



# Iterated Integrals

## 1 Why

We integrate over a product space by integrating one coordinate at a time. TODO: WHY?

## 2 Result

**Proposition 1.** *Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measurable spaces. Let  $f : X \times Y \rightarrow [-\infty, \infty]$  be  $\mathcal{A} \times \mathcal{B}$ -measurable and  $\mu \times \nu$ -integrable. Then*

1. *For  $\mu$ -almost every  $x$  in  $X$  the section  $f_x$  is  $\nu$ -integrable and for  $\nu$ -almost every  $y$  in  $Y$  the section  $f^y$  is  $\mu$ -integrable,*
2. *the functions  $I_f$  and  $J_f$  defined by*

$$I_f(x) = \begin{cases} \int_Y f_x d\nu & \text{if } f_x \text{ is } \nu\text{-integrable,} \\ 0 & \text{otherwise} \end{cases}$$

*and*

$$J_f(y) = \begin{cases} \int_X f^y d\mu & \text{if } f^y \text{ is } \mu\text{-integrable,} \\ 0 & \text{otherwise} \end{cases}$$

*belong to  $\mathcal{L}(X, \mathcal{A}, \mu, R)$  and  $\mathcal{L}(Y, \mathcal{B}, \nu, R)$  respectively, and*

3. *the relation*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X I_f d\mu = \int_Y J_f d\nu$$

*holds.*

The above is called *Fubini's Theorem*