



Why

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1 Definition

Define $S \in \mathbf{R}^{d \times d}$ by

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

For a vector $x \in \mathbf{R}^d$ the *down shift* of x is Sx .

Let $A \in \mathbf{R}^{d \times d}$ be a matrix with columns a_1, \dots, a_d . A is a *circulant matrix* if $a_1 = Sa_d$, $a_2 = Sa_1$, and $a_i = Sa_{i-1}$ for $i = 2, \dots, d$.

Example

For example, the matrix

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is a circulant matrix.

Characterization

A matrix $C \in \mathbf{R}^{d \times d}$ is circulant if and only if there exists c_0, \dots, c_{d-1} so that

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{d-1} S^{d-1}.$$

¹Future sheets will include. These matrices arise in practice and each has the same eigenvectors.

Properties

The sum and product of any two circulant matrix is circulant. In other words, the circulant matrices with the usual matrix addition and multiplication form a commutative ring.

