Why

What are the affine sets in terms of subspaces?

Affine sets which are subspaces

The subspaces of $\mathbf{R}^n$ are the affine sets which contain the origin.

Proposition 1. *$M \subset \mathbf{R}^n$ is a subspace if and only if M is affine and $0 \in M$.*

Proof. ($\Rightarrow$) Suppose M is a subspace. Then $0 \in M$. Also $\alpha x + \beta y \in M$ for all $\alpha, \beta \in \mathbf{R}$ and $x, y \in \mathbf{R}^n$ In particular, $(1-\lambda)x + \lambda y \in M$ for all $\lambda \in \mathbf{R}$, $x, y \in \mathbf{R}^n$. In other words, M contains the line through x and y.

($\Leftarrow$) Suppose M is affine and $0 \in M$. M is closed under scalar multiplication since

$$\alpha x = (1-\alpha)0 + \alpha x$$

is in the line through 0 and x. M is closed under vector addition since

$$(1/2)(x+y) = (1-1/2)x + (1/2)y$$

is in the line through x and y. Thus, $x + y = 2(1/2)(x+y) \in M$. $\square$

Affine sets as translated subspaces

Proposition 2. *Suppose $M \neq \varnothing$ is affine. Then there exists a unique subspace L and vector $a \in \mathbf{R}^n$ for which $M = L + a$. Moreover,*

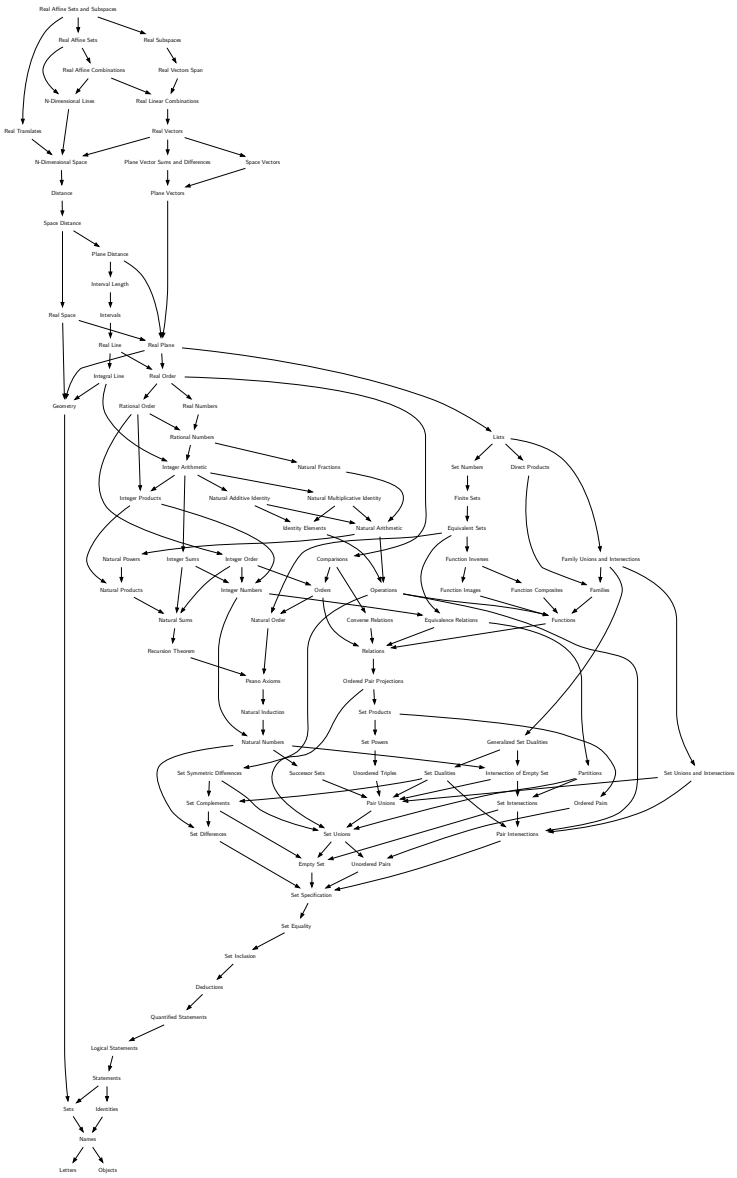
$$L = M - M = \{x - y \mid x, y \in M\}.$$

Proof. First, uniqueness. Suppose L_1 and L_2 are subspaces parallel to M. We will show that $L_2 \supset L_1$ (and similarly, $L_1 \supset L_2$).

Since L_1 and L_2 are both parallel to M, they are also parallel to each other. Consequently, there exists $a \in \mathbf{R}^n$ with $L_2 = L_1 + a$. Since $0 \in L_2$ (it is a subspace, after all), $-a \in L_1$. Since L_1 is a subspace, $a \in L_1$.

So $x + a \in L_1$ for every $a \in L_1$, and so $L_2 = L_1 + a \subset L_1$. A similar argument gives $L_1 \supset L_2$.

If $y \in M$, then $M + (-y) = M - y$ is a translate of M containing zero (since $y - y = 0$). In other words, the affine set $M - y$ is a subspace. This, then, is the unique subspace parallel to M. Since y was arbitrary, the subspace parallel to M is $L = \cup_{y \in M} M - y = \{x - y \mid x, y \in M\}$. $\square$


Real Affine Sets and Subspaces
Real Affine Sets
Real Subspaces
Real Affine Combinations
Real Vectors Span
N-Dimensional Lines
Real Linear Combinations
Real Translates
Real Vectors
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Distance
Plane Vectors
Space Distances
Plane Distance
Interval Length
Real Space
Intervals
Real Line
Real Plane
Integral Line
Real Order
Geometry
Rational Order
Real Numbers
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Natural Arithmetic
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Natural Products
Integer Numbers
Orders
Operations
Function Images
Function Composites
Families
Natural Sums
Natural Order
Converse Relations
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
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Statements
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