Almost Everywhere Measurability

1 Why

Does convergence almost everywhere of a sequence of measurable functions guarantee measurability of the limit function? It does on complete measure spaces, and we can use this result to “weaken" the hypotheses of many theorems.

2 Results

A measure is *complete* if every subset of a measurable set of measure zero is measurable. If the measure is complete, then every negligible set must be measurable.

We begin with a transitivity property: almost everywhere equality of two functions allows us to infer measurability of one from the other.

Proposition 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space Let $f, g : X \to [-\infty, \infty]$ with $f = g$ almost everywhere. If μ is complete and f is $\mathcal{A}$-measurable, then g is $\mathcal{A}$-measurable.*

Proof. □

Proposition 2. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n : X \to [-\infty, \infty]$ for all natural numbers n and $f : X \to [-\infty, \infty]$ with $(f_n)_n$ converging to f almost everywhere. If μ is complete and and f_n is measurable for each n, then f is $\mathcal{A}$-measurable.*

Proof. □