Why

We explore matrix-vector multiplication.

Definition

Given a matrix $A \in \mathbf{R}^{m \times n}$ and a vector $x \in \mathbf{R}^n$, the *product* of A *with* x is the vector $y \in \mathbf{R}^m$ defined by

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \quad i = 1, \ldots, m.$$

Notation

We denote the product of A with x by Ax. With which we concisely write the system linear equations (A, b) as $b = Ax$.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function $x \mapsto Ax$. In other words, we are interested in the existence of an inverse element of A. The geometric interpretation is that A transforms the vector x.

Conversely, we can view x as transforming (acting on) A. Let $a^j \in \mathbf{R}^m$ denote the jth column of A, then

$$Ax = \sum_{j=1}^{n} x_j a^j.$$

In other words, y is linear combination of the columns of A.

Properties

We call the function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ defined by $f(x) = Ax$ the *matrix multiplication function* (or *matrix-vector product function*) associated with A. f is satisfies the following two important properties:

1. $A(x + y) = Ax + Ay$

2. $A(\alpha x) = \alpha Ax$.

We call such a function f *linear*. In other words, the matrix multiplication function is linear. Conversely, if $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is linear, there exists a matrix inducing g.

Proposition 1. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be linear. Then there exists a unique $A \in \mathbf{R}^{m \times n}$ satisfying $f(x) = Ax$ for all $x \in \mathbf{R}^n$.*

Proof. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. $\square$

Real Matrix-Vector Products

Real Linear Combinations
Vectors as Matrices

Real Matrices
Real Inner Product

Linear Equations
Real Norm

Real Vector
Space Vectors
Linear Functions
N-Dimensional Space
Plane Inner Product
Plane Norm

Plane Vector Sums and Differences
Plane Vector
Distance
Right Triangle Sides Relation

Space Distance
Absolute Value
Real Square

Plane Distance
Real Arithmetic
Square Numbers

Interval Length
Real Multiplicative Inverses
Square

Intervals
Real Products

Real Space
Real Additive Inverse

Real Plane
Real Line
Real Summation
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses

Integral Line
Geometry
Real Numbers
Rational Sums
Rational Order
Rational Products

Arrays
Rational Numbers

Lists
Integer Arithmetic
Integer Products

Set Numbers
Natural Additive Identity
Natural Multiplicative Identity

Direct Products
Finite Sets
Identity Elements
Arithmetic
Natural Powers

Family Unions and Intersections
Inverse Elements
Equivalent Sets
Natural Summation
Integer Order
Integer Sums
Natural Products

Comparisons
Function Inverses
Element Functions
Family Operations
Natural Equations
Integer Numbers

Families
Function Images
Function Composites
Operations
Equations
Orders
Natural Sums

Converse Relations
Functions
Natural Order
Equivalence Relations
Recursion Theorem

Relations
Peano Axioms

Set Products
Natural Induction

Set Powers
Generalized Set Dualities
Natural Numbers

Set Unions and Intersections
Unordered Triples
Set Dualities
Intersection of Empty Set
Successor Sets
Set Symmetric Differences

Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Partitions

Pair Intersections
Set Unions
Set Differences

Unordered Pairs
Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets
Identities

Names

Letters
Objects