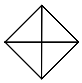


Why

We want to generalize and simplify solving linear equations.

Definition

Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m)$ and $(C \in \mathbf{R}^{m \times n}, d \in \mathbf{R}^m)$ be linear systems. Denote the kth rows of A and C by a^k and c^k, respectively.

(C, d) is a *row reduction* of (A, b) corresponding to row i and variable j if the following two conditions hold: (1) if $k \neq i$ and $A_{ik} \neq 0$ (we say that (C, d) *reduces* (A, b)), then $c^k = a^k - (A_{kj}/A_{ik})a^i$ and $d_k = (A_{kj}/A_{ik})b_i$ and (2) if $k \neq i$ and $A_{ik} = 0$, or if $k = i$, $c^k = a^k$ and $d^k = b^k$. A row reduction is unique, so we call it *the row reduction.*

The key insight is that $x \in \mathbf{R}^d$ is a solution to (A, b) if and only if it is a solution to (C, d).

Proposition 1. *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m)$ be a linear system which row reduces to (C, d). Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d).*

Example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$\begin{aligned} &3x_1 + 2x_2 = 10, \text{ and} \\ &6x_1 + 5x_2 = 20. \end{aligned}$$

We seek solutions to the linear system $(\tilde{A}, \tilde{b})$ where

$$\tilde{A} = \begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix} \text{ and } \tilde{b} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

The row reduction for $(\tilde{A}, \tilde{b})$ for row 1 and variable 1 is

$$\tilde{C} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \tilde{d} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

A solution to the system $(\tilde{C}, \tilde{d})$ satisfies

$$3x_1 + 2x_2 = 10 \text{ and}$$
$$x_2 = 0.$$

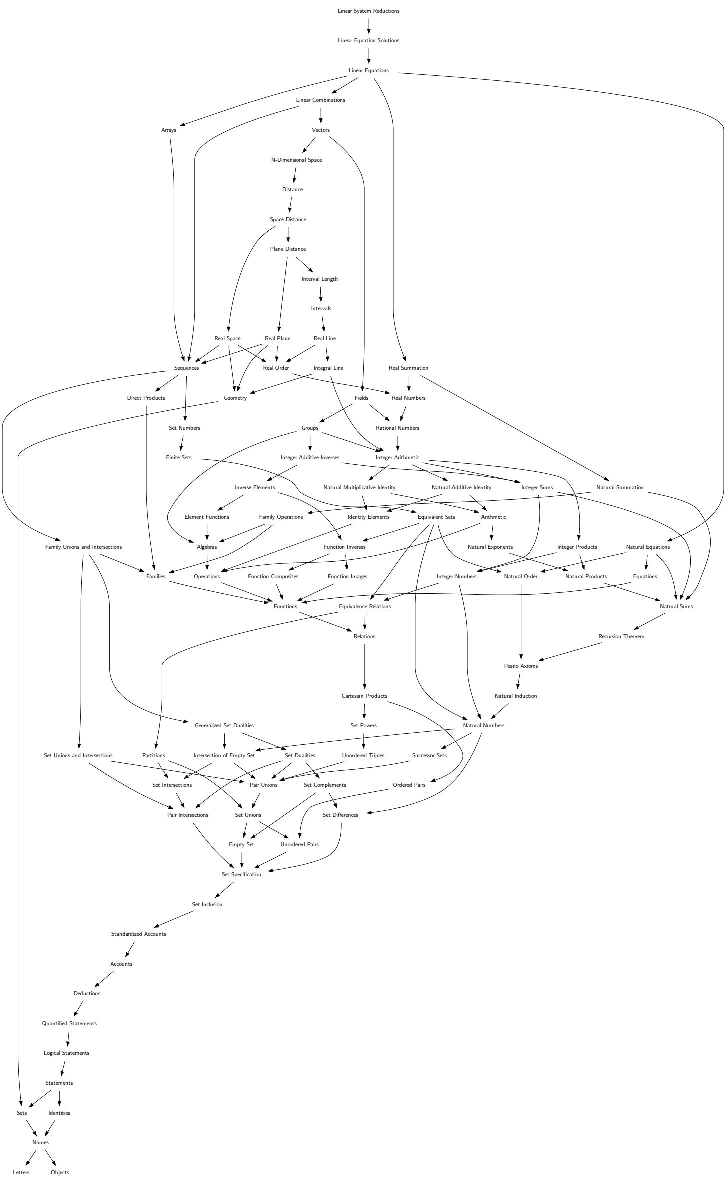
We see that for $x \in \mathsf{R}^2$ to be a solution of $(\tilde{C}, \tilde{d})$, $x_2 = 0$. Using that and the first equation, we have that $x_1 = {}^{10}\!/_3$. This process is called *back-substitution*.

So $(\tilde{C}, \tilde{d})$ has solution set $\{({}^{10}\!/_3, 0)\}$. Proposition 1 says that (A, b) has the same solution set.

Sequence

To any system $(A \in \mathsf{R}n \times n, b \in \mathsf{R}^n)$ there exists a row reduction at row i and column j. Define (A^1, b^1) to be the row reduction of (A, b) at row 1 and column 1. Notice that $A^1_{1j} = 0$ if $j \neq 1$. Similarly, for $i = 2, \dots, n-1$, define (A^i, b^i) to be the row reduction of (A^{i-1}, b^{i-1}) at row i and column i. We call the sequence $((A^1,)$ the *ordinary row reduction* of (A, b) and we call (A^{n-1}, b^{n-1}) the *ordinary row reduction* of (A, b).

Proposition 2. *Let* $(A \in \mathsf{R}^{n\times n}, b \in \mathsf{R}^n)$*. Then* $x \in \mathsf{R}^n$ *is a solution of* (A, b) *if and only if it is a solution of the ordinary row reduction of* (A, b)*.*


Linear System Reductions
Linear Equation Solutions
Linear Equations
Linear Combinations
Arrays
Vectors
N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Plane
Real Line
Sequences
Real Order
Integral Line
Real Summation
Direct Products
Geometry
Fields
Real Numbers
Set Numbers
Groups
Rational Numbers
Finite Sets
Integer Additive Inverses
Integer Arithmetic
Inverse Elements
Natural Multiplicative Identity
Natural Additive Identity
Integer Sums
Natural Summation
Element Functions
Family Operations
Identity Elements
Equivalent Sets
Arithmetic
Family Unions and Intersections
Algebras
Function Inverses
Natural Exponents
Integer Products
Natural Equations
Families
Operations
Function Composites
Function Images
Integer Numbers
Natural Order
Natural Products
Equations
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Peano Axioms
Cartesian Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Dualities
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Complements
Ordered Pairs
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects