

Why

How big can a quadratic form be? How small?

Result

Proposition 1. Suppose $A \in \mathbf{S}^n$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\lambda_n x^\top x \le x^\top A x \le \lambda_1 x^\top x,$$

for all $x \in \mathbb{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ with $A = Q\Lambda Q^{\top}$. Express

$$\begin{split} x^{\top} A x &= x^{\top} Q \Lambda Q^{\top} x = (Q^{\top} x)^{\top} \Lambda (Q^{\top} x) \\ &= \sum_{i=1}^{n} \lambda_{i} (q_{i}^{\top} x)^{2} \\ &= \lambda_{1} \sum_{i=1}^{n} (q_{i}^{\top} x) = \lambda_{1} \|Q^{\top} x\|^{2} = \lambda_{1} \|x\|^{2}. \end{split}$$

Similarly,

$$x^{\top} A x = \sum_{i=1}^{n} \lambda_i (q_i^{\top} x)^2$$

$$\geq \lambda_n \sum_{i=1}^{n} (q_i^{\top} x) = \lambda_n ||Q^{\top} x||^2 = \lambda_n ||x||^2.$$

Notation

For this reason, it is common to order the eigenvalues of $A \in \mathbf{S}^n$ by magnitude with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. λ_1 is sometimes denoted λ_{\max} and λ_n is sometimes denoted λ_{\min} .

Optimization implication

If $z = \alpha x$, then $z^{\top} A z = \alpha^2 x^{\top} A x$. Consider finding $x \in \mathbf{R}^n$ to maximize

Since the objective is $x^{\top}Ax \leq \lambda_1$ for all $x \in \mathbb{R}^n$ with ||x|| = 1, a solution of this problem is the eigenvector $q_1 \in \mathbb{R}^n$ corresponding to λ_1 . In other words, these inequalities are tight.

