



Why

If we model some measured value as a random variable with induced distribution $p : V \rightarrow \mathbf{R}$, then one interpretation of $p(v)$ for $v \in V$ is the *proportion* of times in a large number of trials that we *expect* to measure the value v .¹

Definition

Given a distribution $p : \Omega \rightarrow \mathbf{R}$ and a *real-valued* outcome variable $x : \Omega \rightarrow \mathbf{R}$, the *expectation* (or *mean*) of x under p is $\sum_{\omega \in \Omega} p(\omega)x(\omega)$.

Notation

We denote the expectation of x under p by $\mathbf{E}(x)$. When there is no chance of ambiguity, we write $\mathbf{E}(x)$.

Properties

Let $x, y : \Omega \rightarrow \mathbf{R}$ be two outcome variables and $p : \Omega \rightarrow \mathbf{R}$ a distribution. Let $\alpha, \beta \in \mathbf{R}$. Define $z = \alpha x + \beta y$ by $z(\omega) = \alpha x(\omega) + \beta y(\omega)$. Then $\mathbf{E}(z) = \alpha \mathbf{E}(x) + \beta \mathbf{E}(y)$.

Example: expectation

Suppose $\Omega = \{1, 2, 3, 4, 5\}$ with $p(1) = 0.1$, $p(2) = 0.15$, $p(3) = 0.1$, $p(4) = 0.25$ and $p(5) = 0.4$. Define $x : \Omega \rightarrow \mathbf{R}$ by

$$x(a) = \begin{cases} -1 & \text{if } a = 1 \text{ or } a = 2, \\ 1 & \text{if } a = 3 \text{ or } a = 4, \\ 2 & \text{if } a = 5. \end{cases}$$

The expectation of x under p is

$$\mathbf{E}x = -1 - 0.15 + 0.1 + 0.25 + 2(0.4) = 0.9.$$

¹Future editions may modify this explanation, and take a genetic approach via summary statistics.

Two routes for computation

Denote by $p_x : V \rightarrow \mathbf{R}$ the induced distribution of $x : \Omega \rightarrow V$ (where $V \subset \mathbf{R}$). Then $\mathbf{E}(x) = \sum_{v \in V} p_x(v)v$ since

$$\begin{aligned}\sum_{\omega \in \Omega} p(\omega)x(\omega) &= \sum_{v \in V} \sum_{\omega \in x^{-1}(v)} x(\omega)p(\omega) \\ &= \sum_{v \in V} v \sum_{\omega \in x^{-1}(v)} p(\omega) \\ &= \sum_{v \in V} x(v)p_x(v).\end{aligned}$$

Interpretations

We interpret the mean as the center of mass of the induced distribution.

