Why

The (surprising fact) is that the operation of going from an index list to its induced sublist is *linear*, if the elements of the list are over a field, and thus may be viewed as a vector space.

Definition

The *index matrix* associated with the index sequence α of length r and order n (recall, $r \leq n$) is the $r \times n$ matrix whose i, jth entry is 1 if the sequence's ith coordinate is j, and 0 otherwise.

Examples

Here are some order-5 index lists: $(1, 2, 3)$, $(3, 2, 1)$, $(4, 5, 1)$, $(5, 4, 3, 2, 1)$, $(3,)$.

The matrices corresponding to these examples are:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

for the first two examples, and

$$
\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}
$$

for the last three.

In this case, we refer to the induced sublist as an induced *subvector*. The value of index matrices is that they give induced subvectors via the usual and familiar operation of matrix multiplication. The *subvector* of an n-vector associated with a length-r index sequence is the product of the sequence's $r \times n$ corresponding index matrix with the n-dimensional vector.

For example, define $x = \begin{bmatrix} 6 & 4 & 5 & 3 & 9 \end{bmatrix}^\top$. Then the subvector of x associated with the index sequence $(3, 2, 1)$ is the vector $\begin{bmatrix} 3 & 9 & 6 \end{bmatrix}^\top \in \mathbf{R}^3$, because

$$\begin{bmatrix} 3 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \\ 3 \\ 9 \end{bmatrix}$$

If $r = n$ then the index matrix is a permutation matrix.

Notation

Let $r \le n$ be natural numbers. Let $\alpha : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$ be an index sequence. Denote the index matrix associated with α by P_α. This matrix P_α is an element of $\mathbf{N}^{r \times n}$ and is defined by

$$(P_a)_{ij} = \begin{cases} 1 & j = \alpha(i) \\ 0 & \text{otherwise} \end{cases}$$

Let A be a nonempty set and let $x \in A^n$. then the subvector of x associated with P_α (and so with α) is

$$P_\alpha x = \left(x_{\alpha(1)}, \ldots, x_{\alpha(r)} \right)$$

We denote the product $P_\alpha x$ by x_α.

We denote the product $P_\alpha X P_\alpha^\top$ by $X_{\alpha\alpha}$.

Index Matrices

Permutation Matrices

Real Matrix Inverses

Matrix Transpose

Index Lists

Real Matrix-Matrix Products

Matrices

Real Matrix-Vector Products

Vectors as Matrices

Real Matrices

Real Linear Combinations

Real Inner Product

Real Linear Equations

Vectors

Real Norm

Space Inner Product

Arrays

Linear Functions

Real Vectors

Space Norm

N-Dimensional Space

Plane Vector Sums and Differences

Space Vector

Plane Norm

Plane Inner Product

Distance

Plane Vector

Absolute Value

Right Triangle Sides Relation

Space Distance

Plane Distance

Real Square

Real Arithmetic

Real Multiplicative Inverses

Interval Length

Real Products

Real Space

Real Plane

Real Line

Square Numbers

Real Additive Inverses

Real Summation

Intervals

Permutations

Natural Summation

Lists

Integer Line

Square

Real Order

Real Sums

Rational Arithmetic

Rational Multiplication Inverses

Groups

Set Numbers

Fields

Geometry

Rational Order

Rational Number

Real Numbers

Rational Sum

Rational Products

Inverse Elements

Finite Sets

Direct Products

Rings

Integer Arithmetic

Natural Fractions

Element Functions

Equivalent Sets

Natural Multiplicative Identity

Natural Additive Identity

Integer Products

Family Unions and Intersections

Family Operations

Comparisons

Natural Equations

Function Inverses

Integer Order

Identity Elements

Natural Arithmetic

Integer Sums

Families

Orders

Equation

Function Images

Function Composites

Operations

Integer Numbers

Natural Powers

Natural Order

Equivalence Relations

Converse Relations

Functions

Natural Products

Relations

Natural Sums

Recursion Theorem

Ordered Pair Projections

Peano Axioms

Set Products

Natural Induction

Generalized Set Dualities

Set Powers

Natural Numbers

Set Unions and Intersections

Partitions

Intersection of Empty Set

Set Dualities

Unordered Triples

Successor Sets

Set Symmetric Differences

Set Intersections

Pair Unions

Set Complements

Ordered Pairs

Pair Intersections

Set Unions

Set Differences

Empty Set

Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects