

HOLOMORPHIC FUNCTIONS

Definition

Let Ω be an open set in \mathbf{C} and let $f: \Omega \to \mathbf{C}$. The function f is holomorphic at the point $z_0 \in \mathbf{C}$ if the complex quotient

$$\frac{f(z_0+h)-f(z_0)}{h}$$

has a limit when $h \to \infty$, where $h \in \mathbf{C}$, $h \neq 0$ and $z_0 + h \in \Omega$ so that the quotient is well-defined.

This condition is similar to saying that a function is differentiable, except that the h is complex and so the condition above encomposes all limits approaching z (all angles) in the complex plane. But we emphasize that h is a complex number approaching the complex number (0,0) from any direction. If the limit exists, then we call its value the *derivative of* f at z_0 .

The function f is holomorphic on Ω if f is holomorphic at every point of Ω . If C is a closed subset of \mathbf{C} , we say that f is holomorphic on C if f is holomorphic on some open set containing c. If f is holomorphic on all of C then we call f entire. A holomorphic function is sometimes called regular or complex differentiable. The latter term is used in view of the similarities with the definition of a real derivative.

Notation

In the case that $f: \Omega \to \mathbf{C}$ is holomorphic at z_0 we denote the derivative at z_0 by $f'(z_0)$. We have defined $f'(z_0)$ by

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

¹Future editions will clarify.

