Why

We can characterize the dependence of two events in terms of the rank of a particular matrix.

Definition

Given a probability measure $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ on the finite set Ω and two events $A, B \subset \Omega$, the *joint probability matrix* of A and B is the matrix

$$M = \begin{bmatrix} \mathbf{P}(A \cap B) & \mathbf{P}(A \cap C_\Omega(B)) \\ \mathbf{P}(C_\Omega(A) \cap B) & \mathbf{P}(C_\Omega(A) \cap C_\Omega(B)) \end{bmatrix}.$$

Characterization of independence

If A and B are independent, then so are A and $C_\Omega(B)$, B and $C_\Omega(A)$, and $C_\Omega(A)$ and $C_\Omega(B)$. In other words,

$$M = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_\Omega(A)) \end{bmatrix} \begin{bmatrix} \mathbf{P}(B) & \mathbf{P}(C_\Omega(B)) \end{bmatrix}.$$

In this case, we see that $\mathsf{rank}(M) = 1$.

Conversely, suppose $\mathsf{rank}(M) = 1$. Then, using the law of total probabiliy, each row is a multiple of

$$M1 = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_\Omega(A)) \end{bmatrix}.$$

In particular, we have $\mathbf{P}(A \cap B) = \alpha \mathbf{P}(A)$ and $\mathbf{P}(C_\Omega(A) \cap B) = \alpha P(C_\Omega(A))$. So

$$\mathbf{P}(A \cap B) + \mathbf{P}(C_\Omega(A) \cap B) = \alpha(\mathbf{P}(B) + \mathbf{P}(C_\Omega(A))),$$

from which we deduce $\alpha = \mathbf{P}(B)$ Likewise, the multiplier for the second column of M is $\mathbf{P}(C_\Omega(B))$. In other words, A and B are independent. We conclude that A and B are independent if and only if $\mathsf{rank}(M) = 1$.

Joint Probability Matrices

Real Matrix Rank — Independent Events

Real Subspace — Conditional Event Probabilities

Real Vector Span — Real Matrices

Real Linear Combinations — Linear Equations

Real Vector — Linear Functions — Arrays

Space Vectors — Plane Vector Sums and Differences — N-Dimensional Space

Plane Vectors — Distance

Space Distance

Plane Distance — Event Probabilities

Interval Length — Outcome Probabilities

Real Functions — Real Summation

Real Space — Intervals

Real Line — Real Plane

Integral Line — Geometry — Real Order — Size of Direct Product

Number of Set Products — Rational Order — Real Numbers — Natural Summation

Number of Disjoint Unions — Rational Numbers — Lists — Finite Set Examples

Integer Arithmetic — Set Numbers — Direct Products

Natural Multiplication Identity — Natural Additive Identity — Natural Fractions — Finite Sets

Integer Products — Identity Elements — Natural Arithmetic — Equivalent Sets

Natural Equations — Natural Powers — Integer Sums — Integer Order — Comparisons — Family Operations — Function Inverses — Uncertain Events — Family Unions and Intersections

Equations — Natural Products — Integer Numbers — Orders — Operations — Families — Function Composites — Function Images

Natural Order — Natural Sums — Equivalence Relations — Converse Relations — Functions

Recursion Theorem — Relations

Peano Axioms — Ordered Pair Projections — Uncertain Outcomes

Natural Induction — Natural Numbers — Generalized Set Dualities — Set Products — Set Powers

Set Symmetric Differences — Partitions — Set Dualities — Successor Sets — Intersection of Empty Set — Unordered Triples — Set Unions and Intersections

Set Complements — Set Intersections — Pair Unions — Ordered Pairs

Set Differences — Set Unions — Empty Set — Unordered Pairs — Pair Intersections

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects