The Bourbaki Project

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Contents

Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *symbol* is a written mark. A *script* is a collection symbols called *letters*. In *phonetic* languages the letters correspond to sounds and rules for composing these letters into successions called written words. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into written words which are meant to denote the spoken words of the English language. The written words on this page are several letters one after the other. For example, the word “word” is composed of the letters “w”, “o”, “r”, “d”.

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. The *lower case latin letters.*

a	b	c	d	e	f	g	h	i
j	k	l	m	n	o	p	q	r
s	t	u	v	w	x	y	z	

And the *upper case latin letters.*

A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We also use the *Arabic numerals.*

0 1 2 3 4 5 6 7 8 9

Other symbols

We also use the following symbols.

$$' \quad (\quad) \quad \{ \quad \} \quad \vee \quad \wedge \quad \neg \quad \forall \quad \exists \quad \longrightarrow \quad \longleftrightarrow \quad = \quad \in \quad \to \quad \sim$$

Letters (1) does not immediately need any sheet.

Letters (1) is immediately needed by:

Names (3)

Letters (1) gives the following terms.

language
affections
spoken word
symbol
script
letters
phonetic
lower case latin letters
upper case latin letters
Arabic numerals

Letters

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Objects (2) does not immediately need any sheet.

Objects (2) is immediately needed by:

Definitions (13)

Names (3)

Objects (2) gives the following terms.

object
tangible
intangible

Objects

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an *accent* $'$ or subscripts or superscripts. When we write the symbols we say that the composite symbol formed *denotes* the object. We call it the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{\text{A}}$ or $\boxed{\text{A'}}$ or $\boxed{\text{A}_0}$. The box works well to group the symbols and clarifies that $\boxed{\text{A}}\boxed{\text{A}}$ is different from $\boxed{\text{AA}}$. But experience shows that we need not use boxes.

We indicate a name for an object with italics. Instead of $\boxed{\text{A'}}$ we use A', instead of $\boxed{\text{A}_0}$ we use A_0. Experience shows that this subtlety is enough for clarity and it agrees with traditional and modern practice. Other examples include A'', A''', A'''', B, C, D, E, F, f, f' f_a.

No repetitions

We never use the same name to refer to two different objects. Using the same name for two different objects causes confusion. We make clear when we reuse symbols to mean different objects. We tend to introduce the names used at the beginning of a paragraph or section.

Names are objects

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can reference themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the marks on the page which are meant to look similar are the several uses of a name.

Names as placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is best understood by example.

Names (3) immediately needs:

Letters (1)

Objects (2)

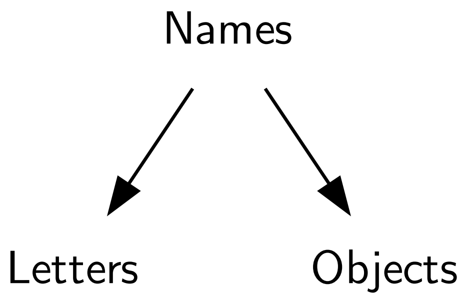
Names (3) is immediately needed by:

Identities (4)

Sets (5)

Names (3) gives the following terms.

accent
denotes
name
assertion
names
accent
letter
terms
relations
placeholder


Names
Letters
Objects

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

Symmetry

Let A denote an object and let B denote an object. "A equals B" means the same as "B equals A". The identity of the names is not dependent on the order in which the names are given. We call this the *symmetry of identity*. It means we can switch the spots of A and B and say the same thing. In other words, there are two ways to make the statement.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself. This fact is called the *reflexivity of identity*. A name is equal to itself because an object is itself.

Identities (4) immediately needs:

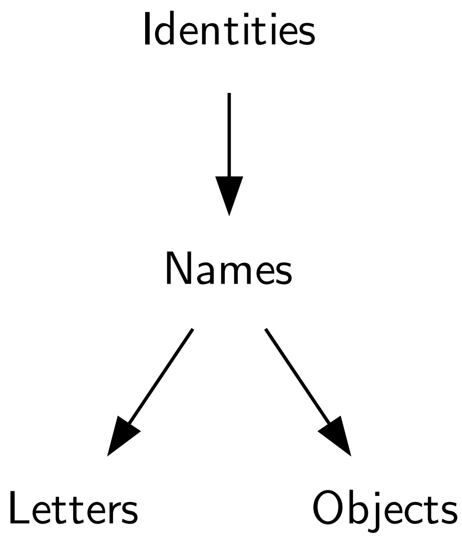
Names (3)

Identities (4) is immediately needed by:

Equation Solutions (**??**)
Statements (7)

Identities (4) gives the following terms.

is
equation
indeterminate
is
equal
name
identity
symmetry of identity
reflexivity of identity
reflexive
symmetric
transitive
equals
reflexive
symmetric
transitive


Identities
Names
Letters
Objects

Why

We want to talk about none, one, or several objects considered together, as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set*. We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

Principle 1 (Existence of Sets)**.** *Intangible groups exist.*

A set may have other sets as its members. This is subtle but becomes familiar. We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object belongs to the set. In other words, the object is a member or an element of the set. We say "The object denoted by a belongs to the set denoted by A".

Not symmetric

Notice that belonging is not symmetric. Saying "the object denoted by a belongs to the set denoted by A" does not mean the same as "the set denoted by A belongs to the object denoted by a" In fact, the latter sentence is nonsensical unless the object denoted by a is also a set.

Not transitive

Let a denote an object and let A and B both denote sets. If the object denoted by a is "a part of" a the set denoted by A, and the set denoted by A is "a part of" the set denoted by B, then usual English usage would suggest that a is "a part of" the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing. The relation of belonging is not quite this. If a thing is an element of a thing, that second thing may be an element of the third thing, but this does not mean that the the first thing is an element of the third thing.

Sets (5) immediately needs:

Names (3)

Sets (5) is immediately needed by:

Set Examples (6)

Statements (7)

Sets (5) gives the following terms.

set
belong
members
elements
in
empty
nonempty

$$
\begin{tikzcd}
 & \text{Sets} \arrow[d] & \\
 & \text{Names} \arrow[dl] \arrow[dr] & \\
\text{Letters} & & \text{Objects}
\end{tikzcd}
$$

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

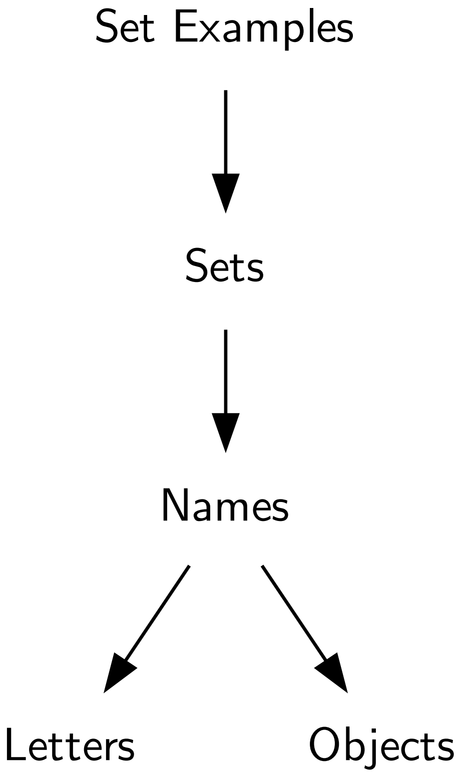
Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of lower case latin letters (introduced in Letters): a, b, c, . . . , x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Set Examples (6) immediately needs:

Sets (5)

Set Examples (6) is not immediately needed by any sheet.

Set Examples (6) gives no terms.


Set Examples
Sets
Names
Letters
Objects

Why

We want symbols to represent identity and belonging.

Definition

In the English language, nouns are words that name people, places and things. In these sheets, names (see Names) serve the role of nouns. In the English language, verbs are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings. As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense. A *statement* is a succession of symbols.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object (see Identities). We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And if we read it aloud, we would read a as "the object denoted by a", then $=$ as "is", then b as "the object denoted by b". Altogether then, "the object denoted by a is the object denoted by b." We might box these three symbols $\boxed{a = b}$ to make clear that they are meant to be read together, but experience shows that (as with English sentences and words) we do not need boxes.

The symbol $=$ is (appropriately) a symmetric symbol. If we flip it left and right, it is the same symbol. This reflects the symmetry of the English sentences represented (see Identities). The symbols $a = b$ mean the same as the symbols $b = a$.

Belonging

As a second example, consider the symbol $\in$. Let a denote an object and let A denote a set. We agree that $\in$ means "belongs to" in the sense of "is an element of" or "is a member of" (see Sets). Then we write $a \in A$. We read these symbols as "the object denoted by a belongs to the set denoted by A".[1]

The symbol $\in$ is not symmetric. If we flip it left and right it looks different. This reflects that $a \in A$ does not the mean the same as $A \in a$ (see Sets). As with english words, the order of symbols is significant. The word "word" is not the same as the word "drow". Our symbolism for belonging reflects the concept's lack of symmetry.

[1]The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for the ancient Greek word $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means, roughly, "belongs". Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of".

Statements (7) immediately needs:

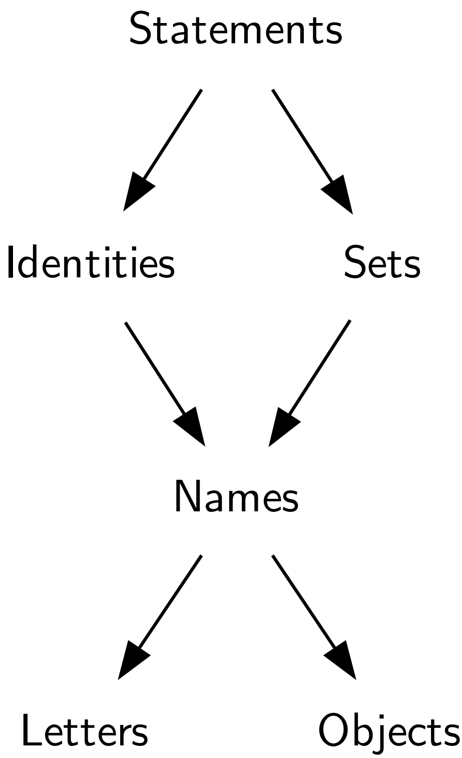
Identities (4)
Sets (5)

Statements (7) is immediately needed by:

Logical Statements (8)

Statements (7) gives the following terms.

statement
relational symbol
name symbol
relational symbol
name symbol
relational symbols
terminal
assertion
membership assertion
identity assertion
primitive sentence
logical form
sentence
belongs to
member


Statements
Identities
Sets
Names
Letters
Objects

Why

We want symbols for “and”, “or”, “not”, and “implies”.[2]

Overview

We call $=$ and $\in$ *relational symbols.* They say how the objects denoted by a pair of placeholder names relate to each other in the sense of being or belonging. We call $\_ = \_$ and $\_ \in \_$ *simple statements.* They denote simple sentences “the object denoted by _ is the object denoted by _” and “the object denoted by _ belongs to the set denoted by _”. The symbols introduced here are *logical symbols* and statements using them are *logical statements.*

Conjunction

Consider the symbol $\land$. We will agree that it means “and”. If we want to make two simple statements like $a = b$ and $a \in A$ at once, we write write $(a = b) \land (a \in A)$. The symbol $\land$ is symmetric, reflecting the fact that a statement like $(a \in A) \land (a = b)$ means the same as $(a = b) \land (a \in A)$.

Disjunction

Consider the symbol $\lor$. We will agree that it means “or” in the sense of either one, the other, or both. If we want to say that

[2]This sheet does not explain logic. In the next edition there will be several more sheets serving this function.

at least one of the simple statements like $a = b$ and $a \in A$, we write write $(a = b) \vee (a \in A)$. The symbol $\vee$ is symmetric, reflecting the fact that a statement like $(a \in A) \vee (a = b)$ means the same as $(a = b) \vee (a \in A)$.

Negation

Consider the symbol $\neg$. We will agree that it means "not". We will use it to say that one object "is not" another object and one object "does not belong to" another object. If we want to say the opposite of a simple statement like $a = b$ we will write $\neg(a = b)$. We read it aloud as "not a is b" or (the more desirable) "a is not b". Similarly, $\neg(a \in A)$ we read as "not, the object denoted by a belongs to the set denoted by A". Again, the more desirable english expression is something like "the object denoted by a does not belong to the set A" For these reasons, we introduce two new symbols $\neq$ and $\notin$. $a \neq b$ means $\neg(a = b)$ and $a \notin A$ means $\neg(a \in A)$.

Implication

Consider the symbol $\longrightarrow$. We will agree that it means "implies". For example $(a \in A) \longrightarrow (a \in B)$ means "the object denoted by a belongs to the object denoted by A implies the object denoted by a belongs to the set denoted by B" It is the same as $(\neg(a \in A)) \vee (a \in B)$. In other words, if $a \in A$, then always $a \in B$. The symbol $\longrightarrow$ is not symmetric, since implication is not symmetric. The symbol $\longleftrightarrow$ means "if and only if".

Logical Statements (8) immediately needs:

Statements (7)

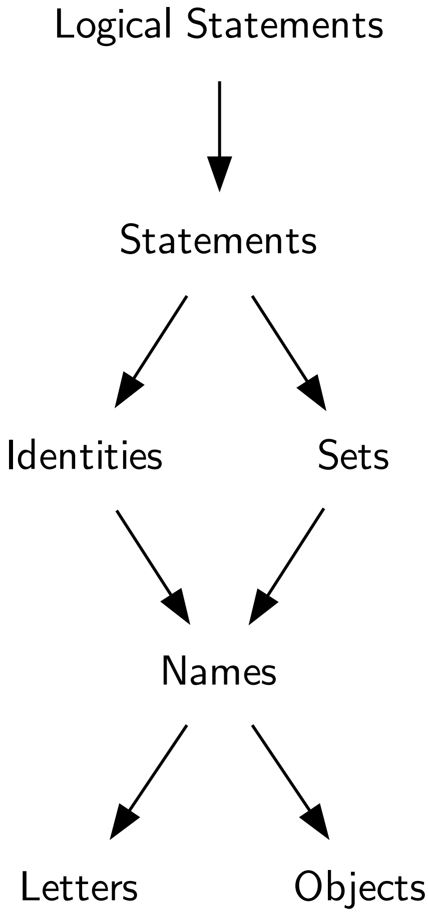
Logical Statements (8) is immediately needed by:

Deductions (9)

Quantified Statements (10)

Logical Statements (8) gives the following terms.

relational symbols
simple statements
logical symbols
logical statements


Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to make conclusions.

Definition

Suppose we have a list of logical statements. We want to write down o

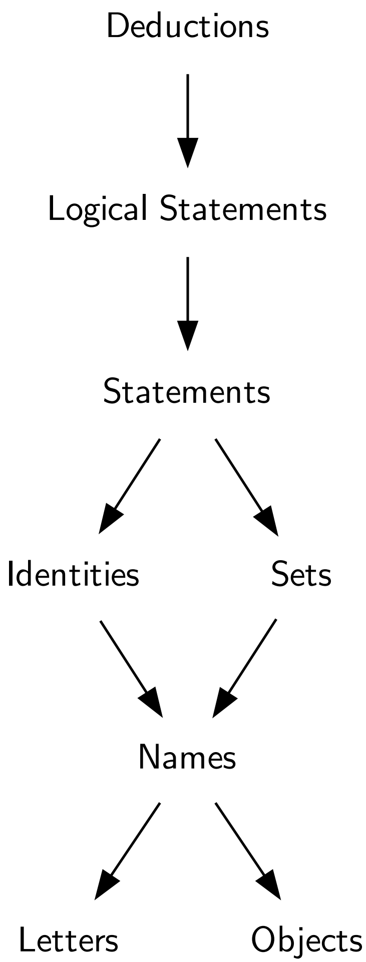
Deductions (9) immediately needs:

Logical Statements (8)

Deductions (9) is immediately needed by:

Accounts (11)

Deductions (9) gives no terms.


Deductions
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want symbols for talking about the existence of objects and for making statements which hold for all objects.[3]

Definition

If we say there exists an object that is blue, we mean the same as if we say that not every object is not blue. If we say that every object is blue, we mean the same as if we say there does not exist an object that is not blue. In other words, "there exists an object so that _" is the same as "not every object is not _". Or, "every object is _" is the same as "there does not exist an object that is _".

When we assert something of every object we also assert the nonexistence of the contrary of that assertion. And likewise when we assert that an object exists with some conditions, we assert that not every object exists without that condition.

The content of our assertions will be logical statements (see Logical Statements) and when we want to make them for all objects or for no object we will use the following symbols. The symbols introduced here are *quantifier symbols* and statements using them are *quantified statements*.

[3] This sheet does not explain quantifiers. In the next edition there will be several more sheets serving this function

Existential Quantifier

Consider the symbol $\exists$. We agree that it means "there exists an object". We write $(\exists x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\exists x)(x \in A)$ to mean "there exists an object in the set denoted by A" We call $\exists$ the *existential quantifier* symbol.

Universal Quantifier

Consider the symbol $\forall$. We agree that it means "for every object". We write $(\forall x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\forall x)((x \in A) \longrightarrow (x \in B)$ to mean, "every object which is in the set denoted by A is in the set denoted by B". We call $\forall$ the *universal quantifier* symbol.

Binding

When we have a name following a $\forall$ or $\exists$ we say that the name is *bound.* If a name is bound, then the statement uses it in one sense but not in another. The name is only used in that single statement. Regular names in statements we call *unbound*

Negations

The statement $\neg(\forall x)(\_)$ is the same as $(\exists x)(\neg(\_))$ and $\neg(\exists x)(\_)$ is the same as $(\forall x)(\neg(\_))$.

Quantified Statements (10) immediately needs:

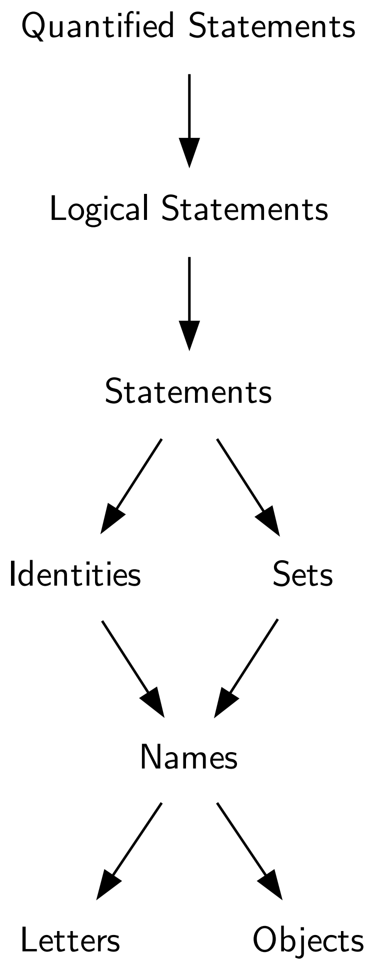
Logical Statements (8)

Quantified Statements (10) is immediately needed by:

Accounts (11)

Quantified Statements (10) gives the following terms.

quantifier symbols
quantified statements
existential quantifier
universal quantifier
bound
unbound


Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to succinctly and clearly make several statements about objects and sets. We want to track the names we use, taking care to avoid using the same name twice.

Definition

An *account*[4] is a list of naming, logical, and quantified statements. We use the words "let _ denote an _" to introduce a name as a placeholder for a thing, and we use the symbols $\_ = \_$ and $\_ \in \_$ to denote statements of identity and belonging. In other words, we have three sentence kinds to record.

1. **Names.** State we are using a name.
2. **Identity.** We want to make statements of identity.
3. **Belonging.** We want to make statements of belonging.

Our main purpose is to keep a list names, of quantified, logical and simple statments about them, and then statements we can deduce from these. In particular we want to group our name usage. In the English language we use paragraphs or sections to do so. In these sheets, we will use accounts. We will list the statements and label each with Arabic numerals (see Letters). which will be a list of statements, each of which is labeled by an Arabic numeral (see Letters).

[4] This sheet will be expanded in future editions.

Experience suggests that we start with an example. Suppose we want to summarize the following english language description of some names and objects.

> Denote an object by a. Also, denote the same object by b Also, denote a set by A. Also, the object denoted by a is an element of the set denoted by A. Also denote an object by c. Also c is the same object as b.

In our usual manner of speaking, we drop the word "also". In these sheets, we translate each of the sentences into our symbols. For names we use, we write `name` in that font followed by the name. For logical statements we `have`, we write `have` followed by the logical statement. For deductions we write `thus` followed by the conclusion and then `by` followed by the Arabic numerals of the premisses. So we write:

Account 1. First Example

1	`name`	a	
2	`name`	b	
3	`have`	$a = b$	
4	`name`	A	
5	`have`	$a \in A$	
6	`name`	c	
7	`have`	$c = b$	
8	`thus`	$a = c$	`by 3,7`

Accounts (11) immediately needs:

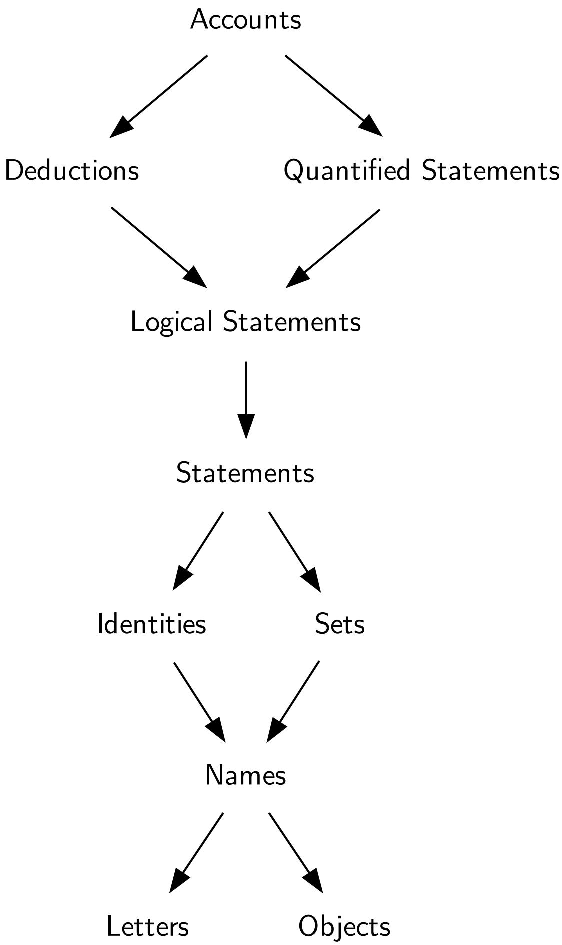
Deductions (9)
Quantified Statements (10)

Accounts (11) is immediately needed by:

Standardized Accounts (12)

Accounts (11) gives the following terms.

$account$


Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to do our best to have only one way to write accounts.

Definition

A *standard account*[5] lists all names, then lists all premisses, then lists all conclusions.

Example

Consider the account.

Account 2. First Example

1	`name`	a		
2	`name`	b		
3	`have`	$a = b$		
4	`name`	c		
5	`have`	$c = b$		
6	`thus`	$a = c$	`by`	`3,5`

[5] This sheet will be expanded in future editions.

Account 3. Standardized First Example

1	`name`	a	
2	`name`	b	
3	`have`	$a = b$	
4	`name`	c	
5	`have`	$c = b$	
6	`thus`	$a = c$	`by 3,5`

We can abbreviate the names:

Account 4. Abbreviated First Example

1-3	`name`	a, b, c	
4	`have`	$a = b$	
5	`have`	$c = b$	
6	`thus`	$a = c$	`by 4,5,IdentityAxioms:1`

Standardized Accounts (12) immediately needs:

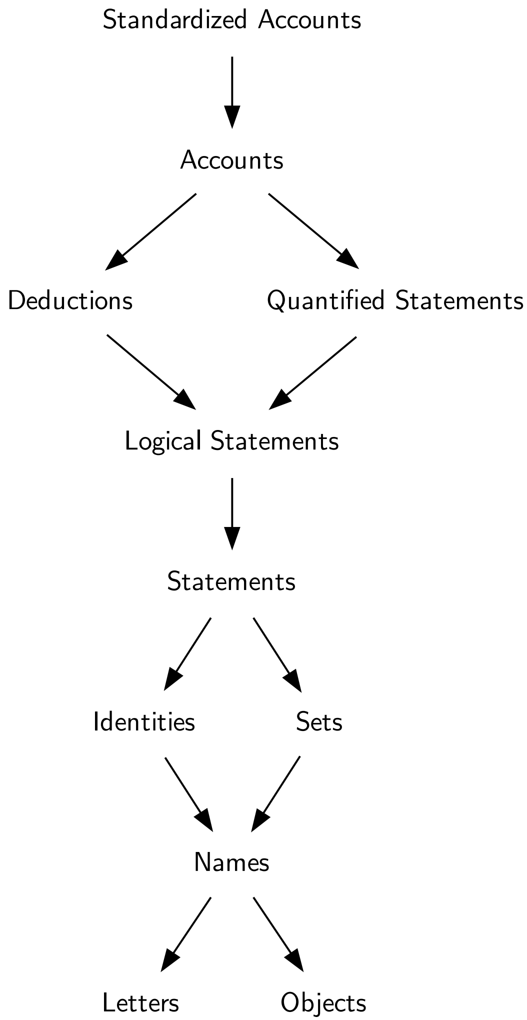
Accounts (11)

Standardized Accounts (12) is immediately needed by:

Set Inclusion (14)

Standardized Accounts (12) gives the following terms.

standard account


Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

All definitions are *nominal.* They are made to give us language and to save space. They are labeled with arabic numerals to enable to us to reference them throughout the project.

They are given set apart from the text with the bold word **Definition #.** and then often with naming or sumamry text between parantheses. The definition body statement is in italics.

Definitions (13) immediately needs:

Objects (2)

Definitions (13) is immediately needed by:

Set Inclusion (14)

Definitions (13) gives the following terms.

$nominal$

Definitions

$$\Big\downarrow$$

Objects

Why

We want language for all of the elements of a first set being the elements of a second set.

Definition

Denote a set by A and a set by B.

Definition 1 (Subsets). If every element of the set denoted by A is an element of the set denoted by B, then we say that the set denoted by A is a *subset* of the set denoted by B.

We say that the set denoted by A is *included* in the set denoted by B. We say that the set denoted by B is a *superset* of the set denoted by A or that the set denoted by B *includes* the set denoted by A.

Every set is included in and includes itself.

Notation

Let A denote a set and B denote a set. We denote that the set A is included in the set B by $A \subset B$. In other words, $A \subset B$ means $(\forall x)((x \in A) \longrightarrow (x \in B))$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A". $B \supset A$ also means $(\forall x)((x \in A) \longrightarrow (x \in B))$.

Properties

There are some properties that our intuition suggests inclusion should have. First, every set should include itself. We describe this fact by saying that inclusion is *reflexive.*

Proposition 1 (Reflexive). *Every set is included in itself*

Proof. `(1) name` A; `(2) have` $(\forall x)(x \in A \longrightarrow x \in A)$; `(3) thus` $A \subset A$ `by` Definition 1. □

Next, we expect that if one set is included in another, This fact is described by saying that inclusion is *transitive*

Proposition 2 (Transitive). *If a one set is included in another, and the latter in yet another, then the former is included in the last.*

Proof. `(1) name` A, B, C; `(2) have` $A \subset B$ `(3) have` $B \subset C$ `(4) thus` $A \subset C$ `by` modus ponens. □

Equality (=) shares these two properties. Let A denote an object. Then $A = A$. Let B and C also denote objects. If $A = B$ and $B = C$, then $A = C$. Of course, inclusion is not symmetric.. Belonging ($\in$) may be, but need not be reflexive and transitive.

Set Inclusion (14) immediately needs:

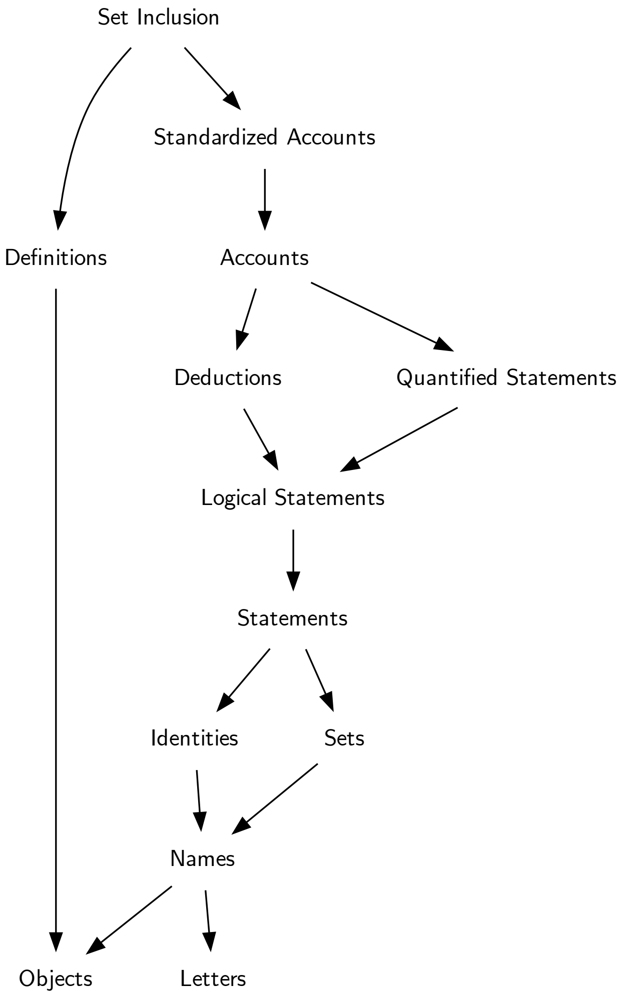
Definitions (13)
Standardized Accounts (12)

Set Inclusion (14) is immediately needed by:

Set Equality (15)
Set Specification (16)

Set Inclusion (14) gives the following terms.

subset
included
superset
includes
improper subsets
proper subsets
reflexive
transitive


Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Objects
Letters

Why

When are two sets the same?

Definition

Let A and B denote sets. If $A = B$ then every element of A is an element of B and every element of B is an element of A. In other words, $(A = B) \longrightarrow ((A \subset B) \land (B \subset A))$.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Then $A = B$? We define it to be so. Sets are determined by their members.

Principle 2 (Extension). *Sets are the same if every member of one is a member of the other and vice versa.*

In other words, two sets are identical if and only if every element of one is an element of the other. This principle is sometimes called the *principle of extension*. We refer to the elements of a set as its *extension*. Roughly speaking, we have declared that if we know the extension then we know the set. A set is determined by its extension.

Deductive principle

We can use this definition to deduce $A = B$ if we first deduce $A \subset B$ and $B \subset A$. With these two implications, we use the principle of extension to conclude that the sets are the same.

In other words, $(A = B) \longleftrightarrow ((A \subset B) \land (B \subset A))$. We also describe this fact by saying that inclusion ($\subset$) is *antisymmetric*.

Belonging and sets compared with ancestry and humans

Compare the principle of extension for identifying sets from their elements with an analogous principle for identifying people from their ancestors.

We can consider a person's ancestors. Namely, the person's parents, grandparents, great grandparents and so on. It is clear that if we label the same human with two names A and B, then A and B have the same ancestors. In other words, same human implies same ancestors. This is the analog of "if two sets are equal they have the same members".

On the other hand, if we have two people denoted by A and B, and we know that A has the same ancestors as B, we can not conclude that A and B denote the same human. For example, siblings have the same ancestors but are different people. This direction, same ancestors implies same human, is the analogue of "if they have the same elements, two sets are the same". It is false for humans and ancestors, but we define it to be true for sets and members.

The principle of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

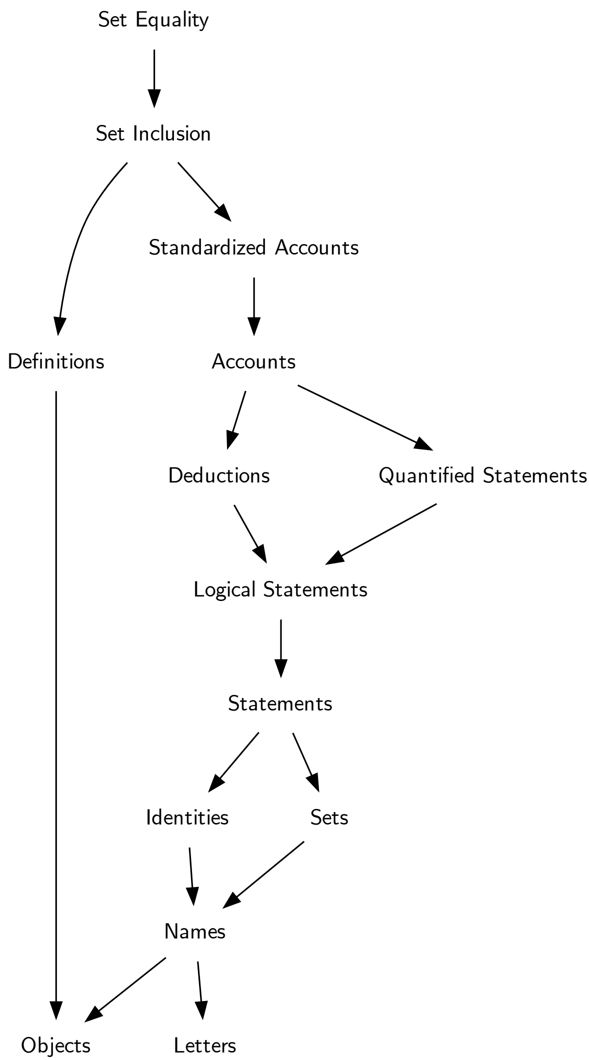
Set Equality (15) immediately needs:

Set Inclusion (14)

Set Equality (15) is not immediately needed by any sheet.

Set Equality (15) gives the following terms.

principle of extension
extension
antisymmetric



Set Equality
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Objects
Letters

Why

We want to construct new sets out of old ones. So, can we always construct subsets?

Definition

We will say that we can. More specifically, if we have a set and some statement which may be true or false for the elements of that set, a set exists containing all and only the elements for which the statement is true.

Roughly speaking, the principle is like this. We have a set which contains some objects. Suppose the set of playing cards in a usual deck exists. We are taking as a principle that the set of all fives exists, so does the set of all fours, as does the set of all hearts, and the set of all face cards. Roughly, the corresponding statements are "it is a five", "it is a four", "it is a heart", and "it is a face card".

Principle 3 (Specification)**.** *For any statement and any set, there is a subset whose elements satisfy the statement.*

We call this the *principle of specification.* We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence. The principle of extension (see Set Equality) says that this set is unique. All basic principles about sets (other than the principle of extension, see Set Equality) assert that we can construct new sets out of old ones in reasonable ways.

Notation

Let A denote a set. Let s denote a statement in which the symbol x and A appear unbound. We assert that there is a set, denote it by B, for which belonging is equivalent to membership in A and s. In other words,

$$(\forall x)((x \in B) \longleftrightarrow ((x \in A) \land s(x))).$$

We denote B by $\{x \in A \mid s(x)\}$. We read the symbol $\mid$ aloud as "such that." We read the whole notation aloud as "a in A such that..." We call it *set-builder notation*.

Nothing contains everything

As an example of the principle of specification and important consequence, consider the statement $x \notin x$. Using this statement and the principle of specification, we can prove that there is not set which contains every thing.

Proposition 3. *No set contains all sets.*[6]

Proof. Suppose there exists a set, denote it A which contains all sets. In other words, suppose $(\exists A)(\forall x)(x \in A)$. Use the principle of specification to construct $B = \{x \in A \mid x \notin x\}$. So $(\forall x)(x \in B \longleftrightarrow (x \in A \land x \notin x))$ In particular, $(B \in B \longleftrightarrow (B \in A \land B \notin B))$. So $B \notin A$. $\square$

[6]We might call such a set, if we admitted its existence, a *universe of discourse* or *universal set*. With the principle of specification, a "principle of a universal set" would give a contradiction (called *Russell's paradox*).

Set Specification (16) immediately needs:

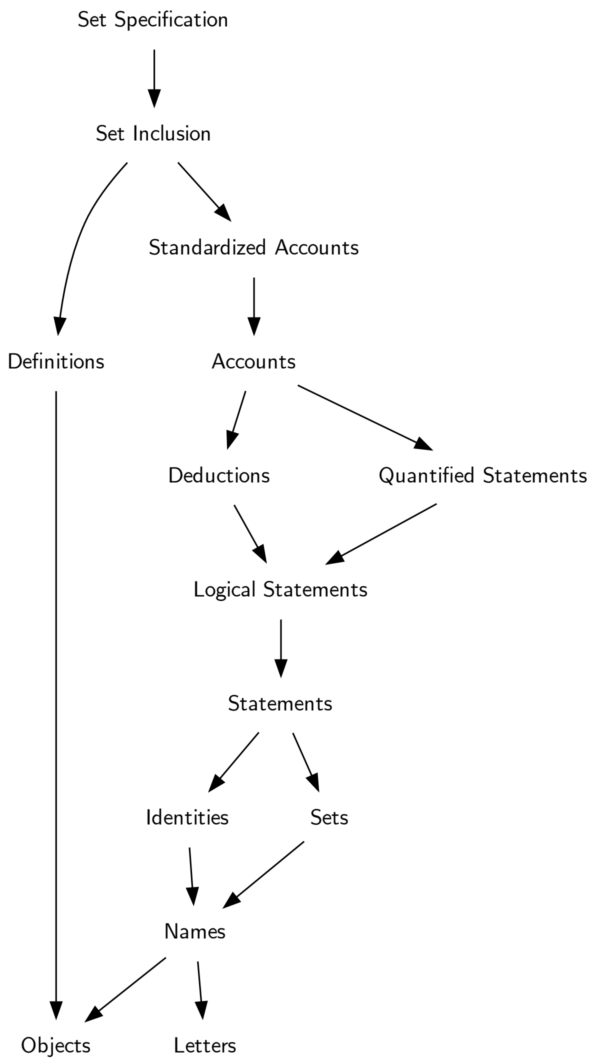
Set Inclusion (14)

Set Specification (16) is immediately needed by:

Empty Set (17)
Pair Intersections (22)
Set Differences (26)
Unordered Pairs (18)

Set Specification (16) gives the following terms.

set-builder notation
principle of specification
specifying
universe of discourse
universal set
Russell's paradox


Set Specification
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
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Why

Can a set have no elements?

Definition

Sure. A set exists by the principle of existence (see Sets); denote it by A. Specify elements (see Set Specification) of any set that exists using the universally false statement $x \neq x$. We denote that set by $\{x \in A \mid x \neq x\}$. It has no elements. In other words, $(\forall x)(x \notin A)$. The principle of extension (see Set Equality) says that the set obtained is unique (contradiction).[7]

Definition 2 (Empty Set). We call the unique set with no elements *the empty set.*

Notation

We denote the empty set by $\varnothing$. In other words, in all future accounts (see Accounts), there are two implicit lines. First, "`name` $\varnothing$" and second "`have` $(\forall x)(x \notin \varnothing)$".

Properties

It is immediate from our definition of the empty set and of the definition of inclusion (see Set Inclusion) that the empty set is included in every set (including itself).

Proposition 4. $(\forall A)(\varnothing \subset A)$

[7] This account will be expanded in the next edition.

Proof. Suppose toward contradiction that $\varnothing \not\subseteq A$. Then there exists $y \in \varnothing$ such that $y \notin A$. But this is impossible, since $(\forall x)(x \notin \varnothing)$. $\square$

Empty Set (17) immediately needs:

Set Specification (16)

Empty Set (17) is immediately needed by:

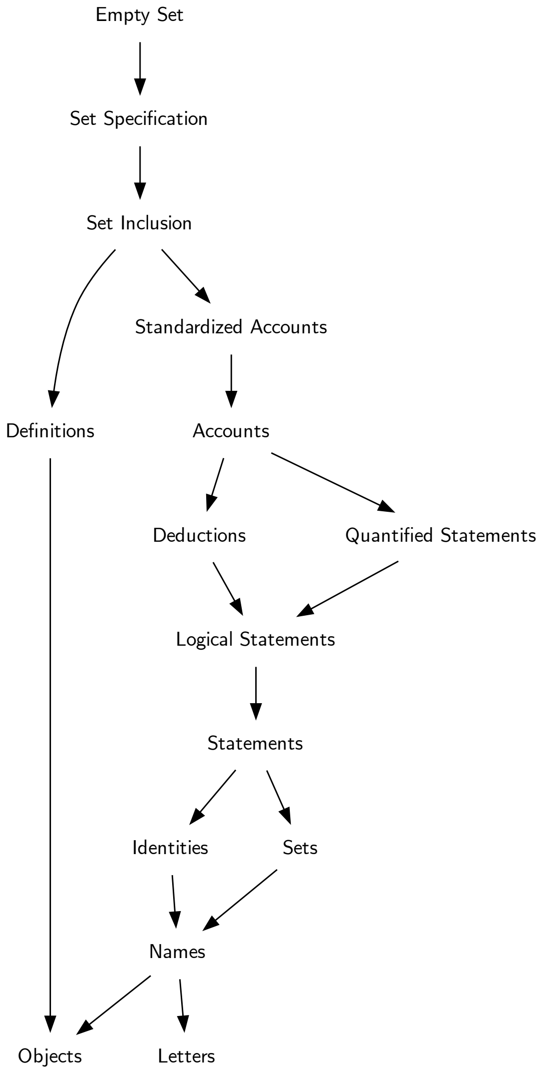
Set Complements (27)

Set Unions (19)

Empty Set (17) gives the following terms.

the empty set

empty set


Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
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Names
Objects
Letters

Why

Can we always make a set out of two objects?

Definition

We say yes.

Principle 4 (Pairing). *Given two objects, there exists a set containing them.*

We refer to this as the *principle of pairing.* Denote one object by a and the other by b. This principle gives us the existence of a set that contains the objects. The principle of specification (see Set Specification) gives use the subset for the statement "$x = a \lor x = b$". The principle of extension (see Set Equality) says this set is unique. We call this set a *pair* or an *unordered pair.*

If the object denoted by a is the object denoted by b, then we call the pair the *singleton* of the object denoted by a. Every element of the singleton of the object denoted by a is a.

In other words, the principle of pairing says that every object is an element of some set. That set may be the singleton, or it may be the pair with any other object. We can construct several sets using this principle: the singleton of the object denoted by a, the singleton of the singleton of the object denoted by a, the singleton of the singleton of the singleton of the object denoted by a, and so on.

Notation

We denote the set which contains a and b as elements and nothing else by $\{a, b\}$. The pair of a with itself is the set $\{a, a\}$ is the singleton of a. We denote it by $\{x\}$. The principle of pairing also says that $\{\{a\}\}$ exists and $\{\{\{a\}\}\}$ exists, as well as $\{a, \{a\}\}$.

Note well that $a \neq \{a\}$. a denotes the object a. $\{a\}$ denotes the set whose only element is a. In other words $(\forall x)(x \in \{a\} \longleftrightarrow x = a)$. The moral is that a sack with a potato is not the same thing as a potato.

Unordered Pairs (18) immediately needs:

Set Specification (16)

Unordered Pairs (18) is immediately needed by:

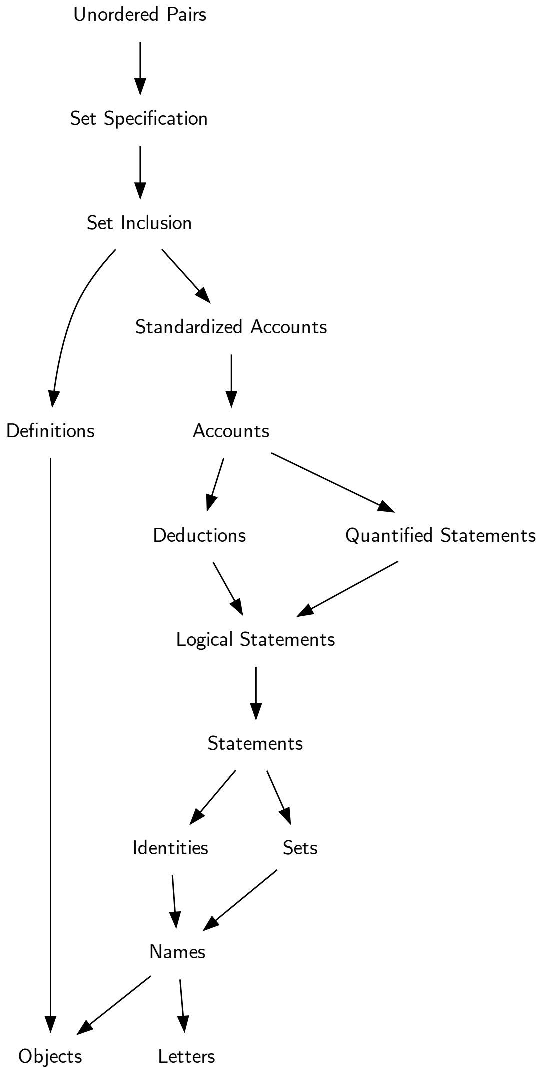
Graphs (**??**)

Ordered Pairs (38)

Set Unions (19)

Unordered Pairs (18) gives the following terms.

principle of pairing
pair
unordered pair
singleton


Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Deductions
Quantified Statements
Logical Statements
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Why

Can we combine sets?

Definition

We say yes. For example, if we have a first set denoted A and a second set denoted B, then we want a third set including all the elements of the set denoted by A and the elements of the set denoted by B. If an object appears in the set denoted by A and in the set denoted by B, it appears in the new set. If an object appears in one set but not the other, it appears in the new set. Indeed, if we have a set of sets, the same should hold.

Principle 5 (Union). *Given a set of sets, there exists a set which contains all elements which belong to any of the sets.*

We call this the *principle of union.* If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the principle of union may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification (see Set Specification) to form the set which contains only those elements which are appear in at least one of any of the sets. The set is unique by the principle of extension. We call that unique set *the union* of the sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\bigcup \mathcal{A}$. So

$$(\forall x)((x \in (\bigcup \mathcal{A})) \longleftrightarrow (\exists A)((A \in \mathcal{A}) \wedge x \in A)).$$

Simple Facts

It is reasonable for the union of the empty set to be empty and for the union of the singleton of a set to be itself.

Proposition 5. $\bigcup \varnothing = \varnothing$

Proof. Immediate[8] $\square$

Proposition 6. $\bigcup \{A\} = A$

Proof. Immediate[9] $\square$

[8] Future editions will include the account.

[9] Future editions will include the account.

Set Unions (19) immediately needs:

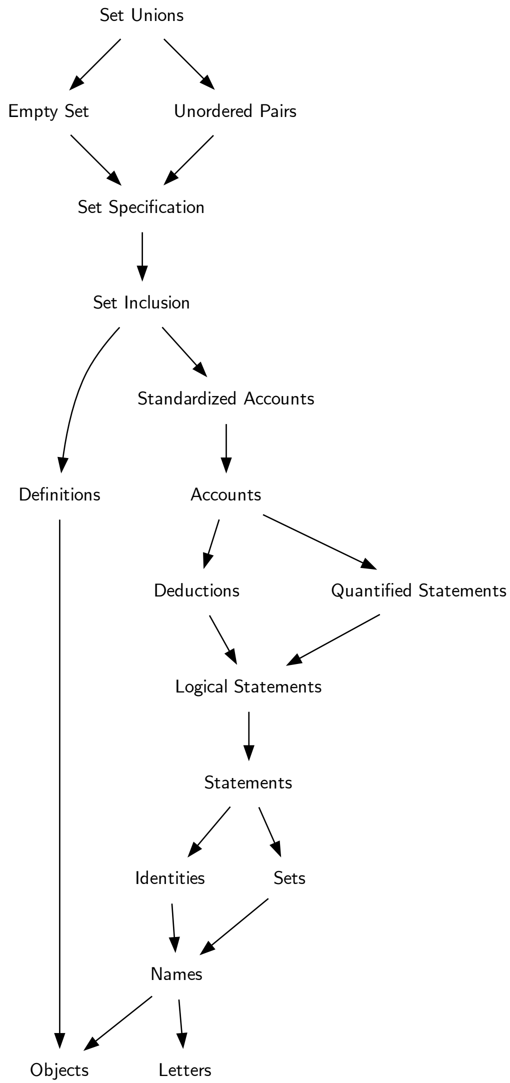
Empty Set (17)
Unordered Pairs (18)

Set Unions (19) is immediately needed by:

Graph Complements (**??**)
Ordered Pair Projections (41)
Pair Unions (20)
Partitions (29)
Set Symmetric Differences (32)
Trees (**??**)

Set Unions (19) gives the following terms.

principle of union
the union


Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
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Why

We often unite the elements of one set with another.

Discussion

Let A and B denote sets. We call $\cup\{A, B\}$ the *pair union* of A and B. We denote the union of the pair $\{A, B\}$ by $A \cup B$. Clearly the pair union does not depend on the order of A and B. In other words, $A \cup B = B \cup A$.

Facts

Here are some basic facts about unions of a pair of sets.[10] Let A and B denote sets.

Proposition 7 (Identity Element). $A \cup \varnothing = A$

Proposition 8 (Commutativity). $A \cup B = B \cup A$

Proposition 9 (Associativity). $(A \cup B) \cup C = A \cup (B \cup C)$

Proposition 10 (Idempotence). $A \cup A = A$.

Proposition 11. $A \subset B \longleftrightarrow A \cup B = B$

[10] Proofs will appear in the next edition.

Pair Unions (20) immediately needs:

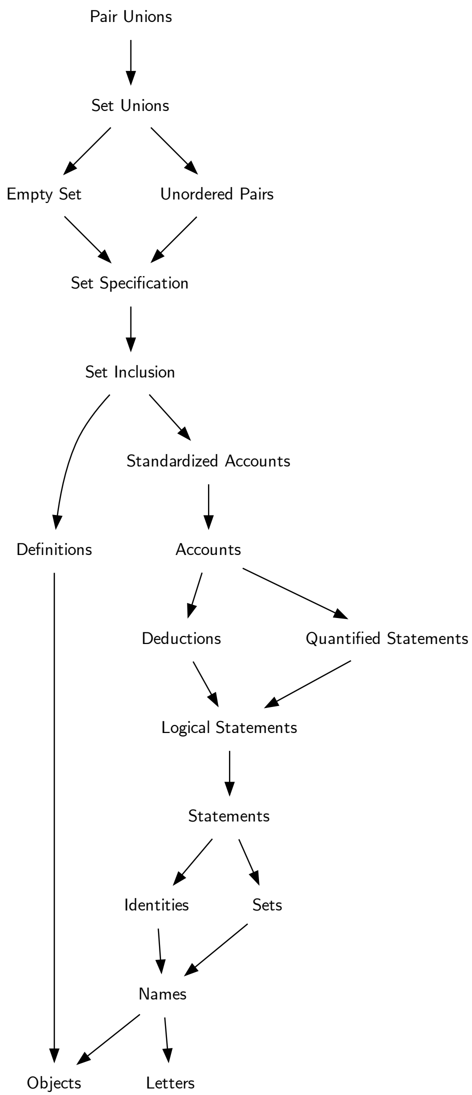
Set Unions (19)

Pair Unions (20) is immediately needed by:

Intersection of Empty Set (24)
Set Decompositions (28)
Set Dualities (30)
Set Unions and Intersections (25)
Successor Sets (58)
Unordered Triples (21)

Pair Unions (20) gives the following terms.

pair union


Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Deductions
Quantified Statements
Logical Statements
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Why

$\{a\} \cup \{b\} = \{a, b\}$

Definition

Let a, b and c denote objects. From the associativity of pair unions (see Pair Unions), we have

$$(\{a\} \cup \{b\}) \cup \{b\} = \{a\} \cup (\{b\} \cup \{c\}).$$

So we will drop the parentheses, and write $\{a\} \cup \{b\} \cup \{c\}$. We call such a set the *unordered triple* of a, b and c. The unordered triple of a, b and c is the set containing these elements and no others.

Notation

Such sets are so commonplace that we denote the unordered triple of a, b and c by $\{a, b, c\}$.

Quadruples

Let d denote an object. Again, the associativity of pair unions allows us to drop the parentheses from

$$(((\{a\} \cup \{b\}) \cup \{c\}) \cup \{d\})).$$

We can therefore write $\{a\} \cup \{b\} \cup \{c\} \cup \{d\}$ without ambiguity. We call this set the *unordered quadruple*. As before, the unordered quadruple contains of a, b, c and d contains a, b, c, and d and nothing besides these.

Notation

We denote the unordered quadruple of the objected denoted by a, b, c and d, denote this set by $\{a, b, c, d\}$.

The case of several named objects

In a similar way we speak of *unordered pentuples*, *unordered sextuples*, *unordered septuples* and so on. If we have several objects named, we denote the set containing these objects be writing their names in between the left brace $\{$ and right brace$\}$, separating the names by commas. For example, if we A, b, x and Y and z denote objects, then we denote the set containing these elements by

$$\{A, b, x, Y, z\}.$$

Unordered Triples (21) immediately needs:

Pair Unions (20)

Unordered Triples (21) is immediately needed by:

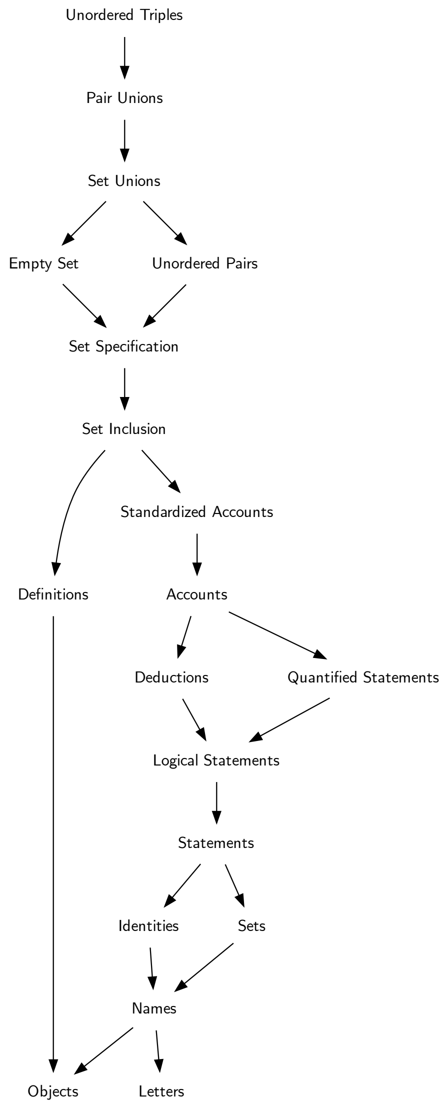
Counts (**??**)

Ordering Sets (37)

Set Powers (33)

Unordered Triples (21) gives the following terms.

unordered triple
unordered quadruple
unordered pentuples
unordered sextuples
unordered septuples


Unordered Triples
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

Does a set exist containing the elements shared between two sets? How might we construct such a set?

Definition

Let A and B denote sets. Consider the set $\{x \in A \mid x \in B\}$. This set exists by the principle of specification (see Set Specification). Moreover $(y \in \{x \in A \mid x \in B\}) \longleftrightarrow (y \in A \land y \in B)$. In other words, $\{x \in A \mid x \in B\}$ contains all the elements of A that are also elements of B.

We can also consider $\{x \in B \mid x \in A\}$, in which we have swapped the positions of A and B. Similarly, the set exists by the principle of specification (see Set Specification) and again $y \in \{x \in B \mid x \in A\} \longleftrightarrow (y \in B \land y \in B)$. Of course, $y \in A \land y \in B$ means the same as[11] $y \in B \land y \in A$ and so by the principle of extension (see Set Equality)

$$\{x \in A \mid x \in B\} = \{x \in B \mid x \in A\}.$$

We call this set the *pair intersection* of the set denoted by A with the set denoted by B.

Notation

We denote the intersection fo the set denoted by A with the set denoted by B by $A \cap B$. We read this notation aloud as "A intersect B".

[11] Future editions will name and cite this rule.

Basic Properties

All the following results are immediate.[12]

Proposition 12. $A \cap \varnothing = \varnothing$

Proposition 13 (Commutativity)**.** $A \cap B = B \cap A$

Proposition 14 (Associativity)**.** $(A \cap B) \cap C = A \cap (B \cap C)$

Proposition 15. $A \cap A = A$

Proposition 16. $(A \subset B) \longleftrightarrow (A \cap B = A)$.

[12]Proofs of these results will appear in the next edition.

Pair Intersections (22) immediately needs:

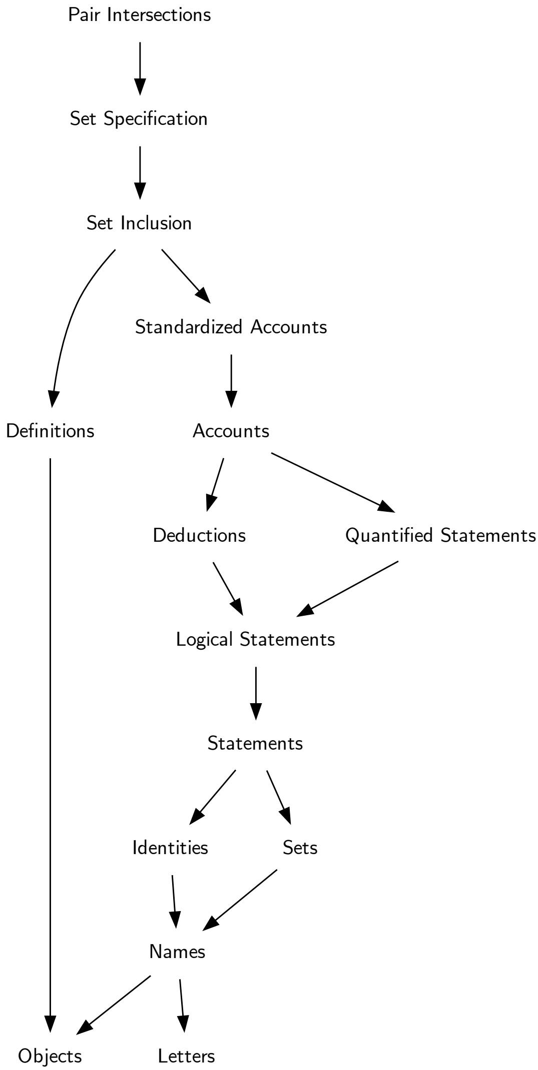
Set Specification (16)

Pair Intersections (22) is immediately needed by:

Set Decompositions (28)
Set Dualities (30)
Set Intersections (23)
Set Operations (78)
Set Unions and Intersections (25)

Pair Intersections (22) gives the following terms.

pair intersection


Pair Intersections
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Why

We can consider intersections of more than two sets.

Definition

Let $\mathcal{A}$ denote a set of sets. In other words, every element of $\mathcal{A}$ is a set. And suppose that $\mathcal{A}$ has at least one set (i.e., $\mathcal{A} \neq \varnothing$). Let C denote a set such that $C \in \mathcal{A}$. Then consider the set,

$$\{x \in C \mid (\forall A)(A \in \mathcal{A} \longrightarrow x \in A)\}.$$

This set exists by the principle of specification (see Set Specification). Moreover, the set does not depend on which set we picked. So the dependence on C does not matter. It is unique by the axiom of extension (see Set Equality). This set is called the *intersection* of $\mathcal{A}$.

Notation

We denote the intersection of $\mathcal{A}$ by $\bigcap \mathcal{A}$.

Equivalence with pair intersections

As desired, the the set denoted by $\mathcal{A}$ is a pair (see Unordered Pairs) of sets, the pair intersection (see Pair Intersections) coincides with intersection as we have defined it in this sheet.[13]

Proposition 17. $\bigcap\{A, B\} = A \cap B$

[13] A full account of the proof will appear in future editions.

Set Intersections (23) immediately needs:

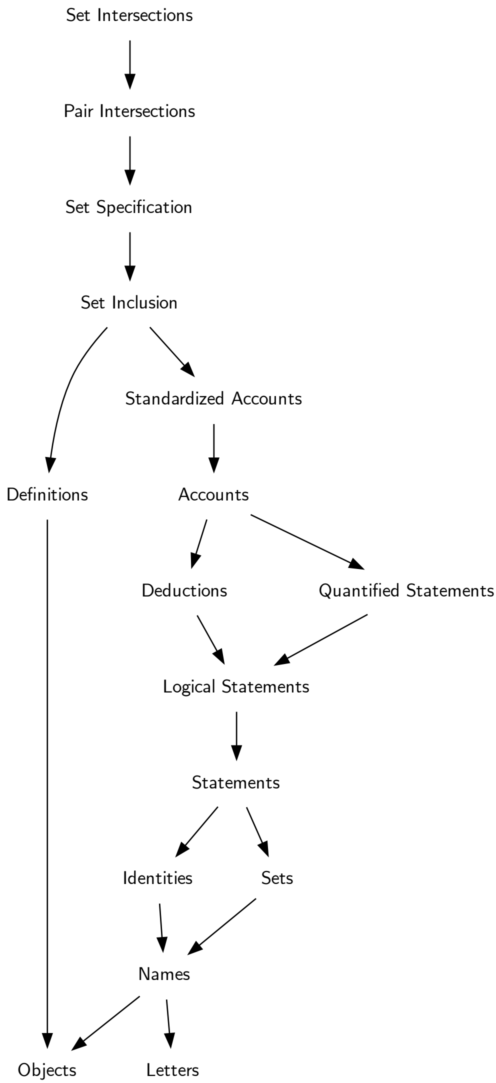
Pair Intersections (22)

Set Intersections (23) is immediately needed by:

Intersection of Empty Set (24)
Partitions (29)
Powers and Intersections (34)

Set Intersections (23) gives the following terms.

intersection


Set Intersections
Pair Intersections
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Why

We only define set intersections for nonempty sets of sets. Why?

Discussion

Which objects are specified by the sentence $(\forall x \in \varnothing)(x \in X)$? Well, since no objects fail to satisfy the statement,[14] the sentence specifies all objects. So in other words, the condition we used to define set intersections (Set Intersections) specifies the "set of everything". In order to maintain other more desirable set principles like selection, we have said that such a set does not exist (see Set Specification).

If, however, all sets under consideration are subsets of one paticular set—denote it E—then we can define intersections as follows. Let $\mathcal{C}$ be a possibly nonempty collection of sets

$$\bigcap \mathcal{C} = \{X \in E \mid (\forall X \in \mathcal{C})(x \in X)\}.$$

This definition agrees with that given in Set Intersections. In particular, it is the intersection of the set $\mathcal{C} \cup \{E\}$

Another definition

This begs the following question. Why not define intersections by selecting from the union. Let $\mathcal{A}$ be a possibly nonempty

[14] Future editions will offer an account of this.

set of sets. Then define:

$$\bigcap \mathcal{A} = \{x \in \bigcup \mathcal{A} \mid (\forall A \in \mathcal{A})(x \in A)\}.$$

If $\mathcal{A}$ is empty, so is $\bigcup \mathcal{A}$ and then there are no elements in the set to select from so $\bigcap \mathcal{A}$ is empty. This does not agree with the previous definitions for the empty set, but does for all other sets of sets.

For these reasons, the intersection of the empty set is a delicate thing.[15]

[15] Future editions will expand on the preference for the former definition.

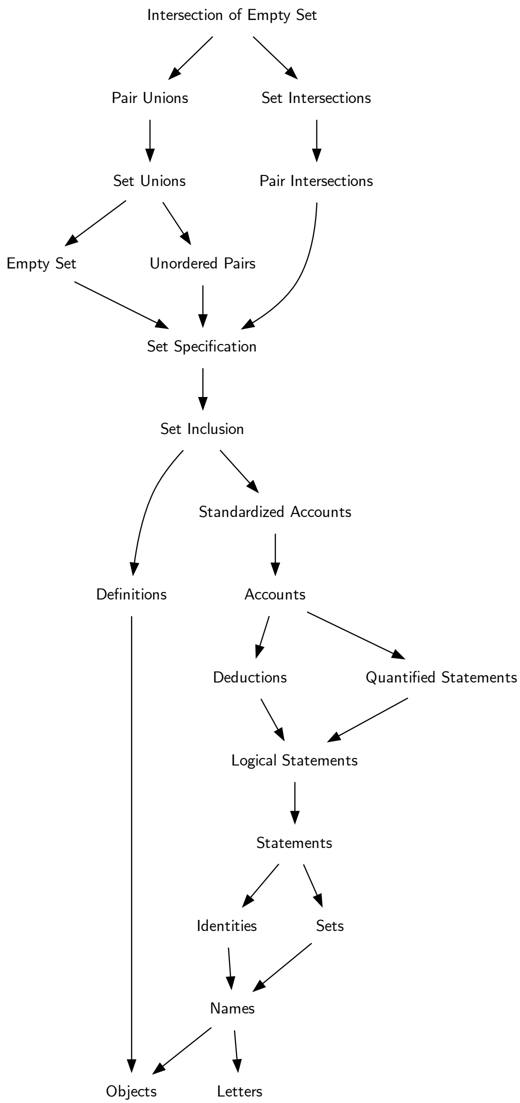
Intersection of Empty Set (24) immediately needs:

Pair Unions (20)
Set Intersections (23)

Intersection of Empty Set (24) is immediately needed by:

Generalized Set Dualities (36)
Natural Numbers (59)

Intersection of Empty Set (24) gives no terms.


Intersection of Empty Set
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Why

We study how intersection and union interact.

Results

The following are easy results.[16] They are known as the *distributive laws.*

Proposition 18. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proposition 19. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

[16] The accounts will appear in future editions.

Set Unions and Intersections (25) immediately needs:

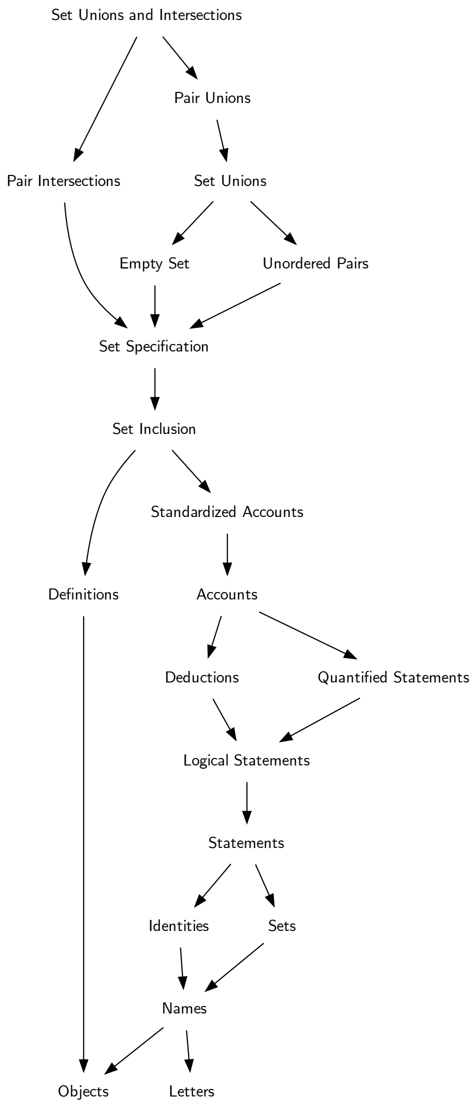
Pair Intersections (22)
Pair Unions (20)

Set Unions and Intersections (25) is immediately needed by:

Family Unions and Intersections (49)

Set Unions and Intersections (25) gives the following terms.

distributive laws


Set Unions and Intersections
Pair Unions
Pair Intersections
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
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Why

We want to consider the elements of one set which are not contained in another set.

Definition

Let A and B denote sets. The *difference* between A and B is the set $\{x \in A \mid x \notin B\}$. It is not necessary that $B \subset A$.

Notation

We denote the difference between A and B by $A - B$.

Properties

The following are straightforward.[17]

Proposition 20. $A - \varnothing = A$

Proposition 21. $A - A = \varnothing$

[17] Accounts will appear in future editions.

Set Differences (26) immediately needs:

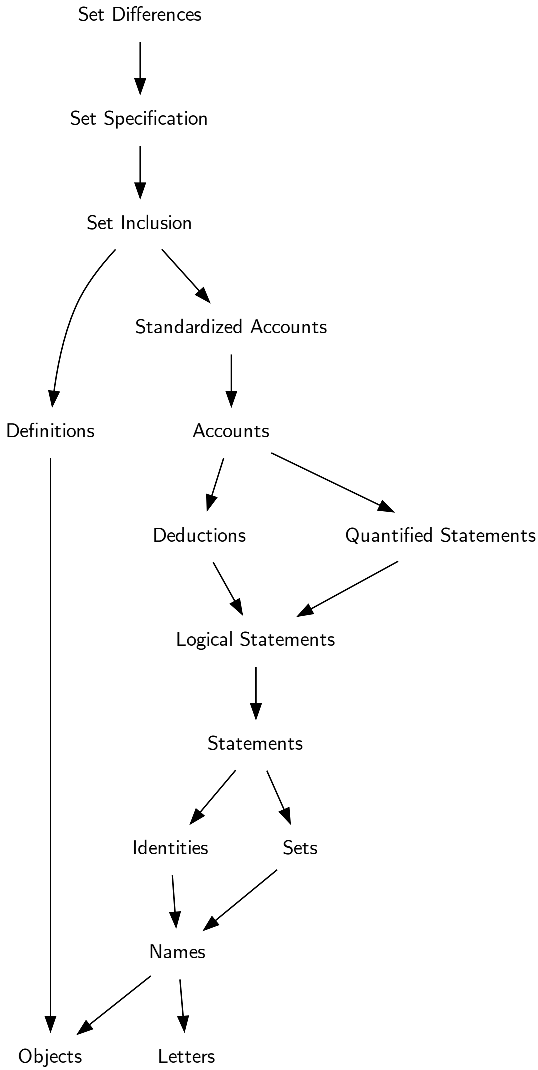
Set Specification (16)

Set Differences (26) is immediately needed by:

Natural Numbers (59)
Set Complements (27)

Set Differences (26) gives the following terms.

difference


Set Differences
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Why

It is often the case in considering set differences that all sets considered are subsets of one set.

Definition

Let A and B denote sets. In many cases, we take the difference between a set and one contained in it. In other words, we assume that $B \subset A$. In this case, we often take complements relative to the same set A. So we do not refer to it, and instead refer to the relative complement of B in A as the *complement* of B.

Notation

Let A denote a set, and let B denote a set for which $B \subset A$. We denote the relative complement of B in A by $C_A(B)$. When we need not mention the set A, and instead speak of the complement of B without qualification, we denote this complement by $C(B)$.

Complement of a complement

One nice property of a complement when $B \subset A$ is:

Proposition 22. $(B \subset A) \longleftrightarrow (C_A(C_A(B)) = B)$

Basic Facts

Let E denote a set and let A and B denote sets satisfying $A, B \subset E$. Then take all complements with respect to E. Here are some immediate consequences of the definition of complements.[18]

Proposition 23. $C(C(A)) = A$

Proposition 24. $C(\varnothing) = E$

Proposition 25. $C(E) = \varnothing$

Proposition 26. $A \subset B \longleftrightarrow C(B) \subset C(A)$

[18] Proofs will appear in future editions.

Set Complements (27) immediately needs:

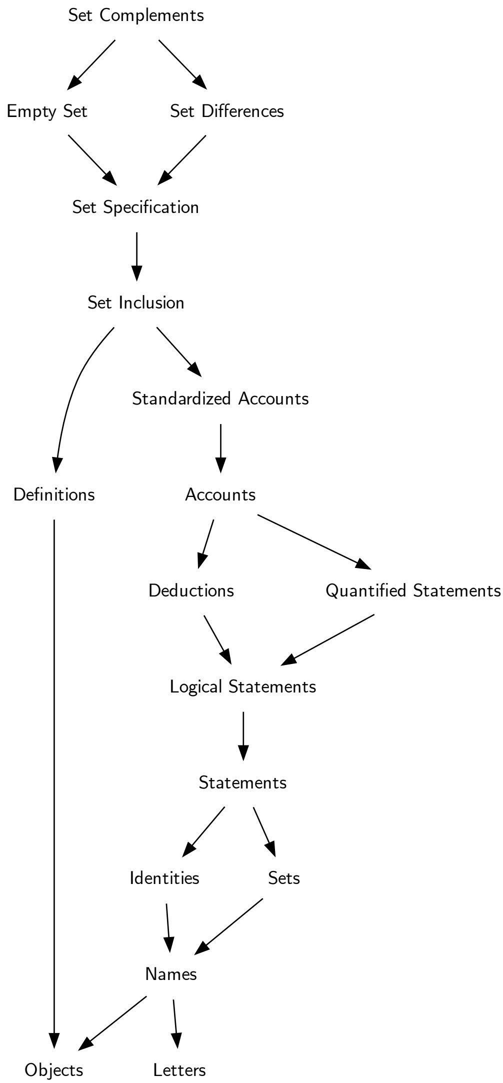
Empty Set (17)
Set Differences (26)

Set Complements (27) is immediately needed by:

Graph Complements (**??**)
Set Decompositions (28)
Set Dualities (30)
Set Symmetric Differences (32)

Set Complements (27) gives the following terms.

complement



Set Complements
Empty Set
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Why

Let E denote a set and let A denote a set with $A \subset E$. A and $C(A)$ as breaking E into two pieces which do not overlap.

Discussion for complements

To make this precise, let us say that by "breaking E into two pieces" we mean that these two pieces are all of E. In other words, every element of E is contained either in A or $C(A)$. We use the language of set unions (Pair Unions).

Proposition 27 (Breaking). $A \cup C(A) = E$

Next, let us say that "do not overlap" means that no elemenet of A is an element of $C(A)$ and vice versa. We use the language of set interserctions (see Pair Intersections).

Proposition 28 (Non-overlapping). $A \cap C(A) = \varnothing$

Definition

We call a pair $\{A, B\}$ a *decomposition* of E if $A \cap B = \varnothing$ and $A \cup B = E$. If $A \cap B$ we say that $\{A, B\}$ are *disjoint.* If we have a set of sets $\mathcal{A}$ satisfying $(A \in \mathcal{A} \land B \in \mathcal{A}) \longrightarrow (A \cap B = \varnothing)$ then we call $\mathcal{A}$ *pairwise disjoint.*

Set Decompositions (28) immediately needs:

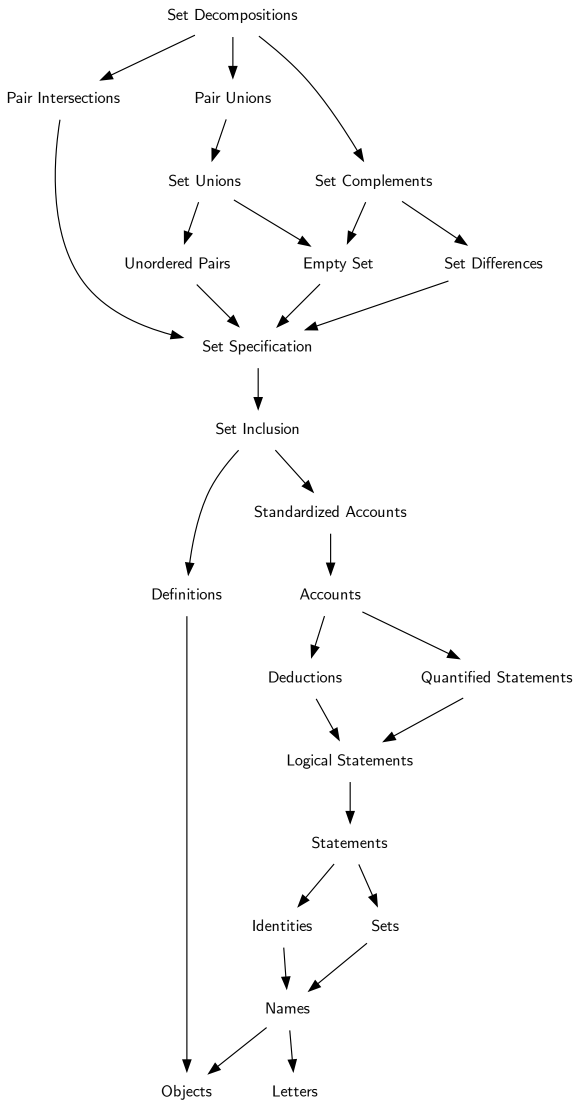
Pair Intersections (22)
Pair Unions (20)
Set Complements (27)

Set Decompositions (28) is immediately needed by:

Set Exercises (31)

Set Decompositions (28) gives the following terms.

decomposition
disjoint
pairwise disjoint


Set Decompositions
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Why

We divide a set into disjoint subsets whose union is the whole set. In this way we can handle each subset of the main set individually, and so handle the entire set piece by piece.

Definition

A *partition* of a set X is a set of pairwise disjoint (see Set Decompositions) subsets of X whose union is X. We call the elements ofa partition the *pieces* of the partition. When speaking of a partition, we commonly call the set of sets *mutually exclusive* (by which we mean that they are pairwise disjoint) and *collectively exhaustive* (by which we mean that their union is full set).

Notation

Let X be a set and $\mathcal{C}$ be a set of subsets of X. $\mathcal{C}$ is a partition of X means $(\forall A)(\forall B)((A \in \mathcal{C} \land A \in \mathcal{C}) \longrightarrow A \cap B = \varnothing)$ and $\bigcup \mathcal{C} = X$.

Partitions (29) immediately needs:

Set Intersections (23)

Set Unions (19)

Partitions (29) is immediately needed by:

Equivalence Relations (43)

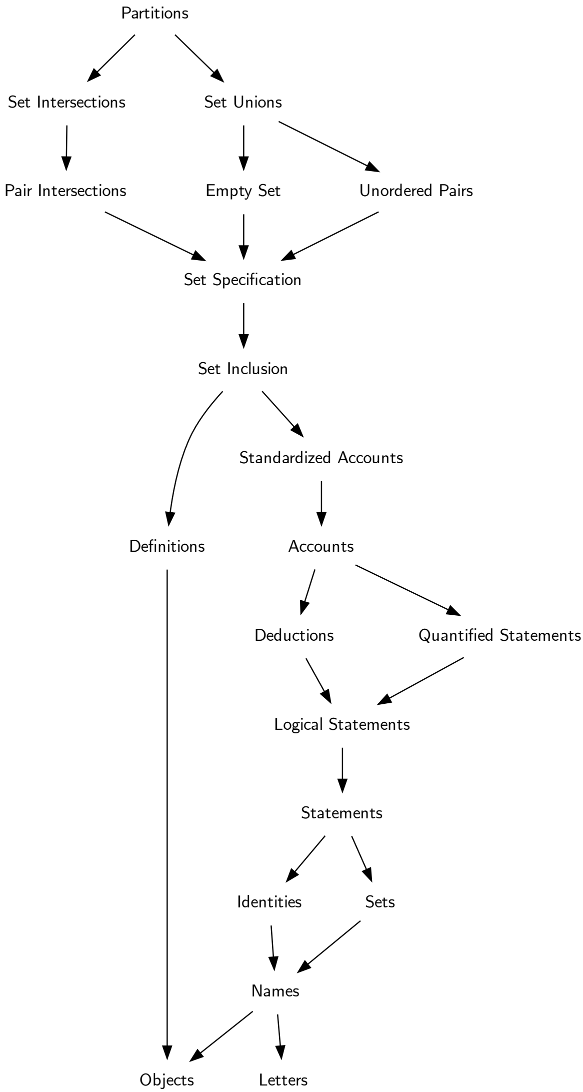
Simple Functions (**??**)

Split Graphs (**??**)

Total Probability (**??**)

Partitions (29) gives the following terms.

partition
pieces
mutually exclusive
collectively exhaustive


Partitions
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Why

How does taking complements relate to forming unions and intersections.

Complements of unions or intersections

Let E denote a set. Let A and B denote sets and $A, B \subset E$. All complements are taken with respect to E. The following are known as *DeMorgan's Laws.*[19]

Proposition 29. $C(A \cup B) = C(A) \cap C(B)$

Proposition 30. $C(A \cap B) = C(A) \cup C(B)$

Principle of duality

As a result of DeMorgan's Laws[20] and basic facts about complements (see Set Complements) theorems having to do with sets come in pairs. In other words, given an inclusion or identity relation involving complements, unions and intersections of some set (above E) if we replace all sets by their complemnets, swap unions and intersections, and flip all inclusions we obtain another result. This is called the *principle of duality for sets.*

[19] Proofs will appear in a future edition.

[20] A future edition will change the name to remove the reference to DeMorgan in accordance with the project's policy.

Set Dualities (30) immediately needs:

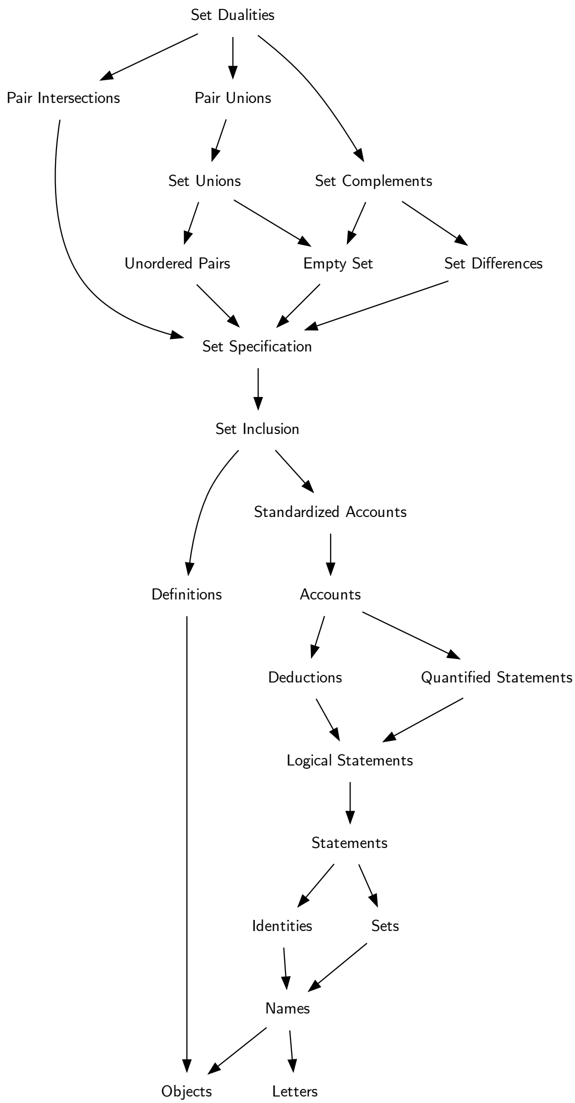
Pair Intersections (22)
Pair Unions (20)
Set Complements (27)

Set Dualities (30) is immediately needed by:

Generalized Set Dualities (36)
Set Exercises (31)

Set Dualities (30) gives the following terms.

DeMorgan's Laws
principle of duality for sets


Set Dualities
Pair Intersections
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Why

Here are some exercises on sets.[21]

Exercise 1. *Let* A, B, C *denote sets. Show* $((A \cap B) \cup C = A \cap (B \cup C)) \longleftrightarrow (C \subset A)$ *Observe that the condition does not involve* B*.*

Exercise 2.

$$A - B = A \cap B'.$$

Exercise 3.

$$A \subset B \textit{ if and only if} A - B = \varnothing.$$

Exercise 4.

$$A - (A - B) = A \cap B.$$

Exercise 5.

$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

Exercise 6.

$$(A \cap B) \subset ((A \cap C) \cup (A \cap C')).$$

Exercise 7.

$$((A \cup C) \cap (B \cup C')) \subset (A \cup B).$$

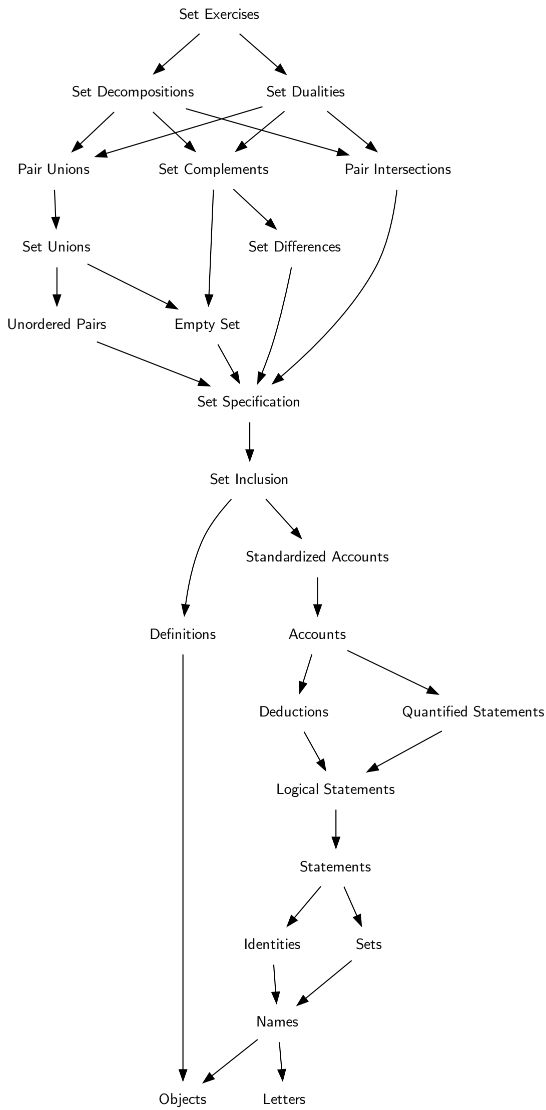
[21]Future editions will give the hypotheses more clearly.

Set Exercises (31) immediately needs:

Set Decompositions (28)
Set Dualities (30)

Set Exercises (31) is not immediately needed by any sheet.

Set Exercises (31) gives no terms.


Set Exercises
Set Decompositions
Set Dualities
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Pair Intersections
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Why

We want to consider the no-overlapping elements of a pair of sets.

Definition

In other words, we want to consider the set of elements which is one or the other but not in both. The *symmetric difference* of a set with another set is the union of the difference between the latter set and the former set and the difference between the former and the latter. The symmetric differences is also called the *Boolean sum* of A and B[22]

Notation

Let A and B denote sets. We denote the symmetric difference by $A + B$.

$$A + B = (A - B) \cup (B - A)$$

Properties

Here are some immediate properties of symmetric differences.[23]

Proposition 31 (Commutative). *$A + B = B + A$.*

Proposition 32 (Associative). *$(A + B) + C = A + (B + C)$.*

[22] Future editions will likely remove or modify this term in accordance with the project's policy on using names.

[23] Future editions will have more detailed (but obvious) hypotheses stated.

Proposition 33 (Identity)**.** $(A + \varnothing) = A$

Proposition 34 (Inverse)**.** $(A + A) = \varnothing$

Set Symmetric Differences (32) immediately needs:

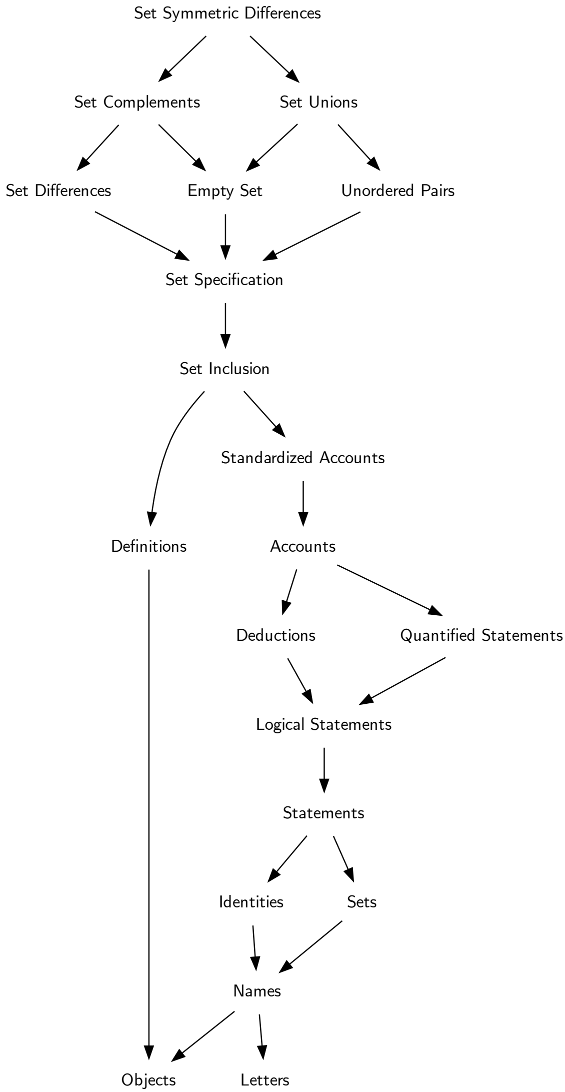
Set Complements (27)
Set Unions (19)

Set Symmetric Differences (32) is immediately needed by:

Set Operations (78)

Set Symmetric Differences (32) gives the following terms.

symmetric difference
Boolean sum



Set Symmetric Differences
Set Complements
Set Unions
Set Differences
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Why

We want to consider all the subsets of a given set.

Definition

We do not yet have a principle stating that such a set exists, but our intuition suggests that it does.

Principle 6 (Powers)**.** *For every set, there exists a set of its subsets.*

We call the existence of this set the *principles of powers* and we call the set the *power set.*[24] As usual, the principle of extension gives uniqueness (see Set Equality). The power set of a set includes the set itself and the empty set.

Notation

Let A denote a set. We denote the power set of A by A^*, read aloud as "powerset of A." $A \in A^*$ and $\varnothing \in A^*$. However, $A \subset A^*$ is false.

Examples

Let a, b, c denote distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in A^*$. We can walk through examples of power sets.

[24]This terminology is standard, but unfortunate. Future editions may change these terms.

Empty Set

Proposition 35. $\varnothing^* = \{\varnothing\}$

Singletons

Proposition 36. $\{a\}^* = \{\varnothing, \{a\}\}$

Pairs

Proposition 37. $\{a, b\}^* = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

Triples

Proposition 38. $\{a, b, c\}^* = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b,$

Properties

We can guess the following easy properties.[25]

Proposition 39. $\varnothing \in A^*$

Proposition 40. $A \in A^*$

We call A and $\varnothing$ the *improper* subsets of A. All other subset we call *proper*.

Basic Fact

Proposition 41. $E \subset F \longrightarrow E^* \subset F^*$

[25]Future editions will expand this account.

Set Powers (33) immediately needs:

Unordered Triples (21)

Set Powers (33) is immediately needed by:

Cartesian Products (40)

Characteristic Functions (**??**)

Powers and Intersections (34)

Powers and Unions (35)

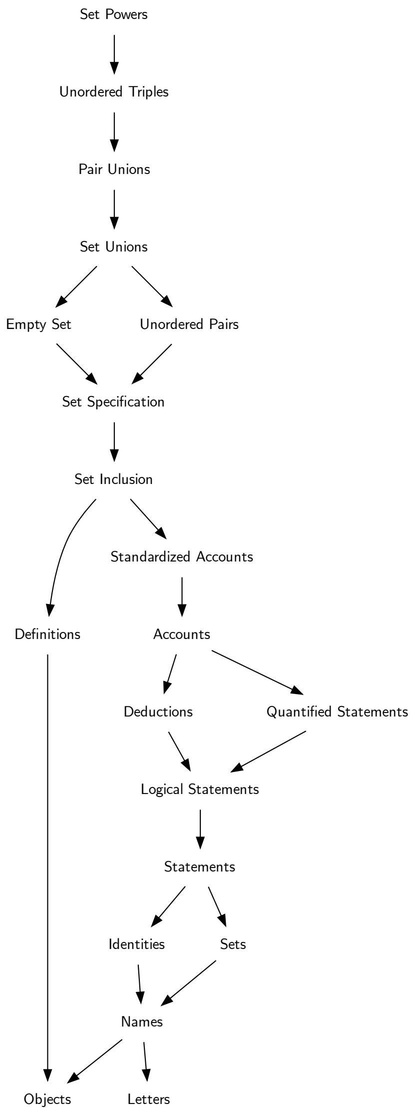
Probability Events (**??**)

Real Length Impossible (**??**)

Subset Systems (**??**)

Set Powers (33) gives the following terms.

principles of powers
power set
improper
proper


Set Powers
Unordered Triples
Pair Unions
Set Unions
Empty Set
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Why

How does the power set relate to an intersection?

Notation Preliminaries

First, if we have a set of sets—denote it $\mathcal{C}$—and all members are subsets of a fixed set—denote it E—then the set of sets is a subset of E^*. In this case, we can write

$$\bigcap\{X \in E^* \mid x \in \mathcal{C}\}$$

Which is a sort of justification for the notation

$$\bigcap_{X \in \mathcal{C}} X.$$

Basic Properties

Here are some basic interactions between the powerset and intersections.[26]

Proposition 42. $A^* \cap F^* = (A \cap F)^*$

Proposition 43. $\bigcap_{X \in \mathcal{A}} A^* = (\bigcap_{X \in \mathcal{A}} A)^*$

Proposition 44. $\bigcap_{X \in E^*} X = \varnothing$

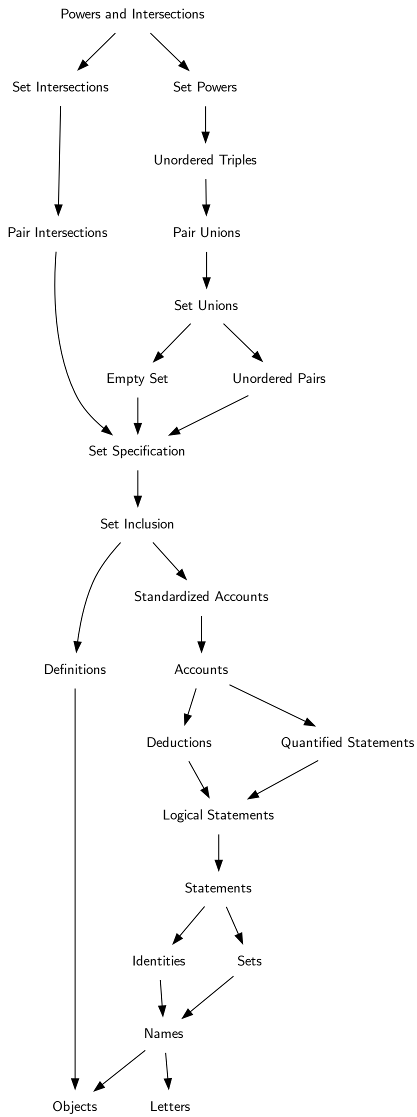
[26]Future editions will expand on these propositions and provide accounts of them.

Powers and Intersections (34) immediately needs:

Set Intersections (23)
Set Powers (33)

Powers and Intersections (34) is not immediately needed by any sheet.

Powers and Intersections (34) gives no terms.


Powers and Intersections
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Why

How does the power set relate to a union?

Notation Preliminaries

Let E denote a set. Let $\mathcal{A}$ denote a set of subsets of the set denoted by E. We define $\bigcup_{A \in \mathcal{A}} A$ to mean $\cap \mathcal{A}$.

Basic Properties

Here are some basic interactions between the powerset and unions.[27]

Proposition 45. $E^* \cup F^* \subset (E \cup F)^*$

Proposition 46. $\bigcup_{X \in \mathcal{C}} X^* \subset (\bigcup_{X \in \mathcal{C}} X)^*$

Proposition 47. $E = \bigcup E^*$

Proposition 48. $(\bigcup E)^* \supset E$.

Typically $E \neq (\bigcup E)^*$, in which case E is a proper subset.

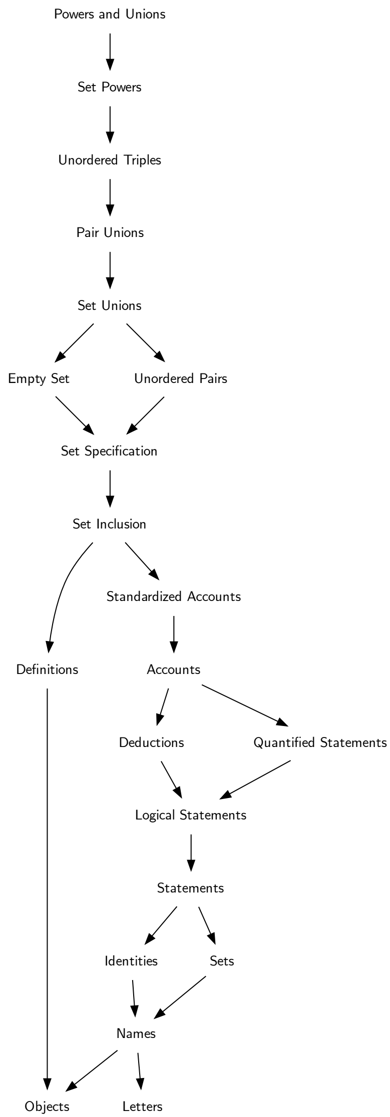
[27]Future editions will expand on these propositions and provide accounts of them.

Powers and Unions (35) immediately needs:

Set Powers (33)

Powers and Unions (35) is not immediately needed by any sheet.

Powers and Unions (35) gives no terms.


Powers and Unions
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Why

If all sets considered in a union or intersection are subsets of a fixed set, then the union and intersection of any set of sets is well defined. We can then derive generalized version of DeMorgan's laws.[28]

New Notation

Let E denote a set. Let $\mathcal{A}$ denote a set of subsets of E. Then define

$$\bigcup_{A \in \mathcal{A}} A := \bigcup \mathcal{A}, \quad \bigcap_{A \in \mathcal{A}} A := \bigcap \mathcal{A}.$$

In this case we have

Proposition 49. $C(\cup_{A \in \mathcal{A}} A) = \cap_{A \in A} C(A)$.

Proposition 50. $C(\cap_{A \in \mathcal{A}} A) = \cup_{A \in A} C(A)$.

[28] In future editions, this sheet may not exist.

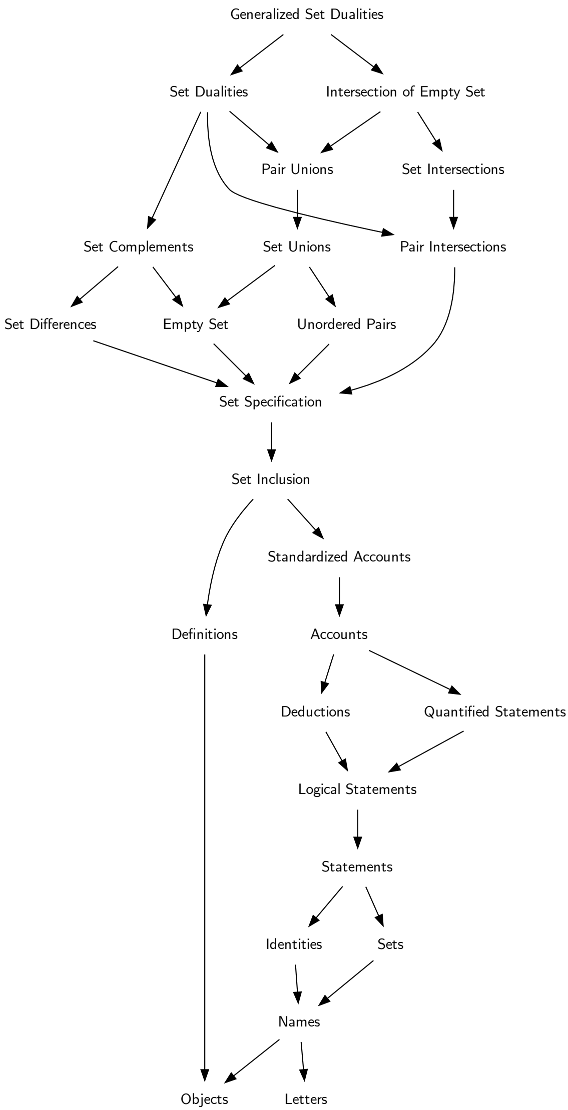
Generalized Set Dualities (36) immediately needs:

Intersection of Empty Set (24)
Set Dualities (30)

Generalized Set Dualities (36) is immediately needed by:

Family Unions and Intersections (49)

Generalized Set Dualities (36) gives no terms.


Generalized Set Dualities
Set Dualities
Intersection of Empty Set
Pair Unions
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Why

We want to arrange the elements of a set in an order using only the concept of sets.

Discussion

What does this mean? Well, we often arrange objects in orders. For example, the letters of this page are arranged into words. Take two such words: 'note' and 'tone'. If letters are objects, what are words?

A first guess is that words seem like groups of letters, and sets seem like groups, and so a word is a set of letters. So, the word 'note' is the set $\{\text{'n'}, \text{'o'}, \text{'t'}, \text{'e'}\}$, and then word 'tone' is the set $\{\text{'t'}, \text{'o'}, \text{'n'}, \text{'e'}\}$. The rub, of course, is that these are the same set.

The trick is that a word is not just the set of letters, it is that set in some order. Since 'tone' and 'note' have the same letters, they have the same set of letters. The question is whether there is a way of saying what a word is in terms of letters by using sets in such a way that the set corresponding to 'tone' is distinguishable from the set corresponding to 'note'.

The way we read English offers a hint. When reading 'tone' we scan from left to right seeing 't', then 'to', then 'ton' then 'tone'. Suppose that for each spot in the ordering of the letters, we consider those leters that appear at or before the spot. In other words, we can consider the sets $\{\text{'t'}\}$,

$\{‘t’, ‘o’\}$, $\{‘t’, ‘o’, ‘n’\}$, $\{‘t’, ‘o’, ‘n’, ‘e’\}$. Let us say that ‘tone’ corresponds to the set of these sets, denoted by $\mathcal{C}$,

$$\mathcal{C} = \{\{‘n’, o, t\}, \{n, o, t, e\}, \{t\}, \{o, t\}\}.$$

Given $\mathcal{C}$, can we recover ‘tone’ (intead of ‘note’)? Sure. First, look for a set contained in all the others. The singleton $\{‘t’\}$ is the only one. So the first letter is ‘t’. Next look for a set distinct from “t'' which is contained in all the rest. The pair $\{‘o’, ‘t’\}$ is the only one. Since we already have ‘t’, the next letter is ‘o’. We do the same twice more, getting ‘n’ and ‘e’, in that order.

There is a certain peculiarity in all these considerations. Every time we write down a set, we write the names (see Names) of the elements in some order. Indeed, whenever we speak of objects, we must say their names in some order. But of course, no matter how we denote or speak of the set, the concept of set has no concept of ordering.

Generally

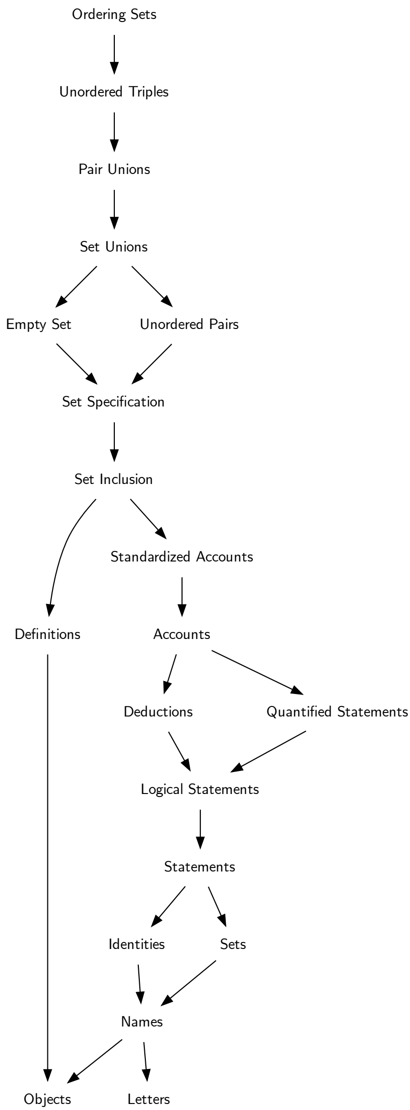
Let a, b, c and d denote objects, no two of which are the same (i.e., $a \neq b$, $b \neq c$, etc.). Suppose we want to consider the elements of the quadruple $\{a, b, c, d\}$ in the order c, b, d, a. We include in the set all objects that that occur at or before that position. For the order c, b, d, a of the objects in the set $\{a, b, c, d\}$ we use $\{c\}$, $\{c, b\}$, $\{c, b, d\}$ and $\{c, b, d, a\}$.

Ordering Sets (37) immediately needs:

Unordered Triples (21)

Ordering Sets (37) is not immediately needed by any sheet.

Ordering Sets (37) gives no terms.


Ordering Sets
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Why

We want to order two objects.

Definition

Let a and b denote objects. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is the object denoted by a and the *second coordinate* is the object denoted by b.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b).

Equality

Our intuition of two objects in order dictates that if we have the same objects in the same order then we have the same ordered pair. Conversely, if we have two identical ordered pairs, they must consist of the same objects in the same location. In other words, two ordered pairs should be equal if and only if they consist of the same objects in the same order. Our definition agrees with this intuition. Indeed,

Proposition 51. $(((a, b) = (x, y)) \longleftrightarrow (a = x \land b = y))$[29]

[29]The proof of this proposition will be found in future editions.

Ordered Pairs (38) immediately needs:

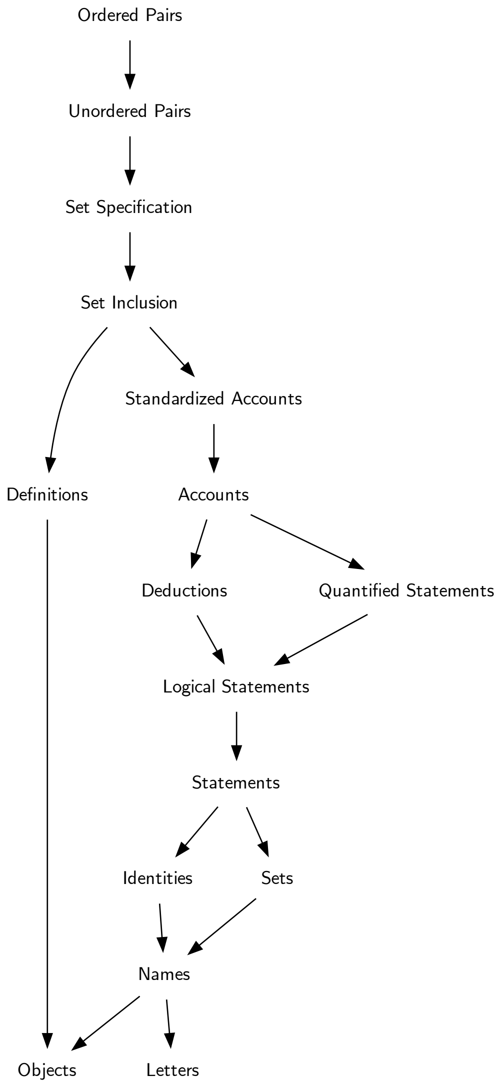
Unordered Pairs (18)

Ordered Pairs (38) is immediately needed by:

Cartesian Products (40)
Counts (**??**)
Ordered Pair Pathologies (39)
Product Sections (**??**)
Subset Systems (**??**)

Ordered Pairs (38) gives the following terms.

ordered pair
first coordinate
second coordinate



Ordered Pairs
Unordered Pairs
Set Specification
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Why

Why define ordered pairs in terms of sets? Why not make them their own type of object?

Pathologies

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second "coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life.

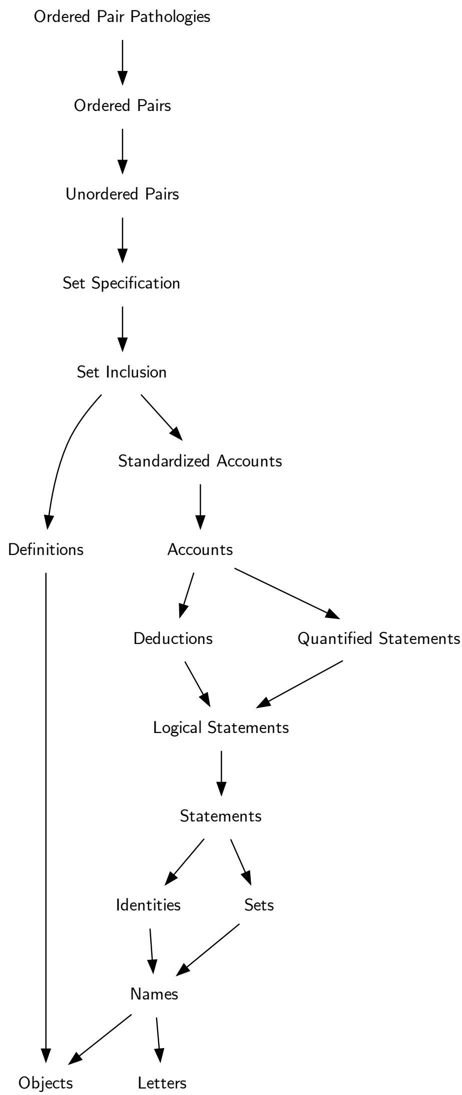
Suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Ordered Pair Pathologies (39) immediately needs:

Ordered Pairs (38)

Ordered Pair Pathologies (39) is not immediately needed by any sheet.

Ordered Pair Pathologies (39) gives no terms.


Ordered Pair Pathologies
Ordered Pairs
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

Does a set exist of all the ordered pairs of elements from an ordered pair of sets?

Definition

Let A and B denote sets. Ordered pairs are sets of singletons and pairs. So to construct the set of all ordered pairs taken from two sets, we want to specify the elements of a set which contains all singletons $\{a\}$ and pairs $\{a, b\}$ for $a \in A$, $b \in B$.

Notice that $a \in A$ and $b \in A$ mean $a, b \in (A \cup B)$. In other words, $\{a\} \subset A$ and $\{b\} \subset B$ and $\{a\}, \{b\} \subset (A \cup B)$. In particular, $\{a\} \in (A \cup B)^*$. Similarly, $\{a, b\} \in (A \cup B)^*$. And so $\{\{a\}, \{a, b\}\} \in (A \cup B)^{**}$.

We define the set of "all ordered pairs" from A and B by specifying the appropriate pairs of this set.[30]

$$\{(a, b) \in (A \cup B)^{**} \mid a \in A \land b \in B\}$$

We name this set the *product* of the set denoted by A and the set denoted by B is the set of all ordered pairs. This set is also called the *cartesian product*.[31] If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

[30] The specific statement used here requires some translation. A discussion of this and the full statement will appear in a future edition.

[31] This is the current name of the sheet, but may change in future editions, in accordance with the project policy on using names.

Notation

We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Cartesian Products (40) immediately needs:

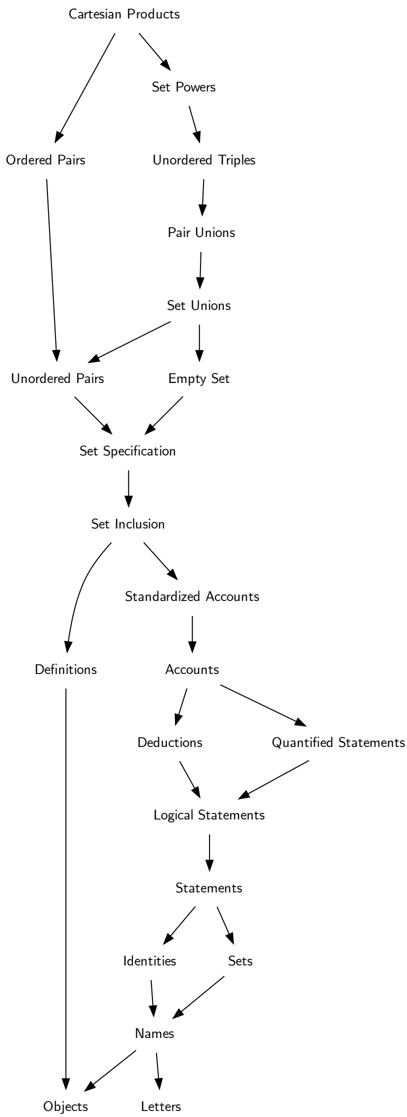
Ordered Pairs (38)
Set Powers (33)

Cartesian Products (40) is immediately needed by:

Ordered Pair Projections (41)
Relations (42)

Cartesian Products (40) gives the following terms.

product
cartesian product



Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
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Why

The product of two sets is a (sub)set of ordered pairs. Is every set or ordered pairs a subset of a product of two sets?

Result

The answer is easily seen to be yes. Let R denote a set or ordered pairs. So for $x \in R$, $x = \{\{a\}, \{a, b\}\}$. First consider $\bigcup R$. Then $\{a\} \in \bigcup R$ and $\{a, b\} \in \bigcup R$. Next consder $\bigcup \bigcup R$. Then $a, b \in \bigcup \bigcup R$. So if we want to sets—denote them by A and B—so that $R \subset A \times B$, we can take both A and B to be the set $\bigcup \bigcup R$.

We often want to shrink the sets A and B to only include the relevant members. In other words, we specify the elements of $\bigcup \bigcup R$ which are actually a first coordinate or second coordinate for some ordered pair in the set R. In other words, we define $A' = \{a \in A \mid (\exists b)((a, b) \in R)\}$ and likewise $B' = \{b \in B \mid (\exists a)((a, b) \in R)\}$. We call A' the *projection of R onto the first coordinate* and B' the *projection of R onto the second coordinate.*

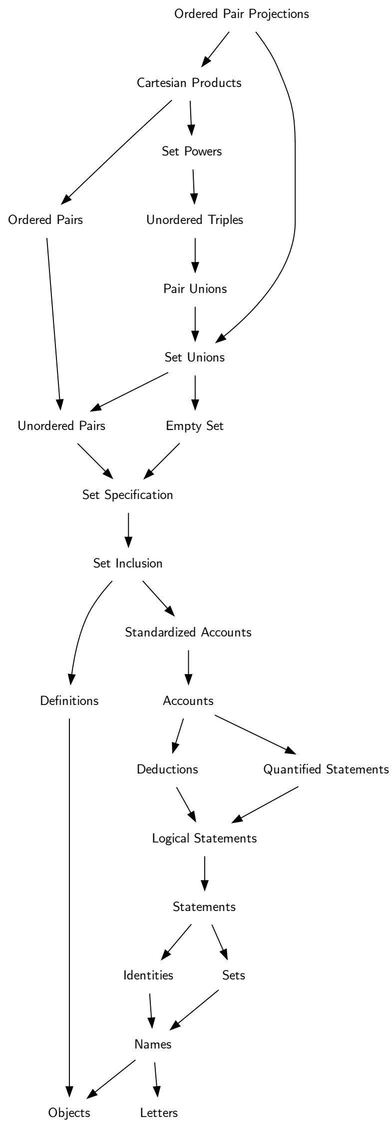
Ordered Pair Projections (41) immediately needs:

Cartesian Products (40)
Set Unions (19)

Ordered Pair Projections (41) is not immediately needed by any sheet.

Ordered Pair Projections (41) gives the following terms.

projection of R onto the first coordinate
projection of R onto the second coordinate.


Ordered Pair Projections
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
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Why

How can we relate the elements of two sets?

Definition

A *relation* is a set of ordered pairs (see Ordered Pairs). So if an object z is an element of a relation, there exists two other objects x, y so that $z = (x, y)$.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation (the projection onto the first coordinate, see Ordered Pair Projections) The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation (the projection onto the second coordinate).

When the domain of a relation R is a subset of X and the range is a subset of Y, we say R is *from* X *to* Y or *between* X and Y. If $X = Y$, then R speak of a relation *in* or *on* X.

Notation

If R is a relation, we express that $(x, y) \in R$ by writing $x \, R \, y$, which we read as "x is in relation R to y". We denote the domain of R by $\operatorname{dom} R$ and the range of R by $\operatorname{ran} R$.

Examples

For an uninteresting relation, consider the empty set. In the empty (set) relation, no object is related to any other. Both the domain and range of $\varnothing$ are $\varnothing$. For another simple relation, consider the product of any two sets X and Y. In $X \times Y$, all objects are related. The domain is X and the range is Y.

For a more interesting example, consider the set

$$R := \{(x, y) \in X \times X \mid x = y\}.$$

This relation is the relation of equality (see Identities) between two objects. Here $x\,R\,y \longleftrightarrow x = y$. dom $R =$ ran $R = X$. Another similar example is if we consider the set X and X^*, and the relation

$$R := \{(x, y) \in X \times X^* \mid x \in y\}.$$

This relation is the relation of belonging (see Sets). Here $x\,R\,y \longleftrightarrow x \in y$. Here dom $R = X$ and ran $R = X^*$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related. Equality is reflexive, symmetric and transitive whereas belonging is neither. Exercise: what is inclusion?

Relations (42) immediately needs:

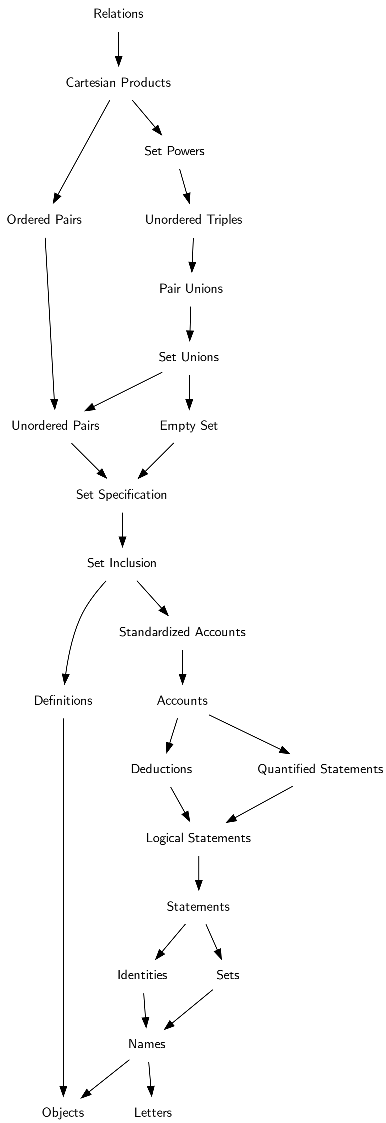
Cartesian Products (40)

Relations (42) is immediately needed by:

Converse Relations (56)
Equivalence Relations (43)
Functions (44)
Partial Orders (**??**)
Relation Composites (55)

Relations (42) gives the following terms.

relation
domain
range
from X to Y
between
in
on
reflexive
symmetric
antisymmetric
transitive


Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
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Why

We want to handle at once all the objects of a set which are indistinguishable or equivalent in some aspect.

Definition

An *equivalence relation* on a set X is a reflexive, symmetric, and transitive relation on X (see Relations). The smallest equivalence relation in a set X is the relation of equality in X. The largest equivalence relation in a set X is $X \times X$.

Equivalence relations are useful because they partition (see Partitions) the set. If R is an equivalence relation on X, the *equivalence class* of an object $x \in X$ is the set $\{y \in X \mid x \; R \; y\}$. We call the set of equivalence classes the *quotient set* of the set under the relation. An equally good name is the divided set of the set under the relation, but this terminology is not standard. The language in both cases reminds us that the relation partitions the set into equivalence classes.

If $\mathcal{C}$ is a partition of X, we can define a relation R on X for which $x \, R \, y \longleftrightarrow (\exists A \in \mathcal{C})(x \in A \land y \in A)$. In other words, if x and y are in the same piece (see Partitions) of $\mathcal{C}$.

The key result is that every equivalence relation partitions the set and every partition of the set is an equivalence relation. Moreover, if we start with an equivalence relation, look for the partition, and then get the relation defined by the partition, we end up with the relation we started with. Likewise, if we start with a partition relation, get the equivalence relation, and then get the partition defined by the relation, we end up with the partition we started with. Before stating and proving this result, we give some notation.

Notation

Let R denote an equivalence relation on a set denoted by X. We denote the equivalence class of $x \in X$ by x/R. We denote the set of equivalence classes of R by X/R, read aloud as "X modulo R" or "x mod R". We denote the equivalence class of an element $x \in X$ by $[x]$.

Main Results

The proofs of these results are straightforward.[32]

Proposition 52. *$X/\mathcal{C}$ is an equivalence relation*

Proposition 53. *X/R is a a partition.*

Proposition 54. *If R is an equivalence relation on X, then $X/(X/R) = R$*

Proposition 55. *If $\mathcal{C}$ is a partition of X, then $X/(X/\mathcal{C}) = \mathcal{C}$.*

These last two propositions make clear the rationale for the notation. Of course, we have not yet had occasion to speak of numbers, much less of division.

[32]Nonetheless, the full accounts will appear in future editions.

Equivalence Relations (43) immediately needs:

Partitions (29)

Relations (42)

Equivalence Relations (43) is immediately needed by:

Canonical Maps (47)

Equivalent Sets (70)

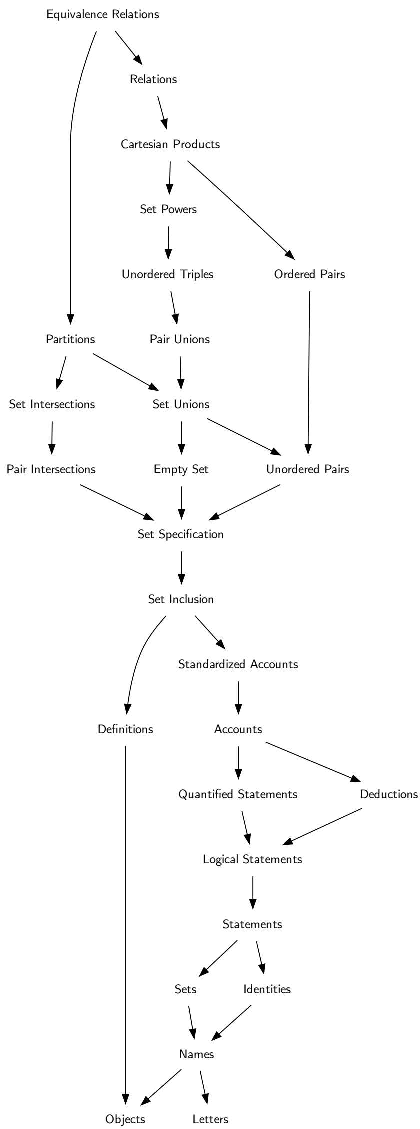
Integer Numbers (84)

Inverses of Composite Relations (57)

Matrix Similarity (**??**)

Equivalence Relations (43) gives the following terms.

quotient set
equivalence class
equivalence relation
equivalence class


Equivalence Relations
Relations
Cartesian Products
Set Powers
Unordered Triples
Ordered Pairs
Partitions
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
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Set Inclusion
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Why

We want a notion for a correspondence between two sets.

Definition

A *function* f *from* a set X *to* a set Y is a relation (see Relations) whose domain is X and whose range is a subset of Y such that for each $x \in X$, there exists a unique $y \in Y$ so that $(x, y) \in f$.

We call the unique $y \in Y$ the *result* of the function *at* the *argument* x. We call Y the *codomain*. If the range is Y we say that f is a function from X *onto* Y (or f is *surjective*). If distinct elements of X are mapped to distinct elements of Y, we say that the function is *one-to-one* (or f is *injective*).

We say that the function *maps* elements from the domain to the codomain. Since the word function and the verb "maps" connote activity, some authors refer to the concept that we have defined as a function as the *graph* of a function—namely, the set of ordered pairs which that function produces—and leave the concept of function undefined.

Notation

Let X and Y denote sets. We denote a function named f whose domain is X and whose codomain is Y by $f : X \to Y$. We read the notation aloud as "f from X to Y". We denote the set of all functions from X to Y (which is a subset of $(X \times Y)^*$) by Y^X. A less standard but equally good notation is $X \to Y$, read aloud as "A to B". Using the notations introduced so far, we denote that $f \in (A \to B)$ by $f : A \to B$. We tend to denote function by lower case latin letters, especially f, g, and h. f is a mnemonic for function and g and h are nearby.

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a." Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

Examples

If $X \subset Y$, the function $\{(x, y) \in X \times Y \mid x = y\}$ is the *inclusion function* of X into Y. We often introduce such a function as "the function from X to Y defined by $f(x) = y$". We mean by this that f is a function and that we are specifying the appropraite ordered pairs using the statement, called *argument-value notation.* The inclusion function of X into X is called the *identity function* of X. If we view the identity function as a relation on X, it is the relation of equality on X.

The functions $f : (X \times Y) \to X$ defined by $f(x, y) = x$ is the *pair projection* of $X \times Y$ ono X. Similarly $g : (X \times Y) \to Y$ defined by $g(x, y) = y$ is the pair projection of $X \times Y$ onto Y. The identity function is one-to-one and onto, the inclusion functions are one-to-one but not always onto, and the pair projections are usually not one-to-one.

Functions (44) immediately needs:

Relations (42)

Functions (44) is immediately needed by:

Canonical Maps (47)

Categories (**??**)

Families (48)

Function Composites (52)

Function Images (46)

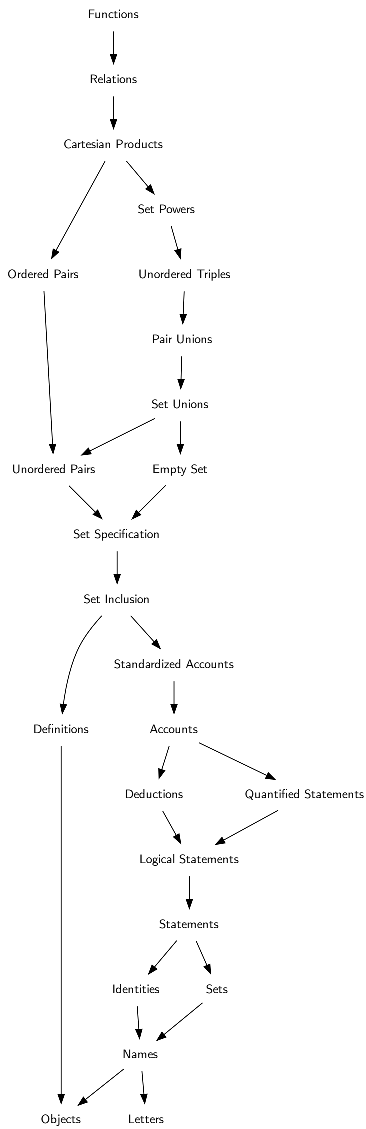
Function Restrictions and Extensions (45)

Operations (75)

Quasiconcave Functions (**??**)

Functions (44) gives the following terms.

function
from
to
result
at
argument
codomain
onto
surjective
one-to-one
injective
maps
graph
inclusion function
argument-value notation
identity function
pair projection


Functions
Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
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Why

The relationship between the inclusion map and the identity map is characteristic of making small functions out of large ones.

Definition

Let $X \subset Y$ and $f : Y \to Z$. There is a natural function $g : X \to Z$, namely the one defined by $g(x) = f(x)$ for all $x \in X$. We call g the *restriction* of f to X. We call f an *extension* of g to Y. Clearly, there may be more than one extension of a function

Notation

We denote the restriction of $f : Y \to Z$ to the set $X \subset Y$ by $f|X$.

Example

A simple example is the that the inclusion mapping from X to Y with $X \subset Y$ is a restriction of the identity map on X

Function Restrictions and Extensions (45) immediately needs:

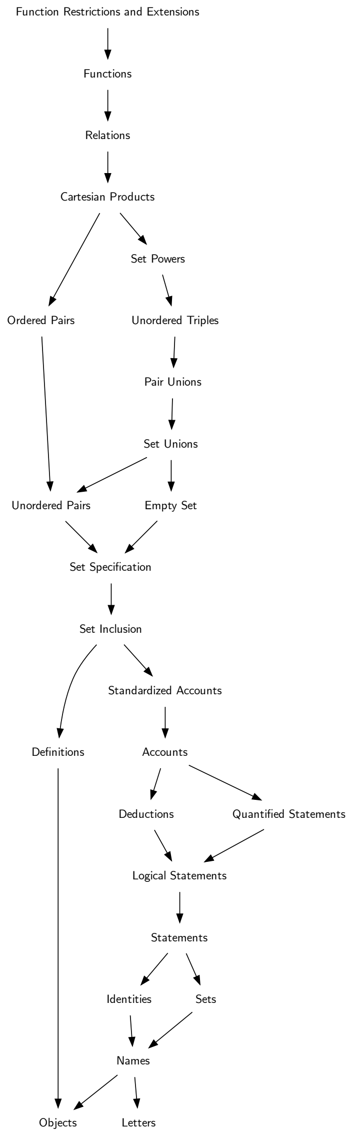
Functions (44)

Function Restrictions and Extensions (45) is immediately needed by:

Natural Integer Isomorphism (93)

Function Restrictions and Extensions (45) gives the following terms.

restriction
extension


Function Restrictions and Extensions
Functions
Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
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Why

We consider the set of results of a set of domain elements.

Definition

The *image* of a set of domain elements under a function is the set of their results. Though the set of domain elements may include several distinct elements, the image may still be a singleton, since the function may map all of elements to the same result.

Using this language, the range (see Functions) of a function is the image of its domain. The range includes all possible results of the function. If the range does not include some element of the codomain, then the function maps no domain elements to that codomain element.

Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for every $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. Following this notation for function images, we denote the range of f by $f(A)$. In this notation, we can record that f maps X onto Y by $f(X) = Y$.

Notational ambiguity

The notation $f(A)$ is can be ambiguous in the case that A is both an element and a set of elements of the domain of f. For example, consider $f : \{\{a\}, \{b\}, \{a, b\}\} \to X$. Then $f(\{a, b\})$ is ambiguous. We will avoid this ambiguity by making clear which we mean in particular cases.

Inverse Images

Similarly to how we can define $f : X^* \to Y^*$ for $A \subset X$

$$f(A) = \{y \in Y \mid (\exists x)(x \in a \wedge y = f(x))\},$$

we can define $f^{-1} : Y^* \to X^*$ for $B \subset X$

$$f^{-1}(B) = \{x \in X \mid (\exists y)(y \in B \wedge y = f(x))\}.$$

In other words, $f^{-1}(B)$ is the set of all elements of the domain which give the elements in B of the range. We call $f^{-1}(B)$ the *inverse image* of B.

Connections

Here are some connections.[33]

Proposition 56. *Let $f : X \to Y$ and $B \subset Y$. $f(f^{-1}(B)) \subset B$. If f is onto, then $f(f^{-1}(B)) \subset B$.*

Proposition 57. *Let $f : X \to Y$ and $A \subset X$. $A \subset f^{-1}(f(A))$. If f is one-to-one, then $A = f^{-1}(f(A))$.*

[33] The proofs are straightfoward, and will appear in future editions.

Function Images (46) immediately needs:

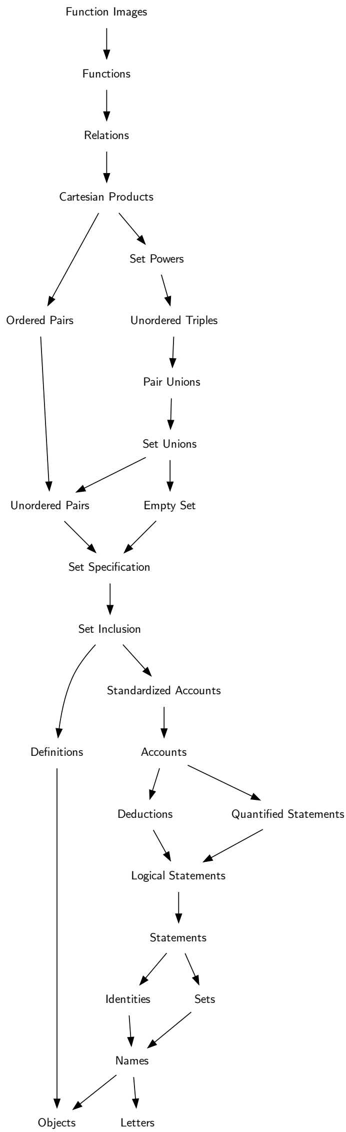
Functions (44)

Function Images (46) is immediately needed by:

Function Inverses (53)

Function Images (46) gives the following terms.

image
inverse image


Function Images
Functions
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Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
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Set Inclusion
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Why

How do equivalence classes and functions relate

Definition

We can associate to each element of a set its equivalence class under an equivalence relation. Let X denote a set and R an equivalence relation. We call the function $f : X \to X/R$ defined by $f(x) = x/R$ the *canonical map* from X to X/R.

Conversely, if f is an arbitrary function from X onto Y, we can naturally define an equivalence relation R in X so that for $a, b \in X$, $a\,R\,b \longleftrightarrow f(a) = f(b)$ f was onto, so for each $y \in Y$, there exists an $x \in X$ with $f(x) = y$. Now let $g : Y \to X/R$ be defined by $g(y) = x/R$. The values of g are the subset X which are mapped to the same value under f. Moreover, the function g is one-to-one.

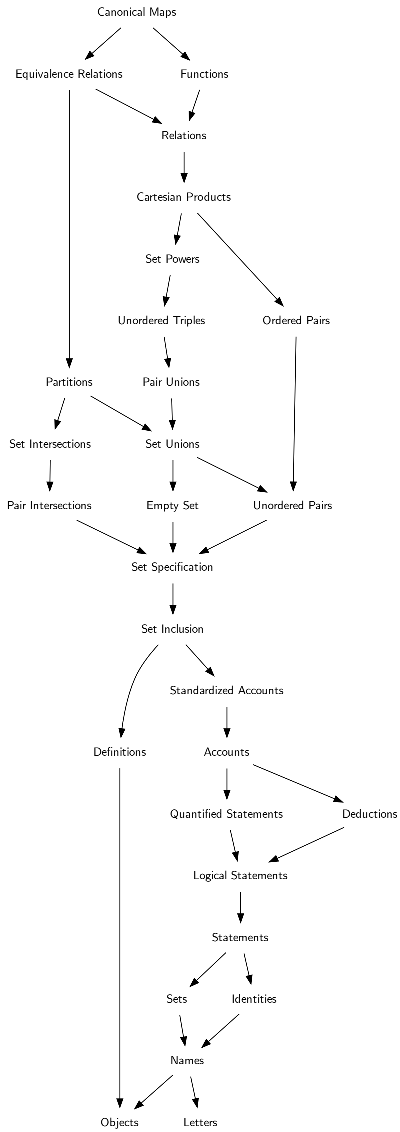
Canonical Maps (47) immediately needs:

Equivalence Relations (43)
Functions (44)

Canonical Maps (47) is not immediately needed by any sheet.

Canonical Maps (47) gives the following terms.

canonical map


Canonical Maps
Equivalence Relations
Functions
Relations
Cartesian Products
Set Powers
Unordered Triples
Ordered Pairs
Partitions
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
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Why

We often use functions to keep track of several objects by the objects of some well-known set with which they correspond. In this case, we use specific language and notation.

Definition

Let I and X denote sets. A *family* is a function from I to X. We call an element of I an *index* and we call I the *index set*. Of course, the letter I was picked here to be a mnemonic for "index". We call the range of the family the *indexed set* and we call the value of the family at an index i a *term* of the family at i or the *ith term* of the family.

Experience shows that it is useful to discuss sets using indices, especially when discussing a set of sets. If the values of the family are sets, we speak of a *family of sets*. Indeed, we often speak of a *family of* whatever object the values of the function are. So for instance, a family of subsets of X is understood to be a function from some index set into X^*.

Notation

Let $x : I \to X$ be a family. We denote the ith term of x by x_i. We sometimes denote the family by $\{x_i\}_{i \in I}$.

Families (48) immediately needs:

Functions (44)

Families (48) is immediately needed by:

Direct Products (50)

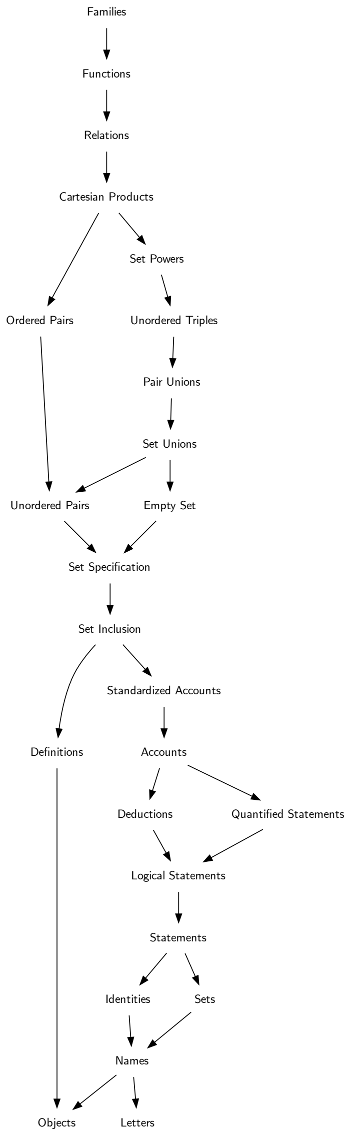
Family Operations (**??**)

Family Unions and Intersections (49)

Sequences (60)

Families (48) gives the following terms.

family of sets
ordered family
family
index
index set
indexed set
term
ith term
family of sets
family of


Families
Functions
Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
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Why

We can use families to think about unions and intersections.

Family Unions

Let $A : I \to X^*$ be a family of subsets. We refer to the union (see Set Unions) of the range (see Relations) of the family the *family union.* We denote it $\cup_{i \in I} A_i$.

Proposition 58. $(x \in \cup_{i \in I} A_i) \longleftrightarrow (\exists i)(x \in A_i)$

If $I = \{a, b\}$ is a pair with $a \neq b$, then $\cup_{i \in I} = A_a \cup A_b$.

There is no loss of generality in considering family unions. Every set of sets is a family: consider the identity function from the set of sets to itself.

We can also show generalized associative and commutative law[34] for unions.

Proposition 59. *Let $\{I_j\}$ be a family of sets and define $K = \cup_j I_j$. Then $\cup_{k \in K} A_k = \cup_{j \in J}(\cup_{i \in I_j} A_i)$.*[35]

Family Intersection

If we have a nonempty family of subsets $A : I \to X^*$, we call the intersection (see Set Intersections) of the range of the family the *family intersection.* We denote it $\cap_{i \in I} A_i$.

[34] The commutative law will appear in future editions.

[35] An account will appear in future editions.

Proposition 60. $x \in \cap_{i\in I} A_i \longleftrightarrow (\forall i)(x \in A_i)$

Similarly we can derive associative and commutative laws for intersection[36]. They can be derived as for unions, or from the facts of unions using generalized DeMorgan's laws (see Generalized SSet Dualities.

Connections

The following are easy[37]

Let $\{A_i\}$ be a family of subsets of X and let $B \subset X$.

Proposition 61. $B \cap \bigcup_i A_i = \bigcup_i (B \cap A_i)$

Proposition 62. $B \cup \bigcap_i A_i = \bigcap_i (B \cup A_i)$

Let $\{A_i\}$ and $\{B_j\}$ be families of sets.[38]

Proposition 63. $(\bigcup_i A_i) \cap (\bigcup_j B_j) = \bigcup_{i,j} (A_i \cap B_j)$

Proposition 64. $(\bigcap_i A_u) \cup (\bigcap_j B_j) = \bigcap_{i,j} (A_i \cup B_j)$.

Proposition 65. $\cap_i X_i \subset X_j \subset \cup_i X_i$ *for each* j*.*

[36]Statements of these will be given in future editions.

[37]Accounts will appear in future editions.

[38]An account of the notation used and the proofs will appear in future editions.

Family Unions and Intersections (49) immediately needs:

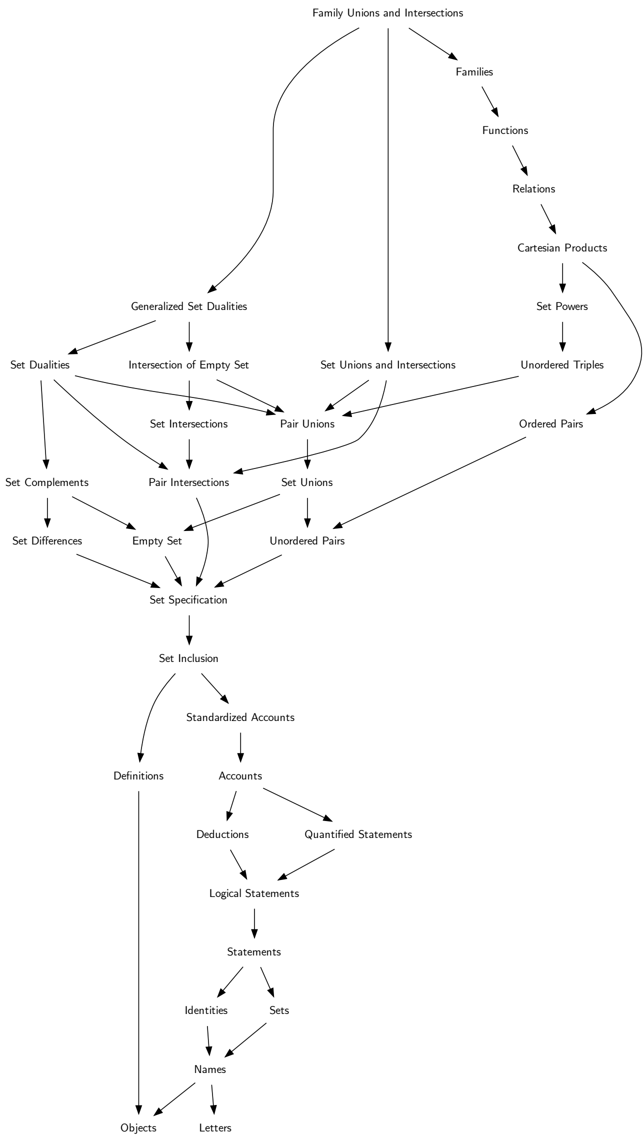
Families (48)
Generalized Set Dualities (36)
Set Unions and Intersections (25)

Family Unions and Intersections (49) is immediately needed by:

Family Products and Unions (51)
Inverses Unions Intersections and Complements (54)

Family Unions and Intersections (49) gives the following terms.

family union
family intersection



Family Unions and Intersections
Families
Functions
Relations
Cartesian Products
Generalized Set Dualities
Set Powers
Set Dualities
Intersection of Empty Set
Set Unions and Intersections
Unordered Triples
Set Intersections
Pair Unions
Ordered Pairs
Set Complements
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
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Why

We can generalize the product of two sets to a product of a family of sets.

Discussion for pairs

Let A and B be sets. There is a natural correspondence between $A \times B$ (see Cartesian Products) and a particular set of families. The particular set of families $z : \{i, j\} \to (A \cup B)$ with $z_i \in A$ and $z_b \in B$. The family z corresponds with the pair (z_i, z_i). The pair (a, b) corresponds to the family $z : \{i, j\} \in (A \cup B)$ defined by $z(i) = a$ and $z(j) = b$. In other words, we can think about ordered pairs as special families. The generalization of Cartesian products to families generalizes the notion for families.

Direct Products

Let X be a set. Let $A : I \to X$ be a family of subsets of X. The *direct product* or *family Cartesian product* of A is the set of all families $a : I \to X$ which satisfy $a_i \in A_i$ for every $i \in I$. A function on a product is called a *function of several variables* and, in particular, a function on the product $X \times Y$ is called a *funciton of two variables.*

Notation

We denote the product of the family $\{A_i\}$ by

$$\prod_{i \in I} A_i$$

We read this notation as "product over i in I of A sub-i."

Projections

The word "projection" is used in two senses with families. Let I be a set, and let $\{A_i\}$ be a family of sets. Define $A = \prod_{i \in I} A_i$. Then if $J \subset I$, there is a natural correspondence between the elements of X and those of $\prod_{j \in J} A_j$. To each element $x \in X$, we restrict x to J and this is an element of $\prod_{j \in J} A_j$. The correspondence is called the *projection* of X onto $\prod_{i \in J} X_i$ Also, the value of x at j is called the *projection of x onto index* j or the j*-coordinate* of x.

Direct Products (50) immediately needs:

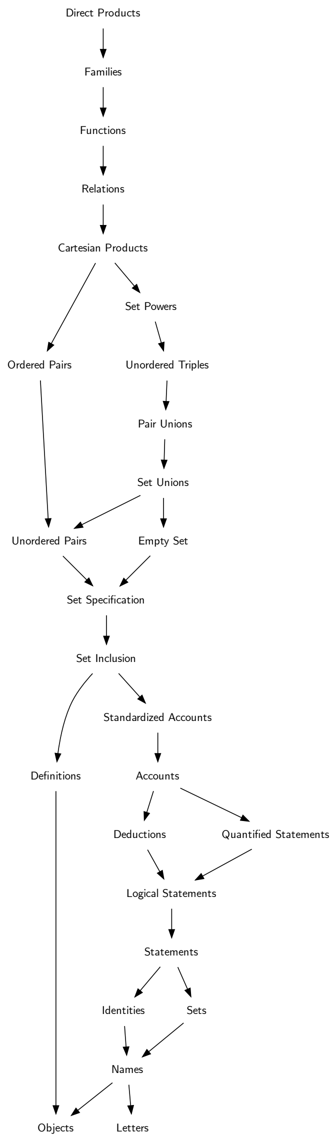
Families (48)

Direct Products (50) is immediately needed by:

Datasets (??)
Family Products and Unions (51)
Joint Distributions (??)
Marginal Distributions (??)
Natural Direct Products (61)
Product Metrics (??)
Product Sigma Algebras (??)
Random Variable Independence (??)
Undirected Paths (??)

Direct Products (50) gives the following terms.

direct product
n-tuples
sequences
family Cartesian product
function of several variables
funciton of two variables.
consecutive
projection
projection of x onto index j
j-coordinate


Direct Products
Families
Functions
Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
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Why

We study how family unions and direct products interact.

Result

The following is easy.[39]

Proposition 66. $(\cup_i A_i) \times (\cup_j B_j) = \cup_{i,j}(A_i \times B_j)$.

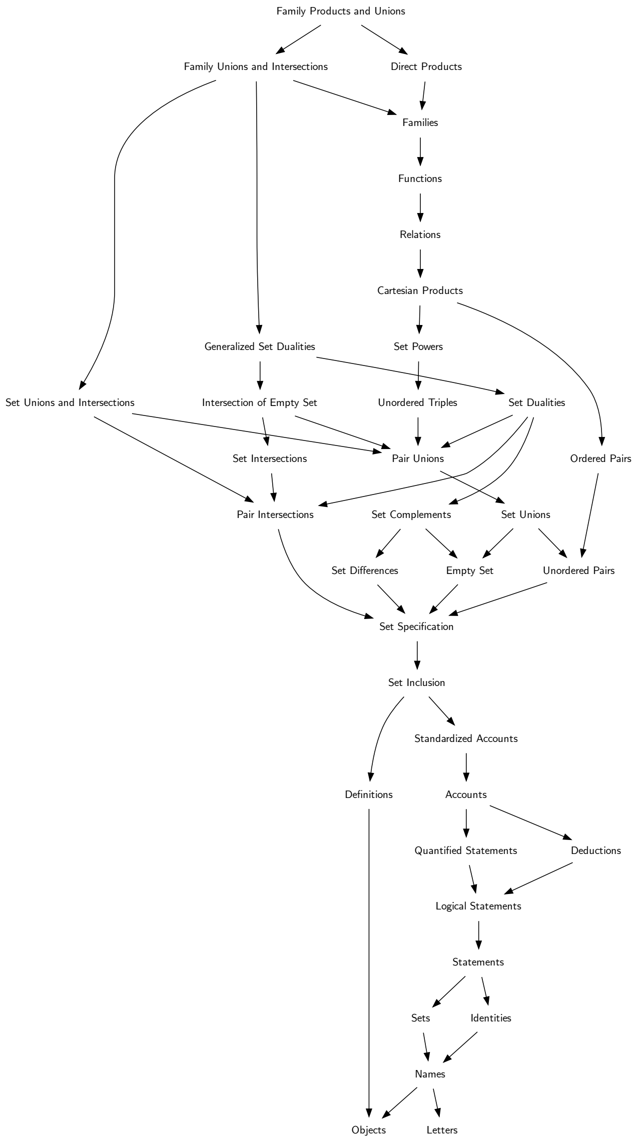
[39] An account will appear in future editions.

Family Products and Unions (51) immediately needs:

Direct Products (50)
Family Unions and Intersections (49)

Family Products and Unions (51) is not immediately needed by any sheet.

Family Products and Unions (51) gives no terms.


Family Products and Unions
Family Unions and Intersections
Direct Products
Families
Functions
Relations
Cartesian Products
Generalized Set Dualities
Set Powers
Set Unions and Intersections
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Intersections
Pair Unions
Ordered Pairs
Pair Intersections
Set Complements
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Why

We want a notion for applying two functions one after the other. We apply a first function then a second function.

Definition

Consider two functions. And suppose the range of the first is a subset of the domain of the second. In other words, every value of the first is in the domain (and so can be used as an argument) for the second.

The *composite* or *composition* of the second function with the first function is the function which associates each element in the first's domain with the element in the second's codomain that the second function associates with the result of the first function.

The idea is that we take an element in the first domain. We apply the first function to it. We obtain an element in the first's codomain. This result is an element of the second's domain. We apply the second function to this result. We obtain an element in the second's codomain. The composition of the second function with the first is the function so constructed. Of course the order of composition is important.

Notation

Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$. We denote the composition of g with f by $g \circ f$ read aloud as "g composed with f." To make clear the domain and comdomain, we denote the composition $g \circ f : A \to C$. $g \circ f$ is defined by

$$(g \circ f)(a) = g(f(a)).$$

for all $a \in A$. Sometimes the notation gf is used for $g \circ f$.

Basic Properties

Function composiiton is associative but not commutative.[40] Indeed, even if $f \circ g$ is defined, $g \circ f$ may not be.

Proposition 67 (Associative)**.** *Let* $f : X \to Y$, $g : Y \to Z$ *and* $h : Z \to U$ *Then* $(f \circ g) \circ h = f \circ (g \circ h)$[41]

[40] Future editions will include a counterexample.

[41] The proof is straightforward. Future editions will include it.

Function Composites (52) immediately needs:

Functions (44)

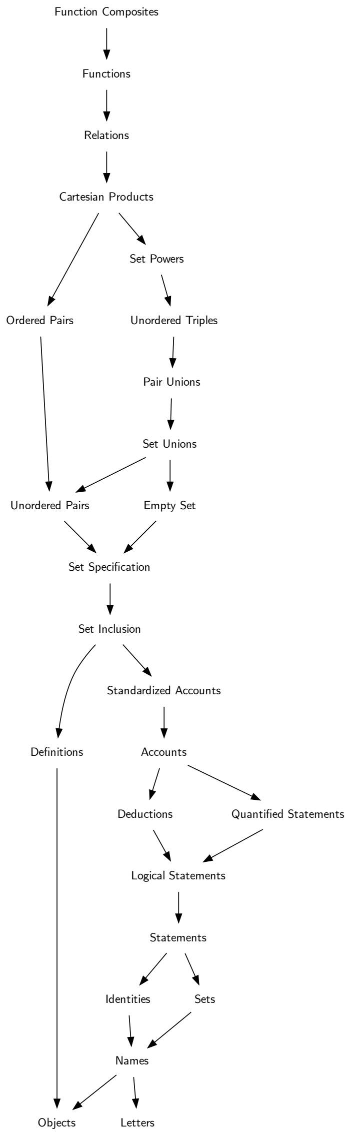
Function Composites (52) is immediately needed by:

Function Inverses (53)

Subsequences (74)

Function Composites (52) gives the following terms.

composite
composition


Function Composites
Functions
Relations
Cartesian Products
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Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
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Why

We want a notion of reversing functions.

Definition

Reversing functions does not make sense if the function is not one-to-one. Let $f : X \to Y$. If x_1 goes to y and x_2 goes to y (i.e., $f(x_1) = f(x_2) = y$), then what should y go to. One answer is that we should have a function which gives all the domain values which could lead to y. This is the inverse image (see Function Images) $f^{-1}(\{y\})$. Nor does reversing funcitons make sense if f is not onto. If there does not exist $x \in X$ so that $y = f(x)$, then $f^{-1}(\{y\}) = \varnothing$.

In the case, however, that the function is one-to-one and onto, then each element of the domain corresponds to one and only one element of the codomain and vice versa. In this case, for all $y \in Y$, $f^{-1}(\{y\})$ is a singleton $\{x\}$ where $f(x) = y$. In this case, we define a fucntion $g : Y \to X$ so that $g(y) = x$ if and only if $f(x) = y$.

In general, if we have two functions, where the codomain of the first is the domain of the second, and the codomain of the second is the domain of the first, we call them *inverse functions* if the composition of the second with the first is the identity function on the first's domain and the composition of the first with the second is the identity function on the second's domain (see Functions and Function Composites).

In this case we say that the second function is an *inverse* of the second, and vice versa. When an inverse exists, it is unique,[42] so we refer to *the inverse* of a function. We call the first function *invertible*. Other names for an invertible function include *bijection*.

Notation

Let A a non-empty set. We denote the identity function on A by id_A, read aloud as "identity on A." id_A maps A onto A. Let A, B be non-empty sets. Let $f : A \to B$ and $g : B \to A$ be functions. f and g are inverse functions if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

The Inverse

Proposition 68 (Uniqueness)**.** *Let $f : A \to B$, $g : B \to A$, and $h : B \to A$. If g and h are both inverse functions of f, then $g = h$.*

Proposition 69 (Existence)**.** *If a function is one-to-one and onto, it has an inverse; and conversely.*

Composites and Inveres

Let $f : X \to Y$ and $g : Y \to Z$. Then g^{-1} maps Z^* to Y^* and f^{-1} maps Y^* to X^*. Then the following is immediate

Proposition 70. $(gf)^{-1} = f^{-1}g^{-1}$

[42] Future editions will prove this assertion and all unproven propositions herein.

Function Inverses (53) immediately needs:

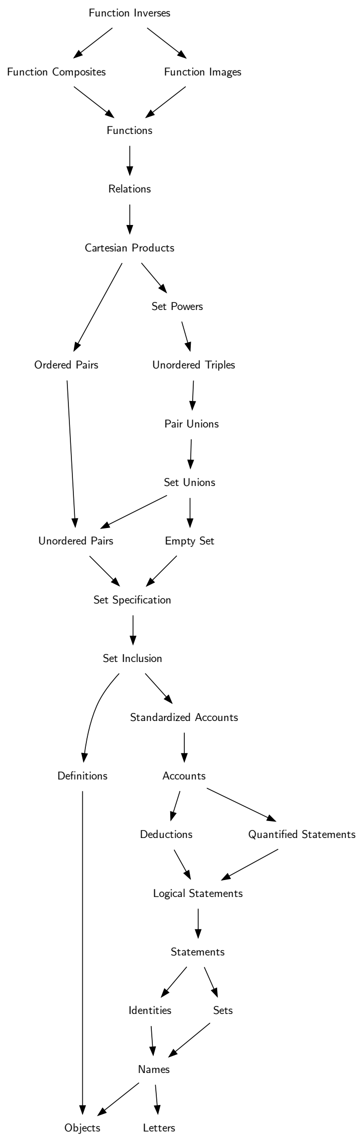
Function Composites (52)
Function Images (46)

Function Inverses (53) is immediately needed by:

Equivalent Sets (70)
Inverse Elements (83)
Isometries (**??**)
Permutations (**??**)

Function Inverses (53) gives the following terms.

inverse functions
inverse
the inverse
invertible
bijection



Function Inverses
Function Composites
Function Images
Functions
Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
Empty Set
Set Specification
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Why

The inverse of a function interacts nicely with family unions, family intersections and complements.

Results

Let $f : X \to Y$. Throughout this sheet, let $f^{-1} : Y^* \to X^*$. And take $\{B_i\}$ to be a family of subsets of Y.[43]

Proposition 71. $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$

Proposition 72. $f^{-1}(\cup_i B_i) = \cap_i f^{-1}(B_i)$

Proposition 73. $f^{-1}(Y - B) = X - f^{-1}(B)$

Properties for Function Image

Notice that $f(\cup_i A_i) = \cup_i f(A_i)$ but not for interesctions. Nor is there a similar correspondence for complements. There are some relations, which we list below.[44]

Proposition 74. *$f(A \cap B) = f(A) \cap f(B)$ if and only if f is one-to-one.*

Proposition 75. *For all $A \subset X$, $f(X - A) = Y - f(A)$ if and only if f is one-to-one.*

Proposition 76. *For all $A \subset X$, $Y - f(A) \subset f(X - A)$ if and only if f is onto.*

[43] The proofs of the following will appear in future editions.

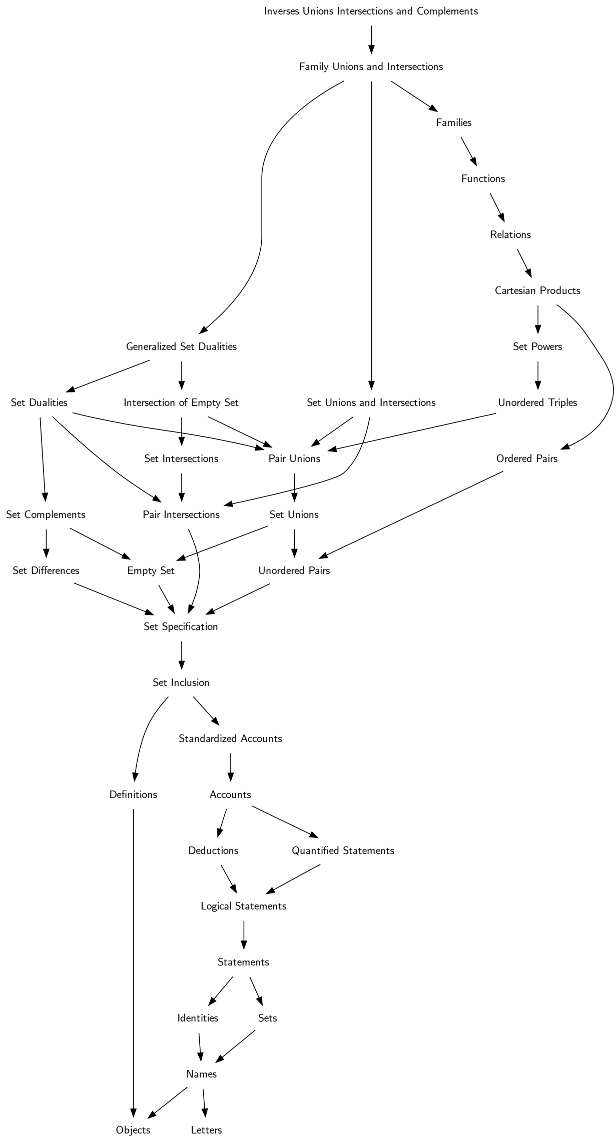
[44] Accounts of these facts will appear in future editions.

Inverses Unions Intersections and Complements (54) immediately needs:

Family Unions and Intersections (49)

Inverses Unions Intersections and Complements (54) is not immediately needed by any sheet.

Inverses Unions Intersections and Complements (54) gives no terms.


Inverses Unions Intersections and Complements
Family Unions and Intersections
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Generalized Set Dualities
Set Powers
Set Dualities
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Set Unions and Intersections
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Why

If x is related to y and y to z, then x and z are related.

Definition

Let R be a relation from X to Y and S a relation from Y to Z. The *composite relation* from X to Z contains the pair $(x, z) \in (X \times Z)$ if and only if there exists a $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S$. This composite relation is sometimes called the *relative product.*

Notation

We denote the composite relation of R and S by $R \circ S$ or RS.

Example

Let X be the set of people and let R be the relation in X "is a brother of" and S be the relation in X "is a father of". Then RS is the relation "is an uncle of".

Properties

Composition of relation is associative but not commutative.[45]

[45] A fuler account will appear in future editions.

Relation Composites (55) immediately needs:

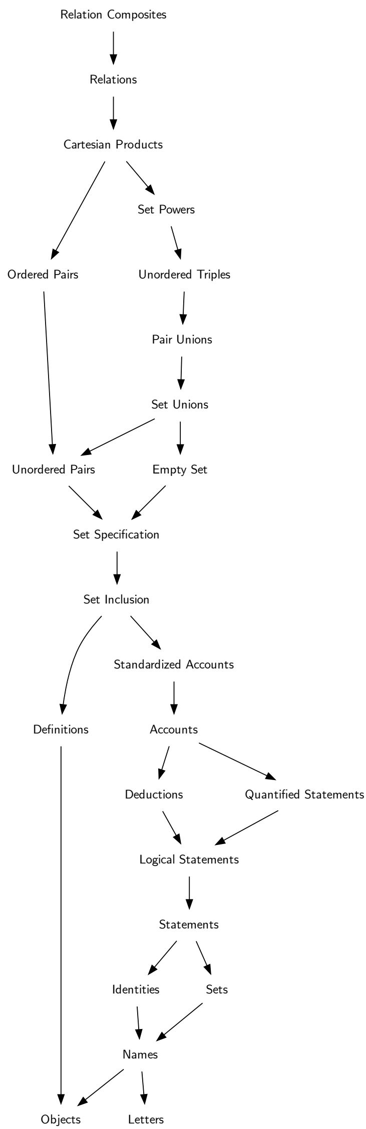
Relations (42)

Relation Composites (55) is immediately needed by:

Inverses of Composite Relations (57)

Relation Composites (55) gives the following terms.

composite relation
relative product


Relation Composites
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Unordered Triples
Pair Unions
Set Unions
Unordered Pairs
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Why

If x is related to y, the y is related to x, but how?

Definition

If R is a relation between X and Y, then the *converse* or *inverse* relation of R is a relation on Y and X relating $y \in Y$ to $x \in X$ if and only if $x \, R \, y$. If $R = R^{-1}$ then R is symmetric.

Notation

We denote the converse relation of R by R^{-1}.

Example

Let X be the set of people and let R be a relation in X. If R is "is a father of", then R^{-1} is "is a son of". If R is "is a mother of", then R^{-1} is "is a daughter of". If R is "is a brother of", then R^{-1} is "is a brother of". The relation "is a brother of" is symmetric.

Converse Relations (56) immediately needs:

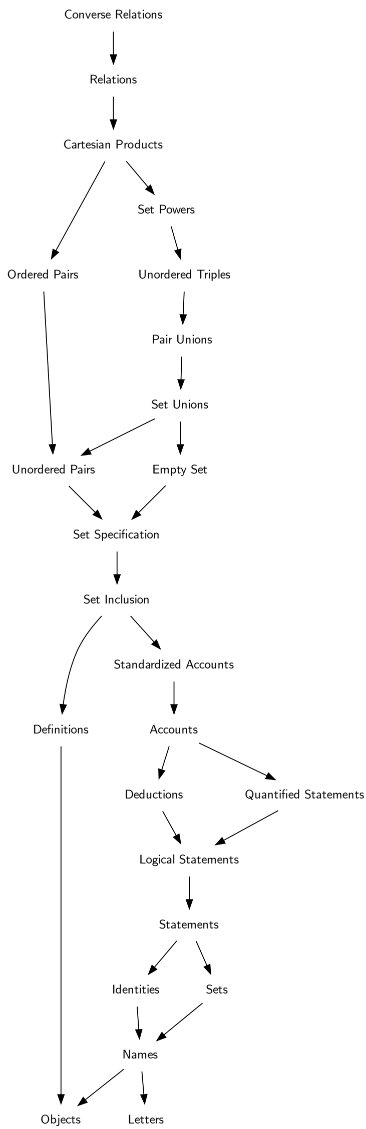
Relations (42)

Converse Relations (56) is immediately needed by:

Inverses of Composite Relations (57)

Converse Relations (56) gives the following terms.

converse
inverse


Converse Relations
Relations
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Why

How do inverse and converse relations interact.

Results

Let R be a relation between X and Y and let S be a relation between Y and Z.

Proposition 77. $(RS)^{-1} = S^{-1}R^{-1}$

Identity Relations

Recall that I is the identity relation on X if $x\,I\,y$ if and only if $x = y$.

Proposition 78. *Let R be a relation on X. Let I be the identity relation on X. Then $RI = IR = R$.*

One would like $RR^{-1} \supset I$, $R^{-1}R \supset I$. The father of the son is the father and the son of the father is the son. But the empty relation violates these claims.

Relation Properties

Proposition 79. *R is symmetric if and only if $R \subset R^{-1}$*

Proposition 80. *R is reflextive if and only if $I \subset R$*

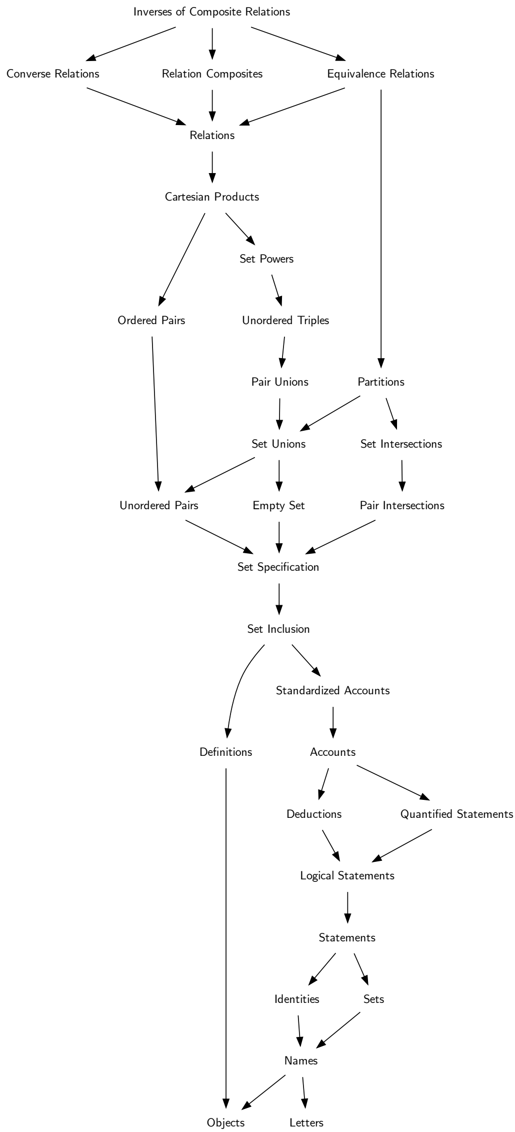
Proposition 81. *R is transitive if and only if $RR \subset R$.*

Inverses of Composite Relations (57) immediately needs:

Converse Relations (56)
Equivalence Relations (43)
Relation Composites (55)

Inverses of Composite Relations (57) is not immediately needed by any sheet.

Inverses of Composite Relations (57) gives no terms.



Inverses of Composite Relations
Converse Relations
Relation Composites
Equivalence Relations
Relations
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Partitions
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
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Why

We want numbers to count with.[46]

Definition

The *successor* of a set is the set which is the union of the set with the singleton of the set. In other words, the successor of a set A is $A \cup \{A\}$. This definition has sense for any set, but is of interest only for those particular sets introduced here.

These sets are the following (and their successors): We call the empty set *zero*.[47] We call the successor of the empty set *one*. In other words, one is $\varnothing \cup \{\varnothing\} = \{\varnothing\}$. We call the successor of one *two*. In other words, two is $\{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$. Likewise, the successor of two we call *three* and the successor of three we call *four*. And we continue as usual,[48] using the English language in the typical way.

A set is a *successor set* if it contains zero and if it contains the successor of each of its elements.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

[46]Future editions will expand on this sheet with a more justified why.

[47]In future editions, zero may be a separate sheet.

[48]Future editions will assume less in the introduction of natural numbers.

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4. So

$$
\begin{aligned}
0 &= \varnothing \\
1 &= 0^+ = \{0\} \\
2 &= 1^+ = \{0, 1\} \\
3 &= 2^+ = \{0, 1, 2\} \\
4 &= 3^+ = \{0, 1, 2, 3\}
\end{aligned}
$$

Successor Sets (58) immediately needs:

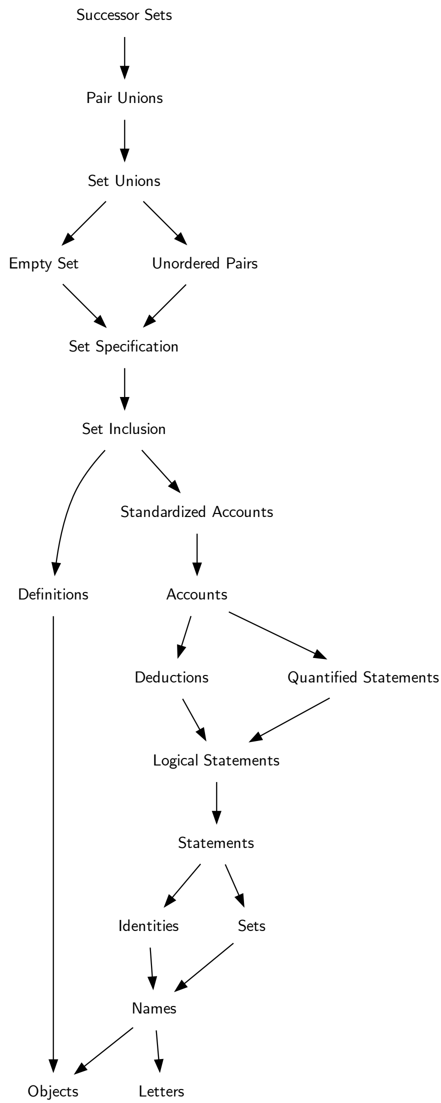
Pair Unions (20)

Successor Sets (58) is immediately needed by:

Natural Numbers (59)

Successor Sets (58) gives the following terms.

successor
zero
one
two
three
four
successor set


Successor Sets
Pair Unions
Set Unions
Empty Set
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Why

Does a set exist which contains zero, and on and two, and three, and all the rest?

Definition

In Successor Sets, we said "and we continue as usual using the English language..." in our definition of zero, and one and two and three. Can this really be carried on and on? We will say yes. We will say that there exists a set which contains zero and contains the successor of each of its elements.

Principle 7 (Natural Numbers)**.** *A set which contains 0 and contains the successor of each of its elements exists.*

This principle is sometimes also called the *principle of infinity.*

We want this set to be unique. The principle says one successor set exists, but not that it is unique. To see that it is unique, notice that the intersection of a nonempty family of successor sets is a successor set.[49] Consider the intersection of the family of all successor sets. The intersection is nonempty by the principle of infinity (see Intersection of Empty Set for this subtlety). The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique. We summarize:

[49] This account will be expanded in future editions.

Proposition 82. *There exists a unique successor smallest successor set.*

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

We denote the set which exists by Proposition 82 by ω. [50] We denote the set of natural numbers without 0 by N, a mnemonic for natural. In other words $\mathsf{N} = \omega - \{0\}$. We often denote elements of ω or N by n, a mnemonic for number, or m, a letter close to n.

We denote the first natural numbers up to n by $\{1, 2, \dots, n\}$. We have defined n so that $n - \{0\} = \{1, 2, \dots, n\}$.

[50] We use this notation to follow many authorities on the subject, and to meet the exigencies of time in producing this first edition. Future editions are likely to rework the treatment.

Natural Numbers (59) immediately needs:

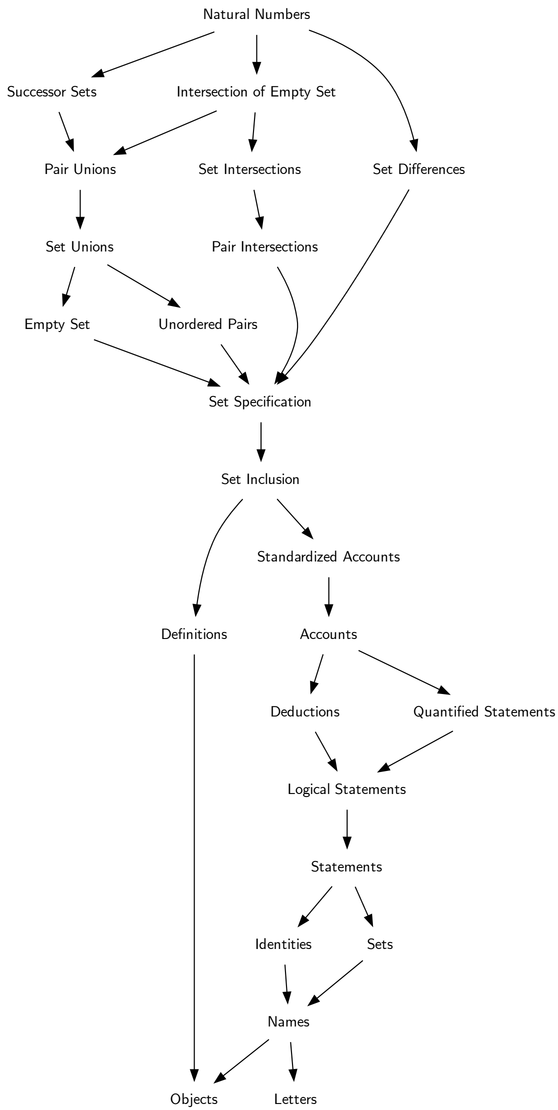
- Intersection of Empty Set (24)
- Set Differences (26)
- Successor Sets (58)

Natural Numbers (59) is immediately needed by:

- Cardinality (**??**)
- Characteristic Functions (**??**)
- Coins (**??**)
- Dice (**??**)
- Equivalent Sets (70)
- Integer Numbers (84)
- Natural Induction (62)
- Natural Numbers Exercises (**??**)
- Sequences (60)

Natural Numbers (59) gives the following terms.

principle of infinity
natural number
number
natural
set of natural numbers
natural numbers
naturals
numbers
zero
natural numbers with zero
addition



Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
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Why

We introduce language for the steps of an infinite process.

Definition

A *finite sequence* is a family whose index set is a natural number (excluding zero). An *infinite sequence* is a family whose index set is the set of natural numbers (without zero). The *nth term* of a sequence (finite or infinite) is the result of the nth natural number. Let A be a non-empty set. A sequence *in A* is a function from the natural numbers to the set.

Notation

Let A be a non-empty set. Let $a : \mathsf{N} \to A$ Then a is a sequence in A. $a(n)$ is the nth term. We also denote a by $(a_n)_n$ and $a(n)$ by a_n.

Natural Unions and intersections

If $\{A_i\}$ is a finite sequence of sets indexed by $\{1, 2, \dots, n\}$, then we denote the union of the family by

$$\cup_{i=1}^{n} A_i$$

If $\{A_i\}$ is an infinite sequence of sets, then we denote the union of the family by

$$\cup_{i=1}^{\infty} A_i.$$

Similarly, we denote the intersections of a finite and infinite sequence of sets $\{A_i\}$ by

$$\cap_{i=1}^{n} A_i \quad \text{and} \quad \cap_{i=1}^{\infty} A_i.$$

respectively.

Sequences (60) immediately needs:

Families (48)

Natural Numbers (59)

Sequences (60) is immediately needed by:

Almost Everywhere (??)

Central Limit Theorem (??)

Egoprox Sequences (??)

Monotone Algebras (??)

Monotone Classes (??)

Monotone Sequences (??)

Natural Direct Products (61)

Nets (??)

Random Variable Sigma Algebra (??)

Real Continuity (??)

Real Integral Series Convergence (??)

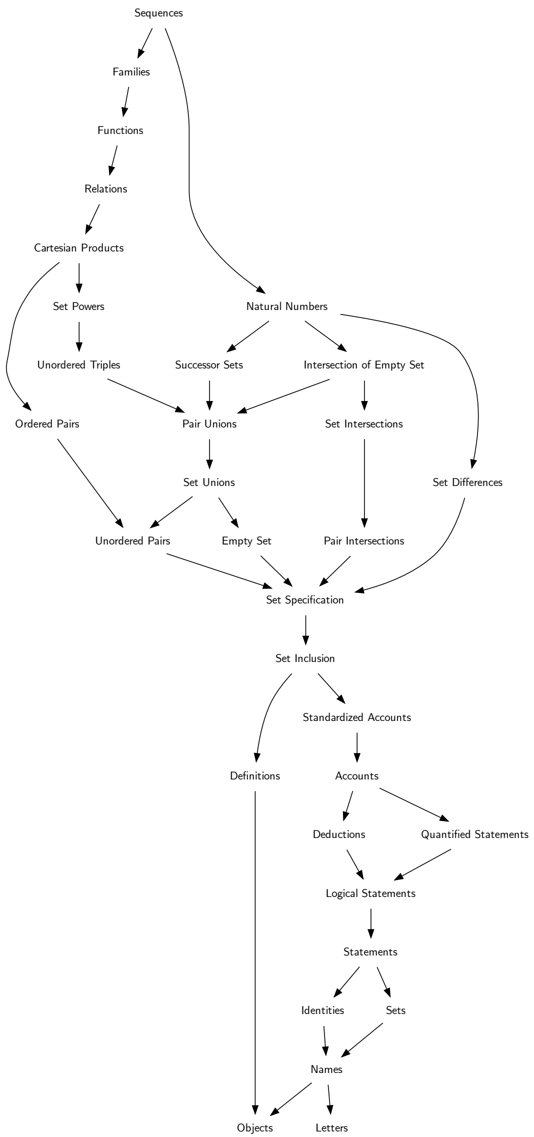
Real Sequences (??)

Subsequences (74)

Tail Sigma Algebra (??)

Sequences (60) gives the following terms.

finite sequence
infinite sequence
nth term
in


Sequences
Families
Functions
Relations
Cartesian Products
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Ordered Pairs
Pair Unions
Set Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
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Why

We want notation for the Cartesian (direct) product of a sequence of sets

Definition

A *natural direct product* is the cartesian product of a sequence.

Notation

Let $\{A_i\}$ be a sequence of sets. If $\{A_i\}$ is finite and indexed by $n - \{\varnothing 0\}$ we denote the product of the sequence by

$$\prod_{i=1}^{n} A_i$$

and if infinite, then by

$$\prod_{i=1}^{\infty} A_i.$$

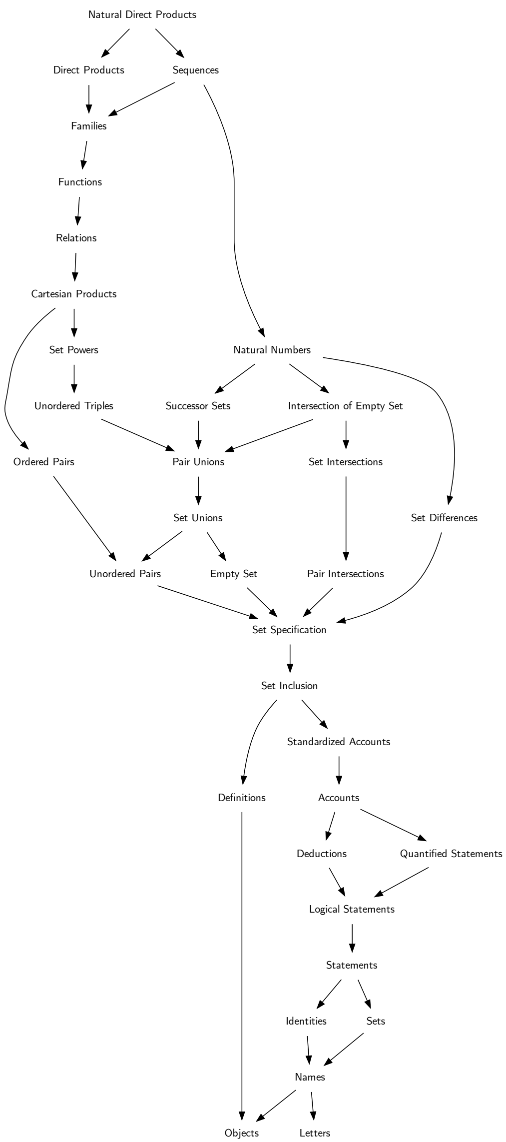
Natural Direct Products (61) immediately needs:

Direct Products (50)
Sequences (60)

Natural Direct Products (61) is not immediately needed by any sheet.

Natural Direct Products (61) gives the following terms.

natural direct product


Natural Direct Products
Direct Products
Sequences
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Set Powers
Natural Numbers
Unordered Triples
Successor Sets
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Ordered Pairs
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Why

We want to show something holds for every natural number.[51]

Definition

The most important property of the set of natural numbers is that it is the unique smallest successor set. In other words, if S is a successor set contained in ω (see Natural Numbers), then $S = \omega$. This is useful for proving that a particular property holds for the set of natural numbers.

To do so we follow standard routine. First, we define the set S to be the set of natural numbers for which the property holds. This step uses the principle of selection (see Set Selection) and ensures that $S \subset \omega$. Next we show that this set S is indeed a successor set. The first part of this step is to show that $0 \in S$. The second part is to show that $n \in S \longrightarrow n^+ \in S$. These two together mean that S is a successor set, and since $S \subset \omega$ by definition, that $S = \omega$. In other words, the set of natural numbers for which the property holds is the entire set of natural numbers. We call this the *principle of mathematical induction.*

[51] Future editions will modify this superficial why.

Natural Induction (62) immediately needs:

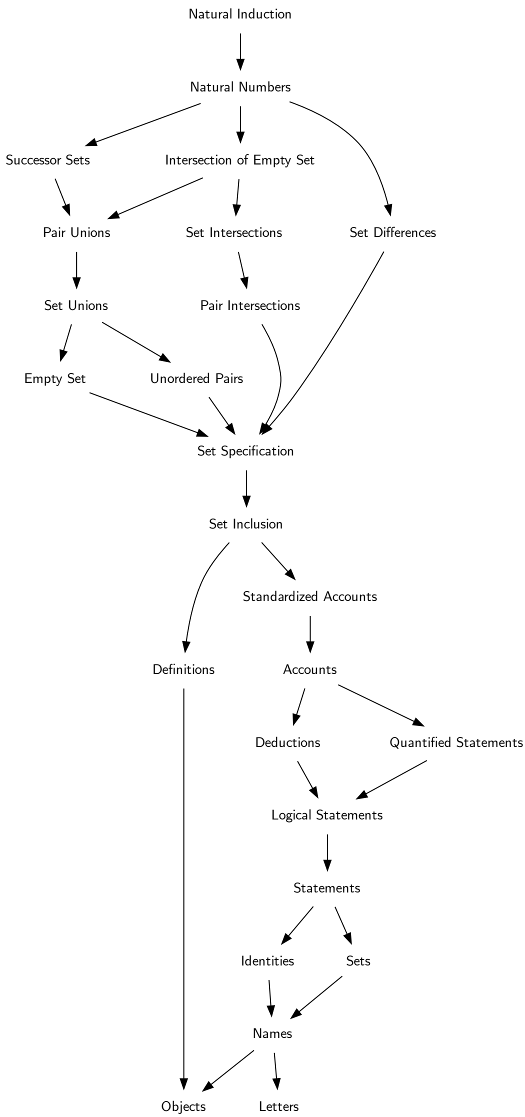
Natural Numbers (59)

Natural Induction (62) is immediately needed by:

Peano Axioms (63)

Natural Induction (62) gives the following terms.

Peano's axioms
principle of mathematical induction.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Why

Historically considered a fountainhead for all of mathematics.

Discussion

So far we know that ω is the unique smallest successor set. In other words, we know that $0 \in \omega$, $n \in \omega \longrightarrow n^+ \in \omega$ and that if these two properties hold of some $S \subset \omega$, then $S = \omega$. We can add two important statements to this list. First, that 0 has no successor. I.e., $n^+ \neq 0$ for all $n \in \omega$. Second, that if two numbers have the same successor, then they are the same number I.e., $n^+ = m^+ \longrightarrow n = m$

These five properties were historically considered the fountainhead of all of mathematics. One by the name of Peano used them to show the elementary properties of arithmetic. They are:

1. $0 \in \omega$.
2. $n \in \omega \longrightarrow n^+ \in \omega$ for all $n \in \omega$.
3. If S is a successor set contained in ω, then $S = \omega$.
4. $n^+ \neq 0$ for all $n \in \omega$
5. $n^+ = m^+ \longrightarrow n = m$ for all $n, m \in \omega$.

These are collectively known as the *Peano axioms*. Recall that the third statement in this list is the *principle of mathematical induction.*

Statements

Here are the statements.[52]

Proposition 83 (Peano's First Axiom)**.** $0 \in \omega$.

Proposition 84 (Peano's Second Axiom)**.** $n \in \omega \longrightarrow n^+ \in \omega$.

Proposition 85 (Peano's Third Axiom)**.** *Suppose* $S \subset \omega$, $0 \in S$, *and* $(n \in S \longrightarrow n^+ \in S$. *Then* $S = \omega$.

Proposition 86 (Peano's Fourth Axiom)**.** $n^+ \neq 0$ *for all* $n \in \omega$.

The last one uses the following two useful facts.

Proposition 87. $x \in n \longrightarrow n \not\subset x$.

Proposition 88. $(x \in y \wedge y \in n) \longrightarrow x \in n$

This latter proposition is sometimes described by saying that n is a *transitive set*. This notion of transitivity is not the same as that described in Relations. Using these one can show:

Proposition 89 (Peano's Fifth Axiom)**.** *Suppose* $n, m \in \omega$ *with* $n^+ = m^+$. *Then* $n = m$.

[52]Accounts of all of these will appear in future editions.

Peano Axioms (63) immediately needs:

Natural Induction (62)

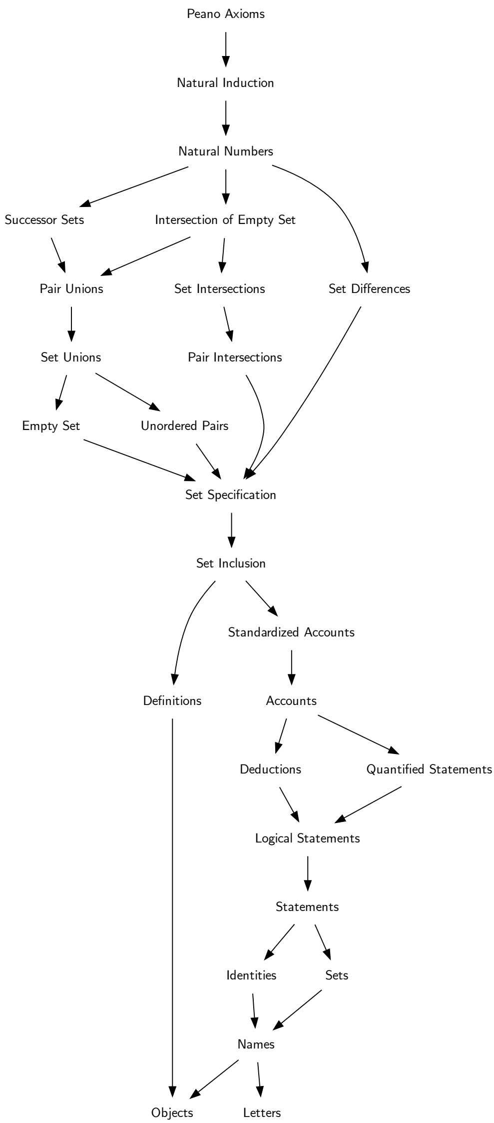
Peano Axioms (63) is immediately needed by:

Natural Order (68)

Recursion Theorem (64)

Peano Axioms (63) gives the following terms.

Peano axioms
principle of mathematical induction
transitive set


Peano Axioms
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Why

It is natural to want to define a sequence by giving its first term and then giving its later terms as functions of its earlier ones. In other words, we want to define sequences inductively.[53]

Main Result

The following is often referred to as the *recurion theorem.*

Proposition 90 (Recursion Theorem[54]). *Let X be a set, let $a \in X$ and let $f : X \to X$. There exists a unique function u so that $u(0) = a$ and $u(\succ (n)) = f(u(n))$.*[55]

When one uses the recursion theorem to assert the existence of a function with the desired properties, it is called *definition by induction.*

[53] Future editions will expand on this. We are really headed toward natural addition, multiplication and exponentiation.

[54] Future editions will likely change this name.

[55] The account is somewhat straightforward, given a good understanding of the results of Peano Axioms. The full account will appear in future editions.

Recursion Theorem (64) immediately needs:

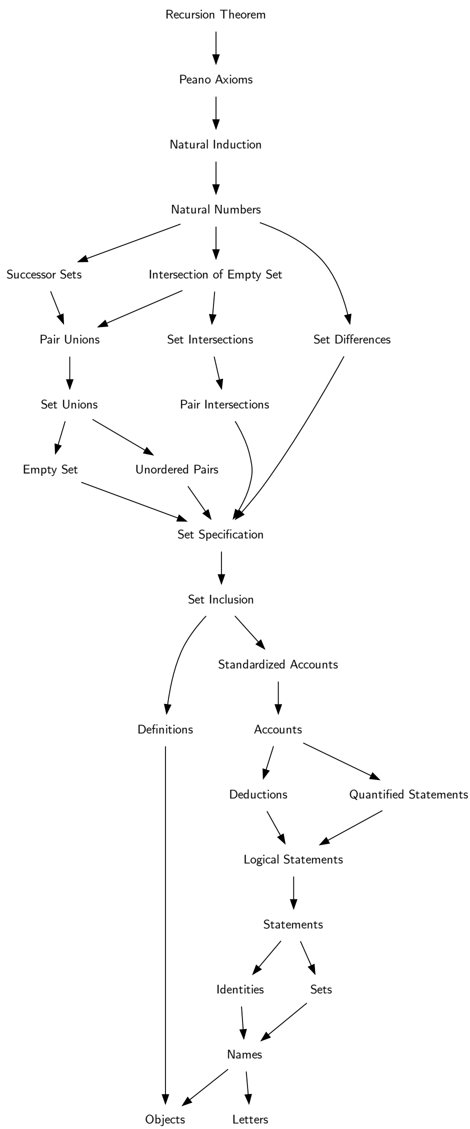
Peano Axioms (63)

Recursion Theorem (64) is immediately needed by:

Natural Sums (65)

Recursion Theorem (64) gives the following terms.

recurion theorem
definition by induction


Recursion Theorem
Peano Axioms
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Why

We want to combine two groups.[56]

Defining Result

Proposition 91. *For each natural number m, there exists a function $s_m : \omega \to \omega$ which satisfies*

$$s_m(0) = m \quad \text{and} \quad s_m(n^+) = (s_m(n))^+$$

for every natural number n.

Proof. The proof uses the recursion theorem (see Recursion Theorem).[57] □

Let m and n be natural numbers. The value $s_m(n)$ is the *sum* of m with n.

Notation

We denote the sum $s_m(n)$ by $m + n$.

Properties

The properties of sums are direct applications of the principle of mathematical induction (see Natural Induction).[58]

[56] Future editions will change this section.
[57] Future editions will give the entire account.
[58] Future editions will include the accounts.

Proposition 92 (Associative)**.** *Let k, m, and n be natural numbers. Then*

$$(k+m)+n = k+(m+n).$$

Proposition 93 (Commutative)**.** *Let m and n be natural numbers. Then*

$$m+n = n+m.$$

Relation to Addition

Proposition 94 (Distributive)**.** *Let k, m, and n be natural numbers. Then*

$$k \cdot (m+n) = (k \cdot m) + (k \cdot n).$$

Natural Sums (65) immediately needs:

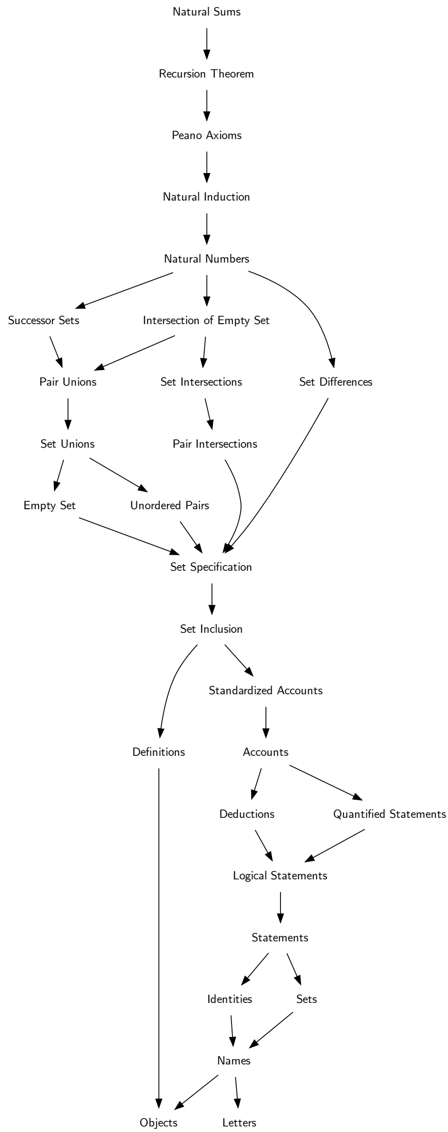
Recursion Theorem (64)

Natural Sums (65) is immediately needed by:

Integer Order (87)
Integer Sums (85)
Natural Differences (??)
Natural Products (66)
Natural Summation (??)

Natural Sums (65) gives the following terms.

sum


Natural Sums
Recursion Theorem
Peano Axioms
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Why

We want to add repeatedly.

Defining Result

Proposition 95. *For each natural number m, there exists a function $p_m : \omega \to \omega$ which satisfies*

$$p_m(0) = 0 \quad \textit{and} \quad p_m(n^+) = (p_m(n))^+ + m$$

for every natural number n.

Proof. The proof uses the recursion theorem (see Recursion Theorem).[59] □

Let m and n be natural numbers. The value $p_m(n)$ is the *product* of m with n.

Notation

We denote the product $p_m(n)$ by $m \cdot n$. We often drop the $\cdot$ and write $m \cdot n$ as mn.

Properties

The properties of products are direct applications of the principle of mathematical induction (see Natural Induction).[60]

[59] Future editions will give the entire account.
[60] Future editions will include the accounts.

Proposition 96 (Associativity)**.** *Let k, m, and n be natural numbers. Then*

$$(k \cdot m) \cdot n = k \cdot (m \cdot n).$$

Proposition 97. *Let m and n be natural numbers. Then*

$$m \cdot n = n \cdot m.$$

Natural Products (66) immediately needs:

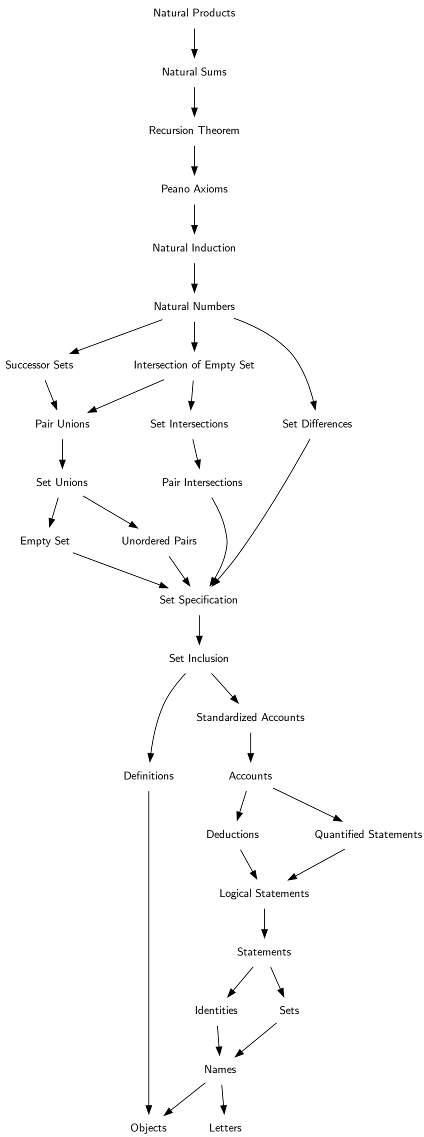
Natural Sums (65)

Natural Products (66) is immediately needed by:

Integer Products (86)
Natural Exponents (67)
Order and Arithmetic (69)
Square Numbers (**??**)

Natural Products (66) gives the following terms.

product
sum
add
addition
product
multiply
multiplication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Why

We want to repeatedly multiply.

Defining Result

Proposition 98. *For each natural number m, there exists a function $e_m : \omega \to \omega$ which satisfies*

$$e_m(0) = 1 \quad \textit{and} \quad e_m(n^+) = (e_m(n))^+ \cdot m$$

for every natural number n.

Proof. The proof uses the recursion theorem (see Recursion Theorem).[61] □

Let m and n be natural numbers. The value $p_m(n)$ is the *power* of m with n. Or the *nth power of m*

Notation

We denote the nth power of m by m^n.

Properties

Here are some basic properties of powers.

Proposition 99. *Let k, m, and n be natural numbers. Then*

$$m^n m^k = m^{k+k}.$$

[61] Future editions will give the entire account.

Proposition 100. *Let k, m, and n be natural numbers. Then*

$$(m^n)^k = m^{nk}.$$

Natural Exponents (67) immediately needs:

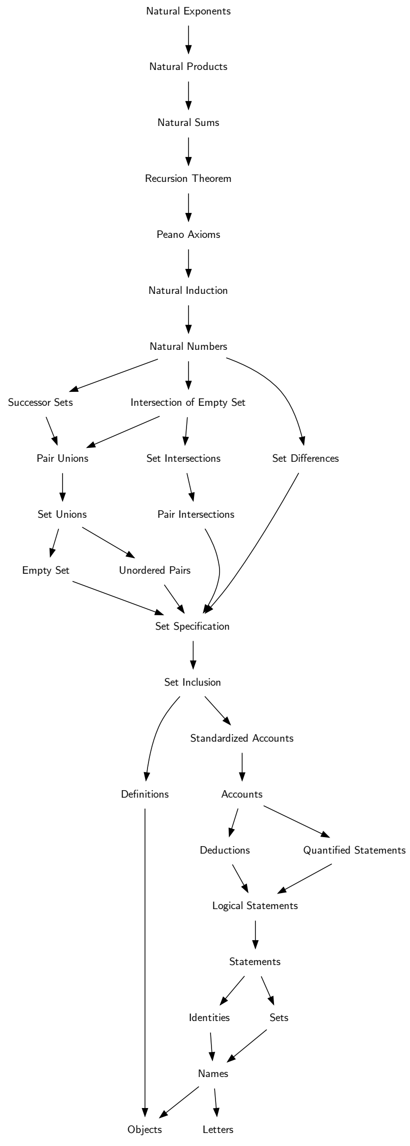
Natural Products (66)

Natural Exponents (67) is immediately needed by:

Arithmetic (77)
Set Numbers and Arithmetic (73)

Natural Exponents (67) gives the following terms.

power
nth power of m


Natural Exponents
Natural Products
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Why

We count in order.[62]

Defining Result

We say that two natural numbers m and n are *comparable* if $m \in n$ or $m = n$ or $n \in m$.

Proposition 101. *Any two natural numbers are comparable.*[63]

In fact, more is true.

Proposition 102. *For any two natural numbers, exactly one of $m \in n$, $m = n$ and $n \in m$ is true.*[64]

Proposition 103. $m \in n \longleftrightarrow m \subset n$.

If $m \in n$, then we say that m is *less than* n. We also say in this case that m is *smaller than* n. If we know that $m = n$ or m is less than n, we say that m is *less than or equal to* n.

Notation

If m is less than n we write $m < n$, read aloud "m less than n." If m is less than or equal to n, we write $m \leqq n$, read alout "m less than or equal to n."

[62] Future editions will expand.

[63] Future editions will include an account.

[64] Use the fact that no natural number is a subset of itself. Future editions will expand this account. See Peano Axioms).

Properties

Notice that $<$ and $\leqq$ are relations on ω (see Relations).[65]

Proposition 104 (Reflexivity). *$\leqq$ is reflexive, but $<$ is not.*

Proposition 105 (Symmetry). *Both $\leqq$ and $<$ are not symmetric.*

Proposition 106 (Transitivity). *Both $\leqq$ and $<$ are transitive.*

Proposition 107 (Antisymmetry). *If $m \leqq n$ and $n \leqq n$, then $m = n$.*

[65] Proofs of the following propositions will appear in future editions.

Natural Order (68) immediately needs:

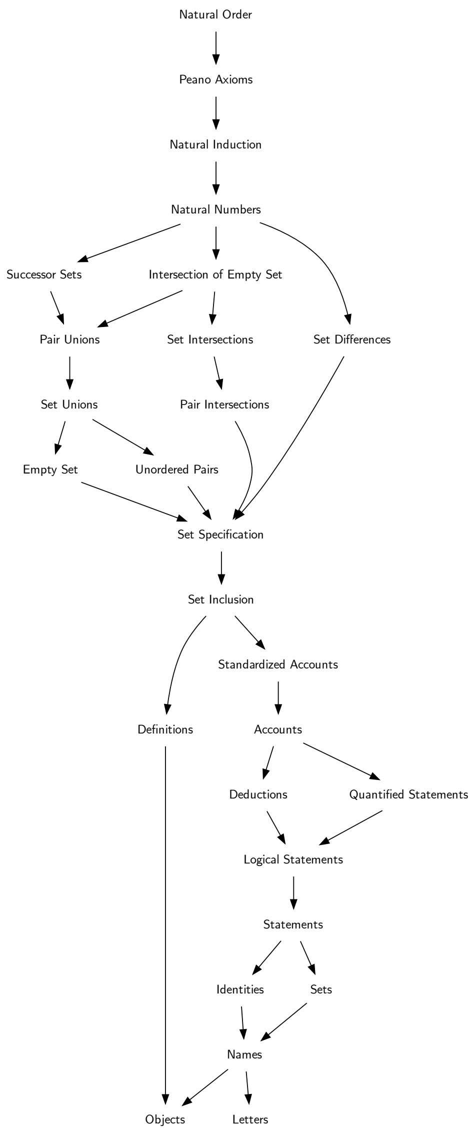
Peano Axioms (63)

Natural Order (68) is immediately needed by:

Equivalent Sets (70)
Order and Arithmetic (69)

Natural Order (68) gives the following terms.

Peano's axioms
comparable
less than
smaller than
less than or equal to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Why

How does arithmetic preserve order?

Results

The following are standard useful results.[66]

Proposition 108. *If $m < n$, then $m + k < n + k$ for all k.*

Proposition 109. *If $m < n$ and $k \neq 0$, then $m \cdot k < n \cdot k$.*

Proposition 110 (Least Element)**.** *If E is a nonempty set of natural numbers, there exists $k \in E$ such that $k \leq m$ for all $m \in E$.*

Proposition 111 (Greatest Element)**.** *If E is a nonempty set of natural numbers, there exists $k \in E$ such that $m \leq k$ for all $m \in E$.*

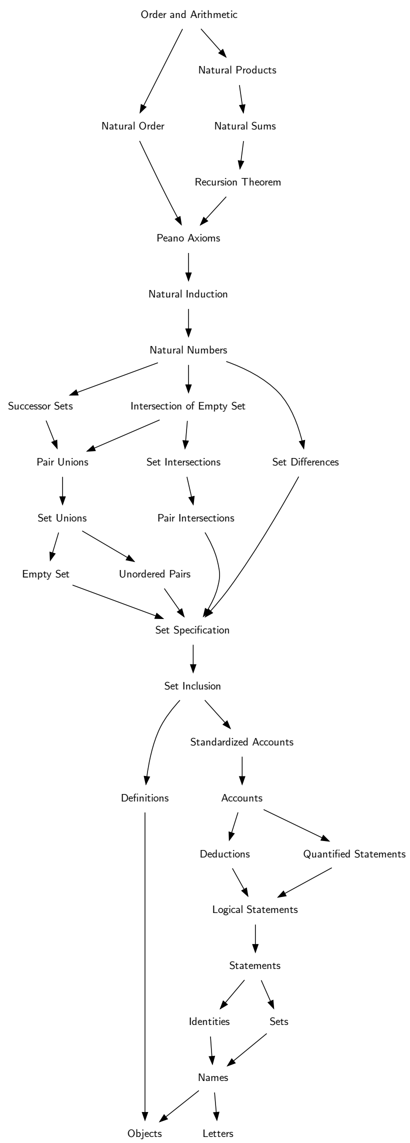
[66] The accounts of which will appear in future editions.

Order and Arithmetic (69) immediately needs:

Natural Order (68)
Natural Products (66)

Order and Arithmetic (69) is not immediately needed by any sheet.

Order and Arithmetic (69) gives no terms.


Order and Arithmetic
Natural Products
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Why

We want to talk about the size of a set.

Definition

Two sets are *equivalent* if there exists a bijection between them. Let X be a set. Then set equivalence as a relation in X^* is an equivalence relation (see Equivalence Relations).

Notation

If A and B are sets and they are equivalent, then we write $A \sim B$, read aloud as "A is equivalent to B."

Basic Result

Every set is equivalent to itself, whether two sets are equivalent does not depend on the order in which we consider them, and if two sets are equivalent to the same set then they are equivalent to each other. These facts can be summarized by the following proposition.

Proposition 112. *Let X a set. Then $\sim$ is an equivalence relation on X^*.*[67]

For natural numbers

Proposition 113. *Every proper subset of a natural number is equivalent to some smaller natural number.*[68]

[67] The proof is direct and will appear in future editions.
[68] The proof, which uses induction, will appear in future editions.

Equivalence to subsets

It is unusual that a set can be equivalents to a proper subset of itself.

Proposition 114. *A set may be equivalent to a proper subset of itself.*

Proof. The example is the set of natural numbers and the function $f(n) = n^+$. It is a bijection from ω onto $\mathbf{N}$.[69] $\square$

However, this never holds for natural numbers.

Proposition 115. *If $n \in \omega$ then $n \not\sim x$ for any $x \subset n$ and $x \neq n$.*

[69] The account will be expanded in future editions.

Equivalent Sets (70) immediately needs:

Equivalence Relations (43)

Function Inverses (53)

Natural Numbers (59)

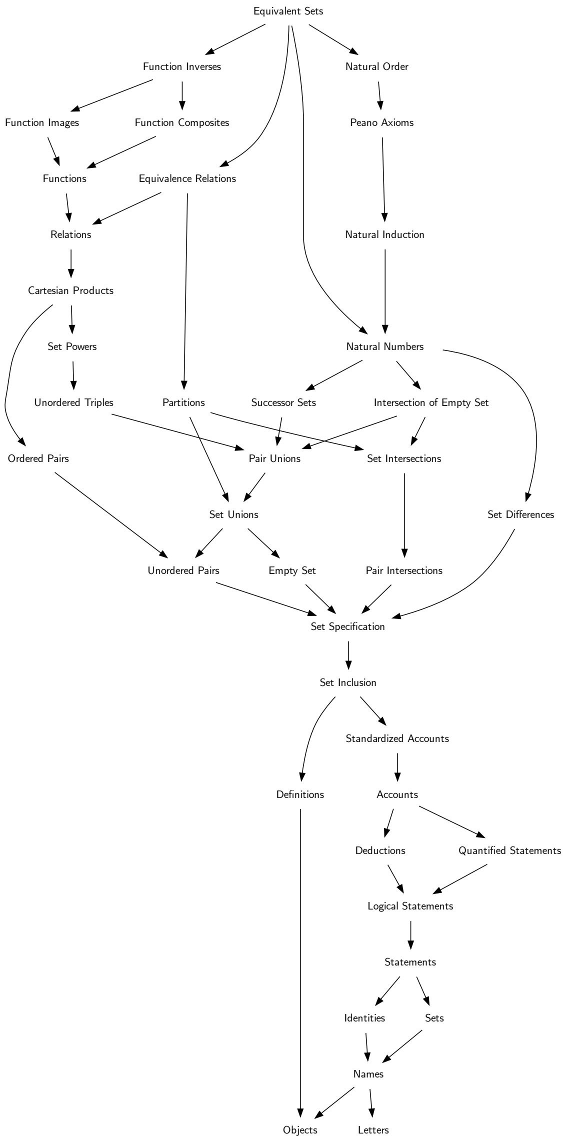
Natural Order (68)

Equivalent Sets (70) is immediately needed by:

Finite Sets (71)

Equivalent Sets (70) gives the following terms.

equivalent


Equivalent Sets
Function Inverses
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Function Images
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Why

We want to talk about the size of a set.[70]

Definition

A *finite* set is one that is equivalent to some natural number; an infinite set is one which is not finite. From this we can show that ω is infinite. This justifies the language we used in Natural Numbers about the principle of infinity. The principle of infinity asserts the existence of a particular infinite set; namely ω.

Motivation for set number

It happens that if a set is equivalent to a natural number, it is equivalent to only one natural number.

Proposition 116. *A set can be equivalent to at most one natural number.*[71]

A consequence is that a finite set is never equivalent to a proper subset of itself. So long as we are considering finite sets, a piece (subset) is always less than than the whole (original set).

Proposition 117. *A finite set is never equivalent to a proper subset of itself.*

[70]Will be expanded in future editions.

[71]Future edition will include proof, which uses comparability of numbers and the results of Equivalent Sets).

Subsets of finite sets

Every subset of a natural number is equivalent to a natural number.[72] A consequence is:

Proposition 118. *Every subset of a finite set is finite.*[73]

Unions of finite sets

Proposition 119. *if A and B are finite, then $A \cup B$ is finite.*

Products of finite sets

Proposition 120. *If A and B are finite, then $A \times B$ is finite.*

Powers of finite sets

Proposition 121. *If A is finite then A^* is finite.*

Functions between finite sets

Proposition 122. *If A and B are finite, then A^B is finite.*

---

[72] This requires proof, and may become a proposition in future editions.

[73] An account will appear in future editions.

Finite Sets (71) immediately needs:

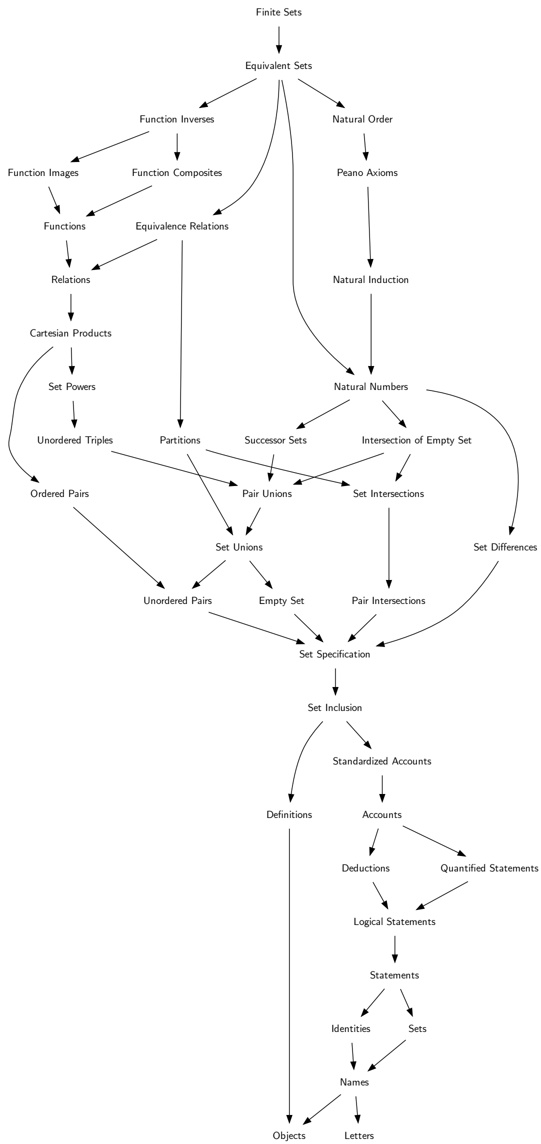
Equivalent Sets (70)

Finite Sets (71) is immediately needed by:

Number of Elements (72)
Numberings (**??**)
Undirected Graphs (**??**)

Finite Sets (71) gives the following terms.

finite



Finite Sets
Equivalent Sets
Function Inverses
Natural Order
Function Images
Function Composites
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Why

We want to count the number of elements in a set.[74]

Defining Result

Proposition 123. *A set can be equivalent to at most one natural number.*

The *number* of a finite set is the unique natural number equivalent to it. We also call this the *size* of the set.

Notation

We denote the number of a set by $|A|$.

Restriction to a finite set

If we restrict the function $E \mapsto |E|$ to the domain X^* of some set X then $|:| X^* \to \omega$ is a function.

Properties

Proposition 124. $A \subset B \longrightarrow |a| \leq |B|$

[74]In future editions, this sheet will likely be called "Set numbers".

Number of Elements (72) immediately needs:

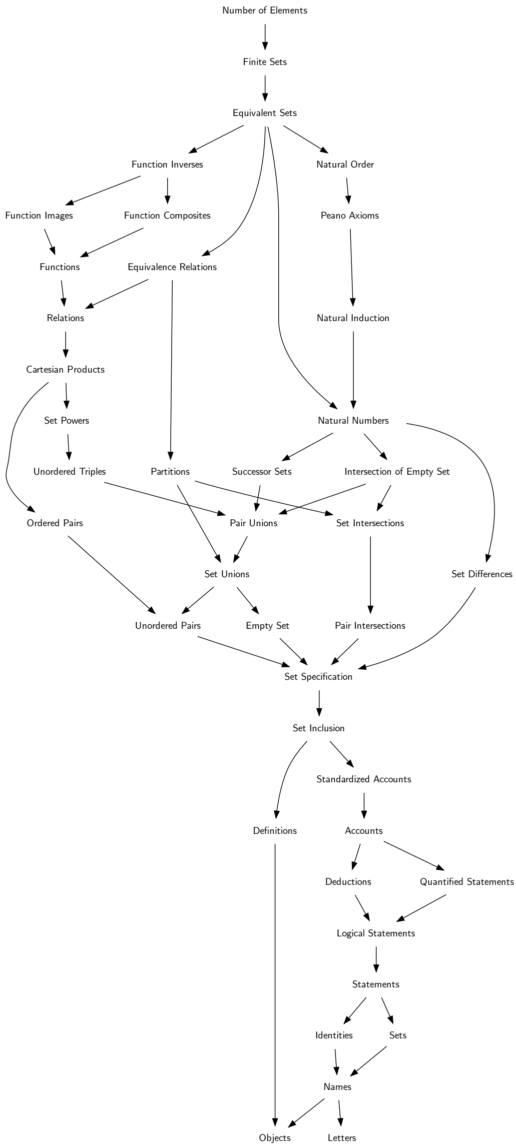
Finite Sets (71)

Number of Elements (72) is immediately needed by:

Set Numbers and Arithmetic (73)

Number of Elements (72) gives the following terms.

number
size


Number of Elements
Finite Sets
Equivalent Sets
Function Inverses
Natural Order
Function Images
Function Composites
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Why

How does the number of elements change with unions, and products.

Results

There are a few nice relations.[75] Recall from Finite Sets that the union and product of finite sets is finite. Also, the power of a finite set is finite.

Proposition 125. *Let A and B be finite sets with $A \cap B = \varnothing$. Then $|A \cup B| = |A| + |B|$.*

Proposition 126. *Let A and B be a finite sets Then $|A \times B| = |A| \cdot |B|$.*

Proposition 127. *Let A and B be a finite sets Then $\left|A^B\right| = |A|^{|B|}$.*

Proposition 128. *Let A be a finite set. Then $|A^*| = 2^{|A|}$.*

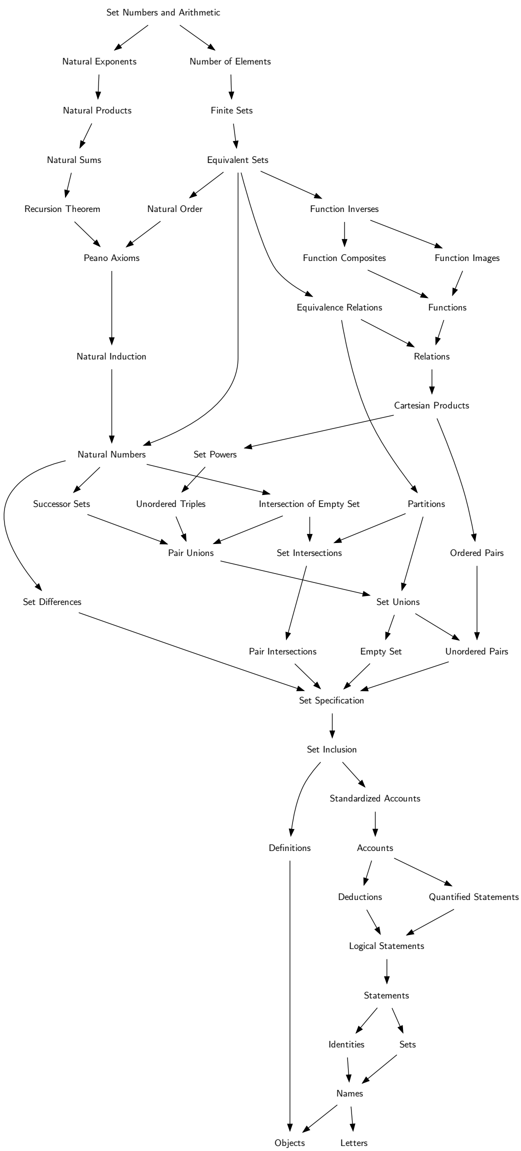
[75] Proofs will appear in future editions.

Set Numbers and Arithmetic (73) immediately needs:

Natural Exponents (67)
Number of Elements (72)

Set Numbers and Arithmetic (73) is not immediately needed by any sheet.

Set Numbers and Arithmetic (73) gives no terms.


Set Numbers and Arithmetic
Natural Exponents
Number of Elements
Natural Products
Finite Sets
Natural Sums
Equivalent Sets
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Natural Order
Function Inverses
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Function Composites
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Why

We want to select particular terms of sequence.

Definition

A *subindex* is a monotonically increasing function from and to the natural numbers. Roughly, it selects some ordered infinite subset of natural numbers. A *subsequence* of a first sequence is any second sequence which is the composition of the first sequence with a subindex.

Notation

Let $i : N \to N$ such that $n < m \longrightarrow i(n) < i(m)$. Then i is a subindex. Let $b = a \circ i$. Then b is a subsequence of a. We denote it by $\{b_{i(n)}\}_n$ and the nth term by $b_{i(n)}$.

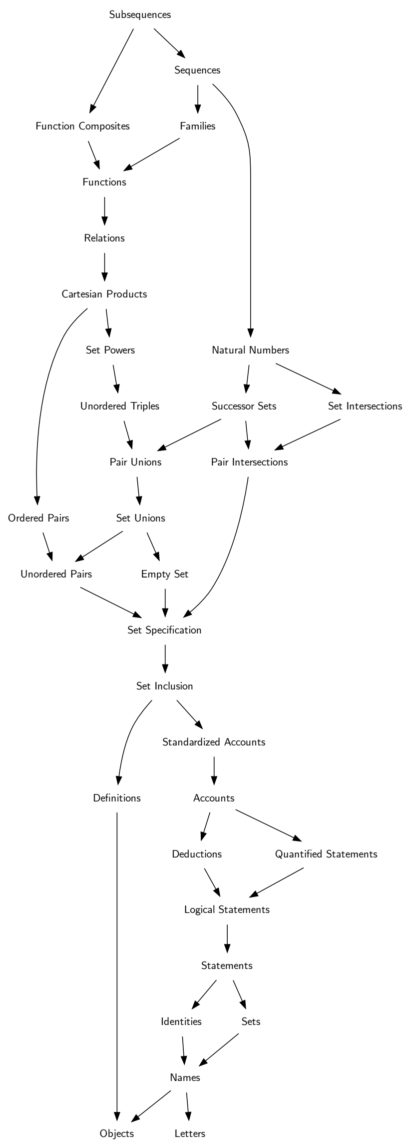
Subsequences (74) immediately needs:

Function Composites (52)
Sequences (60)

Subsequences (74) is not immediately needed by any sheet.

Subsequences (74) gives the following terms.

subindex
subsequence


Subsequences
Sequences
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Why

We want to "combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\,g\,b$ instead. We call this *infix notation.*

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Example

A first example of an operation is if we consider the set A as the power set of some set X. Then the pair union (see Pair Unions) is an operation. For if $E \in X^*$ and $F \in X^*$ then $E \cup F \in F^*$ and so $\cup$ can be viewed as an operation on X^*.

Properties

Recall that $\cup$ has several nice properties. For one $A \cup B = B \cup A$ and $(A \cup B) \cup C = A \cup (B \cup C)$.

An operation with the first property, that the ordered pair (A, B) and (B, A) have the same result is called *commutative*. An operation with the second property, that when given three objects the order in which we operate does not matter is called *associative*.

Operations (75) immediately needs:

Functions (44)

Operations (75) is immediately needed by:

Algebras (76)

Arithmetic (77)

Associative Operations (**??**)

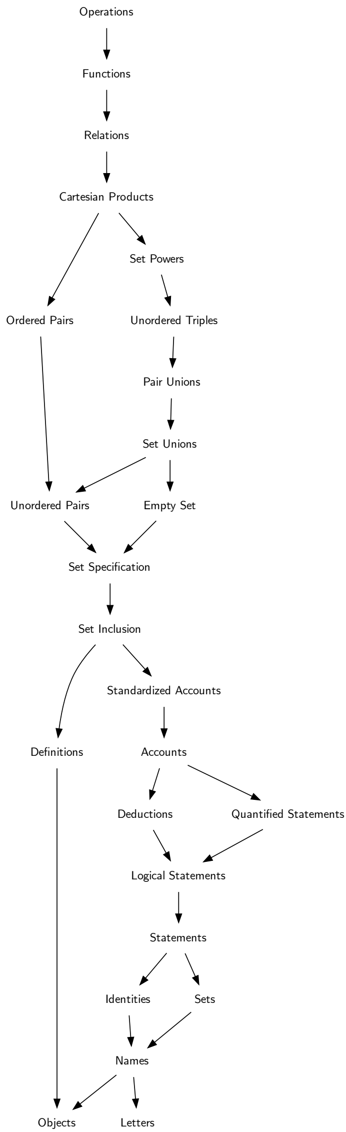
Commutative Operations (**??**)

Identity Elements (80)

Set Operations (78)

Operations (75) gives the following terms.

operation
combine
operate
infix notation
commutative
associative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Why

We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Algebras (76) immediately needs:

Operations (75)

Algebras (76) is immediately needed by:

Element Functions (79)

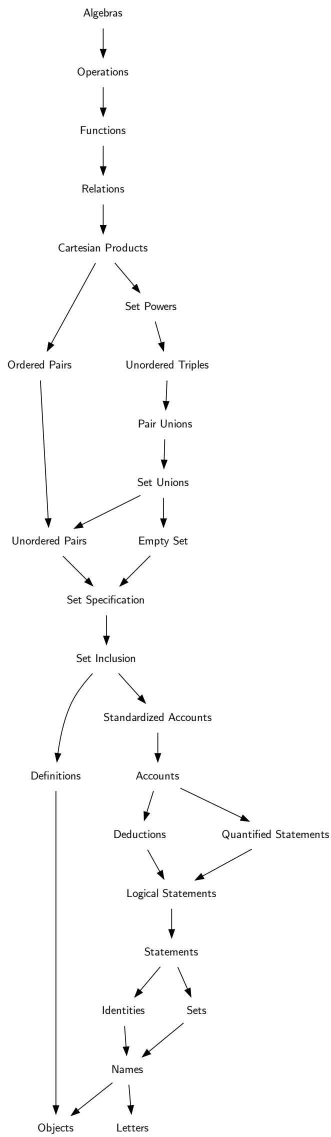
Family Operations (**??**)

Groups (91)

Isomorphisms (90)

Algebras (76) gives the following terms.

algebra
ground set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Why

We name the operations which produce natural sums, products and powers.

Definition

Consider the set of natural numbers. The we can define three functions corresponding to sums, products and powers which are operations (see Operations) on this set.

We call *addition* the function $+ : \omega \times \omega \to \omega$, which maps two natural numbers m and n to their sum $m+n$. We call *multiplication* the function $\cdot : \omega \times \omega \to \omega$, which maps two natural numbers m and n to their sum $m \cdot n$. We call *exponentiation* the function $(m, n) \mapsto m^n$.

In other words, we can think of sums, produces, and powers as obtainable by applying a function to pairs of natural numbers. This function gives another natural numbers We call these three operations the operations of *arithmetic*.

Arithmetic (77) immediately needs:

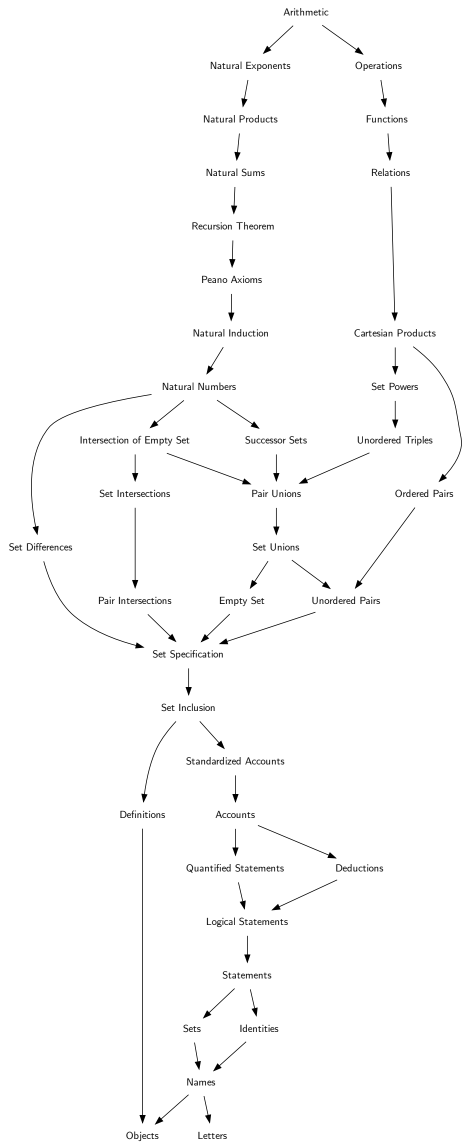
Natural Exponents (67)
Operations (75)

Arithmetic (77) is immediately needed by:

Natural Additive Identity (81)
Natural Multiplicative Identity (82)

Arithmetic (77) gives the following terms.

addition
multiplication
exponentiation
arithmetic


Arithmetic
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Why

We want to consider the elements of two sets together at once, and other sets created from two sets.

Definitions

We have already mentioned that set unions is an operation when considered on the powerset of some given set (see Operations). It is natural to expect the same for intersections (see Pair Intersections) and symmetric differences (see Symmetric Differences).

We call the operation of *forming unions* the function $(A, B) \mapsto A \cup B$. We call the operation of *forming intersections* the function $(A, B) \mapsto A \cap B$. We call the operation of *forming symmetric differences* the function $(A, B) \mapsto A + B$.

We have seen that forming unions commutes and is associative and likewise with forming intersections. As a result of the commutativity of unions and intersections, forming symmetric differences also commutes.

We call these three operations the *set operations*.

Set Operations (78) immediately needs:

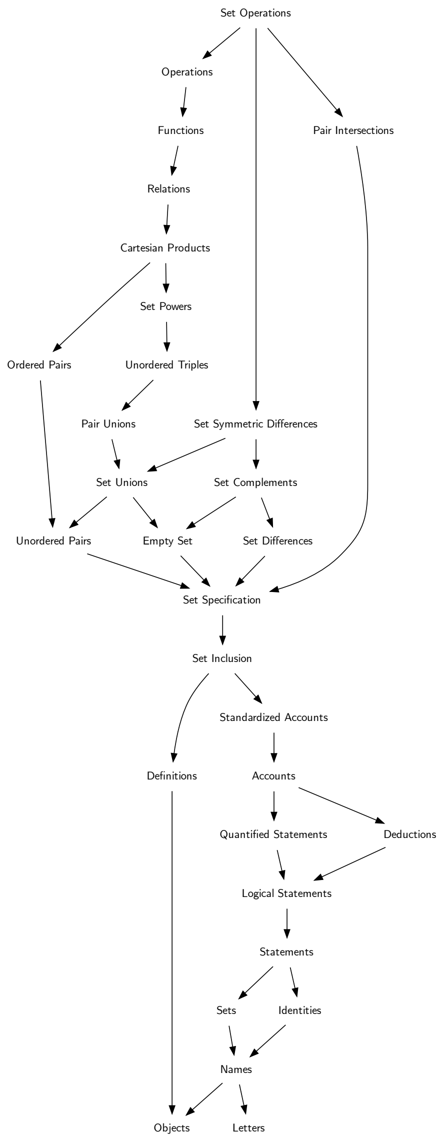
Operations (75)
Pair Intersections (22)
Set Symmetric Differences (32)

Set Operations (78) is immediately needed by:

Convex Sets (??)
Event Probabilities (??)
Extended Real Numbers (??)
Generated Sigma Algebra (??)
Monotone Classes (??)
Pointwise vs Measure Limits (??)
Real Length Impossible (??)
Subset Algebras (??)
Tail Sigma Algebra (??)
Topological Spaces (??)

Set Operations (78) gives the following terms.

intersection
symmetric difference
forming unions
forming intersections
forming symmetric differences
set operations


Set Operations
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Why

Take an element of an algebra, and consider the function defined on the ground set which maps elements to the result of the operation applied to the fixed element and the given element.

Definition

Let $(A, +)$ be an algebra. For each $a \in A$, denote by $+_a : A \to A$ the function defined by

$$+_a(b) = a + b.$$

We call $+_a$ the *left element function*f a.

Similarly, denote by $+^a : A \to A$ the function defined by

$$+^a(b) = b + a.$$

We call $+^a$ the *right element function* of a

The idea is that elements of an algebra can always be associated with functions.

Element Functions (79) immediately needs:

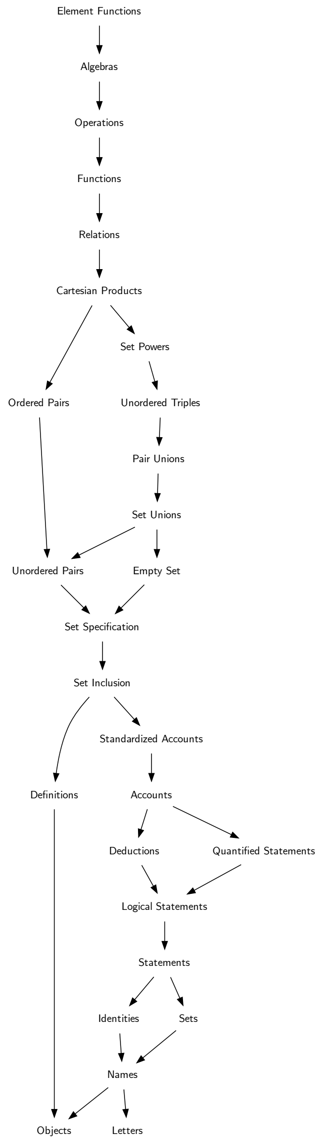
Algebras (76)

Element Functions (79) is immediately needed by:

Inverse Elements (83)

Element Functions (79) gives the following terms.

left element function
right element function


Element Functions
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Why

We can construct functions on the ground set of an algebra by fixing an element in the ground set and defining a function which maps elements to the result of the operation applied to the fixed element and the given element.

Definition

Let $(A, +)$ be an algebra. For each $a \in A$, denote by $+_a : A \to A$ the function defined by

$$+_a(b) = a + b.$$

If $+_a$ is the identity function on A then we call a a *left identity element* of the algebra.

Similarly, denote by $+^a : A \to A$ the function defined by

$$+^a(b) = b + a.$$

If $+^a$ is the identity function on A then we call a a *right identity element* of the algebra.

An *identity element* of the algebra is an element which is both a left and right identity. If the operation commutes, then a left identity and right identities are the same.

Identity Elements (80) immediately needs:

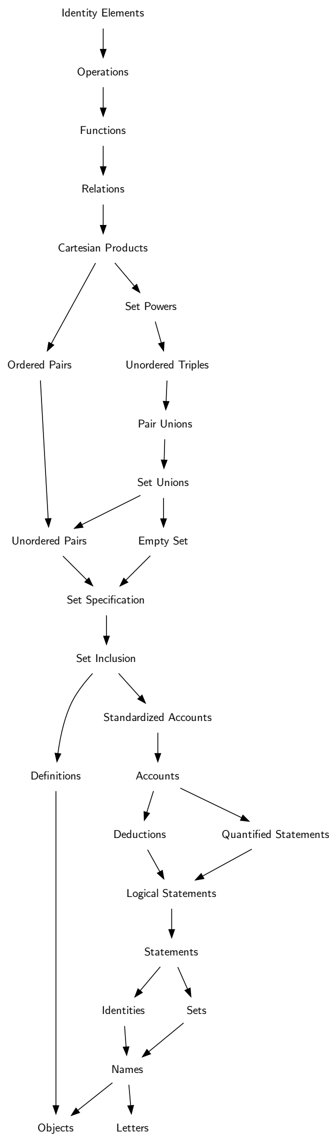
Operations (75)

Identity Elements (80) is immediately needed by:

Natural Additive Identity (81)
Natural Multiplicative Identity (82)

Identity Elements (80) gives the following terms.

left identity element
right identity element
identity element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Why

What is the identity element of addition of the natural numbers.

Result

Proposition 129. *0 is the identity element of ω under $+$.*

Proof. By definition $0 + n = n$ (see Natural Sums). □

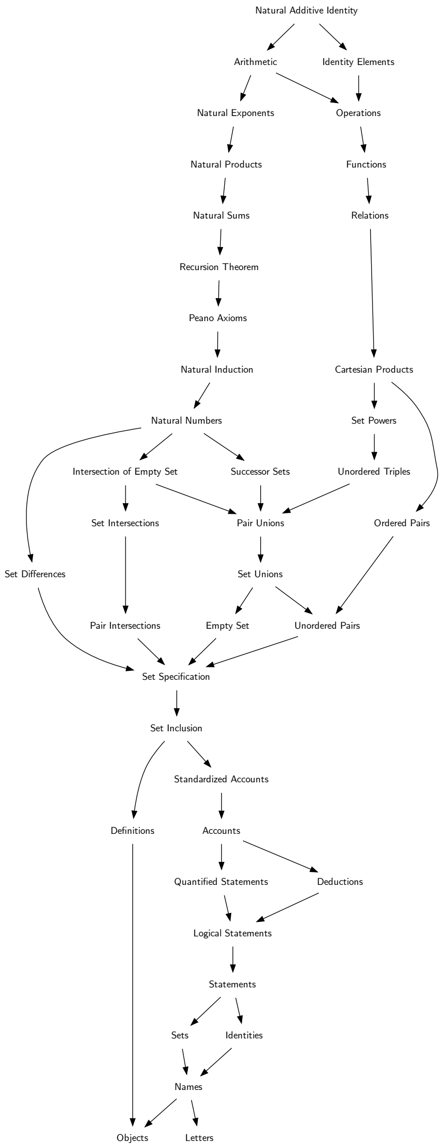
Natural Additive Identity (81) immediately needs:

Arithmetic (77)
Identity Elements (80)

Natural Additive Identity (81) is immediately needed by:

Integer Arithmetic (88)

Natural Additive Identity (81) gives no terms.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Why

What is the identity element of natural multiplication?

Proposition 130. *1 is the identity element of* ω *under* $\cdot$.

Proof. By definition $1 \cdot n = n$ (see Natural Products). $\square$

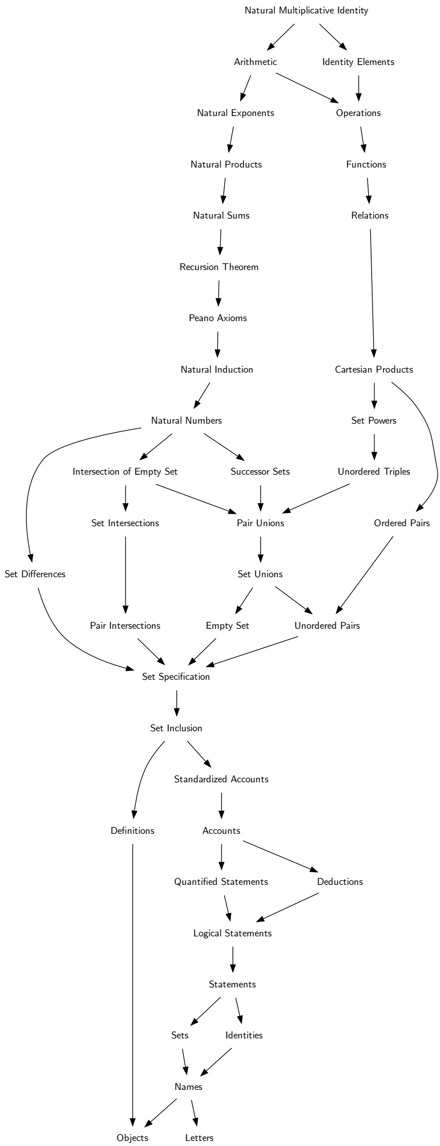
Natural Multiplicative Identity (82) immediately needs:

Arithmetic (77)
Identity Elements (80)

Natural Multiplicative Identity (82) is immediately needed by:

Integer Arithmetic (88)

Natural Multiplicative Identity (82) gives no terms.


Natural Multiplicative Identity
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Why

Is the inverse of an element function the element function of a different element?

Definition

The *inverse* of an element of an algebra (also called the *inverse element*) is the element (if it exists) whose corresponding element function under the operation is the inverse of the first element's function.

Notation

Let $(A, +)$ be an algebra. Let $a \in A$. If the inverse element for a exists and is unique we denote it by a^{-1}. In other words $+^{a^{-1}} \circ +^{a} = \mathrm{id}_A$

Inverse Elements (83) immediately needs:

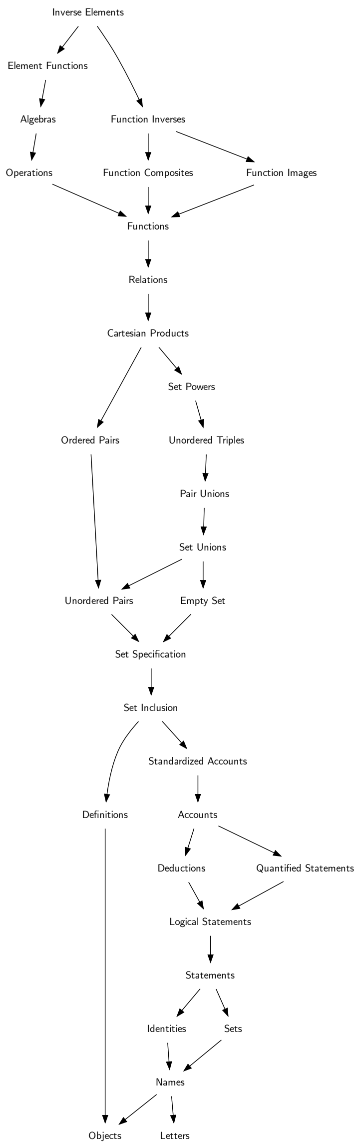
Element Functions (79)
Function Inverses (53)

Inverse Elements (83) is immediately needed by:

Integer Additive Inverses (94)
Matrix Inverses (**??**)
Rational Multiplicative Inverses (100)

Inverse Elements (83) gives the following terms.

inverse
inverse element


Inverse Elements
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Why

We want to do subtraction.[76]

Definition

Consider the set $\omega \times \omega$. This set is the set of ordered pairs of ω. In other words, the ordered pairs of natural numbers.

We say that two of these ordered pairs (a, b) and (c, d) is *integer equivalent* the $a + d = b + c$. Briefly, the intuition is that (a, b) represents a less b, or in the usual notation "$a - b$".[77] So this equivalence relation says these two are the same if $a - b = c - d$ or else $a + d = b + c$.

Proposition 131. *Integer equivalence is an equivalence relation.*[78]

We define the *set of integer numbers* to be the set of equivalence classes (see Equivalence Relations) under integer equivalence on $\omega \times \omega$. We call an element of the set of integer numbers an *integer number* or an *integer*. We call the set of integer numbers the *set of integers* or *integers* for short.

---

[76] Future editions will change this why. In particular, by referencing Inverse Elements and the lack thereof in ω.

[77] This account will be expanded in future editions.

[78] The proof is straightforward. It will be included in future editions.

Notation

We denote the set of integers by Z. If we denote integer equivalence by $\sim$ then $\mathsf{Z} = (\omega \times \omega)/\sim$.

Integer Numbers (84) immediately needs:

Equivalence Relations (43)

Natural Numbers (59)

Integer Numbers (84) is immediately needed by:

Integer Order (87)

Integer Products (86)

Integer Sums (85)

Rational Numbers (95)

Integer Numbers (84) gives the following terms.

integer equivalent
set of integer numbers
integer number
integer
set of integers
integers


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Why

We want sums to follow those of natural numbers.[79]

Definition

Consider $[(a, b)], [(b, c)] \in \mathsf{Z}$. We define the *integer sum* of $[(a, b)]$ with $[(b, c)]$ as $[(a + c, b + d)]$.[80]

Notation

We denote the sum of $[(a, b)]$ and $[(c, d)]$ by $[(a, b)] + [(b, c)]$ So if $x, y \in \mathsf{Z}$ then the sum of x and y is $x + y$.

[79]Future editions will modify this.

[80]One needs to show that this is well-defined. The account will appear in future editions.

Integer Sums (85) immediately needs:

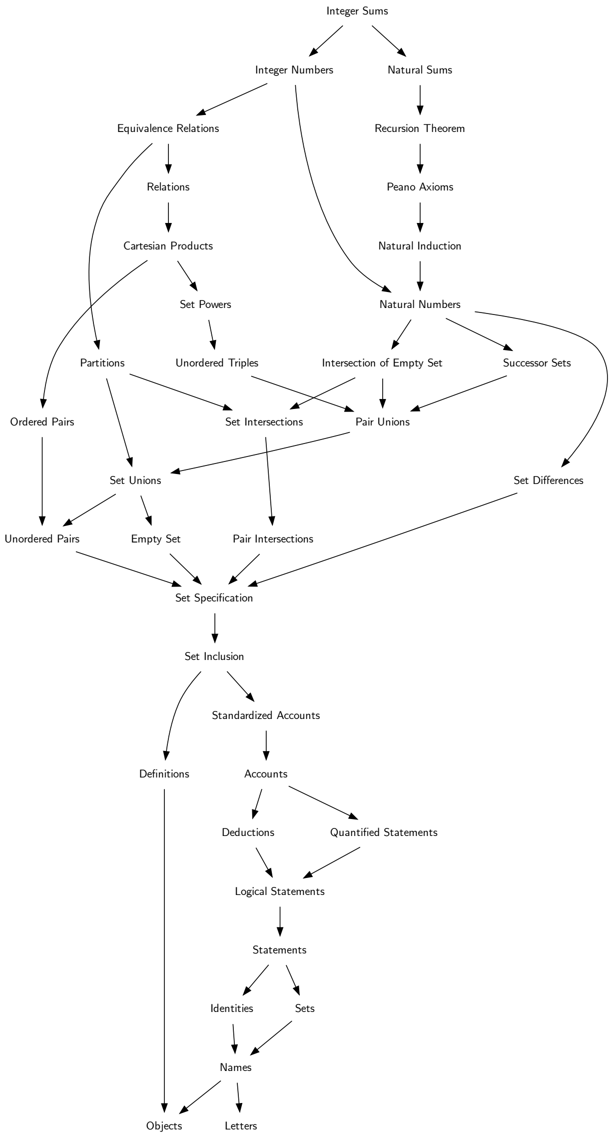
Integer Numbers (84)
Natural Sums (65)

Integer Sums (85) is immediately needed by:

Integer Additive Inverses (94)
Integer Arithmetic (88)

Integer Sums (85) gives the following terms.

integer sum


Integer Sums
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Why

We want sums to follow those of natural numbers.[81]

Definition

Consider $[(a, b)], [(b, c)] \in \mathsf{Z}$. We define *integer product* of $[(a, b)]$ with $[(b, c)]$ as $[(ac + bd, ad + bc)]$.[82]

Notation

We denote the product of $[(a, b)]$ and $[(c, d)]$ by $[(a, b)] \cdot [(b, c)]$ So if $x, y \in \mathsf{Z}$ then the sum of x and y is $x \cdot y$.

[81]Future editions will modify this.

[82]One needs to show that this is well-defined. The account will appear in future editions.

Integer Products (86) immediately needs:

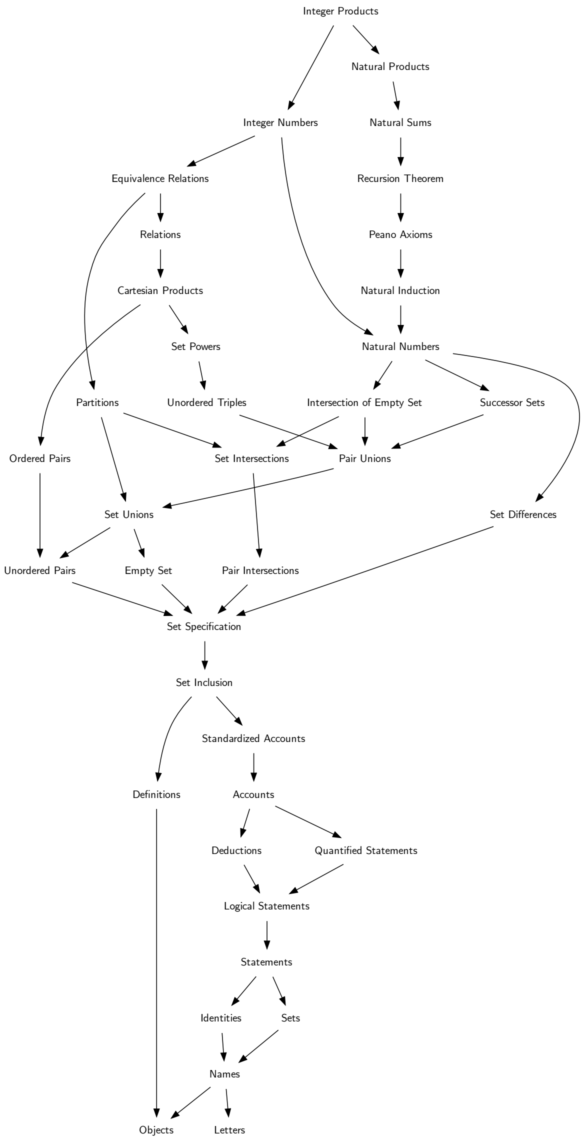
Integer Numbers (84)
Natural Products (66)

Integer Products (86) is immediately needed by:

Integer Arithmetic (88)
Rational Order (101)
Rational Products (97)

Integer Products (86) gives the following terms.

integer product


Integer Products
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Why

We want to order the integers.

Definition

Consider $[(a, b)], [(b, c)] \in \mathsf{Z}$. If $a + d < b + c$, then we say that $[(a, b)]$ is *less than* $[(b, c)]$.[83] If $[(a, b)]$ is less than $[(b, c)]$ or equal, then we say that $[(a, b)]$ is *less than or equal to* $[(b, c)]$.

Notation

If $x, y \in \mathsf{Z}$ and x is less than y, then we write $x < y$. If x is less than or equal to y, we write $x \leq y$.

[83] One needs to show that this is well-defined. The account will appear in future editions.

Integer Order (87) immediately needs:

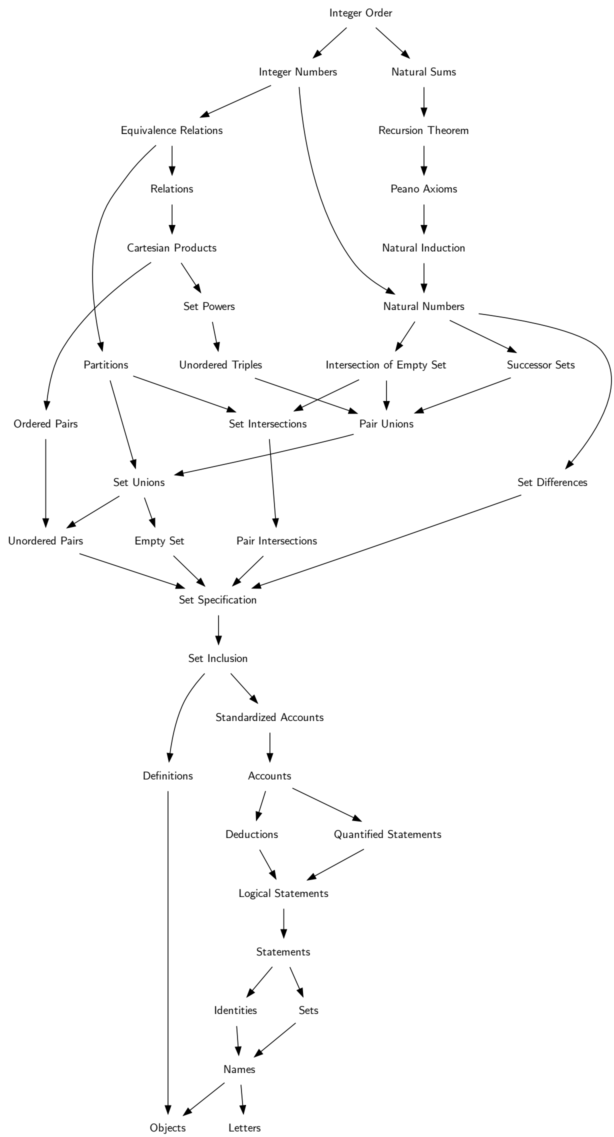
Integer Numbers (84)
Natural Sums (65)

Integer Order (87) is immediately needed by:

Integer Arithmetic and Order (89)
Rational Order (101)

Integer Order (87) gives the following terms.

less than
less than or equal to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Why

What are addition and multiplication for integers? What are the identity elements?

Definition

We call the operation of forming integer sums *integer addition.* We call the operation of forming integer products *integer multiplication.*

Results

It is easy to see the following.[84]

Proposition 132. *The additive identity for* Z *is* $[(0, 0)]$.

Proposition 133. *The multiplicative identity for* Z *is* $[(0, 0)]$.

Notation

We denote the additive identity of Z by 0_{Z} and the multiplicative identity by 1_{Z}. We denote the set $\{z \in \mathsf{Z} \mid z \geq 0_Z\}$ by Z_+.

Distributive

Proposition 134. *For integers* $x, y, z \in \mathsf{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.[85]

[84] Nonetheless, the full accounts will appear in future editions.
[85] An account will appear in future editions.

Integer Arithmetic (88) immediately needs:

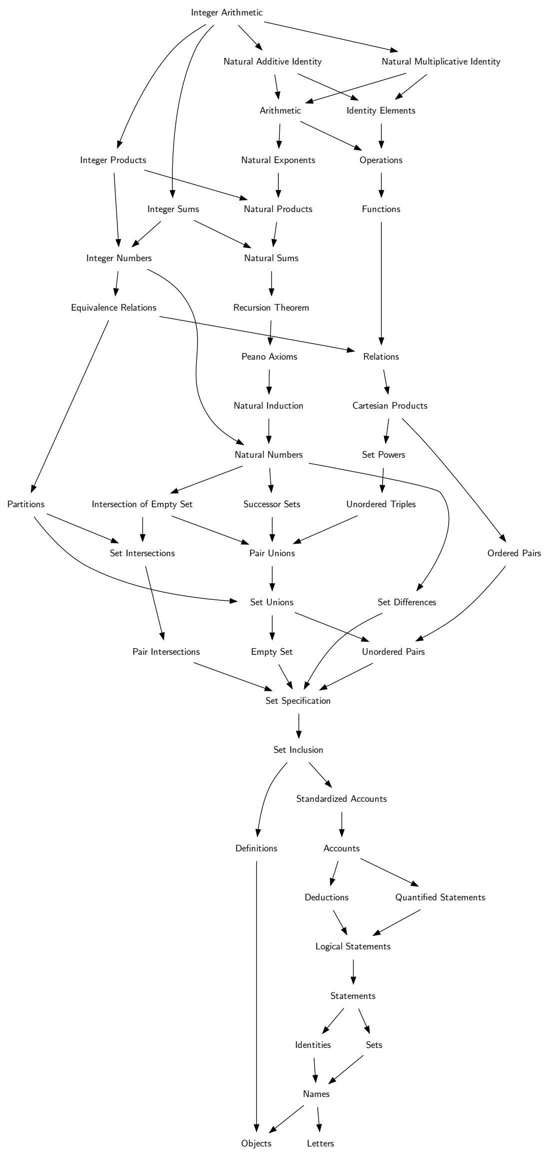
Integer Products (86)
Integer Sums (85)
Natural Additive Identity (81)
Natural Multiplicative Identity (82)

Integer Arithmetic (88) is immediately needed by:

Groups (91)
Integer Arithmetic and Order (89)
Natural Integer Isomorphism (93)
Rational Arithmetic (98)
Rational Multiplicative Inverses (100)
Rational Numbers (95)
Rational Order (101)
Rings (92)

Integer Arithmetic (88) gives the following terms.

integer addition
integer multiplication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Why

How does arithmetic interact with integers.

Results

We can show the following.[86]

Proposition 135. *Let $a, b, c, d \in \mathsf{Z}$. If $a \leq b$ and $c \leq d$, then $a + b \leq c + d$.*

Proposition 136. *Let $a, b, c, d \in \mathsf{Z}$ with $a, b \geq 0_{\mathsf{Z}}$. If $a \leq b$ and $c \leq d$, then $a \cdot c \leq a \cdot d$.*

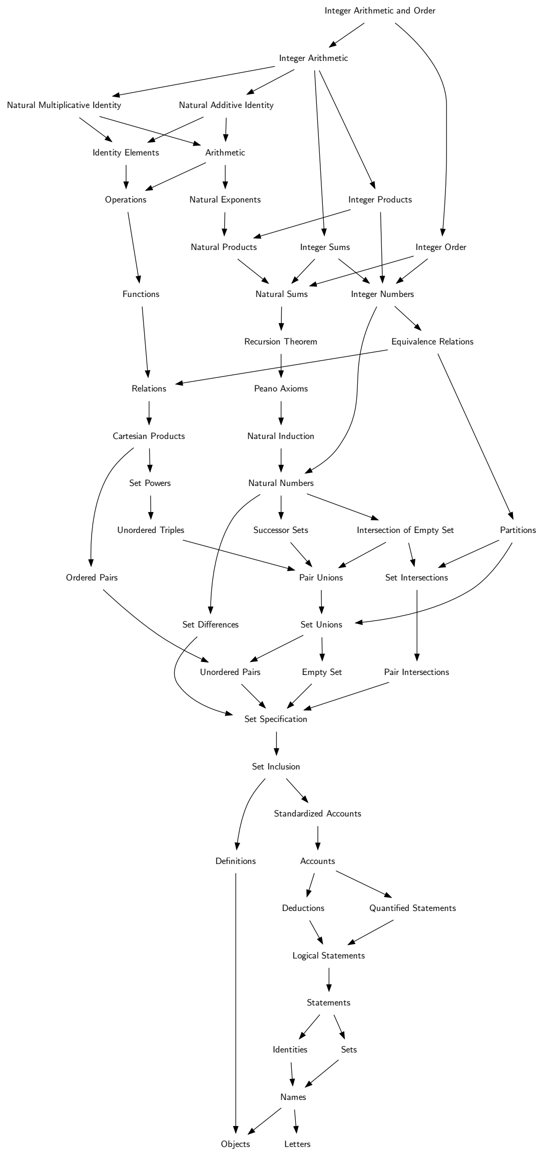
[86] Accounts will appear in future editions.

Integer Arithmetic and Order (89) immediately needs:

Integer Arithmetic (88)
Integer Order (87)

Integer Arithmetic and Order (89) is not immediately needed by any sheet.

Integer Arithmetic and Order (89) gives no terms.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Why

We often have two algebras for which we can identify elements of the ground set.

Definition

Let $(A, +_A)$ and $(B, +_B)$ be two algebras.[87]

An *isomorphism* between these two algebras is a bijection $f : A \to B$ satisfying:

$$f(a +_A a') = f(a) +_B f(a')$$

and

$$f^{-1}(b +_B b') = f^{-1}(b) +_A f^{-1}(b').$$

If there exists an isomorphism between two algebras we say that the algebras are *isomorphic*.

[87] Future editions will change this notation to avoid clashes with right and left identity elements (see Identity Elements).

Isomorphisms (90) immediately needs:

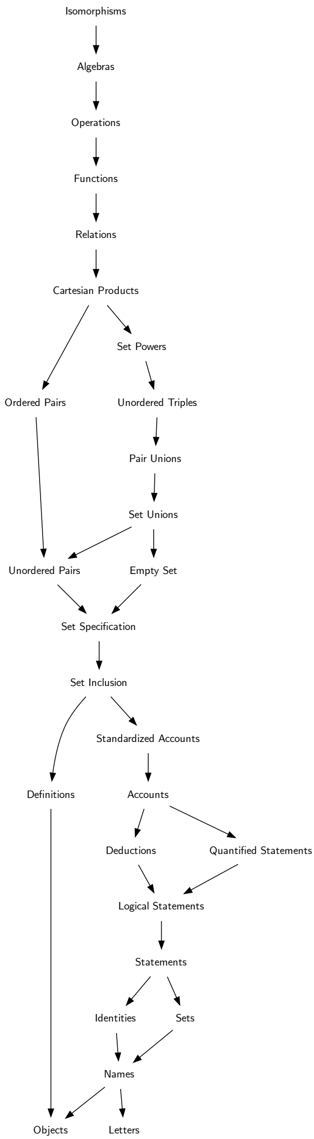
Algebras (76)

Isomorphisms (90) is immediately needed by:

Natural Integer Isomorphism (93)

Isomorphisms (90) gives the following terms.

isomorphism
isomorphic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Why

We generalize the algebraic structure of addition over the integers.

Definition

A *group* is an algebraith: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra *group addition*. A *commutative group* is a group whose operation commutes. A commutative group is also sometimes called an *Abelian group*.

Groups (91) immediately needs:

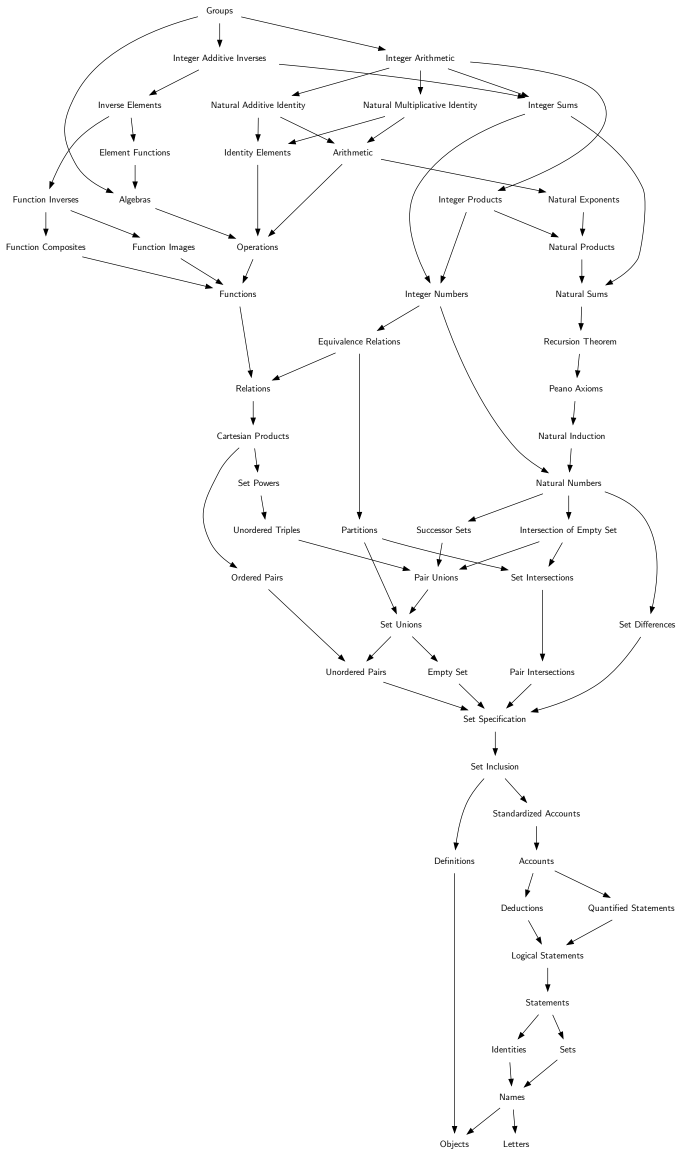
Algebras (76)
Integer Additive Inverses (94)
Integer Arithmetic (88)

Groups (91) is immediately needed by:

Fields (102)
Homomorphisms (103)

Groups (91) gives the following terms.

group
group addition
commutative group
Abelian group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Why

We generalize the algebraic structure of addition and multiplication over the integers.

Definition

A *ring* is two algebras over the same ground set with: (1) the first algebra a commutative group (2) an identity element in the second algebra, and (3) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *ring addition*. We call the operation of the second algebra *ring multiplication*.

Example

Of course, $\mathbf{Z}$ with the usual operations is a ring.

Rings (92) immediately needs:

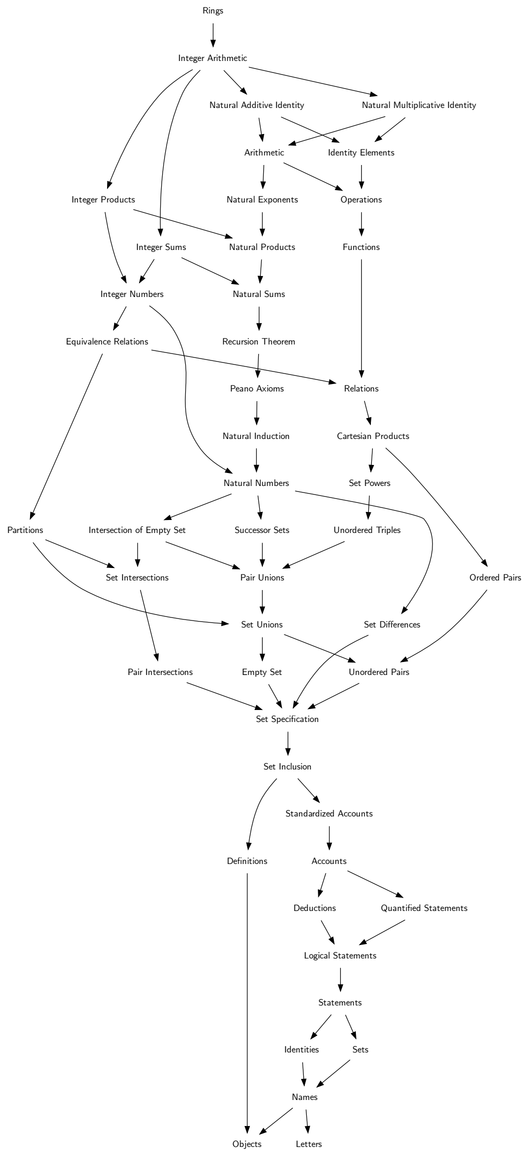
Integer Arithmetic (88)

Rings (92) is immediately needed by:

Homomorphisms (103)

Rings (92) gives the following terms.

ring
ring addition
ring multiplication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Why

Do the natural numbers correspond (in the sense Isomorphisms) to elements of integers.

Main Result

Indeed, the natural numbers correspond to the Z_+.

Proposition 137. *$(\mathbf{Z}_+, + \mid Z_+)$ and $(\omega, +)$ are isomorphic.*

Proof. The function is $f(n) = [(n, 0)]$.[88] $\square$

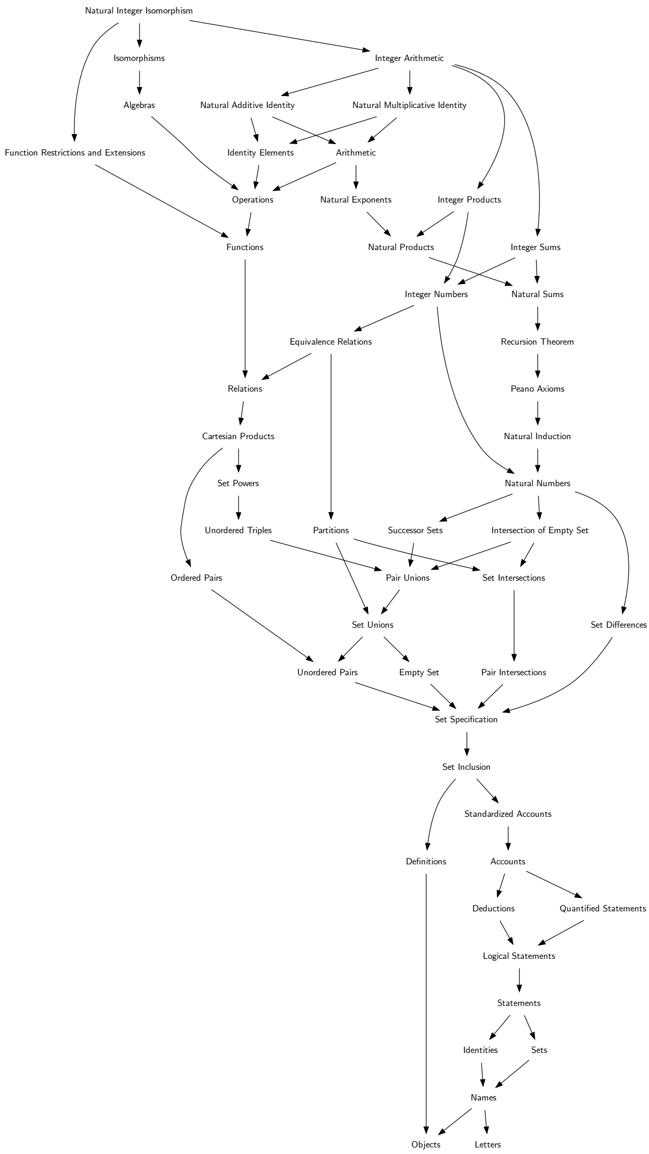
[88] The full account will appear in future editions.

Natural Integer Isomorphism (93) immediately needs:

Function Restrictions and Extensions (45)
Integer Arithmetic (88)
Isomorphisms (90)

Natural Integer Isomorphism (93) is not immediately needed by any sheet.

Natural Integer Isomorphism (93) gives no terms.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Why

What is the additive inverse of $[(a, b)]$ in the integers?

Result

Proposition 138. *The additive inverse of* $[(a, b)] \in \mathsf{Z}$ *is* $[(b, A)]$.

Notation

We denote the additive inverse of $z \in \mathsf{Z}$ by $-z$. We denote $a + (-b)$ by $a - b$.

Subtraction

We call the operation on $(a, b) \mapsto a - b$ *subtraction.*

Integer Additive Inverses (94) immediately needs:

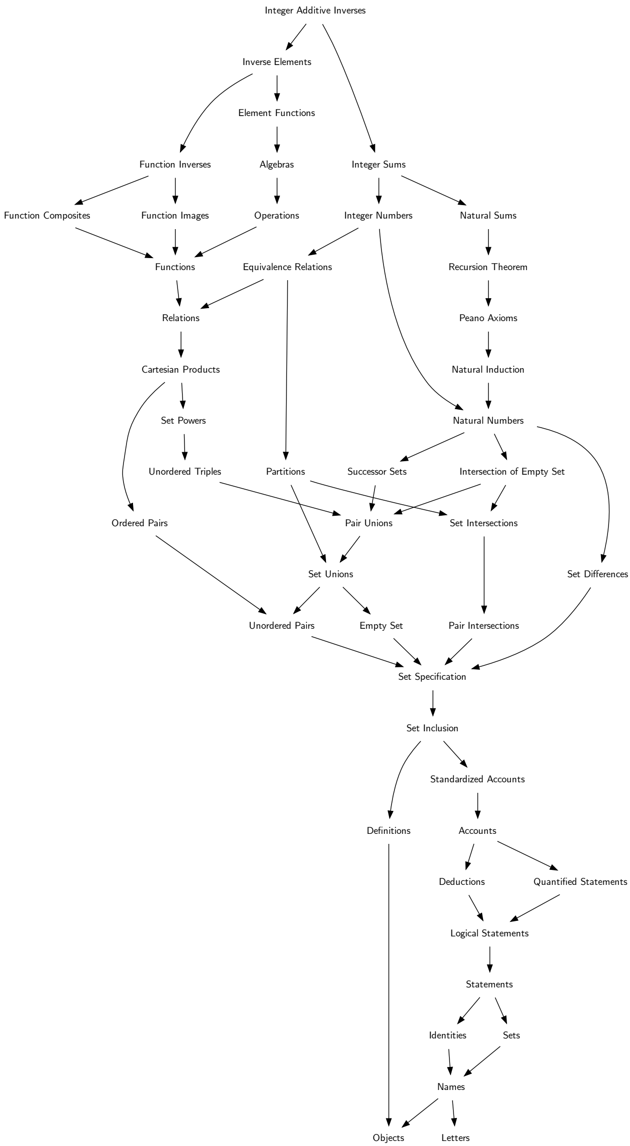
Integer Sums (85)
Inverse Elements (83)

Integer Additive Inverses (94) is immediately needed by:

Groups (91)
Rational Additive Inverses (99)

Integer Additive Inverses (94) gives the following terms.

subtraction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Why

We want fractions.[89]

Definition

Consider $\mathsf{Z} \times (\mathsf{Z} - \{0_{\mathsf{Z}}\})$. We say that the elements (a, b) and (c, d) of this set are *rational equivalent* if $ad = bc$. Briefly, the intuition is that (a, b) represents a over b, or in the usual notation "a/b". So this equivalence relation says these two are the same if $a/b = c/d$ or else $ad = bc$.

Proposition 139. *Rational equivalence is an equivalence relation on* $\mathsf{Z} \times (\mathsf{Z} - \{0_{\mathsf{Z}}\})$.

We define the *set of rational numbers* to be the set of equivalence classes (see Equivalence Classes) under ratioanl equivalence on $\mathsf{Z} \times (\mathsf{Z} - \{0_{\mathsf{Z}}\})$. We call an element of the set of ratioanl numbers a *rational number* or *rational*. We call the set of rational numbers the *set of rationals* or *rationals* for short.

Notation

We denote the set of rationals by Q.[90] If we denote rational equivalence by $\sim$ then $\mathsf{Q} = (\mathsf{Z} \times (\mathsf{Z} - \{0_{\mathsf{Z}}\}))/\sim$.

[89] This why will be expanded in future editions.

[90] From what we can tell so far, Q is a mnemonic for "quantity", from the latin "quantitas".

Rational Numbers (95) immediately needs:

Integer Arithmetic (88)

Integer Numbers (84)

Rational Numbers (95) is immediately needed by:

Fields (102)

Rational Order (101)

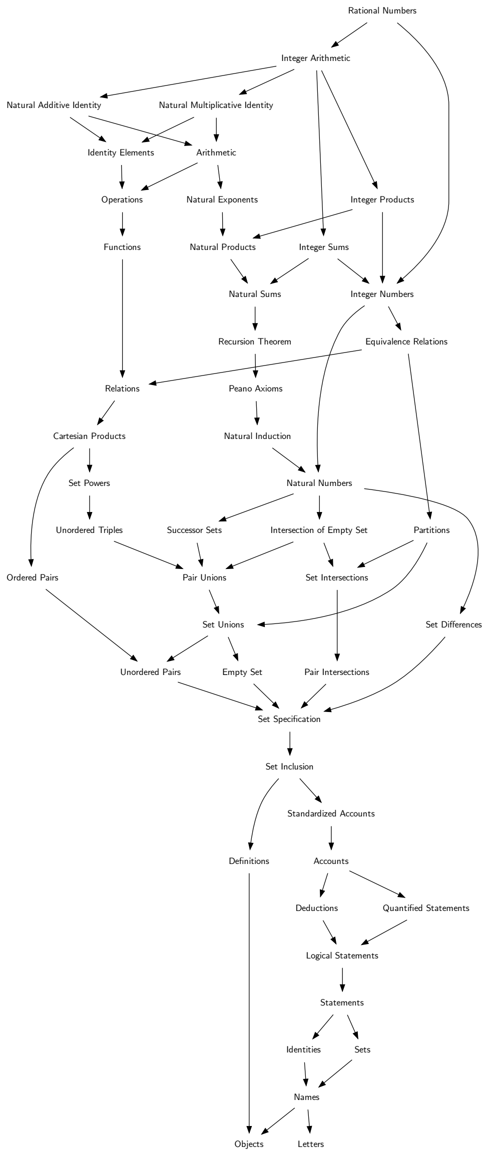
Rational Products (97)

Rational Sums (96)

Real Numbers (105)

Rational Numbers (95) gives the following terms.

rational equivalent
set of rational numbers
rational number
rational
set of rationals
rationals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Why

We want to add rationals.[91]

Definition

Let $[(a, b)], [(b, c)] \in \mathbf{Q}$. The *rational sum* of $[(a, b)]$ with $[(b, c)]$ is $[(ad + bc, bd)]$.[92]

Notation

We denote the rational sum of $q, r \in \mathbf{Q}$ by $q + r$.

[91]Future editions will expand on this why.

[92]An account that this is well-defined will appear in future editions.

Rational Sums (96) immediately needs:

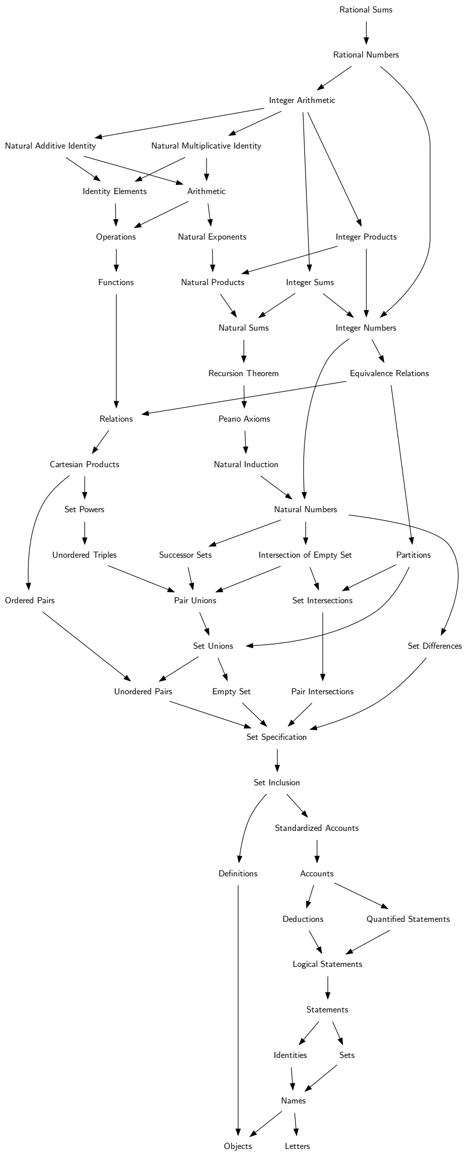
Rational Numbers (95)

Rational Sums (96) is immediately needed by:

Rational Additive Inverses (99)
Rational Arithmetic (98)

Rational Sums (96) gives the following terms.

rational sum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Why

We want to multiply rationals.[93]

Definition

Let $[(a, b)], [(b, c)] \in \mathbf{Q}$. The *rational product* of $[(a, b)]$ with $[(b, c)]$ is $[(ac, bd)]$.[94]

Notation

We denote the rational product of $q, r \in \mathbf{Q}$ by $q \cdot r$.

[93] Future editions will expand on this why.
[94] An account that this is well-defined will appear in future editions.

Rational Products (97) immediately needs:

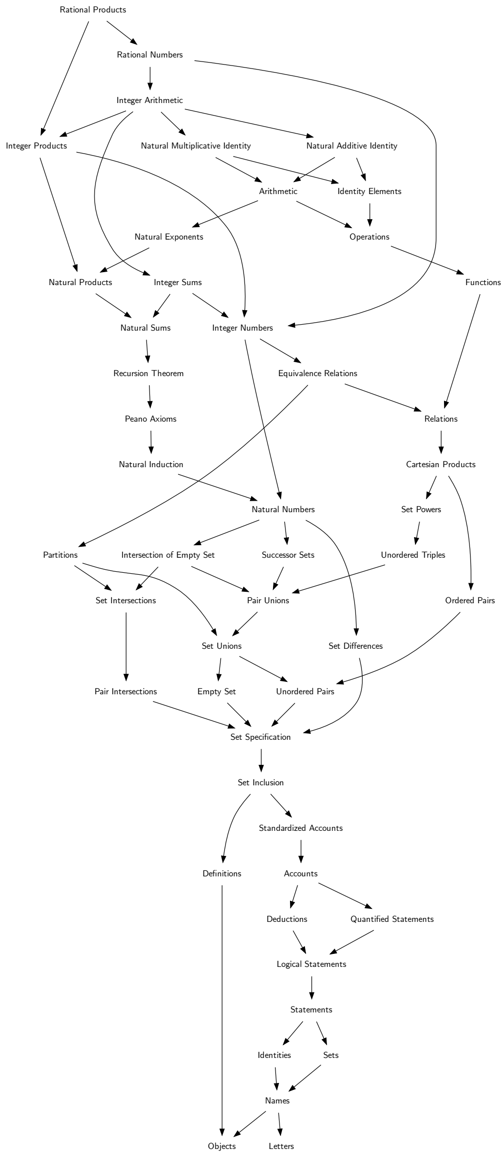
Integer Products (86)
Rational Numbers (95)

Rational Products (97) is immediately needed by:

Rational Arithmetic (98)
Rational Multiplicative Inverses (100)

Rational Products (97) gives the following terms.

rational product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Why

What are addition and multiplication for rationals? What are the identity elements?

Definition

We call the operation of forming rationals sums *rational addition*. We call the operation of forming rational products *rational multiplication*.

Results

It is easy to see the following.[95]

Proposition 140. *The additive identity for* Q *is* $[(0_{\mathsf{Z}}, 1_{\mathsf{Z}})]$.

Proposition 141. *The multiplicative identity for* Z *is* $[(1_{\mathsf{Z}}, 1_{\mathsf{Z}})]$.

Notation

We denote the additive identity of Q by 0_{Q} and the multiplicative identity by 1_{Q}. We denote the set $\{q \in \mathsf{Q} \mid q \geq 0_Q\}$ by Q_+.

Distributive

Proposition 142. *For rationals* $x, y, z \in \mathsf{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.[96]

[95] Nonetheless, the full accounts will appear in future editions.
[96] An account will appear in future editions.

Rational Arithmetic (98) immediately needs:

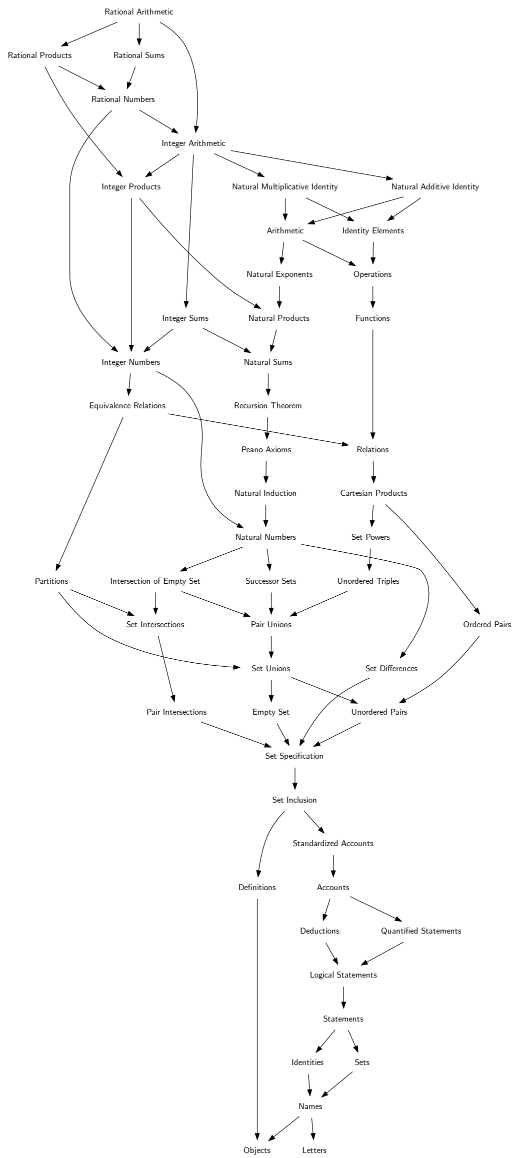
Integer Arithmetic (88)
Rational Products (97)
Rational Sums (96)

Rational Arithmetic (98) is immediately needed by:

Integer Rational Homomorphism (104)
Real Products (109)

Rational Arithmetic (98) gives the following terms.

rational addition
rational multiplication



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Why

What is the additive inverse of $[(a, b)]$ in the rationals?

Result

Proposition 143. *The additive inverse of* $[(a, b)] \in \mathbf{Q}$ *is* $[(-a, b)]$.

Notation

We denote the additive inverse of $q \in \mathbf{Q}$ by $-q$. We denote $a + (-b)$ by $a - b$.

Subtraction

We call the operation $(a, b) \mapsto a - b$ *subtraction.*

Rational Additive Inverses (99) immediately needs:

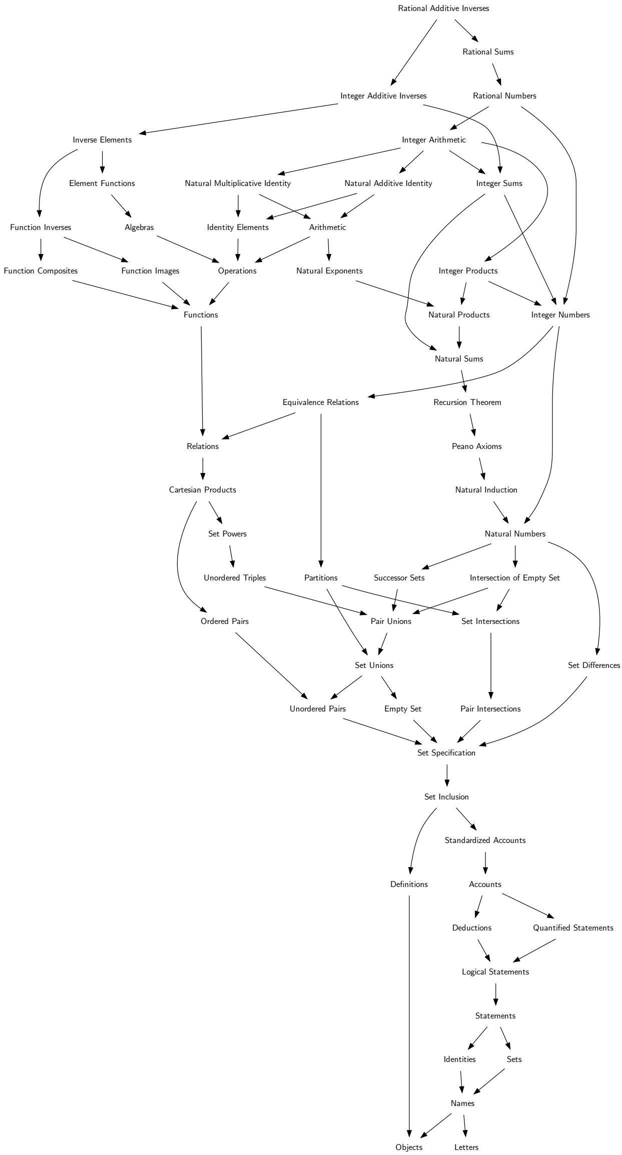
Integer Additive Inverses (94)
Rational Sums (96)

Rational Additive Inverses (99) is immediately needed by:

Integer Rational Homomorphism (104)

Rational Additive Inverses (99) gives the following terms.

subtraction



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Why

What is the multiplicative inverse of $[(a, b)]$ in the rationals?

Result

Proposition 144. *The multiplicative inverse of* $[(a, b)] \in \mathsf{Q}$ *if* $b \neq 0_{\mathsf{Z}}$ *is* $[(b, a)]$.

Notation

We denote the multiplicative inverse of $q \in \mathsf{Q}$ by q^{-1}. We denote $q \cdot (r^{-1})$ by q/r.

Division

We call the operation $(a, b) \mapsto a/b$ *rational division.*

Rational Multiplicative Inverses (100) immediately needs:

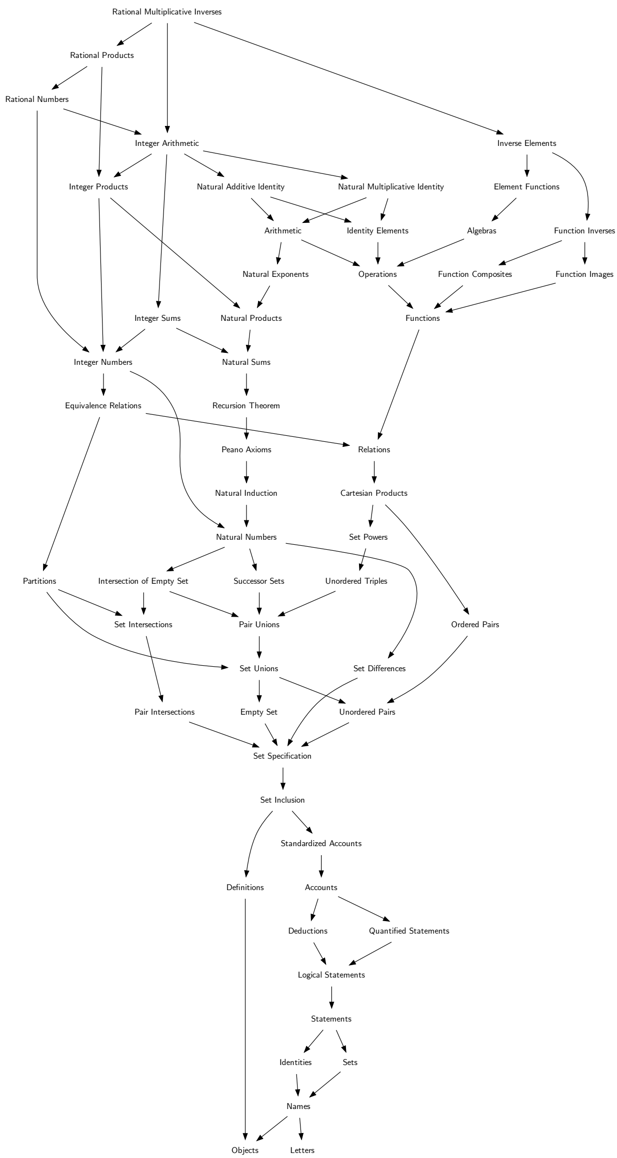
Integer Arithmetic (88)
Inverse Elements (83)
Rational Products (97)

Rational Multiplicative Inverses (100) is immediately needed by:

Integer Rational Homomorphism (104)
Real Multiplicative Inverses (110)

Rational Multiplicative Inverses (100) gives the following terms.

rational division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Why

We want to order the rationals.

Definition

Consider $[(a, b)], [(b, c)] \in \mathbf{Q}$ with $0_{\mathbf{Z}} < b, d$ If $ad < bc$, then we say that $[(a, b)]$ is *less than* $[(b, c)]$.[97] If $[(a, b)]$ is less than $[(b, c)]$ or equal, then we say that $[(a, b)]$ is *less than or equal to* $[(b, c)]$.

Notation

If $x, y \in \mathbf{Q}$ and x is less than y, then we write $x < y$. If x is less than or equal to y, we write $x \leq y$.

[97] One needs to show that this is well-defined. The account will appear in future editions.

Rational Order (101) immediately needs:

Integer Arithmetic (88)

Integer Order (87)

Integer Products (86)

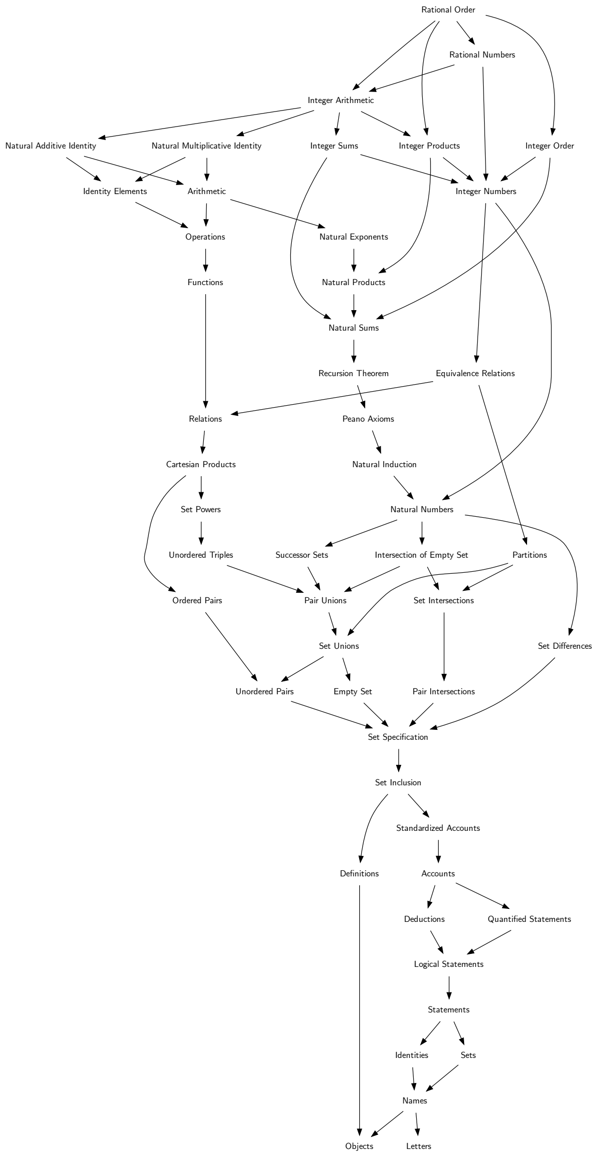
Rational Numbers (95)

Rational Order (101) is immediately needed by:

Complete Fields (113)

Rational Order (101) gives the following terms.

less than
less than or equal to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Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *field addition*. We call the operation of the second algebra *field multiplication*.

Notation

We tend to denote an arbitrary field by F, a mnemonic for "field."

103 Examples

Of course, Q with the usual addition (see Rational Sums) and multiplication (see Rational Products) and the inverse elements (see Rational Additive Inverse) and Ratioanl Multiplicative Inverses) is a field.

Proposition 145. Q *is a field.*

Fields (102) immediately needs:

Groups (91)

Rational Numbers (95)

Fields (102) is immediately needed by:

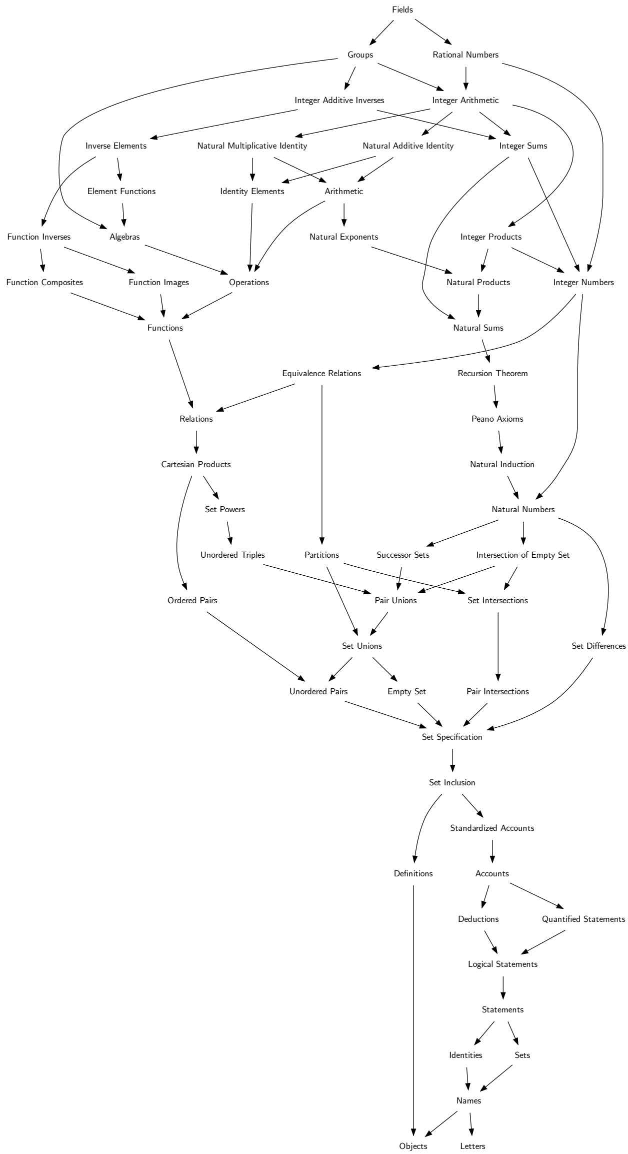
Complete Fields (113)

Homomorphisms (103)

Vectors (**??**)

Fields (102) gives the following terms.

field
field addition
field multiplication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Why

We name a function which preserves algebraic structure.

Definition

A *group homomorphism* between two groups is a function $(A, +)$ and $(B, \tilde{+})$ is a bijection $f : A \to B$ such that $f(1_A) = f(1_B)$ for $1_A \in A$ and $1_B \in B$ and $f(a + a') = f(a) \tilde{+} f(a')$ for all $a, a' \in A$. Similarlly we define *ring homomorphism* and *field homomorphisms*.

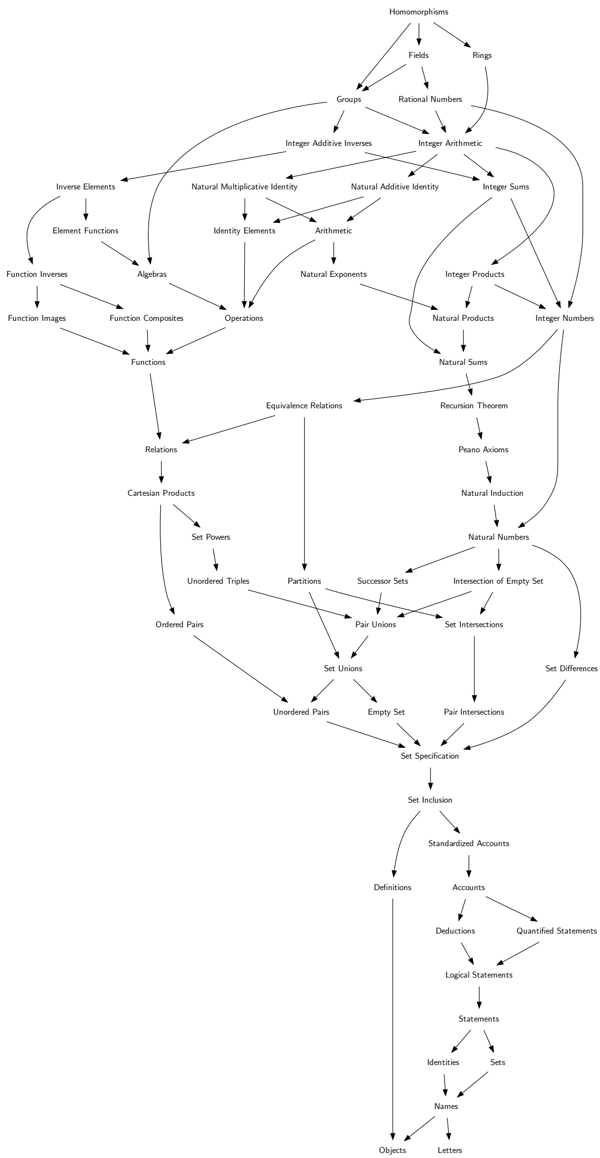
Homomorphisms (103) immediately needs:

Fields (102)
Groups (91)
Rings (92)

Homomorphisms (103) is not immediately needed by any sheet.

Homomorphisms (103) gives the following terms.

group homomorphism
ring homomorphism
field homomorphisms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Why

Do the integer numbers correspond (in the sense Homomorphisms) to elements of the rationals.

Main Result

Indeed, roughly speaking the integers correspond to rationals whose denominator is 1. Denote by $\tilde{\mathsf{Q}}$ the set $\{[(a,b)] \in Q \mid b = 1_{\mathsf{Z}}\}$.

Proposition 146. *The rings* $(\tilde{\mathsf{Q}}, +_{\mathsf{Q}} \mid \tilde{\mathsf{Q}}, \cdot_{\mathsf{Q}} \mid \tilde{\mathsf{Q}})$ *and* $(Z, +_{\mathsf{Z}}, \cdot_Z)$ *are homomorphic.*[98]

Proof. The function is $f : \mathsf{Z} \to \mathsf{Q}$ with $f(z) = [(z,1)]$.[99] $\square$

[98] Indeed, more is true and will be included in future editions. There is an *order perserving* ring homomorphism.

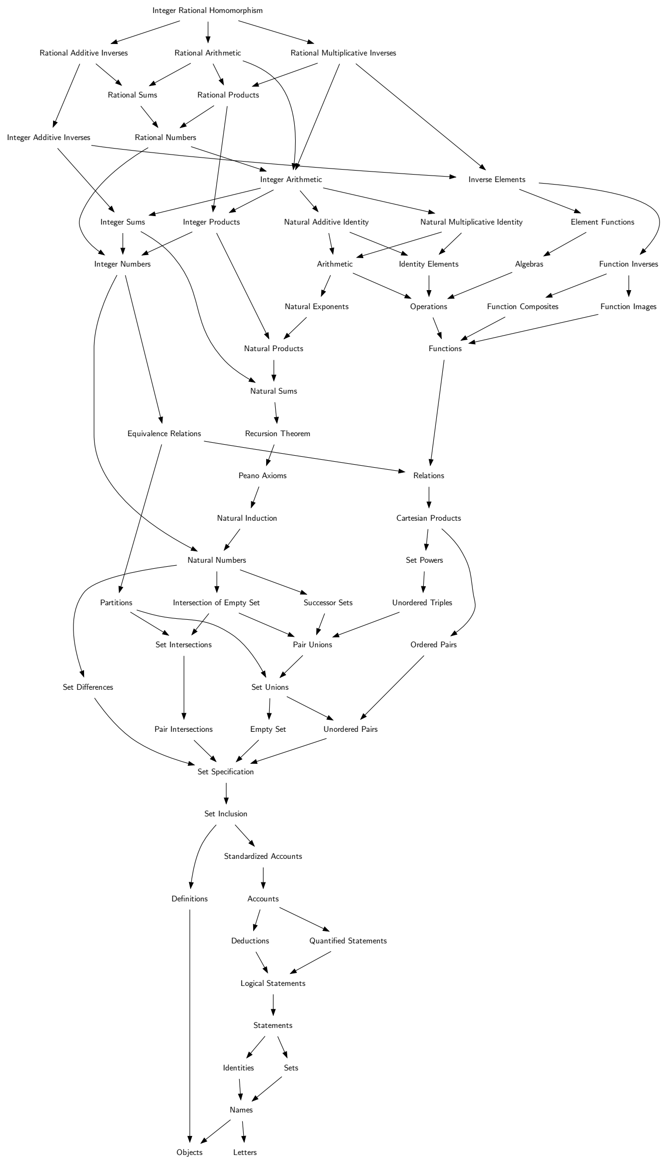
[99] The full account will appear in future editions.

Integer Rational Homomorphism (104) immediately needs:

Rational Additive Inverses (99)
Rational Arithmetic (98)
Rational Multiplicative Inverses (100)

Integer Rational Homomorphism (104) is not immediately needed by any sheet.

Integer Rational Homomorphism (104) gives no terms.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Why

We want a set which corresponds to our notion of points on a line.[100]

Definition

First, call a subset R of $\mathbf{Q}$ a *rational cut* if $R \neq \varnothing$, $R \neq \mathbf{Q}$, for all $q \in R$, $r \leq q \longrightarrow r \in R$, and R has no greatest element. Briefly, the intuition is that the point is the set of all rationals to the left.[101]

The *set of real numbers* is the set of all rational cuts. This set exists by an application of the principle of selection (see Sect Selection to the power set (see Set Powers) of $\mathbf{Q}$. We call an element of the set of real numbers a *real number* or a *real*. We call the set of real numbers the *set of reals* or *reals* for short.

Notation

We follow tradition and denote the set of real numbers by $\mathbf{R}$, likely a mnemonic for "real."

[100]Future editions will modify and expand this justification.
[101]This brief intuition will be expanded upon in future sheets.

Real Numbers (105) immediately needs:

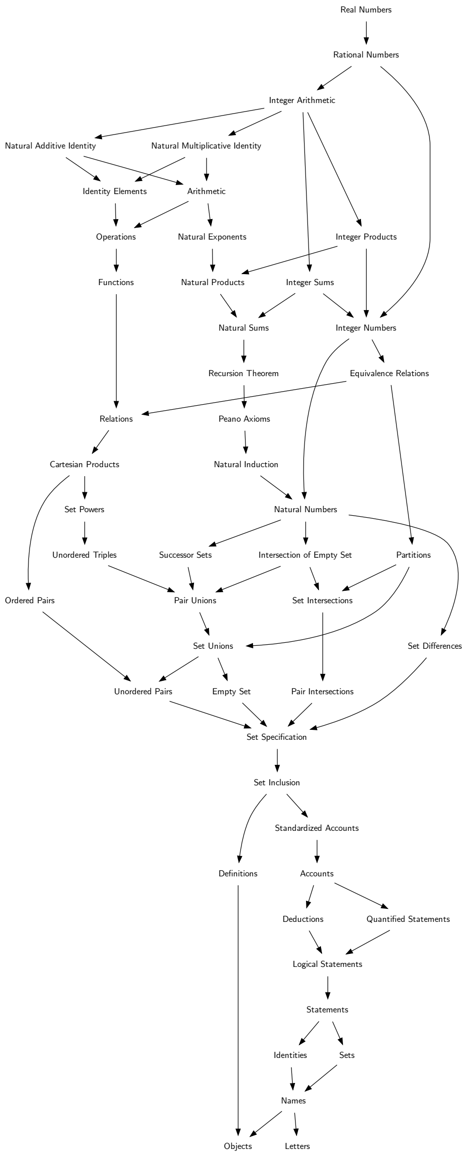
Rational Numbers (95)

Real Numbers (105) is immediately needed by:

Absolute Value (116)
Complex Numbers (??)
Intervals (117)
Logarithm (??)
Loss Functions (??)
Metrics (121)
N-Dimensional Space (??)
Real Continuity (??)
Real Length Impossible (??)
Real Optimizers (??)
Real Order (108)
Real Sequences (??)
Real Square Roots (??)
Real Summation (??)
Real Sums (106)
Real Vectors (??)
Weighted Graphs (??)

Real Numbers (105) gives the following terms.

rational cut
set of real numbers
real number
real
set of reals
reals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Why

We want to add real numbers.[102]

Definition

The *real sum* of two real numbers R and S is the set

$$\{t \in \mathsf{Q} \mid \exists r \in R, s \in S \text{ with } t = r + s\}.$$

Notation

We denote the sum of two real numbers x and y by $x + y$.

Properties

We can show the following.[103]

Proposition 147 (Associative). $x + (y + z) = (x + y) + z$

Proposition 148 (Commutative). $x + y = y + x$

Proposition 149 (Identity). *The set of negative rational numbers is the additive identity.*

We denote the additive identity of R under $+$ by 0_{R}.

[102] Future editions will expand.
[103] Accounts will appear in future editions.

Real Sums (106) immediately needs:

Real Numbers (105)

Real Sums (106) is immediately needed by:

Real Additive Inverses (107)

Real Sums (106) gives the following terms.

real sum


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Why

What is the additive inverse for reals.[104]

Main Result

Proposition 150. *The set* $\{-r \mid r \in R, s \notin R\}$*is an additive inverse of* $R \in \mathsf{R}$.

Notation

We denote the additive inverse of $R \in \mathsf{R}$ by $-R$.

[104]Future editions will expand.

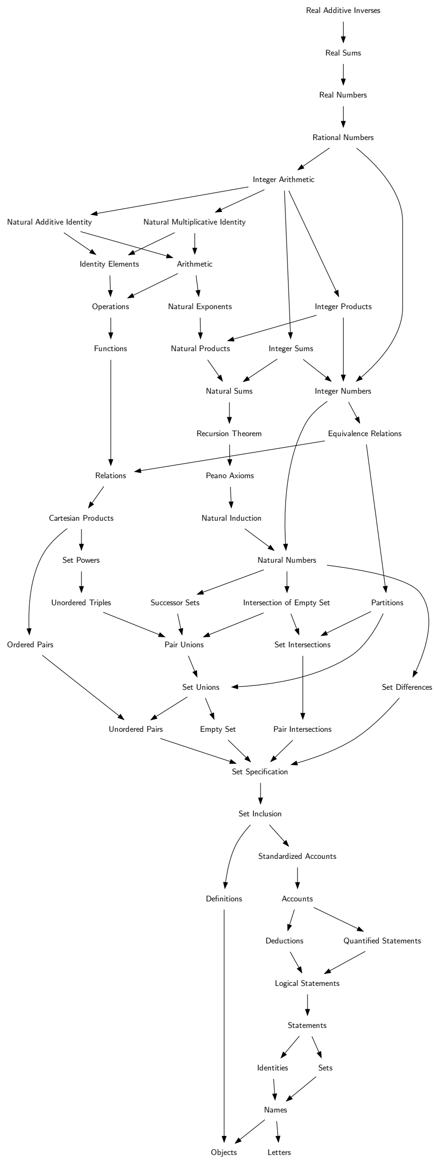
Real Additive Inverses (107) immediately needs:

Real Sums (106)

Real Additive Inverses (107) is immediately needed by:

Real Products (109)

Real Additive Inverses (107) gives no terms.



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Why

We want to order the real numbers.[105]

Definition

Let $R, S \in \mathsf{R}$. If $R \subset S$ and $R \neq S$ then we say that R is *less than* S. If $R \subset S$ then we say that R is *less than or equal to* S.

Notation

If R is less than S we write $R < S$. If R is less than or equal to S we write $R \leq S$.

[105] Future editions will expand

Real Order (108) immediately needs:

Real Numbers (105)

Real Order (108) is immediately needed by:

Complete Fields (113)
Least Upper Bounds (112)
Real Products (109)

Real Order (108) gives the following terms.

less than
less than or equal to

Real Order
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Why

We want to multiply real numbers.[106]

Definition

The *real product* of two real numbers R and S is defined

1. if R or S is $\{q \in Q \mid q < 0_{\mathbf{Q}}\}$, then the $\{q \in Q \mid q < 0_{\mathbf{Q}}\}$
2. otherwise,
   (a) if R or S is $0_{\mathbf{R}}$, then $0_{\mathbf{R}}$.
   (b) if $R, S \neq 0_{\mathbf{R}}$ and $0_{\$} \in R, S$, let T be $\{t \in \mathbf{Q} \mid r \in R, s \in S, r, s \geq 0_{\mathbf{Q}}, t = r \cdot s\}$ then $T \cup \{q \in Q \mid q \leq 0_{\mathbf{Q}}\}$[107]
   (c) If $R, S \neq 0_{\mathbf{R}}$, $0_{\mathbf{R}} \in R$ and $0_{\mathbf{R}} \notin S$, then the additive inverse of the product of $-R$ with S.
   (d) If $R, S \neq 0_{\mathbf{R}}$, $0_{\mathbf{R}} \notin R$ and $0_{\mathbf{R}} \in S$, then the additive inverse of the product of R with $-S$.
   (e) If $R, S \neq 0_{\mathbf{R}}$, and $0_{\mathbf{R}} \notin R, S$, then the product of $-R$ with $-S$.

Notation

We denote the product of two real numbers x and y by $x \cdot y$.

[106] Future editions will expand.

[107] We use $\geq$ in the usual way, it will be defined earlier in future editions.

Properties

Proposition 151 (Associative). $x + (y + z) = (x + y) + z$

Proposition 152 (Commutative). $x + y = y + x$

Proposition 153 (Identity). *The set of all rationals less than* 1_{Q} *is the multiplicative identity.*

We denote the the multiplicative identity by 1_{R}.

Real Products (109) immediately needs:

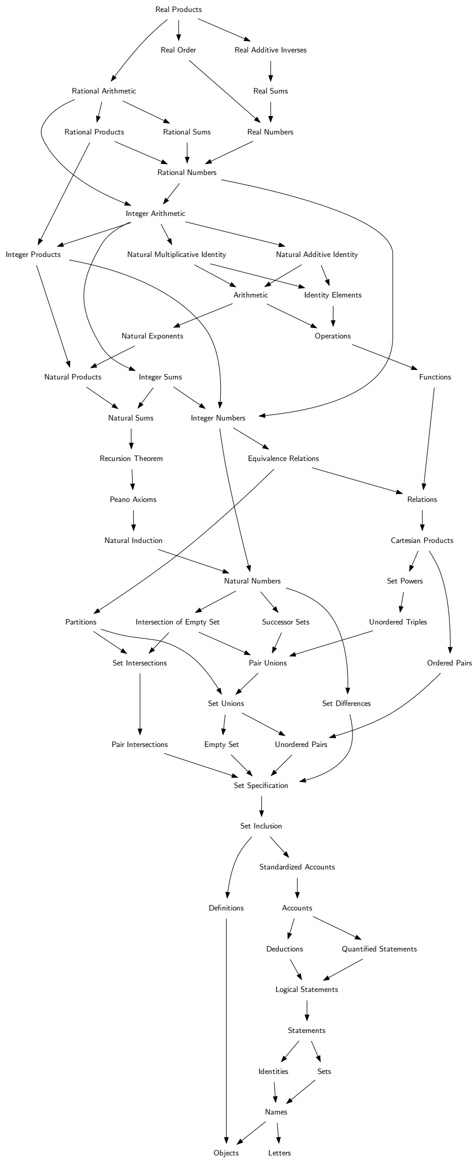
Rational Arithmetic (98)
Real Additive Inverses (107)
Real Order (108)

Real Products (109) is immediately needed by:

Real Multiplicative Inverses (110)

Real Products (109) gives the following terms.

real product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Why

What is the multiplicative inverse in the reals?

Result

We can show the following.[108]

Proposition 154. *The multiplicative inverse of R is, if $R \neq 0_{\mathsf{R}}$,*

1. *if $0_{\mathsf{Q}} \in \mathsf{R}$, then $\{q \in \mathsf{Q} \mid q \leq 0_{\mathsf{Q}}\} \cup \{r^{-1}\}\exists s < r, (r \notin \mathsf{R})$*
2. *If $0_{\mathsf{Q}} \notin \mathsf{R}$, then the additive inverse of the multiplicative inverse of the additive inverse of R.*

Notation

We denote the multiplicative inverse of $r \in \mathsf{R}$ by r^{-1}. We denote $q \cdot (r^{-1})$ by q/r.

Division

We call the operation $(a, b) \mapsto a/b$ *real division.*

[108] The account will appear in future editions.

Real Multiplicative Inverses (110) immediately needs:

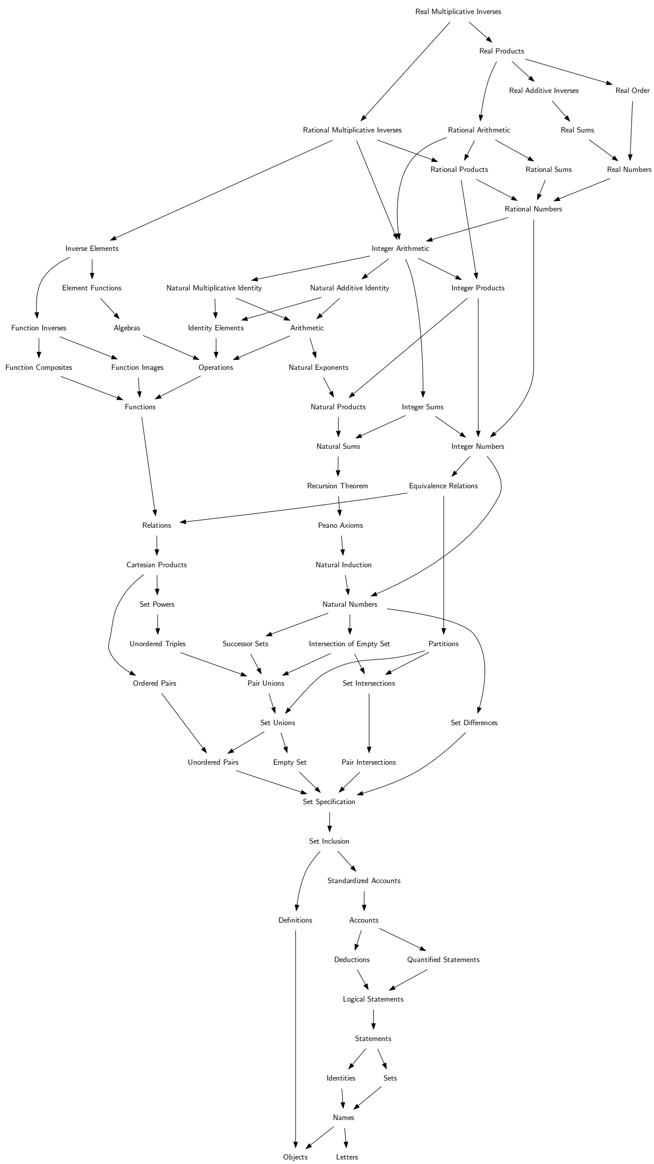
Rational Multiplicative Inverses (100)
Real Products (109)

Real Multiplicative Inverses (110) is immediately needed by:

Real Arithmetic (111)

Real Multiplicative Inverses (110) gives the following terms.

real division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Why

What are addition and multiplication for reals? What are the identity elements?

Definition

We call the operation of forming real sums *real addition.* We call the operation of forming real products *real multiplication.*

Results

It is easy to see the following.[109]

Distributive

Proposition 155. *For reals $x, y, z \in \mathbf{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.*[110]

[109]Nonetheless, the full accounts will appear in future editions.
[110]An account will appear in future editions.

Real Arithmetic (111) immediately needs:

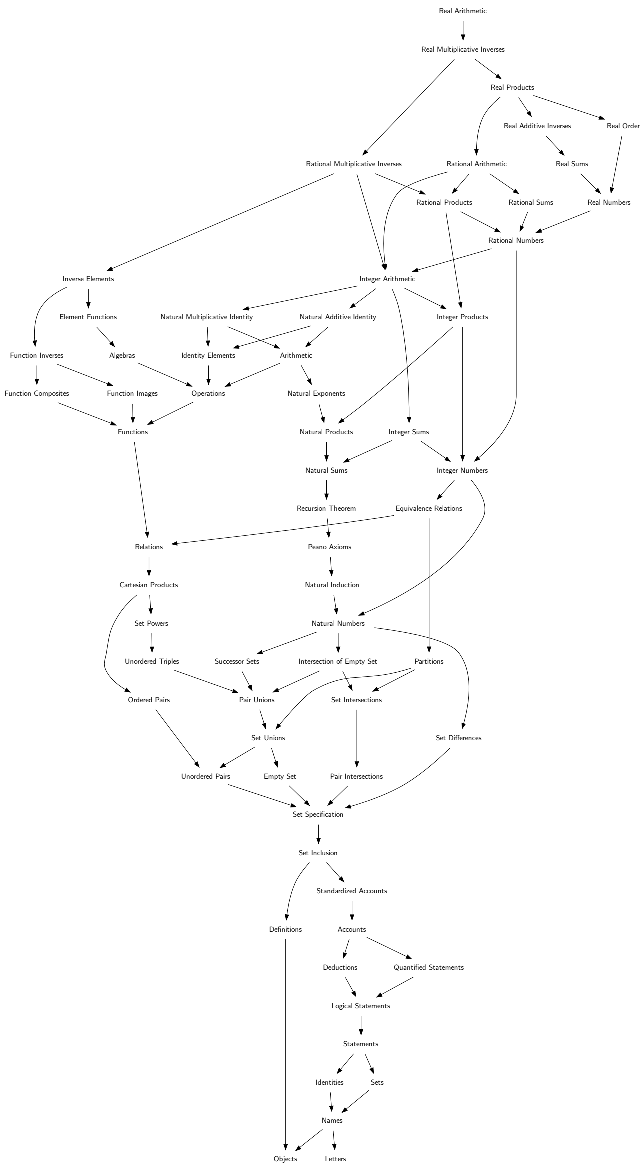
Real Multiplicative Inverses (110)

Real Arithmetic (111) is immediately needed by:

Rational Real Homomorphism (115)

Real Arithmetic (111) gives the following terms.

real addition
real multiplication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Why

Definition

Let A be a set and let $\leq$ be an order[111] on A.

An *upper bound* for $B \subset A$ is an element $a \in A$ so that $b \leq a$ for all $b \in B$. A set is *bounded from above* if it has a least upper bound. A *least upper bound* for B is an element $c \in A$ so that c is an upper bound and $c < a$ for all other upper bounds a.

Proposition 156. *If there is a least upper bound it is unique.*[112]

We call the unique least upper bound of a set (if it exists) the *supremum*.

Properties

We denote the supremum of a set $B \subset A$ by $\sup A$.

[111] To be defined in future editions, but understood in the usual way. See Natural Order or Integer Order or Rational Order etc.
[112] Proof in future editions.

Least Upper Bounds (112) immediately needs:

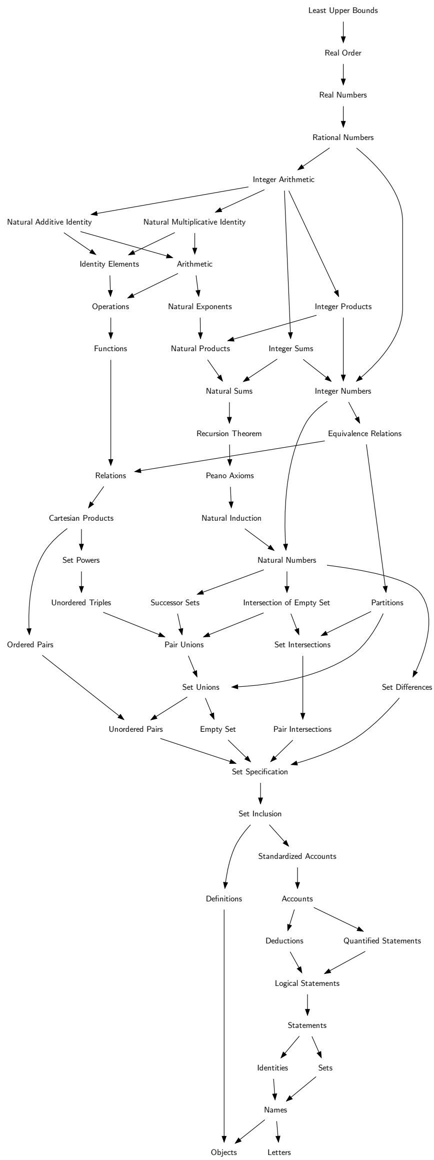
Real Order (108)

Least Upper Bounds (112) is immediately needed by:

Approximate Real Optimizers (**??**)
Complete Fields (113)
Supremum Norm (**??**)

Least Upper Bounds (112) gives the following terms.

upper bound
bounded from above
least upper bound
supremum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Why

We want the a field which corresponds to points on the real line.[113]

Definition

An ordered field[114] is *complete* if every nonempty subset bounded from above has a least upper bound.

[113]Future editions are likely to modify this why.

[114]To be defined in future editions, but we take the usual definition of a field with an order. See, for example Rational Order or Real Order).

Complete Fields (113) immediately needs:

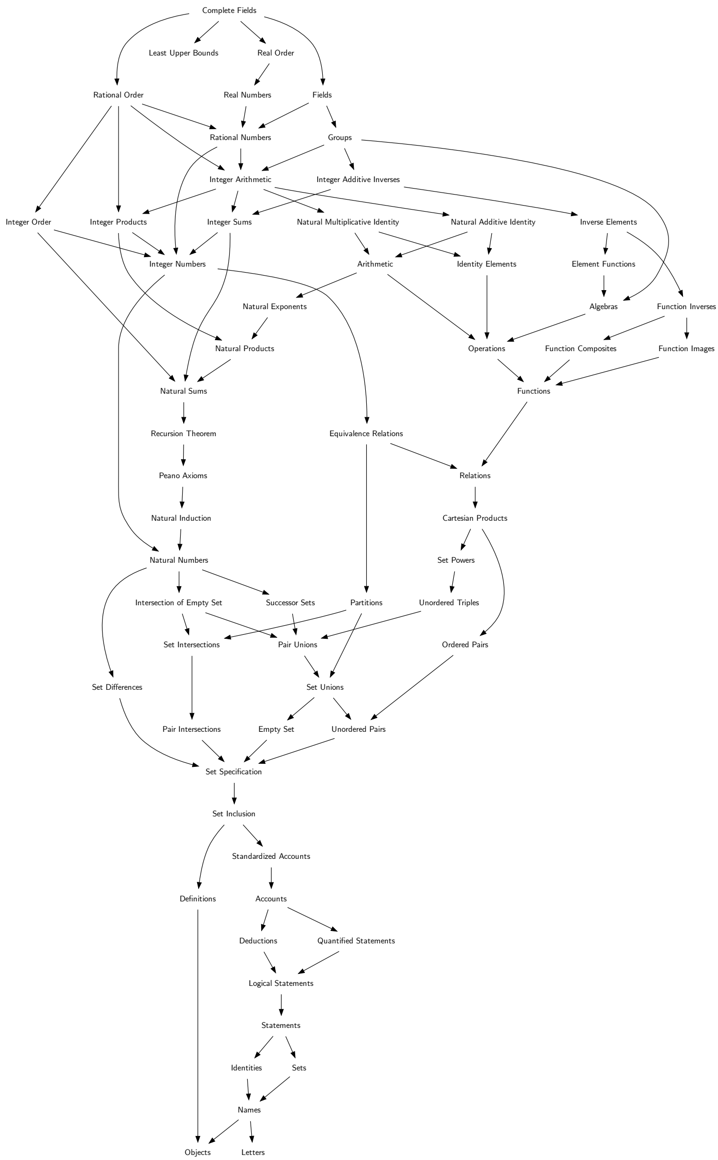
- Fields (102)
- Least Upper Bounds (112)
- Rational Order (101)
- Real Order (108)

Complete Fields (113) is immediately needed by:

- Real Completeness (114)

Complete Fields (113) gives the following terms.

complete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Why

Is the set of real numbers a complete ordered field (in the sense of Complete Fields?

Main Result

Proposition 157. $(\mathsf{R}, +, \cdot, <)$ *is a complete ordered field.*[115]

Proof. The supremum of a set of nonempty real numbers bounded from above R is $\cup R$. $\square$

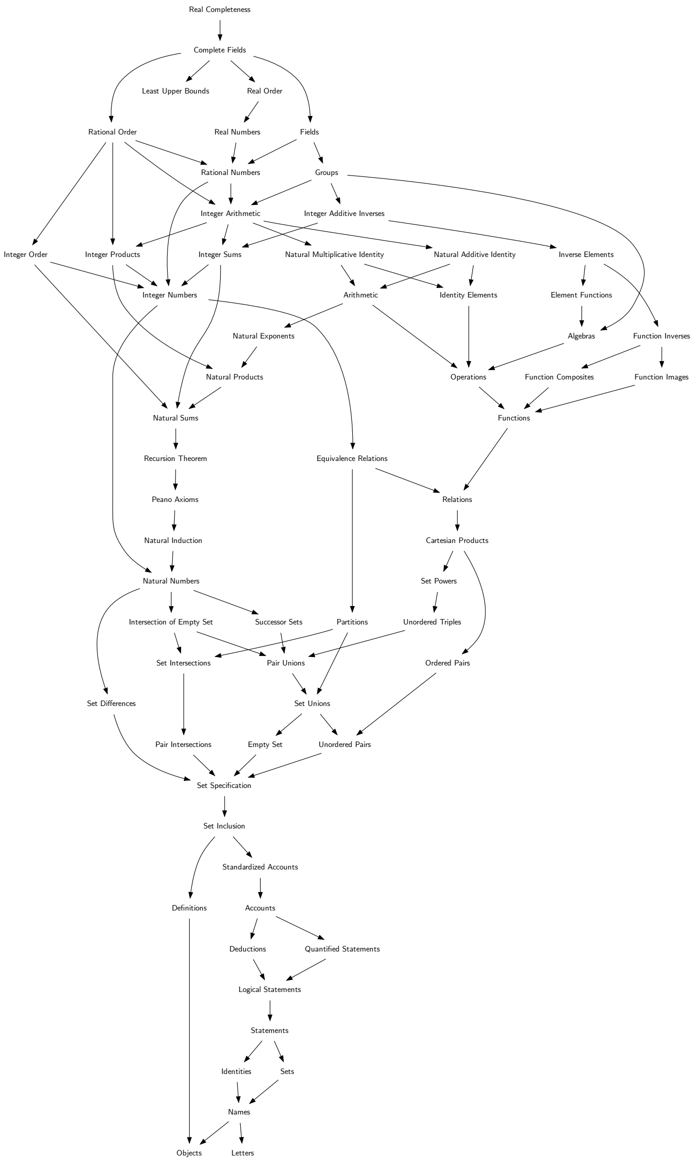
[115]The account will appear in future editions.

Real Completeness (114) immediately needs:

Complete Fields (113)

Real Completeness (114) is not immediately needed by any sheet.

Real Completeness (114) gives no terms.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Why

Do the rational numbers correspond (in the sense Homomorphisms) to elements of the reals.

Main Result

Indeed, roughly speaking the rationals correspond to elements of the reals which are bounded above by that rational. Denote by $\tilde{\mathsf{R}}$ the set $\{q \in \mathsf{R} \mid \exists s \in \mathsf{Q}, q = \{t \in \mathsf{Q} \mid t < s\}\}$.

Proposition 158. *The fields* $(\tilde{\mathsf{R}}, +_{\mathsf{R}} \mid \tilde{\mathsf{R}}, \cdot_{\mathsf{R}} \mid \tilde{\mathsf{R}})$ *and* $(Q, +_{\mathsf{Q}}, \cdot_Q)$ *are homomorphic.*[116]

Proof. The function is $f : \mathsf{Q} \to \mathsf{R}$ with $f(q) = \{(r \in \mathsf{Q} \mid r < q\}$ □

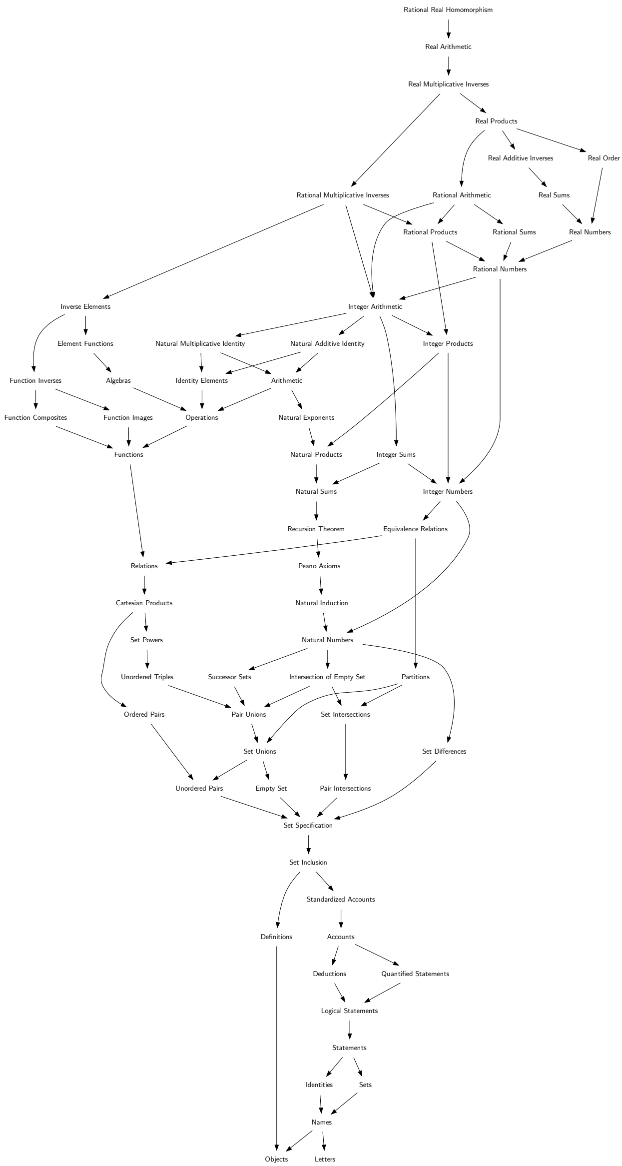
[116] Indeed, more is true and will be included in future editions. There is an *order perserving* field homomorphism.

Rational Real Homomorphism (115) immediately needs:

Real Arithmetic (111)

Rational Real Homomorphism (115) is not immediately needed by any sheet.

Rational Real Homomorphism (115) gives no terms.


Rational Real Homomorphism
Real Arithmetic
Real Multiplicative Inverses
Real Products
Real Additive Inverses
Real Order
Rational Multiplicative Inverses
Rational Arithmetic
Real Sums
Rational Products
Rational Sums
Real Numbers
Rational Numbers
Inverse Elements
Integer Arithmetic
Element Functions
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Function Inverses
Algebras
Identity Elements
Arithmetic
Function Composites
Function Images
Operations
Natural Exponents
Functions
Natural Products
Integer Sums
Natural Sums
Integer Numbers
Recursion Theorem
Equivalence Relations
Relations
Peano Axioms
Cartesian Products
Natural Induction
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Ordered Pairs
Pair Unions
Set Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Pair Intersections
Set Specification
Set Inclusion
Standardized Accounts
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Accounts
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Quantified Statements
Logical Statements
Statements
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Sets
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Objects
Letters

Why

We want a notion of distance between elements of the real line.

Definition

We define a function mapping a real number to its length from zero.

Notation

We denote the absolute value of a real number $a \in \mathsf{R}$ by $|a|$. Thus $|\cdot| : \mathsf{R} \to \mathsf{R}$ can be viewed as a real-valued function on the real numbers which is nonnegative.

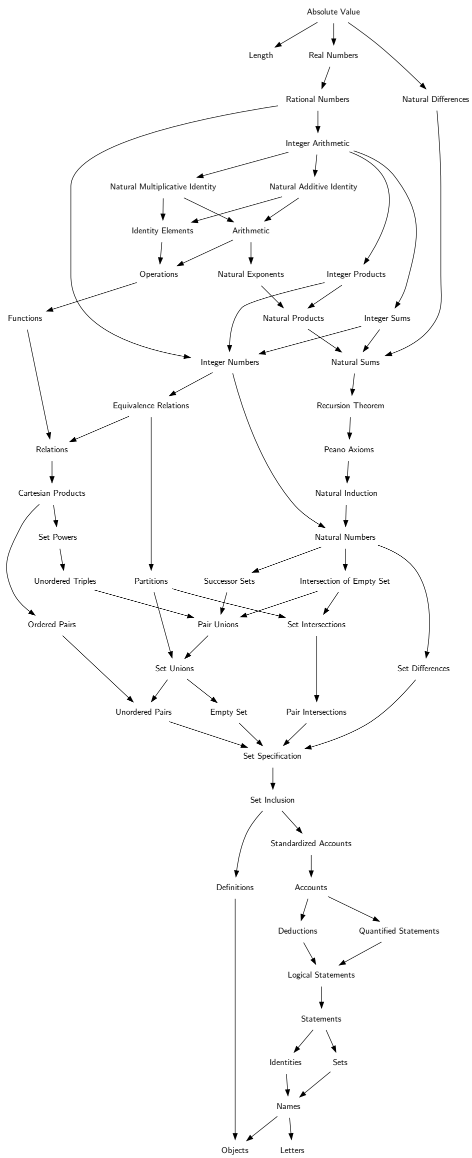
Absolute Value (116) immediately needs:

Length (??)
Natural Differences (??)
Real Numbers (105)

Absolute Value (116) is immediately needed by:

Chordal Graphs (??)
Complex Integrals (??)
Convergence In Measure (??)
Convergence In Probability (??)
Expectation Deviation Upper Bound (??)
Functionals (??)
Integrable Function Spaces (??)
Metric Space Examples (??)
Pointwise vs Measure Limits (??)
Real Continuity (??)
Real Convergence (??)
Real Integral Monotone Convergence (??)
Singular Measures (??)
Supremum Norm (??)
Variation Measure (??)

Absolute Value (116) gives no terms.


Absolute Value
Length
Real Numbers
Rational Numbers
Natural Differences
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Identity Elements
Arithmetic
Operations
Natural Exponents
Integer Products
Functions
Natural Products
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Why

We name and denote subsets of the set of real numbers which correspond to segments of a line.

Definition

Take two real numbers, with the first less than the second.

An *interval* is one of four sets:

1. the set of real numbers larger than the first number and smaller than the second; we call the interval *open*.
2. the set of real numbers larger than or equal to the first number and smaller than or equal to the second number; we call the interval *closed*.
3. the set of real numbers larger than the first number and smaller than or equal to the second; we call the interval *open on the left* and *closed on the right*.
4. the set of real numbers larger than or equal to the first number and smaller than the second; we call the interval *closed on the left* and *open on the right*.

If an interval is neither open nor closed we call it *half-open* or *half-closed*

We call the two numbers the *endpoints* of the interval. An open interval does not contain its endpoints. A closed interval contains its endpoints. A half-open/half-closed interval contains only one of its endpoints. We say that the endpoints *delimit* the interval.

Notation

Let a, b be two real numbers which satisfy the relation $a < b$.

We denote the open interval from a to b by (a, b). This notation, although standard, is the same as that for ordered pairs; no confusion arises with adequate context.

We denote the closed interval from a to b by $[a, b]$. We record the fact $(a, b) \subset [a, b]$ in our new notation.

We denote the half-open interval from a to b, closed on the right, by $(a, b]$ and the half-open interval from a to b, closed on the left, by $[a, b)$.

Intervals (117) immediately needs:

Real Numbers (105)

Intervals (117) is immediately needed by:

Convex Sets (**??**)

Extended Real Numbers (**??**)

Interval Graphs (**??**)

Interval Length (**??**)

Interval Partitions (**??**)

Probability Distributions (**??**)

Product Sections (**??**)

Real Functions (**??**)

Uniform Densities (**??**)

Intervals (117) gives the following terms.

interval
open
closed
open on the left
closed on the right
closed on the left
open on the right
half-open
half-closed
endpoints
delimit

Intervals
Real Numbers
Rational Numbers
Integer Arithmetic
Natural Additive Identity
Natural Multiplicative Identity
Identity Elements
Arithmetic
Operations
Natural Exponents
Integer Products
Functions
Natural Products
Integer Sums
Natural Sums
Integer Numbers
Recursion Theorem
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Why

We want to define the length of a subset of real numbers.

Notions

We take two common notions:

1. The length of a whole is the sum of the lengths of its parts; the *additivity principle*.

2. The length of a whole is the at least the length of any whole it contains the *containment principle*.

The task is to make precise the use of “whole,”, “parts,” and “contains.” We start with intervals.

Definition

By whole we mean set. By part we mean an element of a partition. By contains we mean set containment.

The *length* of an interval is the difference of its endpoints: the larger minus the smaller.

Two intervals are *non-overlapping* if their intersection is a single point or empty. The *length* of the union of two non-overlapping intervals is the sum of their lengths.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A *countably simple* subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A *cover* of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

Cuts

If a, b are real numbers and $a < b$, then we *cut* an interval with a and b as its endpoints by selecting c such that $a < c$ and $c < b$. We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the *cut pieces*.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

All sets

Proposition 159. *Not all subsets of real numbers are simple.*

Exhibit: R is not finite.

Proposition 160. *Not all subsets of real numbers are countably simple.*

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

Notation

Length Common Notions (118) does not immediately need any sheet.

Length Common Notions (118) is immediately needed by:

Distance (119)

Length Common Notions (118) gives the following terms.

additivity principle
containment principle
length
non-overlapping
length
simple
countably simple
cover
cut
cut pieces

Length Common Notions

Why

We want to talk about the "distance" between objects in a set.

Common Notions

Our inspiration is the notion of distance in the plane of geometry. The objects are points and the distance between them is the length of the line segment joining them. We note a few properties of this notion of distance:

1. The distance between any two distinct objects is not zero.
2. The distance between any two objects does not depend on the order in which we consider them.
3. The distance between two objects is no larger than the sum of the distances of each with any third object

The first observation is natural: if two points are not the same, then they are some distance apart. In other words, the line segment between them has length.

The second observation is natural: the line segment connecting two points does not depend on the order specifying the points. This observation justifies the word "between." If it were not the case, then we should use different words, and be careful to speak of the distance "from" a first point "to" a second point.

The third property is a non-obvious property of distance in the plane. It says, in other words, that the length of any side of a triangle is no larger than the sum of the lengths of the two other sides. With experience in geometry, the observation may become natural. But it does not seem to be superficially so.

A more muddled but superficially natural justification for our concern with third observation is that it says something about the transitivity of closeness. Two objects are close if their distance is small. Small is a relative concept, and needs some standard of comparison. Let us fix two points, take the distance between them, and call it a unit. We call two objects close with respect to our unit if their distance is less than a unit.

In this language, the third observation says that if we know two objects are each half of a unit distance from a third object, then the two objects are close (their distance is less than a unit). We might call this third object the reference object. Here, then, is the usefulness of the third property: we can infer closeness of two objects if we know their distance to a reference object.

Distance (119) immediately needs:

Length Common Notions (118)

Distance (119) is immediately needed by:

Distance Asymmetry (120)
Metrics (121)

Distance (119) gives no terms.

$$\begin{tikzcd}
\text{Distance} \arrow[d] \\
\text{Length Common Notions}
\end{tikzcd}$$

Why

Sometimes "distance" as used in the English language refers to an asymmetric concept. This apparent paradox further illuminates the symmetry property.

Apparent Paradox

Distance in the plane is symmetric: the distance from one point to another does not depend on the order of the points so considered. We took this observation as a definiting property of our abstract notion of distance. The meaning, strength, and limitation of this property is clarified by considering an asymmetric case.

Contrast walking up a hill with walking down it. The "distance" between these two points, the top of the hill and a point on its base, may not be symmetric with respect to the time taken or the effort involved. Experience suggests that it will take longer to walk up the hill than to walk down it. A superficial justification may include reference to the some notion of uphill walking requiring more effort.

If we were going to model the top and base of the hill as points in space, however, the distance between them is the same: it is symmetric. It is even the same if we take into account that some specific path, a trail say, must be followed.

If planning a backpacking trip, such symmetry appears foolish. The distance between two locations must not be considered symmetric. Going up the mountain takes longer than going down. It may justify, in the English phrase, "going around, ather than going over."

Distance Asymmetry (120) immediately needs:

Distance (119)

Distance Asymmetry (120) is not immediately needed by any sheet.

Distance Asymmetry (120) gives no terms.

$$\begin{tikzcd}
\text{Distance Asymmetry} \arrow[d] \\
\text{Distance} \arrow[d] \\
\text{Length Common Notions}
\end{tikzcd}$$

Why

We want to talk about a set with a prescribed quantitative degree of closeness (or distance) between its elements.

Definition

The correspondences which serve as a degree of closeness, or measure of distance, must satisify our notions of distances previously developed.

A function on ordered pairs which does not depend on the order of the elements so considered is *symmetric*. A function into the real numbers which takes only non-negative values is *non-negative*. A repeated pair is an ordered pair of the same element twice. A function which satisfies a triangle inequality for any three elements is *triangularly transitive*.

A *metric* (or *distance function*) is a function on ordered pairs of elements of a set which is symmetric, non-negative, zero only on repeated pairs, and triangularly transitive. A *metric space* is an ordered pair: a nonempty set with a metric on the set.

In a metric space, we say that one pair of objects is *closer* together if the metric of the first pair is smaller than the metric value of the second pair.

Notice that a set can be made into different metric spaces by using different metrics.

Notation

Let A be a set and let R be the set of real numbers. We commonly denote a metric by the letter d, as a mnemonic for "distance." Let $d : A \times A \to R$. Then d is a metric if:

1. it is non-negative, which we tend to denote by
$$d(a, b) \geq 0, \quad \forall a, b \in A.$$
2. it is 0 only on repeated pairs, which we tend to denote by
$$d(a, b) = 0 \Leftrightarrow a = b, \quad \forall a, b \in A.$$
3. it is symmetric, which we tend to denote by:
$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$
4. it is triangularly transitive, which we tend to denote by
$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of A with d by (A, d).

Metrics (121) immediately needs:

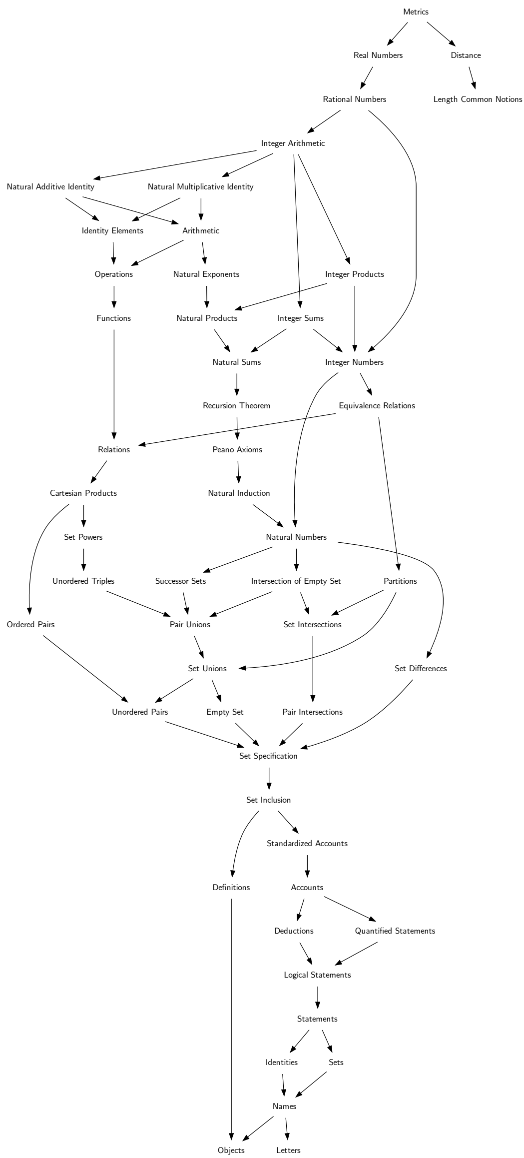
Distance (119)
Real Numbers (105)

Metrics (121) is immediately needed by:

Egoprox Sequences (??)
Isometries (??)
Metric Balls (??)
Metric Continuity (??)
Metric Convergence (??)
Metric Space Examples (??)
Metric Space Functions (??)
Norm Metrics (??)
Product Metrics (??)
Similarity Functions (??)
Topological Spaces (??)

Metrics (121) gives the following terms.

symmetric
non-negative
triangularly transitive
metric
distance function
metric space
closer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Note on Printing

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