



PIVOTED ROW REDUCTIONS

Why

We want to modify ordinary row reduction to handle the case in which a pivot is zero by selecting another suitable pivot.

Example

Let $A \in \mathbf{R}^{5 \times 5}$. If $A_{11} \neq 0$, we may subtract multiples of row 1 from row $2, \dots, 5$ to eliminate variable x_1 from those equations. If A reduces to $C \in \mathbf{R}^{5 \times 5}$ and $C_{22} \neq 0$, then step 2 moves from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & C_{22} & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \text{ to } \begin{bmatrix} \times & \times & \times & \times & \times \\ & C_{22} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \end{bmatrix}.$$

What if $C_{22} = 0$? In this case suppose we pick a different row. For example, if $C_{42} \neq 0$ we can move from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & C_{42} & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \text{ to } \begin{bmatrix} \times & \times & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & C_{42} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \end{bmatrix}.$$

Alternatively, we could introduce zeros in column 3 rather than column 2. For example, if we pick the pivot C_{43} we move from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & C_{43} & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \text{ to } \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \mathbf{0} & \times & \times \\ \times & \times & \mathbf{0} & \times & \times \\ \times & C_{43} & \times & \times & \times \\ \times & \times & \mathbf{0} & \times & \times \end{bmatrix}.$$

We can choose any nonzero entry in $C_{k:m,k:m}$ as the pivot.

Suppose we pick pivot $C_{st} \neq 0$ for $k \leq s, t \leq m$. Define \tilde{C} by swapping row s of C with row k of C and column t of C

with column k of C . Then $\tilde{C}_{kk} = C_{st} \neq 0$ and there exists an ordinary row reduction for \tilde{C} . We call this reduction of (\tilde{C}, \tilde{d}) a *pivoted row reduction* of C or the *st-reduction* of C .

If all remaining pivots are zero, then there is no viable pivot. In this case, at least one variable is free and we do not have a unique solution. For convenience, in this case, we still call the system an *st-reduction* of itself.

Definition

At step k of ordinary elimination, multiples of row k are subtracted from rows $k + 1, \dots, m$ to introduce zeros in entry k of the rows. If we denote the matrix at the beginning of that step by X , then row k of X , column k of X and especially the pivot X_{kk} play a role. Ordinarily, we subtract from every entry in the submatrix $X_{k+1:m, k:m}$ the product of a number in row k and a number in column k , divided by the pivot X_{kk} . Generally, however, we can choose as pivot any nonzero entry of $X_{k:m, k:m}$.

An m -variable system (A, b) is *pivot reducible* (or *reducible*) if there exists a sequence of systems S_1, \dots, S_{m-1} so that S_1 is a reduction of (A, b) and S_i is a reduction of S_{i-1} for $i = 1, \dots, m - 1$. We call S_{m-1} the *final reduction* (or *reduction*) of (A, b) . An immediate consequence of our definition is

Proposition 1. *All systems are reducible.*

