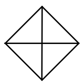


Why

We have precepts in R^d and want to predict postcepts in R. We put a probability measure on a set of linear statistical models.[1]

Setup

We work over a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We have n precepts in R^d. So let $x^1, \dots, x^n \in \mathsf{R}^d$ with data matrix $X \in \mathsf{R}^{n \times d}$.

Let $\theta : \Omega \to \mathsf{R}^d$ and $e : \Omega \to \mathsf{R}^n$ be random vectors with normal density (mean zero and covariances Σ_θ and Σ_e respectively). For each $\omega \in \Omega$, define the map $f : \Omega \to (\mathsf{R}^d \to \mathsf{R})$ by $f(\omega)(x) = \theta(\omega)^\top x^i + e_i(\omega)$.

Define $y : \Omega \to \mathsf{R}^n$ by $y(\omega) = X\theta(\omega) + e$. Let $g : \mathsf{R}^d \times \mathsf{R}^n \to \mathsf{R}$ be the density of (θ, y). Let $g_{\theta|y} : \mathsf{R}^d \times \mathsf{R}^n \to \mathsf{R}$ be the conditional density of θ given y.

Proposition 1. *(θ, y) has covariance* $\begin{pmatrix} \Sigma_\theta & \Sigma_\theta X^\top \\ X\Sigma_\theta & X\Sigma_\theta X^\top + \Sigma_e \end{pmatrix}$

Proposition 2. *There exists $c \in \mathsf{R}$, so that for all $\alpha \in \mathsf{R}^d$ and $\gamma \in \mathsf{R}^n$, $\log g(\alpha, \gamma)$ is*

$$-\frac{1}{2}(\alpha^\top \Sigma \alpha + \alpha^\top \Sigma X^\top \gamma + \gamma^\top X \Sigma \alpha + \gamma^\top X \Sigma X^\top \gamma) + c.$$

Proposition 3. *A solution to maximize $g(\alpha, \gamma)$ with respect to α is $\alpha = -\Sigma^{-1}\Sigma X^\top \gamma$.*

[1] The name of this sheet will change in future editions. And future editions will include accounts.

Proposition 4. *$g_{\theta|y}(\alpha, \gamma)$ is normal with mean*

$$\tilde{\mu}(\gamma) = \Sigma X^\top \left(X\Sigma X^\top\right)^{-1} \gamma$$

and covariance

$$\tilde{\Sigma} = \Sigma - \Sigma X^\top \left(X\Sigma X^\top\right)^{-1} X\Sigma.$$

Proposition 5. *A solution to maximize $g_{\theta|y}(\alpha, \gamma)$ w.r.t. α is*

$$\tilde{\Sigma}\tilde{\Sigma}^{-1}\tilde{\mu}(\gamma).$$

But, of course, y also has a density. Denote the density of y by $g : \mathsf{R}^n \to \mathsf{R}$. In other words, $g \geq 0$ and $\int g = 1$.

Proposition 6.

$$\log g(\gamma) = -{}^1\!/\!{}_2(\gamma^\top \left(X\Sigma X^\top\right)^{-1} \gamma) - \frac{d}{2}\log 2\pi - \frac{1}{2}\log \mathbf{det}\left(X\Sigma X^\top\right)$$

Test

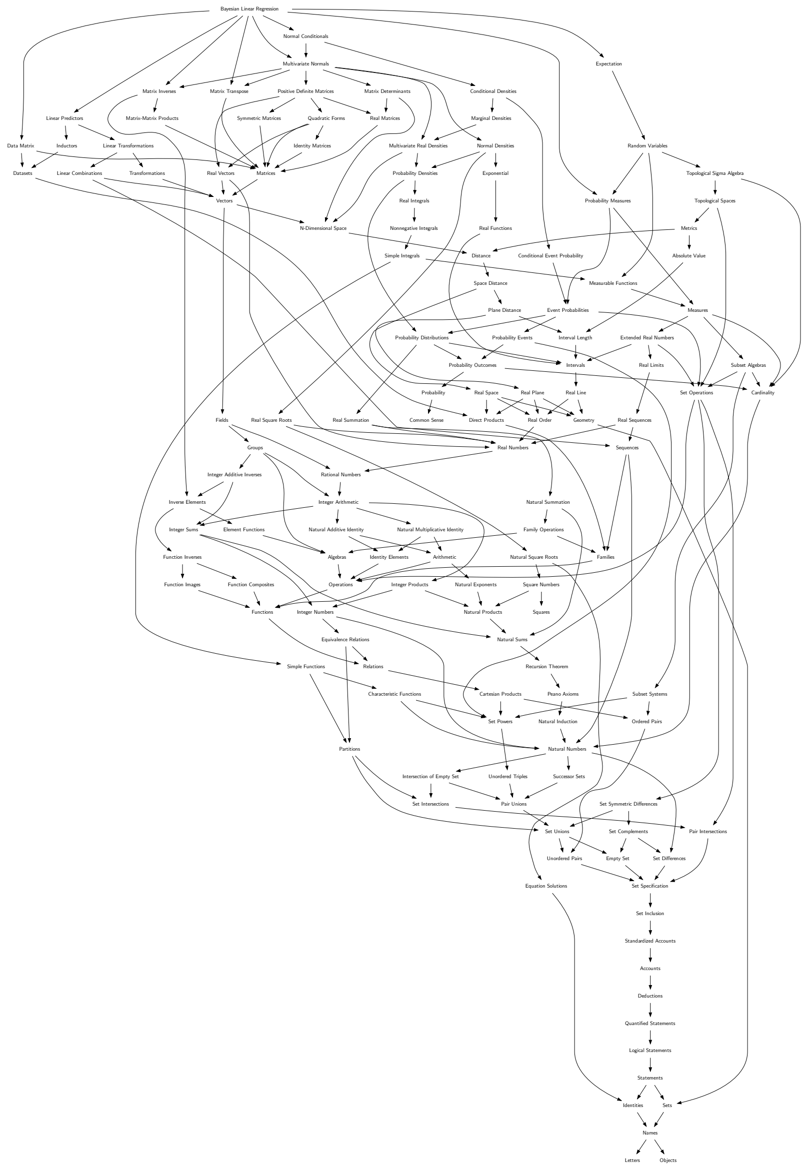
This expression makes clear that y is has a normal density with mean $X\,\mathbf{E}(x)$ and covariance $X\,\mathbf{E}(x)X^\top$.

Let $w : \Omega \to \mathsf{R}^d$ be a random vector with mean 0 and covariance ηI. Let $x^1, \dots, x^n \in \mathsf{R}^d$ Define $y^i : \Omega \to \mathsf{R}$ by $y_i(\omega) = w(\omega)^\top x^i$ for $i = 1, \dots, d$.

Noise setup

Let $e : \Omega \to \mathsf{R}^n$ be a normal random vector with mean 0 and covariance σI. Define $\tilde{y} : \Omega \to \mathsf{R}^n$ by $\tilde{y} = y(\omega) + e(\omega)$.

Proposition 7. *$\tilde{y}$ is a normal random vector with mean zero and covariance $X\Sigma X^\top + \sigma I$.*


Bayesian Linear Regression
Normal Conditionals
Multivariate Normals
Expectation
Matrix Inverses
Matrix Transpose
Positive Definite Matrices
Matrix Determinants
Conditional Densities
Linear Predictors
Matrix-Matrix Products
Symmetric Matrices
Quadratic Forms
Real Matrices
Marginal Densities
Data Matrix
Inductors
Linear Transformations
Identity Matrices
Multivariate Real Densities
Normal Densities
Random Variables
Datasets
Linear Combinations
Transformations
Real Vectors
Matrices
Probability Densities
Exponential
Topological Sigma Algebra
Vectors
Real Integrals
Probability Measures
Topological Spaces
N-Dimensional Space
Nonnegative Integrals
Real Functions
Metrics
Simple Integrals
Distance
Conditional Event Probability
Absolute Value
Space Distance
Measurable Functions
Plane Distance
Event Probabilities
Measures
Probability Distributions
Probability Events
Interval Length
Extended Real Numbers
Probability Outcomes
Intervals
Real Limits
Subset Algebras
Probability
Real Space
Real Plane
Real Line
Set Operations
Cardinality
Fields
Real Square Roots
Real Summation
Common Sense
Direct Products
Real Order
Geometry
Real Sequences
Groups
Real Numbers
Sequences
Integer Additive Inverses
Rational Numbers
Inverse Elements
Integer Arithmetic
Natural Summation
Integer Sums
Element Functions
Natural Additive Identity
Natural Multiplicative Identity
Family Operations
Function Inverses
Algebras
Identity Elements
Arithmetic
Natural Square Roots
Families
Function Images
Function Composites
Operations
Integer Products
Natural Exponents
Square Numbers
Functions
Integer Numbers
Natural Products
Squares
Equivalence Relations
Natural Sums
Simple Functions
Relations
Recursion Theorem
Characteristic Functions
Cartesian Products
Peano Axioms
Subset Systems
Set Powers
Natural Induction
Ordered Pairs
Partitions
Natural Numbers
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Equation Solutions
Set Specification
Set Inclusion
Standardized Accounts
Accounts
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Logical Statements
Statements
Identities
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Names
Letters
Objects