



Why

We generalize the product of two sets to a product of a family of sets. To do so we discuss sets of families.

Discussion for pairs

Suppose X and Y are nonempty sets. There is a natural correspondence between the product $X \times Y$ (see **Set Products**) and the set of families

$$Z = \{z : \{i, j\} \rightarrow (A \cup B) \mid z_i \in A \text{ and } z_j \in B\}$$

where $\{i, j\}$ is any unordered pair with $i \neq j$.

The set Z can be put in one-to-one correspondence with $X \times Y$. The family $z \in Z$ corresponds with the pair (z_i, z_j) . The pair (a, b) corresponds to the family $z \in Z$ defined by $z(i) = a$ and $z(j) = b$. So, ordered pairs can be put in one-to-one correspondence with families. The generalization of Cartesian products to more than two sets generalizes the notion for families.

Definition

Suppose $\{X_i\}_{i \in I}$ is a family of sets. The *direct product* (or *Cartesian product*, *family Cartesian product*) of A is the set of all families (i.e., functions) $a : I \rightarrow X$ which satisfy $a_i \in A_i$ for every $i \in I$.

A function on a product is called a *function of several variables* and, in particular, a function on the product $X \times Y$ is called a *function of two variables*.

Notation

We denote the product of the family $\{A_i\}_{i \in I}$ by

$$\prod_{i \in I} A_i$$

We read this notation as “product over i in I of A sub- i .” Other notation in use includes $\times_{i \in I} A_i$.

Projections

The word “projection” is used in two senses with families. Let I be a set, and let $\{A_i\}_{i \in I}$ be a family of sets. Define $A = \prod_{i \in I} A_i$.

First, let $J \subset I$. There is a natural correspondence between the elements of A and those of $\prod_{j \in J} A_j$. To each element $a \in A$, we restrict a to J and this restriction is an element of $\prod_{j \in J} A_j$. The correspondence is called the *projection* of A onto $\prod_{j \in J} A_j$. The projection in this sense is a set of families.

Second, consider the value of a family $a \in A$ at j . We call a_j the *projection of a onto index j* or the *j -coordinate* of a . This word *coordinate* is meant to follow the language used in defining ordered pairs. The projection in this sense is an element of A_j . The j th projection is a function mapping $\prod_{i \in I} X_i$ to X_j .

