Relations

1 Why

We want a precise notion for how the elements of one set relate to elements of another set, or how elements of a set relate to other elements of the same set.

2 Definition

A **relation** between two non-empty sets A and B is a subset of $A \times B$. So then, naturally, a relation on a single set C is a subset of $C \times C$.

2.1 Notation

As relations are sets, our de facto protocol is to denote them by upper case capital letters, for example, the letter R. Let R a relation on A and B. If $(a, b) \in R$, we often write aRb, read aloud as "a in relation R to b."

In many cases, though, we eschew the set notation and use particular symbols. Often the symbols we use are meant to be suggestive of the relation. Some examples include $\sim$, $=$, $<$, $\leq$, and $\prec$.

3 Equivalence Relations

Here we survey a special relation on a set. Let R a relation on the non-empty set A. If aRa, then we call R **reflexive**. If aRb if and only if bRa then we call R **symmetric**. If aRb and bRc together imply aRc, then we call R **transitive**. If R is reflexive, symmetric, and transitive we call it an **equivalence relation**.

For an element $a \in A$, we call the set of elements in relation R to a the **equivalence class** of a. The key observation, recorded and proven below, is that the equivalence classes partition the set A. A frequent technique is to define an appropriate equivalence relation on a large set A and then to work with the set of equivalence classes of A.

We call the set of equivalence classes the **quotient set** of A under R. An equally good name is the divided set of A under R, but this terminology is not standard. The language in both cases reminds us that $\sim$ partitions the set A into equivalence classes.

3.1 Notation

If R is an equivalence relation on a set A, we use the symbol $\sim$. When alone, $\sim$ is read aloud as "sim," but we still read $a \sim b$ aloud as "a equivalent to b." We denote the quotient set of A under $\sim$ by $A/\sim$, read aloud as "A quotient sim".

3.2 Results

4 Orders

Here we survey a two other special relation on a set. Let R a relation on the non-empty set A. We call R **anti-symmetric** if for two nonequal elements $a, b \in A$, $(a, b) \in R \implies (b, a) \notin R$. If R is reflexive, transitive, and anti-symmetric then we call R a **partial order** on A.

A **partially ordered set** is a set together with a partial order. The language partial is meant to suggest that two elements need not be comparable.j For example, suppose R is $\{(a, a) \mid a \in A\}$; we may justifiably call this no order at all and call A totally unordered, but it is a partial order by our definition.

Often we want all elements of the set A to be comparable. We call R **connexive** if for all $a, b \in A$, $(a, b) \in R$ or $(b, a) \in R$. If R is a partial order and connexive, we call it a **total order**.

A **totally ordered set** is a set together with a total order. The language is a faithful guide: we can compare any two elements. Still, we prefer one word to three, and so we will use the shorter term **chain** for a totally ordered set; other terms include **simply ordered set** and **linearly ordered set**.

4.1 Notation

We denote total and partial orders on a set A by $\preceq$. We read $\preceq$ aloud as "precedes or equal to" and so read $a \preceq b$ aloud as "a precedes or is equal to b." If $a \preceq b$ but $a \neq b$, we write $a \prec b$, read aloud as "a precedes b."