



Why

We can approximate a density with a tree density similar to how we can approximate a distribution with a tree distribution.

Definition

We use the differential relative entropy as a criterion of approximation. An *optimal tree approximator* of density for a tree is a density which factors according to a tree and minimizes its differential relative entropy with the given density.

Notation

Let $g : \mathbf{R}^n \rightarrow \mathbf{R}$ be a density and T be a tree on $\{1, \dots, n\}$. An optimal tree approximator of g for T is a density f that factors according to T and minimizes $d(g, f)$. In other words, given g and T we want to find f to

$$\begin{aligned} & \text{minimize} && d(g, f) \\ & \text{subject to} && f \text{ factors according to } T. \end{aligned}$$

Result

PROPOSITION 1. *Let $g : \mathbf{R}^n \rightarrow \mathbf{R}$ be a density and T be a tree on $\{1, \dots, n\}$. The density $f_T^* : \mathbf{R}^d \rightarrow \mathbf{R}$ defined by*

$$f_T^* = g_1 \prod_{i \neq 1} g_{i|\mathbf{pa}_i}$$

minimizes the differential relative entropy with g among all densities on \mathbf{R}^n which factor according to T (\mathbf{pa}_i is the parent of i in T rooted at vertex 1, $i = 2, \dots, n$).

Proof. Let $f : \mathbf{R}^d \rightarrow \mathbf{R}$ be a density factoring according to T . First, express

$$f = f_1 \prod_{i=1} f_{i|\mathbf{pa}_i}.$$

Second, recall that $d(g, f) = h(g, f) - h(g)$. Since $h(g)$ does not depend on f , f is a minimizer of $d(g, f)$ if and only if f is a minimizer of $h(g, f)$.

Third, express

$$\begin{aligned} h(g, f) &= - \int_{\mathbf{R}^d} g \log f \\ &= - \int_{\mathbf{R}^d} g(x) \left(\log f_1(x_1) + \sum_{i \neq 1} \log f_{i|\mathbf{pa}_i}(x_i, x_{\mathbf{pa}_i}) \right) dx \\ &= h(g_1, f_1) + \sum_{i \neq 1} \left(\int_{\mathbf{R}} g_{\mathbf{pa}_i}(\xi) h(g_{i|\mathbf{pa}_i}(\cdot, \xi), f_{i|\mathbf{pa}_i}(\cdot, \xi)) d\xi \right) \end{aligned}$$

which separates across f_1 and $f_{i|\mathbf{pa}_i}(\cdot, \xi)$ for $i = 1, \dots, n$ and $\xi \in \mathbf{R}$. In particular, since $g_{\mathbf{pa}_i} \geq 0$, we can minimize the integrand pointwise.

Fourth, recall $h(\phi, \psi) \geq 0$ for densities ϕ, ψ of any dimension, and zero if $\phi = \psi$. So $f_1 = g_1$ and $f_{i|\mathbf{pa}_i} = g_{i|\mathbf{pa}_i}$ are solutions.

□

