Linear Combinations

1 Why

We want to build vectors out of other vectors using scalar multiplication and vector addition.

2 Definition

A *linear combination from* a vector space is an ordered pair: the first coordinate is a sequence of vectors and the second is a sequence of scalars. The *result* of a linear combination is the sum of the results of scaling each vector by the corresponding scalar in the sequence; itself a vector in the space. A *linear combination of a set* of vectors a linear combination consisting of a numbering of them and a scalar sequence.

A *trivial linear combination* is one whose sequence of scalars is zero at each coordinate. The result of any trivial linear combination is the zero vector. A *nontrivial linear combination* is one which is not trivial. In other words, to be nontrivial, there must exist at least one index of the scalar sequence whose corresponding value is nonzero.

We say that a given vector *can be written as a linear combination of* a sequence of vectors if there exists a sequence of scalars such that the result of the linear combination of that sequence of vectors and scalars is the given vector. In other words, a vector can be written as a linear combination of some other vectors if there exists scalars for those other vectors such that scaling them and adding the results gives the vector.

2.1 Notation

Let $(V, \mathbf{F})$ be a vector space. Let $v = (v_1, \ldots, v_n)$ be a sequence of vectors in V and $a = (a_1, \ldots, a_n)$ be a sequence of scalars in $\mathbf{F}$. Then (v, a) is a linear combination and we can express its result by

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

If (v, a) is trivial, then $a_i = 0$ for $i \in \{1, 2, \ldots, n\}$ and the result of (v, a) is 0 (the zero vector). Otherwise, there exists $i \in \{1, 2, \ldots, n\}$ such that $a_i \neq 0$; of course, the result of (v, a) may still be 0.

We say that a vector u can be written as a linear combination of the vectors $v_1, v_2, \ldots, v_n$ if there exists scalars $a_1, a_2, \ldots, a_n$ such that the result of the linear combination (v, a) is u. Which we express

$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

If U is a finite set of vectors, then we often say "Let $\{u_1, \ldots, u_n\}$ be a (finite) set of vectors." In this notation, we imply a numbering, and the set notation is intended to indicate that $u_i \neq u_j$ if $i \neq j$ for $i, j \in \{1, 2, \ldots, n\}$. If $u = (u_1, \ldots, u_n)$ then a linear combination of $\{u_1, \ldots, u_n\}$ is any pair (u, a) where $a = (a_1, \ldots, a_n)$ is a scalar sequence.

3 Relationships