Why

We want to solve linear equations. Our approach is to "eliminate" variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

Two-variable example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$\begin{aligned} 3x_1 + 2x_2 &= 10, \text{ and} \\ 6x_1 + 5x_2 &= 20. \end{aligned}$$

We can list the coefficients in a two-dimensional array $A = (3, 2; 6, 5)$ and $b = (10, 20)$. We can eliminate x_1 from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$\begin{aligned} 3x_1 + 2x_2 &= 10 \text{ and} \\ x_2 &= 0. \end{aligned}$$

The key insight is that this system has the *same solution set*. We call the process of moving between these two systems a *row reduction*.

Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate x_1 (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear

system $S_1 = (A^1, b^1)$ in which

$$A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \text{ and } b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The trick is that, since $A'_{22} \neq 0$, we can take the same route to eliminate x_2, to obtain the system $S_2 = (A^2, b^2)$ in which

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \text{ and } b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Likewise for x_3, we obtain $S_3 = (A^3, b^3)$ in which

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one, (A^3, b^3), is easy to solve. We solve it by *back substitution*. First, since $2x_4 = 3$, we find $x_4 = 3/2$. Second, since $2x_3 + 2x_4 = -1$, we find $x_3 = -2$. Similarly we find $x_2 = 1/2$ and $x_3 = 5/4$.

Definition

Let $S = (A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system. The lower *row reduction* of S for index i with $A_{ii} \neq 0$ (or the i-row reduction) is the linear system $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$ if $i < s \leq m$ and A_{st} otherwise. We say that the system (A, b) is *ordinarily reducible*.

Let $a^k, \tilde{a}^k \in \mathbf{R}^n$ denote the kth row of A and $\tilde{A}$, respectively. Then if $k \neq i$, $\tilde{a}^k = a^k - \alpha_k a^i$ where $\alpha_k = A_{kj}/A_{ij}$. In other words, a row k

of the matrix $\tilde{A}$ is obtained by subtracting a multiple of the ith row of matrix A from row k of matrix A. We are "reducing" the rows of A.

Proposition 1. *Let $(A \in \mathbf{R}^{m\times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to (C, d). Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d).*[1]

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = \left(\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

We then subtract three times row 2 from row 3 and four times row 2 from row 4 to obtain

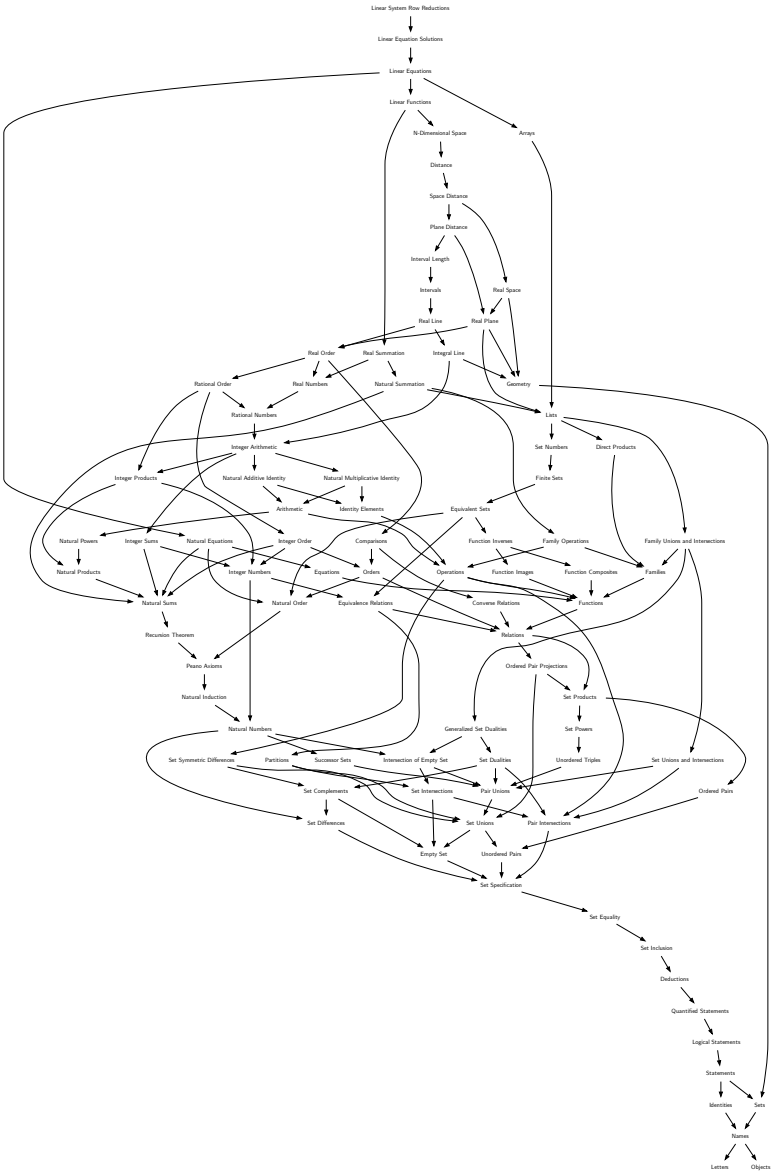
$$S_2 = \left(\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

Finally, we subtract two times row 3 from row 4 to obtain S_4, which we can write as

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ x_2 + x_3 + x_4 &= 0, \\ 2x_3 + 2x_4 &= -1, \quad \text{and} \\ 2x_4 &= 3. \end{aligned}$$

We can now back-substitute to find $x_4 = 3/2$, $x_3 = -2$, $x_2 = 1/2$ and $x_1 = 5/4$. The above proposition says that this is the only solution of S, as well.

[1] Future editions will include an account.


Linear System Row Reductions
Linear Equation Solutions
Linear Equations
Linear Functions
N-Dimensional Space
Arrays
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Real Summation
Integral Line
Rational Order
Real Numbers
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Integer Arithmetic
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Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Natural Powers
Integer Sums
Natural Equations
Integer Order
Comparisons
Function Inverses
Family Operations
Family Unions and Intersections
Natural Products
Integer Numbers
Equations
Orders
Operations
Function Images
Function Composites
Families
Natural Sums
Natural Order
Equivalence Relations
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects