Why

We want a complete norm on the vector space of continuous functions.

Result

Proposition **1.** *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $(f_n)_n$ be an egoprox sequence in $C[a,b]$.

Candidate. $(f_n)_n$ is egoprox means $\forall \varepsilon > 0, \exists N$ so that

$$m, n > N \implies |f_n - f_m|_{\sup} < \varepsilon.$$

Since $|f_n - f_m|_{\sup} < \varepsilon \implies |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in [a,b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a,b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit $l_x \in R$ such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a,b]$. Define $f : [a,b] \to R$ by $f(x) = l_x$ for each $x \in [a,b]$.

Candidate is Limit. First, we argue that $|f_n - f|_{\sup} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $(f_n)_n$ is an egoprox sequence, there exists n_0 so that

$$n, m \geq n_0 \implies |f_n - f_m|_{\sup} < \varepsilon/2.$$

So for all $x \in [a,b]$,

$$n, m \geq n_0 \implies |f_n(x) - f_m(x)| < \varepsilon/2.$$

For all $x \in [a, b]$, and $n \geq n_0$,

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon.$$

The sequence $\{f_k(x)\}_{k=m}^{\infty}$ is a final part of $\{f_k(x)\}_{k=1}^{\infty}$, and so has the same limit, $f(x)$. Therefore, using continuity of subtraction and the absolute value,

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

We conclude that for $n \geq n_0$, $x \in [a, b]$, $|f_n(x) - f(x)| < \varepsilon$, from which we deduce $|f_n - f|_{\sup} < \varepsilon$. Thus $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

Limit is Continuous. Next, we argue that f is continuous. Let $x_0 \in [a, b]$. Let $\varepsilon > 0$. Since $f_n \longrightarrow f$ there exists n_0 so that

$$|f_{n_0} - f|_{\sup} < \varepsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a, b]$. Using $|f(x_0) - f_{n_0}(x_0)| < \varepsilon/3$,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f(x)|,$$

for all $x \in [a, b]$. Using the triangle inequality,

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)|$$

for all $x \in [a, b]$. Using $|f_{n_0}(x_0) - f(x)| < \varepsilon/3$

$$|f(x_0) - f(x)| < \varepsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + \varepsilon/3$$

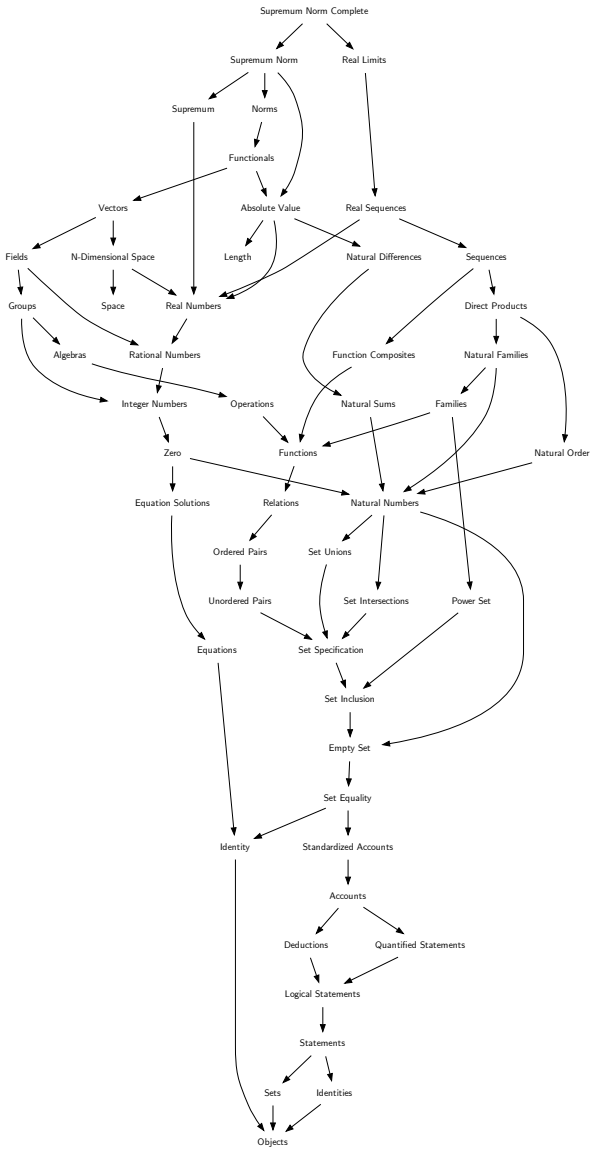
for all $x \in [a, b]$. Since f_{n_0} is continuous, there exists $\delta > 0$ so that

$$|x_0 - x| < \delta \implies |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon/3,$$

for $x \in [a, b]$. In this case,

$$|f(x_0) - f(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since ε was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere. Some call the above the *three epsilon argument*. $\square$


Supremum Norm Complete
Supremum Norm
Real Limits
Supremum
Norms
Functionals
Vectors
Absolute Value
Real Sequences
Fields
N-Dimensional Space
Length
Natural Differences
Sequences
Groups
Space
Real Numbers
Direct Products
Algebras
Rational Numbers
Function Composites
Natural Families
Integer Numbers
Operations
Natural Sums
Families
Zero
Functions
Natural Order
Equation Solutions
Relations
Natural Numbers
Ordered Pairs
Set Unions
Unordered Pairs
Set Intersections
Power Set
Equations
Set Specification
Set Inclusion
Empty Set
Set Equality
Identity
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Deductions
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Statements
Sets
Identities
Objects