Bourbaki

Contents

1 Sets

1.1 Why

We want to speak of a collection of objects, pre-specified or possessing some similar defining property.

1.2 Definition

A **set** is a collection of objects. We use the term **object** in the usual sense of the English language. So a set is itself an object, but of the peculiar nature that it contains other objects.

In thinking of a set, then, we regularly consider the objects it contains. We call the objects contained in a set the **members** or **elements** of the set. So we say that an object contained in a set is a **member of** or an **element of** the set.

1.2.1 Notation

We denote sets by upper case latin letters: for example, A, B, and C. We denote elements of sets by lower case latin letters: for example, a, b, and c. We denote that an object A is an element of a set A by $a \in A$. We read the notation $a \in A$ aloud as "a in A." The $\in$ is a stylized ϵ, which possesses the mnemonic for element. We write $a \notin A$, read aloud as "a not in A," if a is not an element of A.

If we can write down the elements of A, we do so using a brace notation. For example, if the set A is such that it contains only the elements a, b, c, we denote A by $\{a, b, c\}$. If the elements of a set are well-known we introduce the set in English and name it; often we select the name mnemonically. For example, let L be the set of latin letters.

If the elements of a set are such that they satisfy some common condition, we use the braces and include the condition. For example, if V is the set of vowels we denote V by $\{l \in L \mid l \text{ is a vowel}\}$. The $\mid$ is read aloud as "such that," the notation reads aloud as "l in L such that l is a vowel." We call the notation $\{l \in L \mid l \text{ is a vowel}\}$ **set-builder notation**. Set-builder notation is indispensable for sets defiined implicitly by some condition. Here we could have alternatively denoted V by $\{``a", ``e", ``i", ``o", ``u"\}$. We prefer the former, slighly more concise notation.

1.3 Two Sets

If every element of a first set is also an element of second set, we say that the first set is a **subset** of or is **contained in** the second set. Conversely, we say that the second set is a **superset** of or **contains** the first set. If a first set is a subset of a second set and the second set is a subset of the first set, we say the two sets are **equal**

We call the set of subsets of a set A the **powerset** of A. We call the set which has no members the **empty set**. The empty set is contained in every other set.

1.3.1 Notation

Let A and B be sets. We denote that A is a subset of B by $A \subset B$. We read the notation $A \subset B$ aloud as "A subset B". We denote that A is equal to B by $A = B$. We read the notation $A = B$ aloud as "A equals B". We denote the empty set by $\varnothing$, read aloud as "empty." We denote the power set of A by 2^A, read aloud as "two to the A."

2 Ordered Pairs

2.1 Why

We want to handle objects composed of elements from different sets.

2.2 Definition

Let A and B be sets. An ordered pair of a first element a the set of two Construct a new set that is the ordered pairs of elements of A and B: the first element of the pair is an element of A and the second element of the pair is an element of B. The **cartesian product** of A with B is the set of all such ordered pairs.

We call an ordered pair **tuple**. Two tuples are equal if they have equal elements in the same order. Because of the ordering, if $A \neq B$, the cartesian product of A with B is not the same as the cartesian product of B with A. Only in the case that $A = B$ does this symmetry hold.

2.2.1 Notation

For sets A and B we denote the cartesian product of A with B by $A \times B$. We read the notation $A \times B$ as "A cross B." We denote elements of $A \times B$ by (a, b) with the understanding that $a \in A$ and $b \in B$. In this notation, we can write the observation that $A \times B \neq B \times A$, unless $A = B$.

3 Relations

3.1 Why

We want to relate elements of two sets.

3.2 Definition

A **relation** between two non-empty sets A and B is a subset of $A \times B$. So then, naturally, a relation on a single set C is a subset of $C \times C$.

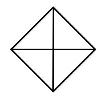
3.2.1 Notation

As relations are sets, our de facto protocol is to denote them by upper case capital letters, for example, the letter R. Let R a relation on A and B. If $(a, b) \in R$, we often write aRb, read aloud as "a in relation R to b."

In many cases, though, we forego the set notation and use particular symbols. Often the symbols we use are meant to be suggestive of the relation. Some examples include $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

3.3 Properties

Let R a relation on a non-empty set A. R is **reflexive** if $(a, a) \in R$ for all $a \in A$. R is **transitive** if $(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$ for all $a, b, c \in A$. R is **symmetric** if $(a, b) \in R \implies (b, a) \in R$ for all $a, b \in A$. R is **anti-symmetric** if $(a, b) \in R \implies (b, a) \notin R$ for all $a, b \in A$.



4 Graphs

4.1 Why

We want to visualize relations.

4.2 Definition

A **graph** is a set and a relation on the set. The graph is **undirected** if the relation is symmetric; otherwise the graph is **directed**.

A **vertex** of the graph is an element of the set. The set is called the **vertex set**. An **edge** of the graph is an element of the relation. The relation is called the **edge set**.

4.2.1 Notation

We denote the vertex set by V, a mnemonic for vertex. We denote the edge set by E, a mnemonic for edge. We denote a graph by (V, E). If the vertex set is assumed we can unambiguously refer to the graph by E.

4.2.2 Visualization

We visualize the graph by drawing a point for each vertex. If two vertices u and v are in relation, we draw a line from the point corresponding to u to the point corresponding to v with an arrow at the point corresponding to v. If the graph is undirected, we omit arrows.

4.3 Paths

A path in a relation is a sequence of elements in which consecutive elements are related. A path **cycles** if an element appears more than once. A path is **finite** if the sequence is finite. A finite path is a **loop** if it cycles once.

5 Functions

5.1 Why

We want a notion for a correspondence between two sets.

5.2 Definition

A **functional** relation on two sets relates each element of the first set with a unique element of the second set. A **function** is a functional relation.

The **domain** of the function is the first set and **codomain** of the function is the second set. The function **maps** elements **from** the domain **to** the codomain. We call the codomain element associated with the domain element the **result** of **applying** the function to the domain element.

5.2.1 Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

A function is an element of th set $A \to B$, so we denote them by lower case latin letters, especially f, g, and h. Of course, f is a mnemonic for function; g and h follow f in the alphabet. We denote that $f \in A \to B$, by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

5.3 Properties

Let $f : A \to B$. The **image** of a set $C \subset A$ is the set $\{f(c) \in B \mid c \in C\}$. The **range** of f is the image of the domain. The **inverse image** of a set $D \subset B$ is the set $\{a \in A \mid f(a) \in B\}$.

The range need not equal the codomain; though it, like every other image, is a subset of the codomain. The function maps to domain **on** to the codomain if the range and codomain are equal; in this case we call the function **onto**. This language suggests that every element of the codomain is used by f. It means that for each element b of the codomain, we can find an element a of the domain so that $f(a) = b$.

An element of the codomain may be the result of several elements of the domain. This overlapping, using an element of the codomain more than once, is a regular occurrence. If a function is a unique correspondence in that every domain element has a different result, we call it **one-to-one**. This language is meant to suggest that each element of the domain corresponds to one and exactly one element of the codomain, and vice versa.

5.3.1 Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. The property that f is onto can be written succintly as $f(A) = B$. We denote the inverse image of $D \subset B$ by $f^{-1}(D)$, read aloud as "f inverse D."

6 Order Relations

6.1 Why

We want to handle elements of a set in a particular order.

6.2 Definition

Let R be a relation on a non-empty set A. R is a **partial order** if it is reflexive, transitive, and anti-symmetric.

A **partially ordered set** is a set and a partial order. The language partial is meant to suggest that two elements need not be comparable. For example, suppose R is $\{(a, a) \mid a \in A\}$; we may justifiably call this no order at all and call A totally unordered, but it is a partial order by our definition.

Often we want all elements of the set A to be comparable. We call R **connexive** if for all $a, b \in A$, $(a, b) \in R$ or $(b, a) \in R$. If R is a partial order and connexive, we call it a **total order**.

A **totally ordered set** is a set together with a total order. The language is a faithful guide: we can compare any two elements. Still, we prefer one word to three, and so we will use the shorter term **chain** for a totally ordered set; other terms include **simply ordered set** and **linearly ordered set**.

6.2.1 Notation

We denote total and partial orders on a set A by $\preceq$. We read $\preceq$ aloud as "precedes or equal to" and so read $a \preceq b$ aloud as "a precedes or is equal to b." If $a \preceq b$ but $a \neq b$, we write $a \prec b$, read aloud as "a precedes b."

7 Natural Numbers

7.1 Why

We want to count.

7.2 Definition

We define the set of **natural numbers** implicitly. There is an element of the set which we call **one**. Then we say that for each element n of the set, there is a unique corresponding element called the **successor** of n which is also in the set. The **successor function** is the implicitly defined a function from the set into itself associating elements with their successors. We call the elements **numbers** and the refer to the set itself as the **naturals**.

To recap, we start by knowing that one is in the set, and the successor of one is in the set. We call the successor of one **two**. We call the successor of two **three**. And so on using the English language in the usual manner. We are saying, in the language of sets, that the essence of counting is starting with one and adding one repeatedly.

7.2.1 Notation

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n. We denote the element called one by 1.

7.3 Induction

We assert two additonal self-evident and indispensable properties of these natural numbers. First, one is the successor of no other element. Second, if we have a subset of the naturals containing one with the property that it contains successors of its elements, then that set is equal to the natural numbers. We call this second property the **principle of mathematical induction.**

These two properties, along with the existence and uniqueness of successors are together called **Peano's axioms** for the natural numbers. When in familiar company, we freely assume Peano's axioms.

7.4 Notation

As an exercise in the notation assumed so far, we can write Peano's axioms: N is a set along with a function $s : N \to N$ such that

1. $s(n)$ is the successor of n for all $n \in N$.
2. s is one-to-one; $s(n) = s(m) \implies m = n$ for all $m, n \in N$.
3. There does not exist $n \in N$ such that $s(n) = 1$.
4. If $T \subset N$, $1 \in T$, and $s(n) \in T$ for all $n \in T$, then $T = N$.

7.5 Order

Let $\preceq$ be a relation on N where $a \preceq b$, $a, b \in N$ if we can obtain b by applying the successor function to a finitely many times. It happens that $\preceq$ is a total order, so $(N, \preceq)$ is a lattice.

8 Algebra

8.1 Why

We want to combine set elements to get other set elements.

8.2 Basics

An **operation** on a set is a function from ordered pairs of elements in the set to the same set. We use operations to combine the elements. We operate on pairs. An **algebra** is a set and an operation. We call the set the **ground set**.

8.2.1 Notation

Let A a set and $g : A \times A \to A$. We commonly forego the notation $g(a, b)$ and instead write $a\,g\,b$. We call this style **infix notation**.

Using lower case latin letters for every the elements and for the operation is confusing, but we often have special symbols for particular operations. Examples of such symbols include $+$, $-$, $\cdot$, $\circ$, and $\star$.

If we had a set A and an operation $+ : A \times A \to A$, we would write $a + b$ for the result of applying $+$ to (a, b). In denoting the algebra, we would say let $(A, +)$ be an algebra.

8.3 Operation Properties

An operation **commutes** if the result of two elements is the same regardless of their order; we call the operation **commutative**

An operation **associates** if given any three elements in order it doesn't matter whether we first operate on the first two and then with the result of the first two the third, or the second two and with the result of the second two the first.

A first operation over a set **distributes** over a second operation over the same set if the result of applying the first operation to an element and a result of the second operation is the same as applying the second operation to the results of the first operation with the arguments of the second operation.

8.3.1 Notation

Let $(A, +)$ an algebra.

We denote that $+$ commutes by asserting

$$a + b = b + a$$

for all $a, b \in A$. We denote that $+$ associates by asserting

$$(a + b) + c = (a + b) + c$$

for all $a, b, c \in A$. Let $(A, \cdot)$ a second algebra over the same set. We denote that $\cdot$ distributes over $+$ by

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

for all $a, b, c \in A$.

8.4 Identity Elements

We call $e \in A$ an **identity element** if (1) $e + a = e$ and (2) $a + e = e$ for all $a \in A$. If only (1) holds, we call e a **left identity**. If only (2) holds, we call e a **right identity**.

8.5 Inverse Elements

We call $b \in A$ an **inverse element** of $a \in A$ if (1) $b + a = e$ and (2) $a + b = e$. If only (1) holds, we call e a **left inverse**. If only (2) holds, we call e a **right inverse**.

9 Set Operations

9.1 Why

We want to consider the elements of two sets together at once, and other sets created from two sets.

9.2 Definitions

Let A and B be two sets.

The **union** of A with B is the set whose elements are in either A *or* B *or* both. The key word in the definition is *or*.

The **intersection** of A with B is the set whose elements are in both A *and* B. The keyword in the definition is *and*.

Viewed as operations, both union and intersection commute; this property justifies the language "with." The intersection is a subset of A, of B, and of the union of A with B.

The **symmetric difference** of A and B is the set whose elements are in the union but not in the intersection. The symmetric difference commutes because both union and intersection commute; this property justifies the language "and." The symmetric difference is a subset of the union.

Let C be a set containing A. The **complement** of A in C is the symmetric difference of A and C. Since $A \subset C$, the union is C and the intersection is A. So the complement is the "left-over" elements of B after removing the elements of A.

We call these four operations **set-algebraic operations**.

9.2.1 Notation

Let A, B be sets. We denote the union of A with B by $A \cup B$, read aloud as "A union B." $\cup$ is a stylized U. We denote the intersection of A with B by $A \cap B$, read aloud as "A intersect B." We denote the symmetric difference of A and B by $A \Delta B$, read aloud as "A symdiff B." "Delta" is a mnemonic for difference.

Let C be a set containing A. We denote the complement of A in C by $C - A$, read aloud as "C minus A."

9.2.2 Results

Proposition 1 *For all sets A and B the operations $\cup$, $\cap$, and Δ commute.*

Proposition 2 *Let S a set. For all sets $A, B \subset S$,*

$$\begin{aligned} &(1) \quad S - (A \cup B) = (S - A) \cap (S - B) \\ &(2) \quad S - (A \cap B) = (S - A) \cup (S - B). \end{aligned}$$

10 Arithmetic

10.1 Why

Counting one by one is slow so we define an algebra on the naturals.

10.2 Sums and Addition

Let m and n be two natural numbers. If we apply the successor function to m n times we obtain a number. If we apply the successor function to n m times we obtain a number. Indeed, we obtain the same number in both cases. We call this number the **sum** of m and n. We say we **add** m to n, or vice versa. We call this symmetric operation mapping (m, n) to their sum **addition**.

10.2.1 Notation

We denote the operation of addition by $+$ and so denote the sum of the naturals m and n by $m + n$.

10.3 Products and Multiplication

Let m and n naturals. If we add n copies of m we obtain a number. If we add m copies of n we obtain a number. Indeed, we obtain the same number in both cases. We call this number the **product** of m and n. We say we **multiply** m to n, or vice versa. We call this symmetric operation mapping (m, n) to their product **multiplication**.

10.3.1 Notation

We denote the operation of multiplication by $\cdot$ and so denote the product of the naturals m and n by $m \cdot n$.

11 Equivalence Relations

11.1 Why

We want to handle at once all elements which are indistinguishable or equivalent in some aspect.

11.2 Definition

Let R be a relation on A R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

For an element $a \in A$, we call the set of elements in relation R to a the **equivalence class** of a. The key observation, recorded and proven below, is that the equivalence classes partition the set A. A frequent technique is to define an appropriate equivalence relation on a large set A and then to work with the set of equivalence classes of A.

We call the set of equivalence classes the **quotient set** of A under R. An equally good name is the divided set of A under R, but this terminology is not standard. The language in both cases reminds us that $\sim$ partitions the set A into equivalence classes.

11.2.1 Notation

If R is an equivalence relation on a set A, we use the symbol $\sim$. When alone, $\sim$ is read aloud as "sim," but we still read $a \sim b$ aloud as "a equivalent to b." We denote the quotient set of A under $\sim$ by $A/\sim$, read aloud as "A quotient sim".

11.2.2 Results

$\boxed{TODO}$

12 Families

12.1 Why

We want to generalize operations beyond two objects.

12.2 Definition

Let A, B be non-empty sets. A **family** of elements of a first set **indexed** by elements of a second set is the range of a function from the second set to the first set. We call second set the **index set**.

If the index set is a finite set, we call the family a **finite family**. If the index set a countable set, we call the family a **countable family**. If the index set is an uncountable set, we call the family a **uncountable family**.

If the codomain is a set of sets, we call the family a **family of sets**. We often use a subset of the whole natural numbers as the index set. In this case, and for other indexed sets with orders, we call the family an **ordered family**

12.2.1 Notation

Let A be a non-empty set. We denote the index set by I, a mnemonic for index. For $iinI$, let we denote the result of applying the function to i by a_i; the notation evokes evokes function notation but avoids naming the function.

We denote the family of a_α indexed with I by $\{a_\alpha\}_{\alpha \in I}$, which is shorthand for set-builder notation. We read this notation "a sub-alpha, alpha in I."

12.3 Operations

The **pairwise extension** of a commutative operation is the function from finite families of the ground set to the ground set obtained by applying the operation pairwise to elements.

The **ordered pairwise extension** of an operation is the function from finite families ground set to the ground set obtained by applying the operation pairwise to elements in order.

12.3.1 Notation

Let $(A, +)$ be an algebra and $\{A_i\}_{i=1}^{n}$ a finite family of elements of A. We denote the pairwise extension by

$$\mathop{+}_{i=1}^{n} A_i$$

12.4 Family Set Algebra

We define the set whose elements are the objects which are contained in at least one family member the **family union**. We define the set whose elements are the objects which are contained in all of the family members the **family intersection**.

12.4.1 Notation

We denote the family union by $\cup_{\alpha \in I} A_\alpha$. We read this notation as "union over alpha in I of A sub-alpha." We denote family intersection by $\cap_{\alpha \in I} A_\alpha$. We read this notation as "intersection over alpha in I of A sub-alpha."

12.4.2 Results

Proposition 3 *For an indexed family $\{A_\alpha\}_{\alpha\in I}$ in S, if $I = \{i, j\}$ then*

$$\cup_{\alpha\in I} A_\alpha = A_i \cup A_j$$

and

$$\cap_{\alpha\in I} A_\alpha = A_i \cap A_j.$$

Proposition 4 *For an indexed family $\{A_\alpha\}_{\alpha\in I}$ in S, if $I = \varnothing$, then*

$$\cup_{\alpha\in I} A_\alpha = \varnothing$$

and

$$\cap_{\alpha\in I} A_\alpha = S.$$

Proposition 5 *For an indexed family $\{A_\alpha\}_{\alpha\in I}$ in S.*

$$C_S(\cup_{\alpha\in I} A_\alpha) = \cap_{\alpha\in I} C_S(A_\alpha)$$

and

$$C_S(\cap_{\alpha\in I} A_\alpha) = \cup_{\alpha\in I} C_S(A_\alpha).$$

13 Direct Products

13.1 Why

We can profitably generalize the notion of cartesian product to families of sets indexed by the natural numbers.

13.2 Direct Products

The **direct product** of family indexed by a subset of the naturals is the set whose elements are ordered sequences of elements from each set in the family. The ordering on the sequences comes from the natural ordering on N. If the index set is finite, we call the elements of the direct product **n-tuples**. If the index set is the natural numbers, and every set in the family is the same set A, we call the elements of the direct product the **sequences** in A.

13.2.1 Notation

For a family $\{A_\alpha\}_{\alpha \in I}$ of S with $I = \{1, \dots, n\}$, we denote the direct product by

$$\prod_{i=1}^{n} A_i.$$

We read this notation as "product over alpha in I of A sub-alpha." We denote an element of $\prod_{i=1}^{n} A_i$ by $(a_1, a_2, \dots, a_n)$ with the understanding that $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$.

If I is the set of natural numbers we denote the direct product by

$$\prod_{i=1}^{\infty} A_i.$$

We denote an element of $\prod_{i=1}^{\infty} A_i$ by (a_i) with the understanding that $a_i \in A_i$ for all $i = 1, 2, 3, \ldots$. If $A_i = A$ for all $i = 1, 2, 3, \ldots$, then (a_i) is a sequence in A.

14 Nets

14.1 Why

We want to generalize the notion of sequence.

14.2 Definition

Recall that a sequence is a function on the naturals. The naturals are ordered and have the property that we can always go further out. If handed two natural numbers m and n, we can always find another, for example $\max\{m, n\} + 1$, larger than m and n. We might think of larger as being further out from the first natural number, namely 1. These observations motivate definining a directed set.

Definition 6 *A* ***directed set*** *is a set D with a partial order $\preceq$ satisfying one additional property: for all $a, b \in D$, there exists $c \in D$ such that $a \preceq c$ and $b \preceq c$.*

Definition 7 *A* ***net*** *is a function on a directed set.*

A sequence, then, is a net. The directed set is the set of natural numbers and the partial order is $m \preceq n$ if $m \leq n$.

14.2.1 Notation

Directed sets involve a set and a partial order. We commonly assume the partial order, and just denote the set. We use the letter D as a mnemonic for directed.

For nets, we use function notation and generalize sequence notation. We denote the net $x : D \to A$ by $\{a_\alpha\}$, emulating notation for sequences. The use of α rather than n reminds us that D need not be the set of natural numbers.

15 Categories

15.1 Why

We generalize the notion of sets and functions.

15.2 Definition

A **category** is a collection of objects together with a set of **category maps** for each ordered pair of objects. The set of maps has a binary operation called **category composition**, whose induced algebra is associative and contains identities.

As the fundamental example, consider the category whose objects are sets and whose maps are functions. The sets are the objects of the category. The functions are the maps. The rule of composition is ordinary function composition. The map identities are the identity functions. We call this category the **category of sets**.

15.2.1 Notation

Our notation for categories is guided by our generalizing the notions of set and functions.

We denote categories with upper-case latin letters in script; for example, $\mathcal{C}$. We read $\mathcal{C}$ aloud as "script C." Upper case latin letters remind that the category is a set of objects. The script form reminds that these objects may themselves be sets.

We denote the objects of a category by upper-case latin letters, for example A, B, C; an allusion to the idea that these generalize sets. We denote the set of maps for an ordered pair of objects (A, B) by $A \to B$; an allusion to the function notation. We denote members of $A \to B$ using lower case latin letters, for example f, g, h; an allusion to our function notation.

16 Groups

16.1 Why

We generalize the algebraic structure of addition over the integers.

16.2 Definition

A **group** is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra **group addition**. A **commutative group** is a group whose operation commutes.

16.2.1 Notation

TODO

17 Rings

17.1 Why

We generalize the algebraic structure of addition and multiplication over the integers.

17.2 Definition

A **ring** is two algebras over the same ground set with: (1) the first algebra a commutative group (2) an identity element in the second algebra, and (3) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra **ring addition**. We call the operation of the second algebra **ring multiplication**.

17.2.1 Notation

$\boxed{\textit{TODO}}$

18 Fields

18.1 Why

We generalize the algebraic structure of addition and multiplication over the rationals.

18.2 Definition

A **field** is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra **field addition**. We call the operation of the second algebra **field multiplication**.

18.2.1 Notation

TODO

19 Homomorphism

19.1 Why

We name a function which preserves group structure.

19.2 Definition

A **homomorphism** from group $(A, +)$ to group $(B, \tilde{+})$ is a function $f : A \to B$ such that $f(e_A) = f(e_B)$ for identities $e_A \in A$ and $e_B \in B$ and $f(a + a') = f(a) \tilde{+} f(a')$ for all $a, a' \in A$.

19.2.1 Notation

$\boxed{\textit{TODO}}$

20 Cardinality

20.1 Why

We want to speak of the number of elements of a set. Subtetly arises when we can not finish counting the set's elements.

20.2 Finite Definition

If a set A is contained in a set B and not equal to B, we say that B is a **larger set** than A. Conversely, we say that A is a **smaller set** than B. We reason that we could pair the elements of B with themselves in A and still have some elements of B left over.

A **finite set** is one whose elements we can count and the process terminates. For example, $\{1, 2, 3\}$ or $\{a, b, c, d\}$. The **cardinality** of a finite set is the number of elements it contains. The cardinality of $\{1, 2, 3\}$ is 3 and the cardinality of $\{a, b, c, d\}$ is 4.

20.2.1 Notation

Let A be a non-empty set. We denote the cardinality of A by $|A|$.

20.3 Infinite Definition

Suppose we know that the counting process could never terminate. This situation superficially seems bizarre, but is in fact built in to some of our fundamental notions: namely, the natural numbers. We defined the natural numbers in a manner which made them not finite.

If we had a bag of natural numbers, we could use the total order to find the largest, and then use the existence of a successor to add a new largest number. Therefore, bizarrely, the process of counting the natural numbers can not terminate.

An **infinite set** is a non-empty set which is not finite. So the natural numbers are an infinite set. Alternatively we say that there are **infinitely many** natural numbers. The negating prefix "in" emphasizes that we have defined the nature of the size of the naturals indirectly: their size is not something we understand from the simple intuition of counting, but in contrast to the simple intuition of counting.

Still, we imagine that if we could go on forever, we could count the natural numbers; so in an infinite sense, they are countable. A **countable** set is one which is either (a) finite or (b) one for which there exists a one-to-one function mapping the natural numbers onto the set.

The natural numbers are countable: we exhibit the identity function. Less obviously the integer numbers and rational numbers are countable. Even more bizarre, the real numbers are not countable. An **uncountable** set is one which is not countable.

20.3.1 Notation

We denote the cardinality of the natural numbers by $\aleph_0$.

21 Subset Algebra

21.1 Why

We often talk about a set and a set of its subsets satisfying properties.

21.2 Definition

A **subset algbera** is two sets: the second is a set of subsets of the first.

We call the first set the **base set**. If the base set is finite, we call the subset algebra a **finite subset algebra**. We call an element of the second set a **distinguished subset**. A subset is an **undistingished subset** if it is not distinguished.

Useful subset algebras are those for which the distinguished subsets satisfy some set-algebraic properties. For one example, the distinguished sets may be closed under set union or set intersection. As a second example, the distinguished sets may include the base set. As a third example, the distinguished sets may be closed under complements or under subsets.

21.2.1 Notation

Let A a set and $\mathcal{A} \subset 2^A$. We denote the subset algebra of A and $\mathcal{A}$ by $(A, \mathcal{A})$, read aloud as “A, script A.”

22 Topological Space

22.1 Why

We want to generalize the notion of continuity.

22.2 Definition

A **topological space** is a subset algebra for which: (1) the empty set and the base set are distinguished, (2) the intersection of a finite family of distinguished subsets is distinguished, and (3) the union of a family of distinguished subsets is distinguished. We call the set of distinguished subsets the **topology**. We call the distinguished subsets the **open sets**.

22.2.1 Notation

Let A a non-empty set. For the set of distinguished sets, we use $\mathcal{T}$, a mnemonic for topology, read aloud as "script T". We denote elements of $\mathcal{T}$ by O, a mnemonic for open. We denote the topological space with base set A and topology $\mathcal{T}$ by $(A, \mathcal{T})$. We denote the properties satisfied by elements of $\mathcal{T}$:

1. $X, \varnothing \in \mathcal{T}$
2. $\{O_i\}_{i=1}^n \subset \mathcal{T} \implies \cap_{i=1}^n O_i \in \mathcal{T}$
3. $\{O_\alpha\}_{\alpha \in I} \subset \mathcal{T} \implies \cup_{\alpha \in I} \in \mathcal{T}$

23 Measurable Space

23.1 Why

We want to generalize the notions of length, area, and volume.

23.2 Definition

A **measurable space** is a a subset algebra where (1) the distinguished subsets include the empty set and the base set, (2) the distinguished subsets are closed under complements, and (3) the distinguished subsets are closed under family unions. We call the set of distinguished subsets the **metrology**. We call the distinguished subsets the **measurable sets**.

23.2.1 Notation

Let A be a non-empty set. For the set of distinguished sets, we use $\mathcal{M}$, a mnemonic for metrology, read aloud as "script M." We denote elements of $\mathcal{M}$ by M, a mnemonic for measure. We denote the measurable space with base set A and metrology $\mathcal{M}$ by $(A, \mathcal{M})$. We denote the properties satified by elements of $\mathcal{M}$ by

1. $\varnothing \in \mathcal{M}$
2. $M \in \mathcal{M} \implies C_A(M) \in \mathcal{M}$
3. $\{M_n\}_{n \in N} \subset \mathcal{M} \implies \cup_{n \in N} M_n \in \mathcal{M}$

23.2.2 Properties

We observe the imediate consequence that the intersection of a family of measurable sets is measurable.

Proposition 8 *If* $\{M_i\}_{\alpha\in I} \subset \mathcal{M}$*, then* $\cap_{\alpha\in I} \in \mathcal{M}$*.*

24 Matroids

24.1 Why

We generalize the notion of linear dependence.

24.2 Definition

A **matroid** is a finite subset algebra satisfying:

1. The subset of a distinguished set is distinguished.
2. For two distinguished subsets of nonequal cardinality, there is an element of the base set in the complement of the smaller set in the bigger set whose singleton union with the smaller set is a distinguished set.

An **independent subset** of a matroid is a distinguished subset. A **depenedent subset** of a matroid is an undistinguished subset.

24.2.1 Notation

We follow the notation of subset algebras, but use M for the base set, a mnemonic for matroid, and $\mathcal{I}$ for the distinguished sets, a menomic for independent.

Let $(M, \mathcal{I})$ a matroid. We denote the properties by

1. $A \in \mathcal{I} \land B \subset A \implies B \in \mathcal{I}$.
2. $A, B \in \mathcal{I} \land |A| < |B| \implies \exists x \in M : (A \cup \{x\}) \in \mathcal{I}$