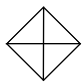


Why

Take a vector space, and consider the set of continuous linear functionals on that space. Given a suitable norm, this space is a complete normed space.

Defining result

Proposition 1. *Let $(V, \|\cdot\|)$ be a normed space. The set V^* of all continuous linear functionals on V is a complete normed space with respect to pointwise algebraic operations and norm $\|\cdot\|_* : V \to \mathbf{R}$ defined by*

$$\|F\|_* = \sup_{x \in V,\ \|x\| \leq 1} |F(x)|.$$

Proof. We argue (1) V^* is a vector space, (2) $\|\cdot\|_*$ is a norm, and (3) $(V, \|\cdot\|_*)$ is complete.[1] □

We call $(V^*, \|\cdot\|_*)$ the *dual space* (or *conjugate dual*, or *Banach dual of* V). Notice that $(V^*, \|\cdot\|_*)$ is complete regardless of whether the original normed space $(V, \|\cdot\|)$ is complete.

Basic dual norm property

Notice that the dual norm satisfies a familiar property.

Proposition 2. *For any vector x in a normed space $(V, \|\cdot\|)$ and any continuous linear functional F on E,*

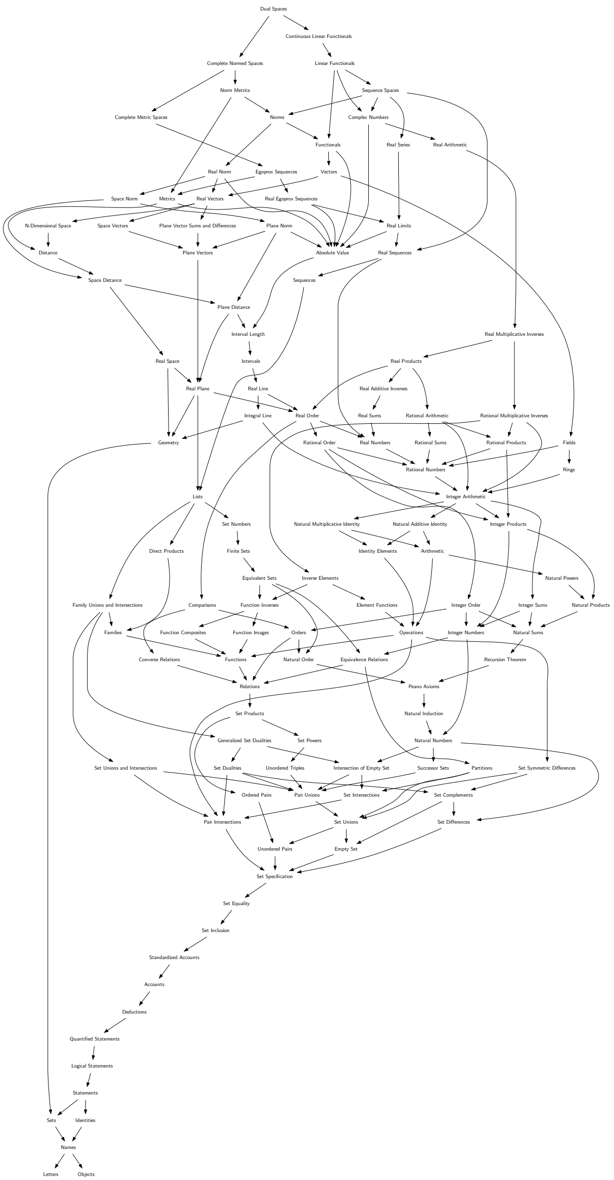
$$|F(x)| \leq \|F\|_* \|x\|.$$

Proof. If $x = 0$, then $\|x\| = 0$ and $F(x) = 0$ (F is linear). Otherwise, $x/\|x\|$ is a unit vector and so

$$\|F\|_* \geq |F(x/\|x\|)| = \frac{|F(x)|}{\|x\|}.$$

[1] Future editions will include an account.

where the inequality is from the definition of $\|\cdot\|_*$ (as a supremum) and the equality follows from the linearity of F. $\square$


Dual Spaces
Continuous Linear Functionals
Complete Normed Spaces
Linear Functionals
Norm Metrics
Sequence Spaces
Complete Metric Spaces
Norms
Complex Numbers
Functionals
Real Series
Real Arithmetic
Real Norm
Egoprox Sequences
Vectors
Space Norm
Metrics
Real Vectors
Real Egoprox Sequences
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Plane Norm
Real Limits
Distance
Plane Vectors
Absolute Value
Real Sequences
Space Distance
Sequences
Plane Distance
Interval Length
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Integral Line
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Fields
Rational Numbers
Rings
Lists
Integer Arithmetic
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Direct Products
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Inverse Elements
Natural Powers
Family Unions and Intersections
Comparisons
Function Inverses
Element Functions
Integer Order
Integer Sums
Natural Products
Families
Function Composites
Function Images
Orders
Operations
Integer Numbers
Natural Sums
Converse Relations
Functions
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Intersection of Empty Set
Successor Sets
Partitions
Set Symmetric Differences
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects