Why

That some sequences grow without bound leads us to add two elements to the set of real numbers.

Definition

The set of *extended real numbers* is the union of the set of real numbers with a set containing two elements: one which we call *positive infinity* and the other we call *negative infinity*. We call an element of the extended real numbers *finite* if it is not positive or negative infinity. Otherwise, we call it *infinite*.

The only two infinite extended real numbers are exactly the two elements positive and negative infinity.

Notation

We denote positive infinity by $+\infty$ and the negative infinity by $-\infty$. We denote the set of extended real numbers by $\bar{\mathbf{R}}$ so

$$\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$$

Intervals

We extend intervals in the obvious way. We define

$$[-\infty, \alpha) := \{x \in \mathbf{R} \mid x < \alpha\} \cup \{-\infty\}$$

$$[-\infty, \alpha] := \{x \in \mathbf{R} \mid x \leq \alpha\} \cup \{-\infty\}$$

Similarly,

$$(\alpha, +\infty] := \{x \in \mathbf{R} \mid x > \alpha\} \cup \{+\infty\}$$

$$[\alpha, +\infty] := \{x \in \mathbf{R} \mid x \geq \alpha\} \cup \{+\infty\}$$

We to obtain useful notation if we make these open

$$(-\infty, \alpha) := \{x \in \mathbf{R} \mid x < \alpha\}$$

$$(-\infty, \alpha] := \{x \in \mathbf{R} \mid x \leq \alpha\}$$
$$(\alpha, +\infty) := \{x \in \mathbf{R} \mid x > \alpha\}$$
$$(\alpha, +\infty) := \{x \in \mathbf{R} \mid x \geq \alpha\}$$

Finally, we sometimes denote $\mathbf{R}$ by $(-\infty, \infty)$ and $\bar{\mathbf{R}}$ by $[-\infty, +\infty]$, referring to the *real line* and *extended real line* respectively. We refer to any of the aforementioned sets as *extended real intervals*. The following notations are sometime useful

$$\bar{\mathbf{R}}_+ := [0, \infty] \quad \text{and} \quad \bar{\mathbf{R}}_{++} := (0, \infty].$$

Extended order

Naturally, we define $-\infty < a < +\infty$ for every $\alpha \in \mathbf{R}$. This new extended order can be put to good use, mostly notationally. For example, the intervals above can be rewritten by selecting x from $\bar{\mathbf{R}}$. As a particular example, we can write

$$[-\infty, \alpha] = \{x \in \bar{\mathbf{R}} \mid -\infty \leq x \leq \alpha\}$$

and likewise for the other sets.

It is logically consistent to define the greatest lower bound of the empty set to be $+\infty$ and likwise the least upper bound of the empty set to be $-\infty$. In symbols,

$$\inf \varnothing := +\infty \quad \text{and} \quad \sup \varnothing = -\infty$$

Extended arithmetic

We extend addition to all but one ordered pair of elements of the new set.

1. The sum of any real number with a real number is defined as before.

2. The sum of positive infinity with positive infinity is positive infinity. The sum any real number with positive infinity is positive infinity. In symbols,

$$\alpha + \infty = \infty + \alpha = \infty \quad \text{for } -\infty < \alpha \leq \infty$$

3. The sum any real number with negative infinity is negative infinity. The sum of negative infinity with negative infinity is negative infinity. In symbols,

$$\alpha - \infty = -\infty + \alpha = -\infty \quad \text{for} \ -\infty \leq x < \infty$$

We do not define the sum of positive infinity and negative infinity. In other words, the combinations $\infty - \infty$ and $-\infty + \infty$ are undefined and are avoided.

Similarly, we extend multiplication

$$\alpha\infty = \infty\alpha = \infty, \quad \alpha(-\infty) = (-\infty)\alpha = -\infty \quad \text{for} \ 0 < \alpha \leq \infty,$$

$$\alpha\infty = \infty\alpha = -\infty, \quad \alpha(-\infty) = (-\infty)\alpha = \infty \quad \text{for} \ -\infty \leq< \alpha < 0,$$

$$0\infty = \infty 0 = 0 = 0(-\infty) = (-\infty)0, \quad -(-\infty) = \infty$$

Under these rules, the familiar laws remain commutative, associative, and multiplication distributes over addition. This is provided, of course, that we never form the sum $\infty - \infty$ or $-\infty + \infty$. These can be verified by testing all possible finte combinations for the values in the laws. Avoiding a forbidden sum requires some cautious attention, like dividing by zero, and in practice one or the other of the infinities tends to be excluded by hypothesis.

Extended Real Numbers

Real Limits

Absolute Value

Interval Length

Intervals

Real Line

Real Order

Real Sequences

Rational Order

Real Numbers

Sequences

Integral Line

Rational Numbers

Lists

Set Numbers

Direct Products

Integer Arithmetic

Natural Fractions

Integer Products

Natural Multiplicative Identity

Natural Additive Identity

Finite Sets

Identity Elements

Natural Arithmetic

Equivalent Sets

Family Unions and Intersections

Integer Sums

Integer Order

Natural Powers

Comparisons

Operations

Function Inverses

Integer Numbers

Natural Products

Orders

Function Composites

Function Images

Families

Natural Sums

Natural Order

Equivalence Relations

Converse Relations

Functions

Recursion Theorem

Relations

Peano Axioms

Ordered Pair Projections

Natural Induction

Set Products

Natural Numbers

Set Powers

Generalized Set Dualities

Set Symmetric Differences

Partitions

Successor Sets

Intersection of Empty Set

Unordered Triples

Set Dualities

Set Unions and Intersections

Set Complements

Set Intersections

Pair Unions

Ordered Pairs

Set Differences

Set Unions

Pair Intersections

Empty Set

Unordered Pairs

Geometry

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects