

## INNER PRODUCT LINEAR FUNCTIONAL REPRESENTATIONS

## Why

We can identify any linear functional  $F: \mathbb{R}^n \to \mathbb{R}$  with a vector  $y \in \mathbb{R}^n$  so that  $F(x) = \langle x, y \rangle$ . We generalize this result to complete inner product spaces.

## Motivating result

The following is known as the Riesz representation theorem (or Riesz-Fréchet representation theorem, or Riesz theorem, or Riesz-Fréchet theorem).

**Proposition 1.** Let  $((V,k), \langle \cdot, \cdot \rangle)$  be a complete inner product space and let  $F: V \to k$  be a continuous linear functional on V. There exists a unique  $y \in V$  so that

$$F(x) = \langle x, y \rangle$$

for all  $x \in V$ . Moreover  $||y|| = ||F||_*$ .

Clearly  $\mathbb{R}^n$  is a complete inner product space, and so this this theorem says the expected. We can identify linear functionals on  $\mathbb{R}^n$  with elements (vectors) in  $\mathbb{R}^n$ .<sup>1</sup>

<sup>&</sup>lt;sup>1</sup>Future editions will expand further.

