



MONOTONIC MATRIX FUNCTIONS

Why

Since we have a partial order on the set of positive semidefinite matrices, we can study which familiar functions are have order-preserving or order-reversing properties.

Norms

It would be nice if the matrix norm induced by the matrix scalar produce (see **Matrix Scalar Product**) was an isotonic function. In other words, if $A, B \in \mathbf{S}^d$ satisfy $A \geq B$, does $\|A\| \geq \|B\|$?

Since $\|A\|^2 = \mathbf{tr} A^2$, we should study the trace first..

Trace

Proposition 1. *Let $f : \mathbf{S}^d \rightarrow \mathbf{R}$ defined by $f(A) = \mathbf{tr} A$.*

In other words, the function f is the restriction of the trace function onto the set of symmetric matrices.

Proposition 2. *Let $B \in \mathbf{S}^d$ Let $f_B : \mathbf{S}^d \rightarrow \mathbf{R}$ defined by $f(A) = \mathbf{tr} AB$.*

Inversion

Proposition 3. *Let $A \in \mathbf{S}_{++}^d$. Then the map $f : \mathbf{S}_{++}^d \rightarrow \mathbf{S}_{++}^d$ satisfying $f(A) = A^{-1}$ is an isotonic function mapping the (open) positive definite cone into itself.¹*

¹Future editions will include a proof.

