Gaussian Maximum Likelihood

1 Why

2 Formulation

Let $x^1, \dots, x^n$ be a sequence of records in R. We want to select a density from among gaussian densities. A gaussian density is parameterized by its mean and positive standard deviation.

Following the principle of maximum likelihood, we want to solve

$$\begin{array}{rl} \textbf{find} & \mu, \sigma \in \mathsf{R} \\ \textbf{to maximize} & \displaystyle\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{1}{2\sigma^2}(x-\mu)^2\right) \\ \textbf{subject to} & \sigma > 0 \end{array}$$

We call a solution to the above problem a **maximum likelihood gaussian density**.

Let f be a gaussian density with parameters $\mu \in \mathsf{R}$ and $sigma \in \mathsf{R}_+$. We want to find μ and σ to

maximize

3 Solution

Proposition 1. *Let* $(x^1, \ldots, x^n)$ *be a dataset in* R*. The gaussian density with mean*

$$\mu = \frac{1}{n} \sum_{k=1}^{n} x^k$$

and covariance

$$\sigma^2 = \frac{1}{n} \sum_{k=1}^{n} (x^k - \mu)^2$$

is a maximum likelihood gaussian density.

Proof. We can maximize the log density likelihood

$$\sum_{k=1}^{n} \frac{1}{2\sigma^2}(x^k - \mu)^2 - \frac{1}{2} \log 2\pi\sigma^2.$$

The partial derivative of the log density likelihood with respect to μ is

$$-\sum_{k=1}^{n} \frac{1}{\sigma^2}(x - \mu)^2.$$

For all σ, this partial derivative is zero when $\mu = \frac{1}{n} \sum_{k=1}^{n} x^k$. The partial derivative of the log density likelihood with respect to σ^2 is

$$\left(\frac{1}{2} \sum_{k=1}^{n} (x^k - \mu)^2 \right) (-2\sigma^{-3}) - \frac{1}{2} \frac{1}{2\pi\sigma^2} 4\pi\sigma$$

$\square$