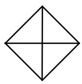


Why

It is believable that $1/2, 1/4, 1/8, \dots$ has a convergent series. And likewise with $1/3, 1/9, 1/27, \dots$. What of $a_k = x^k$ for $x \in \mathbf{R}$.

Definition

Let $x \in \mathbf{R}$. The *geometric series* of x is the series of the sequence (a_k) defined by $a_k = x^k$.

Characterization of convergence

Does the geometric series of x converge? In other words, does (s_n) defined by $s_n = \sum_{k=1}^{n} x^k$ have a limit.

For $x = 1$ and $x = -1$, we have see (see Real Series) that the series diverges. However for the cases $x = 1/2$ and $x = 1/3$ the geometric series converges.

Proposition 1. *If $|x| < 1$, then the geometric series of x converges and*

$$\lim_{n\to\infty} \sum_{k=1}^{n} x^k = \frac{x}{1-x}$$

If $|x| \geq 1$ then the geometric series of x diverges.

Proof. Define $s_n = \sum_{k=1}^{n} x^k$. Then

$$\begin{aligned} x \cdot s_n &= x \cdot (x^1 + x^2 + \cdots + x^n) \\ &= x^2 + x^3 + \cdots + x^{n+1} \\ &= s_n - x + x^{n+1}. \end{aligned}$$

From which we deduce, $s_n(1-x) = x(1-x^n)$. If $x \neq 1$, then

$$s_n = \frac{1}{1-x}(1-x^n)$$

If $|x| < 1$, then using the algebra of limits (see Real Limit Algebra) we deduce

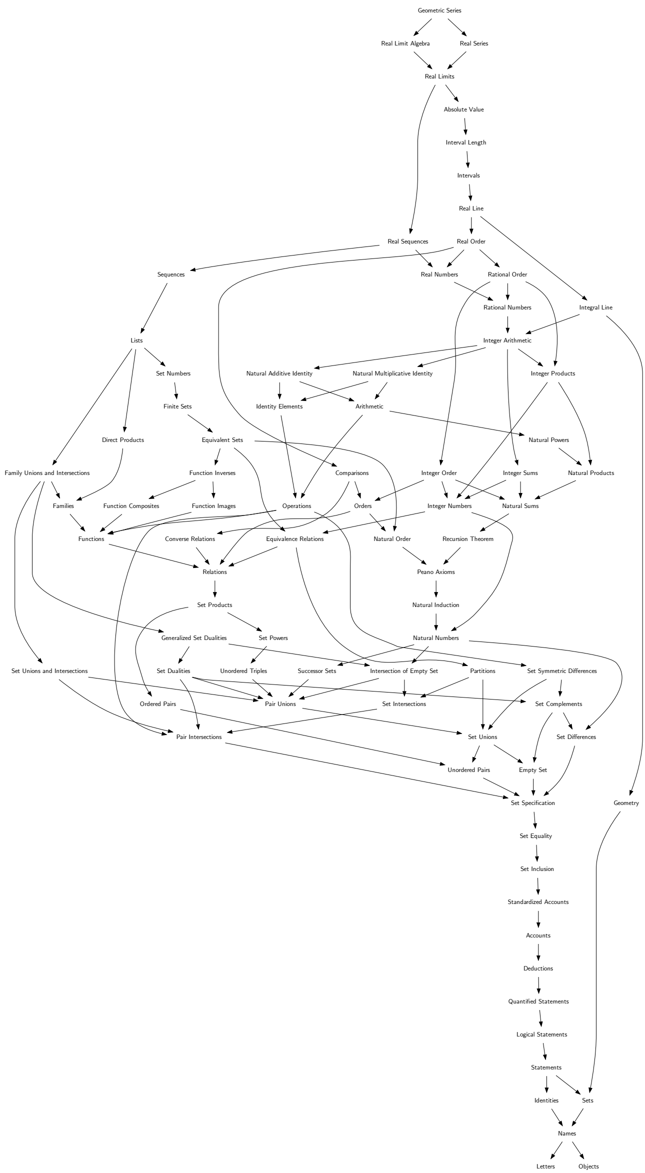
$$\lim_{n\to\infty} \frac{1}{1-x}(1-x^n) = \frac{1}{1-x}(1-0) = \frac{1}{1-x},$$

since $\lim_{k\to\infty} x^k = 0$ for $|x| < 1$.

If $x = 1$ or $x = -1$, then we have seen that (s_n) diverges.[1]

$\square$

[1] Future editions will include the trivial account about the case $|x| > 1$.


Geometric Series
Real Limit Algebra
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Sequences
Real Order
Sequences
Real Numbers
Rational Order
Rational Numbers
Integral Line
Lists
Integer Arithmetic
Set Numbers
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Finite Sets
Identity Elements
Arithmetic
Direct Products
Equivalent Sets
Natural Powers
Family Unions and Intersections
Function Inverses
Comparisons
Integer Order
Integer Sums
Natural Products
Families
Function Composites
Function Images
Operations
Orders
Integer Numbers
Natural Sums
Functions
Converse Relations
Equivalence Relations
Natural Order
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Set Symmetric Differences
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Geometry
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects