The Bourbaki Project

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Why

We want to talk about things.

Definition

We use the word *object* with its usual sense in the English language. An object may be tangible, in that we can hold or touch it, or an object may be abstract, in that we can do neither.

To discuss objects we give them *names*. When discussing abstract objects, we tend to give them short names. A single Latin letter regularly suffices. We use italics when writing the name. For example, we will refer to the object named “a" by a. To aid our memory, we tend to choose the letter mnemonically.

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object that two names refer to is the same, then we say that the first name *equals* the second name.

Notation

We denote that the object named a and the object named b refer to the same object by $a = b$. We read this notation aloud as: “a is b" or “a equals b". We denote that the object a and b refer to different objects by $a \neq b$. We read this aloud as “a is not b" or “a does not equal b".

Other English readings of $a = b$ include: “a is the same as b", “a is equivalent to b", “a refers to the same object as b."

Properties

Given an object a, $a = a$ is true. We say that equivalence is *reflexive*. Given objects a and b, $a = b$ implues $b = a$. We say that equality is *symmetric*. Given objects a, b, and c, $a = b$ and $b = c$ implies $a = c$. We say that equality is *transitive*.

Why

We want to talk about none, one, or several objects considered as an abstract whole.

Definition

A *set* is an abstract object. We think of it as several objects considered as a whole. The central primitive notion is that of *belonging*. A set *contains* the objects so considered. These objects are the *members* or *elements* of the set. They belong to the set.

The objects a set contains may be other sets. In other words, an element of a set may be another set. This may be subtle at first glance, but becomes familiar with experience.

We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Notation

We tend to denote sets by upper case Latin letters: for example, A, B, and C. To aid our memory, we tend to use the lower case form of the letter for an element of the set. For example, let A and B be nonempty sets. We tend to denote by a an element of A. And similarly, we tend to denote by b an element of B

We denote that an object a is an element of a set A by $a \in A$. We read the notation $a \in A$ aloud as “a in A." The symbol $\in$ is a stylized lower case Greek letter ε. Since ε is read aloud "ehp-sih-lawn," $\in$ is a mnemonic for "element of". We denote that an object a is not an element of the set A by $a \notin A$. We read this notation aloud as "a not in A."

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, ..., x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Why

When are two sets the same?

Definition

Given sets A and B, if $A = B$ then every element of A is an element of B and every element of B is an element of A.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Is $A = B$ true? We define it to be so.

Two sets are *equal* if and only if every element of one is an element of the other. In other words, two sets are the same if they have the same elements. This statement is sometimes called the *axiom of extension*. Roughly speaking, if we refer to the elements of a set as its *extension*, then we have declared that if we know the extension then we know the set. A set is determined by its extension.

This definition gives us a way to argue that $A = B$ from the properties of the elements of A and B. It may not be obvious that the sets are the same. We first argue that each element of A is an element of B and then argue that each element of B is an element of A. With these two implications, we use the axiom of extension to conclude that the sets are the same.

An immediate consequence of the axiom of extension is that there is a unique set that is empty. Suppose A and B are both empty. Then $A = B$. Suppose toward contradiction that there exists an element $a \in A$ with $a \notin B$ or $b \in B$ with $b \notin B$. But then A or B would be nonempty. Thus there is a set which is empty and any other empty set is this set. In other words, this set is unique. We call it the *empty set*.

Notation

As with any objects, we denote that A and B are equal by $A = B$. We denote that they are not equal by $A \neq B$. We denote the unique empty set by $\varnothing$.

The axiom of extension is

$$A = B \Leftrightarrow (a \in A \implies a \in B) \wedge (b \in B \implies b \in A).$$

A Contrast

We can compare the axiom of extension for sets and their elements with an analogous statement for human beings and their ancestors.

On the one hand, if two human beings are equal then they have the same ancestors. The ancestors being the person's parents, grandparents, greatgrandparents, and so on. This direction, same human implies same ancestors, is the analogue of the "only if" part of the axiom of extension. It is true. On the other hand, if two human beings have the same set of ancestors, they need not be the same human. This direction, same ancestors implies same human, is the analogue of the "if" part of the axiom of extension. It is false. For example, siblings have the same ancestors but are different people.

We conclude that the axiom of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Why

We want language for all of the elements of a first set being the elements of a second set.

Definisions

Given two sets A and B, if every element of A is an element of B then we call that A is a *subset* of the B. We say that A is *included* in B. We say that B is a *superset* of A or that B *includes* A. A set A includes and is included in itself.

If $A = B$, then A includes B and B includes A. The axiom of extension asserts the converse also holds. If A includes B and B includes A, then $A = B$. In other words, if A is a subset of B and B a subset of A, then $A = B$.

The empty set is a subset of every other set. Suppose toward contradiction that A were a set which did not include the empty set. Then there would exist an element in the empty set which is not in A. But then the empty set would not be empty. We call the empty set and A *improper subsets* of A. All other subsets we call *proper subsets.* In other words, B is an improper subset of A if and only if A includes B, $B \neq A$ and $B \neq \varnothing$.

Notation

Given two sets A and B, we denote that A is included in B by $A \subset B$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A".

In this notation, we express the axiom of extension

$$A = B \Leftrightarrow (A \supset B) \land (A \subset B).$$

The notation $A \subset B$ is a concise symbolism for the sentence "every element of A is an element of B." Or for the alternative notation $a \in A \implies a \in B$.

Properties

Given a set A, $A \subset A$. Like equality, we say that inclusion is *reflexive*. Given sets A and B, if $A \subset B$ and $B \subset C$ then $A \subset C$. Like equality, we say that inclusion is *transitive*. If $A \subset B$ and $B \subset A$, then $A = B$ (by the axiom of extension). Unlike equality, which is symmetric, we say that inclusion is *antisymmetric*.

Comparison with belonging

Given a set A inclusion is reflexive. $A \subset A$ is always true. Is $A \in A$ ever true? Also, inclusion is transitive. Whereas belonging is not.

Why

Can we always construct subsets?

Definition

We will say that we can. We assert that to every set and every sentence predicated of elements of the set there exists a second set (a subset of the first) whose elements satisfy the sentence. It is an consequence of the axiom of extension that this set is unique. The *axiom of specification* is this assertion. We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence.

Notation

Let A be a set. Let $S(a)$ be a sentence. We use the notation

$$\{a \in A \mid \; S(a) \}$$

to denote the subset of A specified by S. We read the symbol $\mid$ aloud as "such that." We read the whole notation aloud as "a in A such that..."

We call the notation *set-builder notation*. Set-builder notation avoids enumerating elements. This notation is really indispensable for sets which have many members, too many to reasonably write down.

Example

For example, let a, b, c, d be distinct objects. Let $A = \{a, b, c, d\}$. Then $\{x \in A \mid x \neq a\}$ is the set $\{b, c, d\}$

Now let B be an arbitrary set. The set $\{b \in B \mid b \neq b\}$ specifies the empty set. Since the statement $b \neq b$ is false for all objects b.

Why

We want to consider the elements of two sets together at one. Does a set exist which contains all elements which appear in either of one set or another?

Definition

We say yes. For every set of sets there exists a sets which contains all the elements that belong to at least one set of the given collection. We refer to this as the *axiom of unions*. If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the axiom of unions may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification to form the set which contains only those elements which are appear in at least one of any of the sets. As a result of the axiom of extension, this set is unique. We call it the *union* of the set of sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\cup \mathcal{A}$.

Simple Facts

Proposition **1.** $\cup \varnothing = \varnothing$

PROPOSITION **2.** $\cup\{A\} = A$

Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

Definition

Let A and B be nonempty sets. Let $a \in A$ and $b \in B$. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is a and the *second coordinate* is b.

The *product* of A and B is the set of all ordered pairs. This set is also called the *cartesian product.* If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Taste

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second "coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life. Plus, suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Equality

Proposition **3.** *$(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.*

Proof. TODO □

Why

How can we relate the elements of two sets?

Definition

A *relation* between two nonempty sets is a subset of their cross product. A relation on a single set is a subset of the cross product of it with itself.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation. The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation.

Notation

Let A and B be two nonempty sets. A relation on A and B is a subset of $A \times B$. Let C be a nonempty set. A relation on a C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The ordered pair (a, b) may or may not be in a relation on A and B. Also notice that if $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as “a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *antisymmetric*antisymmetric if two different objects are related only in one order, and never both. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related.

Notation

Let R be a relation on a non-empty set A. R is reflexive if

$$(a, a) \in R$$

for all $a \in A$. R is transitive if

$$(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

for all $a, b, c \in A$. R is symmetric if

$$(a, b) \in R \implies (b, a) \in R$$

for all $a, b \in A$. R is anti-symmetric if

$$(a, b) \in R \implies (b, a) \notin R$$

for all $a, b \in A$.

Why

We want a notion for a correspondence between two sets.

Definition

A *functional* relation on two sets relates each element of the first set with a unique element of the second set. A *function* is a functional relation.

The *domain* of the function is the first set and *codomain* of the function is the second set. The function *maps* elements *from* the domain *to* the codomain. We call the codomain element associated with the domain element the *result* of *applying* the function to the domain element.

Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in (A \to B)$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as “g of a and b". We read g_{ab} aloud as “g sub a b."

Why

We want to “combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations to *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\, g\, b$ instead. We call this *infix notation*.

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Why

We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Why

We want to define the natural numbers. TODO: better why

Definition

The *successor* of a set is the union of the set with the singleton whose element is the set. This definition holds for any set, but is of interest only for the sets which will be defined in this sheet.

These sets are the following (and their successors): *One* is the successor of the empty set. *Two* is the successor of one. *Three* is the successor of two. *Four* is the successor of three. And so on; using the English language in the usual manner.

Can this be carried on and on? We will say yes. We will say that there exists a set which contains one and contains the successor of each of its elements. So, this set contains one. Since it contains one, it contains two. Since it contains two, it contains three. And so on. We call this assertion the *axiom of infinity*.

A set is a *successor set* if it contains one and if it contains the successor of each of its elements. In these words, the axiom of infinity asserts the existence of a successor set. We want this set to be unique. So we have a successor set. By the axiom of specification, the intersection of all the successor sets included in this first successor set exists. Moreover, this intersection is a successor set. Even more, this intersection is unique. For this, take a second successor set. Its intersection with the first successor set is contained in the first successor set. Thus, this intersection of two sets is one of the successor sets contained in the first set, and so, is contained in the intersection of all such sets. So then, that first intersection is contained in second intersection of two sets, which is, of course, contained in the second successor set. In other words, we start with a successor set. Use it to construct a succesor set contained in it, in such a way that every other successor set also contains this successor set so constructed. The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique.

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4.

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n.

Why

Definition

integer numbers *integers*

TODO

Why

We generalize the algebraic structure of addition over the integers.

Definition

A *group* is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra *group addition*. A *commutative group* is a group whose operation commutes.

Notation

TODO

Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *field addition*. We call the operation of the second algebra *field multiplication*.

Notation

We denote an arbitrary field by F, a mnemonic for “field."

TODO

Why

Definition

Why

We want a notion of distance between elements of the real line.

Definition

We define a function mapping a real number to its length from zero.

Notation

We denote the absolute value of a real number $a \in \mathsf{R}$ by $|a|$. Thus $|\cdot| : \mathsf{R} \to \mathsf{R}$ can be viewed as a real-valued function on the real numbers which is nonnegative.