

## Sigma Algebra Event Independence

### 1 Why

TODO

#### 2 Definition

The sigma algebra *generated by an event* is the sigma algebra consisting of the empty set, the event, the complement (in the base set) of the event, and the base set.

A family of events events are *independent* if the sigma algebras generated by the events are independent.

### 2.1 Notation

Let  $(X, \mathcal{A}, \mu)$  be a probability space. Let  $A \in \mathcal{A}$  be an event. The sigma algebra generated by A is  $\{\emptyset, A, X - A, X\}$ . We denote it by  $\sigma(A)$ .

Let  $B \in \mathcal{A}$ . If A is independent of B we write  $A \perp B$ .

# 3 Equivalent Condition

**Proposition 1.** Two events are independent if and only if the measure of their intersection is the product of their measures.

*Proof.* Let  $(X, \mathcal{A}, \mu)$  be a probability space. Let  $A, B \in \mathcal{A}$ .

 $(\Rightarrow)$  If  $A \perp B$ , then by definition  $A \in \sigma(A)$  and  $B \in \sigma(B)$  and so:

$$\mu(A \cap B) = \mu(A)\mu(B).$$

 $(\Leftarrow)$  Conversely, let  $a \in \sigma(A)$  and  $b \in \sigma(B)$ . If  $a = \emptyset$  or  $b = \emptyset$  then  $a \cap b = \emptyset$ . So

$$\mu(a \cap b) = \mu(\varnothing) = \mu(a)\mu(b),$$

since one of the two measures on the right hand side is zero. On the other hand, if a = X, then  $a \cap b = b$  and so

$$\mu(a \cap b) = \mu(b) = \mu(a)\mu(b),$$

since  $\mu(a) = \mu(X) = 1$ . Likewise if b = X.

So it remains to verify  $\mu(a \cap b) = \mu(a)\mu(b)$  for the cases  $a \in \{A, X - A\}$  and  $b \in \{B, X - B\}$ . If a = A, and b = B, then the identity follows by hypothesis. Next, observe that  $A \cap (X - B) = A - (A \cap B)$  and  $(A \cap B) \subset A$  so  $\mu(X) < \infty$  allows us to deduce:

$$\mu(A \cap (X - B)) = \mu(A - (A \cap B))$$
$$= \mu(A) - \mu(A \cap B)$$
$$= \mu(A)(1 - \mu(B))$$
$$= \mu(A)\mu(X - A).$$

Similar for X-A and B. Finally, recall that  $\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B)$ . So then,

$$\mu((X - A) \cap (X - B)) = 1 - \mu(A \cup B)$$

$$= 1 - \mu(A) - \mu(B) + \mu(A \cap B)$$

$$= 1 - \mu(A) - \mu(B) + \mu(A)\mu(B)$$

$$= (1 - \mu(A))(1 - \mu(B))$$

$$= \mu(X - A)\mu(X - B).$$