Tree-Structured Density Approximation

1 Why

2 Problem

Tree-structured density approximation is a mathematical optimization problem in which we find a tree-structured density which minimizes its differential entropy relative to a given density over the set of all densities which factor according to trees.

2.1 Notation

Let $g : \mathbf{R}^n \to \mathbf{R}$ be a density. Let d denote the differential relative entropy. We want to find a density $f : \mathbf{R}^n \to \mathbf{R}$ and tree T on $\{1, \ldots, n\}$ to

$$\begin{aligned} \text{minimize} \quad & d(q, p) \\ \text{subject to} \quad & p \text{ factors according to the tree } T. \end{aligned}$$

3 Solution

Proposition 1. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a density. Let T be a tree on $\{1, \ldots, n\}$. Let p_j be the parent of vertex j for the T rooted*

at vertex i, $j = 1, \ldots, n$ *and* $j \neq i$. *Then the density* $f : \mathbf{R}^d \to \mathbf{R}$ *defined by*

$$f = g_i \prod_{j \neq i} g_{j|p_j}$$

achieves minimum differential entropy relative to g *among all densities which factor according to* T.

Proposition 2. *Let* $g : \mathbf{R}^n \to \mathbf{R}$ *be a density. A tree* T *is a solution to the problem above if and only if it is a minimum spanning tree of the differential mutual information graph of* g.