



Why

We extend our notion of length, area, and volume beyond the Lebesgue measure on the product spaces of real numbers.

Definition

Suppose \mathcal{A} is an algebra of sets. A function $f : \mathcal{A} \rightarrow \bar{\mathbf{R}}_+$ is *finitely additive* if

$$f(\cup_{i=1}^n A_i) = \sum_{i=1}^n f(A_i) \quad \text{for all } A_1, \dots, A_n \in \mathcal{A}$$

Similarly, suppose \mathcal{F} is a σ -algebra. Then f is *countably additive* if

$$f(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} f(F_i) \quad \text{for all sequences } \{F_i\}_{i \in \mathbf{N}} \text{ in } \mathcal{F}$$

If, in addition, $f(\emptyset) = 0$, then f is called a *finitely additive measure* or *countably additive measure* respectively. Since a countably additive measure is finitely additive (the converse is false!), when we speak of a *measure* we mean a countable additive one.

When (X, \mathcal{F}) is a countably unital subset algebra and $\mu : \mathcal{F} \rightarrow \bar{\mathbf{R}}_+$, then we call (X, \mathcal{F}) a *measurable space* and call (X, \mathcal{F}, μ) a *measure space*. The word “space” is natural, since the notion of a measure generalized the notion of volume in real space (see **Real Space and N-Dimensional Space**). We often call \mathcal{F} the *measurable sets*. In other words, a measure space is a triple: a base set, a sigma algebra, and a measure.

Notation

We often use μ for a measure since it is a mnemonic for “measure”. We often also use ν and λ since these letters are near μ in the Greek alphabet.

Examples

Example 1. Let (A, \mathcal{A}) a measurable space. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a

measure. We call μ the counting measure.

Example 2. Let (A, \mathcal{A}) measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise. Then μ is a measure. We call μ the point mass concentrated at a .

Example 3. The Lebesgue measure on the measurable space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is a measure.

Example 4. Let \mathcal{A} the co-finite algebra on N . Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.

Example 5. Let (A, \mathcal{A}) a measurable space. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be 0 if $A = \emptyset$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.

Example 6. Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = \mathcal{P}(A)$. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \emptyset$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \emptyset$ then using finite additivity We obtain a contradiction

$$1 = \mu(B \cup C) \neq \mu(B) + \mu(C) = 2$$

□

Properties

Proposition 1 (monotonicity). Suppose (A, \mathcal{A}, μ) is measure space. Then

$$\mu(B) \leq \mu(C) \quad \text{for all } B \subset C \subset A$$

Proposition 2 (subadditivity). Suppose (A, \mathcal{A}, m) is a measure space and $\{A_n\} \subset \mathcal{A}$ is a countable family. Then $m(\cup A_n) \leq \sum_i m(A_i)$.

Proposition 3. *For a measure space (A, \mathcal{A}, m) .*

$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} m(A_i)$$

Proposition 4. *For a measure space (A, \mathcal{A}, m) .*

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} m(A_i)$$

