



SIMPLE INTEGRAL ADDITIVITY

Why

If we stack two rectangles, with equal base lengths but different heights, on top of each other, the additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property, as it ought to.

Result

PROPOSITION 1. *The simple non-negative integral operator is additive.*

Proof. Let (X, \mathcal{A}, μ) be a measure space. Let $\mathcal{SF}_+(X)$ denote the non-negative real-valued simple functions on X . Define $s : \mathcal{SF}_+(X) \rightarrow [0, \infty]$ by $s(f) = \int f d\mu$ for $f \in \mathcal{SF}_+(X)$.

In this notation, we want to show that $s(f+g) = s(f)+s(g)$ for all $f, g \in \mathcal{SF}_+(X)$. Toward this end, let $f, g \in \mathcal{SF}_+(X)$ with the simple partitions:

$$\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n \subset \mathcal{A} \quad \text{and} \quad \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset [0, \infty].$$

We consider the refinement of the two partitions. TODO: this is why you don't do the unique maximal partition business. $\{A_i \cap B_j\}_{i,j=1}^{i=m, j=n}$.

First, let $\alpha \in (0, \infty)$. Then $\alpha f \in \mathcal{SF}_+(X)$, with the simple

partition $\{A_n\} \subset \mathcal{A}$ and $\{\alpha a_n\} \subset [0, \infty]$.

$$s(\alpha f) = \sum_{i=1}^n \alpha a_n \mu(A_i) = \alpha \sum_{i=1}^n a_n \mu(A_i) = \alpha s(f).$$

If $\alpha = 0$, then αf is uniformly zero; it is the non-negative simple with partition $\{X\}$ and $\{0\}$. Regardless of the measure of X , this non-negative simple function is zero. Recall that we define $0 \cdot \infty = \infty \cdot 0 = 0$. □

