



Why

What are the affine sets in terms of subspaces?

Affine sets which are subspaces

The subspaces of \mathbf{R}^n are the affine sets which contain the origin.

Proposition 1. *$M \subset \mathbf{R}^n$ is a subspace if and only if M is affine and $0 \in M$.*

Proof. (\Rightarrow) Suppose M is a subspace. Then $0 \in M$. Also $\alpha x + \beta y \in M$ for all $\alpha, \beta \in \mathbf{R}$ and $x, y \in \mathbf{R}^n$. In particular, $(1 - \lambda)x + \lambda y \in M$ for all $\lambda \in \mathbf{R}$, $x, y \in \mathbf{R}^n$. In other words, M contains the line through x and y .

(\Leftarrow) Suppose M is affine and $0 \in M$. M is closed under scalar multiplication since

$$\alpha x = (1 - \alpha)0 + \alpha x$$

is in the line through 0 and x . M is closed under vector addition since

$$(1/2)(x + y) = (1 - 1/2)x + (1/2)y$$

is in the line through x and y . Thus, $x + y = 2(1/2)(x + y) \in M$. \square

Affine sets as translated subspaces

Proposition 2. *Suppose $M \neq \emptyset$ is affine. Then there exists a unique subspace L and vector $a \in \mathbf{R}^n$ for which $M = L + a$. Moreover,*

$$L = M - M = \{x - y \mid x, y \in M\}.$$

Proof. First, uniqueness. Suppose L_1 and L_2 are subspaces parallel to M . We will show that $L_2 \supset L_1$ (and similarly, $L_1 \supset L_2$).

Since L_1 and L_2 are both parallel to M , they are also parallel to each other. Consequently, there exists $a \in \mathbf{R}^n$ with $L_2 = L_1 + a$. Since $0 \in L_2$ (it is a subspace, after all), $-a \in L_1$. Since L_1 is a subspace, $a \in L_1$.

So $x + a \in L_1$ for every $a \in L_1$, and so $L_2 = L_1 + a \subset L_1$. A similar argument gives $L_1 \supset L_2$.

If $y \in M$, then $M + (-y) = M - y$ is a translate of M containing zero (since $y - y = 0$). In other words, the affine set $M - y$ is a subspace. This, then, is the unique subspace parallel to M . Since y was arbitrary, the subspace parallel to M is $L = \cup_{y \in M} M - y = \{x - y \mid x, y \in M\}$.

□

