Why

When is a linear transformation between V and W one-to-one? In other words, when does

$$Tx = Ty \Rightarrow x = y$$

Rearranging, and using additivity, we ask when

$$T(x - y) = 0 \Rightarrow x - y = 0$$

Clearly we are interested in vectors z for which $Tz = 0$.

Definition

Suppose $T \in \mathcal{L}(V, W)$. The *null space* (or *kernel*) of T is the set of vectors in V which are mapped to 0 under T. In symbols, the null space of T is the set

$$T = \{v \in V \mid Tv = 0\}$$

The word "null" means "zero" in German.

A subspace

Why use the term *space*? Well, T is a *subspace* of V.

Proposition 1. *Suppose $T \in \mathcal{L}(V, W)$. Then (T) is a subspace of V.*

Proof. We verify that (T) contains 0 and is closed under vector addition and scalar multiplication. First, $0 \in T$ since $T0 = 0$ by homogeneity. Second, by additivity, if $x, y \in T$, then

$$T(x + y) = Tx + Ty = 0 + 0 = 0$$

Third, if $u \in T$ and $\alpha \in \mathbf{F}$, then

$$T(\lambda u) = \lambda(Tu) = \lambda 0 = 0$$

□

Characterization of injectivity

Proposition 2. *Suppose $T \in \mathcal{L}(V, W)$. Then*

$$T = \{0\} \longleftrightarrow T \text{ is one-to-one}$$

If $T = \{0\}$ we say that T has *zero nullspace* or *trivial nullspace.*

Examples

Zero map. Suppose T is the zero map from V to W. In other words,

$$Tv = 0 \quad \text{for all } v \in V$$

Then $T = V$. I.e., the null space is the whole space.

Simple function on $\mathbf{C}^3$. Define $\phi \in \mathcal{L}(\mathbf{C}^3, \mathbf{C})$ by

$$\phi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$$

Then ϕ is

$$\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1 + 2z_2 + 3z_3 = 0\}$$

This is the *solution set* of a linear equation.

Polynomial differentiation. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ is the linear map defined by

$$Dp = p' \quad \text{for all } p \in \mathcal{P}(\mathbf{R})$$

In other words, Dp is the derivative of the polynomial p. Then (D) is the set of constant functions.

Multiplication by x^2. Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbf{R} \text{ and } p \in \mathcal{P}(\mathbf{R})$$

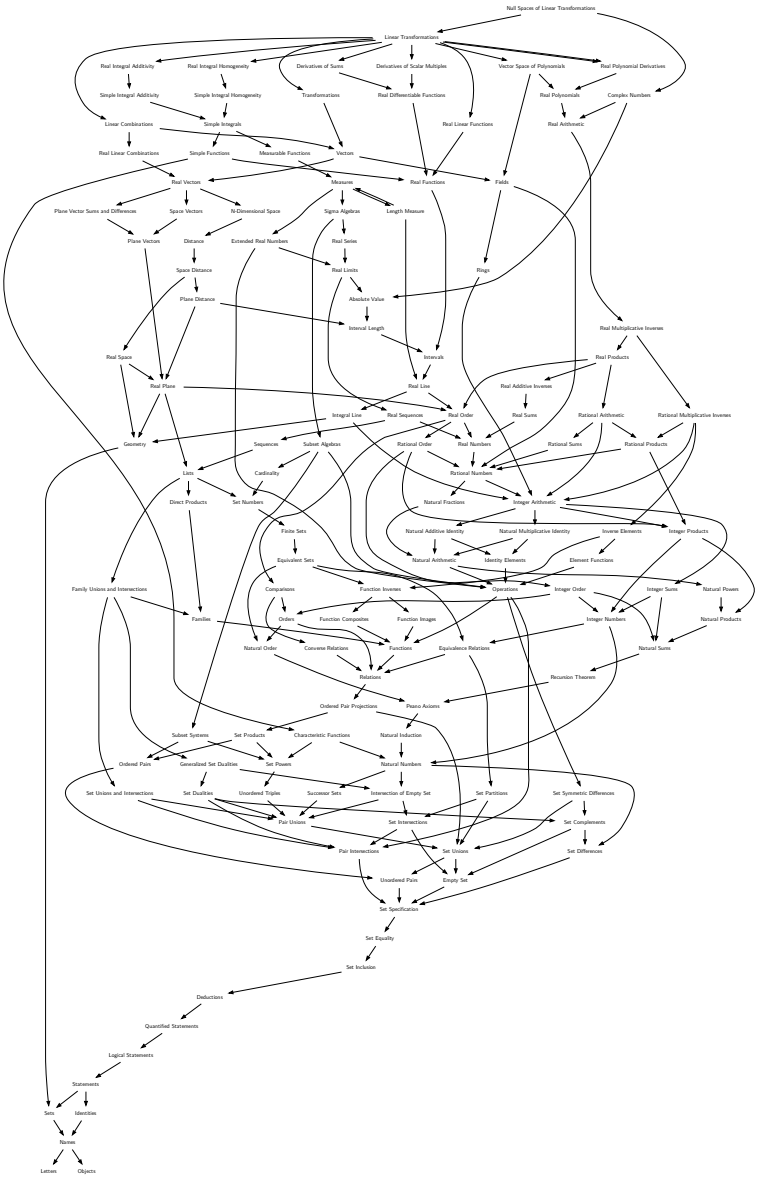
Then $(T) = \{0\}$, since no other polynomial satisfies $x^2 p(x) = 0$ for all $x \in \mathbf{R}$.

Backward shift. Define $T \in \mathcal{L}(\mathbf{F}^{\mathbf{N}}, \mathbf{F}^{\mathbf{N}})$ by

$$T(x_1, x_2, x_3, \ldots,) = (x_2, x_3, \ldots,)$$

so that T is the backward shift Then $T(x_1, x_2, x_3, \dots) = 0$ if and only $x_2 = x_3 = \cdots = 0$. So

$$T = \{(\alpha, 0, 0, \dots,) \mid \alpha \in \mathbf{F}\}$$


Null Spaces of Linear Transformations
Linear Transformations
Real Integral Additivity
Real Integral Homogeneity
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Absolute Value
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Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Integral Line
Real Sequences
Real Order
Real Sums
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Set Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
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