Why

For each value of a random variable's codomain, the set of outcomes corresponding to that value is the inverse image of the random variable. We can speak of the probability that a random variable takes a value then, by assigning it the probability of the set of outcomes corresponding to that value.

Definition

Let $p : \Omega \to \mathbf{R}$ be a probability distribution with corresponding probability measure $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$. Suppose $x : \Omega \to V$ is an outcome variable. The *probability* $x = a$, for $a \in \Omega$, is

$$\mathbf{P}(\{\omega \in \Omega \mid x(\omega) = a\}).$$

From the definition of $\mathbf{P}$, we express the above as

$$\textstyle\sum_{\omega \in \Omega \mid x(\omega) = a} p(\omega).$$

We refer to the probability of the *event* that $x = a$.

Notation

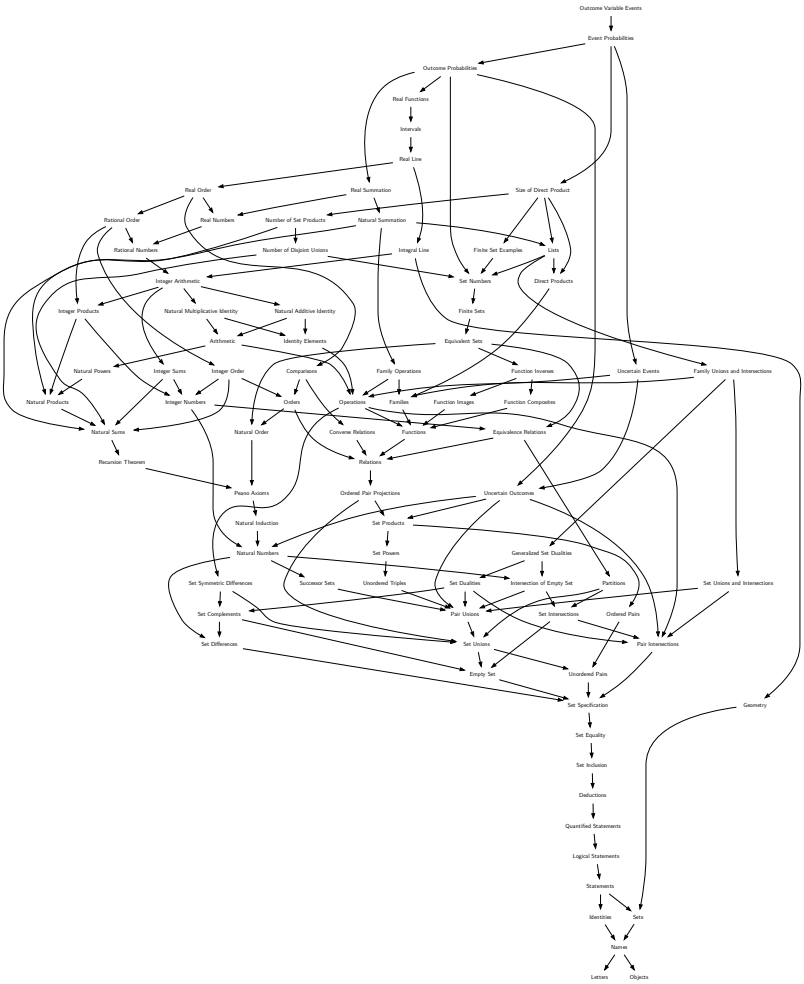
We denote the probability that $x = a$ by $\mathbf{P}[x = a]$. Our square brackets deviate from the slightly slippery but universally standard notation $\mathbf{P}(x = a)$. We prefer the square brackets, since $x = a$ is not itself an argument to $\mathbf{P}$, but shorthand for the set $\{\omega \in \Omega \mid x(\omega) = a\}$.

There are many similar notations. For example, $\mathbf{P}[x \in C]$ means $\mathbf{P}(\{x \in \Omega \mid x(\omega) \in C\})$. In particular, if $x : \Omega \to \mathbf{R}$, $\mathbf{P}[x \geq a]$ means $\mathbf{P}(\{\omega \in \Omega \mid x(\omega) \geq a\})$. Since the *event* that $x = a$ is the inverse image of $\{a\}$ under x, we also use the notations $\mathbf{P}(x^{-1}(a))$ and $\mathbf{P}(x^{-1}(C))$.

Example: sum of two dice

Define $\Omega = \{1, \ldots, 6\}^2$ and define $p : \Omega \to \mathbf{R}$ with $p(\omega) = 1/36$ for each $\omega \in \Omega$. Define $x : \Omega \to \mathbf{N}$ by $x(\omega_1, \omega_2) = \omega_1 + \omega_2$. Then

$$\mathbf{P}[x = 4] = p((2, 2)) + p(1, 3) + p(3, 1) = 1/12.$$


Outcome Variable Events
Event Probabilities
Outcome Probabilities
Real Functions
Intervals
Real Line
Real Order
Real Summation
Size of Direct Product
Rational Order
Real Numbers
Number of Set Products
Natural Summation
Rational Numbers
Number of Disjoint Unions
Integral Line
Finite Set Examples
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparison
Family Operations
Function Inverses
Uncertain Events
Family Unions and Intersections
Natural Products
Integer Numbers
Orders
Operations
Families
Function Images
Function Composites
Natural Sums
Natural Order
Converse Relations
Functions
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Uncertain Outcomes
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Geometry
Set Equality
Set Inclusion
Deductions
Quantified Statements
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Statements
Identities
Sets
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