Why

The (surprising fact) is that the operation of going from an index list to its induced sublist is *linear*, if the elements of the list are over a field, and thus may be viewed as a vector space.

Definition

The *index matrix* associated with the index sequence α of length r and order n (recall, $r \leq n$) is the $r \times n$ matrix whose i,jth entry is 1 if the sequence's ith coordinate is j, and 0 otherwise.

Examples

Here are some order-5 index lists: $(1,2,3)$, $(3,2,1)$, $(4,5,1)$, $(5,4,3,2,1)$, $(3,)$.

The matrices corresponding to these examples are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for the first two examples, and

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

for the last three.

In this case, we refer to the induced sublist as an induced *subvector*. The value of index matrices is that they give induced subvectors via the usual and familiar operation of matrix multiplication. The *subvector* of an n-vector associated with a length-r index sequence is the product of the sequence's $r \times n$ corresponding index matrix with the n-dimensional vector.

For example, define $x = \begin{bmatrix} 6 & 4 & 5 & 3 & 9 \end{bmatrix}^\top$. Then the subvector of x associated with the index sequence $(3, 2, 1)$ is the vector $\begin{bmatrix} 3 & 9 & 6 \end{bmatrix}^\top \in \mathbf{R}^3$, because

$$
\begin{bmatrix} 3 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \\ 3 \\ 9 \end{bmatrix}
$$

If $r = n$ then the index matrix is a permutation matrix.

Notation

Let $r \leq n$ be natural numbers. Let $\alpha : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$ be an index sequence. Denote the index matrix associated with α by P_α. This matrix P_α is an element of $\mathbf{N}^{r \times n}$ and is defined by

$$
(P_a)_{ij} = \begin{cases} 1 & j = \alpha(i) \\ 0 & \text{otherwise} \end{cases}
$$

Let A be a nonempty set and let $x \in A^n$. then the subvector of x associated with P_α (and so with α) is

$$
P_\alpha x = \left(x_{\alpha(1)}, \ldots, x_{\alpha(r)} \right)
$$

We denote the product $P_\alpha x$ by x_α.

We denote the product $P_\alpha X P_\alpha^\top$ by $X_{\alpha\alpha}$.

Index Matrices

Permutation Matrices

Matrix Transpose

Real Matrix Inverses

Index Lists

Matrices

Real Matrix-Matrix Products

Real Matrix-Vector Products

Real Linear Combinations

Vectors as Matrices

Real Inner Product

Space Inner Product

Real Matrix

Vectors

Real Norm

Space Norm

Linear Equations

Linear Functions

Arrays

Real Vectors

Space Vectors

Plane Vector Sums and Differences

N-Dimensional Space

Plane Norm

Plane Inner Product

Distance

Absolute Value

Plane Vectors

Right Triangle Sides Relation

Space Distance

Real Squares

Plane Distance

Real Arithmetic

Interval Length

Real Multiplication Inverses

Intervals

Real Space

Real Products

Square Numbers

Real Summation

Natural Summation

Real Order

Real Line

Real Additive Inverses

Natural Arithmetic

Integral Line

Square

Real Plane

Real Sums

Rational Multiplication Inverse

Geometry

Permutation

Groups

Lists

Inverse Elements

Fields

Rational Order

Real Numbers

Rational Sums

Rational Products

Set Numbers

Direct Products

Rings

Rational Numbers

Integer Arithmetic

Finite Sets

Integer Products

Equivalent Sets

Element Functions

Natural Multiplication Identity

Natural Additive Identity

Function Inverses

Family Unions and Intersections

Family Operations

Compatible

Identity Elements

Natural Equations

Arithmetic

Integer Order

Integer Sums

Function Compatible

Function Images

Families

Operations

Equations

Orders

Natural Powers

Integer Numbers

Natural Products

Equivalence Relation

Functions

Converse Relations

Natural Order

Relations

Natural Sums

Ordered Pair Projections

Recursion Theorems

Set Products

Peano Axioms

Set Powers

Natural Induction

Generalized Set Dualities

Natural Numbers

Set Unions and Intersection

Partitions

Unordered Triples

Intersection of Empty Set

Successor Sets

Set Dualities

Set Symmetric Differences

Ordered Pairs

Set Intersection

Pair Unions

Set Complements

Pair Intersections

Set Unions

Unordered Pairs

Empty Set

Set Differences

Set Specification

Set Equality

Set Inclusion

Deduction

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects