

## ORTHOGONAL REAL SUBSPACES

## Definition

Two subspaces  $S,T\subset \mathbf{R}^n$  are orthogonal if

$$x^{\top}y = 0$$
 for all  $x \in S, y \in T$ .

For any set  $S \subset \mathbb{R}^n$  (not necessarily a subspace), the *orthogonal complement* of S is the set

$$S^{\perp} = \{ x \in \mathbf{R}^n \mid x^{\top} y = 0 \text{ for all } y \in S \}.$$

 $S^{\perp}$  is the set of all vectors which are orthogonal to every vector in S.

## Orthgonal complement is a subspace

Notice that  $S^{\perp}$  is always a subspace. If  $x \in S^{\perp}$ , then  $x^{\top}y = 0$  for all  $y \in S$ . So then  $(\alpha x)^{\top}y = \alpha(x^{\top}y) = 0$  for all  $\alpha \in \mathbf{R}$  and  $y \in S$ . We conclude  $\alpha x \in S^{\perp}$  for all  $\alpha \in \mathbf{R}$ . In other words,  $S^{\perp}$  is closed under scalar multiplication. If  $x, z \in S^{\perp}$ , then  $(x+z)^{\top}y = x^{\top}y + z^{\top}y = 0 + 0 = 0$ . We conclude that  $x+z \in S^{\perp}$  for all  $x, z \in S^{\perp}$ . In other words,  $S^{\perp}$  is closed under vector addition. Consequently,  $S^{\perp}$  is a subspace.

