Definition

Two subspaces $S, T \subset \mathbf{R}^n$ are *orthogonal* if

$$x^\top y = 0 \text{ for all } x \in S, y \in T.$$

For any set $S \subset \mathbf{R}^n$ (not necessarily a subspace), the *orthogonal complement* of S is the set

$$S^\perp = \{x \in \mathbf{R}^n \mid x^\top y = 0 \text{ for all } y \in S\}.$$

$S^\perp$ is the set of all vectors which are orthogonal to every vector in S.

Orthgonal complement is a subspace

Notice that $S^\perp$ is always a subspace. If $x \in S^\perp$, then $x^\top y = 0$ for all $y \in S$. So then $(\alpha x)^\top y = \alpha(x^\top y) = 0$ for all $\alpha \in \mathbf{R}$ and $y \in S$. We conclude $\alpha x \in S^\perp$ for all $\alpha \in \mathbf{R}$. In other words, $S^\perp$ is closed under scalar multiplication. If $x, z \in S^\perp$, then $(x + z)^\top y = x^\top y + z^\top y = 0 + 0 = 0$. We conclude that $x + z \in S^\perp$ for all $x, z \in S^\perp$. In other words, $S^\perp$ is closed under vector addition. Consequently, $S^\perp$ is a subspace.

Orthogonal Real Subspaces
Real Subspaces
Vectors as Matrices
Real Vectors Span
Real Inner Product
Real Matrices
Real Linear Combinations
Space Inner Product
Real Norm
Linear Equations
Real Vectors
Space Norm
Linear Functions
Arrays
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Plane Inner Product
Plane Norm
Right Triangle Side Relation
Distance
Plane Vectors
Absolute Value
Real Squares
Space Distance
Real Arithmetic
Plane Distance
Real Multiplicative Inverses
Interval Length
Real Products
Real Space
Intervals
Square Numbers
Real Additive Inverses
Real Line
Real Plane
Squares
Integral Line
Rational Multiplicative Inverses
Rational Arithmetic
Real Sums
Real Order
Real Summation
Geometry
Rational Products
Rational Sums
Rational Order
Real Numbers
Rational Numbers
Integer Arithmetic
Lists
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Set Numbers
Arithmetic
Identity Elements
Finite Sets
Direct Products
Natural Powers
Natural Summation
Equivalent Sets
Inverse Elements
Integer Sums
Natural Products
Integer Order
Natural Equations
Function Inverses
Element Functions
Family Operations
Comparisons
Family Unions and Intersections
Integer Numbers
Natural Sums
Orders
Equations
Function Composition
Function Images
Operations
Families
Recursion Theorem
Equivalence Relations
Natural Order
Functions
Converse Relations
Peano Axioms
Relations
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Duality
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Duality
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects