Why

We want a complete norm on the vector space of continuous functions.

Result

Proposition 1. *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $(f_n)_n$ be an egoprox sequence in $C[a,b]$.

Candidate. $(f_n)_n$ is egoprox means $\forall \varepsilon > 0, \exists N$ so that

$$m, n > N \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon.$$

Since $\|f_n - f_m\|_{\sup} < \varepsilon \longrightarrow |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in [a,b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a,b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit $l_x \in R$ such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a,b]$. Define $f : [a,b] \to R$ by $f(x) = l_x$ for each $x \in [a,b]$.

Candidate is Limit. First, we argue that $\|f_n - f\|_{\sup} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $(f_n)_n$ is an egoprox sequence, there exists n_0 so that

$$n, m \geq n_0 \longrightarrow \|f_n - f_m\|_{\sup} < \varepsilon/2.$$

So for all $x \in [a,b]$,

$$n, m \geq n_0 \longrightarrow |f_n(x) - f_m(x)| < \varepsilon/2.$$

For all $x \in [a,b]$, and $n \geq n_0$,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

The sequence $\{f_k(x)\}_{k=m}^{\infty}$ is a final part of $\{f_k(x)\}_{k=1}^{\infty}$, and so has the same limit, $f(x)$. Therefore, using continuity of subtraction and the absolute value,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for $n \geq n_0$, $x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$, from which we deduce $\|f_n - f\|_{\sup} < \varepsilon$. Thus $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

Limit is Continuous. Next, we argue that f is continuous. Let $x_0 \in [a,b]$. Let $\varepsilon > 0$. Since $f_n \longrightarrow f$ there exists n_0 so that

$$\|f_{n_0} - f\|_{\sup} < \varepsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a,b]$. Using $|f(x_0) - f_{n_0}(x_0)| < \varepsilon/3$,

$$|f(x_0) - f(x)|\& < \varepsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a,b]$. Using the triangle inequality,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

for all $x \in [a,b]$. Using $|f_{n_0}(x_0) - f(x)| < \varepsilon/3$

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \varepsilon/3$$

for all $x \in [a,b]$. Since f_{n_0} is continuous, there exists $\delta > 0$ so that

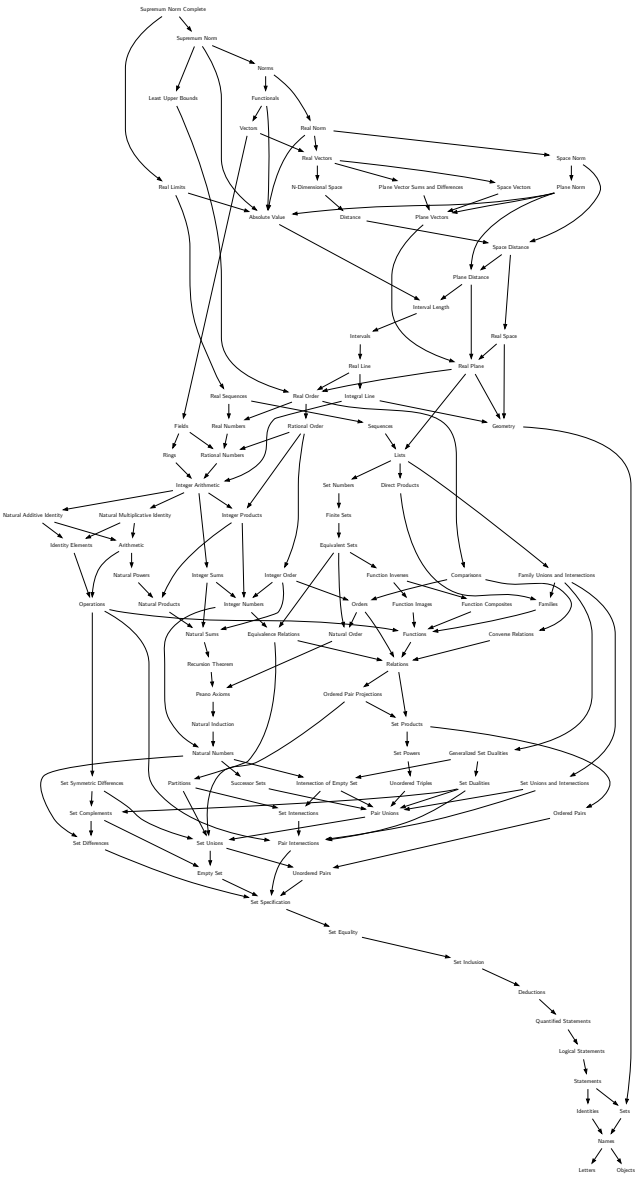
$$|x_0 - x| < \delta \longrightarrow |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3,$$

for $x \in [a,b]$. In this case,

$$|f(x_0) - f(x)|\& < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since ε was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere. Some call the above the *three epsilon argument.*

$\square$


Supremum Norm Complete
Supremum Norm
Norms
Least Upper Bounds
Functionals
Vectors
Real Norm
Real Vectors
Space Norm
Real Limits
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Plane Norm
Absolute Value
Distance
Plane Vectors
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Real Sequences
Geometry
Fields
Real Numbers
Rational Order
Sequences
Rings
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Direct Products
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Function Inverses
Comparisons
Family Unions and Intersections
Operations
Natural Products
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Equivalence Relations
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Converse Relations
Recursion Theorem
Relations
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Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
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Names
Letters
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