



# Multivariate Normal Maximum Likelihood

## 1 Why

What of the generalization to a multivariate normal.

## 2 Result

**Proposition 1.** *Let  $(x^1, \dots, x^n)$  be a dataset in  $\mathbf{R}^d$ . Let  $f$  be a multivariate normal density with mean*

$$\frac{1}{n} \sum_{k=1}^d x^k$$

*and covariance*

$$\frac{1}{n} \sum_{k=1}^n \left( x^k - \frac{1}{n} \sum_{k=1}^n x^k \right) \left( x^k - \frac{1}{n} \sum_{k=1}^n x^k \right)^\top.$$

*Then  $f$  is a maximum likelihood multivariate normal density.*

*Proof.* We express the log likelihood

$$\sum_{k=1}^n -\frac{1}{2} (x^k - \mu)^\top \Sigma^{-1} (x^k - \mu) - \frac{1}{2} \log(2\pi)^d - \frac{1}{2} \log \mathbf{det} \Sigma$$

Let  $P = \Sigma^{-1}$ . The  $\log \mathbf{det} \Sigma$  is  $\log \mathbf{det} P^{-1}$  is  $\log (\mathbf{det} P)^{-1}$  is  $-\log \mathbf{det} P$ . Use matrix calculus to get

$$\frac{\partial \ell}{\partial P} = \sum_{k=1}^n (x^k - \mu)(x^k - \mu)^\top - P^{-1}.$$

□

We call these two objects the *maximum likelihood mean* or *empirical mean* and *maximum likelihood covariance* or *empirical covariance* of the dataset. We call the normal density with the empirical mean and empirical covariance the *empirical normal* of the dataset.