The Bourbaki Project

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Contents

Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *script* is a collection of written marks or symbols called *letters*. In *phonetic* languages, the letters correspond to sounds. A *written word* is a succession of letters. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into *written words* which are meant to denote the *spoken words* of the English language. The written words on this page are several letters one after the other. For example, the word “word” is composed of the letters “w”, “o”, “r”, “d”.

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. We call these the *lower case latin letters.*

$$\begin{array}{ccccccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} & \mathrm{i} \\ \mathrm{j} & \mathrm{k} & \mathrm{l} & \mathrm{m} & \mathrm{n} & \mathrm{o} & \mathrm{p} & \mathrm{q} & \mathrm{r} \\ \mathrm{s} & \mathrm{t} & \mathrm{u} & \mathrm{v} & \mathrm{w} & \mathrm{x} & \mathrm{y} & \mathrm{z} & \end{array}$$

And we call these the *upper case latin letters.*

$$\begin{array}{ccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H} & \mathrm{I} \\ \mathrm{J} & \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{O} & \mathrm{P} & \mathrm{Q} & \mathrm{R} \\ \mathrm{S} & \mathrm{T} & \mathrm{U} & \mathrm{V} & \mathrm{W} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \end{array}$$

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We will also use the following symbols. We call these the *Arabic numerals.*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Other symbols

We will also use the following symbols We call thes the *logical symbols.*

$$(\quad) \quad \vee \quad \wedge \quad \neg \quad \forall \quad \exists \quad \implies \quad \iff \quad = \quad \in$$

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an accent $'$ or subscripts or superscripts. When we write the symbol, we say that it *denotes* the object. We call the symbols the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{\text{A}}$ or $\boxed{\text{A'}}$. The box works well to group in the accent, and also clarifies that $\boxed{\text{A}}\boxed{\text{A}}$ is different from $\boxed{\text{AA}}$. But experience shows that the boxes are mostly unnecessary.

We indicate a name for an object with italics. Instead of $\boxed{\text{A'}}$ we use A'. Experience shows that this subtelty is enough for clarity, and it agrees with traditional and modern practice.

No repetitions

We will also agree that we will never use the same name to refer to two different objects. It is in the nature of things—and of names in particular—that we can not do this without confusion.

Names are objects

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can references themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the several marks on the page, all of which are meant to look similar, which ar meant to denote the object, are uses of the name.

Placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is beset understood by example.

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

Symmetry

"A equals B" means the same as "B equals A". This is because the identity of the object is not changed by the order in which the names are given.

This fact is called the *symmetry of identity*. It is obvious. Not subtle in the slightest. We can switch the spots of A and B and say the same thing. There are two ways to say the same thing.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself.

This fact is called the *reflexivity of identity*. It too is obvious. And not subtle. We can always declare that the same symbol denotes the same object. We agreed upon this in *Names*.

Why

We want to talk about none, one, or several objects considered as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set*. We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

The objects in a set may be other sets. In other words, an element of a set may be another set. This may be subtle at first glance, but becomes familiar with experience.

We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object belongs to the set. In other words, the object is a member or an element of the set. We say "The object denoted by a belongs to the set denoted by A".

Asymmetry

Notice that belonging is not symmetric. Saying "the object denoted by a belongs to the set denoted by A" does not mean the same as "the set denoted by A belongs to the object denoted by a" In fact, the latter sentence is nonsensical unless the object a is also a set.

Nontransitive

Let a denote an object and let A denote a set and B denote a set. If the object denoted by a is *a part of* a the set denoted by A, and the set denoted by A is *a part of* the set denoted by B, then usual English usage would suggest that a is a *a part of* the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing. The relation of belonging is not the same. We do not allow this with sets. If a thing is an element of a thing, that second thing may be an element of the third thing, but this does not mean that the.

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, $\ldots$, x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Why

We want to write about objects belonging to sets.

Definition

Let A denote a set; in other words, an intangible object which has some objects as members. Let a denote an object. Recall that if two names refer to the same object, the names are equal. Similarly, if the object denoted by a is an element of the set denoted by A, then we say that the former name belongs to the latter name. We write that the name a belongs to the name A by $a \in A$.

We read this sequence of symbols aloud as "a in A." The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means "belongs" in ancient greek. Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of". Of course, we must take care. The first name is not an element on the second name. Rather, the object denoted by the first name is an element of the set (object) denoted by the second name.

We tend to denote sets by upper case latin letters: for example, A, B, and C. To aid our memory, we tend to use the lower case form of the letter for an element of the set. For example, let A and B denote nonempty sets. We tend to denote by a an object which is an element of A. And similarly, we tend to denote by b an object which is an element of B

Why

We want symbols to represent identity and belonging.

Definition

In the English language, *nouns* are words that name people, places and things. In these sheets, *names* (see *Names*) serve the role of nouns. In the English language, *verbs* are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense. A *statement* is several symbols.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings. As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object in the set of the sheet *Identity*. We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And if we read it aloud, we would read a as "the object denoted by a", then $=$ as "is", then b as "the object denoted by b". Altogether then, "the object denoted by a is the object denoted by b." We might box these three symbols $\boxed{a \sim b}$ to make clear that they are meant to be read together, but experience shows that (as with English sentences and words) we do not need boxes.

The symbol $=$ is (appropriately) a symmetric symbol. If we flip it left and right, it is the same symbol. This reflects the symmetry of the English sentences represented. $A = B$ means the same as $B = A$.

Belonging

As a second example, consider the symbol $\in$. Let a denote an object and let A denote a set. Let us suppose that the object denoted by a belongs to the set denoted by A. We agree that $\in$ means "belongs to" in the sense of "is an element of" or "is a member of" as given in *Sets*. Then we write $a \in A$. We read these symbols as "the object denoted by a belongs to the set denoted by A".[1]

The symbol $\in$ is not symmetric. If we flip it left and right it looks different. And as we discussed in *Sets*, $a \in A$ does not the mean the same as $A \in a$.

[1]The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for the ancient Greek word $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means, roughly, "belongs". Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of".

Why

We want symbols for “and”, “or”, “not”, and “implies”.

Definition

In *Statements* we discussed that nouns are names and that we will only use the present tense of the verbs “is” and “belongs”. We had statements like $a = b$ (identity) and $a \in A$ (belonging).

We call $=$ and $\in$ *relational symbols*. They say how the objects denoted by a pair of placeholder names relate to each other in the sense of being or belonging. We call $\_ = \_$ and $\_ \in \_$ *simple statements*. They denote simple sentences “the object denoted by _ is the object denoted by _” and “the object denoted by _ belongs to the set denoted by _”. We want want to assert that one make more complicated statements.

Conjunction

Consider the symbol $\land$. We will agree that it means “and”. If we want to make two simple statements like $a = b$ and $a \in A$ at once, we write write $(a = b) \land (a \in A)$. The symbol $\land$ is symmetric, reflecting the fact that a statement like $(a \in A) \land (a = b)$ means the same as $(a = b) \land (a \in A)$.

Disjunction

Consider the symbol $\lor$. We will agree that it means “or” in the sense of either one, the other, or both. If we want to say that possibly only one, but at least one of the simple statements like $a = b$ and $a \in A$, we write write $(a = b) \vee (a \in A)$. The symbol $\vee$ is symmetric, reflecting the fact that a statement like $(a \in A) \vee (a = b)$ means the same as $(a = b) \vee (a \in A)$.

Negation

Consider the simple $\neg$. We will agree that it means "not", in the sense of negating whatever it follows. If we want to say the opposite of a simple statement like $a = b$ we will write $\neg(a = b)$. We read it aloud as "not a is b". Of course, the more desirable english expression is "a is not b". Similarly, $\neg(a \in A)$ we read as "not, the object denoted by a belongs to the set denoted by A". Again, the more desirable english expression is something like "the object denoted by a does not belong to the set A" For these reasons, we introduce two new symbols $\neq$ and $\notin$. $a \neq b$ means $\neg(a = b)$ and $a \notin A$ means $\neg(a \in A)$.

Implication

Consider the symbol $\implies$. We will agree that it means "implies". For example $(a \in A) \implies (a \in B)$ means "the object denoted by a belongs to the object denoted by A implies the obejcted denoted by a belongs to the set denoted by B" It is the same as $(\neg(a \in A)) \vee (a \in B)$. In other words, if $a \in A$, then always $a \in B$. The symbol $\implies$ is not symmetric, since implication is not symmetric.

Why

We want to make conclusions.

Definition

Suppose we have a list of logical statements. We want to write down o

Why

We want symbols for talking about some object of a set or all objets of a set.

Definition

Existential Quantifier

Consider the symbol $\exists$. We agree that it means "there exists an object". We write $\exists a \in A$. We read this as "there exists an object in the set denoted by A, and denote that very same object by a". We call $\exists$ the existential quantifier.

Universal Quantifier

Consider the symbol $\forall$. We agree that it means "for every object". We write $\forall a \in A$. We read this as "for every object in the set denoted by A, and denote one such object by a".

Why

We want to succinctly and clearly make several statements about objects and sets. We want to track the names we use, taking care to avoid using the same name twice.

Definition

An *account* is a list of naming, logical, and quantified statements. We use the words "let _ denote an _" to introduce a name as a placeholder for a thing, and we use the symbols $\_ = \_$ and $\_ \in \_$ to denote statements of identity and belonging. In other words, we have three sentence kinds to record.

1. **Names.** State we are using a name.
2. **Identity.** We want to make statements of identity.
3. **Belonging.** We want to make statements of belonging.

Our main purpose is to keep a list names and logical statements about them and then deductions. We want to group our usage of names. In the English language we use paragraphs or sections to do so. In these sheets, we will use *accounts*, which will be a list of statements, each of which is labeled by an Arabic numeral (see *Letters*).

Experience suggests that we start with an example. Suppose we want to summarize the following english language description of some names and objects.

> Denote an object by a. Also, denote the same object by b Also, denote a set by A. Also, the object denoted by a is an element of the set denoted by A. Also denote an object by c. Also c is the same object as b.

In our usual manner of speaking, we drop the word "also". In these sheets, we translate each of the sentences into our symbols. For names we use, we write `name` in that font followed by the name. For logical statements we "have", we write `have` followed by the logical statement. So we write:

Account 1. First Example

1	`name`	a
2	`name`	b
3	`have`	$a = b$
4	`name`	A
5	`have`	$a \in A$
6	`name`	c
7	`have`	$c = b$
8	`thus`	$a = c$ `by 3,7`

Why

We want to do our best to have only one way to write accounts.

Discussion

Consider the account.

Account 2. First Example

1	`name`	a	
2	`name`	b	
3	`have`	$a = b$	
4	`name`	c	
5	`have`	$c = b$	
6	`thus`	$a = c$	`by 4,5`

We standardize it:

Account 3. Standardized First Example

1-3	`name`	a, b, c	
4	`have`	$a = b$	
5	`have`	$c = b$	
6	`thus`	$a = c$	`by 4,5,IdentityAxioms:1`

Why

When are two sets the same?

Definition

Given sets A and B, if $A = B$ then every element of A is an element of B and every element of B is an element of A.

Account 4. Joint Membership

1-3	`name`	A, B, x	
4	`have`	$A = B$	
5	`have`	$x \in B$	
6	`thus`	$x \in A$	`by 4,5`

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Is $A = B$ true? We define it to be so. Two sets are *equal* if and only if every element of one is an element of the other. In other words, two sets are the same if they have the same elements. This statement is sometimes called the *axiom of extension*. Roughly speaking, if we refer to the elements of a set as its *extension*, then we have declared that if we know the extension then we know the set. A set is determined by its extension.

This definition gives us a way to argue that $A = B$ from the properties of the elements of A and B. It may not be obvious that the sets are the same. We first argue that each element of A is an element of B and then argue that each element of B is an element of A. With these two implications, we use the axiom of extension to conclude that the sets are the same.

The logical statement is: $(((\forall x)(x \in A \implies x \in B) \wedge (\forall x)(x \in B \implies x \in A))) \implies (A = B)$ Here is an example of applying that:

Account 5. Extension

1-2	`name`	A, B	
3	`have`	$(\forall x)((x \in A) \implies (x \in B))$	
4	`have`	$(\forall x)((x \in B) \implies (x \in A))$	
5	`thus`	$A = B$	`by 3,4`

A Contrast

We can compare the axiom of extension for sets and their elements with an analogous statement for human beings and their ancestors.

On the one hand, if two human beings are equal then they have the same ancestors. The ancestors being the person's parents, grandparents, greatgrandparents, and so on. This direction, same human implies same ancestors, is the analogue of the "only if" part of the axiom of extension. It is true. On the other hand, if two human beings have the same set of ancestors, they need not be the same human. This direction, same ancestors implies same human, is the analogue of the "if" part of the axiom of extension. It is false. For example, siblings have the same ancestors but are different people.

We conclude that the axiom of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Why

If there is a set, there is an empty set. Are there many such sets? How do they (or it) relate to other sets?

Empty Set

An immediate consequence of the axiom of extension is that there is a unique set that is empty.

Account 6.

1-2	`name`	A, B		
3	`have`	$\neg((\exists a)(a \in A))$		
4	`have`	$\neg((\exists b)(b \in B))$		
5	`thus`	$(\forall x)(x \in A \implies b \in A)$	`by`	3
6	`thus`	$(\forall x)(x \in B \implies b \in B)$	`by`	4
7	`thus`	$A = B$	`by`	5,6

Definition

First, we assume there exists a set. As a consequence, there exists a set which contains no elements at all. We use the axiom of specification with a condition that is always false, and so selects no elements.

As a result of the axiom of extension, this set with no elements is unique. We call this empty set *the empty set.*

Notation

We denote the empty set by $\varnothing$.

Why

We want language for all of the elements of a first set being the elements of a second set.

Definition

Denote a set by A and a set by B. If every element of the set denoted by A is an element of the set denoted by B, then we say that the set denoted by A is a *subset* of the set denoted by B. We say that the set denoted by A is *included* in the set denoted by B. We say that the set denoted by B is a *superset* of the set denoted by A or that the set denoted by B *includes* the set denoted by A. A set includes and is included in itself.

If the sets denoted by A and B are identical, then each contains the other. If $A = B$, then the set denoted by A includes the set denoted by B and the set denoted by B includes the set denoted by A. The axiom of extension asserts the converse also holds. If the set denoted by A includes the set denoted by B and the set denoted by$B$ includes the set denoted by A, then A and B denote the same set. In other words, if the set denoted by A is a subset of the set denoted by B and the set denoted by B a subset of the set denoted by A, then $A = B$.

The empty set is a subset of every other set.

Account 7. Empty Set Inclusion

1-2	`name`	A, $\varnothing$	
3	`have`	$\neg((\exists x)(x \in \varnothing))$	
4	`thus`	$(\forall x)((x \in \varnothing) \implies (x \in A))$	`by`
5	`idet`	$\varnothing \subset A$	`of`

Suppose toward contradiction that A were a set which did not include the empty set. Then there would exist an element in the empty set which is not in A. But then the empty set would not be empty. We call the empty set and A *improper subsets* of A. All other subsets we call *proper subsets*. In other words, B is an improper subset of A if and only if A includes B, $B \neq A$ and $B \neq \varnothing$.

Notation

Given two sets A and B, we denote that A is included in B by $A \subset B$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A".

In this notation, we express the axiom of extension

$$A = B \Leftrightarrow (A \supset B) \land (A \subset B).$$

The notation $A \subset B$ is a concise symbolism for the sentence "every element of A is an element of B." Or for the alternative notation $a \in A \implies a \in B$.

Properties

Given a set A, $A \subset A$. Like equality, we say that inclusion is *reflexive*. Given sets A and B, if $A \subset B$ and $B \subset C$ then $A \subset C$. Like equality, we say that inclusion is *transitive*. If $A \subset B$ and $B \subset A$, then $A = B$ (by the axiom of extension). Unlike equality, which is symmetric, we say that inclusion is *antisymmetric*.

Comparison with belonging

Given a set A inclusion is reflexive. $A \subset A$ is always true. Is $A \in A$ ever true? Also, inclusion is transitive. Whereas belonging is not.

Why

Can we always construct subsets?

Definition

We will say that we can. We assert that to every set and every sentence predicated of elements of the set there exists a second set (a subset of the first) whose elements satisfy the sentence. It is an consequence of the axiom of extension that this set is unique. The *axiom of specification* is this assertion. We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence.

Notation

Let A be a set. Let $S(a)$ be a sentence. We use the notation

$$\{a \in A \mid \; S(a)\ \}$$

to denote the subset of A specified by S. We read the symbol $|$ aloud as "such that." We read the whole notation aloud as "a in A such that..."

We call the notation *set-builder notation*. Set-builder notation avoids enumerating elements. This notation is really indispensable for sets which have many members, too many to reasonably write down.

Example

For example, let a, b, c, d be distinct objects. Let $A = \{a, b, c, d\}$. Then $\{x \in A \mid x \neq a\}$ is the set $\{b, c, d\}$

Now let B be an arbitrary set. The set $\{b \in B \mid b \neq b\}$ specifies the empty set. Since the statement $b \neq b$ is false for all objects b.

Why

We want to consider the elements of two sets together at one. Does a set exist which contains all elements which appear in either of one set or another?

Definition

We say yes. For every set of sets there exists a sets which contains all the elements that belong to at least one set of the given collection. We refer to this as the *axiom of unions*. If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the axiom of unions may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification to form the set which contains only those elements which are appear in at least one of any of the sets. As a result of the axiom of extension, this set is unique. We call it the *union* of the set of sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\cup \mathcal{A}$.

Simple Facts

Proposition **1.** $\cup \varnothing = \varnothing$

Proposition **2.** $\cup\{A\} = A$

Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

Definition

Let A and B be nonempty sets. Let $a \in A$ and $b \in B$. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is a and the *second coordinate* is b.

The *product* of A and B is the set of all ordered pairs. This set is also called the *cartesian product*. If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Taste

Notice that $a \not\in (a, b)$ and similarly $b \not\in (a, b)$. These facts led us to use the terms first and second "coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life. Plus, suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Equality

PROPOSITION **3.** *$(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.*

Proof. TODO $\square$

Why

How can we relate the elements of two sets?

Definition

A *relation* between two nonempty sets is a subset of their cross product. A relation on a single set is a subset of the cross product of it with itself.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation. The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation.

Notation

Let A and B be two nonempty sets. A relation on A and B is a subset of $A \times B$. Let C be a nonempty set. A relation on a C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The ordered pair (a, b) may or may not be in a relation on A and B. Also notice that if $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as "a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *antisymmetric*antisymmetric if two different objects are related only in one order, and never both. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related.

Notation

Let R be a relation on a non-empty set A. R is reflexive if

$$(a, a) \in R$$

for all $a \in A$. R is transitive if

$$(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

for all $a, b, c \in A$. R is symmetric if

$$(a, b) \in R \implies (b, a) \in R$$

for all $a, b \in A$. R is anti-symmetric if

$$(a, b) \in R \implies (b, a) \notin R$$

for all $a, b \in A$.

Why

We want a notion for a correspondence between two sets.

Definition

A *functional* relation on two sets relates each element of the first set with a unique element of the second set. A *function* is a functional relation.

The *domain* of the function is the first set and *codomain* of the function is the second set. The function *maps* elements *from* the domain *to* the codomain. We call the codomain element associated with the domain element the *result* of *applying* the function to the domain element.

Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in (A \to B)$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

Why

We want to "combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations to *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\, g\, b$ instead. We call this *infix notation.*

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Why

We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Why

We want to define the natural numbers. TODO: better why

Definition

The *successor* of a set is the union of the set with the singleton whose element is the set. This definition holds for any set, but is of interest only for the sets which will be defined in this sheet.

These sets are the following (and their successors): *One* is the successor of the empty set. *Two* is the successor of one. *Three* is the successor of two. *Four* is the successor of three. And so on; using the English language in the usual manner.

Can this be carried on and on? We will say yes. We will say that there exists a set which contains one and contains the successor of each of its elements. So, this set contains one. Since it contains one, it contains two. Since it contains two, it contains three. And so on. We call this assertion the *axiom of infinity*.

A set is a *successor set* if it contains one and if it contains the successor of each of its elements. In these words, the axiom of infinity asserts the existence of a successor set. We want this set to be unique. So we have a successor set. By the axiom of specification, the intersection of all the successor sets included in this first successor set exists. Moreover, this intersection is a successor set. Even more, this intersection is unique. For this, take a second successor set. Its intersection with the first successor set is contained in the first successor set. Thus, this intersection of two sets is one of the successor sets contained in the first set, and so, is contained in the intersection of all such sets. So then, that first intersection is contained in second intersection of two sets, which is, of course, contained in the second successor set. In other words, we start with a successor set. Use it to construct a succesor set contained in it, in such a way that every other successor set also contains this successor set so constructed. The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique.

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4.

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n.

Why

Definition

integer numbers integers

TODO

Why

We generalize the algebraic structure of addition over the integers.

Definition

A *group* is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra *group addition*. A *commutative group* is a group whose operation commutes.

Notation

TODO

Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *field addition*. We call the operation of the second algebra *field multiplication*.

Notation

We denote an arbitrary field by F, a mnemonic for "field."

TODO

Why

Definition

Why

We want a notion of distance between elements of the real line.

Definition

We define a function mapping a real number to its length from zero.

Notation

We denote the absolute value of a real number $a \in \mathsf{R}$ by $|a|$. Thus $|\cdot| : \mathsf{R} \to \mathsf{R}$ can be viewed as a real-valued function on the real numbers which is nonnegative.

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