



## SEQUENCES

### Why

The most important families are those indexed by (subsets of) the natural numbers.

### Definition

A *finite sequence* (or *list*) is a family whose index set is  $\{1, \dots, n\}$  for some  $n \in \mathbf{N}$ . The *length* of a finite sequence is the size of its index set. If the codomain of a sequence is  $A$ , we say the sequence is *in*  $A$ .

Let  $A$  be a set with  $|A| = n$ . In this case, another term for a finite sequence is a *string* (or *list*). A sequence  $a : \{1, \dots, n\} \rightarrow A$  is an *ordering* of  $A$  if  $a$  is invertible. In this case, we call the inverse a *numbering* of  $A$ . An ordering associates with each number a unique object and a numbering associates with each object a unique number (the object's *index*).

### Notation

Since the natural numbers are ordered, we regularly denote finite sequences from left to right between parentheses. For example, we denote  $a : \{1, \dots, 4\} \rightarrow A$  by  $(a_1, a_2, a_3, a_4)$ .

### Relation to Direct Products

A *natural direct product* is a product of a sequence of sets. We denote the direct product of a sequence of sets  $A_1, \dots, A_n$  by  $\prod_{i=1}^n A_i$ . If each  $A_i$  is the same set  $A$ , then we denote the

product  $\prod_{i=1}^n A_i$  by  $A^n$ . In this case, we call an element (the sequence  $a = (a_1, a_2, \dots, a_n) \in A^n$ ) an *n-tuple* or *tuple*. The set of sequences in a set  $A$  is the direct product  $A^n$ .

## Infinite Sequences

An *infinite sequence* is a family whose index set is  $\mathbf{N}$  (the set of natural numbers without zero). The *nth term* or *coordinate* of a sequence is the result of the *nth* natural number,  $n \in \mathbf{N}$ .<sup>1</sup>

### Notation

Let  $A$  be a non-empty set and  $a : \mathbf{N} \rightarrow A$ . Then  $a$  is a (infinite) sequence in  $A$ .  $a(n)$  is the *nth* term. We also denote  $a$  by  $(a_n)_n$  and  $a(n)$  by  $a_n$ . If  $\{A_n\}_{n \in \mathbf{N}}$  is an infinite sequence of sets, then we denote the direct product of the sequence by  $\prod_{i=1}^{\infty} A_i$ .

### Natural unions and intersections

We denote the family union of the finite sequence of sets  $A_1, \dots, A_n$  by  $\cup_{i=1}^n A_i$ . We denote the family of the infinite sequence of sets  $(A_n)_n$  by  $\cup_{i=1}^{\infty} A_i$ . Similarly, we denote the intersections of a finite and infinite sequence of sets  $\{A_i\}$  by  $\cap_{i=1}^n A_i$  and  $\cap_{i=1}^{\infty} A_i$ , respectively.

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<sup>1</sup>Future editions may also comment that we are introducing language for the steps of an infinite process.

## Slices<sup>2</sup>

An *index range* for a length- $n$  sequence  $s$  is a pair  $(i, j)$  for which  $1 \leq i < j \leq n$ . The *slice* corresponding to the index range  $(i, j)$  is the length  $j - i$  sequence  $s'$  defined by  $s'_1 = s_i$ ,  $s'_2 = s_{i+1}, \dots, s'_j = s_{i+j-1}$ . We denote the  $(i, j)$ -slice of  $s$  by  $s_{i:j}$ . If  $i = 1$  we use  $s_{:j}$  and if  $j = n$  we use  $s_{i:}$  as shorthands.

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<sup>2</sup>Future editions may break sequences in finite sequences and infinite sequences. This would simplify the sheet and remove the dependence on the principle of infinity. Future editions may also use the term “list” instead of finite sequence.

