Why

Definition

The *event sigma algebra* of $A \in \mathcal{F}$ where $(\Omega, \mathcal{F})$ is a measurable space is the set sub-σ-algebra $\{\varnothing, A, A^c, \Omega\}$.

A family of events events are *independent* if the event sigma algebras are independent.

Notation

Let $(X, \mathcal{A}, \mu)$ be a probability space. Let $A \in \mathcal{A}$ be an event. The sigma algebra generated by A is $\{\varnothing, A, X - A, X\}$. We denote it by $\sigma(A)$.

Let $B \in \mathcal{A}$. If A is independent of B we write $A \perp B$.

Equivalent Condition

Proposition 1. *Two events are independent if and only if the measure of their intersection is the product of their measures.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a probability space. Let $A, B \in \mathcal{A}$.

($\Rightarrow$) If $A \perp B$, then by definition $A \in \sigma(A)$ and $B \in \sigma(B)$ and so:

$$\mu(A \cap B) = \mu(A)\mu(B).$$

($\Leftarrow$) Conversely, let $a \in \sigma(A)$ and $b \in \sigma(B)$. If $a = \varnothing$ or $b = \varnothing$

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[1]Future editions will include

then $a \cap b = \emptyset$. So

$$\mu(a \cap b) = \mu(\emptyset) = \mu(a)\mu(b),$$

since one of the two measures on the right hand side is zero. On the other hand, if $a = X$, then $a \cap b = b$ and so

$$\mu(a \cap b) = \mu(b) = \mu(a)\mu(b),$$

since $\mu(a) = \mu(X) = 1$. Likewise if $b = X$.

So it remains to verify $\mu(a \cap b) = \mu(a)\mu(b)$ for the cases $a \in \{A, X - A\}$ and $b \in \{B, X - B\}$. If $a = A$, and $b = B$, then the identity follows by hypothesis. Next, observe that $A \cap (X - B) = A - (A \cap B)$ and $(A \cap B) \subset A$ so $\mu(X) < \infty$ allows us to deduce:

$$\begin{aligned}
\mu(A \cap (X - B)) &= \mu(A - (A \cap B)) \\
&= \mu(A) - \mu(A \cap B) \\
&= \mu(A)(1 - \mu(B)) \\
&= \mu(A)\mu(X - A).
\end{aligned}$$

Similar for $X - A$ and B. Finally, recall that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$. So then,

$$\begin{aligned}
\mu((X - A) \cap (X - B)) &= 1 - \mu(A \cup B) \\
&= 1 - \mu(A) - \mu(B) + \mu(A \cap B) \\
&= 1 - \mu(A) - \mu(B) + \mu(A)\mu(B) \\
&= (1 - \mu(A))(1 - \mu(B)) \\
&= \mu(X - A)\mu(X - B).
\end{aligned}$$

$\square$

Independent Event Sigma Algebras

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Probability Measures

Dependent Events

Measures

Conditional Event Probabilities

Sigma Algebra

Event Probabilities

Real Series

Extended Real Numbers

Real Limits

Absolute Value

Interval Length

Probability Distributions

Intervals

Real Line

Real Sequences

Real Order

Real Summation

Sequences

Real Numbers

Rational Order

Integral Line

Rational Numbers

Subset Algebra

Lists

Integer Arithmetic

Cardinality

Set Numbers

Natural Multiplicative Identity

Natural Additive Identity

Natural Summation

Direct Products

Finite Sets

Identity Elements

Arithmetic

Family Operations

Equivalent Sets

Operations

Natural Powers

Integer Products

Function Inverse

Family Unions and Intersections

Comparisons

Integer Order

Integer Sums

Natural Products

Function Images

Function Composites

Families

Orders

Integer Numbers

Natural Sum

Functions

Converse Relations

Equivalence Relations

Natural Order

Recursion Theorem

Relations

Peano Axioms

Subset Systems

Set Products

Generalized Set Dualities

Natural Induction

Uncertain Outcomes

Ordered Pairs

Set Powers

Unordered Triple

Intersection of Empty Set

Natural Numbers

Successor Sets

Set Dualities

Set Symmetric Differences

Set Unions and Intersections

Partitions

Pair Unions

Set Complements

Set Intersections

Set Differences

Pair Intersections

Set Unions

Unordered Pairs

Empty Set

Geometry

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects