Functions

1 Why

We want a notion for a correspondence between two sets.

2 Definition

To each element of a first set we associate an element of a second set. We call this correspondence a **function**. We call the first set the **domain** and the second set the **codomain**.

A function is a relation on the domain and codomain. We call the union of the domain and codomain the **sumdomain**. The function is a relation on the sumdomain and so we refer to the set of ordered pairs whose first element is in the domain and whose second element is the corresponding result the graph of the function.

We say that the function **maps** elements from the domain into the codomain. We call the codomain element associated with the domain element the **result** of **applying** the function to the domain element.

2.1 Notation

We often denote functions by lower case latin letters, especially f, g, and h. Of course, f is a mnemonic for function; g and h follow f in the alphabet.

Let A and B be two non-empty sets. When we want to be explicit that the domain of a function f is A and its codomain is B we write $f : A \to B$, read aloud as "f from A to B." For each element a in the domain, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a." The set $\{(a, f(a)) \in A \times B \mid a \in A\}$ of ordered pairs is the graph of f. We often denote it by Γ_f; "gamma" is a mnemonic for graph.

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

3 Properties

Let $f : A \to B$. The **image** of a set $C \subset A$ is the set $\{f(c) \in B \mid c \in C\}$. The **range** of f is the image of the domain. The **inverse image** of a set $D \subset B$ is the set $\{a \in A \mid f(a) \in B\}$.

The range need not equal the codomain; though it, like every other image, is a subset of the codomain. If the range and codomain are equal, we call the function **onto**. We say that the function maps the domain onto the range. This language suggests that every element of the codomain is used by f. It means that for each element b of the codomain, we can find an element a of the domain so that $f(a) = b$.

An element of the codomain may be the result of several elements of the domain. This overlapping, using an element of the codomain more than once, is a regular occurrence. If a function is a unique correspondence in that every domain element has a different result, we call it **one-to-one**. This language is meant to suggest that each element of the domain corresponds to one and exactly one element of the codomain, and vice versa.

3.1 Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. The property that f is onto can be written succintly as $f(A) = B$. We denote the inverse image of $D \subset B$ by $f^{-1}(D)$, read aloud as "f inverse D."