



**Why**

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**1 Definition**

Define  $S \in \mathbf{R}^{d \times d}$  by

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

For a vector  $x \in \mathbf{R}^d$  the *down shift* of  $x$  is  $Sx$ .

Let  $A \in \mathbf{R}^{d \times d}$  be a matrix with columns  $a_1, \dots, a_d$ .  $A$  is a *circulant matrix* if  $a_1 = Sa_d$ ,  $a_2 = Sa_1$ , and  $a_i = Sa_{i-1}$  for  $i = 2, \dots, d$ .

**Example**

For example, the matrix

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is a circulant matrix.

**Characterization**

A matrix  $C \in \mathbf{R}^{d \times d}$  is circulant if and only if there exists  $c_0, \dots, c_{d-1}$  so that

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{d-1} S^{d-1}.$$

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<sup>1</sup>Future sheets will include. These matrices arise in practice and each has the same eigenvectors.

## **Properties**

The sum and product of any two circulant matrix is circulant. In other words, the circulant matrices with the usual matrix addition and multiplication form a commutative ring.



