



Pointwise vs Measure Limits

1 Why

How does convergence pointwise (or almost everywhere pointwise) relate to convergence in measure?

2 Results

Proposition 1. *There exists a measure space and a sequence of measurable real-valued functions on that space converging everywhere (and so almost everywhere) but not converging in measure.*

Proposition 2. *There exists a measure space and a sequence of measurable real-valued functions on that space converging in measure but not converging almost everywhere (nor everywhere).*

Proposition 3. *On finite measure spaces, all sequences of measurable real-valued functions converging almost everywhere converge in measure.*

Proof. Let (X, \mathcal{A}, μ) be a measure space. Let $(f_n)_n$ be a sequence of measurable functions on X such that $f_n \rightarrow f$ almost everywhere. Let $\varepsilon > 0$.

For each $x \in X$, if $|f_n(x) - f(x)| > \varepsilon$ for infinitely many n , then $f_n(x) \not\rightarrow f(x)$. Let A be the set of such x . and let $B = \{x \in X \mid f_n(x) \not\rightarrow f(x)\}$. A is a subset of B . The measure of B is

zero since $f_n \rightarrow f$. Use the the monotonicity of measure to conclude. $\mu(A) \leq \mu(B) = 0$. Since $\mu(A) \geq 0$, $\mu(A) = 0$.

For natural k , let E_k be the $\{x \in X \mid |f_k(x) - f(x)| > \varepsilon\}$. Then $x \in A$ means that for every natural n , there exists a $k \geq n$ such that $x \in E_k$. In particular, for every n , x is in $\cup_{k=n}^{\infty} E_k$; denote this set by B_n . If x is in B_n for every n , then $x \in \cap_{n=1}^{\infty} B_n$. So we can write

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} B_n.$$

The sequence of sets $(B_n)_n$ is decreasing. So since μ is finite,

$$\lim_{n \rightarrow \infty} \mu(B_n) = \mu(A) = 0.$$

For every n , the set B_n contains $\{x \in X \mid |f_n(x) - f(x)| > \varepsilon\}$, namely E_n , the first set in the union. So then $\mu(E_n) \leq \mu(B_n)$ by monotonicity and so

$$0 \leq \lim_{n \rightarrow \infty} \mu(E_n) \leq \lim_{n \rightarrow \infty} \mu(B_n) = 0,$$

and we conclude $\lim_n \mu(E_n) = 0$. Since ε was arbitrary, we conclude $f_n \rightarrow f$ in measure.

□

Proposition 4. *On any measure space, for a sequence of measurable real-valued functions converging in measure to a measurable real-valued limit function, there exists a subsequence converging to the limit function almost everywhere.*

Proof. Let (X, \mathcal{A}, μ) be a measure space. Let $(f_n)_n$ be a sequence of measurable functions on X such that $f_n \rightarrow f$ in measure.

There exists n_1 so that

$$\mu(\{x \in X \mid |f_{n_1}(x) - f(x)| > 1\}) < \frac{1}{2}.$$

Can find $n_2 > n_1$ so that

$$\mu\left(\left\{x \in X \mid |f_{n_2}(x) - f(x)| > \frac{1}{2}\right\}\right) < \frac{1}{4}.$$

We can inductively find a sequence $\{n_k\}_k$ so that:

$$\mu\left(\left\{x \in X \mid |f_{n_k}(x) - f(x)| > \frac{1}{k}\right\}\right) \leq \frac{1}{2^k}.$$

□