



# Product Sections

## 1 Why

Toward a theory of iterated integrals, we need to generalize rectangular strips to arbitrary products.

## 2 Definition

Consider the product of two non-empty sets.

First, consider a subset of this product. For a specified element in the first set, the **set section** of the subset with respect to that element is the set of elements in the second set for which the ordered pair of the specified element and that element is in the subset; the section is a subset of the second set. For elements of the second set, we define sections similarly.

Second, consider a function on the product. For a specified element in the first set, the **function section** of the function for that element is the function from the second set to the codomain of the function which maps elements of the second set to the result of the function applied to the ordered pair of the specified element and the element of the second set. For elements of the second set, we define sections similarly.

## 2.1 Notation

Let  $X, Y$  be non-empty sets.

Let  $E \subset X \times Y$ . For  $x \in X$ , we denote the section of  $E$  with respect to  $x$  by  $E_x$ . For  $y \in Y$ , we denote the section of  $E$  with respect to  $y$  by  $E^y$ . For every  $x \in X$  and  $y \in Y$ ,

$$E_x = \{y \in Y \mid (x, y) \in E\} \quad \text{and} \quad E^y = \{x \in X \mid (x, y) \in E\}.$$

$E_x \subset Y$  and  $E^y \subset X$ .

Let  $f : X \times Y \rightarrow Z$ . For  $x \in X$ , we denote the section of  $f$  with respect to  $x$  by  $f_x : Y \rightarrow Z$ . For  $y \in Y$ , we denote the section of  $f$  with respect to  $y$  by  $f^y : X \rightarrow Z$ . For every  $x \in X$  and  $y \in Y$ ,

$$f_x(y) = f(x, y) \quad \text{and} \quad f^y(x) = f(x, y).$$