$$\diamondplus$$

Simple Integral Homogeneity

1 Why

If we stack a rectangle on top of itself we have a rectangle twice the height. The additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property.

2 Result

Proposition 1. *The simple non-negative integral operator is homogenous over non-negative real values.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{SF}_+(X)$ denote the non-negative real-valued simple functions on X. Define $s : \mathcal{SF}_+(X) \to [0, \infty]$ by $s(f) = \int f d\mu$ for $f \in \mathcal{SF}_+(X)$.

In this notation, we want to show that $s(\alpha f) = \alpha s(f)$ for all $\alpha \in [0, \infty)$ and $f \in \mathcal{SF}_+(X)$. Toward this end, let $f \in \mathcal{SF}_+(X)$ with the simple partition $\{A_n\} \subset \mathcal{A}$ and $\{a_n\} \subset [0, \infty]$.

First, let $\alpha \in (0, \infty)$. Then $\alpha f \in \mathcal{SF}_+(X)$, with the simple partition $\{A_n\} \subset \mathcal{A}$ and $\{\alpha a_n\} \subset [0, \infty]$.

$$s(\alpha f) = \sum_{i=1}^{n} \alpha a_n \mu(A_i) \quad = \alpha \sum_{i=1}^{n} a_n \mu(A_i) = \alpha s(f).$$

If $\alpha = 0$, then αf is uniformly zero; it is the non-negative simple with partition $\{X\}$ and $\{0\}$. Regardless of the measure of X, this non-

negative simple function is zero Recall that we define $0 \cdot \infty = \infty \cdot 0 = 0$. $\qquad\square$