Definition

The *nullspace* (or *kernel*) of a matrix $A \in \mathbf{R}^{m \times n}$ is the set

$$\{x \in \mathbf{R}^n \mid Ax = 0\}.$$

It is the set of vectors mapped to zero by A. Equivalently, it is the set of vectors orthogonal to the rows of A.

Notation

We denote the nullspace of $A \in \mathbf{R}^{m \times n}$ by $\mathrm{null}(A) \subset \mathbf{R}^n$. Some authors denote the nullspace of A by $\mathcal{N}(A)$.

A subspace

The nullspace of a matrix is a subspace (this justifies the terminology null*space*!). There are a few routes to see this.

The first is direct. If $w, z \in \mathrm{null}(A)$, then $Aw = 0$ and $Az = 0$. So then $A(w + z) = Aw + Az = 0$. So $\mathrm{null}(A)$ is closed under vector addition. Also $A(\alpha w) = \alpha(Aw) = 0$ for all $\alpha \in \mathbf{R}$. [In particular $A0 = 0$, so $0 \in \mathrm{null}(A)$; i.e., $\mathrm{null}(A)$ contains the origin.] So $\mathrm{null}(A)$ is closed under scalar multiplication.

The second is by thinking about orthogonal complements. Second, we can view the $\mathrm{null}(A)$ as the set of vectors orthgonal to all the rows of A. In other words, $\mathrm{null}(A) = \{\tilde{a}_1, \ldots, \tilde{a}_m\}^\perp$. The orthogonal complement of any set is a subspace (see Orthogonal Real Subspaces).

Ambiguity in solutions

Suppose we have a solution to the system of linear equation with data (A, y). In other words, we have a vector $x \in \mathbf{R}^n$ so that $y = Ax$. If we have a vector $z \in \mathrm{null}(A)$, then $x + z$ is also a solution to the system (A, y), since

$$A(x + z) = Ax + Az = Ax + 0 = y$$

Conversely, suppose there were another solution $\tilde{x} \in \mathbf{R}^n$ to the system (A, y). Then $y = Ax = A\tilde{x}$, so

$$0 = y - y = Ax - A\tilde{x} = A(x - \tilde{x}).$$

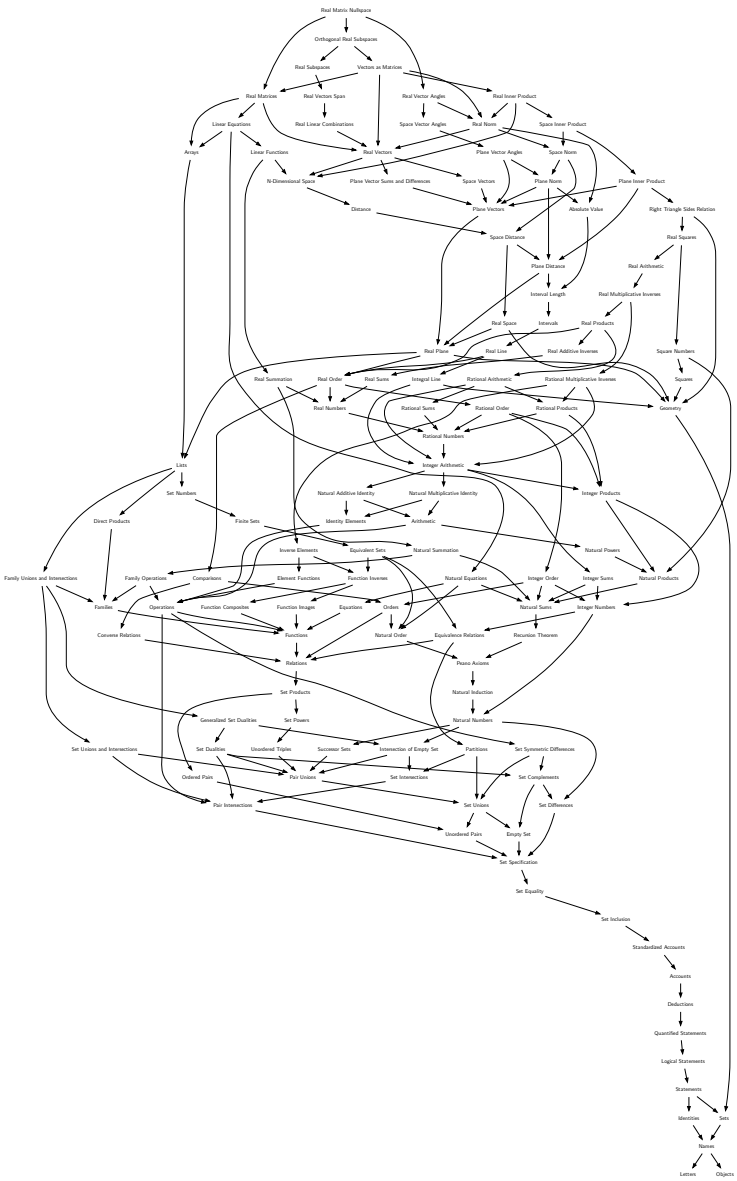
Consequently, $(x - \tilde{x}) \in \text{null}(A)$, and so $\tilde{x}$ is the solution x plus some vector in the null space of A. Consequently we are interested in whether A has vectors in its nullspace.

Zero nullspace

The origin 0 is always in the nullspace of A. However, this vector does not mean that we can find different solutions, since $x + 0 = x$ for all $x \in \mathbf{R}^n$. If, on the other hand, there is a nonzero vector $z \in \text{null}(A)$, then $x + z \neq x$, and $x + z$ is a solution for (A, y). We think about A as a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. In the case that there is a nonzero element in the nullspace, A maps different vectors to the same vector. Here, x and $x+z$ both map to y. In this case, the function is *not invertible*, because it is not one-to-one. If, however, zero is the only element of the null space, the function is one-to-one. So call A *one-to-one* if $\text{null}(A) = 0$.

Equivalent statements

A matrix $A \in \mathbf{R}^{m \times n}$ is *one-to-one* if the linear function $f : \mathbf{R}^n \to \mathbf{R}^m$ defined by $f(x) = Ax$ is one-to-one. In this case, if there exists $x \in \mathbf{R}^n$ so that $y = Ax$, then there is only one such x. Different elements in $\mathbf{R}^n$ map to different elements in $\mathbf{R}^m$.


Real Matrix Nullspace
Orthogonal Real Subspaces
Real Subspaces
Vectors as Matrices
Real Matrices
Real Vectors Span
Real Vector Angles
Real Inner Product
Linear Equations
Real Linear Combinations
Space Vector Angles
Real Norm
Space Inner Product
Arrays
Linear Functions
Real Vectors
Plane Vector Angles
Space Norm
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Plane Norm
Plane Inner Product
Distance
Plane Vectors
Absolute Value
Right Triangle Side Relation
Space Distance
Real Squares
Plane Distance
Real Arithmetic
Interval Length
Real Multiplicative Inverses
Real Spaces
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Square Numbers
Real Summation
Real Order
Real Sums
Integral Line
Rational Arithmetic
Rational Multiplicative Inverses
Squares
Real Numbers
Rational Sums
Rational Order
Rational Products
Geometry
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Direct Products
Finite Sets
Identity Elements
Arithmetic
Inverse Elements
Equivalent Sets
Natural Summation
Natural Powers
Family Unions and Intersections
Family Operations
Comparisons
Element Functions
Function Inverses
Natural Equations
Integer Order
Integer Sums
Natural Products
Families
Operations
Function Composites
Function Images
Equations
Order
Natural Sums
Integer Numbers
Converse Relations
Functions
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Set Symmetric Differences
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects