Why

1

1   Definition

Define $S \in \mathbf{R}^{d \times d}$ by

$$
S = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}
$$

For a vector $x \in \mathbf{R}^d$ the *down shift* of x is Sx.

Let $A \in \mathbf{R}^{d \times d}$ be a matrix with columsn $a_1, \ldots, a_d$. A is a *circulant matrix* if $a_1 = Sa_d$, $a2 = Sa_2$, and $a_i = A_{i-1}$ for $i = 2, \ldots, d$.

Example

For example, the matrix

$$
\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}
$$

is a circulant matrix.

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[1]Future sheets will include. These matrices arise in practice and each has the same eigenvectors.

Characterization

A matrix $C \in \mathbf{R}^{d \times d}$ is circulant if and only if there exists $c_0, \ldots, c_{d-1}$ so that

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{n-1} S^{n-1}.$$

Properties

The sum and product of any two circulant matrix is circulant. In other words, the circulant matrices with the usual matrix addition and multiplication form a commutative ring.

Circulant Matrices → Real Matrices → Matrices → Vectors → N-Dimensional Space → Distance → Space Distance → Plane Distance → Interval Length → Intervals

Real Plane — Real Line — Real Space

Integral Line — Real Order

Geometry — Real Numbers — Fields

Rational Numbers — Groups

Sequences — Integer Arithmetic — Integer Additive Inverses

Set Numbers — Natural Additive Identity — Natural Multiplicative Identity — Inverse Elements — Integer Sums

Direct Products — Finite Sets — Identity Elements — Arithmetic — Element Functions — Natural Powers — Integer Products

Equivalent Sets — Algebras — Operations — Function Inverses — Natural Products — Integer Numbers

Family Unions and Intersections — Families — Equivalence Relations — Function Composites — Function Images — Natural Sums

Natural Order — Functions — Recursion Theorem

Relations — Peano Axioms

Cartesian Products — Natural Induction

Generalized Set Dualities — Set Powers — Natural Numbers

Set Unions and Intersections — Set Dualities — Unordered Triples — Intersection of Empty Set — Successor Sets — Partitions

Set Complements — Pair Unions — Set Intersections — Ordered Pairs

Pair Intersections — Set Unions — Set Differences

Empty Set — Unordered Pairs

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects