The Bourbaki Project

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Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *script* is a collection of written marks called *letters*. In *phonetic* languages, the letters correspond to sounds. A *written word* is a succession of letters. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into *written words* which are meant to denote the *spoken words* of the English language. The written words on this page are several letters one after the other. For example, the word “word" is composed of the letters “w", “o", “r", “d".

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. The lower case latin letters.

a	b	c	d	e	f	g	h	i
j	k	l	m	n	o	p	q	r
s	t	u	v	w	x	y	z	

And the upper case latin letters.

A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We will also use the following symbols. They are called the Arabic numberals.

0 1 2 3 4 5 6 7 8 9

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Why

We (still) want to talk and write about things.

Names

We must use sounds to speak about objects. Likewise we must use symbols to write about objects. If we take some symbols like those in *Letters*, and we write them we say that they symbols *denote* the object. We call the collection of the symbols the *name* of the object. In these sheets, we will mostly tend to use the upper and lower case latin letters to denote objects. Sometimes, however, we will use the Arabic numerals, or add a mark like $'$ to latin letters, or we may use both letters and numerals to denote objects.

We are, however, using these same symbols on these pages for spoken words of the English language. So we need to distinguish when a symbol or group of symbols is meant to denote an object. We could box our symbols, and agree that everything in the box denotes the object. For example, $\boxed{\text{A}}$. Or we could underline our symbols, like $\underline{\text{A}}$. Either would work. The box would work particularly well for using two symbols to denote an object. For example, $\boxed{\text{AA}}$.And $\boxed{\text{A}}\boxed{\text{A}}$ is clearly different from $\boxed{\text{AA}}$.

Experience shows that using two letters twice is often confusing, and if accents are used, not needed. Rather than $\boxed{\text{AA}}$ why not use $\boxed{\text{A'}}$. Instead of $\boxed{\text{AAA}}$, use $\boxed{\text{A''}}$. Then experience also shows that the complications like boxes around symbols are unecessary. In other words, we agree never to use $\boxed{\text{A'B'}}$. If we have $\boxed{\text{A'}}\boxed{\text{B'}}$ there is really no confusion in dropping the boxes and writing A'B'. But we still want some way of distinguishing that we are talking about objects.

In these sheets, then, we will indicate that we are denoting an object by using italics. Instead of $\boxed{\text{A}}$, we will use A. Instead of $\boxed{\text{A'}}$, we will use A'. Experience shows that this practice is subtle, but easy enough to distinguish. This choice has the added benefit of agreeing with the traditional and modern practice. And the practice is several millenia old—so it ought to suit us in these sheets.

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, pathological things like this may happen.

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*.

Let A denote an object and let B denote an object. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

"A equals B" means the same as "B equals A". This is because the identity of the object is not changed by the order in which the names are given.

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself.

Why

We want to talk about none, one, or several objects considered as an aggregate.

Definition

A *set* is an intangible object. We think of it as several objects considered as a whole. We say that these objects *belong* to the set. They are the set's *members* or *elements*.

The objects a set contains may be other sets. In other words, an element of a set may be another set. This may be subtle at first glance, but becomes familiar with experience.

We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, ..., x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Why

We want to write about objects belonging to sets.

Definition

Let A denote a set; in other words, an intangible object which has some objects as members. Let a denote an object. Recall that if two names refer to the same object, the names are equal. Similarly, if the object denoted by a is an element of the set denoted by A, then we say that the former name belongs to the latter name. We write that the name a belongs to the name A by $a \in A$.

We read this sequence of symbols aloud as “a in A." The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means “belongs" in ancient greek. Since in English, ε is read aloud “ehp-sih-lawn," $\in$ is also a mnemonic for “element of". Of course, we must take care. The first name is not an element on the second name. Rather, the object denoted by the first name is an element of the set (object) denoted by the second name.

We tend to denote sets by upper case latin letters: for example, A, B, and C. To aid our memory, we tend to use the lower case form of the letter for an element of the set. For example, let A and B denote nonempty sets. We tend to denote by a an object which is an element of A. And similarly, we tend to denote by b an object which is an element of B

Why

We want to succinctly and unambiguously record statements about the objects and sets of objects that names refer to.

Definition

In the English language we have nouns and verbs. The nouns reference objects and the verbs reflect the relations of these objects to each other. In these sheets the nouns are names (introduced in *Objects*) and we speak only in the present tense. We use only the verbs “is" and “belongs".

We say that these two verbs put two objects

There There will only be two verbs and they are. There are only two verbs: equals We speak only in the present testand we have *relational symbols*. A *terminal* is a relational symbol, a left object and a right object. We will only make use of relational symb

For now, denote a relational symbol by $\boxed{s}$. Denote a name by $\boxed{a}$

Why

We want to succinctly and unambiguously record statements about objects and sets of objects, keeping track of which names we are using.

Definition

We want to say the necessary and not the superfluous. It is common in the history of mathematics to describe the development in English. There are minor pitfalls to this.

We will not describe these here, but rather will give one example. The first is that people of say “let A be such and such". What they mean is “denote such and such by A" with a tacit assumption that you know that such and such means. The practice is similar to using a pronoun. If I say “Ben walks to the store" and then “He bought some brocolli," we understand that “He" refers to Ben. If I say “John walks to the store" and then “He bought some tofu," we undersatnd that “He" refers to John.

The importance of using pronouns is relevant when we do not have proper names like Ben or John. For example, suppose I say “The tall brown-haired thin man walks to the store" it saves quite a bit of sound to say “He" in the next thought, or even, “the man". These sheets include many other examples of using pronouns

Instead of “Let A be such and such", we will just write name A. The idea is that we are introducing a name, A, that is some symbols to denote some object. The use of a different font and the symbol name before A is an abbreviation of the following: “Look at this symbol here A. This symbol is a reference for some object. It will always reference that object."

For the statement that the object denoted by A *is* or *is the same* or *is the very same object* as the object denoted by B, we will use =. So =we will write $A = B$.

If all we want to say is that two objects

We also need some way of identifying that two names

Account 1.

1	name	x
2	name	y
3	have	$x = y$

Account 2.

1	name	x		
2	name	y		
3	name	z		
4	have	$x \subset y$		
5	have	$y \subset z$		
6	thus	$(\forall a)(a \in x \implies a \in y)$	by	4
7	thus	$(\forall a)(a \in y \implies a \in z)$	by	5
8	thus	$x \subset z$	by	6,7

Account 3.

1-3	`name`	$x,\, y,\, z$		
4	`have`	$x \subset y$		
5	`have`	$y \subset z$		
6	`thus`	$(\forall a)(a \in x \implies a \in y)$	`by`	`4`
7	`thus`	$(\forall a)(a \in y \implies a \in z)$	`by`	`5`
8	`thus`	$x \subset z$	`by`	`6,7`

Why

We want to talk about objects and sets of objects.

Symbols and Words

On this page are the Latin letters forming words of the English language. The letters of the English language are a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u , v, w, x, y, z. A word is several letters next to each other. The words are many, and listing them is no task for this sheet. An example will suffice. The word “example" consists of the letters e, x, a, m, p, l, e.

It is an old truth that the words are visual marks corresponding to sounds, and that the sounds themselves are auditory marks corresponding to thoughts. It is an old debate whether thought needs language, but that will not concern us. If x is an object. $\in \varepsilon$

We ask for a standard way of denoting statements about objects and sets. We will often have a set, and another set, and another set, and we we will very quickly want to give these sets names. Sometimes I am referencing a particular set, and sometimes I am referenc

a standard way of denoting statements about objects and sets of objects. It will be succinct, easily read, and easily verifiable.

Similarly we will use a formal language to succinctly denote mathematics statements about objects and sets of objects.

A *symbol* is

Discussion

An *assertion* is a sequence of symbols which is assumed to be true.

Let a be an object. Let A be a set. A *membership assertion* is $a \in A$. Notice that $\in$ is not symmetric. $a \in A$ does not assert the same meaning as $A \in a$.

Let b be an object. An *identity assertion* is $a = b$. Notice that $a = b$ asserts the same as $b = a$.

A *primitive sentence* is a belonging assertion or an equality assertion. The symbolism used includes three pieces: the names of the two objects and the symbols $\in$ or $=$.

A *logical form* is one of several structures:

1. and
2. or (in the sense of "— or — or both")
3. not
4. implies (in the sense of "if —, then —"
5. if and only if
6. for some

7. for all

This list is redundant.

A *sentence* primitive sentence or a logical form with a primitive sentence or a logical form with sentences.

Why

When are two sets the same?

Definition

Given sets A and B, if $A = B$ then every element of A is an element of B and every element of B is an element of A.

Account 4.

`name` A
`name` B
`have` $A = B$

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Is $A = B$ true? We define it to be so.

Two sets are *equal* if and only if every element of one is an element of the other. In other words, two sets are the same if they have the same elements. This statement is sometimes called the *axiom of extension*. Roughly speaking, if we refer to the elements of a set as its *extension*, then we have declared that if we know the extension then we know the set. A set is determined by its extension.

This definition gives us a way to argue that $A = B$ from the properties of the elements of A and B. It may not be obvious that the sets are the same. We first argue that each element of A is an element of B and then argue that each element of

B is an element of A. With these two implications, we use the axiom of extension to conclude that the sets are the same.

An immediate consequence of the axiom of extension is that there is a unique set that is empty. Suppose A and B are both empty. Then $A = B$. Suppose toward contradiction that there exists an element $a \in A$ with $a \notin B$ or $b \in B$ with $b \notin B$. But then A or B would be nonempty. Thus there is a set which is empty and any other empty set is this set. In other words, this set is unique. We call it the *empty set*.

Notation

As with any objects, we denote that A and B are equal by $A = B$. We denote that they are not equal by $A \neq B$. We denote the unique empty set by $\varnothing$.

The axiom of extension is

$$A = B \Leftrightarrow (a \in A \implies a \in B) \land (b \in B \implies b \in A).$$

A Contrast

We can compare the axiom of extension for sets and their elements with an analogous statement for human beings and their ancestors.

On the one hand, if two human beings are equal then they have the same ancestors. The ancestors being the person's parents, grandparents, greatgrandparents, and so on. This direction, same human implies same ancestors, is the analogue of the "only if" part of the axiom of extension. It is true.

On the other hand, if two human beings have the same set of ancestors, they need not be the same human. This direction, same ancestors implies same human, is the analogue of the "if" part of the axiom of extension. It is false. For example, siblings have the same ancestors but are different people.

We conclude that the axiom of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Why

We want language for all of the elements of a first set being the elements of a second set.

Definition

Given two sets A and B, if every element of A is an element of B then we call that A is a *subset* of the B. We say that A is *included* in B. We say that B is a *superset* of A or that B *includes* A. A set A includes and is included in itself.

If $A = B$, then A includes B and B includes A. The axiom of extension asserts the converse also holds. If A includes B and B includes A, then $A = B$. In other words, if A is a subset of B and B a subset of A, then $A = B$.

The empty set is a subset of every other set. Suppose toward contradiction that A were a set which did not include the empty set. Then there would exist an element in the empty set which is not in A. But then the empty set would not be empty. We call the empty set and A *improper subsets* of A. All other subsets we call *proper subsets.* In other words, B is an improper subset of A if and only if A includes B, $B \neq A$ and $B \neq \varnothing$.

Notation

Given two sets A and B, we denote that A is included in B by $A \subset B$. We read the notation $A \subset B$ aloud as "A is included in B" or “A subset B". Or we write $B \supset A$, and read it aloud “B includes A" or “B superset A".

In this notation, we express the axiom of extension

$$A = B \Leftrightarrow (A \supset B) \wedge (A \subset B).$$

The notation $A \subset B$ is a concise symbolism for the sentence “every element of A is an element of B." Or for the alternative notation $a \in A \implies a \in B$.

Properties

Given a set A, $A \subset A$. Like equality, we say that inclusion is *reflexive*. Given sets A and B, if $A \subset B$ and $B \subset C$ then $A \subset C$. Like equality, we say that inclusion is *transitive*. If $A \subset B$ and $B \subset A$, then $A = B$ (by the axiom of extension). Unlike equality, which is symmetric, we say that inclusion is *antisymmetric*.

Comparison with belonging

Given a set A inclusion is reflexive. $A \subset A$ is always true. Is $A \in A$ ever true? Also, inclusion is transitive. Whereas belonging is not.

Why

Can we always construct subsets?

Definition

We will say that we can. We assert that to every set and every sentence predicated of elements of the set there exists a second set (a subset of the first) whose elements satisfy the sentence. It is an consequence of the axiom of extension that this set is unique. The *axiom of specification* is this assertion. We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence.

Notation

Let A be a set. Let $S(a)$ be a sentence. We use the notation

$$\{a \in A \mid \; S(a) \}$$

to denote the subset of A specified by S. We read the symbol $|$ aloud as "such that." We read the whole notation aloud as "a in A such that..."

We call the notation *set-builder notation.* Set-builder notation avoids enumerating elements. This notation is really indispensable for sets which have many members, too many to reasonably write down.

Example

For example, let a, b, c, d be distinct objects. Let $A = \{a, b, c, d\}$. Then $\{x \in A \mid x \neq a\}$ is the set $\{b, c, d\}$

Now let B be an arbitrary set. The set $\{b \in B \mid b \neq b\}$ specifies the empty set. Since the statement $b \neq b$ is false for all objects b.

Why

We want to consider the elements of two sets together at one. Does a set exist which contains all elements which appear in either of one set or another?

Definition

We say yes. For every set of sets there exists a sets which contains all the elements that belong to at least one set of the given collection. We refer to this as the *axiom of unions.* If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the axiom of unions may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification to form the set which contains only those elements which are appear in at least one of any of the sets. As a result of the axiom of extension, this set is unique. We call it the *union* of the set of sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\cup\mathcal{A}$.

Simple Facts

Proposition 1. $\cup\varnothing = \varnothing$

Proposition **2.** $\cup\{A\} = A$

Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

Definition

Let A and B be nonempty sets. Let $a \in A$ and $b \in B$. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is a and the *second coordinate* is b.

The *product* of A and B is the set of all ordered pairs. This set is also called the *cartesian product.* If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the product of A with B by $A \times B$, read aloud as “A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Taste

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second “coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life. Plus, suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Equality

PROPOSITION **3.** *$(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.*

Proof. TODO $\square$

Why

How can we relate the elements of two sets?

Definition

A *relation* between two nonempty sets is a subset of their cross product. A relation on a single set is a subset of the cross product of it with itself.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation. The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation.

Notation

Let A and B be two nonempty sets. A relation on A and B is a subset of $A \times B$. Let C be a nonempty set. A relation on a C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The ordered pair (a, b) may or may not be in a relation on A and B. Also notice that if $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as “a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *antisymmetric*antisymmetric if two different objects are related only in one order, and never both. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related.

Notation

Let R be a relation on a non-empty set A. R is reflexive if

$$(a, a) \in R$$

for all $a \in A$. R is transitive if

$$(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

for all $a, b, c \in A$. R is symmetric if

$$(a, b) \in R \implies (b, a) \in R$$

for all $a, b \in A$. R is anti-symmetric if

$$(a, b) \in R \implies (b, a) \notin R$$

for all $a, b \in A$.

Why

We want a notion for a correspondence between two sets.

Definition

A *functional* relation on two sets relates each element of the first set with a unique element of the second set. A *function* is a functional relation.

The *domain* of the function is the first set and *codomain* of the function is the second set. The function *maps* elements *from* the domain *to* the codomain. We call the codomain element associated with the domain element the *result* of *applying* the function to the domain element.

Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in (A \to B)$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as “g of a and b". We read g_{ab} aloud as “g sub a b."

Why

We want to “combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations to *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a \, g \, b$ instead. We call this *infix notation.*

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Why

We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Why

We want to define the natural numbers. TODO: better why

Definition

The *successor* of a set is the union of the set with the singleton whose element is the set. This definition holds for any set, but is of interest only for the sets which will be defined in this sheet.

These sets are the following (and their successors): *One* is the successor of the empty set. *Two* is the successor of one. *Three* is the successor of two. *Four* is the successor of three. And so on; using the English language in the usual manner.

Can this be carried on and on? We will say yes. We will say that there exists a set which contains one and contains the successor of each of its elements. So, this set contains one. Since it contains one, it contains two. Since it contains two, it contains three. And so on. We call this assertion the *axiom of infinity*.

A set is a *successor set* if it contains one and if it contains the successor of each of its elements. In these words, the axiom of infinity asserts the existence of a successor set. We want this set to be unique. So we have a successor set. By the axiom of specification, the intersection of all the successor sets included in this first successor set exists. Moreover, this intersection is a successor set. Even more, this intersection is unique. For this, take a second successor set. Its intersection with the first successor set is contained in the first successor set. Thus, this intersection of two sets is one of the successor sets contained in the first set, and so, is contained in the intersection of all such sets. So then, that first intersection is contained in second intersection of two sets, which is, of course, contained in the second successor set. In other words, we start with a successor set. Use it to construct a succesor set contained in it, in such a way that every other successor set also contains this successor set so constructed. The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique.

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4.

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n.

Why

Definition

integer numbers integers

TODO

Why

We generalize the algebraic structure of addition over the integers.

Definition

A *group* is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra *group addition*. A *commutative group* is a group whose operation commutes.

Notation

TODO

Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *field addition.* We call the operation of the second algebra *field multiplication.*

Notation

We denote an arbitrary field by F, a mnemonic for "field."

TODO

Why

Definition

Why

We want a notion of distance between elements of the real line.

Definition

We define a function mapping a real number to its length from zero.

Notation

We denote the absolute value of a real number $a \in \mathsf{R}$ by $|a|$. Thus $|\cdot| : \mathsf{R} \to \mathsf{R}$ can be viewed as a real-valued function on the real numbers which is nonnegative.

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