Definition

The *nullspace* (or *kernel*) of a matrix $A \in \mathbf{R}^{m \times n}$ is the set

$$\{x \in \mathbf{R}^n \mid Ax = 0\}.$$

It is the set of vectors mapped to zero by A. Equivalently, it is the set of vectors orthogonal to the rows of A.

Notation

We denote the nullspace of $A \in \mathbf{R}^{m \times n}$ by $\mathrm{null}(A) \subset \mathbf{R}^n$. Some authors denote the nullspace of A by $\mathcal{N}(A)$.

A subspace

The nullspace of a matrix is a subspace (this justifies the terminology null*space*!). There are a few routes to see this.

The first is direct. If $w, z \in \mathrm{null}(A)$, then $Aw = 0$ and $Az = 0$. So then $A(w + z) = Aw + Az = 0$. So $\mathrm{null}(A)$ is closed under vector addition. Also $A(\alpha w) = \alpha(Aw) = 0$ for all $\alpha \in \mathbf{R}$. [In particular $A0 = 0$, so $0 \in \mathrm{null}(A)$; i.e., $\mathrm{null}(A)$ contains the origin.] So $\mathrm{null}(A)$ is closed under scalar multiplication.

The second is by thinking about orthogonal complements. Second, we can view the $\mathrm{null}(A)$ as the set of vectors orthgonal to all the rows of A. In other words, $\mathrm{null}(A) = \{\tilde{a}_1, \ldots, \tilde{a}_m\}^{\perp}$. The orthogonal complement of any set is a subspace (see Orthogonal Real Subspaces).

Ambiguity in solutions

Suppose we have a solution to the system of linear equation with data (A, y). In other words, we have a vector $x \in \mathbf{R}^n$ so that $y = Ax$. If we have a vector $z \in \mathrm{null}(A)$, then $x + z$ is also a solution to the system (A, y), since

$$A(x + z) = Ax + Az = Ax + 0 = y$$

Conversely, suppose there were another solution $\tilde{x} \in \mathbf{R}^n$ to the system (A, y). Then $y = Ax = A\tilde{x}$, so

$$0 = y - y = Ax - A\tilde{x} = A(x - \tilde{x}).$$

Consequently, $(x - \tilde{x}) \in \mathrm{null}(A)$, and so $\tilde{x}$ is the solution x plus some vector in the null space of A. Consequently we are interested in whether A has vectors in its nullspace.

Zero nullspace

The origin 0 is always in the nullspace of A. However, this vector does not mean that we can find different solutions, since $x + 0 = x$ for all $x \in \mathbf{R}^n$. If, on the other hand, there is a nonzero vector $z \in \mathrm{null}(A)$, then $x + z \neq x$, and $x + z$ is a solution for (A, y). We think about A as a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. In the case that there is a nonzero element in the nullspace, A maps different vectors to the same vector. Here, x and $x + z$ both map to y. In this case, the function is *not invertible*, because it is not one-to-one. If, however, zero is the only element of the null space, the function is one-to-one. So call A *one-to-one* if $\mathrm{null}(A) = 0$.

Equivalent statements

A matrix $A \in \mathbf{R}^{m \times n}$ is *one-to-one* if the linear function $f : \mathbf{R}^n \to \mathbf{R}^m$ defined by $f(x) = Ax$ is one-to-one. In this case, if there exists $x \in \mathbf{R}^n$ so that $y = Ax$, then there is only one such x. Different elements in $\mathbf{R}^n$ map to different elements in $\mathbf{R}^m$.

Real Matrix Nullspace

Orthogonal Real Subspaces

Real Subspaces — Vectors as Matrices

Real Matrices — Real Vectors Span — Real Vector Angles — Real Inner Product

Linear Equation — Real Linear Combinations — Space Vector Angles — Real Norm — Space Inner Product

Arrays — Linear Functions — Real Vectors — Plane Vector Angles — Space Norm

N-Dimensional Space — Plane Vector Sums and Differences — Space Vector — Plane Norm — Plane Inner Product

Distance — Plane Vector — Absolute Value — Right Triangle Side Relation

Space Distance — Real Square

Plane Distance — Real Arithmetic

Interval Length — Real Multiplicative Inverse

Real Space — Intervals — Real Products — Square Numbers

Real Plane — Real Line — Real Additive Inverse — Square

Real Summation — Real Order — Real Sums — Integer Line — Rational Arithmetic — Rational Multiplicative Inverse

Real Numbers — Rational Sums — Rational Order — Rational Products — Geometry

Rational Numbers

Integer Arithmetic

Lists — Natural Additive Identity — Natural Multiplicative Identity — Integer Products

Set Number — Finite Sets — Identity Elements — Arithmetic — Natural Powers

Direct Products — Inverse Elements — Equivalent Sets — Natural Summation

Family Unions and Intersections — Family Operations — Comparisons — Element Functions — Function Inverses — Natural Equations — Integer Order — Integer Sums — Natural Products

Families — Operations — Function Composite — Function Image — Equation — Orders — Natural Sums — Integer Numbers

Converse Relations — Functions — Natural Order — Equivalence Relations — Recursion Theorem

Relations — Peano Axioms

Set Products — Natural Induction

Generalized Set Dualities — Set Powers — Natural Numbers

Set Unions and Intersections — Set Dualities — Unordered Triples — Successor Sets — Intersection of Empty Set — Partitions — Set Symmetric Differences

Ordered Pairs — Pair Unions — Set Intersections — Set Complements

Pair Intersection — Set Unions — Set Differences

Unordered Pairs — Empty Set

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities — Sets

Names

Letters — Objects