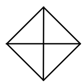


Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Suppose p is a distribution on a *finite* set of outcomes Ω. Given an event $E \subset \Omega$, the *probability of E under p* as the sum of the probabilities of the outcomes in E.

Notation

It is common to define a function $P : \mathcal{P}(\Omega) \to \mathbf{R}$ by

$$P(A) = \sum_{a \in A} p(a) \quad \text{for all } A \subset \Omega$$

We call this function P *the event probability function* (or *the probability measure*) associated with p. Since it depends on the sample space Ω and the distribution p, we occasionally denote this dependence by $P_{\Omega,p}$ or P_p.

It is tempting, and therefore common to write $P(\omega)$ when $\omega \in \Omega$ and one intends to denote $P(\{\omega\})$. Of course, this corresponds with $p(\omega)$. It is therefore easy to see that from P we can compute p, and vice versa.

Examples

Rolling a die. We consider the usual fair die model (see Outcome Probabilities). Here we have $\Omega = \{1, \dots, 6\}$ and a distribution $p : \Omega \to [0, 1]$ defined by

$$p(\omega) = 1/6 \quad \text{for all } \omega \in \Omega$$

Given the model, the probability of the event $E = \{2, 4, 6\}$ is

$$P(E) = \textstyle\sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of event probabilities

The properties of p ensure that P satisfies

1. $P(A) \geq 0$ for all $A \subset \Omega$;
2. $P(\Omega) = 1$ (and $P(\varnothing) = 0$);
3. $P(A) + P(B)$ for all $A, B \subset \Omega$ and $A \cap B = \varnothing$.

The last statement (3) follows from the more general identity

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

for $A, B \subset \Omega$, by using $\mathbf{P}(\varnothing) = 0$ of (2) above. These three conditions are sometimes called the *axioms of probability for finite sets.*

Do all such P satisfying (1)-(3) have a corresponding underlying probability distribution? The answer is easily seen to be yes. Suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satisfies (1)-(3). Define $q : \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an *event probability function* (or a *(finite) probability measure*).

Other basic consequences

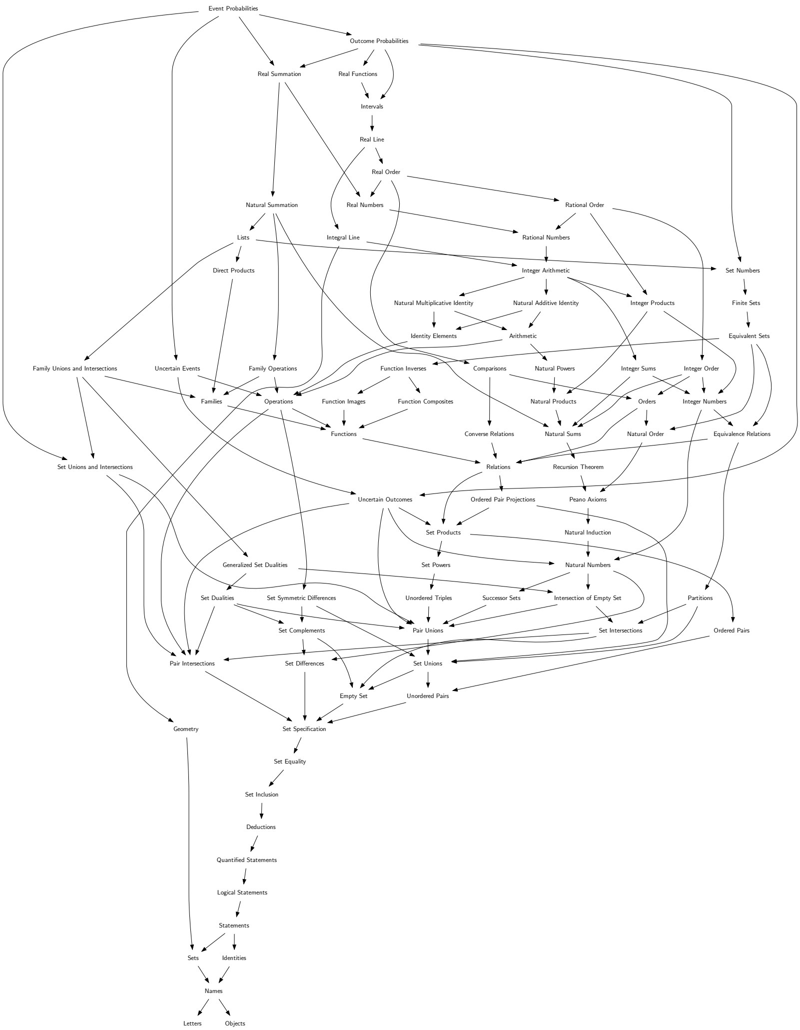
Probability by cases. Suppose $A_1, \dots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$P(B) = \textstyle\sum_{i=1}^{n} P(A_i \cap B).$$

Some authors call this the *law of total probability.* This is easy to see by using the distributive laws of set algebra (see Set Unions and Intersections).

Monotonicity. If $A \subseteq B$, then $P(A) \leq P(B)$. This is easy to see by splitting B into $A \cap B$ and $B - A$, and applying (1) and (3).

Subadditivity. For $A, B \subset \Omega$, $P(A \cup B) \leq P(A) + P(B)$. This is easy to see from the more general identity in (3) above. This is sometimes referred to as a *union bound*, in reference to *bounding* the quantity $P(A \cup B)$.


Event Probabilities
Outcome Probabilities
Real Summation
Real Functions
Intervals
Real Line
Real Order
Natural Summation
Real Numbers
Rational Order
Lists
Integral Line
Rational Numbers
Direct Products
Integer Arithmetic
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Family Unions and Intersections
Uncertain Events
Family Operations
Function Inverses
Comparisons
Natural Powers
Integer Sums
Integer Order
Families
Operations
Function Images
Function Composites
Natural Products
Orders
Integer Numbers
Functions
Converse Relations
Natural Sums
Natural Order
Equivalence Relations
Set Unions and Intersections
Relations
Recursion Theorem
Uncertain Outcomes
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Dualities
Set Symmetric Differences
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Set Complements
Pair Unions
Set Intersections
Ordered Pairs
Pair Intersections
Set Differences
Set Unions
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects