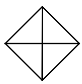


Why

Here's a nice (surprising) example of computing an event probability. Consider the following question: We have n letters to put into n addressed envelopes, but we *randomly* put them into envelopes. What's the chance that no letter is in the correct envelope?

Example

Let us first number the envelopes and letters. Next, suppose we model this uncertain outcome with the sample space $\Omega = S_n$. Here S_n denotes the symmetric group of degree n, as usual (see Permutations). We agree to interpret $\omega \in \Omega$ so that $\omega(i)$ is the number of the *letter* in the *envelope* numbered i, where $i = 1, \dots, n$. Suppose we put a distribution $p : \Omega \to [0, 1]$ on Ω so that every permutation is equally likely:

$$p(\omega) = \frac{1}{n!}$$

We are interested in the event W defined by

$$W = \{\omega \in \Omega \mid \omega(s) \neq s \text{ for all } s = 1, \dots, n\}$$

which we interpret as the event that no letter is in the correct envelope. To get a handle on this event, we express it as smaller events.

Define A_i by

$$A_i = \{\omega \in \Omega \mid \omega(i) = i\}$$

so that A_i is the set of outcomes in which letter i is in envelope i. The even that at least one letter goes into the correct envelope is given

$$\cup_{i=1}^{n} A_i$$

We can compute this probability using the genrealized inclusion-exclusion formula.

First, notice that the event

$$\cap_{i=1}^{n} A_i$$

contains the single outcome in which all letters go into the correct envelope. More generally, for any r between 1 and n, $\cap_{i=1}^n A_i$ contains all outcomes in which the letters $1, \ldots, r$ go into the correct envelope. What is the size of $A_1 \cap \cdots \cap A_r$? Given that the $\omega(1) = 1, \omega(2) = 2, \ldots, \omega(r) = r$, there are $n - r$ envelopes and $n - r$ ways of assigning letters to them. Thus, by the fundamental principle of counting

$$|\cap_{i=1}^r A_i| = (n-r)!$$

Thus the probability of the event is

$$P(\cap_{i=1}^r A_i) = \sum_{\omega \in \cap_{i=1}^r A_i} p(\omega) = \frac{(n-r)!}{n!}.$$

where we have used the fact that $p(\omega) = 1/n!$ for every $\omega \in \Omega$. A similar argument holds for any distinct $i_1, \ldots, i_r$ indices, where i_j are distinct integers between 1 and n. So $P(A_{i-1} \cap \cdots \cap A_{i_r}) = (n-r)!/n!$ Thus, each probability in the rth sum of the inclusion-exclusion formula is $(n-r)!/n!$, since the rth sum as $\binom{n}{r}$ terms, the rth sum is

$$\binom{n}{r} \frac{(n-r)!}{n!} = \frac{n!}{r!(n-r)!} \frac{(n-r)!}{n!} = \frac{1}{r!}$$

Finally, we apply the generalized inclusion-exclusian formula to obtain

$$P(A_1 \cup \cdots \cup A_n) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1} \frac{1}{n!}.$$

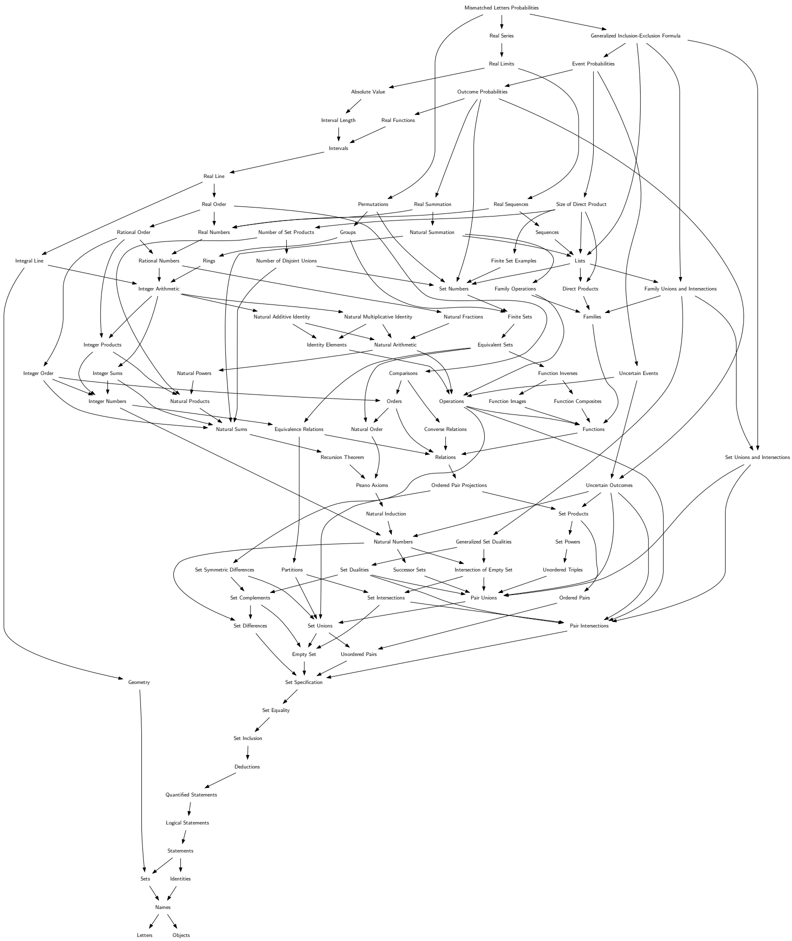
Hence, the probablty that no letter goes into the correct envelope $W = \Omega - \cup_{i=1}^n A_i$ is

$$1 - P(A_1 \cup \cdots \cup A_n) = 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

If we take $n \to \infty$, the above series series converges to $1/e \approx 0.37$.[1]

This is sometimes called the *secretary problem.*

[1]Future editions will define e.


Mismatched Letters Probabilities
Real Series
Generalized Inclusion-Exclusion Formula
Real Limits
Event Probabilities
Absolute Value
Outcome Probabilities
Interval Length
Real Functions
Intervals
Real Line
Real Order
Permutations
Real Summation
Real Sequences
Size of Direct Product
Rational Order
Real Numbers
Number of Set Products
Groups
Natural Summation
Sequences
Integral Line
Rational Numbers
Rings
Number of Disjoint Unions
Finite Set Examples
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Integer Arithmetic
Set Numbers
Family Operations
Direct Products
Family Unions and Intersections
Natural Additive Identity
Natural Multiplicative Identity
Natural Fractions
Finite Sets
Families
Integer Products
Identity Elements
Natural Arithmetic
Equivalent Sets
Integer Order
Integer Sums
Natural Powers
Comparisons
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Uncertain Events
Integer Numbers
Natural Products
Orders
Operations
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Function Composites
Natural Sums
Equivalence Relations
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Unordered Triples
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects