



## Why

How big can a quadratic form be? How small?

## Result

**Proposition 1.** *Suppose  $A \in \mathbf{S}^n$  has real eigenvalues  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ . Then*

$$\lambda_n x^\top x \leq x^\top A x \leq \lambda_1 x^\top x,$$

for all  $x \in \mathbf{R}^n$ .

*Proof.* Since  $A$  is symmetric, there exists an orthogonal matrix  $Q \in \mathbf{R}^{n \times n}$  with  $A = Q\Lambda Q^\top$ . Express

$$\begin{aligned} x^\top A x &= x^\top Q\Lambda Q^\top x = (Q^\top x)^\top \Lambda (Q^\top x) \\ &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &= \lambda_1 \sum_{i=1}^n (q_i^\top x)^2 = \lambda_1 \|Q^\top x\|^2 = \lambda_1 \|x\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} x^\top A x &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &\geq \lambda_n \sum_{i=1}^n (q_i^\top x)^2 = \lambda_n \|Q^\top x\|^2 = \lambda_n \|x\|^2. \end{aligned}$$

□

## Notation

For this reason, it is common to order the eigenvalues of  $A \in \mathbf{S}^n$  by magnitude with  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ .  $\lambda_1$  is sometimes denoted  $\lambda_{\max}$  and  $\lambda_n$  is sometimes denoted  $\lambda_{\min}$ .

## Optimization implication

If  $z = \alpha x$ , then  $z^\top A z = \alpha^2 x^\top A x$ . Consider finding  $x \in \mathbf{R}^n$  to maximize

$$\begin{aligned} & \text{maximize} && x^\top A x \\ & \text{subject to} && \|x\| = 1. \end{aligned}$$

Since the objective is  $x^\top A x \leq \lambda_1$  for all  $x \in \mathbf{R}^n$  with  $\|x\| = 1$ , a solution of this problem is the eigenvector  $q_1 \in \mathbf{R}^n$  corresponding to  $\lambda_1$ . In other words, these inequalities are tight.

