**Result**

**Proposition 1.** *Suppose $P$ is a* **probability measure** *on a finite set of outcomes* $\Omega$. *For any two* **events** $A, B$ *with $P(A), P(B) > 0$, we have*

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

*Proof.* By definition, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

And also symmetrically,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

From this second equation we have $P(A \cap B) = P(B \mid A)P(A)$, which we can substitute into the numerator of the first expression to obtain the result. $\square$

This result is known by many names including *Bayes' rule*, *Bayes rule* (no possessive), *Bayes' formula*, and *Bayes' theorem*.

It is a *basic* consequence of the *definition* of conditional probability, but it is *useful* in the case that we are given problem data in terms of the probabilities on the right hand side of the above equation.

**Examples**

**Diagnostic test**

Suppose we want to model the situation in which a rare disease afflicts 0.5% of a population and we have a diagnostic test that is 99% accurate.

We consider the population $\Omega$ of people. We agree to partition the population into $D$ and $H$ so that

$$D \cup H = \Omega \quad \text{and} \quad D \cap H = \varnothing$$

$D$ is the set of people with the disease, and $H$ is the set of *healthy* people without the disease. Similarly, we agree to partition the population into $R$ and $N$ so that

$$R \cup N = \Omega \quad \text{and} \quad P \cap N = \varnothing$$

$R$ is the set of people who test positive, and $N$ is the set of people who test negative.

We agree that 0.5% of the the population being afflicted means, $P(D) = 0.005$. We agree that having a 99% accurate test means *means*

$$P(D \mid R) = 0.99 \quad \text{and } P(H \mid N) = 0.99$$

Now, what is the conditional probability of having the disease given that the test is positive? Using Bayes rule,

$$P(D \mid R) = \frac{P(R \mid D)P(D)}{P(R \mid D)P(D) + P(R \mid H)P(H)}$$

Using our supposed values,

$$P(D \mid R) = \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995} \approx 0.33$$

This may be viewed as suprising, since the test is perceived to be accurate.

The frequentist interpretation is clear: if have many outcomes, say a thousand individuals, we may expect that about five of these thousand have the disease. The test is likely to diagnose these correctly. However, of the other 995 people, about 1% of them—say 10 people—will be mis-diagnosed. Thus we may expect to see about 15 positive test results, but only five of which correspond to individuals with the disease.

**Compound form**

More is true.

**Proposition 2.** *Suppose $P$ is a finite probability measure on a set of outcomes $\Omega$. For any three events $A, B, C$ with $P(A), P(B), P(C) > 0$, we have*

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C)P(A \mid C)}{P(B \mid C)}.$$

*Proof.* Future editions will include, the strategy is the same as above. □

Posterior Event Probabilities

Conditional Event Probabilities

Event Probabilities

Outcome Probabilities

Real Functions

Intervals

Real Line

Real Order

Real Summation

Size of Direct Product

Rational Order

Real Numbers

Natural Summation

Number of Set Products

Integral Line

Rational Numbers

Number of Disjoint Unions

Finite Set Examples

Lists

Integer Arithmetic

Natural Additive Identity

Natural Multiplicative Identity

Natural Fractions

Set Numbers

Direct Products

Integer Products

Identity Elements

Natural Arithmetic

Finite Sets

Integer Sums

Natural Powers

Integer Order

Comparisons

Uncertain Events

Family Operations

Equivalent Sets

Function Inverse

Family Unions and Intersections

Integer Numbers

Natural Products

Orders

Converse Relations

Equivalence Relations

Natural Order

Function Image

Function Composition

Families

Functions

Recursion Theorem

Relations

Ordered Pair Projections

Uncertain Outcomes

Peano Axioms

Set Products

Natural Induction

Generalized Set Dualities

Set Powers

Natural Numbers

Unordered Triples

Successor Sets

Set Symmetric Differences

Intersection of Empty Set

Set Dualities

Set Partitions

Set Unions and Intersection

Ordered Pairs

Pair Unions

Set Complements

Set Intersection

Set Differences

Set Unions

Pair Intersections

Unordered Pairs

Empty Set

Geometry

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects