Why

We want to model uncertain outcomes in dynamical systems.[1]

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{T-1}$ be sets. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Let $\mathcal{W}_0, \ldots, \mathcal{W}_T$. Let $w_t : \Omega \to \mathcal{W}_t$ for $t = 0, \ldots, T$ be random variables. For $t = 0, \ldots, T-1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathcal{X}_{t+1}$.

We call the sequence

$$\mathcal{D} = ((\mathcal{X}_t)_{t=0}^{T}), (\mathcal{U}_t)_{t=0}^{T-1}, (w_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1})$$

a *stochastic discrete-time dynamical system.* We call w_t the *noise* variables.

Problem

Let $x_0 : \Omega \to \mathcal{X}_0$ be a random variable. Define $x_1 : \Omega \to \mathcal{X}_1, \ldots, x_T : \Omega \to \mathcal{X}_t$ by

$$x_{t+1} = f_t(x_t, u_t, w_t)$$

for $t = 0, \ldots, T-1$. Roughly speaking, the state transition functions are nondeterministic. In other words, it is uncertain which state we will arrive in given our current state and action. The choice u_t only determines the distribution of x_{t+1}. Here x_0 is (still) called the *initial state* and is a random variable, usually assumed independent of the w_t.

Let $g_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathbf{R} \cup \{\infty\}$ for $t = 0, \ldots, T-1$ and $g_T : \mathcal{X}_T \times \mathcal{W}_T \to \mathbf{R} \cup \{\infty\}$. We call $(x_0, \mathcal{D}, (g_t)_{t=0}^{T})$ a *stochastic dynamic optimization problem.* As with dynamic optimization problems, we call g_t the *stage cost function* and g_T the *terminal cost function.* It is common for these to not depend on w_T (in other words, to be deterministic). It is also common for these to take infinite values to encode constraints.

[1] Future editions will expand.

As before, a stochastic dynamic optimization problem is just an optimization problem. Define $U = \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_{T-1}$ and let $u \in U$. Define $C : \Omega \to \mathbf{R}$ by

$$C = \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T).$$

We call C the *total cost* for actions u. It is a random variable.

Define $J : U \to \mathbf{R} \cup \{\infty\}$ by

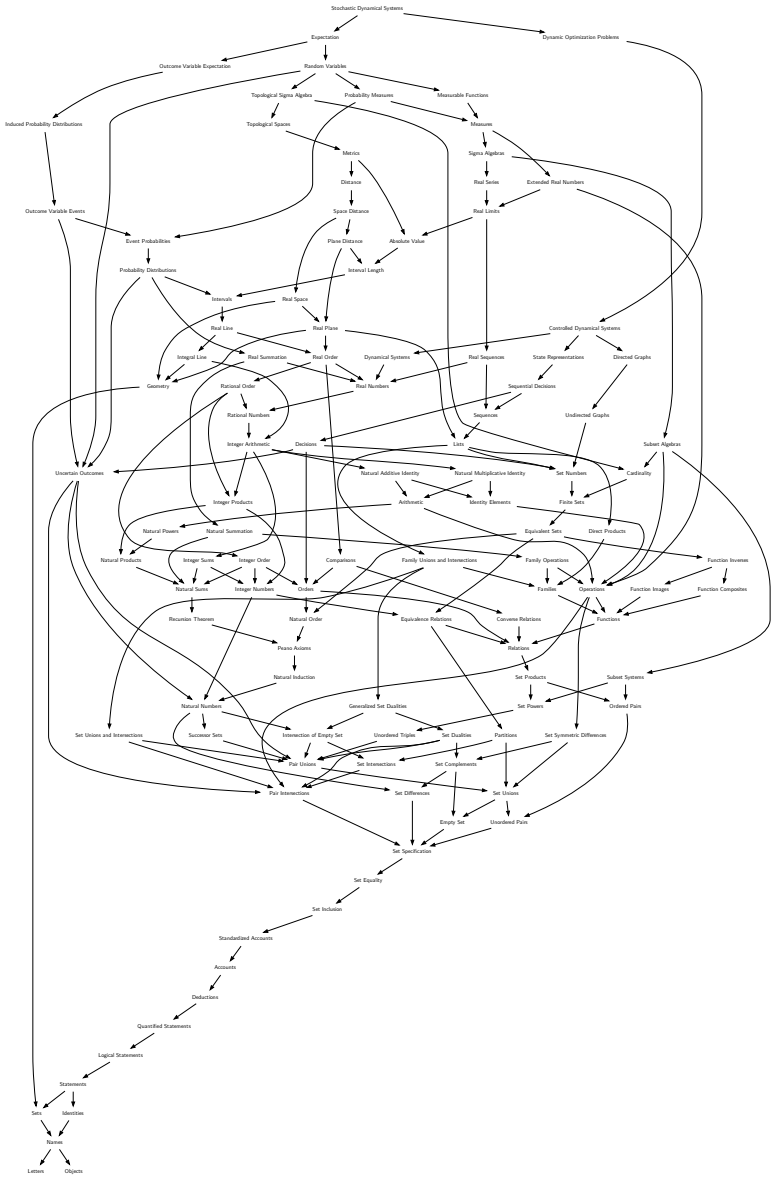
$$J = \mathbf{E}\left(\sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T)\right).$$

$J(u)$ is the *expected total cost* for inputs u.

The optimization problem is (U, J). In other words, the objective is the mean total stage cost plus the terminal cost.

Other terminology

Stochastic dynamic optimization problems are frequently called *stochastic control problems*.


Stochastic Dynamical Systems
Dynamic Optimization Problems
Expectation
Outcome Variable Expectation
Random Variables
Induced Probability Distributions
Topological Sigma Algebra
Probability Measure
Measurable Function
Topological Space
Measure
Metrics
Sigma Algebra
Distance
Real Series
Extended Real Numbers
Outcome Variable Events
Space Distance
Real Limits
Event Probabilities
Plane Distance
Absolute Value
Interval Length
Probability Distributions
Intervals
Real Space
Real Line
Real Plane
Controlled Dynamical Systems
Integral Line
Real Summation
Real Order
Dynamical Systems
Real Sequences
State Representations
Directed Graphs
Geometry
Rational Order
Real Numbers
Sequential Decisions
Rational Numbers
Sequences
Undirected Graphs
Integer Arithmetic
Decisions
Lists
Subset Algebra
Uncertain Outcomes
Natural Additive Identity
Natural Multiplicative Identity
Set Numbers
Cardinality
Integer Products
Arithmetic
Identity Elements
Finite Sets
Natural Powers
Natural Summation
Equivalent Sets
Direct Products
Natural Products
Integer Sums
Integer Order
Comparisons
Family Unions and Intersections
Family Operations
Function Inverses
Natural Sums
Integer Numbers
Orders
Families
Operations
Function Images
Function Composites
Recursion Theorem
Natural Order
Equivalence Relations
Converse Relations
Functions
Peano Axioms
Relations
Natural Induction
Set Products
Subset Systems
Natural Numbers
Generalized Set Dualities
Set Powers
Ordered Pairs
Set Unions and Intersections
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Differences
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects