Why

How big can a quadratic form be? How small?

Result

Proposition 1. *Suppose $A \in \mathbf{S}^n$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_n x^\top x \leq x^\top A x \leq \lambda_1 x^\top x,$$

for all $x \in \mathbf{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ with $A = Q \Lambda Q^\top$. Express

$$\begin{aligned} x^\top A x = x^\top Q \Lambda Q^\top x &= (Q^\top x)^\top \Lambda (Q^\top x) \\ &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &= \lambda_1 \sum_{i=1}^n (q_i^\top x) = \lambda_1 \|Q^\top x\|^2 = \lambda_1 \|x\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} x^\top A x &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &\geq \lambda_n \sum_{i=1}^n (q_i^\top x) = \lambda_n \|Q^\top x\|^2 = \lambda_n \|x\|^2. \end{aligned}$$

□

Notation

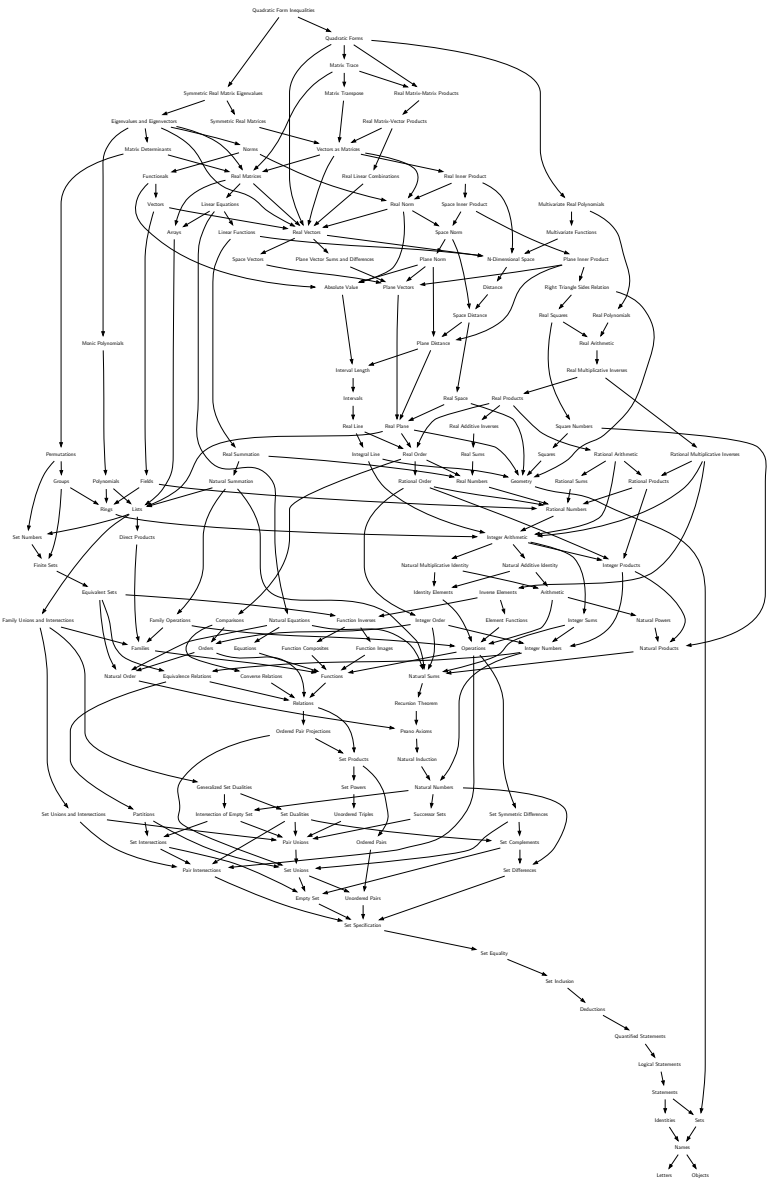
For this reason, it is common to order the eigenvalues of $A \in \mathbf{S}^n$ by magnitude with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. λ_1 is sometimes denoted $\lambda_{\max}$ and λ_n is sometimes denoted $\lambda_{\min}$.

Optimization implication

If $z = \alpha x$, then $z^\top A z = \alpha^2 x^\top A x$. Consider finding $x \in \mathbf{R}^n$ to maximize

$$\begin{array}{ll} \text{maximize} & x^\top A x \\ \text{subject to} & \|x\| = 1. \end{array}$$

Since the objective is $x^\top A x \leq \lambda_1$ for all $x \in \mathbf{R}^n$ with $\|x\| = 1$, a solution of this problem is the eigenvector $q_1 \in \mathbf{R}^n$ corresponding to λ_1. In other words, these inequalities are tight.


Quadratic Form Inequalities
Quadratic Forms
Matrix Trace
Symmetric Real Matrix Eigenvalues
Matrix Transpose
Real Matrix-Matrix Products
Eigenvalues and Eigenvectors
Symmetric Real Matrices
Real Matrix-Vector Products
Matrix Determinants
Norms
Vectors as Matrices
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Multivariate Real Polynomials
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Plane Vector Sums and Differences
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Absolute Value
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Distance
Right Triangle Sides Relation
Space Distance
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Real Arithmetic
Interval Length
Real Multiplicative Inverses
Intervals
Real Space
Real Products
Real Line
Real Plane
Real Additive Inverses
Square Numbers
Permutations
Real Summation
Integral Line
Real Order
Real Sums
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Rational Arithmetic
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Rational Numbers
Set Numbers
Direct Products
Integer Arithmetic
Finite Sets
Natural Multiplication Identity
Natural Addition Identity
Integer Products
Equivalent Sets
Identity Elements
Inverse Elements
Arithmetic
Family Unions and Intersections
Family Operations
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Function Inverses
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Integer Sums
Natural Powers
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Equivalence Relations
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Recursion Theorem
Ordered Pair Projections
Peano Axioms
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Natural Induction
Generalized Set Dualities
Set Powers
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Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Dualities
Unordered Triples
Successor Sets
Set Symmetric Differences
Set Intersections
Pair Unions
Ordered Pairs
Set Complements
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Types
Lemas
Objects