Why

We discuss a decomposition using eigenvalues and eigenvectors.[1]

Defining result

An *eigenvalue decomposition* of a matrix $A \in \mathbf{R}^{n \times n}$ is an ordered pair (X, Λ) in which X is invertible, Λ is diagonal, and $A = X\Lambda X^{-1}$.

In this case, $AX = X\Lambda$, in other words,

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

in which x_i is the ith column of X and λ_i is the ith diagonal element of Λ. We have $Ax_i = \lambda_i x_i$ for $i = 1, \ldots, n$. In other words, the ith column of X is an eigenvector of A and the jth entry of Λ is the corresponding eigenvalue. If X is orthonormal, so that $X^{-1} = X^\top$, then we can interpret such a decomposition as a change of basis to *eigenvector coordinates*. If $Ax = b$, and $A = X\Lambda X^{-1}$ then $(X^{-1}b) = \Lambda(X^{-1}x)$. Here, $X^{-1}x$ expands x is the basis of columns of X. So to compute Ax , we first expand into the basis of columns of X, scale by Λ, and then interpret the result as the coefficients of a linear combination of the columns of X.

In this case that $A = X\Lambda X^\top$ for an eigenvalue decomposition (X, Λ) of A, we can also write

$$A = X\Lambda X^\top = \sum_{i=1}^{n} \Lambda_{ii} x_i x_i^\top.$$

Proposition 1. *Every real symmetric matrix has an eigenvalue decomposition (X, Λ) in which X is orthonormal.*[2]

[1] Future editions will expand.

[2] In future editions, this may be the motivating result for the definition of eigenvalues.

Eigenvalue Decomposition
Orthogonal Matrices
Matrix Transpose
Orthonormal Set of Vectors
Real Inner Product Norms
Eigenvalues and Eigenvectors
Orthonormal Set of Real Vectors
Real Inner Products
Norms
Vectors as Matrices
Matrices
Matrix Determinants
Real Vector Angles
Real Inner Product
Independent Set of Real Vectors
Functionals
Monic Polynomials
Real Matrices
Space Vector Angles
Space Inner Product
Real Norm
Real Linear Combinations
Vectors
Real Linear Equations
Plane Vector Angles
Space Norm
Real Vectors
Linear Functions
Arcos
Plane Inner Product
Plane Norm
Plane Vector Sums and Differences
Space Vectors
N-Dimensional Space
Real Square Roots
Right Triangle Sides Relation
Plane Vectors
Absolute Value
Distance
Polynomials
Real Squares
Space Distance
Real Polynomials
Plane Distance
Real Arithmetic
Interval Length
Real Multiplicative Inverses
Natural Square Roots
Intervals
Real Space
Real Products
Square Numbers
Real Line
Real Plane
Real Additive Inverses
Real Summation
Integral Line
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Natural Summation
Permutations
Squares
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Fields
Lists
Groups
Rational Numbers
Rings
Direct Products
Set Numbers
Natural Fractions
Integer Arithmetic
Finite Sets
Integer Products
Natural Addition Identity
Natural Multiplication Identity
Inverse Elements
Equivalent Sets
Integer Order
Comparisons
Natural Equations
Integer Sums
Natural Arithmetic
Identity Elements
Element Functions
Family Operations
Function Inverses
Family Unions and Intersections
Orders
Equations
Integer Numbers
Natural Powers
Operations
Function Composition
Function Images
Families
Natural Order
Converse Relations
Natural Products
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Successor Sets
Unordered Triples
Set Symmetric Difference
Intersection of Empty Set
Set Dualities
Partitions
Set Unions and Intersections
Ordered Pairs
Pair Unions
Set Complements
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Name
Letters
Objects