Area

1 Why

We list some principles which are notion of area in planar geometry satisfies.

2 Common Notions

We take two common notions; these are analagous to those we developed for length.

1. The area of the whole is the sum of the area of the parts; the *additivity principle*.
2. If one whole contains another, the first's area at least as large as the second's area; the *containment principle*.

Again, the task is to make precise the use of “whole,", “parts," and “contains." We start with rectangles.

3 Definition

The *area* of an rectangle is the sum of the lengths of its sides.

Two rectangles are *non-overlapping* if their intersection is a single point or empty. The *area* of the union of two non-overlapping intervals is the sum of their areas.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A *countably simple* subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A *cover* of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

3.1 Cuts

If a, b are real numbers and $a < b$, then we *cut* an interval with a and b as its endpoints by selecting c such that $a < c$ and $c < b$.

We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the *cut pieces*.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

4 All sets

Proposition 1. *Not all subsets of real numbers are simple.*

Exhibit: R is not finite.

Proposition 2. *Not all subsets of real numbers are countably simple.*

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

4.1 Notation