



## Why

We generalize the product of two sets to a product of a family of sets. To do so we discuss sets of families.

## Discussion for pairs

Let  $A$  and  $B$  be sets. There is a natural correspondence between the product set  $A \times B$  (see **Cartesian Products**) and the set of families

$$Z = \{z : \{i, j\} \rightarrow (A \cup B) \mid z_i \in A \text{ and } z_j \in B\}.$$

The family  $z \in Z$  corresponds with the pair  $(z_i, z_j)$ . The pair  $(a, b)$  corresponds to the family  $z \in Z$  defined by  $z(i) = a$  and  $z(j) = b$ . So, ordered pairs can be put in one-to-one correspondence with families. The generalization of Cartesian products to more than two sets generalizes the notion for families.

## Definition

Let  $X$  be a set. Let  $A : I \rightarrow X$  be a family of subsets of  $X$ . The *direct product* or *family Cartesian product* of  $A$  is the set of all families  $a : I \rightarrow X$  which satisfy  $a_i \in A_i$  for every  $i \in I$ .

A function on a product is called a *function of several variables* and, in particular, a function on the product  $X \times Y$  is called a *function of two variables*.

## Notation

We denote the product of the family  $\{A_i\}$  by

$$\prod_{i \in I} A_i$$

We read this notation as “product over  $i$  in  $I$  of  $A$  sub- $i$ .”

## Projections

The word “projection” is used in two senses with families. Let  $I$  be a set, and let  $\{A_i\}_{i \in I}$  be a family of sets. Define  $A = \prod_{i \in I} A_i$ .

First, let  $J \subset I$ . There is a natural correspondence between the elements of  $A$  and those of  $\prod_{j \in J} A_j$ . To each element  $a \in A$ , we restrict  $a$  to  $J$  and this restriction is an element of  $\prod_{j \in J} A_j$ . The correspondence is called the *projection* of  $A$  onto  $\prod_{i \in J} A_i$ . The projection in this sense is a set of families.

Second, consider the value of a family  $a \in A$  at  $j$ . We call  $a_j$  the *projection of  $a$  onto index  $j$*  or the  *$j$ -coordinate* of  $a$ . This word *coordinate* is meant to follow the language used in defining ordered pairs. The projection in this sense is an element of  $A_j$ . The  $j$ th projection is a function from  $\prod_{i \in I} X_i \rightarrow X_j$ .

