Why

How big can a quadratic form be? How small?

Result

Proposition 1. *Suppose $A \in \mathbf{S}^n$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_n x^\top x \leq x^\top A x \leq \lambda_1 x^\top x,$$

for all $x \in \mathbf{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ with $A = Q \Lambda Q^\top$. Express

$$x^\top A x = x^\top Q \Lambda Q^\top x = (Q^\top x)^\top \Lambda (Q^\top x)$$

$$= \sum_{i=1}^{n} \lambda_i (q_i^\top x)^2$$

$$= \lambda_1 \sum_{i=1}^{n} (q_i^\top x) = \lambda_1 \|Q^\top x\|^2 = \lambda_1 \|x\|^2.$$

Similarly,

$$x^\top A x = \sum_{i=1}^{n} \lambda_i (q_i^\top x)^2$$

$$\geq \lambda_n \sum_{i=1}^{n} (q_i^\top x) = \lambda_n \|Q^\top x\|^2 = \lambda_n \|x\|^2.$$

$\square$

Notation

For this reason, it is common to order the eigenvalues of $A \in \mathbf{S}^n$ by magnitude with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. λ_1 is sometimes denoted $\lambda_{\max}$ and λ_n is sometimes denoted $\lambda_{\min}$.

Optimization implication

If $z = \alpha x$, then $z^\top A z = \alpha^2 x^\top A x$. Consider finding $x \in \mathbf{R}^n$ to maximize

$$\text{maximize} \quad x^\top A x$$
$$\text{subject to} \quad \|x\| = 1.$$

Since the objective is $x^\top A x \leq \lambda_1$ for all $x \in \mathbf{R}^n$ with $\|x\| = 1$, a solution of this problem is the eigenvector $q_1 \in \mathbf{R}^n$ corresponding to λ_1. In other words, these inequalities are tight.

Quadratic Form Inequalities

Quadratic Forms

Symmetric Real Matrix Eigenvalues — Matrix Trace — Real Matrix-Matrix Products

Eigenvalues and Eigenvectors — Symmetric Real Matrices

Monic Polynomials — Matrix Determinants — Matrix Transpose — Real Matrix-Vector Products

Norms — Real Matrix Determinants — Vectors as Matrices

Functionals — Permutations — Real Matrices — Real Inner Product — Real Linear Combinations

Vectors — Real Linear Equations — Multivariate Real Polynomials — Real Norm — Space Inner Product

Linear Function — Arrays — Multivariate Functions — Real Vectors — Space Norm — Plane Inner Product

N-Dimensional Space — Space Vectors — Plane Vector Sums and Differences — Plane Norm — Right Triangle Sides Relation

Fields — Groups — Polynomials — Absolute Value — Distance — Plane Vectors

Rings — Space Distance — Real Squares

Real Polynomials — Real Arithmetic — Plane Distance

Real Multiplicative Inverses — Interval Length — Real Space

Real Products — Intervals — Real Line

Real Summation — Real Additive Inverses

Rational Arithmetic — Real Sums — Rational Multiplicative Inverses — Real Order — Integral Line

Natural Summation — Rational Sums — Rational Products — Real Numbers — Rational Order — Geometry

Lists — Rational Numbers — Square Numbers

Set Numbers — Direct Products — Squares

Finite Sets — Integer Arithmetic — Natural Fractions

Equivalent Sets — Integer Products — Natural Multiplicative Identity — Natural Additive Identity — Inverse Elements

Natural Equations — Family Unions and Intersections — Family Operations — Function Inverses — Integer Sums — Identity Elements — Natural Arithmetic — Element Functions — Integer Order — Comparisons

Equations — Families — Function Composites — Function Images — Integer Numbers — Operations — Natural Powers — Orders — Converse Relations

Natural Order — Equivalence Relations — Functions — Natural Products

Natural Sums — Relations

Recursion Theorem

Peano Axioms — Ordered Pair Projections

Natural Induction — Set Products

Generalized Set Dualities — Natural Numbers — Set Powers — Set Symmetric Differences

Set Unions and Intersections — Set Dualities — Successor Sets — Intersection of Empty Set — Unordered Triples — Partitions — Ordered Pairs

Pair Unions — Set Intersections — Set Complements

Pair Intersections — Set Differences — Set Unions

Empty Set — Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities — Sets

Names

Letters — Objects