The Bourbaki Project

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Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *script* is a collection of written marks or symbols called *letters*. In *phonetic* languages, the letters correspond to sounds. A *written word* is a succession of letters. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into *written words* which are meant to denote the *spoken words* of the English language. The written words on this page are several letters one after the other. For example, the word “word” is composed of the letters “w”, “o”, “r”, “d”.

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. We call these the *lower case latin letters.*

$$\begin{array}{ccccccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} & \mathrm{i} \\ \mathrm{j} & \mathrm{k} & \mathrm{l} & \mathrm{m} & \mathrm{n} & \mathrm{o} & \mathrm{p} & \mathrm{q} & \mathrm{r} \\ \mathrm{s} & \mathrm{t} & \mathrm{u} & \mathrm{v} & \mathrm{w} & \mathrm{x} & \mathrm{y} & \mathrm{z} & \end{array}$$

And we call these the *upper case latin letters.*

$$\begin{array}{ccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H} & \mathrm{I} \\ \mathrm{J} & \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{O} & \mathrm{P} & \mathrm{Q} & \mathrm{R} \\ \mathrm{S} & \mathrm{T} & \mathrm{U} & \mathrm{V} & \mathrm{W} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \end{array}$$

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We will also use the following symbols. We call these the *Arabic numerals.*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Other symbols

We will also use the following symbols We call thes the *logical symbols.*

$$(\quad) \quad \vee \quad \wedge \quad \neg \quad \forall \quad \exists \quad \Longrightarrow \quad \Longleftrightarrow \quad = \quad \in$$

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an accent $'$ or subscripts or superscripts. When we write the symbol, we say that it *denotes* the object. We call the symbols the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{\text{A}}$ or $\boxed{\text{A'}}$. The box works well to group in the accent, and also clarifies that $\boxed{\text{A}}\boxed{\text{A}}$ is different from $\boxed{\text{AA}}$. But experience shows that the boxes are mostly unnecessary.

We indicate a name for an object with italics. Instead of $\boxed{\text{A'}}$ we use A'. Experience shows that this subtelty is enough for clarity, and it agrees with traditional and modern practice.

No repetitions

We will also agree that we will never use the same name to refer to two different objects. It is in the nature of things—and of names in particular—that we can not do this without confusion.

Names are objects

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can references themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the several marks on the page, all of which are meant to look similar, which ar meant to denote the object, are uses of the name.

Placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is beset understood by example.

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

Symmetry

"A equals B" means the same as "B equals A". This is because the identity of the object is not changed by the order in which the names are given.

This fact is called the *symmetry of identity*. It is obvious. Not subtle in the slightest. We can switch the spots of A and B and say the same thing. There are two ways to say the same thing.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself.

This fact is called the *reflexivity of identity*. It too is obvious. And not subtle. We can always declare that the same symbol denotes the same object. We agreed upon this in *Names*.

Why

We want to talk about none, one, or several objects considered as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set*. We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

The objects in a set may be other sets. In other words, an element of a set may be another set. This may be subtle at first glance, but becomes familiar with experience.

We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object belongs to the set. In other words, the object is a member or an element of the set. We say "The object denoted by a belongs to the set denoted by A".

Asymmetry

Notice that belonging is not symmetric. Saying "the object denoted by a belongs to the set denoted by A" does not mean the same as "the set denoted by A belongs to the object denoted by a" In fact, the latter sentence is nonsensical unless the object a is also a set.

Nontransitive

Let a denote an object and let A denote a set and B denote a set. If the object denoted by a is *a part of* a the set denoted by A, and the set denoted by A is *a part of* the set denoted by B, then usual English usage would suggest that a is a *a part of* the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing. The relation of belonging is not the same. We do not allow this with sets. If a thing is an element of a thing, that second thing may be an element of the third thing, but this does not mean that the.

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, ..., x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Why

We want to write about objects belonging to sets.

Definition

Let A denote a set; in other words, an intangible object which has some objects as members. Let a denote an object. Recall that if two names refer to the same object, the names are equal. Similarly, if the object denoted by a is an element of the set denoted by A, then we say that the former name belongs to the latter name. We write that the name a belongs to the name A by $a \in A$.

We read this sequence of symbols aloud as "a in A." The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means "belongs" in ancient greek. Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of". Of course, we must take care. The first name is not an element on the second name. Rather, the object denoted by the first name is an element of the set (object) denoted by the second name.

We tend to denote sets by upper case latin letters: for example, A, B, and C. To aid our memory, we tend to use the lower case form of the letter for an element of the set. For example, let A and B denote nonempty sets. We tend to denote by a an object which is an element of A. And similarly, we tend to denote by b an object which is an element of B

Why

We want symbols to represent identity and belonging.

Definition

In the English language, *nouns* are words that name people, places and things. In these sheets, *names* (see *Names*) serve the role of nouns. In the English language, *verbs* are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense. A *statement* is several symbols.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings. As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object in the set of the sheet *Identity*. We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And