Supremum Norm

1 Why

We want a norm on the vector space of continuous functions.

2 Definition

Consider a function from a closed real interval to the real numbers. The **absolute supremum** the supremum of the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

2.1 Defining Result

Proposition 1. *The functional mapping $f \in C[a, b]$ to its supremum is a norm.*

Proof. Let R denote the set of real numbers. Define $\phi : C[a, b] \to R$ so that

$$\phi(f) = \sup\{|f(x)| \mid x \in [a, b]\}.$$

1. $|f(x)| \geq 0$ for all $x \in [a, b]$, so $\phi(f) \geq 0$.

2. If $\phi(f) = 0$ then $|f(x)| \leq 0$ for all x and so $f(x) = 0$ for all $x \in [a, b]$. If $f = 0$, then $|f(x)| = 0$ for all $x \in [a, b]$

3. For all α real, $|\alpha f(x)| = |\alpha||f(x)|$. so $\phi(\alpha f) = |\alpha|\phi(f)$

4. For all $f, g \in C[a, b]$, and $x \in [a, b]$, $|f(x)+g(x)| \leq |f(x)|+|g(x)|$ by the triangle inequality for absolute value. Thus,

$$\begin{aligned}
\phi(f+g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a, b]\} \\
&\leq \sup\{|f(x)| \mid x \in [a, b]\} + \sup\{|g(x)| \mid x \in [a, b]\} \\
&= \phi(f) + \phi(g)
\end{aligned}$$

□

As a result of this proposition, we call the functional ϕ defined above the **supremum norm**.

2.2 Notation

Let $f \in C[a, b]$. We denote the supremum norm of f by $|f|_{\text{sup}}$.