Why

We use a normal random function model to make a regressor.

Definition

Let $F : \Omega \to (A \to \mathbf{R})$ be a normal random function with mean function $m : A \to \mathbf{R}$ and covariance function $k : A \times A \to \mathbf{R}$ over the probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let the family of random variables (or stochastic process) of F be $f : A \to (\Omega \to \mathbf{R})$.

Let e be a normal random vector with mean zero and covariance Σ_e. We call e the *error vector* or *noise vector*. Let $a^1, \ldots, a^n \in A$. We sometimes call the sequence $a^1, \ldots, a^n$ the *design*. Define $y : \Omega \to \mathbf{R}^d$ by

$$y_i = f(a^i) + e_i$$

We call y the *observation vector* or *observation random vector*.

Let $b^1, \ldots, b^m \in A$. Define $z : \Omega \to \mathbf{R}^d$ by $z_i = f(b^i)$ for $i = 1, \ldots, n$. So z_i is the random variable corresponding to the family at index $b^i \in A$. Then (y, z) is normal. We call the conditional density of z given y the *predictive density* for b given a.

Proposition 1. *Define* $m_a \in \mathbf{R}^n$ *by* $(m(a^1), \cdots, m(a^n))^\top$ *and*

define m_b *by* $(m(b^1), \cdots, m(b^m))^\top$.[1] *Define* $\Sigma_a \in \mathbf{R}^{n\times n}$ *by*

$$\begin{pmatrix} k(a^1,a^1) & \cdots & k(a^1,a^n) \\ \vdots & \ddots & \vdots \\ k(a^n,a^1) & \cdots & k(a^n,a^n) \end{pmatrix}$$

and define $\Sigma_{ba} \in \mathbf{R}^{m\times n}$ *by*

$$\begin{pmatrix} k(b^1,a^1) & \cdots & k(b^1,a^n) \\ \vdots & \ddots & \vdots \\ k(b^m,a^1) & \cdots & k(b^m,a^n) \end{pmatrix}.$$

The predictive density $g_{z|y}(\cdot,\gamma) : \mathbf{R}^m \to \mathbf{R}$ *of* $b \in A$ *for design* $a^1, \ldots, a^n$ *is normal with mean.*

$$m_b + K_{ba}\left(K_a + \Sigma_e\right)^{-1}(\gamma - m_a)$$

and covariance

$$\Sigma_b - \Sigma_{ba}\left(\Sigma_a + \Sigma_e\right)^{-1}\Sigma_{ab}.$$

Regressor

Ultimately, we want a regressor $h : A \to \mathbf{R}$. We maximize the the predictive density. The *normal random function predictor* or *gaussian process predictor* for dataset $(a^1,\gamma_1), \ldots, (a^n,\gamma_n)$ in $A \times \mathbf{R}$ is $h : A \to \mathbf{R}$ defined by

$$h(x) = m(x) + \Big(k(x,a^1)\cdots k(x,a^n)\Big)\left(\Sigma_a + \Sigma_e\right)^{-1}(\gamma - m_a).$$

[1] Future editions will fix the re-use of the symbol m.

The process of using a normal random function predictor is often called *Gaussian process regression.* The upside is that a gaussian process predicor interpolates the data, is smooth, and the so-called variance increases with the distance from the data.[2]

[2]This last piece is true for certain covariance kernels and will be clarified in future editions.


Normal Random Function Regressors
Normal Random Functions
Random Functions
Multivariate Normals
Matrix Determinants
Positive Definite Matrices
Random Vectors
Matrix Inverse
Real Matrices
Symmetric Matrices
Quadratic Forms
Random Variables
Multivariate Real Densities
Normal Densities
Matrix-Matrix Products
Matrix Transpose
Identity Matrices
Topological Sigma Algebra
Probability Densities
Exponential Function
Matrices
Real Vectors
Topological Spaces
Real Integrals
Vectors
Outcome Variables
Metrics
N-Dimensional Space
Nonnegative Integrals
Absolute Value
Distance
Simple Integrals
Real Functions
Probability Measure
Measurable Functions
Space Distance
Real Square Roots
Event Probabilities
Measures
Plane Distance
Real Arithmetic
Subset Algebras
Extended Real Numbers
Probability Distributions
Interval Length
Real Multiplicative Inverses
Set Operations
Cardinality
Intervals
Real Products
Real Limits
Real Line
Real Plane
Real Space
Real Additive Inverses
Real Sequences
Real Summation
Geometry
Direct Products
Real Order
Real Sums
Rational Multiplicative Inverses
Rational Arithmetic
Fields
Real Numbers
Rational Products
Rational Sums
Groups
Uncertain Outcomes
Rational Numbers
Integer Additive Inverses
Natural Summation
Inverse Elements
Integer Arithmetic
Sequences
Family Operations
Element Functions
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Integer Sums
Families
Function Inverses
Algebras
Identity Elements
Arithmetic
Integer Numbers
Natural Square Roots
Function Images
Function Composites
Operations
Natural Exponents
Square Numbers
Functions
Equivalence Relations
Natural Products
Squares
Simple Functions
Relations
Natural Sums
Characteristic Functions
Subset Systems
Cartesian Products
Recursion Theorems
Set Powers
Ordered Pairs
Peano Axioms
Natural Induction
Partitions
Unordered Triples
Natural Numbers
Equation Solutions
Successor Sets
Intersection of Empty Set
Set Symmetric Differences
Pair Unions
Set Intersections
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Set Complements
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Deduction
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects