



## Why

We explore matrix-vector multiplication.

## Definition

Given a matrix  $A \in \mathbf{R}^{m \times n}$  and a vector  $x \in \mathbf{R}^n$ , the *product* of  $A$  with  $x$  is the vector  $y \in \mathbf{R}^m$  defined by

$$y_i = \sum_{j=1}^n A_{ij}x_j, \quad i = 1, \dots, m.$$

## Notation

We denote the product of  $A$  with  $x$  by  $Ax$ . With which we concisely write the system of linear equations  $(A, b)$  as  $b = Ax$ .

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function  $x \mapsto Ax$ . In other words, we are interested in the existence of an inverse element of  $A$ . The geometric interpretation is that  $A$  transforms the vector  $x$ .

Conversely, we can view  $x$  as transforming (acting on)  $A$ . Let  $a^j \in \mathbf{R}^m$  denote the  $j$ th column of  $A$ , then

$$Ax = \sum_{j=1}^n x_j a^j.$$

In other words,  $y$  is linear combination of the columns of  $A$ .

## Properties

We call the function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  defined by  $f(x) = Ax$  the *matrix multiplication function* (or *matrix-vector product function*) associated with  $A$ .  $f$  satisfies the following two important properties:

1.  $A(x + y) = Ax + Ay$

2.  $A(\alpha x) = \alpha Ax$ .

We call such a function  $f$  *linear*. In other words, the matrix multiplication function is linear. Conversely, if  $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is linear, there exists a matrix inducing  $g$ .

**Proposition 1.** *Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  be linear. Then there exists a unique  $A \in \mathbf{R}^{m \times n}$  satisfying  $f(x) = Ax$  for all  $x \in \mathbf{R}^n$ .*

*Proof.* Evaluate  $f$  at the standard unit vectors  $e_i$ . The  $i$ th component of  $e_i$  is 1 and all other components are 0. □

Moreover, this matrix representation of  $f$  is unique.

**Proposition 2.** *If  $A, B \in \mathbf{R}^{m \times n}$  are two matrices so that  $f(x) = Ax = Bx$ , then  $A = B$ .*

*Proof.* We have  $Ax - Bx = 0$  so  $(A - B)x = 0$  for every  $x$ . In particular  $y^\top (A - B)x = 0$  for every  $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ . In particular,  $e_i^\top (A - B)e_j = 0$ . Conclusion:  $A_{ij} - B_{ij} = 0$ , and conclude that  $A_{ij} = B_{ij}$ . Thus,  $A = B$ . Evaluate  $f$  at the standard unit vectors  $e_i$ . The  $i$ th component of  $e_i$  is 1 and all other components are 0. □

