Why

The (surprising fact) is that the operation of going from an index list to its induced sublist is *linear*, if the elements of the list are over a field, and thus may be viewed as a vector space.

Definition

The *index matrix* associated with the index sequence α of length r and order n (recall, $r \leq n$) is the $r \times n$ matrix whose i, jth entry is 1 if the sequence's ith coordinate is j, and 0 otherwise.

Examples

Here are some order-5 index lists: $(1, 2, 3)$, $(3, 2, 1)$, $(4, 5, 1)$, $(5, 4, 3, 2, 1)$, $(3,)$.

The matrices corresponding to these examples are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for the first two examples, and

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

for the last three.

In this case, we refer to the induced sublist as an induced *subvector*. The value of index matrices is that they give induced subvectors via the usual and familiar operation of matrix multiplication. The *subvector* of an n-vector associated with a length-r index sequence is the product of the sequence's $r \times n$ corresponding index matrix with the n-dimensional vector.

For example, define $x = \begin{bmatrix} 6 & 4 & 5 & 3 & 9 \end{bmatrix}^\top$. Then the subvector of x associated with the index sequence $(3, 2, 1)$ is the vector $\begin{bmatrix} 3 & 9 & 6 \end{bmatrix}^\top \in \mathbf{R}^3$, because

$$\begin{bmatrix} 3 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \\ 3 \\ 9 \end{bmatrix}$$

If $r = n$ then the index matrix is a permutation matrix.

Notation

Let $r \leq n$ be natural numbers. Let $\alpha : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$ be an index sequence. Denote the index matrix associated with α by P_α. This matrix P_α is an element of $\mathbf{N}^{r \times n}$ and is defined by

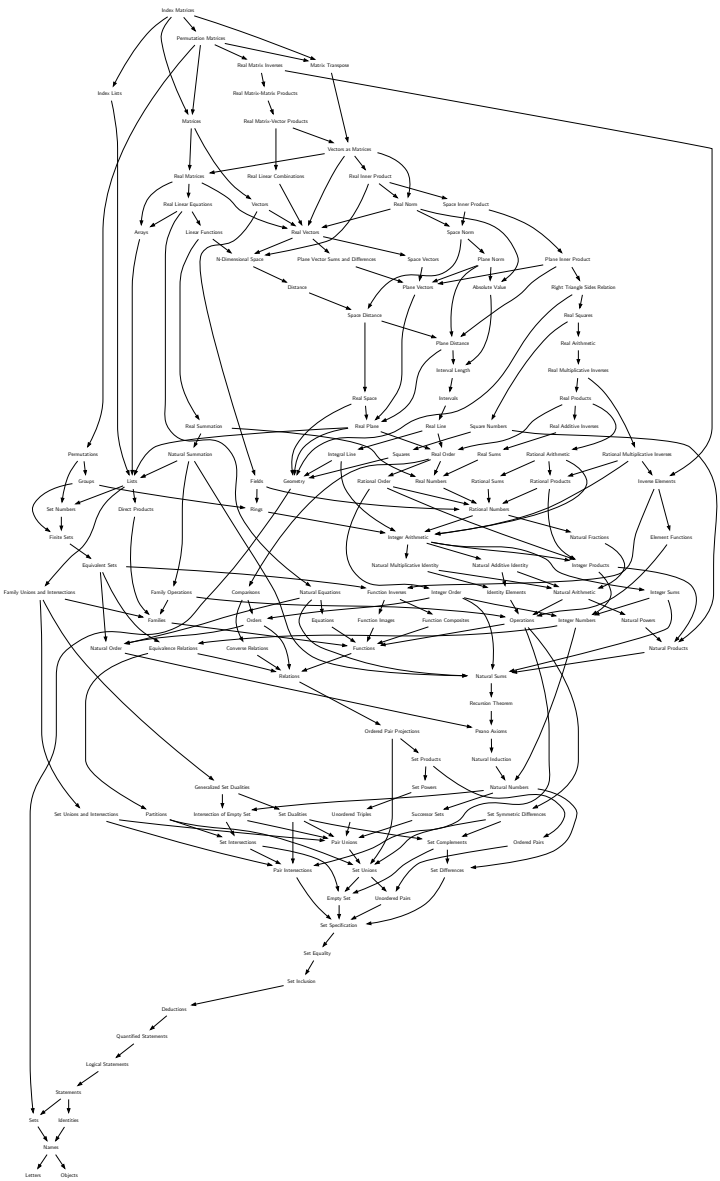
$$(P_a)_{ij} = \begin{cases} 1 & j = \alpha(i) \\ 0 & \text{otherwise} \end{cases}$$

Let A be a nonempty set and let $x \in A^n$. then the subvector of x associated with P_α (and so with α) is

$$P_\alpha x = \left(x_{\alpha(1)}, \ldots, x_{\alpha(r)} \right)$$

We denote the product $P_\alpha x$ by x_α.

We denote the product $P_\alpha X P_\alpha^\top$ by $X_{\alpha\alpha}$.


Index Matrices
Permutation Matrices
Real Matrix Inverses
Matrix Transpose
Index Lists
Real Matrix-Matrix Products
Matrices
Real Matrix-Vector Products
Vectors as Matrices
Real Matrices
Real Linear Combinations
Real Inner Product
Real Norm
Space Inner Product
Real Linear Equations
Vectors
Arrays
Linear Functions
Real Vectors
Space Norm
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Plane Norm
Plane Inner Product
Distance
Plane Vectors
Absolute Value
Right Triangle Sides Relation
Space Distance
Real Squares
Plane Distance
Real Arithmetic
Interval Length
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Summation
Real Plane
Real Line
Square Numbers
Real Additive Inverses
Permutations
Natural Summation
Integral Line
Squares
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Groups
Lists
Fields
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Inverse Elements
Set Numbers
Direct Products
Rings
Rational Numbers
Finite Sets
Integer Arithmetic
Natural Fractions
Element Functions
Equivalent Sets
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Family Unions and Intersections
Family Operations
Comparisons
Natural Equations
Function Inverses
Integer Order
Identity Elements
Natural Arithmetic
Integer Sums
Families
Order
Equations
Function Images
Function Composites
Operations
Integer Numbers
Natural Powers
Natural Order
Equivalence Relations
Converse Relations
Functions
Natural Products
Relations
Natural Sums
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Duality
Unordered Triples
Successor Sets
Set Symmetric Differences
Set Intersections
Pair Unions
Set Complements
Ordered Pairs
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Name
Letters
Objects