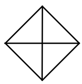


Why

We can characterize the dependence of two events in terms of the rank of a particular matrix.

Definition

Given a probability measure $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ on the finite set Ω and two events $A, B \subset \Omega$, the *joint probability matrix* of A and B is the matrix

$$M = \begin{bmatrix} \mathbf{P}(A \cap B) & \mathbf{P}(A \cap C_\Omega(B)) \\ \mathbf{P}(C_\Omega(A) \cap B) & \mathbf{P}(C_\Omega(A) \cap C_\Omega(B)) \end{bmatrix}.$$

Characterization of independence

If A and B are independent, then so are A and $C_\Omega(B)$, B and $C_\Omega(A)$, and $C_\Omega(A)$ and$C_\Omega(B)$. In other words,

$$M = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_\Omega(A)) \end{bmatrix} \begin{bmatrix} \mathbf{P}(B) & \mathbf{P}(C_\Omega(B) \end{bmatrix}.$$

In this case, we see that $\mathsf{rank}(M) = 1$.

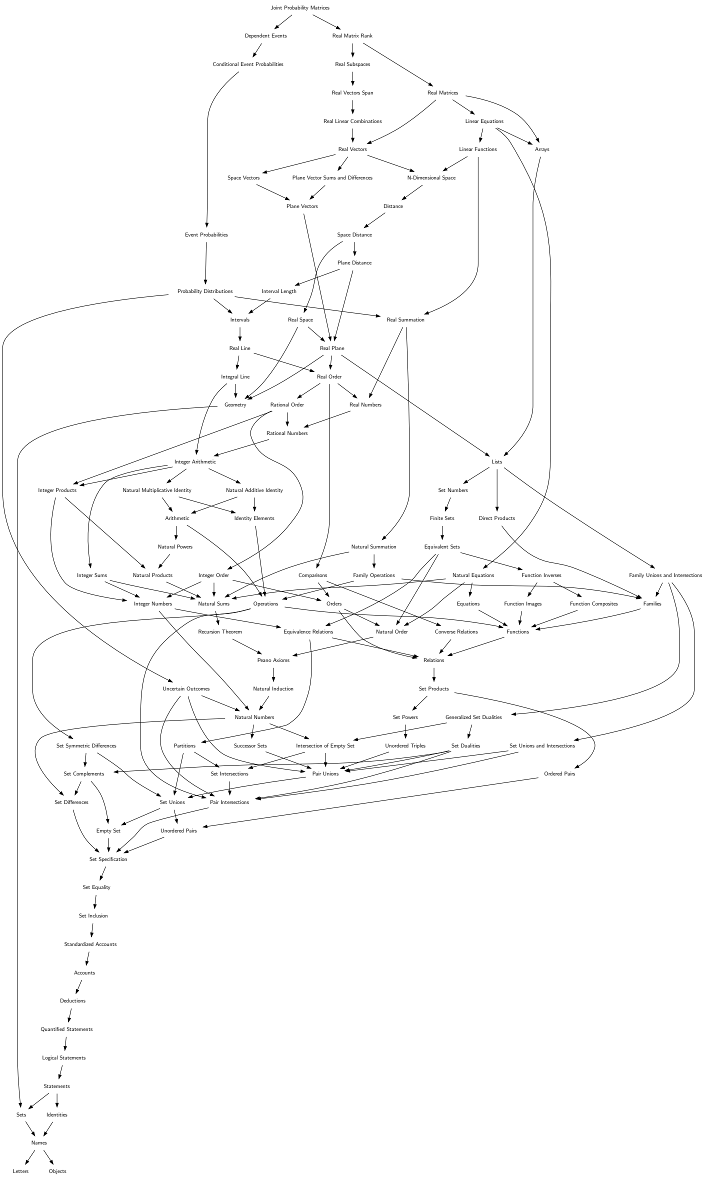
Conversely, suppose $\mathsf{rank}(M) = 1$. Then, using the law of total probabiliy, each row is a multiple of

$$M1 = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_\Omega(A)) \end{bmatrix}.$$

In particular, we have $\mathbf{P}(A \cap B) = \alpha \mathbf{P}(A)$ and $\mathbf{P}(C_\Omega(A) \cap B) = \alpha P(C_\Omega(A))$. So

$$\mathbf{P}(A \cap B) + \mathbf{P}(C_\Omega(A) \cap B) = \alpha(\mathbf{P}(B) + \mathbf{P}(C_\Omega(A))),$$

from which we deduce $\alpha = \mathbf{P}(B)$ Likewise, the multiplier for the second column of M is $\mathbf{P}(C_\Omega(B))$. In other words, A and B are independent. We conclude that A and B are independent if and only if $\mathsf{rank}(M) = 1$.


Joint Probability Matrices
Dependent Events
Real Matrix Rank
Conditional Event Probabilities
Real Subspaces
Real Vectors Span
Real Matrices
Real Linear Combinations
Linear Equations
Real Vectors
Linear Functions
Arrays
Space Vectors
Plane Vector Sums and Differences
N-Dimensional Space
Plane Vectors
Distance
Event Probabilities
Space Distance
Plane Distance
Probability Distributions
Interval Length
Intervals
Real Space
Real Summation
Real Line
Real Plane
Integral Line
Real Order
Geometry
Rational Order
Real Numbers
Rational Numbers
Integer Arithmetic
Lists
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Set Numbers
Arithmetic
Identity Elements
Finite Sets
Direct Products
Natural Powers
Natural Summation
Equivalent Sets
Integer Sums
Natural Products
Integer Order
Comparisons
Family Operations
Natural Equations
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Sums
Operations
Orders
Equations
Function Images
Function Composites
Families
Recursion Theorem
Equivalence Relations
Natural Order
Converse Relations
Functions
Peano Axioms
Relations
Uncertain Outcomes
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects