

#### NORMAL RANDOM FUNCTIONS

# Why

We want to discuss real-valued random functions whose family of random variables have simple densities.<sup>1</sup>

## **Definition**

A normal random function is a real-valued random function whose family of real-valued random variables has the property that any subfamily is jointly normal.

For this reason, we call the family of random variables (or stochastic process) corresponding to the random function a *gaussian process* or *normal process*.

#### Notation

Let  $(\Omega, \mathcal{A}, \mathbf{P})$  be a probability space and A a set. Let  $x : \Omega \to (A \to \mathbf{R})$  be a random function with family  $y : A \to (\Omega \to \mathbf{R})$ .

The random function x is a normal if, for all  $a^1, \ldots, a^m \in A$ ,  $(y(a^1), \ldots, y(a^m))$  is jointly normal.

### Mean and covariance function

**Proposition 1.** Let  $x: \Omega \to (A \to \mathbf{R})$  be a normal random function with family  $X: A \to (\Omega \to \mathbf{R})$ . There exists unique functions  $m: A \to \mathbf{R}$  and  $k: A \times A \to \mathbf{R}$  so that the mean of the random variable  $X_a$  is m(a) for all A and the covariance of the random variables  $X_a$  and  $X_{a'}$  is k(a, a') for all  $a, a' \in A$ .

For this reason, we call m the mean function and k the covariance function of the random function.

Conversely, let  $m: A \to \mathbf{R}$  and  $k: A \times A \to \mathbf{R}$ . Then if k satisfies the

<sup>&</sup>lt;sup>1</sup>Future editions will expand.

<sup>&</sup>lt;sup>2</sup>Future editions may include an account.

property that for all  $a^1, \ldots, a^m$ , the  $m \times m$  matrix

$$\begin{pmatrix} k(a^1, a^1) & \cdots & k(a^1, a^m) \\ \vdots & \ddots & \vdots \\ k(a^m, a^1) & \cdots & k(a^m, a^m) \end{pmatrix}$$

is positive semidefinite, then we can construct a Gaussian process with mean function m and covariance function k.<sup>3</sup> For this reason, we call k with such a property positive semidefinite or a covariance function. Notice, of course, that k is symmetric. The matrix above is sometimes called the  $Gram\ matrix$  for k and  $a^1, \ldots, a^m$ .

## Example

Let  $A = \{1, ..., n\}$  and let  $K \in \mathbf{R}^{n \times n}$  be symmetric positive semidefinite. Define  $m: A \to \mathbf{R}$  to be  $m \equiv 0$  (the constant zero function) and  $k(i,j) = K_{ij}$ . Then the normal random function  $x: \Omega \to (A \to \mathbf{R})$  with mean m and covariance k is in one to one correspondence with the gaussian random vectors with mean zero.

<sup>&</sup>lt;sup>3</sup>Some authors belabor this point because of the natural inclination to want to specify an *inverse* covariance function, which need not satisfy the consistency property. The consistency property ensures that any marginal of a subfamilys density is the density of that further subfamily. Future editions may expand.

