Why

How does convergence pointwise (or almost everywhere pointwise) relate to convergence in measure?

Results

Proposition **1.** *There exists a measure space and a sequence of measurable real-valued functions on that space converging everywhere (and so almost everywhere) but not converging in measure.*

Proposition **2.** *There exists a measure space and a sequence of measurable real-valued functions on that space converging in measure but not converging almost everywhere (nor everywhere).*

Proposition **3.** *On finite measure spaces, all sequences of measurable real-valued functions converging almost everywhere converge in measure.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of measurable functions on X such that $f_n \longrightarrow f$ almost everywhere. Let $\varepsilon > 0$.

For each $x \in X$, if $|f_n(x) - f(x)| > \varepsilon$ for infinitely many n, then $f_n(x) \not\longrightarrow f(x)$. Let A be the set of such x. and let $B = \{x \in X \mid f_n(x) \not\longrightarrow f(x)\}$. A is a subset of B. The measure of B is zero since $f_n \longrightarrow f$. Use the the monotonicity

of measure to conclude. $\mu(A) \leq \mu(B) = 0$. Since $\mu(A) \geq 0$, $\mu(A) = 0$.

For natural k, let E_k be the $\{x \in X \mid |f_k(x) - f(x)| > \varepsilon\}$. Then $x \in A$ means that for every natural n, there exists a $k \geq n$ such that $x \in E_k$. In particular, for every n, x is in $\cup_{k=n}^{\infty} E_k$; denote this set by B_n. If x is in B_n for every n, then $x \in \cap_{n=1}^{\infty} B_n$. So we can write

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} B_n.$$

The sequence of sets $(B_n)_n$ is decreasing. So since μ is finite,

$$\lim_{n\to\infty} \mu(B_n) = \mu(A) = 0.$$

For every n, the set B_n contains $\{x \in X \mid |f_n(x) - f(x)| > \varepsilon\}$, namely E_n, the first set in the union. So then $\mu(E_n) \leq \mu(B_n)$ by monotonicity and so

$$0 \leq \lim_{n\to\infty} \mu(E_n) \leq \lim_{n\to\infty} \mu(B_n) = 0,$$

and we conclude $\lim_n E_n = 0$. Since ε was arbitrary, we conclude $f_n \longrightarrow f$ in measure.

$\square$

Proposition **4.** *On any measure space, for a sequence of measurable real-valued functions converging in measure to a measurable real-valued limit function, there exists a subsequence convergeng to the limit function almost everywhere.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of measurable functions on X such that $f_n \longrightarrow f$ in measure.

There exists n_1 so that

$$\mu(\{x \in X \mid |f_{n_1}(x) - f(x)| > 1\}) < \frac{1}{2}.$$

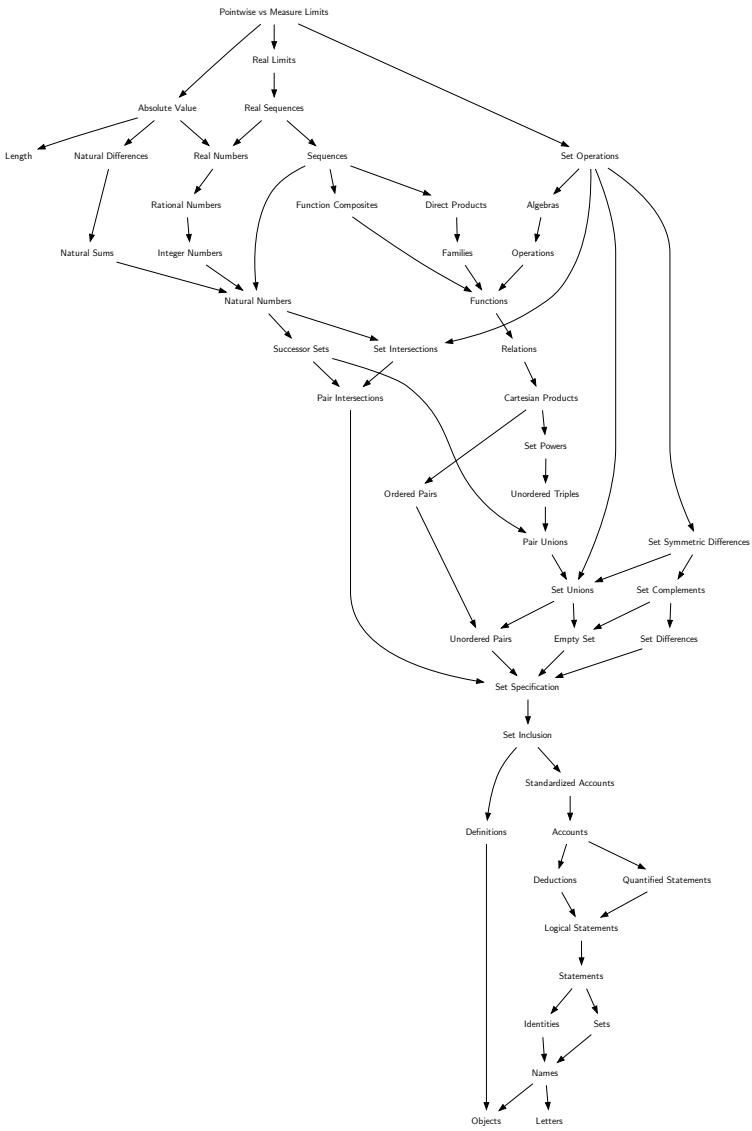
Can find $n_2 > n_1$ so that

$$\mu(\left\{x \in X \;\middle|\; |f_{n_2}(x) - f(x)| > \frac{1}{2}\right\}) < \frac{1}{4}.$$

We can inductively find a sequence $\{n_k\}_k$ so that:

$$\mu\left(\left\{x \in x \;\middle|\; |f_{n_k}(x) - f(x)| > \frac{1}{k}\right\}\right) \leq \frac{1}{2^k}.$$

$\square$


Pointwise vs Measure Limits
Real Limits
Absolute Value
Real Sequences
Length
Natural Differences
Real Numbers
Sequences
Set Operations
Rational Numbers
Function Composites
Direct Products
Algebras
Natural Sums
Integer Numbers
Families
Operations
Natural Numbers
Functions
Successor Sets
Set Intersections
Relations
Pair Intersections
Cartesian Products
Set Powers
Ordered Pairs
Unordered Triples
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Unordered Pairs
Empty Set
Set Differences
Set Specification
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Objects
Letters