



Why

We generalize the algebraic structure of addition and multiplication over the integers.¹

Definition

A *ring* (or *ring with identity*) (A, f, g) is a set A and two operations on A satisfying the following set of conditions.

(A) (i) f is associative. (ii) f is *commutative*, (iii) A has an identity for f (i.e., is $e \in A$ with $f(a, e) = f(e, a) = a$ for all $a \in A$) (iv) A has inverse elements for f (i.e., for any $a \in A$, there is \tilde{a} satisfying $f(a, \tilde{a}) = f(\tilde{a}, a) = e$)

(B) (i) g is associative; (ii) A has an identity element for g (i.e., there is $\tilde{e} \in A$ satisfying $g(a, \tilde{e}) = g(\tilde{e}, a) = a$)

(C) (i) g left distributes: $g(f(a, b), c) = f(g(a, c), g(b, c))$; (ii) g right distributes: $g(c, f(a, b)) = f(g(c, a), g(c, b))$.

Conditions (A) concern f , conditions (B) concern g , and conditions (C) relate the two. Define $\psi : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ by $\psi(a, b) = a + b$ and $\pi : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ by $\pi(a, b) = a \cdot b$. We have defined a ring so that (\mathbf{Z}, ψ, π) is one. The element referred to in (A.2) is $0 \in \mathbf{Z}$, so we refer to this element in any ring as the *additive identity*. That referred to (A.3) is $1 \in \mathbf{Z}$, so we refer to this element in any ring as the *multiplicative identity*. We refer to the elements mentioned in (A.4) as *additive inverses*. We call to f *ring addition* and g *ring multiplication*. Although integer products are commutative, we have not required this aspect (future editions will elaborate).

¹Future editions will likely modify this sheet, and give a genetic treatment involving the solution of polynomial equations by Galois.

Notation

The notation commonly adopted in discussing rings relies on analogy with the set of integers \mathbf{Z} . We denote the ring addition by $+$ and ring multiplication by \cdot . Moreover, we denote the ring's additive identity by 0 and the ring's multiplicative identity by 1 . Finally, we denote the additive inverse of $a \in A$ by $-a$.

Rewriting the conditions (A), (B), (C) in this notation gives familiar-looking relations, from when the objects involved were integers. (A) (1) $a + (b + c) = (a + b) + c$; (2) $a + b = b + a$; (3) $a + 0 = 0 + a = a$; (4) $a + (-a) = 0$. (B) (1) $a(bc) = (ab)c$; (2) $1a = a1 = a$. (C) (1) $(a + b)c = ac + bc$; (2) $c(a + b) = ca + cb$.

Immediate consequences

We need not require that $0x = 0$, because we can deduce it:

$$0x + x = (0 + 1)x = 1x = x.$$

Similarly, $(-a)b = -(ab)$ since

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

Other familiar relations among the integers, e.g. $(-a)(-b) = ab$, may be deduced.

