



### Why

We want to select a normal density which summarizes well a dataset.

### Formulation

Let  $D = (x^1, \dots, x^n)$  be a dataset in  $\mathbf{R}$ . We want to select a density from among normal densities, which require specifying a mean and covariance.

Following the principle of maximum likelihood, we want to solve

$$\begin{aligned} &\textbf{find} \quad \mu, \sigma \in \mathbf{R} \\ &\textbf{to maximize} \quad \prod_{k=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x^k - \mu}{\sigma}\right)^2\right) \\ &\textbf{subject to} \quad \sigma > 0 \end{aligned}$$

We call a solution to the above problem a *maximum likelihood normal density* with respect to the dataset.

### Solution

**Prop. 1.** *Let  $(x^1, \dots, x^n)$  be a dataset in  $\mathbf{R}$ . Let  $f$  be a normal density with mean*

$$\frac{1}{n} \sum_{k=1}^n x^k$$

*and covariance*

$$\frac{1}{n} \sum_{k=1}^n \left( x^k - \frac{1}{n} \sum_{k=1}^n x^k \right)^2.$$

*Then  $f$  is a maximum likelihood normal density.*

*Proof.* Every normal density has two parameters: the mean and the covariance. If the likelihood of one normal is less than or equal to the likelihood of another, then so is are their log likelihoods. Let  $f$  be a normal density with parameter  $\mu$  and  $\sigma^2$ . We express the log likelihood of  $f$  by

$$\sum_{k=1}^n \left( \frac{1}{2\sigma^2} (x^k - \mu)^2 - \frac{1}{2} \log 2\pi\sigma^2 \right)$$

The partial derivative of the log likelihood with respect to the mean  $(\partial_\mu \ell) : \mathbf{R}^2 \rightarrow \mathbf{R}$  is

$$(\partial_\mu \ell)(\mu, \sigma^2) = - \sum_{k=1}^n \frac{1}{\sigma^2} (x - \mu)$$

and with respect to the covariance  $(\partial_{\sigma^2} \ell) : \mathbf{R}^2 \rightarrow \mathbf{R}$  is

$$(\partial_{\sigma^2} \ell)(\mu, \sigma^2) = \left( \frac{-1}{2(\sigma^2)^2} \sum_{k=1}^n (x^k - \mu)^2 \right) - \frac{1}{2\sigma^2}$$

We are interested in finding  $\mu_0 \in \mathbf{R}$  and  $\sigma_0^2 > 0$ , at which  $\partial_\mu \ell(\mu_0, \sigma_0^2) = 0$  and  $\partial_{\sigma^2} \ell(\mu_0, \sigma_0^2) = 0$ . So we have two equations. First, notice that  $\partial_\mu \ell$  is zero if and only if its first argument (the mean) is  $\frac{1}{n} \sum_{k=1}^n x^k$ . Second, notice that for all  $\mu, \sigma^2$ ,  $\partial_{\sigma^2} \ell$  is zero if and only if

$$\sigma^2 = \sum_{k=1}^n (x^k - \mu)^2.$$

So the pair

$$\left( \frac{1}{n} \sum_{k=1}^n x^k, \frac{1}{n} \sum_{k=1}^n (x_k - \frac{1}{n} \sum_{k=1}^n x^k)^2 \right)$$

is a stationary point of  $\ell$ . □

