Definition

An optimization problem $(\mathcal{X}, f)$ is called *linear* (a *linear optimization problem*) if $\mathcal{X} \subset \mathbf{R}^n$ is a polyhedron and $f : \mathbf{R}^n \to \bar{\mathbf{R}}$ is a linear function.

Problem data

Recall that f is linear means there exists $c \in \mathbf{R}^n$ such that

$$f(x) = c^\top x \quad \text{for all } x \in \mathbf{R}^n$$

Also, $\mathcal{X}$ polyhedral means there exists $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^d$ such that

$$\mathcal{X} = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

For this reason, the *problem data* (A, b, c) is sufficient to specify a linear optimization problem. Recall that $Ax \leq b$ means element-wise inequality (i.e., that the inequality holds in each component).

Task

Given data $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}^n$, we want to find $x \in \mathbf{R}^d$ to

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

We either want $x^\star \in \mathbf{R}^n$ so that $Ax^\star \leq b$ and $c^\top x^\star \leq c^\top x$ for all $x \in \mathbf{R}^n$, or we want to know that $\{x \mid Ax \leq b\} = \varnothing$, or we want to know that for all $\alpha \in \mathbf{R}$, there is an $x \in \mathbf{R}^n$ satisfying $Ax \leq b$ and $c^\top x \leq \alpha$. This problem is regularly called *linear programming* (a *linear program*). Many authors define this problem with the goal as maximization: of course, minimizing $c^\top x$ is equivalent to (has the same set of optimal solutions) maximizing $-c^\top x$.

Notation

In the context of linear optimization, $c^\top x$ is often abbreviated cx. As usual, $x \in \mathbf{R}^n$ is called *feasible* (a *feasible solution*) if $Ax \leq b$. $x^\star \in \mathbf{R}^n$

is called *optimal* (an *optimal solution*, *optimum solution*) if $c^\top x^\star \leq c^\top x$ for all $x \in \mathbf{R}^n$. We sometimes denote the rows of a by $\bar{a}_i^\top \in \mathbf{R}^n$ for $i = 1, \ldots, m$, i.e.,

$$A = \begin{bmatrix} - & \bar{a}_1^\top & - \\ & \vdots & \\ - & \bar{a}_m^\top & - \end{bmatrix} \in \mathbf{R}^{m \times n}$$

and refer to the inequality $\bar{a}_i^\top x \leq b_i$ as an *inequality constraint* for $i = 1, \ldots, m$. The set or the expression $Ax \leq b$ are both sometimes called the *inequality constraints* of the problem. For this reason, we can say in English that linear optimization deals with maximizing or minimizing a linear objective function of *finitely* many variables subject to finitely many linear inequalities.

The problem (A, b, c) is *infeasible* if the polyhedron

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

is empty. In symbols, if $P = \varnothing$. The problem is *unbounded* if for all $\alpha \in \mathbf{R}$, there exists $x \in P$ with $c^\top x < \alpha$.

Here's an infeasible instance. Define

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \end{bmatrix}$$

We want $x \in \mathbf{R}^1$ so that $x \leq -1$ and $x \geq 1$. There is no such x.

Here's an unbounded instance: drop the second inequality constraint. Then we are interested in finding x_1 to minimize x_1 subject to $x_1 \leq -1$. Given $\alpha \in \mathbf{R}$, if $\alpha > 0$ pick $x_1 = -1$, else pick 2α. This problem is unbounded.

Here's a simple example with an optimal solution. Modify $b = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$. Now we want to find x_1 so that

$$x_1 \leq 1 \quad \text{and } x_1 \geq 0$$

and we minimize x_1 Clearly $x_1 = 0$ is an optimal solution. Indeed, it is the unique optimal solution in this case.

Linear Optimization Problem
Real Polytopes
Topological Interior
Topological Neighborhoods
Real Balls
Topologies
Real Halfspaces
Real Symplexes
Subspace Orthogonal Complements
Subspace Dimensions
Vector Space Dimensions
Linear Inner Product Spaces Representation
Real Vectors
Continuous Linear Functions
Real Norm Spaces
Complex Real Vector Spaces
Complex Normed Spaces
Complex Inner Product Spaces
Vector Space Basis
Affine Set Dimensions
Linearly Dependent Vectors
Affinely Independent Vectors
Orthogonal Complements
Separation
Real Norms
Real Inner Products
Complex Inner Products
Real Vector Spaces
Real Subspaces Dimensions
Real Affine Sets
Real Affine Sets and Subspaces
Real Affine Sets
Real Affine Spaces
Real Affine Combination
Real Affine Combinations
Real Convex Sets
Real Convex Functions
Real Differentiable Functions
Real Linear Sets
Real Linear Functions
Real Sequence Spaces
Real Sequence Limits
Real Sequences
Real Series
Real Series Sum
Real Series Convergence
Real Series Convergence Tests
Real Series Convergence Tests
Real Series Convergence Tests