



## Real Limits

### 1 Why

We want to speak of an infinite process which, although never arrives, does terminate.

### 2 Definition

A *limit* of a sequence of real numbers is a real number for which we can always find a final part of the sequence wholly contained in an interval around the limit, no matter how small the interval.

You propose a limit for a sequence. To test this proposal, I specify some small positive real number. Then we look for a final part wholly contained in the interval of that width. If we can always find the final part, no matter how small the positive number I specified, then the proposed limit is true.

#### 2.1 Existence

Some sequences have no limits. Consider the sequence which alternates between the  $+1$  and  $-1$ . To show that the limit does not exist, we argue indirectly. We take any real number and test it with the interval length one. No matter which real number we have selected,  $+1$  and  $-1$  are a distance two apart, and so can not both be contained in an interval of width one.

## 2.2 Uniqueness

If a sequence has a limit, it has only one limit. So, from here on, we will speak of *the limit* of the sequence.

To see this uniqueness, suppose that two real numbers satisfy the limiting property. We now argue indirectly: suppose also that they are not equal. Denote the distance between them by  $x$ . Then ask for final parts in intervals of width  $x/2$  for both limits.

## 2.3 Approximation

We use limits to speak about the terminating behavior of infinite processes. We think about the sequence as approximating the limit. The sequence may never actually take the value of its limit, so the limit need be in the set of terms of the sequence, but it does get close.

The definition, moreover, ensures that the sequence will get arbitrarily close. We can operationalize this property, by taking the first element of that final part after which all elements are close to the limit. This element is an element of the sequence approximates the limit value well.

## 2.4 Notation

Let  $(a_n)_n$  be a sequence of real numbers. Let  $a$  be a real number. We denote that  $a$  is the limit of  $(a_n)_n$  by

$$a = \lim_{n \rightarrow \infty} a_n.$$

We read this statement aloud as "a is the limit of a sub n." The above statement asserts two facts: (1) the sequence  $(a_n)_n$  has a limit and (2)

the limit is the real number  $a$ . We sometimes abbreviate the by writing

