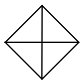


## Why

We want to find matrix square roots.[1]

## Definition

Let $A \in \mathbf{R}^{n \times n}$ be real symmetric. An *lower upper triangular decomposition* $A$ is a pair of matrices $(L, L^\top)$ with $L \in \mathbf{R}^{n \times n}$ lower triangular, satisfying

$$A = LL^\top.$$

Other terminology includes *lower triangular factorization*, *LU decomposition*, *LU factorization*, and (most universally) *Cholesky decomposition* or *Cholesky factorization*. Define $R = L^\top$, then we also have

$$A = R^\top R.$$

## Basic properties

**Proposition 1.** *Let $A \in \mathbf{R}^{m \times m}$ be positive definite. Then there exists unique lower triangular matrix $L \in \mathbf{R}^{n \times n}$ so that*

$$A = LL^\top.$$

In other words, the Cholesky decomposition exists and is unique when the matrix $A$ is positive definite.

**Proposition 2.** *If $A$ is positive semisemidefinite, there exists a permutation matrix $P$ for which there is a unique L so that*

$$P^\top AP = LL^\top.$$

[1]Future editions will expand.

This second proposition says that there is a unique Cholesky decomposition for a particular permutation (or pivoting) of $A$.

### Unitriangular form

A *lower diagonal upper decomposition* (or *lower diagonal upper factorization*) of a matrix $A$ a sequence $(L, D, L^\top)$ where $L \in \mathsf{R}^{n \times n}$ is unit lower triangular, $D \in \mathsf{R}^{n \times n}$ is diagonal and

$$A = LDL^\top.$$

Other terminology includes *LDL decomposition*, *LDL factorization*, *LDU factorization*, *LDU decomposition.*

If $(L \in \mathsf{R}^{n \times n}, D \in \mathsf{R}^{n \times n}, L^\top)$ is a LDU decomposition of $A \in \mathsf{R}^{n \times n}$, then $(L\sqrt{D}A = LDL^\top$ then $(\tilde{L}D^{1/2}, D^{1/2}L^\top)$ is a LU decomposition. Conversely, if $(B, B^\top)$ is a LU decomposition and $S$ is the diagonal matrix satisfying $S_{ii} = B_{ii}$ for $i = 1, \dots, n$, then $(BS^{-1}, S^2, S^{-1}B^\top)$ is a LDU decomposition of $A$.

## 1 Why

An *LDU factorization* of a $n \times n$ positive definite matrix permuted by $\{1, 2, \dots, n\}$ is an ordered pair whose first coordinate is a unit lower triangular matrix and whose second coordinate is a positive diagonal matrix, with the property that the given matrix permuted according to the permutation is equal to the product of the unit lower triangular matrix, the diagonal matrix, and the transpose of the unit lower triangular matrix.

## Notation

Let $(L, D)$ be an LDU factorization of $A \in \mathbf{S}^n_{++}$ permuted by the permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Then

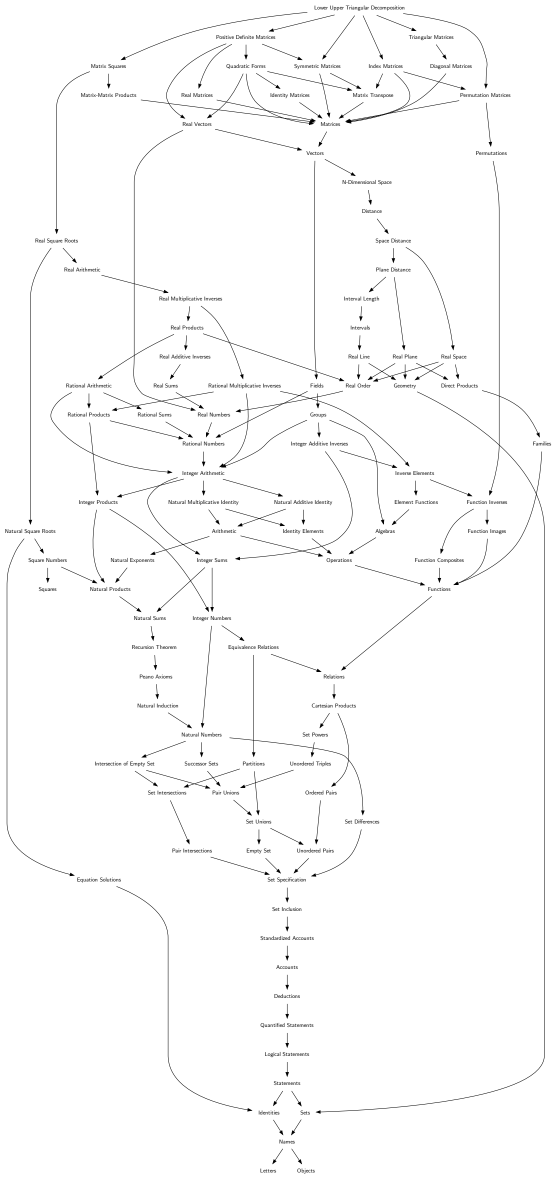
$$P_\sigma A P_\sigma^\top = LDL^\top.$$

Alternatively,

$$A = P_\sigma^\top LDL^\top P_\sigma.$$

## Existence

PROPOSITION **1.** *For every positive definite matrix $A \in \mathbf{S}^n_{++}$ and permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ there exists a unique LDU factorization of $A$ permuted by $\sigma$.*



Lower Upper Triangular Decomposition
Positive Definite Matrices
Triangular Matrices
Matrix Squares
Quadratic Forms
Symmetric Matrices
Index Matrices
Diagonal Matrices
Matrix-Matrix Products
Real Matrices
Identity Matrices
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Plane Distance
Real Multiplicative Inverses
Interval Length
Real Products
Intervals
Real Additive Inverses
Real Line
Real Plane
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Groups
Rational Numbers
Integer Additive Inverses
Families
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Inverse Elements
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Element Functions
Function Inverses
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Intersection of Empty Set
Successor Sets
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