Set Equality

1 Why

When are two sets the same?

2 Definition

Consider the sets A and B. If A is B, then every element of A is an element of B and every element of B is an element of A.

What of the converse? If every element of A is an element of B and vice versa is A the same as B? We declare the affirmative. Thus we can assert equality of sets.

Two sets are *equal* if and only if every element of one is an element of the other. In other words, two sets are the same if they have the same elements. This statement is sometimes called the *axiom of extension*.

The importance is that we have given ourselves a way to argue two sets are equivalent. Argue the consequence of the first paragraph, and the use the axiom of extension to conclude that the sets are the same.

2.1 Notation

Let A and B be two sets. As with any objects, we denote that A and B are equal by $A = B$. The axiom of extension is

$$A = B \Leftrightarrow (a \in A \implies a \in B) \wedge (b \in B \implies b \in A).$$

3 A Contrast

We can compare the axiom of extension for sets and their elements with an analogous statement for human beings and their ancestors.

If two human beings are equal, then they have the same ancestors. The ancestors being the person's parents, grandparents, greatgrandparents, and so on. This direction, same human implies same ancestors, is the analogue of the "only if" part of the axiom of extension. It is true.

In contrast, if two human beings have the same set of ancestors, they need not be be equal. This direction, same ancestors implies same human, is the analogue of the "if" part of the axiom of extension. It is false: siblings have the same ancestors, but are different people.

We conclude that the axiom of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.