



Why

We want a notion for a correspondence between two sets.

Definition

A *function* f (or *correspondence*, *mapping*, *map*) from a set X to a set Y is a relation whose domain is X and whose range is a subset of Y , such that for each $x \in X$,

1. there exists $y \in Y$ so that $(x, y) \in f$
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$; where y and z are in Y

We often summarize these two conditions by saying: to every element $x \in X$ there corresponds a *unique* element $y \in Y$ so that $(x, y) \in f$.

We call this unique element $y \in Y$ the *result* of the function *at* the *argument* x . We call Y a *codomain*—notice our use of the word “a”, since the codomain is not a property of the function. If the range is Y we say that f is a function from X *onto* Y (or call f *onto*, *surjective*). If distinct elements of X are mapped to distinct elements of Y , we say that the function is *one-to-one* (or *injective*).

We say that the function *maps* (or *takes*) elements from the domain to the codomain. Since the word “function” and the verb “maps” connote activity, some authors refer to the set of ordered pairs as the *graph* of a function and avoid defining the term “function” as we have, in terms of sets.

Notation

Given sets X and Y , we abbreviate the statement that the object denoted by f is a function whose domain is a X and whose codomain is a set Y by

$$f : X \rightarrow Y$$

We read the notation aloud as “ f from X to Y .” We emphasize again that the *range* of f need not be Y , but must necessarily be a subset.

We denote by Y^X the set of functions from X to Y . This set is contained in the power set $\mathcal{P}((X \times Y))$. A reasonable but nonstandard notation is $X \rightarrow Y$, read as “ A to B .” All the following three statements have the same meaning:

$$f : X \rightarrow Y, \quad f \in Y^X, \quad f \in (X \rightarrow Y).$$

We tend to denote functions by lower case latin letters; especially f , g , and h . f is a mnemonic for function and g and h are nearby in the usual ordering of the Latin letters.

Suppose $f : A \rightarrow B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud “ f of a .” We sometimes drop the parentheses, and write the result as f_a , read aloud as “ f sub a .” Let $g : A \times B \rightarrow C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as “ g of a and b ”. We read g_{ab} aloud as “ g sub a b .”

Examples

If $X \subset Y$, the function $\{(x, y) \in X \times Y \mid x = y\}$ is the *inclusion function* of X into Y . We often introduce such a function as “the function from X to Y defined by $f(x) = y$ ”. We mean by this that f is a function and that we are specifying the appropriate ordered pairs using the statement, called *argument-value notation*. The inclusion function of X into X is called the *identity function* of X . If we view the identity function as a relation on X , it is the relation of equality on X .

The functions $f : (X \times Y) \rightarrow X$ defined by $f(x, y) = x$ is the *pair projection* of $X \times Y$ onto X . Similarly $g : (X \times Y) \rightarrow Y$ defined by $g(x, y) = y$ is the pair projection of $X \times Y$ onto Y .

The identity function is one-to-one and onto, the inclusion functions are one-to-one but not always onto, and the pair projections are usually not one-to-one.

