Why

We want to modify ordinary row reduction to handle the case in which a pivot is zero by selecting another suitable pivot.

Example

Let $A \in \mathbf{R}^{5 \times 5}$. If $A_{11} \neq 0$, we may subtract multiples of row 1 from row $2, \ldots, 5$ to eliminate variable x_1 from those equations. If A reduces to $C \in \mathbf{R}^{5 \times 5}$ and $C_{22} \neq 0$, then step 2 moves from

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
 & C_{22} & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & \times & \times & \times
\end{bmatrix}
\quad \text{to} \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
 & C_{22} & \times & \times & \times \\
 & \mathbf{0} & \times & \times & \times \\
 & \mathbf{0} & \times & \times & \times \\
 & \mathbf{0} & \times & \times & \times
\end{bmatrix} .
$$

What if $C_{22} = 0$? In this case suppose we pick a different row. For example, if $C_{42} \neq 0$ we can move from

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & C_{42} & \times & \times & \times \\
 & \times & \times & \times & \times
\end{bmatrix}
\quad \text{to} \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
 & \mathbf{0} & \times & \times & \times \\
 & \mathbf{0} & \times & \times & \times \\
 & C_{42} & \times & \times & \times \\
 & \mathbf{0} & \times & \times & \times
\end{bmatrix} .
$$

Alternatively, we could introduce zeros in column 3 rather than column 2. For example, if we pick the pivot C_{43} we move from

$$
\begin{bmatrix}
\times & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & \times & \times & \times \\
 & \times & C_{43} & \times & \times \\
 & \times & \times & \times & \times
\end{bmatrix}
\quad \text{to} \quad
\begin{bmatrix}
\times & \times & \times & \times & \times \\
 & \times & \mathbf{0} & \times & \times \\
 & \times & \mathbf{0} & \times & \times \\
 & \times & C_{43} & \times & \times \\
 & \times & \mathbf{0} & \times & \times
\end{bmatrix} .
$$

We can choose any nonzero entry in $C_{k:m,k:m}$ as the pivot.

Suppose we pick pivot $C_{st} \neq 0$ for $k \leq s, t \leq m$. Define $\tilde{C}$ by swapping row s of C with row k of C and column t of C with column k of C. Then $\tilde{C}_{kk} = C_{st} \neq 0$ and there exists an ordinary

row reduction for $\tilde{C}$. We call this reduction of $(\tilde{C}, \tilde{d})$ a *pivoted row reduction* of C or the *st-reduction* of C.

If all remaining pivots are zero, then there is no viable pivot. In this case, at least one variable is free and we do not have a unique solution. For convenience, in this case, we still call the system an *st*-reduction of itself.

Definition

At step k of ordinary elimination, multiples of row k are subtracted from rows $k+1, \ldots, m$ to introduce zeros in entry k of the rows. If we denote the matrix at the beginning of that step by X, then row k of X, column k of X and especially the pivot X_{kk} play a role. Ordinarily, we subtract from every entry in the submatrix $X_{k+1:m,k:m}$ the product of a number in row k and a number in column k, divided by the pivot X_{kk}. Generally, however, we can choose as pivot any nonzero entry of $X_{k:m,k:m}$.

An m-variable system (A, b) is *pivot reducible* (or *reducible*) if there exists a sequence of systems $S_1, \ldots, S_{m-1}$ so that S_1 is a reduction of (A, b) and S_i is a reduction of S_{i-1} for $i = 1, \ldots, m - 1$. We call S_{m-1} the *final reduction* (or *reduction*) of (A, b). An immediate consequence of our definition is

Proposition 1. *All systems are reducible.*

Pivoted Row Reductions

Ordinary Row Reductions

Linear System Row Reductions

Linear Equation Solutions

Linear Equations

Linear Functions

Arrays

N-Dimensional Space

Distance

Space Distance

Plane Distance

Interval Length

Real Space

Intervals

Real Plane

Real Line

Real Summation

Integral Line

Real Order

Geometry

Real Number

Rational Order

Rational Numbers

Lists

Integer Arithmetic

Set Number

Natural Multiplicative Identity

Natural Additive Identity

Integer Products

Direct Products

Finite Sets

Identity Elements

Arithmetic

Equivalent Sets

Natural Summation

Natural Powers

Family Unions and Intersections

Function Inverse

Family Operations

Comparisons

Integer Order

Integer Sums

Natural Equations

Natural Products

Families

Function Images

Function Composite

Operations

Orders

Integer Numbers

Equations

Natural Sums

Functions

Converse Relations

Equivalence Relations

Natural Order

Recursion Theorem

Relations

Peano Axioms

Set Products

Natural Induction

Generalized Set Dualities

Set Powers

Natural Numbers

Set Unions and Intersections

Set Dualities

Unordered Triples

Intersection of Empty Set

Successor Sets

Partitions

Set Symmetric Differences

Ordered Pairs

Pair Unions

Set Intersections

Set Complements

Pair Intersections

Set Unions

Set Differences

Unordered Pairs

Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects

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