



# Simple Integral Additivity

## 1 Why

If we stack two rectangles, with equal base lengths but different heights, on top of each other, the additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property, as it ought to.

## 2 Result

**Proposition 1.** *The simple non-negative integral operator is additive.*

*Proof.* Let  $(X, \mathcal{A}, \mu)$  be a measure space. Let  $\mathcal{SF}_+(X)$  denote the non-negative real-valued simple functions on  $X$ . Define  $s : \mathcal{SF}_+(X) \rightarrow [0, \infty]$  by  $s(f) = \int f d\mu$  for  $f \in \mathcal{SF}_+(X)$ .

In this notation, we want to show that  $s(f + g) = s(f) + s(g)$  for all  $f, g \in \mathcal{SF}_+(X)$ . Toward this end, let  $f, g \in \mathcal{SF}_+(X)$  with the simple partitions:

$$\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n \subset \mathcal{A} \quad \text{and} \quad \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset [0, \infty].$$

We consider the refinement of the two partitions. TODO: this is why you don't do the unique maximal partition business.  $\{A_i \cap B_j\}_{i,j=1}^{i=m, j=n}$ .

First, let  $\alpha \in (0, \infty)$ . Then  $\alpha f \in \mathcal{SF}_+(X)$ , with the simple partition  $\{A_n\} \subset \mathcal{A}$  and  $\{\alpha a_n\} \subset [0, \infty]$ .

$$s(\alpha f) = \sum_{i=1}^n \alpha a_n \mu(A_i) = \alpha \sum_{i=1}^n a_n \mu(A_i) = \alpha s(f).$$

If  $\alpha = 0$ , then  $\alpha f$  is uniformly zero; it is the non-negative simple with partition  $\{X\}$  and  $\{0\}$ . Regardless of the measure of  $X$ , this non-negative simple function is zero. Recall that we define  $0 \cdot \infty = \infty \cdot 0 = 0$ .  $\square$