Why

When does the technique of row reductions prevail?

Multivariable row reductions

Let $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$. The *kth row reduction* of S is the linear system (C, d) with $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$ if $i < s \leq m$ and $C_{st} = A_{st}$ otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable k from equations $k + 1, \ldots, m$. We are taking the kth column of A from

$$
\begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ A_{k+1,k} \\ \vdots \\ A_{mk} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

We interpret the ith row reduction as *subtracting equations* of the system or *reducing rows* of the array A. If $a^i, c^i \in \mathbf{R}^n$ denote the ith rows of A and C, $c^i = a^i - (A_{ik}/A_{kk})a^k$ for $k < i \leq m$, In other words, we obtain the ith row of matrix C by subtracting a multiple of the kth row of matrix A from the ith row of matrix A, for $k < i \leq m$. The following is an immediate consequence of real arithmetic.

Proposition 1. *Let* $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ *be a linear system*

which row reduces to (C, d). Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d).

Ordinary reductions

We call the system S *ordinarily reducible* if there exists a sequence of systems $S_1, \ldots, S_{m-1}$ so that S_1 is the 11-reduction of S and S_i is the ii-reduction of S_{i-1} for $i = 1, \ldots, n-1$. In this case, we call S_{n-1} the *final ordinary reduction* (or just *ordinary reduction*) of S. The following is an immediate consequence of Proposition 1.

Proposition 2. *Let S' be the (final) ordinary reduction of S. Then S and S' have equivalent solution sets.*

This process of constructing the ordinary reduction is called *Gauss elimination* or *Gaussian elimination*. We call the kkth entry of system S_{k-1} the *pivot*. In an ordinarily reducible system, the pivots are nonzero.

The idea is that a system is ordinarily reducible if we can take row reductions in sequence an end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system (A, b) in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.

Ordinary Row Reductions

Linear System Row Reductions

Linear Equation Solutions

Linear Equations

Linear Combinations

Vectors

Arrays

N-Dimensional Space

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Line — Real Plane — Real Space

Sequences

Integral Line — Real Summation — Real Order

Geometry

Real Numbers

Fields

Direct Products

Rational Numbers

Groups

Set Numbers

Integer Arithmetic

Integer Additive Inverses

Finite Sets

Natural Summation — Natural Additive Identity — Natural Multiplicative Identity — Integer Sums — Inverse Elements

Equivalent Sets

Arithmetic — Identity Elements — Family Operations — Element Functions

Natural Equations — Natural Powers — Integer Products — Algebras — Function Inverses — Family Unions and Intersections

Equations — Natural Products — Natural Order — Integer Numbers — Operations — Families — Function Images — Function Composites

Natural Sums — Functions — Equivalence Relations

Recursion Theorem — Relations

Peano Axioms

Natural Induction — Cartesian Products

Natural Numbers — Set Powers — Generalized Set Dualities

Successor Sets — Unordered Triples — Set Dualities — Intersection of Empty Set — Partitions — Set Unions and Intersections

Ordered Pairs — Set Complements — Pair Unions — Set Intersections

Set Differences — Set Unions — Pair Intersections

Unordered Pairs — Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects