



Why

When does the technique of row reductions prevail?

Multivariable row reductions

Let $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$. The k th row reduction of S is the linear system (C, d) with $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$ if $i < s \leq m$ and $C_{st} = A_{st}$ otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable k from equations $k + 1, \dots, m$. We are taking the k th column of A from

$$\begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ A_{k+1,k} \\ \vdots \\ A_{mk} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We interpret the i th row reduction as *subtracting equations* of the system or *reducing rows* of the array A . If $a^i, c^i \in \mathbf{R}^n$ denote the i th rows of A and C , $c^i = a^i - (A_{ik}/A_{kk})a^k$ for $k < i \leq m$. In other words, we obtain the i th row of matrix C by subtracting a multiple of the k th row of matrix A from the i th row of matrix A , for $k < i \leq m$. The following is an immediate consequence of real arithmetic.

Proposition 1. *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to (C, d) . Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d) .*

Ordinary reductions

We call the system S *ordinarily reducible* if there exists a sequence of systems S_1, \dots, S_{m-1} so that S_1 is the 11-reduction of S and S_i is the ii -reduction of S_{i-1} for $i = 1, \dots, n - 1$. In this case, we call S_{n-1} the

final ordinary reduction (or just *ordinary reduction*) of S . The following is an immediate consequence of Proposition 1.

Proposition 2. *Let S' be the (final) ordinary reduction of S . Then S and S' have equivalent solution sets.*

This process of constructing the ordinary reduction is called *Gauss elimination* or *Gaussian elimination*. We call the kk th entry of system S_{k-1} the *pivot*. In an ordinarily reducible system, the pivots are nonzero.

The idea is that a system is ordinarily reducible if we can take row reductions in sequence an end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system (A, b) in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.

