Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Given a distribution $p : \Omega \to \mathbf{R}$, the *probability of an event $A \subset \Omega$ is* $\sum_{a \in A} p(a)$, the sum of the probabilities of its outcomes.

Define $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ by $\mathbf{P}(A) = \sum_{a \in A} p(a)$. We call $\mathbf{P}$ *the event probability function* (or *the probability measure*) induced by p. Clearly $\mathbf{P}$ depends on the set of outcomes Ω and the distribution $p : \Omega \to \mathbf{R}$. We sometimes denote this dependence by $\mathbf{P}_{\Omega,p}$ or $\mathbf{P}_p$.

Example: die

Define $p : \{1, \ldots, 6\} \to \mathbf{R}$ by $p(\omega) = 1/6$ for $\omega = 1, \ldots, 6$. Define the event $E = \{2, 4, 6\}$. Then

$$\mathbf{P}(E) = \sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of P

As a result of the conditions on p, $\mathbf{P}$ satisfies

1. $\mathbf{P}(A) \geq 0$ for all $A \subset \Omega$;

2. $\mathbf{P}(\Omega) = 1$ (and $\mathbf{P}(\varnothing) = 0$);

3. $\mathbf{P}(A) + \mathbf{P}(B)$ for all $A, B \subset \Omega$ and $A \cap B = \varnothing$. This statement follows from the more general identity

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

for $A, B \subset \Omega$, by using $P(\varnothing) = 0$ of (2) above.

Do all such $\mathbf{P}$ satisfying (1)-(3) have a corresponding underlying probability distribution? In other words, suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satifies (1)-(3). These three conditions are sometimes called the *axioms of probability for finite sets*.

Define $q : \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an *event probability function* (or a *probability measure*).

Probability by cases

Let $\mathbf{P}$ be a probability event function. Suppose $A_1, \ldots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

Some authors call this the *law of total probability*.

Event Probabilities

Probability Distributions

Intervals

Real Line

Real Order

Rational Order   Real Numbers   Real Summation

Integral Line   Rational Numbers

Integer Arithmetic

Integer Products   Natural Additive Identity   Natural Multiplicative Identity

Arithmetic   Identity Elements

Natural Powers   Natural Summation

Comparisons   Integer Order   Integer Sums   Natural Products   Family Operations

Orders   Integer Numbers   Natural Sums   Families   Operations

Converse Relations   Natural Order   Equivalence Relations   Recursion Theorem   Functions

Relations   Peano Axioms

Set Products   Natural Induction   Uncertain Outcomes

Ordered Pairs   Set Powers   Natural Numbers

Unordered Triples   Successor Sets   Intersection of Empty Set   Partitions

Pair Unions   Set Intersections   Set Symmetric Differences

Set Unions   Set Complements   Pair Intersections

Unordered Pairs   Set Differences   Empty Set

Geometry   Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets   Identities

Names

Letters   Objects