



Why

In the case that it is not possible to easily identify (or guess) the limit of a sequence, we are naturally interested in a simple condition on the sequence which is equivalent to convergence.

Definition

A sequence $(x_n)_{n \in \mathbf{N}}$ in \mathbf{R} is said to be *egopox* (or *Cauchy* or a *Cauchy sequence*) if for every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ so that for all $m, n > N$, $|x_m - x_n| < \varepsilon$. We call this property of the sequence (*eventual*) *egoproximity*.

Notation

We sometimes denote this property as

$$|x_n - x_m| \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty.$$

Example

For example, consider $\lim_{N \rightarrow \infty} \sum_{n=1}^N 1/n^3$.

Sufficiency in \mathbf{R}

Clearly a convergent sequence is egoprox.¹ What of the converse? Recall that we think of egoprox sequences as “bunching up.” For the reals, if a sequence is bunching up, then our intuition is that it should be converging. In other words, an

¹Future editions may elaborate here.

egoprox real sequence always converges. The egoprox condition is sufficient. Bunching up is sufficient.

Proposition 1. *If $(x_n)_{n \in \mathbf{N}}$ is egoprox, then there exists $x_0 \in \mathbf{R}$ so that $\lim_{n \rightarrow \infty} x_n = x_0$.*

In other words, in \mathbf{R} egoproximity is equivalent to convergence. The above is sometimes called the *Bolzano-Weierstrass theorem*.

