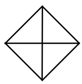


Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Suppose p is a distribution on a *finite* set of outcomes Ω. Given an event $E \subset \Omega$, the *probability of E under p* as the sum of the probabilities of the outcomes in E. The frequentist interpretation is clear—the probability of an event is the proportion of times any of its outcomes will occur in the long run.

Notation

It is common to define a function $P : \mathcal{P}(\Omega) \to \mathbf{R}$ by

$$P(A) = \sum_{a \in A} p(a) \quad \text{for all } A \subset \Omega$$

We call this function P *the event probability function* (or *the probability measure*) associated with p. Since it depends on the sample space Ω and the distribution p, we occasionally denote this dependence by $P_{\Omega,p}$ or P_p.

It is tempting, and therefore common to write $P(\omega)$ when $\omega \in \Omega$ and one intends to denote $P(\{\omega\})$. Of course, this corresponds with $p(\omega)$. It is therefore easy to see that from P we can compute p, and vice versa.

Examples

Rolling a die. We consider the usual fair die model (see Outcome Probabilities). Here we have $\Omega = \{1, \dots, 6\}$ and a distribution $p : \Omega \to [0, 1]$ defined by

$$p(\omega) = 1/6 \quad \text{for all } \omega \in \Omega$$

Given the model, the probability of the event $E = \{2, 4, 6\}$ is

$$P(E) = \textstyle\sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of event probabilities

The properties of p ensure that P satisfies

1. $P(A) \geq 0$ for all $A \subset \Omega$;
2. $P(\Omega) = 1$ (and $P(\varnothing) = 0$);
3. $P(A) + P(B)$ for all $A, B \subset \Omega$ and $A \cap B = \varnothing$.

The last statement (3) follows from the more general identity

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

for $A, B \subset \Omega$, by using $\mathbf{P}(\varnothing) = 0$ of (2) above. These three conditions are sometimes called the *axioms of probability for finite sets.*

Do all such P satisfying (1)-(3) have a corresponding underlying probability distribution? The answer is easily seen to be yes. Suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satisfies (1)-(3). Define $q : \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an *event probability function* (or a *(finite) probability measure*).

Other basic consequences

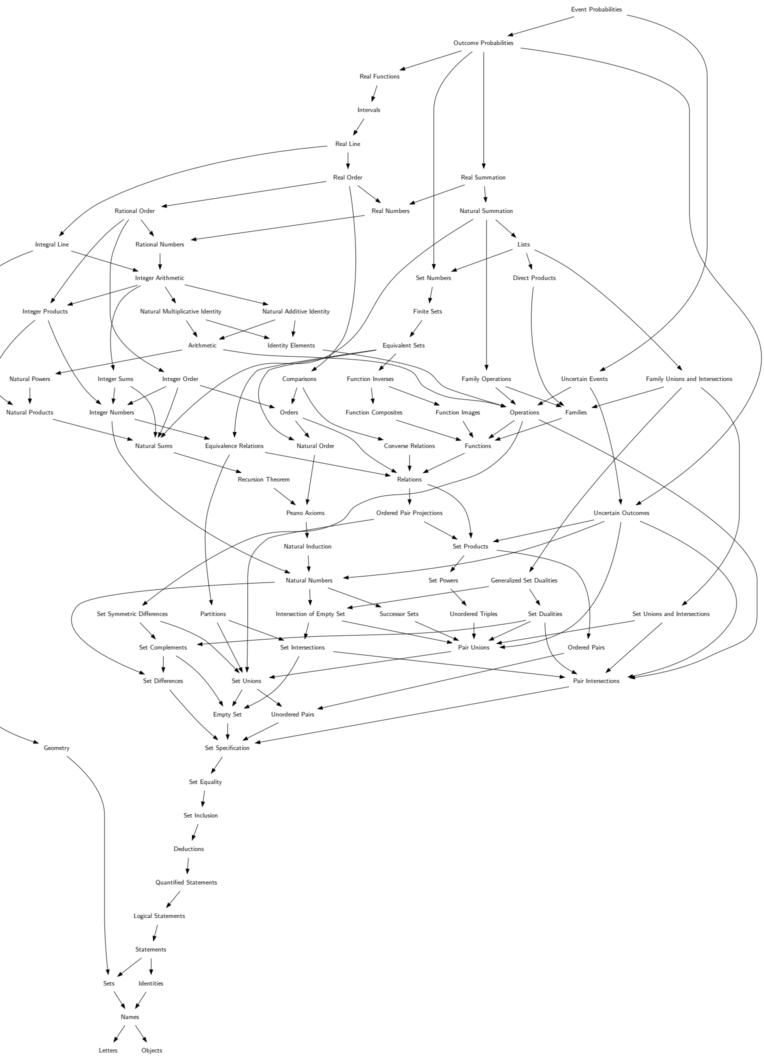
Probability by cases. Suppose $A_1, \dots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$P(B) = \textstyle\sum_{i=1}^{n} P(A_i \cap B).$$

Some authors call this the *law of total probability.* This is easy to see by using the distributive laws of set algebra (see Set Unions and Intersections).

Monotonicity. If $A \subseteq B$, then $P(A) \leq P(B)$. This is easy to see by splitting B into $A \cap B$ and $B - A$, and applying (1) and (3).

Subadditivity. For $A, B \subset \Omega$, $P(A \cup B) \leq P(A) + P(B)$. This is easy to see from the more general identity in (3) above. This is sometimes referred to as a *union bound*, in reference to *bounding* the quantity $P(A \cup B)$.


Event Probabilities
Outcome Probabilities
Real Functions
Intervals
Real Line
Real Order
Real Summation
Rational Order
Real Numbers
Natural Summation
Integral Line
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Operations
Uncertain Events
Family Unions and Intersections
Natural Products
Integer Numbers
Orders
Function Composites
Function Images
Operations
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Uncertain Outcomes
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects