



Definition

A *subspace* (or *linear subspace*, *vector subspace*) of a vector space is a subset of vectors that is a vector space when restriction vector addition and scalar multiplication to the set. In other words, a subspace is a subset of a vector space which is closed under vector addition and scalar multiplication.

For example, the entire set of vectors is a subspace. As a second example, the set consisting only of the zero vector is a subspace; we call this the *zero subspace*. These two subspaces are the *trivial subspaces*. A *nontrivial subspace* is a subspace that is not trivial.

Notation

Let (V, \mathbf{F}) be a vector space. Let $U \subset V$ with

$$\alpha u + \beta v \in U$$

for all $\alpha, \beta \in \mathbf{F}$ and $u, v \in U$. Then U is a subspace of (V, \mathbf{F}) .

Characterization

Proposition 1. *Suppose V is a vector space over a field \mathbf{F} and $U \subset V$. U is a subspace if and only if U satisfies*

1. $0 \in U$ (contains additive identity)
2. $u + w \in U$ for all $u, w \in U$ (closed under addition)
3. $\alpha u \in U$ for all $\alpha \in \mathbf{F}$ and $u \in U$ (closed under scalar addition)

Properties

Proposition 2. *The intersection of a family of subspaces is a subspace.*

Proposition 3. *There exists a family of subspaces whose union is not a subspace;*

Remark 1. *In other words: the union of a family subspaces need not be a subspace.*

Proposition 4. *A subspace must contain the zero vector; in other words, every subspace is nonempty.*

