



# Real Length Impossible

## 1 Why

Given a subset of the real line, what is its length?

## 2 Background

Let  $a, b \in \mathbb{R}$  with  $a \leq b$ . The *length* of the closed interval of the real numbers  $[a, b]$  is  $b - a$ . The length is non-negative.

A family  $\{A_\alpha\}_{\alpha \in I}$  is *disjoint* if for  $\alpha, \beta \in I$ ,  $\alpha \neq \beta$ , then  $A_\alpha \cap A_\beta = \emptyset$ . A set  $A$  can be *partitioned* into a family if there exists a disjoint family whose union is  $A$ . A set  $A \subset \mathbb{R}$  is *simple* if it can be partitioned into a countable family whose members are closed intervals. The above discussion suggests that we should define the length of a simple set as the sum of the lengths of sets which partition it.

The above discussion suggests that if we wish to define a function  $\text{length} : 2^{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ , we should ask that (1)  $\text{length}(A) \geq 0$ , (2)  $\text{length}([a, b]) = b - a$ , (3) for disjoint closed intervals  $\{A_n\}_{n \in \mathbb{N}}$ ,  $\text{length}(\bigcup_i A_i) = \sum_i \text{length}(A_i)$ , and (4) for all  $A \subset \mathbb{R}$  and  $a \in \mathbb{R}$ ,  $\text{length}(A + a) = \text{length}(A)$ .

## 3 Converse

Define the equivalence relation  $\sim$  on  $\mathbb{R}$  by  $x \sim y$  if  $x - y \in \mathbb{Q}$

### 3.1 Notation

Let  $A$  be a set and  $\mathcal{A} \subset A^*$ . We denote the subset algebra of  $A$  and  $\mathcal{A}$  by  $(A, \mathcal{A})$ , read aloud as “A, script A.”

## 4 Properties

**Proposition 1.** *For any set  $A$ ,  $2^A$  is a sigma algebra.*

**Proposition 2.** *The intersection of a family of sigma algebras is a sigma algebra.*

## 5 Generation

**Proposition 3.** *Let  $A$  a set and  $\mathcal{B}$  a set of subsets. There is a unique smallest sigma algebra  $(A, \mathcal{A})$  with  $\mathcal{B} \subset \mathcal{A}$ .*

We call the unique smallest sigma algebra containing  $B$  the *generated sigma algebra* of  $B$ .