Why

We discuss a decomposition using eigenvalues and eigenvectors.[1]

Defining result

An *eigenvalue decomposition* of a matrix $A \in \mathbf{R}^{n \times n}$ is an ordered pair (X, Λ) in which X is invertible, Λ is diagonal, and $A = X\Lambda X^{-1}$.

In this case, $AX = X\Lambda$, in other words,

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

in which x_i is the ith column of X and λ_i is the ith diagonal element of Λ. We have $Ax_i = \lambda_i x_i$ for $i = 1, \ldots, n$. In other words, the ith column of X is an eigenvector of A and the jth entry of Λ is the corresponding eigenvalue.

If X is orthonormal, so that $X^{-1} = X^\top$, then we can interpret such a decomposition as a change of basis to *eigenvector coordinates*. If $Ax = b$, and $A = X\Lambda X^{-1}$ then $(X^{-1}b) = \Lambda(X^{-1}x)$. Here, $X^{-1}x$ expands x is the basis of columns of X. So to compute Ax , we first expand into the basis of columns of X, scale by Λ, and then interpret the result as the coefficients of a linear combination of the columns of X.

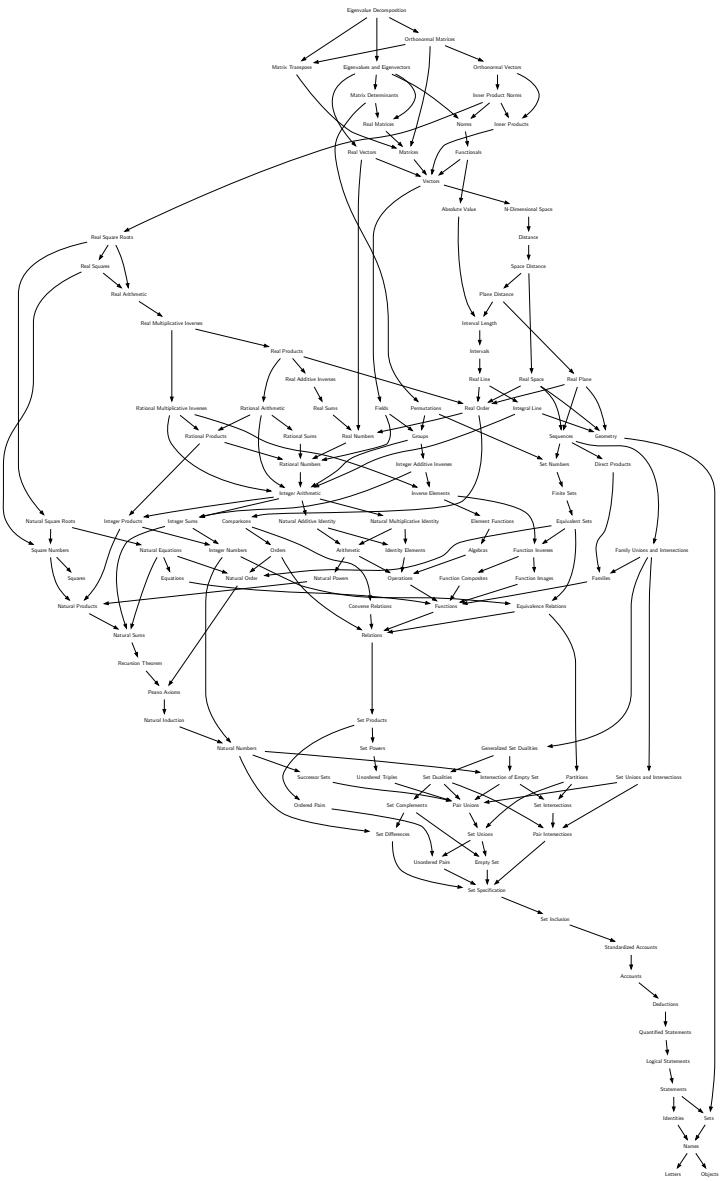
[1] Future editions will expand.

In this case that $A = X\Lambda X^\top$ for an eigenvalue decomposition (X, Λ) of A, we can also write

$$A = X\Lambda X^\top = \sum_{i=1}^{n} \Lambda_{ii} x_i x_i^\top .$$

Proposition 1. *Every real symmetric matrix has an eigenvalue decomposition (X, Λ) in which X is orthonormal.*[2]

[2] In future editions, this may be the motivating result for the definition of eigenvalues.


Eigenvalue Decomposition
Orthonormal Matrices
Matrix Transpose
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Inner Product Norms
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Absolute Value
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Real Square Roots
Distance
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Space Distance
Real Arithmetic
Plane Distance
Real Multiplicative Inverses
Interval Length
Real Products
Intervals
Real Additive Inverses
Real Line
Real Space
Real Plane
Rational Multiplicative Inverses
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