



Why

We want to associate the natural numbers with bit strings for use on digital computers.¹

Definition

A *digital natural* is a bit string. The set of *d-bit digital natural numbers* is the set of length- d bit strings $\{0, 1\}^d$. For example, the set of 8-bit digital naturals is the set $\{0, 1\}^8$.

Correspondence with $\mathbf{N} \cup \{0\}$

We associate $x \in \{0, 1\}^d$ corresponds to the number $\sum_{i=1}^d x_i 2^i$. For example, the bit string $(0, 0, 0) \in \{0, 1\}^3$ corresponds to the natural number $0 \in \omega$. Likewise, $(1, 0, 0)$ corresponds to $1 \in \mathbf{N}$, $(0, 1, 0)$ corresponds to 2, $(1, 1, 0)$ corresponds to 3, etc.

Call the function so defined the *digital natural decoder*, and denote it by $f : \{0, 1\}^d \rightarrow \mathbf{N} \cup \{0\}$. In other words $f((0, 0, 0)) = 0$, $f((0, 1, 0)) = 2$, etc. Call the set $f(\{0, 1\}^d)$ the set of naturals *representable* by length- d bit strings.

Specifically, if, for $n \in \mathbf{N} \cup \{0\}$, there exists $x \in \{0, 1\}^d$ so that $f(x) = n$, we say that x is *representable in d bits*.

Correspondence between d and $k > d$ bit naturals

Let $x \in \{0, 1\}^d$ and $y \in \{0, 1\}^k$ with $k > d$. Although $\{0, 1\}^d \not\subset \{0, 1\}^k$, $f(\{0, 1\}^d) \subset f(\{0, 1\}^k)$. We can identify

¹Future editions will expand.

$x \in \{0, 1\}^d$ with $x' \in \{0, 1\}^k$ where $x' = (x_1, \dots, x_d, 0, \dots, 0)$ so that $f(x) = f(x')$. Clearly then, if x is representable in d bits, it is representable in $k > d$ bits.

Addition

We want to define addition $\oplus : \{0, 1\}^d \times \{0, 1\}^d \rightarrow \{0, 1\}^d$ so that $f(x \oplus x') = f(x) + f(x')$. In general, we are stuck, because $x + x'$ may not be representable in d bits. Suppose, however and for the time being, that it is.²

²Future editions will complete.

