



## Why

We want a notion of reversing functions.

## Definition

Reversing functions does not make sense if the function is not one-to-one. Let  $f : X \rightarrow Y$ . If  $x_1$  goes to  $y$  and  $x_2$  goes to  $y$  (i.e.,  $f(x_1) = f(x_2) = y$ ), then what should  $y$  go to. One answer is that we should have a function which gives all the domain values which could lead to  $y$ . This is the inverse image (see **Function Images**)  $f^{-1}(\{y\})$ . Nor does reversing functions make sense if  $f$  is not onto. If there does not exist  $x \in X$  so that  $y = f(x)$ , then  $f^{-1}(\{y\}) = \emptyset$ .

In the case, however, that the function is one-to-one and onto, then each element of the domain corresponds to one and only one element of the codomain and vice versa. In this case, for all  $y \in Y$ ,  $f^{-1}(\{y\})$  is a singleton  $\{x\}$  where  $f(x) = y$ . In this case, we define a function  $g : Y \rightarrow X$  so that  $g(y) = x$  if and only if  $f(x) = y$ .

**Proposition 1** (Uniqueness). *Let  $f : A \rightarrow B$ ,  $g : B \rightarrow A$ , and  $h : B \rightarrow A$ . If  $g$  and  $h$  are both inverse functions of  $f$ , then  $g = h$ .*

**Proposition 2** (Existence). *If a function is one-to-one and onto, it has an inverse; and conversely.*<sup>1</sup>

## Composites and inverses

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Then  $g^{-1}$  maps  $\mathcal{P}(Z)$  to  $\mathcal{P}(Y)$  and  $f^{-1}$  maps  $\mathcal{P}(Y)$  to  $\mathcal{P}(X)$ . Then the following is immediate

**Proposition 3.**  $(gf)^{-1} = f^{-1}g^{-1}$

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<sup>1</sup>A proof will appear in future editions.

