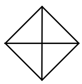


Why

We want to speak of an infinite process which, although never arrives, does terminate.

Definition

A *limit* of a sequence of real numbers is a real number for which we can always find a final part of the sequence wholly contained in an interval around the limit, no matter how small the interval.

You propose a limit for a sequence. To test this proposal, I specify some small positive real number. Then we look for a final part wholly contained in the interval of that width. If we can always find the final part, no matter how small the positive number I specified, then the proposed limit is true.

Existence

Some sequences have no limits. Consider the sequence which alternates between the $+1$ and -1. To show that the limit does not exist, we argue indirectly. We take any real number and test it with the interval length one. No matter which real number we have selected, $+1$ and -1 are a distance two apart, and so can not both be contained in an interval of width one.

Uniqueness

If a sequence has a limit, it has only one limit. So, from here on, we will speak of *the limit* of the sequence.

To see this uniqueness, suppose that two real numbers satisfy the limiting property. We now argue indirectly: suppose also that they are not equal. Denote the distance between them by x. Then ask for final parts in intervals of width $x/2$ for both limits.

Approximation

We use limits to speak about the terminating behavior of infinite processes. We think about the sequence as approximating the limit. The sequence may never actually take the value of its limit, so the limit need not be in the set of terms of the sequence. But the idea, of course, is that the set of terms, especially those "far out" in the sequence, are close to the limit.

The definition, moreover, ensures that the sequence will get arbitrarily close. We can operationalize this property, by taking the first element of that final part after which all elements are close to the limit. This element of the sequence approximates the limit value well.

Notation

Let $(a_n)_n$ be a sequence of real numbers. Let a be a real number. We denote that a is the limit of $(a_n)_n$ by

$$a = \lim_{n \to \infty} a_n.$$

We read this statement aloud as "a is the limit of a sub n." The above statement asserts two facts: (1) the sequence $(a_n)_n$ has a limit and (2) the limit is the real number a. We sometimes abbreviate the by writing $a = \lim_n a_n$.

Why

We want to talk about sequences of real numbers which, as we go further and further in the sequence, get closer and closer to some fixed real number. Not all sequences have this property, but many do.

Definition

Consider a sequence of real numbers. If the sequence is going to end up somewhere in the real numbers, we need to know where. Consider a real number. Suppose that for any positive real number, no matter how small, there exists a natural number so that the any term corresponding to a natural number larger than the first natural number is no more different than that positive number from the real number considered.

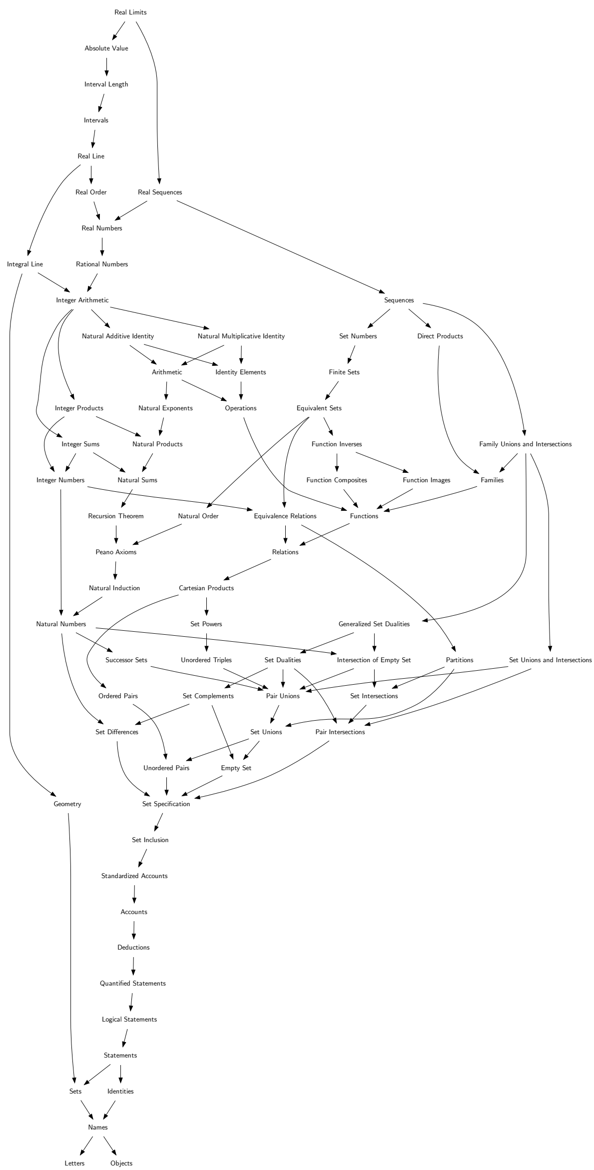
In this case, we call the real number a *limit* of the sequence. We say that the sequence *converges* to that real number. Not all sequences *converge*.

Notation

Let $x : \mathsf{N} \to \mathsf{R}$ be a sequence of real numbers. Let $y \in \mathsf{R}$.

First, suppose that for each $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathsf{N}$ so that $|x_n - y| < \varepsilon$ for each $n \geq n_\varepsilon$. In this case, x converges to y. The real number y is the limit of the sequence x.

Second, suppose that there exists an $\varepsilon > 0$ so that for each $n \in N$, there exists $m_n \geq n$ so that $|x_{m_n} - y| > \varepsilon$ for each $y \in \mathsf{R}$. In this case x does not converge.


Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Real Numbers
Integral Line
Rational Numbers
Integer Arithmetic
Sequences
Natural Additive Identity
Natural Multiplicative Identity
Set Numbers
Direct Products
Arithmetic
Identity Elements
Finite Sets
Integer Products
Natural Exponents
Operations
Equivalent Sets
Integer Sums
Natural Products
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Sums
Function Composites
Function Images
Families
Recursion Theorem
Natural Order
Equivalence Relations
Functions
Peano Axioms
Relations
Natural Induction
Cartesian Products
Natural Numbers
Set Powers
Generalized Set Dualities
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects