Why

Historically considered a fountainhead for all of mathematics.

Discussion

So far we know that ω is the unique smallest successor set. In other words, we know that $0 \in \omega$, $n \in \omega \longrightarrow n^+ \in \omega$ and that if these two properties hold of some $S \subset \omega$, then $S = \omega$. We can add two important statements to this list. First, that 0 is the successor of no number. In other words, $n^+ \neq 0$ for all $n \in \omega$. Second, that if two numbers have the same successor, then they are the same number In other words, $n^+ = m^+ \longrightarrow n = m$

These five properties were historically considered the fountainhead of all of mathematics. One by the name of Peano used them to show the elementary properties of arithmetic. They are:

1. $0 \in \omega$.

2. $n \in \omega \longrightarrow n^+ \in \omega$ for all $n \in \omega$.

3. If S is a successor set contained in ω, then $S = \omega$.

4. $n^+ \neq 0$ for all $n \in \omega$

5. $n^+ = m^+ \longrightarrow n = m$ for all $n, m \in \omega$.

These are collectively known as the *Peano axioms*. Recall that the third statement in this list is the *principle of mathematical induction*.

Statements

Here are the statements.

Proposition 1 (Peano's First Axiom)**.** $0 \in \omega$.

Proposition 2 (Peano's Second Axiom)**.** $n \in \omega \longrightarrow n^+ \in \omega$.

Proposition 3 (Peano's Third Axiom). *Suppose $S \subset \omega$, $0 \in S$, and $(n \in S \longrightarrow n^+ \in S$. Then $S = \omega$.*

Proposition 4 (Peano's Fourth Axiom). $n^+ \neq 0$ *for all $n \in \omega$.*

The last one uses the following two useful facts.

Proposition 5. $x \in n \longrightarrow n \not\subset x$.

Proposition 6. $(x \in y \wedge y \in n) \longrightarrow x \in n$

This latter proposition is sometimes described by saying that n is a *transitive set*. This notion of transitivity is not the same as that described in Relations. Using these one can show:

Proposition 7 (Peano's Fifth Axiom). *Suppose $n, m \in \omega$ with $n^+ = m^+$. Then $n = m$.*

Peano Axioms

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Natural Induction

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Natural Numbers

Successor Sets   Intersection of Empty Set

Pair Unions   Set Differences

Set Unions   Set Intersections

Unordered Pairs   Empty Set   Pair Intersections

Set Specification

↓

Set Equality

↓

Set Inclusion

↓

Deductions

↓

Quantified Statements

↓

Logical Statements

↓

Statements

Identities   Sets

Names

Letters   Objects

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