



Why

We want to sum infinitely many real numbers.

Definition

Let $(a_k)_{k \in \mathbf{N}}$ be a sequence in \mathbf{R} . Define $(s_n)_{n \in \mathbf{N}}$ by

$$s_n = \sum_{k=1}^n a_k.$$

We call s_n the *n*th partial sum of (x_k) . In other words, the first partial sum s_1 is a_1 , the second partial sum s_2 is $a_1 + a_2$, the third partial sum s_3 is $a_1 + a_2 + a_3$ and so on. We call (s_n) the *sequence of partial sums* or *series* of (a_k) . If the *series* converges, then we say that (a_k) is *summable*. Clearly not every series is summable: consider, for example, $a_k = 1$ for all k . It has the divergent series $(1, 2, 3, 4, 5, \dots)$.

Notation

If the sequence is summable, then there exists a unique $s \in \mathbf{R}$ (the limit), which we denote

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

We read these relations aloud as “s is the limit as n goes to infinity of s_n” and “s is the limit as n goes to infinity of the sum of a_k from k equals 1 to n.” We often avoid referencing s_n by abbreviating the above with

$$\sum_{k=1}^{\infty} a_k = s.$$

We read this notation aloud as “the sum from 1 to infinity of a_k is s.” The notation is subtle, and requires justification by the algebra of series.¹

¹Future editions will include such justification.

Convergence

For a series to converge, intuition suggests that the additional terms added should be getting smaller and smaller. Indeed:

Proposition 1. *Let $(a_k)_{k \in \mathbf{N}}$ be a sequence of real numbers. If (a_k) is summable then a_k converges to 0.²*

The converse of this theorem has immediate relevance as a preliminary test for determining whether a series converges.

Proposition 2. *If (a_k) does not converge or converges to $a_0 \neq 0$, then it is not summable.*

²Future editions will include an account.

