



# Length Common Notions

## 1 Why

We want to define the length of a subset of real numbers.

## 2 Notions

We take two common notions:

1. The length of a whole is the sum of the lengths of its parts;  
the *additivity principle*.
2. The length of a whole is the at least the length of any  
whole it contains the *containment principle*.

The task is to make precise the use of “whole,” “parts,” and “contains.” We start with intervals.

## 3 Definition

By whole we mean set. By part we mean an element of a partition. By contains we mean set containment.

The *length* of an interval is the difference of its endpoints: the larger minus the smaller.

Two intervals are *non-overlapping* if their intersection is a single point or empty. The *length* of the union of two non-overlapping intervals is the sum of their lengths.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A *countably simple* subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to  $+\infty$ .

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A *cover* of a set  $A$  of real numbers is a family whose union contains  $A$ . Since a cover always contains the set  $A$ , its length, which we understand, must be larger (containment principles) than  $A$ . So what if we declare that the length of an arbitrary set  $A$  be the greatest lower bound of the lengths of all sequences of intervals covering  $A$ . Will this work?

### 3.1 Cuts

If  $a, b$  are real numbers and  $a < b$ , then we *cut* an interval with  $a$  and  $b$  as its endpoints by selecting  $c$  such that  $a < c$  and  $c < b$ . We obtain two intervals, one with endpoints  $a, c$  and one with endpoints  $c, b$ ; we call these two the *cut pieces*.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

## 4 All sets

**Proposition 1.** *Not all subsets of real numbers are simple.*

*Exhibit:  $\mathbb{R}$  is not finite.*

**Proposition 2.** *Not all subsets of real numbers are countably simple.*

*Exhibit: the rationals.*

Here's the great insight: approximate a set by a countable family of intervals.

### 4.1 Notation