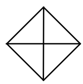


Why

Suppose we want to fill up a backpack by selecting from various objects which gives us differing amounts of comfort. We only have so much space in the backpack.

Definition

Number the objects $1, \dots, n$. Model the amount of space needed for a subset $H \subset \{1, \dots, n\}$ of the n items by $s(H)$; here $s : \mathcal{P}(S) \to \mathsf{R}_+$. Model the comfort (or value) they provide by $v(H)$; here $v : \mathcal{P}(\{1, \dots, n\}) \to \mathsf{R}$. Given a space constraint c, we want to find $H \subset P$ to

$$\begin{aligned} \text{maximize} \quad & v(H) \\ \text{subject to} \quad & s(H) \leq c \end{aligned}$$

In other words, find the susbet of items which will fit in the bag and maximize the value. Such problems are called *knapsack problems.*

Linear formulation

It is natural to expect the space constraint to be additive In other words $H \cap G = \varnothing$ means $s(H \cup G) = s(H) + s(G)$, from which we conclude that the function is monotonic (using the fact that it is nonnegative). I.e., given $G \subset H$, we have $s(G) \leq S(H)$. Suppose we also model the value function as additive. Given $H \cap G = \varnothing$, then $v(H \cup G) = v(H) + v(G)$.

It turns out that additivity is sufficient to have a linear representation for both s and v. For $H \subset P$, denote by $\chi_H : P \to \{0, 1\}$ the characteristic function of H. In other words, χ_H is defined by

$$x_H(i) = \begin{cases} 1 & \text{if } i \in H \\ 0 & \text{otherwise} \end{cases}$$

Then there exists $p : \{1, \dots, n\} \to \mathsf{R}$ and $w : \{1, \dots, n\} \to \mathsf{R}_+$ such that

$$v(H) = \textstyle\sum_{i=1}^{n} p(i)\chi_H(i) \quad \text{for all } H \subset \{1, \dots, n\}$$

and

$$s(H) = \sum_{i=1}^{n} w(i)\chi_H(i) \quad \text{for all } H \subset \{1, \ldots, n\}$$

We can formulate the following problem: given $c \in \mathbf{R}_+$, $w : \{0,1\} \to \mathbf{R}_+$ and $p : \{0,1\}^n \to \mathbf{R}$, find $H \subset \{1, \ldots, n\}$ to

$$\begin{array}{rl} \text{maximize} & \sum_{i=1}^{n} p(i)\chi_H(i) \\ \text{subject to} & \sum_{i=1}^{n} w_i \chi_H(i) \leq c \end{array}$$

It is common to identify χ_H with a list $x \in \{0,1\}^n$ and to find x to

$$\begin{array}{rl} \text{maximize} & \sum_i p_i x_i \\ \text{subject to} & \sum_i w_i x_i \leq c \\ & x_i \in \{0,1\} \text{ for all i } = 1, \ldots, n \end{array}$$

This problem is often called the *zero-one knapsack problem* (or *0-1 knapsack problem*). The *problem data* is the triple (p, w, c).

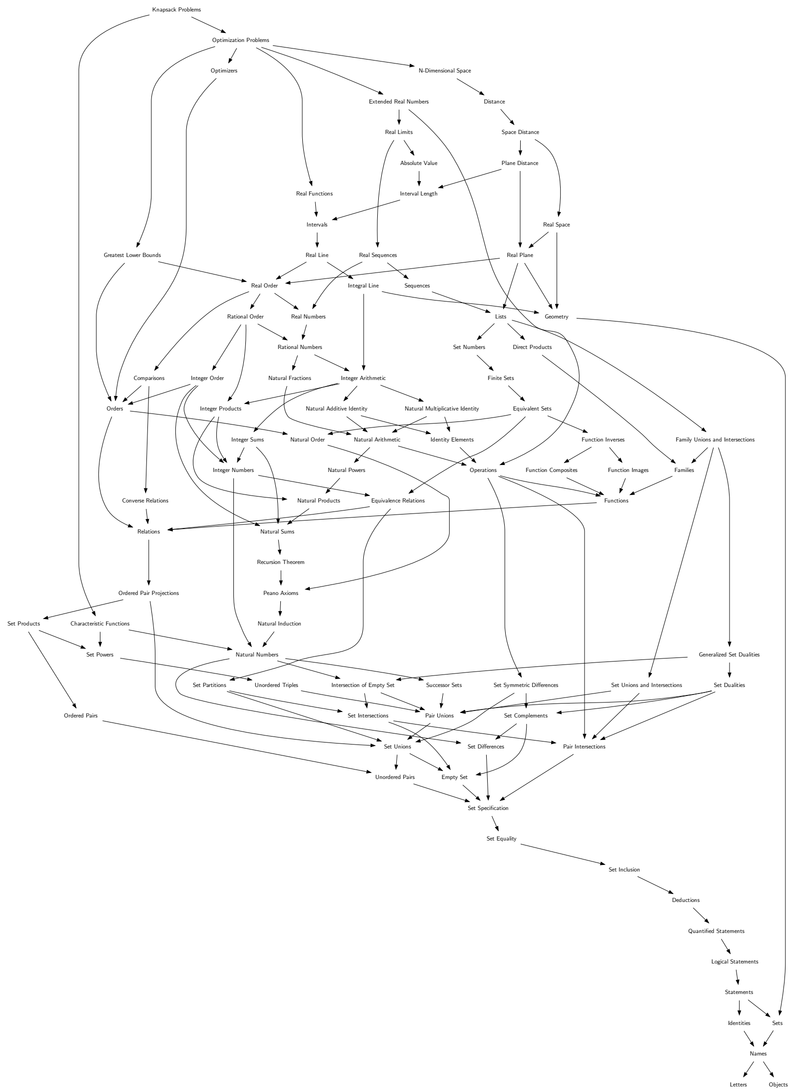
Alternative perspectives

Suppose instead the n objects are investments, the ith investment requiring w_i investment, returning p_i. Given that we have c dollars in capital, how should we allocate the funds to the investments to maximize return.

Other areas in which these problems are used as models include capital budgeting, cargo loading

Terminology

We generally refer to the n *items*, the ith such item having *weight* w_i and profit p_i. Sometimes such problems are called *single knapsack problems* (one container), in contrast with *multiple knapsack problems* (in which there are multiple containers).


Knapsack Problems
Optimization Problems
Optimizers
N-Dimensional Space
Extended Real Numbers
Distance
Real Limits
Space Distance
Absolute Value
Plane Distance
Real Functions
Interval Length
Intervals
Real Space
Greatest Lower Bounds
Real Line
Real Sequences
Real Plane
Real Order
Integral Line
Sequences
Rational Order
Real Numbers
Lists
Geometry
Rational Numbers
Set Numbers
Direct Products
Comparisons
Integer Order
Natural Fractions
Integer Arithmetic
Finite Sets
Orders
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Equivalent Sets
Integer Sums
Natural Order
Natural Arithmetic
Identity Elements
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Powers
Operations
Function Composites
Function Images
Families
Converse Relations
Natural Products
Equivalence Relations
Functions
Relations
Natural Sums
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Characteristic Functions
Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Set Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Symmetric Differences
Set Unions and Intersections
Set Dualities
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects