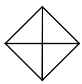

Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering $\geq$ on S^d defined by

$$A \geq B \quad \longleftrightarrow \quad A - B \geq 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_+.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \quad \longleftrightarrow \quad A - B > 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_{++}.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$\begin{aligned} \alpha A \geq 0 &\quad \text{for all } \delta > 0, A \geq 0, \\ A + B \geq 0 &\quad \text{for all } A, B \geq 0, \\ A \geq B \text{ and } B \geq A \longrightarrow A = B &\quad \text{for all } A, B \in \mathsf{S}^d, \\ \lim_{n \to \infty} A_n = A \longrightarrow A \geq 0 &\quad \text{for all sequences } (A_n)_n \text{ in } \mathsf{S}^d_+. \end{aligned}$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that $\geq$ is antisymmetric. Moreover,

$$A \geq A \quad \text{for all } A \in \mathsf{S}^d, \text{ and}$$
$$A \geq B \text{ and } B \geq C \longrightarrow A \geq C \text{ for all } A, B, C \in \mathsf{S}^d.$$

In other words, $\geq$ is also reflexive and transitive. In other words, $\geq$ is a partial order (see Orders).[1]

[1] Future editions will include more formal accounts.

For $d = 1$, $\geq$ reduces to the familiar total order of the real line (see Real Order). The converse perspective is to see the positive semidefinite order as an extension of the order on $\mathbf{R}$ to the space $\mathbf{S}^d$. Of course, the key difference is that two matrices may not be comparable. The order is *partial.*

For example, the matrices $A, B \in \mathbf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.