Why

Take a vector space, and consider the set of continuous linear functionals on that space. Given a suitable norm, this space is a complete normed space.

Defining result

Proposition 1. *Let $(V, \|\cdot\|)$ be a normed space. The set V^* of all continuous linear functionals on V is a complete normed space with respect to pointwise algebraic operations and norm $\|\cdot\|_* : V \to \mathbf{R}$ defined by*

$$\|F\|_* = \sup_{x \in V,\ \|x\| \leq 1} |F(x)|.$$

Proof. We argue (1) V^* is a vector space, (2) $\|\cdot\|_*$ is a norm, and (3) $(V, \|\cdot\|_*)$ is complete.[1] □

We call $(V^*, \|\cdot\|_*)$ the *dual space* (or *conjugate space*, *conjugate dual*, or *Banach dual of* V). Notice that $(V^*, \|\cdot\|_*)$ is complete regardless of whether the original normed space $(V, \|\cdot\|)$ is complete.

Basic dual norm property

Notice that the dual norm satisfies a familiar property.

Proposition 2. *For any vector x in a normed space $(V, \|\cdot\|)$ and any continuous linear functional F on E,*

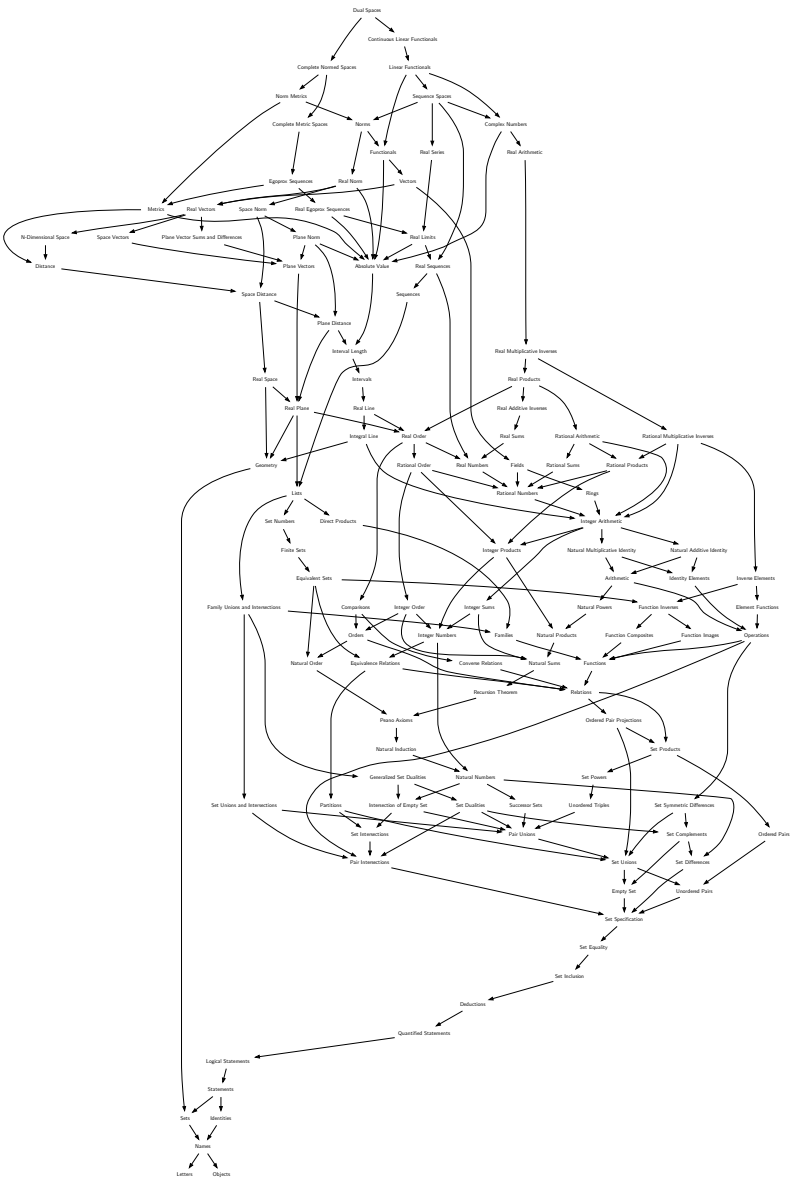
$$|F(x)| \leq \|F\|_* \|x\|.$$

Proof. If $x = 0$, then $\|x\| = 0$ and $F(x) = 0$ (F is linear). Otherwise, $x/\|x\|$ is a unit vector and so

$$\|F\|_* \geq |F(x/\|x\|)| = \frac{|F(x)|}{\|x\|}.$$

[1] Future editions will include an account.

where the inequality is from the definition of $\|\cdot\|_*$ (as a supremum) and the equality follows from the linearity of F. $\square$


Dual Spaces
Continuous Linear Functionals
Complete Normed Spaces
Linear Functionals
Norm Metrics
Sequence Spaces
Complete Metric Spaces
Norms
Complex Numbers
Functionals
Real Series
Real Arithmetic
Epsilon Sequences
Real Norm
Vectors
Metrics
Real Vectors
Space Norm
Real Epsilon Sequences
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Plane Norm
Real Limits
Distance
Plane Vectors
Absolute Value
Real Sequences
Space Distance
Sequences
Plane Distance
Interval Length
Real Multiplicative Inverses
Real Spaces
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Integral Line
Geometry
Rational Order
Real Numbers
Fields
Rational Sums
Rational Products
Rational Numbers
Rings
Lines
Integer Arithmetic
Set Numbers
Direct Products
Finite Sets
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Equivalent Sets
Arithmetic
Identity Elements
Inverse Elements
Family Unions and Intersections
Comparisons
Integer Order
Integer Sums
Natural Powers
Function Inverses
Element Functions
Orders
Integer Numbers
Families
Natural Products
Function Composites
Function Images
Operations
Natural Order
Equivalence Relations
Converse Relations
Natural Sums
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Generalized Set Duality
Natural Numbers
Set Powers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Duality
Successor Sets
Unordered Triples
Set Symmetric Differences
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects