



## Why

We want to discuss when two sets are the same, and to do so we want to say when all the elements of one set are in another set.

## Definition

Denote a set by  $A$  and a set by  $B$ . If every element of the set denoted by  $A$  is an element of the set denoted by  $B$ , then we say that the set denoted by  $A$  is a *subset* of the set denoted by  $B$ .

We say that the set denoted by  $A$  is *included* in the set denoted by  $B$ . We say that the set denoted by  $B$  is a *superset* of the set denoted by  $A$  or that the set denoted by  $B$  *includes* the set denoted by  $A$ .

Every set is included in and includes itself. If the set denoted by  $B$  is a subset of the set denoted by  $A$ , but  $B$  is not  $A$ , we call  $B$  a *proper subset* of  $A$ .

## Notation

Let  $A$  denote a set and  $B$  denote a set. We denote that the set  $A$  is included in the set  $B$  by  $A \subset B$ . In other words,  $A \subset B$  means  $(\forall x)((x \in A) \longrightarrow (x \in B))$ . We read the notation  $A \subset B$  aloud as “A is included in B” or “A subset B”. Or we write  $B \supset A$ , and read it aloud “B includes A” or “B superset A”.  $B \supset A$  also means  $(\forall x)((x \in A) \longrightarrow (x \in B))$ .

Some authors use the notation  $\subseteq$  for  $\subset$ , and use  $B \subsetneq A$  to indicate that the set denoted by  $B$  is a *proper subset* of the set denoted by  $A$ .

## Properties

There are some properties that our intuition suggests inclusion should have. First, every set should include itself. We describe this fact by saying that inclusion is *reflexive*.

**Proposition 1** (Reflexive). *Every set is included in itself.*

*Proof.* Suppose  $A$  is a set. Then we have  $(\forall x)(x \in A \longrightarrow x \in A)$  In other words,  $A \subset A$ .  $\square$

Next, we expect that if one set is included in another, This fact is described by saying that inclusion is *transitive*

**Proposition 2** (Transitive). *If a set is included in another, and the latter in yet another, then the first is included in the last.*

*Proof.* Suppose  $A, B, C$  are sets. If  $A \subset B$  and  $B \subset C$  Thus  $A \subset C$  by modus ponens.  $\square$

Equality ( $=$ ) shares these two properties. Let  $A$  denote an object. Then  $A = A$ . Let  $B$  and  $C$  also denote objects. If  $A = B$  and  $B = C$ , then  $A = C$ . Of course, inclusion is not symmetric.. Belonging ( $\in$ ) may be, but need not be reflexive and transitive.

