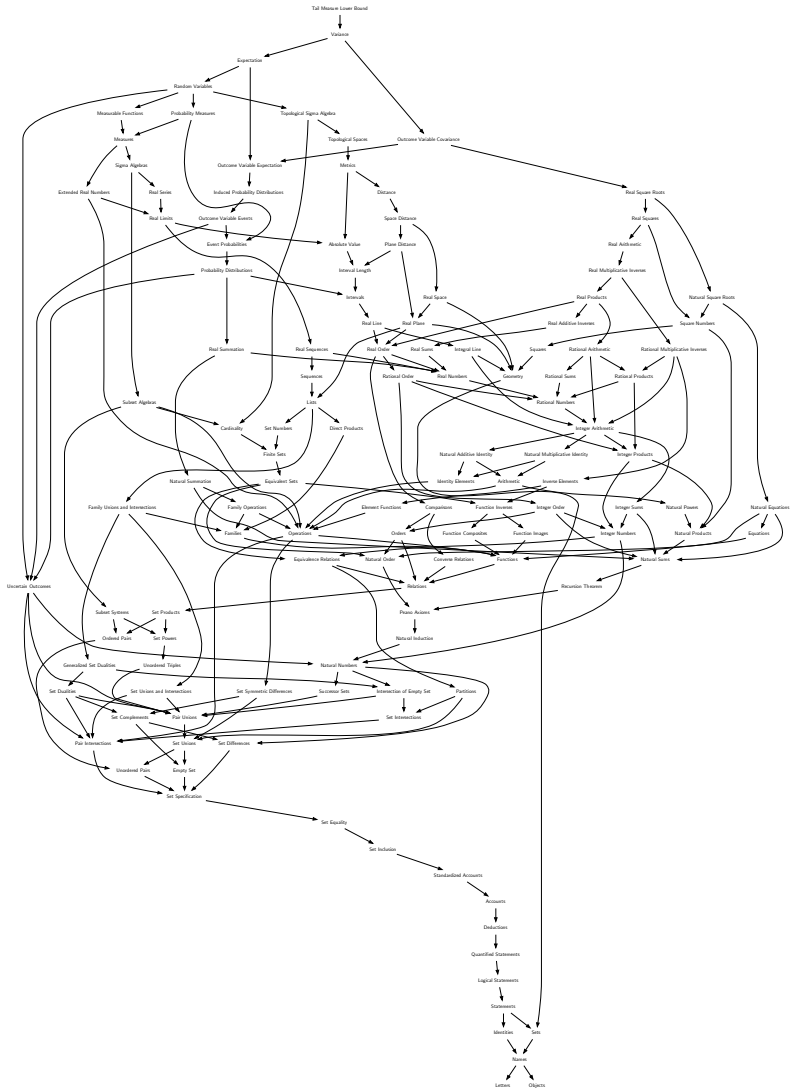
Now $g^2 > h^2$, so $\int g^2 d\mu \geq \int h^2 d\mu$. Also $\chi_A^2 = \chi_A$ so $\int \chi_A^2 = \mu(A)$. $h(x) > 0$ if and only if $g(x) \geq t$ for all x. So $A = \{x \in X \mid g(x) \geq t\}$. Combining we have:

$$\int (g-t) d\mu \leq \sqrt{\left(\int g^2 d\mu\right) \mu(A)}.$$

Prop. 2. *Let X be a random variable with $\mathbf{E}(X^2) \leq \infty$. Then for all $t \in [0, \mathbf{E}(X))$, we have*

$$P(X > t) \geq \frac{(\mathbf{E}(X) - t)^2}{\mathbf{E} X^2}.$$

The above is also called the *Paley-Zygmund Inequality*.


Tail Measure Lower Bound
Variance
Expectation
Random Variables
Measurable Functions
Probability Measures
Topological Sigma Algebra
Topological Spaces
Outcome Variable Covariance
Measures
Outcome Variable Expectation
Metrics
Sigma Algebras
Extended Real Numbers
Real Sets
Induced Probability Distributions
Distance
Real Square Roots
Real Limits
Outcome Variable Events
Space Distance
Real Squares
Event Probabilities
Absolute Value
Plane Distance
Real Arithmetic
Interval Length
Probability Distributions
Real Multiplication Inverses
Intervals
Real Space
Real Products
Natural Square Roots
Real Line
Real Plane
Real Additive Inverses
Square Numbers
Real Summation
Real Sequences
Real Order
Real Sums
Integral Line
Squares
Rational Arithmetic
Rational Multiplication Inverses
Sequences
Rational Order
Real Numbers
Geometry
Rational Sums
Rational Products
Subset Algebras
Lists
Rational Numbers
Cardinality
Set Numbers
Direct Products
Integer Arithmetic
Finite Sets
Natural Additive Identity
Natural Multiplication Identity
Integer Products
Natural Summation
Equivalent Sets
Identity Elements
Arithmetic
Inverse Elements
Family Unions and Intersections
Family Operations
Element Functions
Comparisons
Function Inverses
Integer Order
Integer Sums
Natural Power
Natural Equations
Families
Operations
Order
Function Compositions
Function Images
Integer Numbers
Natural Products
Equations
Equivalence Relations
Natural Order
Converse Relations
Functions
Natural Sums
Uncertain Outcomes
Relations
Recursion Theorem
Subset Systems
Set Products
Proofs Axioms
Ordered Pairs
Set Powers
Natural Induction
Generalized Set Duality
Unordered Triples
Natural Numbers
Set Duality
Set Unions and Intersections
Set Symmetric Differences
Successor Sets
Intersection of Empty Set
Partitions
Set Complements
Pair Unions
Set Intersections
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects