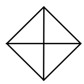


Why

We can characterize the dependence of two events in terms of the rank of a particular matrix.

Definition

Given a probability measure $\mathsf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ on the finite set Ω and two events $A, B \subset \Omega$, the *joint probability matrix* of A and B is the matrix

$$M = \begin{bmatrix} \mathsf{P}(A \cap B) & \mathsf{P}(A \cap C_\Omega(B)) \\ \mathsf{P}(C_\Omega(A) \cap B) & \mathsf{P}(C_\Omega(A) \cap C_\Omega(B)) \end{bmatrix}.$$

Characterization of independence

If A and B are independent, then so are A and $C_\Omega(B)$, B and $C_\Omega(A)$, and $C_\Omega(A)$ and$C_\Omega(B)$. In other words,

$$M = \begin{bmatrix} \mathsf{P}(A) \\ \mathsf{P}(C_\Omega(A)) \end{bmatrix} \begin{bmatrix} \mathsf{P}(B) & \mathsf{P}(C_\Omega(B) \end{bmatrix}.$$

In this case, we see that $\mathsf{rank}(M) = 1$.

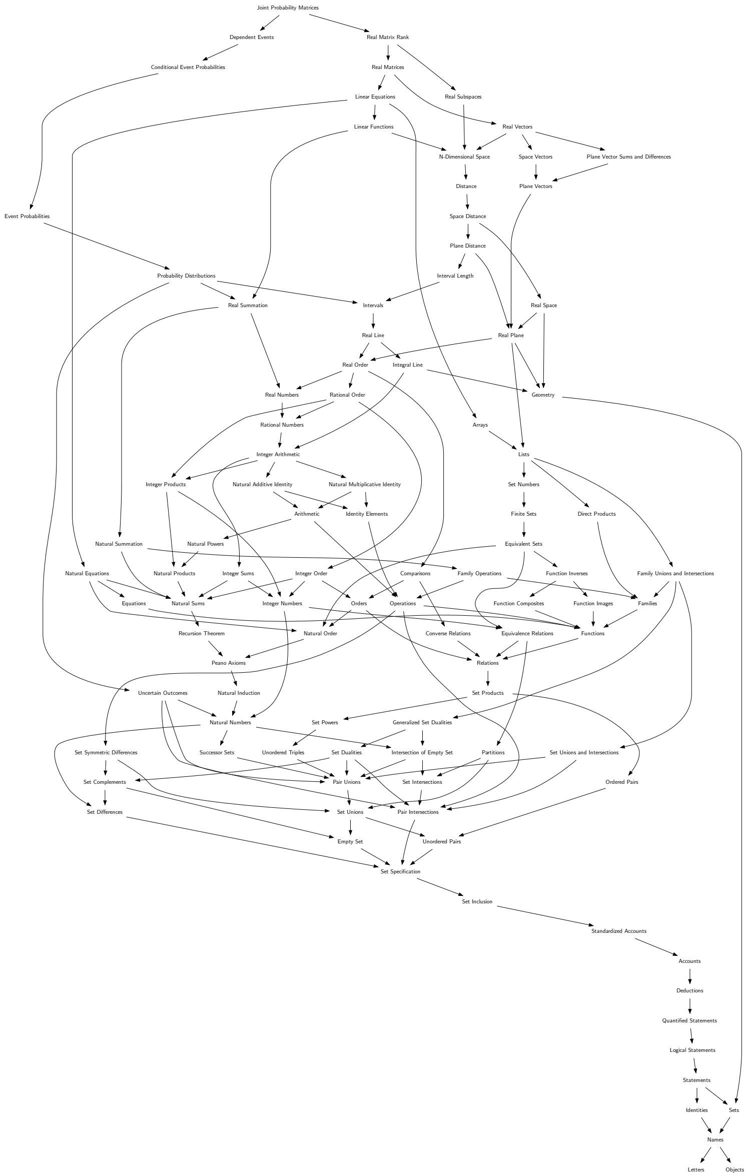
Conversely, suppose $\mathsf{rank}(M) = 1$. Then, using the law of total probabiliy, each row is a multiple of

$$M1 = \begin{bmatrix} \mathsf{P}(A) \\ \mathsf{P}(C_\Omega(A)) \end{bmatrix}.$$

In particular, we have $\mathsf{P}(A \cap B) = \alpha \mathsf{P}(A)$ and $\mathsf{P}(C_\Omega(A) \cap B) = \alpha P(C_\Omega(A))$. So

$$\mathsf{P}(A \cap B) + \mathsf{P}(C_\Omega(A) \cap B) = \alpha(\mathsf{P}(B) + \mathsf{P}(C_\Omega(A))),$$

from which we deduce $\alpha = \mathsf{P}(B)$ Likewise, the multiplier for the second column of M is $\mathsf{P}(C_\Omega(B))$. In other words, A and B are independent. We conclude that A and B are independent if and only if $\mathsf{rank}(M) = 1$.


Joint Probability Matrices
Dependent Events
Real Matrix Rank
Conditional Event Probabilities
Real Matrices
Linear Equations
Real Subspaces
Linear Functions
Real Vectors
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Distance
Plane Vectors
Event Probabilities
Space Distance
Plane Distance
Probability Distributions
Interval Length
Real Summation
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Real Numbers
Rational Order
Geometry
Rational Numbers
Arrays
Integer Arithmetic
Lists
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Set Numbers
Arithmetic
Identity Elements
Finite Sets
Direct Products
Natural Summation
Natural Powers
Equivalent Sets
Natural Equations
Natural Products
Integer Sums
Integer Order
Comparisons
Family Operations
Function Inverses
Family Unions and Intersections
Equations
Natural Sums
Integer Numbers
Orders
Operations
Function Composites
Function Images
Families
Recursion Theorem
Natural Order
Converse Relations
Equivalence Relations
Functions
Peano Axioms
Relations
Uncertain Outcomes
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects