



## Why

We consider the probabilistic linear model in which all random variables are normal.

## Definition

A *normal linear model* is a probabilistic linear model in which the parameter and noise vectors have normal (Gaussian) densities. The model is also called the *Gaussian linear model* or the *linear model with Gaussian noise*.

Let  $(x : \Omega \rightarrow \mathbf{R}^d, A \in \mathbf{R}^{n \times d}, e : \Omega \rightarrow \mathbf{R}^n)$  be a probabilistic linear model over the probability space  $(\Omega, \mathcal{A}, \mathbf{P})$  in which  $x$  and  $e$  have normal densities. Recall that a probabilistic linear model has observation vector  $y : \Omega \rightarrow \mathbf{R}^n$  defined by

$$y = Ax + e.$$

## Conditional density of $x$ on $y$

Since  $x$  and  $e$  are normal and independent,  $y$  is normal.<sup>1</sup> Moreover, the random vector  $(x, y)$  is normal with covariance

$$\begin{pmatrix} \Sigma_x & \Sigma_x A^\top \\ A \Sigma_x & A \Sigma_x A^\top + \Sigma_e \end{pmatrix}.$$

So the conditional density (see **Normal Conditionals**) of  $g_{x|y}(\cdot, \gamma)$  is normal with mean

$$\Sigma_x A^\top (A \Sigma_x A^\top + \Sigma_e)^{-1} \gamma$$

and covariance

$$\Sigma_x - \Sigma_x A^\top (A \Sigma_x A^\top + \Sigma_e)^{-1} A \Sigma_x.$$

This density is sometimes called the posterior for the parameters given the observations. So the parameter posterior of the normal linear model is normal.

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<sup>1</sup>Future editions will include an account.

We can write the conditional mean as

$$(\Sigma_x^{-1} + A^\top \Sigma_e^{-1} A)^{-1} A^\top \Sigma_e^{-1}$$

and the conditional covariance as<sup>2</sup>

$$(\Sigma_x^{-1} + A^\top \Sigma_e^{-1} A)^{-1}.$$

Very frequently we use these forms when  $d < n$ . In other words, in the case that we have fewer unknowns than measurements. In that case  $\Sigma_x$  is smaller than  $A \Sigma_x A^\top$ .

### Maximum conditional estimate of $x$

**Proposition 1.** *The maximum conditional estimate of  $x : \Omega \rightarrow \mathbf{R}^d$  given observed value  $\gamma \in \mathbf{R}^n$  of  $y : \Omega \rightarrow \mathbf{R}^n$  is the conditional mean*

$$\Sigma_x A^\top (A \Sigma_x A^\top + \Sigma_e)^{-1} \gamma.$$

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<sup>2</sup>A proof will appear in future editions. Use the matrix inversion lemma or facts about inverses.

