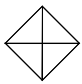


Why

We generalize the algebraic structure of addition and multiplication over the integers.[1]

Definition

A *ring* (or *ring with identity*) (A, f, g) is a set A and two operations on A satisfying the following set of conditions.

(A) (i) f is associative. (ii) f is *commutative*, (iii) A has an identity for f (i.e., is $e \in A$ with $f(a, e) = f(e, a) = a$ for all $a \in A$ (iv) A has inverse elements for f (i.e., for any $a \in A$, there is $\tilde{a}$ satisfying $f(a, \tilde{a}) = f(\tilde{a}, b) = e$)

(B) (i) g is associative; (ii) A has an idenity element for g (i.e., there is $\tilde{e} \in A$ satisfying $g(a, \tilde{e}) = g(\tilde{e}, a) = a$)

(C) (i) g left distributes: $g(f(a, b), c) = f(g(a, c), g(b, c)$; (ii) g right distributes: $g(c, f(a, b)) = f(g(c, a), g(c, b))$.

Conditions (A) concern f, conditions (B) concern g, and conditions (C) relate the two. Define $\psi : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by $\psi(a, b) = a + b$ and $\pi : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by $\pi(a, b) = a \cdot b$. We have defined a ring so that $(\mathbf{Z}, \psi, \pi)$ is one. The element referred to in (A.2) is $0 \in \mathbf{Z}$, so we refer to this element in any ring as the *additive identity*. That referred to (A.3) is $1 \in \mathbf{Z}$, so we refer to this element in any ring as the *multipliciative identity*. We refer to the elements mentioned in (A.4) as *additive inverses*. We call to f *ring addition* and g *ring multiplication*. Although integer products are commutative, we have not required this aspect (future editions will elaborate).

[1] Future editions will likely modify this sheet, and give a genetic treatment involving the solution of polynomial equations by Galois.

Notation

Our notation furthers this analogy with $\mathbf{Z}$. We denote the ring addition by $+$ and ring multiplication by $\cdot$. Moreover, we denote the ring's additive identity by 0 and the ring's multiplicative identity by 1. Finally, we denote the additive inverse of $a \in A$ by $-a$.

These notational conventions make the condtions (A), (B), (C) familiar relations among integers. (A) (1) $a + (b + c) = (a + b) + c$; (2) $a+b = b+a$; (3) $a+0 = 0+a = a$; (4) $a+(-a) = 0$. (B) (1) $a(bc) = (ab)c$; (2) $1a = a1 = a$. (C) (1) $(a + b)c = ac + bc$; (2) $c(a + b) = ca + cb$.

Immediate consequences

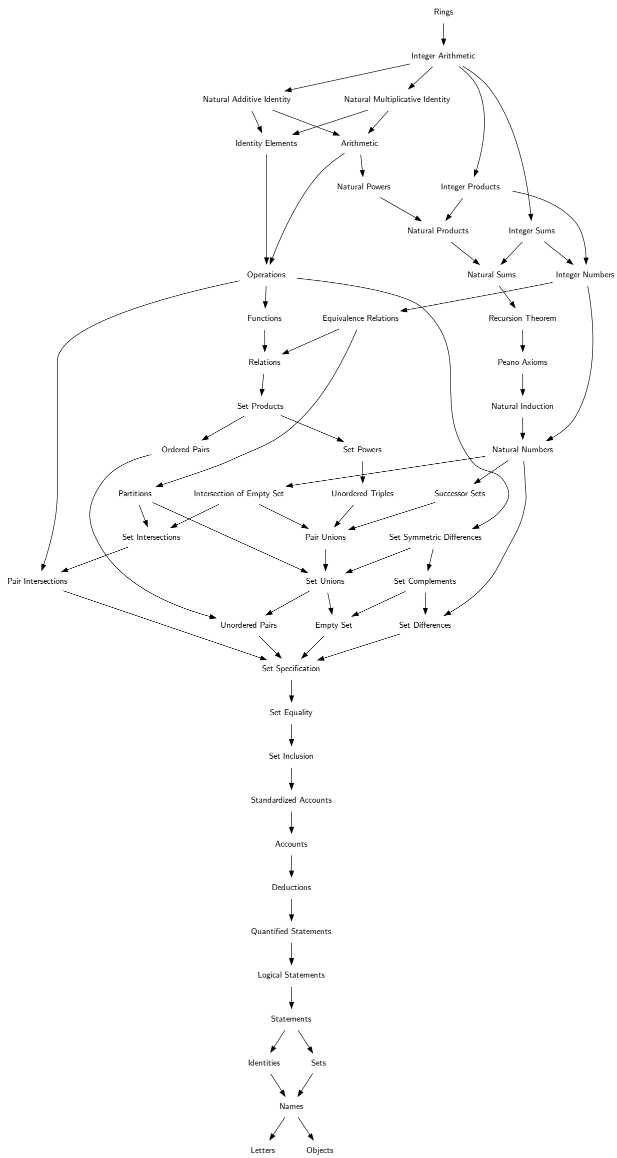
We need not require that $0x = 0$, because we can deduce it:

$$0x + x = (0 + 1)x = 1x = x.$$

Similarly, $(-a)b = -(ab)$ since

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

Other familiar relations among the integers, e.g. $(-a)(-b) = ab$, may be deduced.


Rings
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Identity Elements
Arithmetic
Natural Powers
Integer Products
Natural Products
Integer Sums
Operations
Natural Sums
Integer Numbers
Functions
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Symmetric Differences
Pair Intersections
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects