Families

1 Why

We want to generalize operations beyond two objects.

2 Definition

Let A, B be non-empty sets. A *family* of elements of a first set *indexed* by elements of a second set is the range of a function from the second set to the first set. We call second set the *index set*.

If the index set is a finite set, we call the family a *finite family*. If the index set a countable set, we call the family a *countable family*. If the index set is an uncountable set, we call the family a *uncountable family*.

If the codomain is a set of sets, we call the family a *family of sets*. We often use a subset of the whole natural numbers as the index set. In this case, and for other indexed sets with orders, we call the family an *ordered family*

2.1 Notation

Let A be a non-empty set. We denote the index set by I, a mnemonic for index. For $i \in I$, let we denote the result of applying the function to i by a_i; the notation evokes evokes function notation but avoids naming the function.

We denote the family of a_α indexed with I by $\{a_\alpha\}_{\alpha \in I}$, which is short-hand for set-builder notation. We read this notation "a sub-alpha, alpha in I."

3 Operations

The *pairwise extension* of a commutative operation is the function from finite families of the ground set to the ground set obtained by applying the operation pairwise to elements.

The *ordered pairwise extension* of an operation is the function from finite families ground set to the ground set obtained by applying the operation pairwise to elements in order.

3.1 Notation

Let $(A, +)$ be an algebra and $\{A_i\}_{i=1}^n$ a finite family of elements of A. We denote the pairwise extension by

$$\mathop{+}_{i=1}^{n} A_i$$

4 Family Set Algebra

We define the set whose elements are the objects which are contained in at least one family member the *family union*. We define the set whose elements are the objects which are contained in all of the family members the *family intersection*.

4.1 Notation

We denote the family union by $\cup_{\alpha \in I} A_\alpha$. We read this notation as "union over alpha in I of A sub-alpha." We denote family intersection by $\cap_{\alpha \in I} A_\alpha$. We read this notation as "intersection over alpha in I of A sub-alpha."

4.2 Results

Proposition 1. *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \{i, j\}$ then*

$$\cup_{\alpha \in I} A_\alpha = A_i \cup A_j$$

and

$$\cap_{\alpha \in I} A_\alpha = A_i \cap A_j.$$

Proposition 2. *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \varnothing$, then*

$$\cup_{\alpha \in I} A_\alpha = \varnothing$$

and

$$\cap_{\alpha \in I} A_\alpha = S.$$

Proposition 3. *For an indexed family* $\{A_\alpha\}_{\alpha\in I}$ *in* S.

$$C_S(\cup_{\alpha\in I} A_\alpha) = \cap_{\alpha\in I} C_S(A_\alpha)$$

and

$$C_S(\cap_{\alpha\in I} A_\alpha) = \cup_{\alpha\in I} C_S(A_\alpha).$$