Why

We can approximate a density with a tree density similar to how we can approximate a distribution with a tree distribution.

Definition

We use the differential relative entropy as a criterion of approximation. An *optimal tree approximator* of density for a tree is a density which factors according to a tree and minimizes its differential relative entropy with the given density.

Notation

Let $g : \mathbf{R}^n \to \mathbf{R}$ be a density and T be a tree on $\{1, \ldots, n\}$. An optimal tree approximator of g for T is a density f that factors according to T and minimizes $d(g, f)$. In other words, given g and T we want to find f to

$$\begin{aligned} &\text{minimize} && d(g, f) \\ &\text{subject to} && f \text{ factors according to } T. \end{aligned}$$

Result

Prop. 1. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a density and T be a tree on $\{1, \ldots, n\}$. The density $f_T^* : \mathbf{R}^d \to \mathbf{R}$ defined by*

$$f_T^* = g_1 \prod_{i \neq 1} g_{i|\mathrm{pa}\, i}$$

minimizes the differential relative entropy with g among all densities on $\mathbf{R}^n$ which factor according to T ($\mathrm{pa}\, i$ is the parent

of i in T rooted at vertex 1, $i = 2, \ldots, n$).

Proof. Let $f : \mathbf{R}^d \to \mathbf{R}$ be a density factoring according to T. First, express

$$f = f_1 \prod_{i=1} f_{i|\mathrm{pa}\, i}.$$

Second, recall that $d(g, f) = h(g, f) - h(g)$. Since $h(g)$ does not depend on f, f is a minimizer of $d(g, f)$ if and only if f is a minimizer of $h(g, f)$.

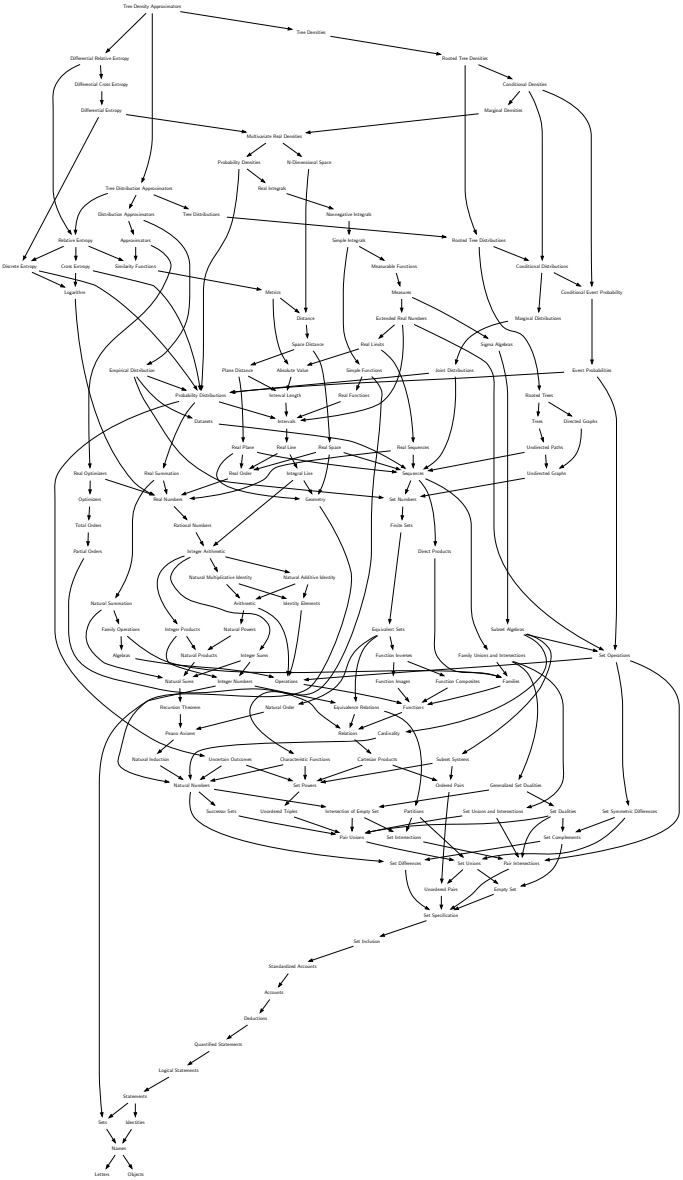
Third, express

$$\begin{aligned} h(g, f) &= -\int_{\mathbf{R}^d} g \log f \\ &= -\int_{\mathbf{R}^d} g(x) \left(\log f_i(x_i) + \sum_{i \neq 1} \log f_{i|\mathrm{pa}\, i}(x_i, x_{\mathrm{pa}\, i}) \right) dx \\ &= h(g_1, f_1) + \sum_{i \neq 1} \left(\int_{\mathbf{R}} g_{\mathrm{pa}\, i}(\xi) h\big(g_{i|\mathrm{pa}\, i}(\cdot, \xi), f_{i|\mathrm{pa}\, i}(\cdot, \xi)\big) d\xi \right) \end{aligned}$$

which separates across f_1 and $f_{i|\mathrm{pa}\, i}(\cdot, \xi)$ for $i = 1, \ldots, n$ and $\xi \in \mathbf{R}$. In particular, since $g_{pai} \geq 0$, we can minimize the integrand pointwise.

Fourth, recall $h(\phi, \psi) \geq 0$ for densities ϕ, ψ of any dimension, and zero if $\phi = \psi$. So $f_1 = g_1$ and $f_{i|\mathrm{pa}\, i} = g_{i|\mathrm{pa}\, i}$ are solutions.

□


Tree Density Approximation
Tree Densities
Rooted Tree Densities
Differential Relative Entropy
Differential Cross Entropy
Differential Entropy
Conditional Densities
Marginal Densities
Multivariate Real Densities
Probability Densities
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Real Integrals
Tree Distribution Approximation
Distribution Approximation
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Relative Entropy
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Similarity Functions
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Space Distance
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Real Plane
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Real Space
Real Sequences
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Real Order
Integral Line
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Real Optimizers
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Finite Sets
Total Orders
Partial Orders
Direct Products
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Natural Summation
Arithmetic
Identity Elements
Family Operations
Integer Products
Natural Powers
Equivalent Sets
Subset Algebras
Set Operations
Algebras
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Characteristic Functions
Cartesian Products
Subset Systems
Natural Numbers
Set Powers
Ordered Pairs
Generalized Set Dualities
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Dualities
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deduction
Quantified Statements
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Statements
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