Why

Which is the optimal tree to use for tree density approximation?

Definition

We want to choose the tree whose corresponding approximator for the given density achieves minimum relative entropy with the given density among all tree density approximators. We call such a density an *optimal tree approximator* of the given density. We call a tree according to which an optimal tree approximator factors and optimal approximator tree.

Result

Proposition **1.** *Let $g : \mathbf{R}^n \to [0,1]$ be a density. A tree T on $\{1,\ldots,n\}$ is an optimal approximator tree if and only if it is a maximal spanning tree of the differential mutual information graph of q.*

Proof. First, denote the optimal approximator of g for tree T by f_T^*. Recall

$$f_T^* = f_1 \prod_{i \neq 1} f_{i|\mathbf{pa}_i}$$

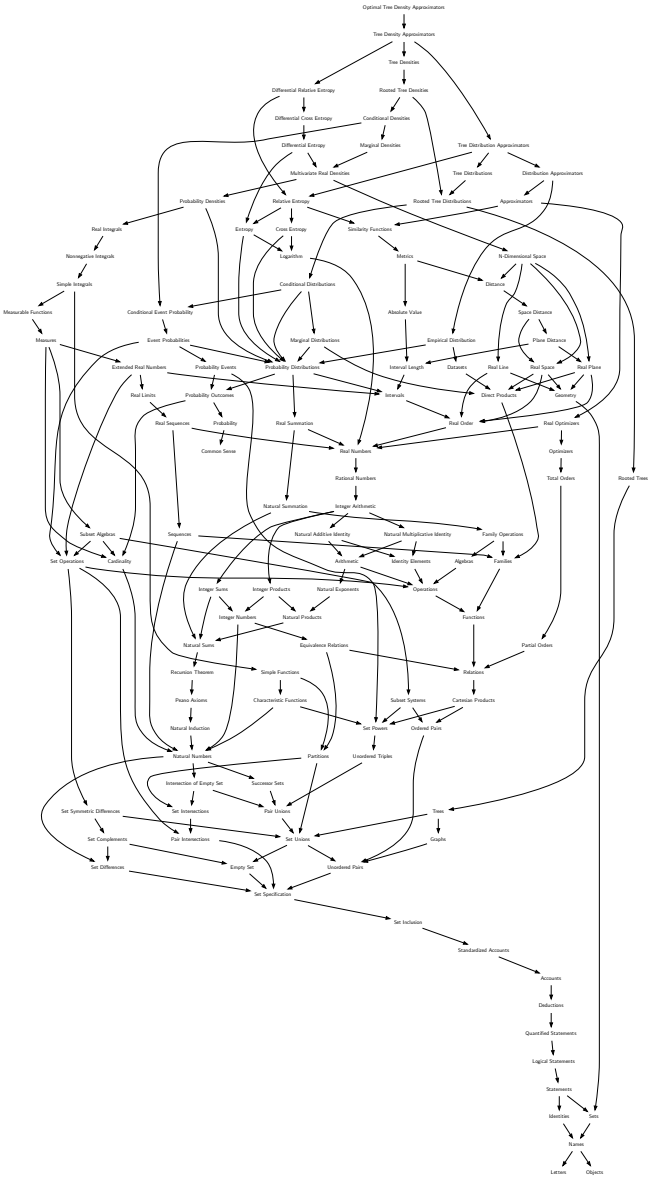
Second, recall $d(g,f) = H(g,f) - H(g)$. Since $H(g)$ does not depend on f, f is a minimizer of $d(g,f)$ if and only if it is a minimizer of $H(g,f)$.

Third, express the cross entropy of f_T^* relative to g as

$$\begin{aligned}
H(q, p_T^*) &= h(q_1) - \sum_{j \neq i} \left(\int_{\mathbf{R}^d} g(x) \log g_{i|pai}(x_i, x_{\mathbf{pa}_i}) dx \right) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i,\mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) - \log q_{\mathbf{pa}_i}(a_{\mathbf{pa}_i})) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i,\mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) - \log q_{\mathbf{pa}_i}(a_{\mathbf{pa}_i}) - \log q_i \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{i \neq 1} I(q_i, q_{\mathbf{pa}_i}) \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{\{i,j\} \in T} I(q_i, q_j)
\end{aligned}$$

where $\mathbf{pa}_i$ denotes the parent of vertex i in T ($i = 2, \ldots, n$). $H(g_i)$ does not depend on the choice of tree. Choosing a tree to minimize the second term in the final expression above is equivalent to choosing a maximal spanning tree from the weighted graph with differential mutual information edge weights; namely, the mutual information graph of g.

□


Optimal Tree Density Approximators
Tree Density Approximators
Tree Densities
Differential Relative Entropy
Rooted Tree Densities
Differential Cross Entropy
Conditional Densities
Differential Entropy
Marginal Densities
Tree Distribution Approximators
Multivariate Real Densities
Tree Distributions
Distribution Approximators
Probability Densities
Relative Entropy
Rooted Tree Distributions
Approximators
Real Integrals
Entropy
Cross Entropy
Similarity Functions
Nonnegative Integrals
Logarithm
Metrics
N-Dimensional Space
Simple Integrals
Conditional Distributions
Distance
Measurable Functions
Conditional Event Probability
Absolute Value
Space Distance
Measures
Event Probabilities
Marginal Distributions
Empirical Distribution
Plane Distance
Extended Real Numbers
Probability Events
Probability Distributions
Interval Length
Datasets
Real Line
Real Space
Real Plane
Real Limits
Probability Outcomes
Intervals
Direct Products
Geometry
Real Sequences
Probability
Real Summation
Real Order
Real Optimizers
Common Sense
Real Numbers
Optimizers
Rational Numbers
Total Orders
Rooted Trees
Natural Summation
Integer Arithmetic
Subset Algebras
Sequences
Natural Additive Identity
Natural Multiplicative Identity
Family Operations
Set Operations
Cardinality
Arithmetic
Identity Elements
Algebras
Families
Operations
Integer Sums
Integer Products
Natural Exponents
Integer Numbers
Natural Products
Functions
Natural Sums
Equivalence Relations
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Recursion Theorem
Simple Functions
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Peano Axioms
Characteristic Functions
Subset Systems
Cartesian Products
Natural Induction
Set Powers
Ordered Pairs
Natural Numbers
Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Intersections
Pair Unions
Trees
Set Symmetric Differences
Set Complements
Pair Intersections
set Unions
Graphs
Empty Set
Unordered Pairs
Set Differences
Set Specification
Set Inclusion
Standardized Accounts
Accounts
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Statements
Identities
Sets
Names
Lemas
Objects