Functions

1 Why

We want a notion for a correspondence between two sets.

2 Definition

To each element of a first set we associate an element of a second set. We call this correspondence a **function**. We call the first set the **domain** and the second set the **codomain**. We say that the function **maps** elements from the domain into the codomain.

We call the codomain element associated with the domain element the **result** of **applying** the function to the domain element. We call the subset of ordered pairs whose first element is in the domain and whose second element is the corresponding result the **graph** of the function. The graph is a relation between the domain and codomain. So a function can be viewed as or specified as a relation between these two sets.

2.1 Notation

We often denote functions by lower case latin letters, especially f, g, and h. Of course, f is a mnemonic for function; g and h follow f in the alphabet. Let A and B be two non-empty sets. When we want to be explicit that the domain of a function f is A and its codomain is B we write $f : A \to B$, read aloud as "f from A to B."

For each element a in the domain, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

For $f : A \to B$, the set $\{(a, f(a)) \in A \times B \mid a \in A\}$ of ordered pairs is the graph of f. We often denote it by Γ_f; "gamma" is a mnemonic for graph.

3 Properties

For a function $f : A \to B$ and a set $C \subset A$, we call the set $\{f(c) \in B \mid c \in C\}$ the **image** of C. We call the image of the domain the **range** of the function.

The range need not equal the codomain; though it, like every other image, is a subset of the codomain. If the range and codomain are equal, we call the function **onto**. We say that the function maps the domain onto the range. This language suggests that every element of the codomain is used by f. It means that for each element b of the codomain, we can find an element a of the domain so that $f(a) = b$.

An element of the codomain may be the result of several elements of the domain. This overlapping, using an element of the codomain more than once, is a regular occurrence. If a function is a unique correspondence in that every domain element has a different result, we call it **one-to-one**. This language is meant to suggest that each element of the domain corresponds to one and exactly one element of the codomain, and vice versa.

3.1 Notation

Let $f : A \to B$ and $C \subset A$. We denote the image of C by $f(C)$, read aloud as "f of C." The property that f is onto can be written succintly as $f(A) = B$.

Let $c \in C$. Notice that we have defined notation for $f(c)$ and $f(C)$. In overloading notation, we have introduced ambiguity. $f(c)$ is an element of the codomain. $f(C)$ is a subset of the codomain. Read aloud, the two are indistinguishable, must be careful to have specified whether we mean an element c or a set C. There is no trouble when we take this precaution.