Why

We want a norm on the vector space of continuous functions.

Definition

Consider a function from a closed real interval to the real numbers. The *absolute supremum* of the function is the supremum of the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

Proposition 1. *The functional mapping $f \in C[a, b]$ to its absolute supremum is a norm.*

Proof. Let R denote the set of real numbers. Define $\phi : C[a, b] \to R$ by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a, b]\}.$$

1. $|f(x)| \geq 0$ for all $x \in [a, b]$, so $\phi(f) \geq 0$.
2. If $\phi(f) = 0$ then $|f(x)| \leq 0$ for all x and so $f(x) = 0$ for all $x \in [a, b]$. If $f = 0$, then $|f(x)| = 0$ for all $x \in [a, b]$
3. For all α real, $|\alpha f(x)| = |\alpha||f(x)|$. So $\phi(\alpha f) = |\alpha|\phi(f)$
4. For all $f, g \in C[a, b]$, and $x \in [a, b]$, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ by the triangle inequality for absolute value. Thus,

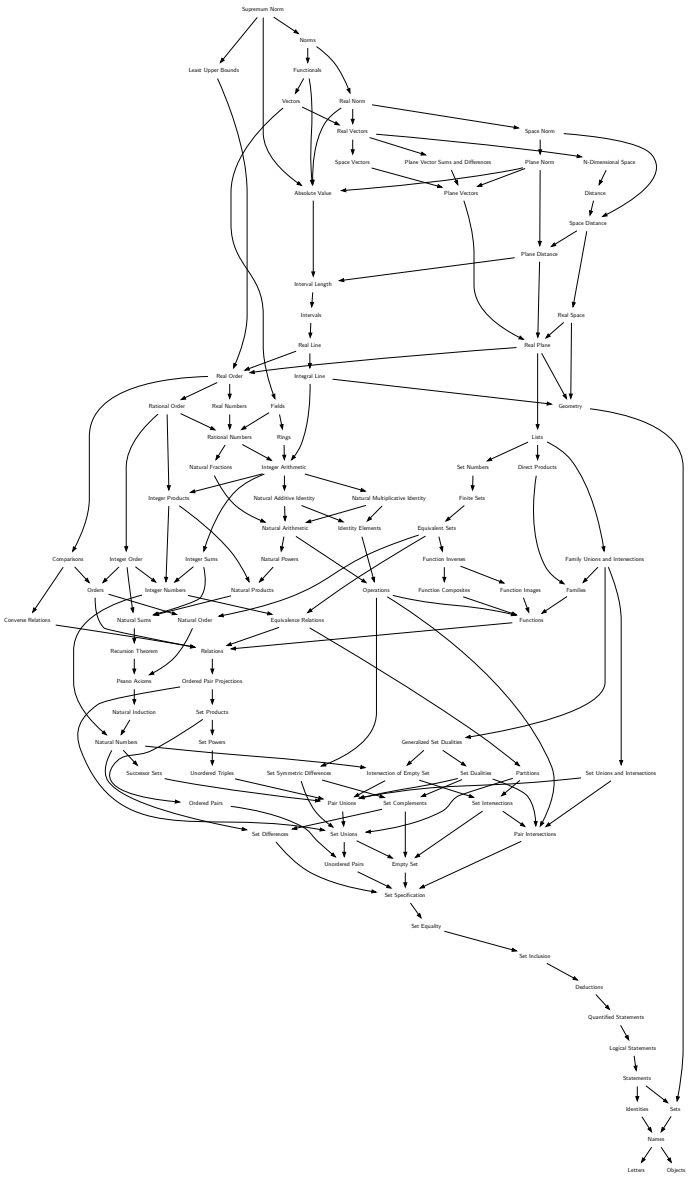
$$\begin{aligned}\phi(f + g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a, b]\} \\ &\leq \sup\{|f(x)| \mid x \in [a, b]\} + \sup\{|g(x)| \mid x \in [a, b]\} \\ &= \phi(f) + \phi(g)\end{aligned}$$

□

We call the functional ϕ defined above the *supremum norm.*

Notation

Let $f \in C[a, b]$. We denote the supremum norm of f by $\|f\|_{\sup}$.


Supremum Norm
Norms
Least Upper Bounds
Functionals
Vectors
Real Norm
Real Vectors
Space Norm
Space Vectors
Plane Vector Sums and Differences
Plane Norm
N-Dimensional Space
Absolute Value
Plane Vectors
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Geometry
Rational Order
Real Numbers
Fields
Rational Numbers
Rings
Lists
Natural Fractions
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Natural Arithmetic
Identity Elements
Equivalent Sets
Comparisons
Integer Order
Integer Sums
Natural Powers
Function Inverses
Family Unions and Intersections
Orders
Integer Numbers
Natural Products
Operations
Function Composites
Function Images
Families
Converse Relations
Natural Sums
Natural Order
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Successor Sets
Unordered Triples
Set Symmetric Differences
Intersection of Empty Set
Set Dualities
Partitions
Set Unions and Intersections
Ordered Pairs
Pair Unions
Set Complements
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects