



Size of a direct product of sets

Proposition 1. *Suppose A_1, \dots, A_n is a list of finite sets. Then*

$$|\prod_{i=1}^n A_i| = \prod_{i=1}^n |A_i|$$

Proof. As before, the proof will be by induction. Future editions will include. \square

We highlight one simple consequence: if $|A|$ is finite, then $|A^n| = |A|^n$.

Principle of counting

Proposition 2. *Suppose X_1, \dots, X_k is a list of nonempty sets and f_2, \dots, f_k are functions mapping $X_1, X_1 \times X_2, \dots, X_1 \times \dots \times X_{k-1}$ to $\mathcal{P}(X_2), \mathcal{P}(X_3), \dots, \mathcal{P}(X_k)$ such that there exists natural numbers n_1, n_2, \dots, n_k ,*

$$n_1 = |X_1|$$

$$n_2 = |f(x_1)| \quad \text{for all } x_1 \in X_1$$

$$n_3 = |f(x_1, x_2)| = n_3 \quad \text{for all } (x_1, x_2) \in X_1 \times f(x_1)$$

$$n_4 = |f(x_1, x_2, x_3)| = n_3 \quad \text{for all } (x_1, x_2, x_3) \in X_1 \times f_2(x_1) \times f_3(x_1, x_2)$$

\vdots

$$n_k = |f(x_1, x_2, \dots, x_{k-1})|$$

$$\text{for all } (x_1, x_2, \dots, x_{k-1}) \in X_1 \times f_2(x_1) \times \dots \times f_k(x_1, \dots, x_{k-1})$$

Then

$$|\{x : \{1, \dots, k\} \rightarrow \cup_{i=1}^k | x_1 \in X_1, x_2 \in f_2(x_1), \dots, x_k \in f_k(x_1, \dots, x_{k-1})\}|$$

$$\text{is } \prod_{i=1}^k n_i$$

Proof. Future editions will include the proof. \square

The English is much clearer. Suppose we have to do k tasks (here, the term task is left undefined). The first task can be done in n_1 ways,

and after it has been completed, no matter how, the second task can be done in n_2 ways. Further, after the first two tasks have been completed, in whatever manner, there are n_3 ways of doing the third task. And so on. Then the number of ways to complete all tasks is $n_1 n_2 \cdots n_k$. This conclusion, often taken as a principle, or axiom, is called the *fundamental principle of counting*.

Example

Consider the usual 52 playing cards. How many ways are there to stack them as a deck? First the bottom card, we have 52 choices. For the next to bottom card, we have 51 choices—we can pick any card except the bottom one. For the third to bottom card, we have 50 choices—and so on. By applying the above proposition, we can deduce that there are $52 \cdot 51 \cdot 50 \cdots 1$ ways of arranging the deck.

Denote the set of cards by C . For the above proposition, $k = 52$ and X_1, \dots, X_k are so that $X_i = C$ for all $i = 1, \dots, 52$. The function $f_2(c_1) = C - \{c_1\}$. The function $f_3(c_1, c_2) = C - \{c_1, c_2\}$. And so on, so that the function $f_k(c_1, \dots, c_{k-1}) = C - \{c_1, \dots, c_{k-1}\}$. Notice that given distinct arguments to these functions, the size of the sets $f_1(c_1)$ is always 51. Likewise, the size of $f_2(c_1, c_2)$ is 50, $f_3(c_1, c_2, c_3)$ is 49 and so on, so that the size of $f_k(c_1, \dots, c_k)$ is 1, regardless of the choice of c_i , $i = 1, \dots, 51$.

