Why

Can we characterize positive (semi-)definite matrices in terms of their eigenvalues?

Main Result

Using eigenvalue decompositions, we can answer in the affirmative.

Proposition 1. *Let $A \in \mathbf{S}^d$ with smallest eigenvalue $\lambda_{\min}(A)$. Then*

$$
\begin{aligned}
A \in \mathbf{S}_+^d \quad &\longleftrightarrow \quad \lambda_{\min}(A) \geq 0 \\
&\longleftrightarrow \quad \operatorname{tr} AB \geq 0 \text{ for all } B \in \mathbf{S}_+^d.
\end{aligned}
$$

and

$$
\begin{aligned}
A \in \mathbf{S}_{++}^d \quad &\longleftrightarrow \quad \lambda_{\min}(A) > 0 \\
&\longleftrightarrow \quad \operatorname{tr} AB > 0 \text{ for all nonzero } B \in \mathbf{S}_{++}^d.
\end{aligned}
$$

Eigenvalues and Definiteness

Positive Definite Matrices · Eigenvalue Decomposition

Quadratic Forms · Symmetric Matrices · Orthonormal Matrices

Orthonormal Vectors · Identity Matrices · Matrix Trace · Eigenvalues and Eigenvectors

Inner Product Norms · Matrix-Matrix Products · Matrix Transpose · Matrix Determinants

Inner Products · Norms · Matrix-Vector Products · Real Matrices

Real Vectors · Functionals · Linear Combinations · Matrices

Vectors

Absolute Value · N-Dimensional Space

Distance

Space Distance

Real Square Roots

Real Squares · Plane Distance

Real Arithmetic

Real Multiplicative Inverses

Real Products · Interval Length

Real Additive Inverses · Intervals

Real Sums · Real Order · Real Line · Real Plane · Real Space

Rational Arithmetic · Rational Multiplicative Inverses · Fields · Permutations · Integral Line · Sequences

Rational Products · Rational Sums · Real Numbers · Groups · Set Numbers · Geometry

Rational Numbers · Integer Additive Inverses · Finite Sets

Integer Arithmetic · Inverse Elements · Direct Products

Natural Square Roots · Integer Products · Integer Sums · Natural Additive Identity · Natural Multiplicative Identity · Element Functions · Equivalent Sets · Family Unions and Intersections

Square Numbers · Natural Equations · Integer Numbers · Arithmetic · Identity Elements · Algebras · Function Inverses · Families

Squares · Natural Powers · Equations · Natural Order · Operations · Function Composites · Function Images

Natural Products · Functions · Equivalence Relations

Natural Sums · Relations

Recursion Theorem

Peano Axioms

Natural Induction · Cartesian Products

Natural Numbers · Set Powers · Generalized Set Dualities

Successor Sets · Unordered Triples · Set Dualities · Intersection of Empty Set · Partitions · Set Unions and Intersections

Ordered Pairs · Set Complements · Pair Unions · Set Intersections

Set Differences · Set Unions · Pair Intersections

Unordered Pairs · Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects