The Bourbaki Project

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Contents

Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *symbol* is a written mark. A *script* is a collection symbols called *letters*. In *phonetic* languages the letters correspond to sounds and rules for composing these letters into successions called written words. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into written words which are meant to denote the spoken words of the English language. The written words on this page are several letters one after the other. For example, the word “word” is composed of the letters “w”, “o”, “r”, “d”.

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. The *lower case latin letters.*

$$\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \\ j & k & l & m & n & o & p & q & r \\ s & t & u & v & w & x & y & z & \end{array}$$

And the *upper case latin letters.*

$$\begin{array}{ccccccccc} A & B & C & D & E & F & G & H & I \\ J & K & L & M & N & O & P & Q & R \\ S & T & U & V & W & X & Y & Z & \end{array}$$

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We also use the *Arabic numerals.*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Other symbols

We also use the following symbols.

$$' \quad (\quad) \quad \{ \quad \} \quad \vee \quad \wedge \quad \neg \quad \forall \quad \exists \quad \longrightarrow \quad \longleftrightarrow \quad = \quad \in \quad \to$$

Letters (1) does not immediately need any sheet.

Letters (1) is immediately needed by:

Names (3)

Letters (1) gives the following terms.

language

affections

spoken word

symbol

script

letters

phonetic

lower case latin letters

upper case latin letters

Arabic numerals

Letters

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Objects (2) does not immediately need any sheet.

Objects (2) is immediately needed by:

Definitions (13)
Names (3)

Objects (2) gives the following terms.

object
tangible
intangible

Objects

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an *accent* $'$ or subscripts or superscripts. When we write the symbols we say that the composite symbol formed *denotes* the object. We call it the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{\mathrm{A}}$ or $\boxed{\mathrm{A'}}$ or $\boxed{\mathrm{A}_0}$. The box works well to group the symbols and clarifies that $\boxed{\mathrm{A}}\boxed{\mathrm{A}}$ is different from $\boxed{\mathrm{AA}}$. But experience shows that we need not use boxes.

We indicate a name for an object with italics. Instead of $\boxed{\mathrm{A'}}$ we use A', instead of $\boxed{\mathrm{A}_0}$ we use A_0. Experience shows that this subtlety is enough for clarity and it agrees with traditional and modern practice. Other examples include A'', A''', A'''', B, C, D, E, F, f, f' f_a.

No repetitions

We never use the same name to refer to two different objects. Using the same name for two different objects causes confusion. We make clear when we reuse symbols to mean different objects. We tend to introduce the names used at the beginning of a paragraph or section.

Names are objects

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can reference themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the marks on the page which are meant to look similar are the several uses of a name.

Names as placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is best understood by example.

Names (3) immediately needs:

Letters (1)

Objects (2)

Names (3) is immediately needed by:

Identities (4)

Sets (5)

Names (3) gives the following terms.

accent

denotes

name

assertion

names

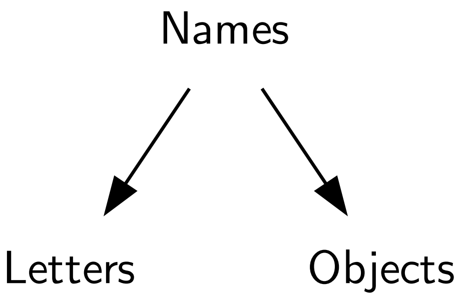
accent

letter

terms

relations

placeholder


Names
Letters
Objects

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

Symmetry

Let A denote an object and let B denote an object. "A equals B" means the same as "B equals A". The identity of the names is not dependent on the order in which the names are given. We call this the *symmetry of identity*. It means we can switch the spots of A and B and say the same thing. In other words, there are two ways to make the statement.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself. This fact is called the *reflexivity of identity*. A name is equal to itself because an object is itself.

Identities (4) immediately needs:

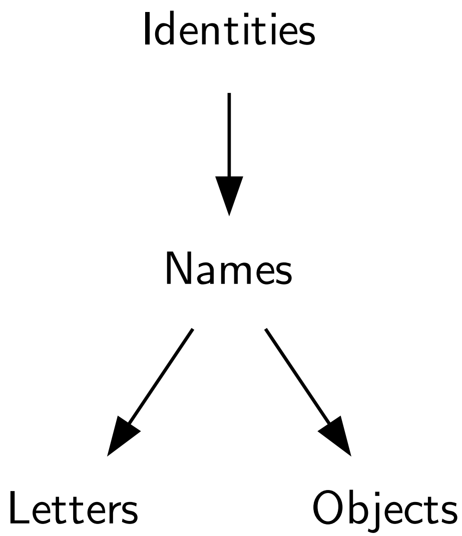
Names (3)

Identities (4) is immediately needed by:

Equation Solutions (**??**)
Statements (7)

Identities (4) gives the following terms.

is
equation
indeterminate
is
equal
name
identity
symmetry of identity
reflexivity of identity
reflexive
symmetric
transitive
equals
reflexive
symmetric
transitive


Identities
Names
Letters
Objects

Why

We want to talk about none, one, or several objects considered together, as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set*. We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

Principle 1 (Existence of Sets)**.** *Intangible groups exist.*

A set may have other sets as its members. This is subtle but becomes familiar. We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object belongs to the set. In other words, the object is a member or an element of the set. We say "The object denoted by a belongs to the set denoted by A".

Not symmetric

Notice that belonging is not symmetric. Saying "the object denoted by a belongs to the set denoted by A" does not mean the same as "the set denoted by A belongs to the object denoted by a" In fact, the latter sentence is nonsensical unless the object denoted by a is also a set.

Not transitive

Let a denote an object and let A and B both denote sets. If the object denoted by a is "a part of" a the set denoted by A, and the set denoted by A is "a part of" the set denoted by B, then usual English usage would suggest that a is "a part of" the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing. The relation of belonging is not quite this. If a thing is an element of a thing, that second thing may be an element of the third thing, but this does not mean that the the first thing is an element of the third thing.

Sets (5) immediately needs:

Names (3)

Sets (5) is immediately needed by:

Set Examples (6)

Statements (7)

Sets (5) gives the following terms.

set

belong

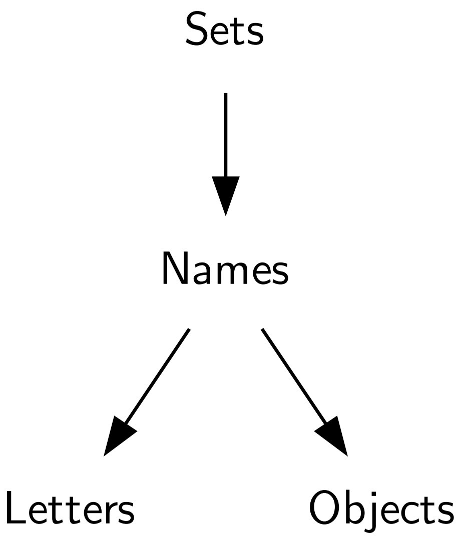
members

elements

in

empty

nonempty


Sets
Names
Letters
Objects

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

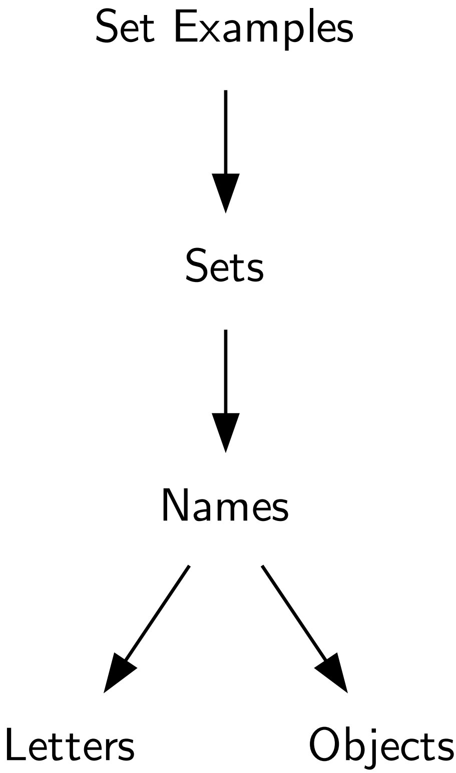
Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of lower case latin letters (introduced in Letters): a, b, c, . . . , x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Set Examples (6) immediately needs:

Sets (5)

Set Examples (6) is not immediately needed by any sheet.

Set Examples (6) gives no terms.


Set Examples
Sets
Names
Letters
Objects

Why

We want symbols to represent identity and belonging.

Definition

In the English language, nouns are words that name people, places and things. In these sheets, names (see Names) serve the role of nouns. In the English language, verbs are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings. As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense. A *statement* is a succession of symbols.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object (see Identities). We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And if we read it aloud, we would read a as "the object denoted by a", then $=$ as "is", then b as "the object denoted by b". Altogether then, "the object denoted by a is the object denoted by b." We might box these three symbols $\boxed{a = b}$ to make clear that they are meant to be read together, but experience shows that (as with English sentences and words) we do not need boxes.

The symbol $=$ is (appropriately) a symmetric symbol. If we flip it left and right, it is the same symbol. This reflects the symmetry of the English sentences represented (see Identities). The symbols $a = b$ mean the same as the symbols $b = a$.

Belonging

As a second example, consider the symbol $\in$. Let a denote an object and let A denote a set. We agree that $\in$ means "belongs to" in the sense of "is an element of" or "is a member of" (see Sets). Then we write $a \in A$. We read these symbols as "the object denoted by a belongs to the set denoted by A".[1]

The symbol $\in$ is not symmetric. If we flip it left and right it looks different. This reflects that $a \in A$ does not the mean the same as $A \in a$ (see Sets). As with english words, the order of symbols is significant. The word "word" is not the same as the word "drow". Our symbolism for belonging reflects the concept's lack of symmetry.

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[1]The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for the ancient Greek word ἐστί which means, roughly, "belongs". Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of".

Statements (7) immediately needs:

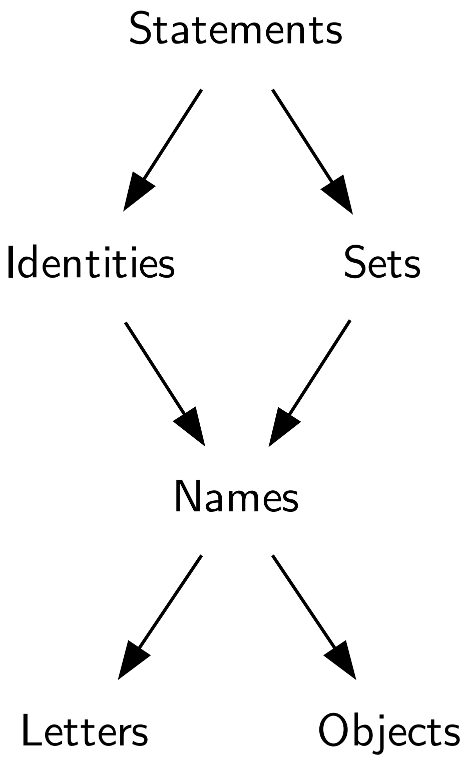
Identities (4)
Sets (5)

Statements (7) is immediately needed by:

Logical Statements (8)

Statements (7) gives the following terms.

statement
relational symbol
name symbol
relational symbol
name symbol
relational symbols
terminal
assertion
membership assertion
identity assertion
primitive sentence
logical form
sentence
belongs to
member


Statements
Identities
Sets
Names
Letters
Objects

Why

We want symbols for "and", "or", "not", and "implies".[2]

Overview

We call $=$ and $\in$ *relational symbols*. They say how the objects denoted by a pair of placeholder names relate to each other in the sense of being or belonging. We call $\_ = \_$ and $\_ \in \_$ *simple statements*. They denote simple sentences "the object denoted by _ is the object denoted by _" and "the object denoted by _ belongs to the set denoted by _". The symbols introduced here are *logical symbols* and statements using them are *logical statements*.

Conjunction

Consider the symbol $\land$. We will agree that it means "and". If we want to make two simple statements like $a = b$ and $a \in A$ at once, we write write $(a = b) \land (a \in A)$. The symbol $\land$ is symmetric, reflecting the fact that a statement like $(a \in A) \land (a = b)$ means the same as $(a = b) \land (a \in A)$.

Disjunction

Consider the symbol $\lor$. We will agree that it means "or" in the sense of either one, the other, or both. If we want to say that

[2] This sheet does not explain logic. In the next edition there will be several more sheets serving this function.

at least one of the simple statements like $a = b$ and $a \in A$, we write write $(a = b) \vee (a \in A)$. The symbol $\vee$ is symmetric, reflecting the fact that a statement like $(a \in A) \vee (a = b)$ means the same as $(a = b) \vee (a \in A)$.

Negation

Consider the symbol $\neg$. We will agree that it means "not". We will use it to say that one object "is not" another object and one object "does not belong to" another object. If we want to say the opposite of a simple statement like $a = b$ we will write $\neg(a = b)$. We read it aloud as "not a is b" or (the more desirable) "a is not b". Similarly, $\neg(a \in A)$ we read as "not, the object denoted by a belongs to the set denoted by A". Again, the more desirable english expression is something like "the object denoted by a does not belong to the set A" For these reasons, we introduce two new symbols $\neq$ and $\notin$. $a \neq b$ means $\neg(a = b)$ and $a \notin A$ means $\neg(a \in A)$.

Implication

Consider the symbol $\longrightarrow$. We will agree that it means "implies". For example $(a \in A) \longrightarrow (a \in B)$ means "the object denoted by a belongs to the object denoted by A implies the object denoted by a belongs to the set denoted by B" It is the same as $(\neg(a \in A)) \vee (a \in B)$. In other words, if $a \in A$, then always $a \in B$. The symbol $\longrightarrow$ is not symmetric, since implication is not symmetric. The symbol $\longleftrightarrow$ means "if and only if".

Logical Statements (8) immediately needs:

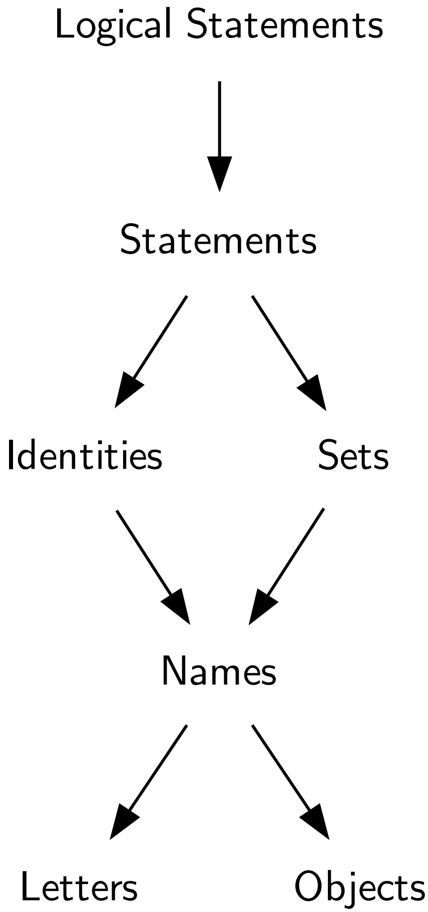
Statements (7)

Logical Statements (8) is immediately needed by:

Deductions (9)
Quantified Statements (10)

Logical Statements (8) gives the following terms.

relational symbols
simple statements
logical symbols
logical statements


Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to make conclusions.

Definition

Suppose we have a list of logical statements. We want to write down o

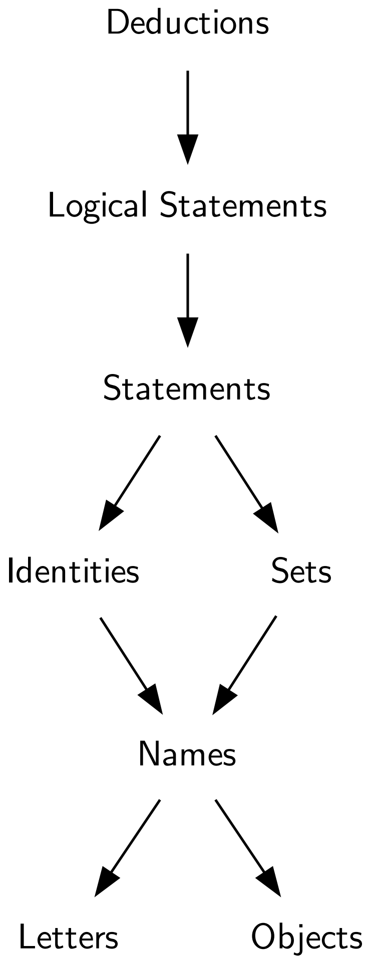
Deductions (9) immediately needs:

Logical Statements (8)

Deductions (9) is immediately needed by:

Accounts (11)

Deductions (9) gives no terms.


Deductions
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want symbols for talking about the existence of objects and for making statements which hold for all objects.[3]

Definition

If we say there exists an object that is blue, we mean the same as if we say that not every object is not blue. If we say that every object is blue, we mean the same as if we say there does not exist an object that is not blue. In other words, "there exists an object so that _" is the same as "not every object is not _". Or, "every object is _" is the same as "there does not exist an object that is _".

When we assert something of every object we also assert the nonexistence of the contrary of that assertion. And likewise when we assert that an object exists with some conditions, we assert that not every object exists without that condition.

The content of our assertions will be logical statements (see Logical Statements) and when we want to make them for all objects or for no object we will use the following symbols. The symbols introduced here are *quantifier symbols* and statements using them are *quantified statements*.

[3] This sheet does not explain quantifiers. In the next edition there will be several more sheets serving this function

Existential Quantifier

Consider the symbol $\exists$. We agree that it means "there exists an object". We write $(\exists x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\exists x)(x \in A)$ to mean "there exists an object in the set denoted by A" We call $\exists$ the *existential quantifier* symbol.

Universal Quantifier

Consider the symbol $\forall$. We agree that it means "for every object". We write $(\forall x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\forall x)((x \in A) \longrightarrow (x \in B)$ to mean, "every object which is in the set denoted by A is in the set denoted by B". We call $\forall$ the *universal quantifier* symbol.

Binding

When we have a name following a $\forall$ or $\exists$ we say that the name is *bound*. If a name is bound, then the statement uses it in one sense but not in another. The name is only used in that single statement. Regular names in statements we call *unbound*

Negations

The statement $\neg(\forall x)(\_)$ is the same as $(\exists x)(\neg(\_))$ and $\neg(\exists x)(\_)$ is the same as $(\forall x)(\neg(\_))$.

Quantified Statements (10) immediately needs:

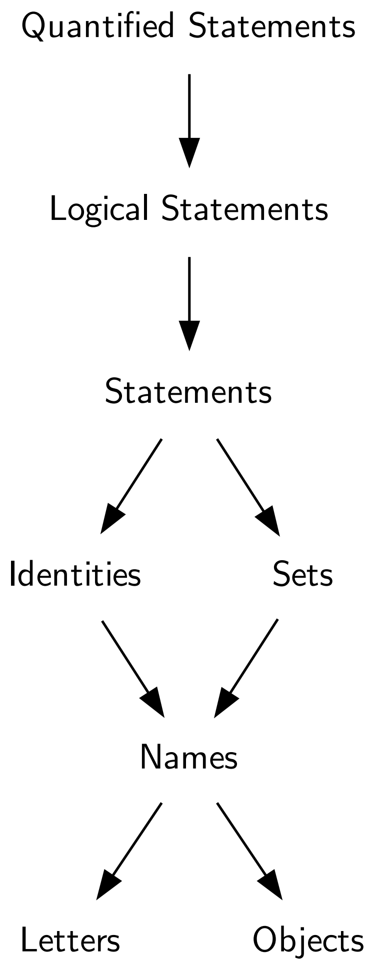
Logical Statements (8)

Quantified Statements (10) is immediately needed by:

Accounts (11)

Quantified Statements (10) gives the following terms.

quantifier symbols
quantified statements
existential quantifier
universal quantifier
bound
unbound


Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to succinctly and clearly make several statements about objects and sets. We want to track the names we use, taking care to avoid using the same name twice.

Definition

An *account*[4] is a list of naming, logical, and quantified statements. We use the words "let _ denote an _" to introduce a name as a placeholder for a thing, and we use the symbols $\_ = \_$ and $\_ \in \_$ to denote statements of identity and belonging. In other words, we have three sentence kinds to record.

1. **Names.** State we are using a name.
2. **Identity.** We want to make statements of identity.
3. **Belonging.** We want to make statements of belonging.

Our main purpose is to keep a list names, of quantified, logical and simple statments about them, and then statements we can deduce from these. In particular we want to group our name usage. In the English language we use paragraphs or sections to do so. In these sheets, we will use accounts. We will list the statements and label each with Arabic numerals (see Letters). which will be a list of statements, each of which is labeled by an Arabic numeral (see Letters).

[4]This sheet will be expanded in future editions.

Experience suggests that we start with an example. Suppose we want to summarize the following english language description of some names and objects.

> Denote an object by a. Also, denote the same object by b Also, denote a set by A. Also, the object denoted by a is an element of the set denoted by A. Also denote an object by c. Also c is the same object as b.

In our usual manner of speaking, we drop the word "also". In these sheets, we translate each of the sentences into our symbols. For names we use, we write `name` in that font followed by the name. For logical statements we `have`, we write `have` followed by the logical statement. For deductions we write `thus` followed by the conclusion and then `by` followed by the Arabic numerals of the premisses. So we write:

Account 1. First Example

1	`name`	a	
2	`name`	b	
3	`have`	$a = b$	
4	`name`	A	
5	`have`	$a \in A$	
6	`name`	c	
7	`have`	$c = b$	
8	`thus`	$a = c$	`by 3,7`

Accounts (11) immediately needs:

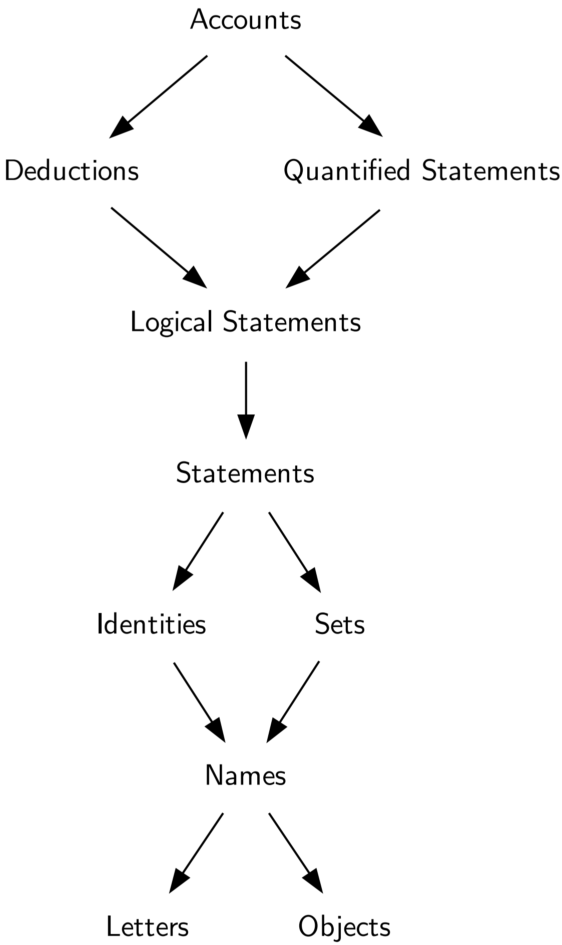
Deductions (9)
Quantified Statements (10)

Accounts (11) is immediately needed by:

Standardized Accounts (12)

Accounts (11) gives the following terms.

account



Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

Why

We want to do our best to have only one way to write accounts.

Definition

A *standard account*[5] lists all names, then lists all premisses, then lists all conclusions.

Example

Consider the account.

Account 2. First Example

1	`name`	a		
2	`name`	b		
3	`have`	$a = b$		
4	`name`	c		
5	`have`	$c = b$		
6	`thus`	$a = c$	`by`	`3,5`

[5] This sheet will be expanded in future editions.

Account 3. Standardized First Example

1	`name`	a	
2	`name`	b	
3	`have`	$a = b$	
4	`name`	c	
5	`have`	$c = b$	
6	`thus`	$a = c$	`by 3,5`

We can abbreviate the names:

Account 4. Abbreviated First Example

1-3	`name`	a, b, c	
4	`have`	$a = b$	
5	`have`	$c = b$	
6	`thus`	$a = c$	`by 4,5,IdentityAxioms:1`

Standardized Accounts (12) immediately needs:

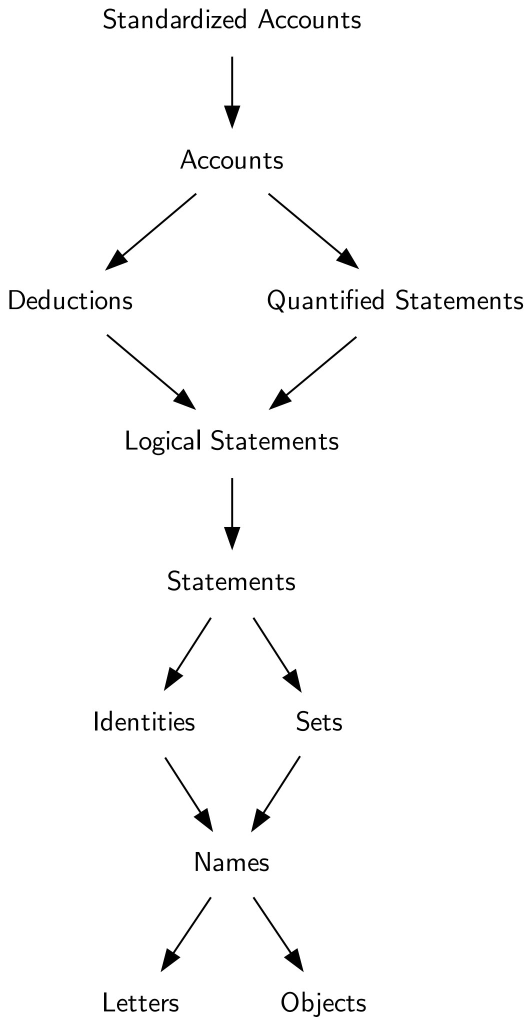
Accounts (11)

Standardized Accounts (12) is immediately needed by:

Set Inclusion (14)

Standardized Accounts (12) gives the following terms.

standard account


Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects

All definitions are nominal. They are made to give us language and to save space.

Definitions (13) immediately needs:

Objects (2)

Definitions (13) is immediately needed by:

Set Inclusion (14)

Definitions (13) gives no terms.

$$\begin{tikzcd}
\text{Definitions} \arrow[d] \\
\text{Objects}
\end{tikzcd}$$

Why

We want language for all of the elements of a first set being the elements of a second set.

Definition

Denote a set by A and a set by B.

Definition 1 (Subsets). If every element of the set denoted by A is an element of the set denoted by B, then we say that the set denoted by A is a *subset* of the set denoted by B.

We say that the set denoted by A is *included* in the set denoted by B. We say that the set denoted by B is a *superset* of the set denoted by A or that the set denoted by B *includes* the set denoted by A.

Every set is included in and includes itself.

Notation

Let A denote a set and B denote a set. We denote that the set A is included in the set B by $A \subset B$. In other words, $A \subset B$ means $(\forall x)((x \in A) \longrightarrow (x \in B))$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A". $B \supset A$ also means $(\forall x)((x \in A) \longrightarrow (x \in B))$.

Properties

There are some properties that our intuition suggests inclusion should have. First, every set should include itself. We describe this fact by saying that inclusion is *reflexive.*

Proposition 1 (Reflexive)**.** *Every set is included in itself*

Proof. `(1) name` A; `(2) have` $(\forall x)(x \in A \longrightarrow x \in A)$; `(3) thus` $A \subset A$ `by` Definition 1. $\square$

Next, we expect that if one set is included in another, This fact is described by saying that inclusion is *transitive*

Proposition 2 (Transitive)**.** *If a one set is included in another, and the latter in yet another, then the former is included in the last.*

Proof. `(1) name` A, B, C; `(2) have` $A \subset B$ `(3) have` $B \subset C$ `(4) thus` $A \subset C$ `by` modus ponens. $\square$

Equality $(=)$ shares these two properties. Let A denote an object. Then $A = A$. Let B and C also denote objects. If $A = B$ and $B = C$, then $A = C$. Of course, inclusion is not symmetric.. Belonging $(\in)$ may be, but need not be reflexive and transitive.

Set Inclusion (14) immediately needs:

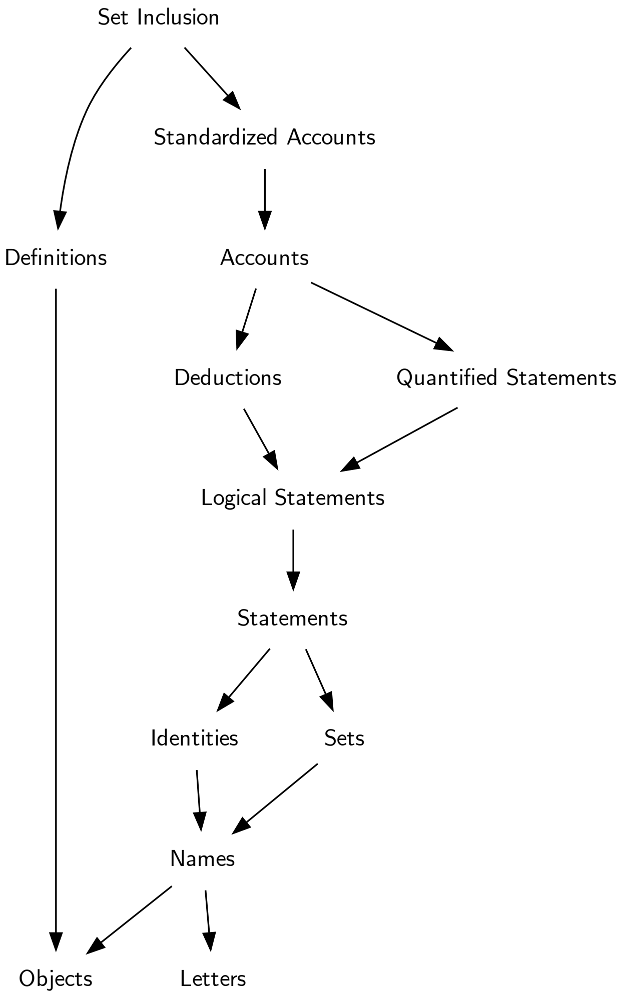
Definitions (13)
Standardized Accounts (12)

Set Inclusion (14) is immediately needed by:

Set Equality (15)
Set Powers (31)
Set Specification (16)

Set Inclusion (14) gives the following terms.

subset
included
superset
includes
improper subsets
proper subsets
reflexive
transitive


Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Objects
Letters

Why

When are two sets the same?

Definition

Let A and B denote sets. If $A = B$ then every element of A is an element of B and every element of B is an element of A. In other words, $(A = B) \longrightarrow ((A \subset B) \land (B \subset A))$.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Then $A = B$? We define it to be so. Sets are determined by their members.

Principle 2 (Extension)**.** *Sets are the same if every member of one is a member of the other and vice versa.*

In other words, two sets are identical if and only if every element of one is an element of the other. This principle is sometimes called the *principle of extension*. We refer to the elements of a set as its *extension*. Roughly speaking, we have declared that if we know the extension then we know the set. A set is determined by its extension.

Deductive principle

We can use this definition to deduce $A = B$ if we first deduce $A \subset B$ and $B \subset A$. With these two implications, we use the principle of extension to conclude that the sets are the same.

In other words, $(A = B) \longleftrightarrow ((A \subset B) \wedge (B \subset A))$. We also describe this fact by saying that inclusion ($\subset$) is *antisymmetric*.

Belonging and sets compared with ancestry and humans

Compare the principle of extension for identifying sets from their elements with an analogous principle for identifying people from their ancestors.

We can consider a person's ancestors. Namely, the person's parents, grandparents, great grandparents and so on. It is clear that if we label the same human with two names A and B, then A and B have the same ancestors. In other words, same human implies same ancestors. This is the analog of "if two sets are equal they have the same members".

On the other hand, if we have two people denoted by A and B, and we know that A has the same ancestors as B, we can not conclude that A and B denote the same human. For example, siblings have the same ancestors but are different people. This direction, same ancestors implies same human, is the analogue of "if they have the same elements, two sets are the same". It is false for humans and ancestors, but we define it to be true for sets and members.

The principle of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Set Equality (15) immediately needs:

Set Inclusion (14)

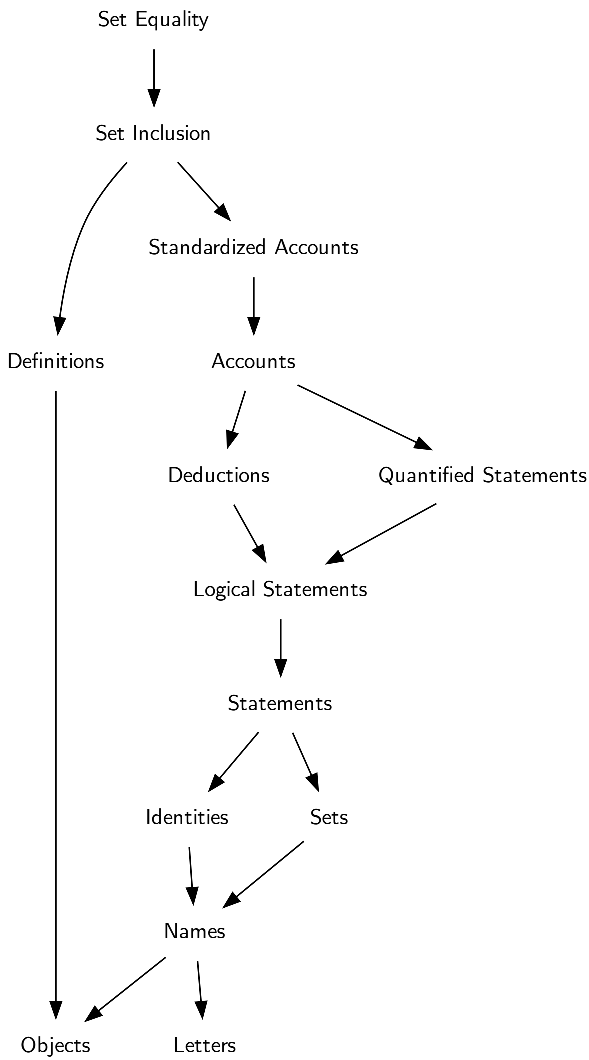
Set Equality (15) is not immediately needed by any sheet.

Set Equality (15) gives the following terms.

principle of extension

extension

antisymmetric


Set Equality
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
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Identities
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Why

We want to construct new sets out of old ones. So, can we always construct subsets?

Definition

We will say that we can. More specifically, if we have a set and some statement which may be true or false for the elements of that set, a set exists containing all and only the elements for which the statement is true.

Roughly speaking, the principle is like this. We have a set which contains some objects. Suppose the set of playing cards in a usual deck exists. We are taking as a principle that the set of all fives exists, so does the set of all fours, as does the set of all hearts, and the set of all face cards. Roughly, the corresponding statements are “it is a five”, “it is a four”, “it is a heart”, and “it is a face card”.

Principle 3 (Specification)**.** *For any statement and any set, there is a subset whose elements satisfy the statement.*

We call this the *principle of specification.* We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence. The principle of extension (see Set Equality) says that this set is unique. All basic principles about sets (other than the principle of extension, see Set Equality) assert that we can construct new sets out of old ones in reasonable ways.

Notation

Let A denote a set. Let s denote a statement in which the symbol x and A appear unbound. We assert that there is a set, denote it by B, for which belonging is equivalent to membership in A and s. In other words,

$$(\forall x)((x \in B) \longleftrightarrow ((x \in A) \land s(x))).$$

We denote B by $\{x \in A \mid s(x)\}$. We read the symbol $\mid$ aloud as "such that." We read the whole notation aloud as "a in A such that..." We call it *set-builder notation*.

Nothing contains everything

As an example of the principle of specification and important consequence, consider the statement $x \notin x$. Using this statement and the principle of specification, we can prove that there is not set which contains every thing.

Proposition 3. *No set contains all sets.*[6]

Proof. Suppose there exists a set, denote it A which contains all sets. In other words, suppose $(\exists A)(\forall x)(x \in A)$. Use the principle of specification to construct $B = \{x \in A \mid x \notin x\}$. So $(\forall x)(x \in B \longleftrightarrow (x \in A \land x \notin x))$ In particular, $(B \in B \longleftrightarrow (B \in A \land B \notin B))$. So $B \notin A$. $\square$

[6]We might call such a set, if we admitted its existence, a *universe of discourse* or *universal set*. With the principle of specification, a "principle of a universal set" would give a contradiction (called *Russell's paradox*).

Set Specification (16) immediately needs:

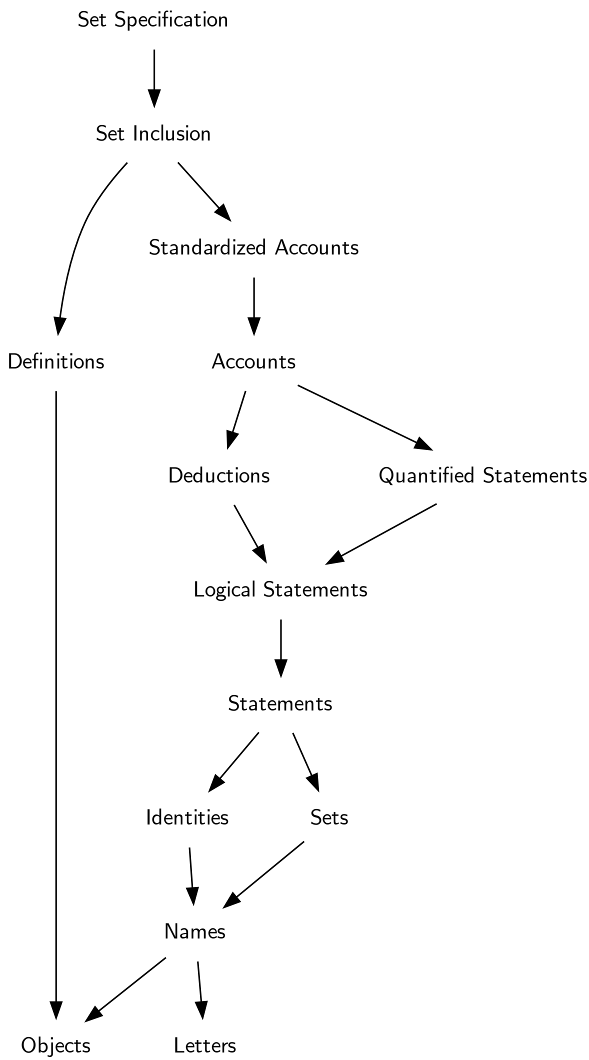
Set Inclusion (14)

Set Specification (16) is immediately needed by:

Empty Set (17)
Pair Intersections (22)
Set Differences (25)
Unordered Pairs (18)

Set Specification (16) gives the following terms.

set-builder notation
principle of specification
specifying
universe of discourse
universal set
Russell's paradox


Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Deductions
Quantified Statements
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Why

Can a set have no elements?

Definition

Sure. A set exists by the principle of existence (see Sets); denote it by A. Specify elements (see Set Specification) of any set that exists using the universally false statement $x \neq x$. We denote that set by $\{x \in A \mid x \neq x\}$. It has no elements. In other words, $(\forall x)(x \notin A)$. The principle of extension (see Set Equality) says that the set obtained is unique (contradiction).[7]

Definition 2 (Empty Set). We call the unique set with no elements *the empty set.*

Notation

We denote the empty set by $\varnothing$. In other words, in all future accounts (see Accounts), there are two implicit lines. First, "`name` $\varnothing$" and second "`have` $(\forall x)(x \notin \varnothing)$".

Properties

It is immediate from our definition of the empty set and of the definition of inclusion (see Set Inclusion) that the empty set is included in every set (including itself).

Proposition 4. $(\forall A)(\varnothing \subset A)$

[7]This account will be expanded in the next edition.

Proof. Suppose toward contradiction that $\varnothing \not\subseteq A$. Then there exists $y \in \varnothing$ such that $y \notin A$. But this is impossible, since $(\forall x)(x \notin \varnothing)$. $\square$

Empty Set (17) immediately needs:

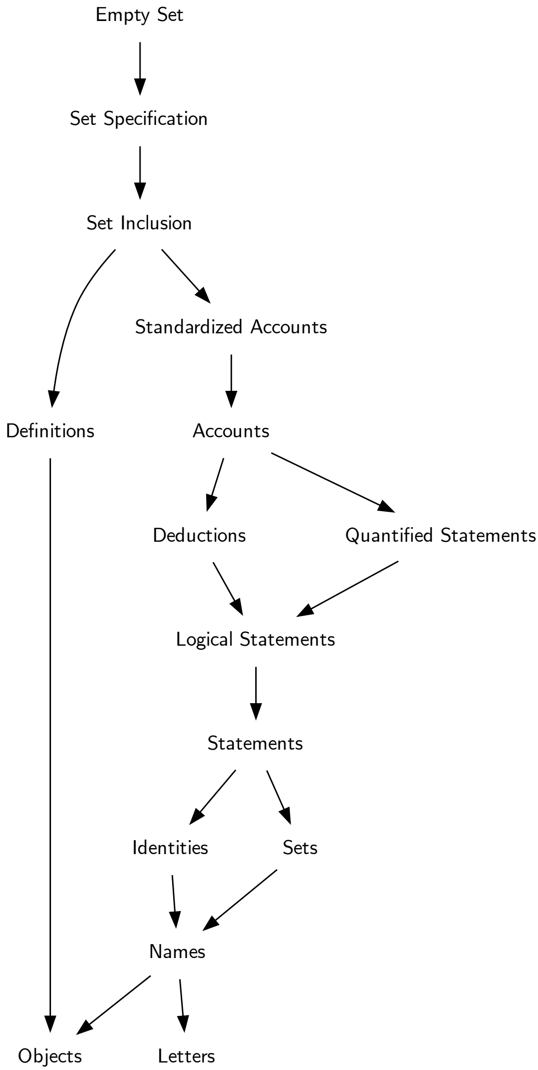
Set Specification (16)

Empty Set (17) is immediately needed by:

Natural Numbers (50)
Set Complements (26)
Set Unions (19)

Empty Set (17) gives the following terms.

the empty set
empty set


Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
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Names
Objects
Letters

Why

Can we always make a set out of two objects?

Definition

We say yes.

Principle 4 (Pairing). *Given two objects, there exists a set containing them.*

We refer to this as the *principle of pairing.* Denote one object by a and the other by b. This principle gives us the existence of a set that contains the objects. The principle of specification (see Set Specification) gives use the subset for the statement "$x = a \lor x = b$". The principle of extension (see Set Equality) says this set is unique. We call this set a *pair* or an *unordered pair.*

If the object denoted by a is the object denoted by b, then we call the pair the *singleton* of the object denoted by a. Every element of the singleton of the object denoted by a is a.

In other words, the principle of pairing says that every object is an element of some set. That set may be the singleton, or it may be the pair with any other object. We can construct several sets using this principle: the singleton of the object denoted by a, the singleton of the singleton of the object denoted by a, the singleton of the singleton of the singleton of the object denoted by a, and so on.

Notation

We denote the set which contains a and b as elements and nothing else by $\{a, b\}$. The pair of a with itself is the set $\{a, a\}$ is the singleton of a. We denote it by $\{x\}$. The principle of pairing also says that $\{\{a\}\}$ exists and $\{\{\{a\}\}\}$ exists, as well as $\{a, \{a\}\}$.

Note well that $a \neq \{a\}$. a denotes the object a. $\{a\}$ denotes the set whose only element is a. In other words $(\forall x)(x \in \{a\} \longleftrightarrow x = a)$. The moral is that a sack with a potato is not the same thing as a potato.

Unordered Pairs (18) immediately needs:

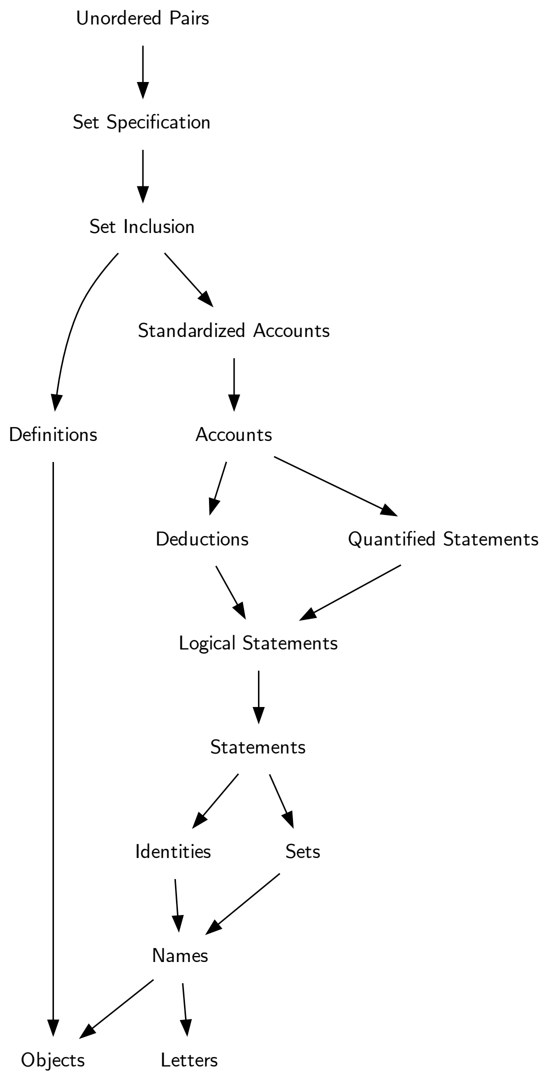
Set Specification (16)

Unordered Pairs (18) is immediately needed by:

Graphs (67)
Ordered Pairs (32)
Set Unions (19)

Unordered Pairs (18) gives the following terms.

principle of pairing
pair
unordered pair
singleton


Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Deductions
Quantified Statements
Logical Statements
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Why

Can we combine sets?

Definition

We say yes. For example, if we have a first set denoted A and a second set denoted B, then we want a third set including all the elements of the set denoted by A and the elements of the set denoted by B. If an object appears in the set denoted by A and in the set denoted by B, it appears in the new set. If an object appears in one set but not the other, it appears in the new set. Indeed, if we have a set of sets, the same should hold.

Principle 5 (Union)**.** *Given a set of sets, there exists a set which contains all elements which belong to any of the sets.*

We call this the *principle of union.* If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the principle of union may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification (see Set Specification) to form the set which contains only those elements which are appear in at least one of any of the sets. The set is unique by the principle of extension. We call that unique set *the union* of the sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\bigcup \mathcal{A}$. So

$$(\forall x)((x \in (\bigcup \mathcal{A})) \longleftrightarrow (\exists A)((A \in \mathcal{A}) \land x \in A)).$$

Simple Facts

It is reasonable for the union of the empty set to be empty and for the union of the singleton of a set to be itself.

PROPOSITION **1.** $\bigcup \varnothing = \varnothing$

Proof. Immediate[8] □

PROPOSITION **2.** $\bigcup \{A\} = A$

Proof. Immediate[9] □

[8] Future editions will include the account.
[9] Future editions will include the account.

Set Unions (19) immediately needs:

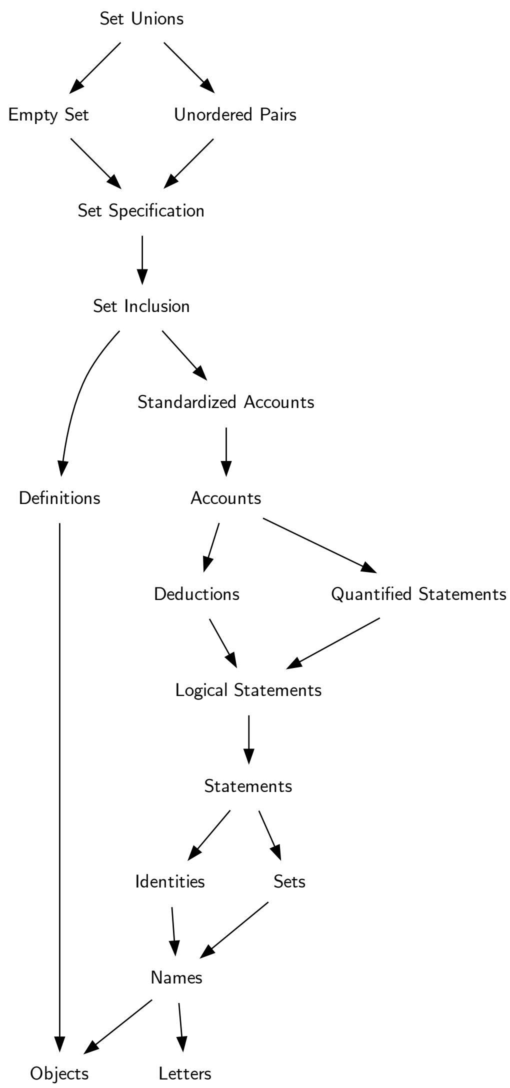
Empty Set (17)
Unordered Pairs (18)

Set Unions (19) is immediately needed by:

Graph Complements (**??**)
Natural Numbers (50)
Pair Unions (20)
Set Operations (40)
Set Symmetric Differences (30)
Trees (68)

Set Unions (19) gives the following terms.

principle of union
the union


Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

We often unite the elements of one set with another.

Discussion

Let A and B denote sets. We call $\cup\{A, b\}$ the *pair union* of A and B. We denote the union of the pair $\{A, B\}$ by $A \cup B$. Clearly the pair union does not depend on the order of A and B. In other words, $A \cup B = B \cup A$.

Facts

Here are some basic facts about unions of a pair of sets.[10] Let A and B denote sets.

Proposition 5 (Identity Element). $A \cup \varnothing = A$

Proposition 6 (Commutativity). $A \cup B = B \cup A$

Proposition 7 (Associativity). $(A \cup B) \cup C = A \cup (B \cup C)$

Proposition 8 (Idempotence). $A \cup A = A.$

Proposition 9. $A \subset B \longleftrightarrow A \cup B = B$

[10] Proofs will appear in the next edition.

Pair Unions (20) immediately needs:

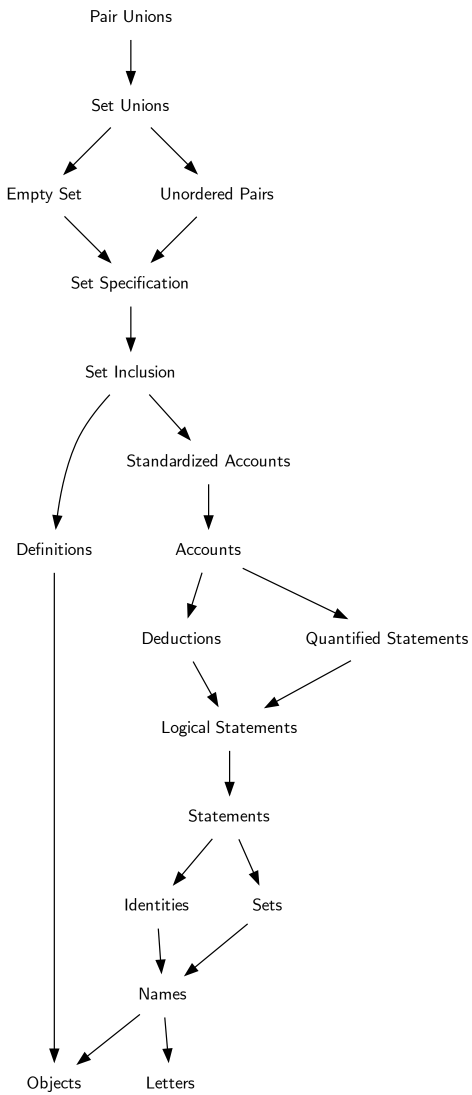
Set Unions (19)

Pair Unions (20) is immediately needed by:

Intersection of Empty Set (24)
Set Decompositions (27)
Set Dualities (28)
Unordered Triples (21)

Pair Unions (20) gives the following terms.

pair union


Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

$\{a\} \cup \{b\} = \{a, b\}$

Definition

Let a, b and c denote objects. From the associativity of pair unions (see Pair Unions), we have

$$(\{a\} \cup \{b\}) \cup \{b\} = \{a\} \cup (\{b\} \cup \{c\}).$$

So we will drop the parentheses, and write $\{a\} \cup \{b\} \cup \{c\}$. We call such a set the *unordered triple* of a, b and c. The unordered triple of a, b and c is the set containing these elements and no others.

Notation

Such sets are so commonplace that we denote the unordered triple of a, b and c by $\{a, b, c\}$.

Quadruples

Let d denote an object. Again, the associativity of pair unions allows us to drop the parentheses from

$$(((\{a\} \cup \{b\}) \cup \{c\}) \cup \{d\})).$$

We can therefore write $\{a\} \cup \{b\} \cup \{c\} \cup \{d\}$ without ambiguity. We call this set the *unordered quadruple*. As before, the unordered quadruple contains of a, b, c and d contains a, b, c, and d and nothing besides these.

Notation

We denote the unordered quadruple of the objected denoted by a, b, c and d, denote this set by $\{a, b, c, d\}$.

The case of several named objects

In a similar way we speak of *unordered pentuples*, *unordered sextuples*, *unordered septuples* and so on. If we have several objects named, we denote the set containing these objects be writing their names in between the left brace { and right brace}, separating the names by commas. For example, if we A, b, x and Y and z denote objects, then we denote the set containing these elements by

$$\{A, b, x, Y, z\}.$$

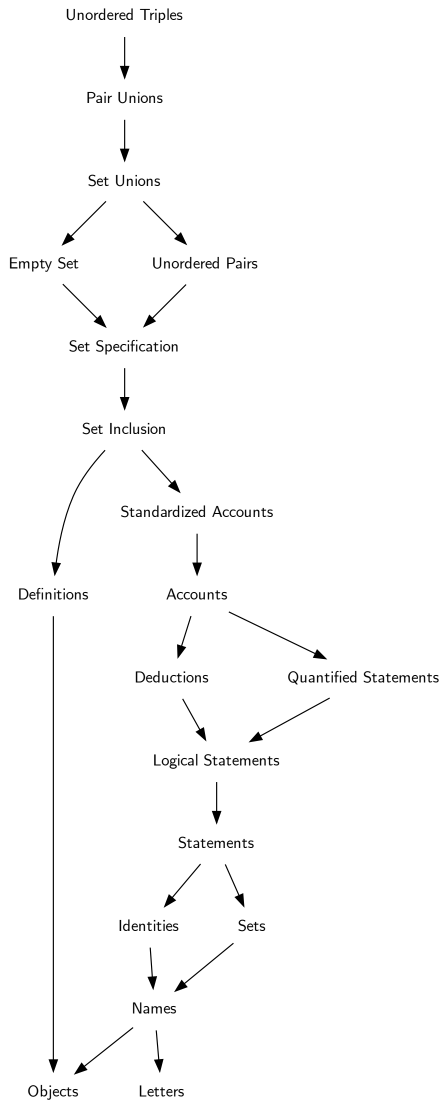
Unordered Triples (21) immediately needs:

Pair Unions (20)

Unordered Triples (21) is not immediately needed by any sheet.

Unordered Triples (21) gives the following terms.

unordered triple
unordered quadruple
unordered pentuples
unordered sextuples
unordered septuples


Unordered Triples
Pair Unions
Set Unions
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

Does a set exist containing the elements shared between two sets? How might we construct such a set?

Definition

Let A and B denote sets. Consider the set $\{x \in A \mid x \in B\}$. This set exists by the principle of specification (see Set Specification). Moreover $(y \in \{x \in A \mid x \in B\}) \longleftrightarrow (y \in A \land y \in B)$. In other words, $\{x \in A \mid x \in B\}$ contains all the elements of A that are also elements of B.

We can also consider $\{x \in B \mid x \in A\}$, in which we have swapped the positions of A and B. Similarly, the set exists by the principle of specification (see Set Specification) and again $y \in \{x \in B \mid x \in A\} \longleftrightarrow (y \in B \land y \in B)$. Of course, $y \in A \land y \in B$ means the same as[11] $y \in B \land y \in A$ and so by the principle of extension (see Set Equality)

$$\{x \in A \mid x \in B\} = \{x \in B \mid x \in A\}.$$

We call this set the *pair intersection* of the set denoted by A with the set denoted by B.

Notation

We denote the intersection fo the set denoted by A with the set denoted by B by $A \cap B$. We read this notation aloud as "A intersect B".

[11]Future editions will name and cite this rule.

Basic Properties

All the following results are immediate.[12]

Proposition 10. $A \cap \varnothing = \varnothing$

Proposition 11 (Commutativity)**.** $A \cap B = B \cap A$

Proposition 12 (Associativity)**.** $(A \cap B) \cap C = A \cap (B \cap C)$

Proposition 13. $A \cap A = A$

Proposition 14. $(A \subset B) \longleftrightarrow (A \cap B = A)$.

[12]Proofs of these results will appear in the next edition.

Pair Intersections (22) immediately needs:

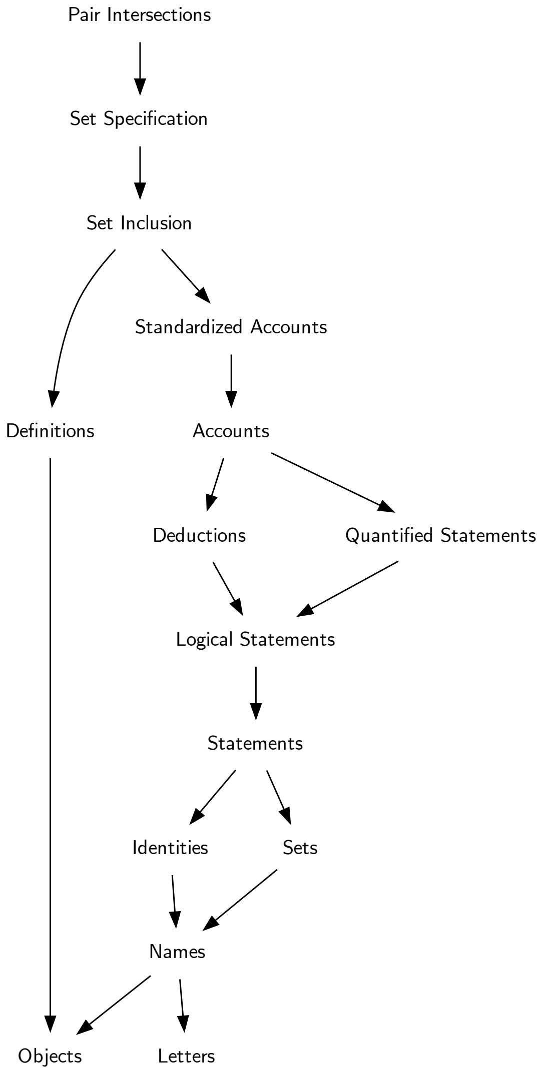
Set Specification (16)

Pair Intersections (22) is immediately needed by:

Set Decompositions (27)
Set Dualities (28)
Set Intersections (23)

Pair Intersections (22) gives the following terms.

pair intersection


Pair Intersections
Set Specification
Set Inclusion
Standardized Accounts
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Why

We can consider intersections of more than two sets.

Definition

Let $\mathcal{A}$ denote a set of sets. In other words, every element of $\mathcal{A}$ is a set. And suppose that $\mathcal{A}$ has at least one set (i.e., $\mathcal{A} \neq \varnothing$). Let C denote a set such that $C \in \mathcal{A}$. Then consider the set,

$$\{x \in C \mid (\forall A)(A \in \mathcal{A} \longrightarrow x \in A)\}.$$

This set exists by the principle of specification (see Set Specification). Moreover, the set does not depend on which set we picked. So the dependence on C does not matter. It is unique by the axiom of extension (see Set Equality). This set is called the *intersection* of $\mathcal{A}$.

Notation

We denote the intersection of $\mathcal{A}$ by $\bigcap \mathcal{A}$.

Equivalence with pair intersections

As desired, the the set denoted by $\mathcal{A}$ is a pair (see Unordered Pairs) of sets, the pair intersection (see Pair Intersections) coincides with intersection as we have defined it in this sheet.[13]

Proposition 15. $\bigcap\{A, B\} = A \cap B$

[13]A full account of the proof will appear in future editions.

Set Intersections (23) immediately needs:

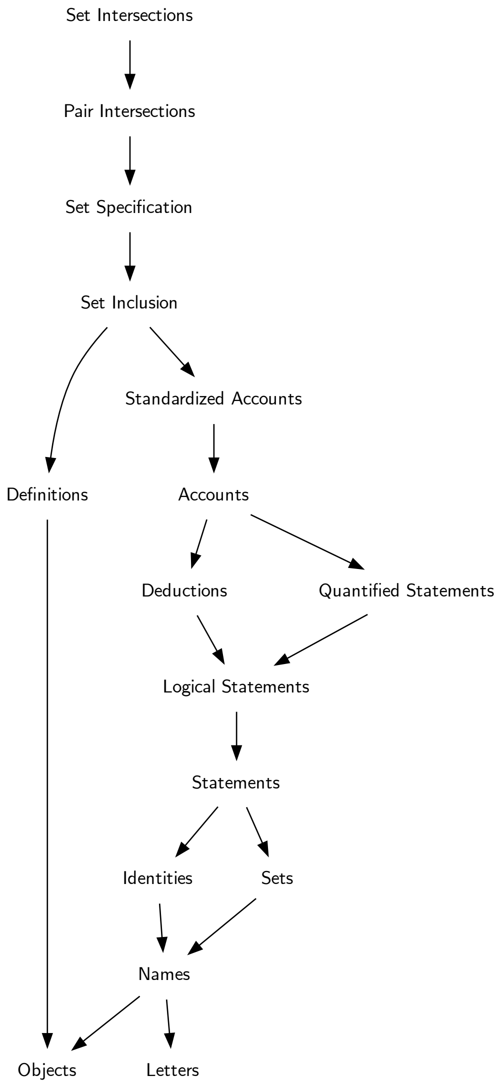
Pair Intersections (22)

Set Intersections (23) is immediately needed by:

Intersection of Empty Set (24)
Natural Numbers (50)
Set Operations (40)

Set Intersections (23) gives the following terms.

intersection



Set Intersections
Pair Intersections
Set Specification
Set Inclusion
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Why

We only define set intersections for nonempty sets of sets. Why?

Discussion

Which objects are specified by the sentence $(\forall x \in \varnothing)(x \in X)$? Well, since no objects fail to satisfy the statement,[14] the sentence specifies all objects. So in other words, the condition we used to define set intersections (Set Intersections) specifies the "set of everything". In order to maintain other more desirable set principles like selection, we have said that such a set does not exist (see Set Specification).

If, however, all sets under consideration are subsets of one paticular set—denote it E—then we can define intersections as follows. Let $\mathcal{C}$ be a possibly nonempty collection of sets

$$\bigcap \mathcal{C} = \{X \in E\}(\forall X \in \mathcal{C})(x \in X).$$

This definition agrees with that given in Set Intersections. In particular, it is the intersection of the set $\mathcal{C} \cup \{E\}$

Another definition

This begs the following question. Why not define intersections by selecting from the union. Let $\mathcal{A}$ be a possibly nonempty

[14]Future editions will offer an account of this.

set of sets. Then define:

$$\bigcap \mathcal{A} = \{x \in \bigcup \mathcal{A} \mid (\forall A \in \mathcal{A})(x \in A)\}.$$

If $\mathcal{A}$ is empty, so is $\bigcup \mathcal{A}$ and then there are no elements in the set to select from so $\bigcap \mathcal{A}$ is empty. This does not agree with the previous definitions for the empty set, but does for all other sets of sets.

For these reasons, the intersection of the empty set is a delicate thing.[15]

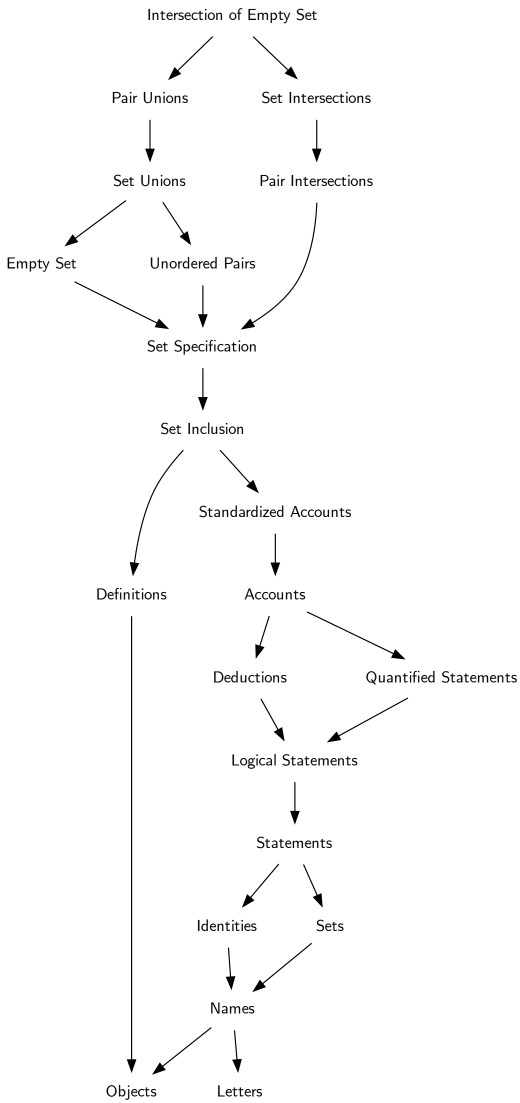
[15] Future editions are likely to expand on, and perhaps mention why we prefer the former definition.

Intersection of Empty Set (24) immediately needs:

Pair Unions (20)
Set Intersections (23)

Intersection of Empty Set (24) is not immediately needed by any sheet.

Intersection of Empty Set (24) gives no terms.


Intersection of Empty Set
Pair Unions
Set Intersections
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Why

We want to consider the elements of one set which are not contained in another set.

Definition

Let A and B denote sets. The *difference* between A and B is the set $\{x \in A \mid x \notin B\}$. It is not necessary that $B \subset A$.

Notation

We denote the difference between A and B by $A - B$.

Properties

The following are straightforward.

Proposition 16. $A - \varnothing = A$

Proposition 17. $A - A = \varnothing$

Set Differences (25) immediately needs:

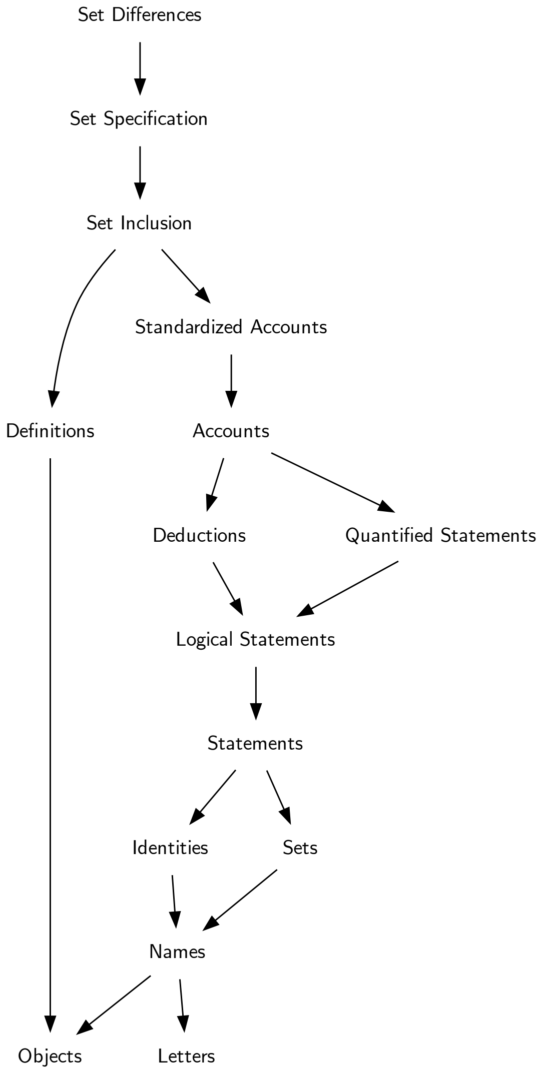
Set Specification (16)

Set Differences (25) is immediately needed by:

Set Complements (26)

Set Differences (25) gives the following terms.

difference


Set Differences
Set Specification
Set Inclusion
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Why

It is often the case in considering set differences that all sets considered are subsets of one set.

Definition

Let A and B denote sets. In many cases, we take the difference between a set and one contained in it. In other words, we assume that $B \subset A$. In this case, we often take complements relative to the same set A. So we do not refer to it, and instead refer to the relative complement of B in A as the *complement* of B.

Notation

Let A denote a set, and let B denote a set for which $B \subset A$. We denote the relative complement of B in A by $C_A(B)$. When we need not mention the set A, and instead speak of the complement of B without qualification, we denote this complement by $C(B)$.

Complement of a complement

One nice property of a complement when $B \subset A$ is:

Proposition 18. $(B \subset A) \longleftrightarrow (C_A(C_A(B)) = B)$

Basic Facts

Let E denote a set and let A and B denote sets satisfying $A, B \subset E$. Then take all complements with respect to E. Here are some immediate consequences of the definition of complements.[16]

Proposition 19. $C(C(A)) = A$

Proposition 20. $C(\varnothing) = E$

Proposition 21. $C(E) = \varnothing$

Proposition 22. $A \subset B \longleftrightarrow C(B) \subset C(A)$

[16] Proofs will appear in future editions.

Set Complements (26) immediately needs:

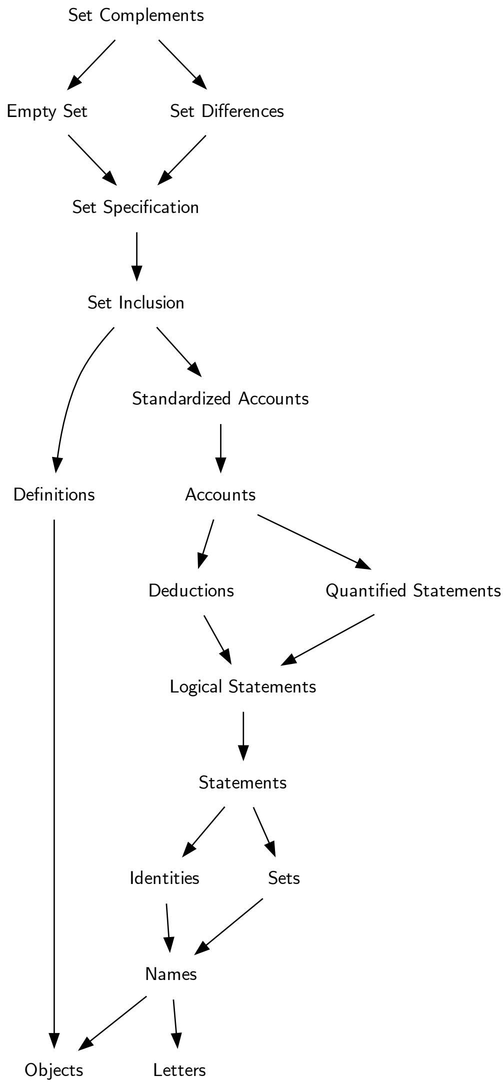
Empty Set (17)
Set Differences (25)

Set Complements (26) is immediately needed by:

Graph Complements (**??**)
Set Decompositions (27)
Set Dualities (28)
Set Symmetric Differences (30)

Set Complements (26) gives the following terms.

complement


Set Complements
Empty Set
Set Differences
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Why

Let E denote a set and let A denote a set with $A \subset E$. A and $C(A)$ as breaking E into two pieces which do not overlap.

Discussion for complements

To make this precise, let us say that by "breaking E into two pieces" we mean that these two pieces are all of E. In other words, every element of E is contained either in A or $C(A)$. We use the language of set unions (Pair Unions).

Proposition 23 (Breaking). $A \cup C(A) = E$

Next, let us say that "do not overlap" means that no elemenet of A is an element of $C(A)$ and vice versa. We use the language of set interserctions (see Pair Intersections).

Proposition 24 (Non-overlapping). $A \cap C(A) = \varnothing$

Definition

We call a pair $\{A, B\}$ a *decomposition* of E if $A \cap B = \varnothing$ and $A \cup B = E$.

Set Decompositions (27) immediately needs:

Pair Intersections (22)

Pair Unions (20)

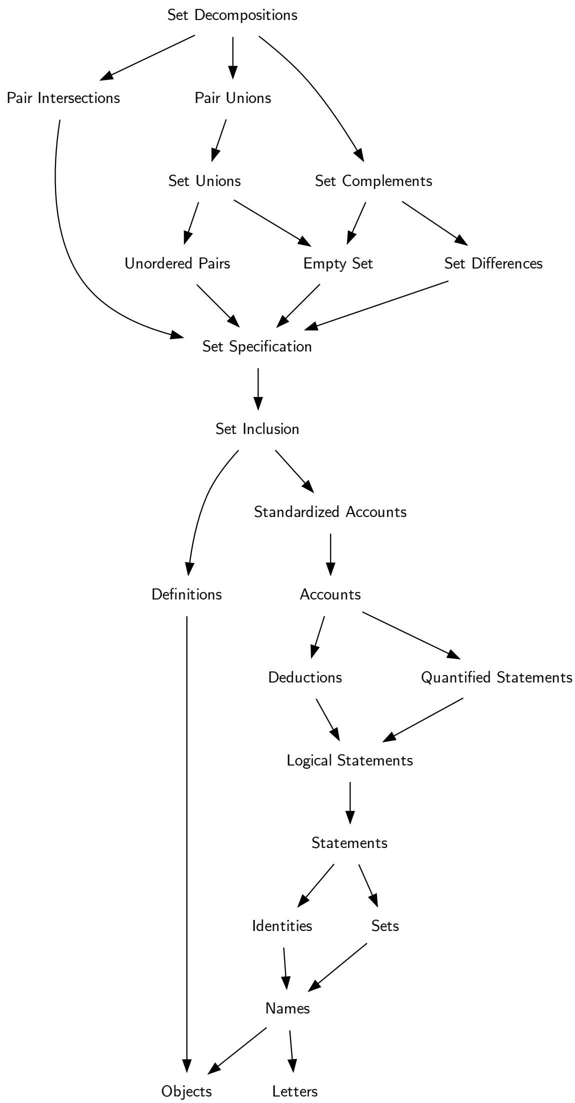
Set Complements (26)

Set Decompositions (27) is immediately needed by:

Set Exercises (29)

Set Decompositions (27) gives the following terms.

decomposition


Set Decompositions
Pair Intersections
Pair Unions
Set Unions
Set Complements
Unordered Pairs
Empty Set
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Why

How does taking complements relate to forming unions and intersections.

Complements of unions or intersections

Let E denote a set. Let A and B denote sets and $A, B \subset E$. All complements are taken with respect to E. The following are known as *DeMorgan's Laws*.[17]

Proposition 25. $C(A \cup B) = C(A) \cap C(B)$

Proposition 26. $C(A \cap B) = C(A) \cup C(B)$

Principle of duality

As a result of DeMorgan's Laws[18] and basic facts about complements (see Set Complements) theorems having to do with sets come in pairs. In other words, given an inclusion or identity relation involving complements, unions and intersections of some set (above E) if we replace all sets by their complemnets, swap unions and intersections, and flip all inclusions we obtain another result. This is called the *principle of duality* for sets.

[17] Proofs will appear in a future edition.

[18] A future edition will change the name to remove the reference to DeMorgan in accordance with the project's policy.

Set Dualities (28) immediately needs:

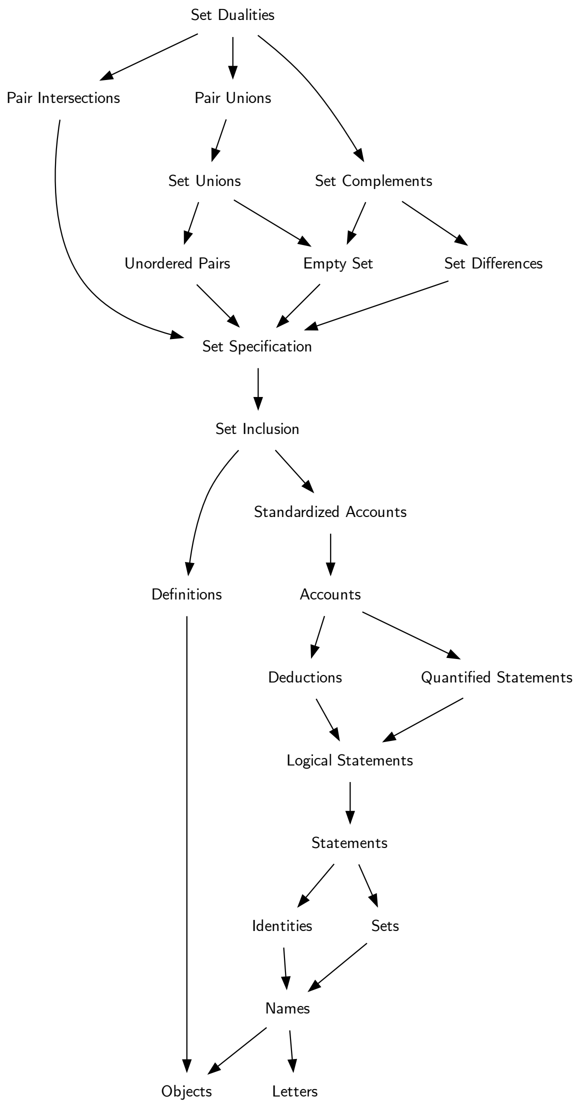
Pair Intersections (22)
Pair Unions (20)
Set Complements (26)

Set Dualities (28) is immediately needed by:

Set Exercises (29)

Set Dualities (28) gives the following terms.

DeMorgan's Laws
principle of duality


Set Dualities
Pair Intersections
Pair Unions
Set Unions
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Why

Here are some exercises on sets.[19]

Exercise 1. *Let A, B, C denote sets. Show $((A \cap B) \cup C = A \cap (B \cup C)) \longleftrightarrow (C \subset A)$ Observe that the condition does not involve B.*

Exercise 2.

$$A - B = A \cap B'.$$

Exercise 3.

$$A \subset B \textit{ if and only if} A - B = \varnothing.$$

Exercise 4.

$$A - (A - B) = A \cap B.$$

Exercise 5.

$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

Exercise 6.

$$(A \cap B) \subset ((A \cap C) \cup (A \cap C')).$$

Exercise 7.

$$((A \cup C) \cap (B \cup C')) \subset (A \cup B).$$

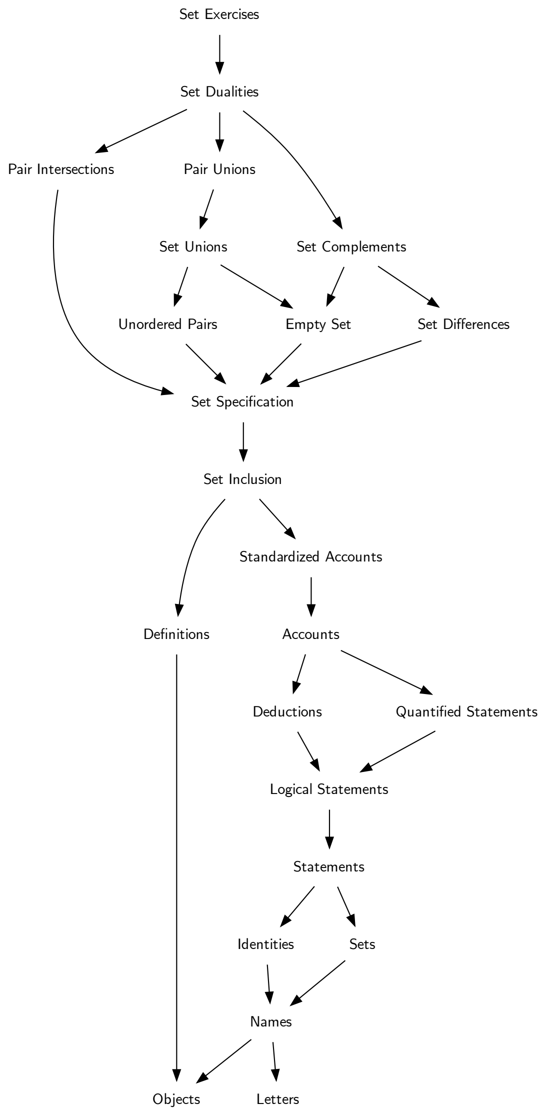
[19] Future editions will give the hypotheses more clearly.

Set Exercises (29) immediately needs:

Set Decompositions (27)
Set Dualities (28)

Set Exercises (29) is not immediately needed by any sheet.

Set Exercises (29) gives no terms.


Set Exercises
Set Dualities
Pair Intersections
Pair Unions
Set Unions
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Why

We want to consider the no-overlapping elements of a pair of sets.

Definition

In other words, we want to consider the set of elements which is one or the other but not in both. The *symmetric difference* of a set with another set is the union of the difference between the latter set and the former set and the difference between the former and the latter. The symmetric differences is also called the *Boolean sum* of A and B[20]

Notation

Let A and B denote sets. We denote the symmetric difference by $A + B$.

$$A + B = (A - B) \cup (B - A)$$

Properties

Here are some immmediate properties of symmetric differences.[21]

Proposition 27 (Commutative)**.** *$A + B = B + A$.*

[20]Future editions will likely remove or modify this term in accordance with the project's policy on using names.

[21]Future editions will have more detailed hypotheses here; for this edition they are obvious.

Proposition 28 (Associative)**.** $(A + B) + C = A + (B + C)$.

Proposition 29 (Identity)**.** $(A + \varnothing) = A$

Proposition 30 (Inverse)**.** $(A + A) = \varnothing$

Set Symmetric Differences (30) immediately needs:

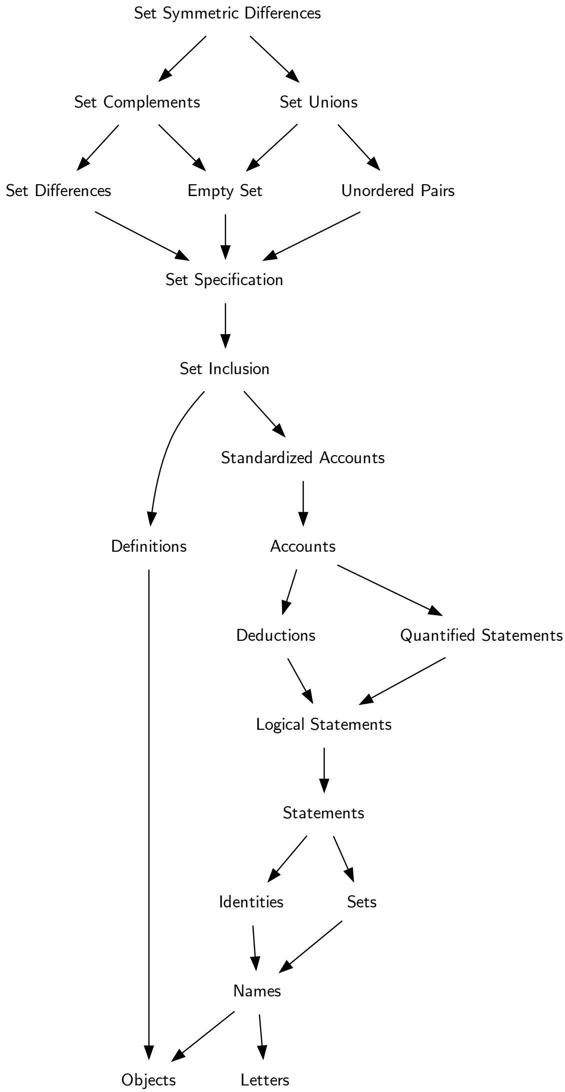
Set Complements (26)
Set Unions (19)

Set Symmetric Differences (30) is immediately needed by:

Set Operations (40)

Set Symmetric Differences (30) gives the following terms.

symmetric difference
Boolean sum


Set Symmetric Differences
Set Complements
Set Unions
Set Differences
Empty Set
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Why

We want to consider all the subsets of a given set.

Principle 6 (Powers). *For every set, there exists a set of its subsets.*

We call the existence of this set the *principles of powers* and we call the set the *power set*.[22] As usual, the principle of extension gives uniqueness (see Set Equality). The power set of a set includes the set itself and the empty set.

Notation

Let A denote a set. We denote the power set of A by A^*, read aloud as "powerset of A." $A \in A^*$ and $\varnothing \in A^*$. However, $A \subset A^*$ is false.

Examples

Let a, b, c denote distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in A^*$. We can walk through examples of power sets.

Empty Set

Proposition 31. $\varnothing^* = \{\varnothing\}$

[22] This terminology is standard, but unfortunate. Future editions may change these terms.

Singletons

Proposition 32. $\{a\}^* = \{\varnothing, \{a\}\}$

Pairs

Proposition 33. $\{a,b\}^* = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$

Triples

Proposition 34. $\{a,b,c\}^*\{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

Properties

We can guess the following easy properties.[23]

Proposition 35. $\varnothing \in A^*$

Proposition 36. $A \in A^*$

We call A and $\varnothing$ the *improper* subsets of A. All other subset we call *proper*.

[23] Future editions will expand this account.

Set Powers (31) immediately needs:

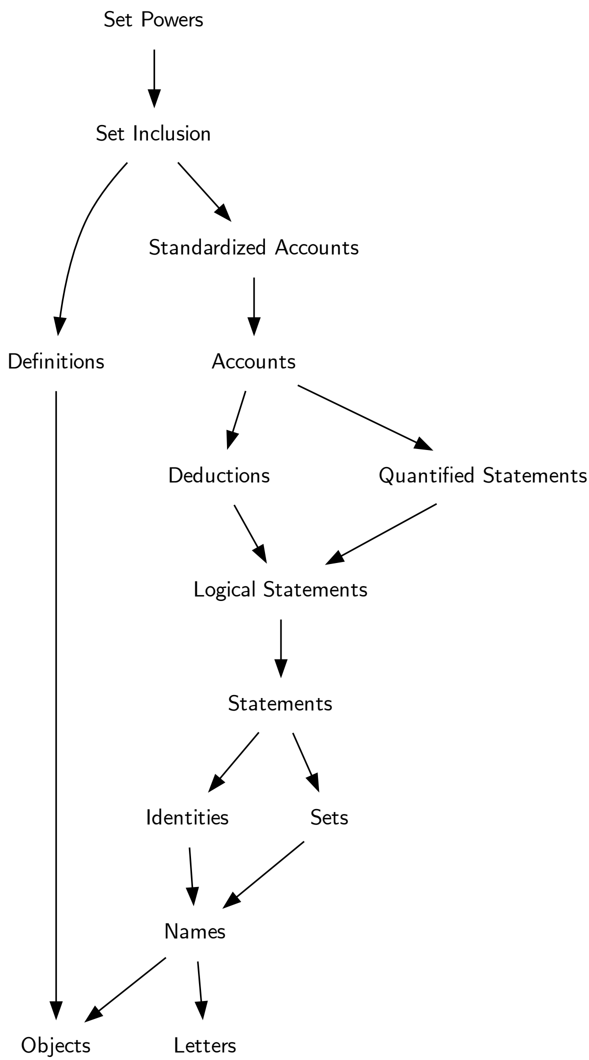
Set Inclusion (14)

Set Powers (31) is immediately needed by:

Families (42)
Probability Events (**??**)
Real Length Impossible (**??**)
Subset Systems (**??**)

Set Powers (31) gives the following terms.

principles of powers
power set
improper
proper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Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

Definition

Let A and B be nonempty sets. Let $a \in A$ and $b \in B$. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is a and the *second coordinate* is b.

The *product* of A and B is the set of all ordered pairs. This set is also called the *cartesian product*. If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Taste

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second "coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life. Plus, suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Equality

PROPOSITION **3.** *$(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.*

Proof. TODO $\square$

Ordered Pairs (32) immediately needs:

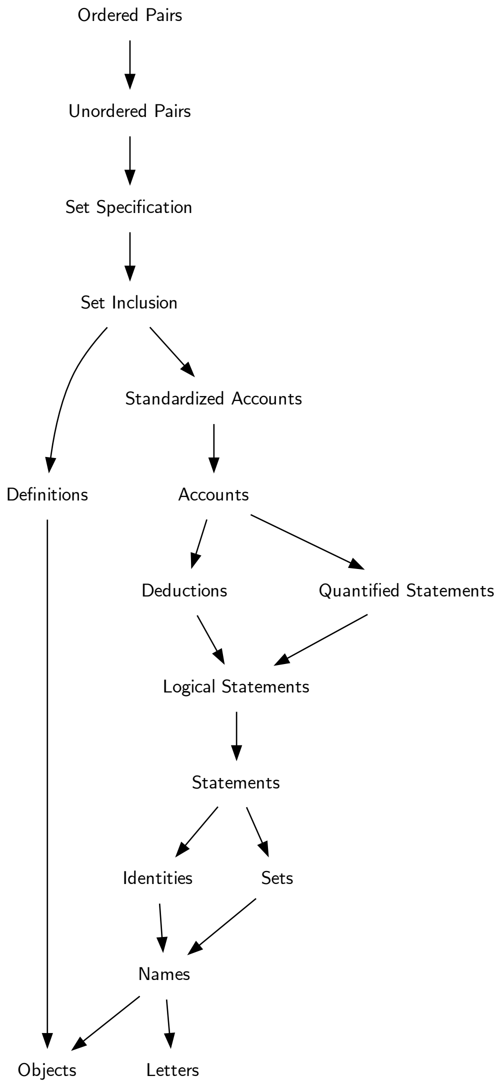
Unordered Pairs (18)

Ordered Pairs (32) is immediately needed by:

Product Sections (**??**)
Relations (33)
Subset Systems (**??**)

Ordered Pairs (32) gives the following terms.

ordered pair
first coordinate
second coordinate
product
cartesian product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Why

How can we relate the elements of two sets?

Definition

A *relation* between two nonempty sets is a subset of their cross product. A relation on a single set is a subset of the cross product of it with itself.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation. The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation.

Notation

Let A and B be two nonempty sets. A relation on A and B is a subset of $A \times B$. Let C be a nonempty set. A relation on a C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The ordered pair (a, b) may or may not be in a relation on A and B. Also notice that if $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as "a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *antisymmetric*antisymmetric if two different objects are related only in one order, and never both. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related.

Notation

Let R be a relation on a non-empty set A. R is reflexive if

$$(a, a) \in R$$

for all $a \in A$. R is transitive if

$$(a, b) \in R \wedge (b, c) \in R \longrightarrow (a, c) \in R$$

for all $a, b, c \in A$. R is symmetric if

$$(a, b) \in R \longrightarrow (b, a) \in R$$

for all $a, b \in A$. R is anti-symmetric if

$$(a, b) \in R \longrightarrow (b, a) \notin R$$

for all $a, b \in A$.

Relations (33) immediately needs:

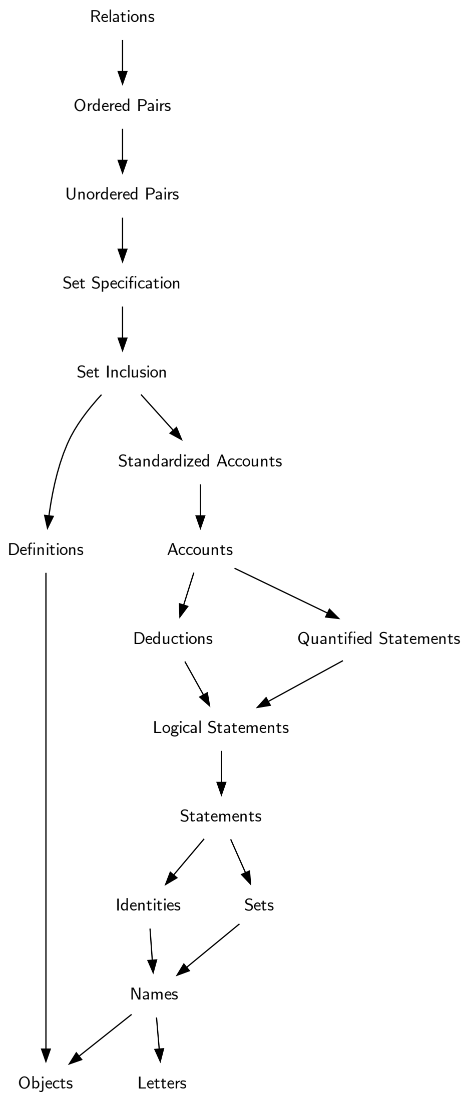
Ordered Pairs (32)

Relations (33) is immediately needed by:

Equivalence Relations (**??**)
Functions (34)
Partial Orders (**??**)

Relations (33) gives the following terms.

relation
domain
range
reflexive
symmetric
antisymmetric
transitive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Why

We want a notion for a correspondence between two sets.

Definition

A *functional* relation on two sets relates each element of the first set with a unique element of the second set. A *function* is a functional relation.

The *domain* of the function is the first set and *codomain* of the function is the second set. The function *maps* elements *from* the domain *to* the codomain. We call the codomain element associated with the domain element the *result* of *applying* the function to the domain element.

Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in (A \to B)$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

Functions (34) immediately needs:

Relations (33)

Functions (34) is immediately needed by:

Categories (**??**)
Families (42)
Function Composites (45)
Function Extensions (38)
Function Graphs (35)
Function Images (36)
Function Restrictions (37)
Identity Functions (**??**)
Injective Functions (46)
Operations (39)
Quasiconcave Functions (**??**)
Surjective Functions (47)

Functions (34) gives the following terms.

functional

function

domain

codomain

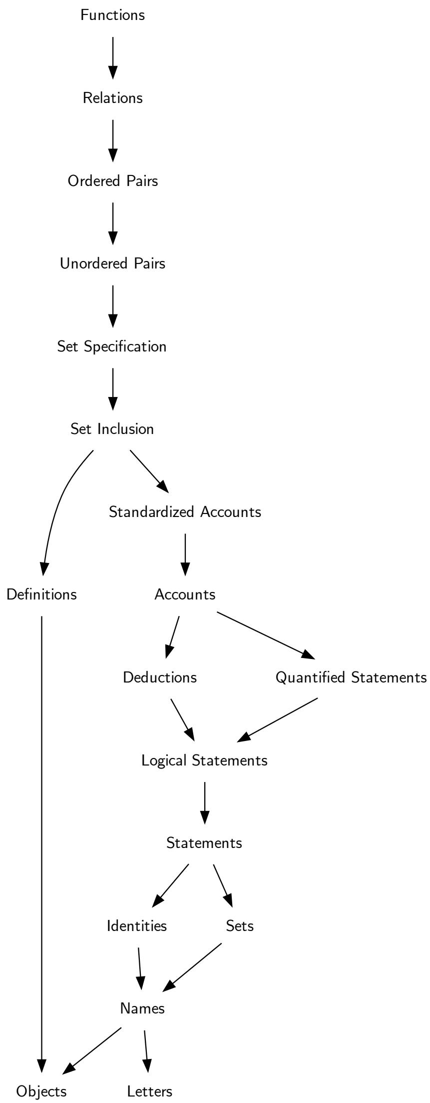
maps

from

to

result

applying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The set $\{(a, f(a)) \in A \times B \mid a \in A\}$ of ordered pairs is the *graph* of f.

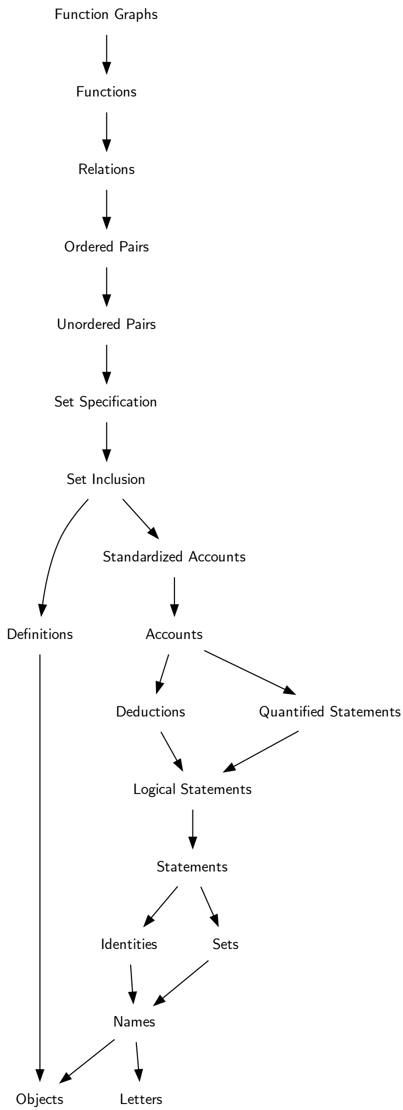
Function Graphs (35) immediately needs:

Functions (34)

Function Graphs (35) is not immediately needed by any sheet.

Function Graphs (35) gives the following terms.

graph


Function Graphs
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Why

We consider the set of results of a set of domain elements.

Definition

The *image* of a set of domain elements under a function is the set of their results. Though the set of domain elements may include several distinct elements, the image may still be a singleton, since the function may map all of elements to the same result.

The *range* of a function is the image of its domain. The range includes all possible results of the function. If the range does not include some element of the codomain, then the function maps no domain elements to that codomain element.

Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for every $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. Following this notation for function images, we denote the range of f by $f(A)$.

Function Images (36) immediately needs:

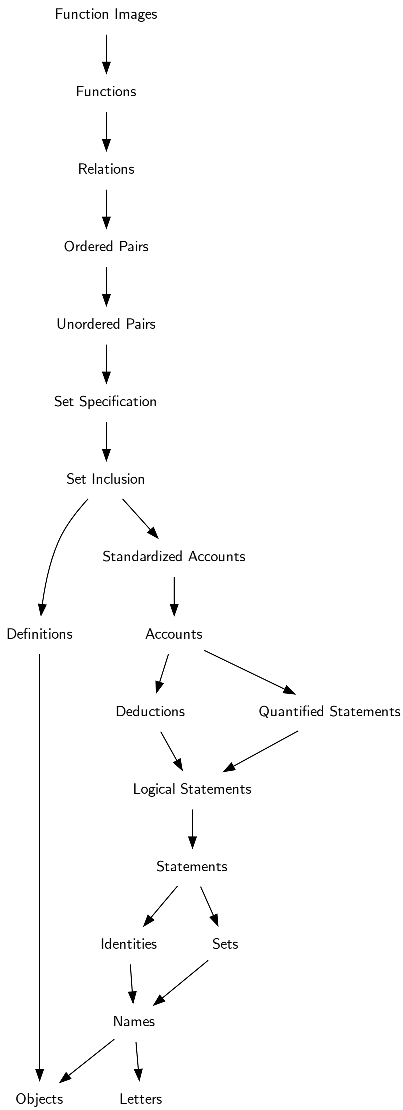
Functions (34)

Function Images (36) is not immediately needed by any sheet.

Function Images (36) gives the following terms.

image

range


Function Images
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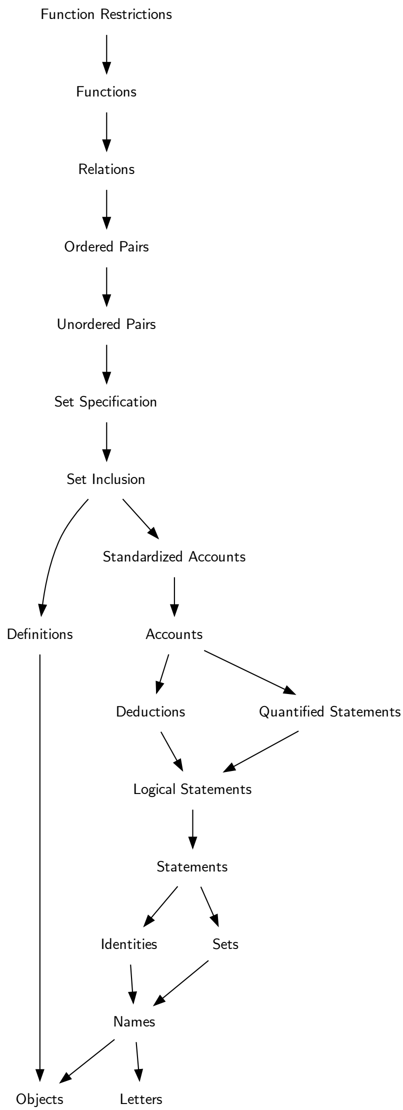
Function Restrictions (37) immediately needs:

Functions (34)

Function Restrictions (37) is not immediately needed by any sheet.

Function Restrictions (37) gives no terms.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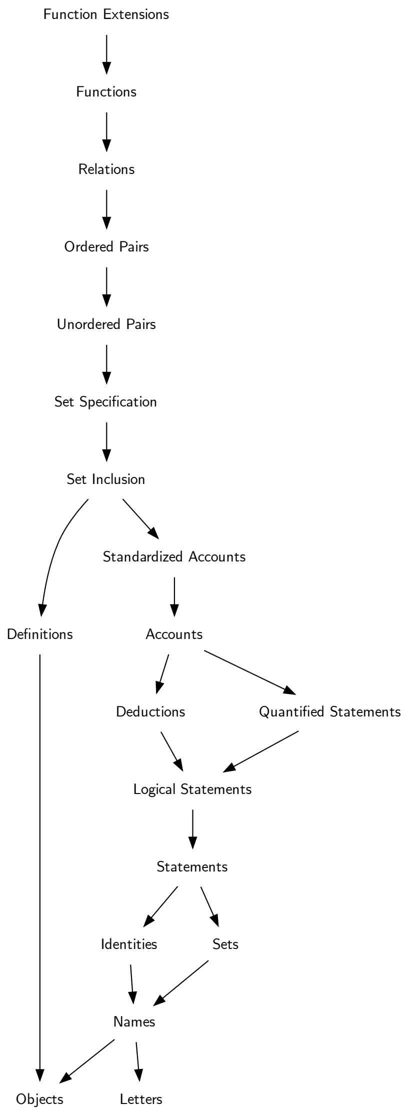
Function Extensions (38) immediately needs:

Functions (34)

Function Extensions (38) is not immediately needed by any sheet.

Function Extensions (38) gives no terms.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Why

We want to "combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations to *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a \, g \, b$ instead. We call this *infix notation*.

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Operations (39) immediately needs:

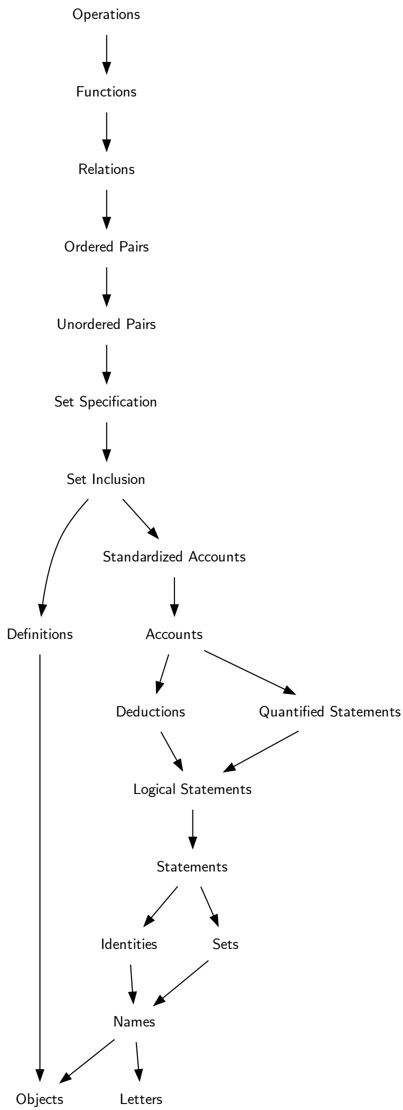
Functions (34)

Operations (39) is immediately needed by:

Algebras (41)
Associative Operations (**??**)
Commutative Operations (**??**)

Operations (39) gives the following terms.

operation
combine
operate
infix notation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Why

We want to consider the elements of two sets together at once, and other sets created from two sets.

Definitions

Let A and B be two sets.

The *union* of A with B is the set whose elements are in either A *or* B *or* both. The key word in the definition is *or*.

The *intersection* of A with B is the set whose elements are in both A *and* B. The keyword in the definition is *and*.

Viewed as operations, both union and intersection commute; this property justifies the language "with." The intersection is a subset of A, of B, and of the union of A with B.

The *symmetric difference* of A and B is the set whose elements are in the union but not in the intersection. The symmetric difference commutes because both union and intersection commute; this property justifies the language "and." The symmetric difference is a subset of the union.

Let C be a set containing A. The *complement* of A in C is the symmetric difference of A and C. Since $A \subset C$, the union is C and the intersection is A. So the complement is the "left-over" elements of B after removing the elements of A.

We call these four operations *set-algebraic operations*.

Notation

Let A, B be sets. We denote the union of A with B by $A \cup B$, read aloud as "A union B." $\cup$ is a stylized U. We denote the intersection of A with B by $A \cap B$, read aloud as "A intersect B." We denote the symmetric difference of A and B by $A + B$, read aloud as "A symdiff B." "Delta" is a mnemonic for difference.

Let C be a set containing A. We denote the complement of A in C by $C - A$, read aloud as "C minus A."

Results

Proposition 4. *For all sets A and B the operations $\cup$, $\cap$, and $+$ commute.*

Proposition 5. *Let S a set. For all sets $A, B \subset S$,*

$$\text{(1)} \quad S - (A \cup B) = (S - A) \cap (S - B)$$

$$\text{(2)} \quad S - (A \cap B) = (S - A) \cup (S - B).$$

Proposition 6. *Let S a set. For all sets $A, B \subset S$,*

$$A + B = (A \cup B) \cap C_S(A \cap B)$$

$\boxed{TODO : notation}$

Set Operations (40) immediately needs:

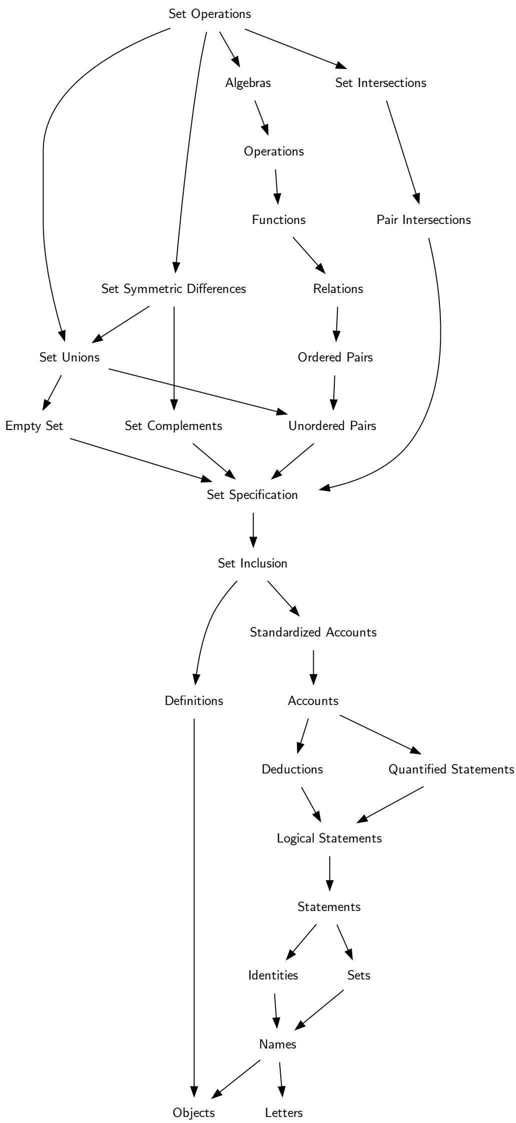
Algebras (41)
Set Intersections (23)
Set Symmetric Differences (30)
Set Unions (19)

Set Operations (40) is immediately needed by:

Convex Sets (??)
Event Probabilities (??)
Extended Real Numbers (??)
Family Set Operations (44)
Generated Sigma Algebra (??)
Monotone Classes (??)
Partitions (??)
Pointwise vs Measure Limits (??)
Real Length Impossible (??)
Subset Algebras (??)
Tail Sigma Algebra (??)
Topological Spaces (??)

Set Operations (40) gives the following terms.

intersection
symmetric difference
complement
set-algebraic operations
union



Set Operations
Algebras
Set Intersections
Operations
Functions
Pair Intersections
Set Symmetric Differences
Relations
Set Unions
Ordered Pairs
Empty Set
Set Complements
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Algebras (41) immediately needs:

Operations (39)

Algebras (41) is immediately needed by:

Element Functions (??)

Family Operations (43)

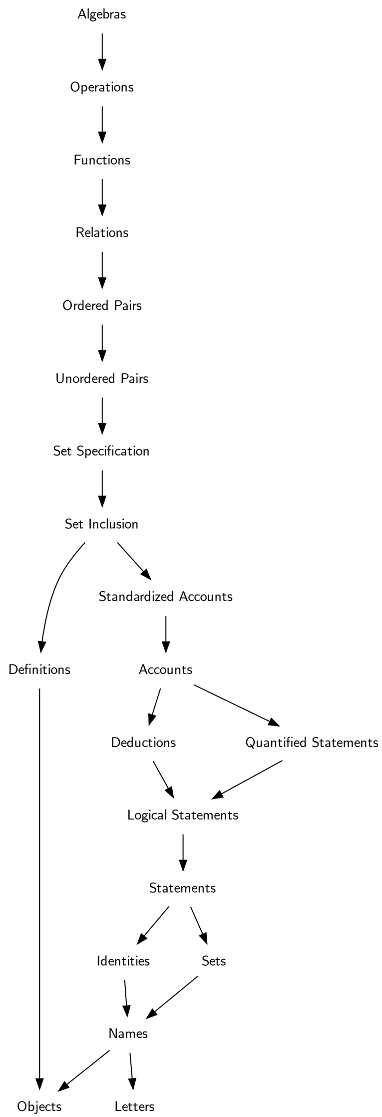
Groups (56)

Set Operations (40)

Algebras (41) gives the following terms.

algebra

ground set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Why

Halmos: "There are occasions when the range of a functionis deemed to be more important than the function itself. When that is the case, both the terminology and the notation undergo radical alterations." It is useful to have some language and notation for talking about a set of sets.

Definition

A *family* of sets is a set of sets. Experience shows that it is useful to have these associated with the elements of a well-known second set.

An *indexed family of sets* is a function from one set to the power set of a second set. We call the first set the *index set*. We call the second set the *base set*. The range of the indexed family of sets is, of course, a family.

Notation

Let A and I be be a non-empty sets. We use I as a mnemonic for "index" set. Let $a : I \to A^*$ be a family. For $i \in I$, we follow the function notation and denote the result of applying a to i by a_i.

We denote the range of the family by family of a_α indexed with I by $\{a_\alpha\}_{\alpha \in I}$, which is short-hand for set-builder notation. We read this notation "a sub-alpha, alpha in I."

Families (42) immediately needs:

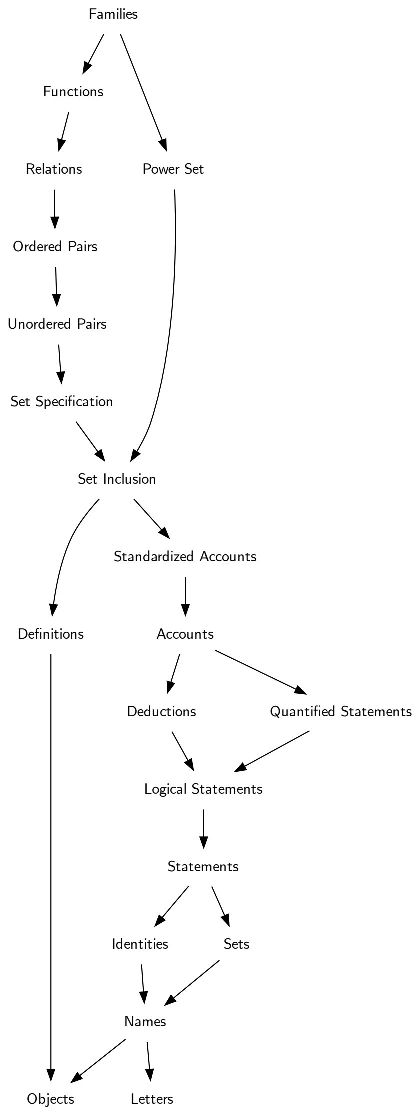
Functions (34)
Set Powers (31)

Families (42) is immediately needed by:

Family Operations (43)
Natural Families (**??**)

Families (42) gives the following terms.

finite family
countable family
uncountable family
family of sets
ordered family
family
indexed family of sets
index set
base set



Families
Functions
Relations
Power Set
Ordered Pairs
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
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Why

We want to generalize operations beyond two objects.

Operations

The *pairwise extension* of a commutative operation is the function from finite families of the ground set to the ground set obtained by applying the operation pairwise to elements. TODO: this is not a function if the operation is not commutative.

The *ordered pairwise extension* of an operation is the function from finite families ground set to the ground set obtained by applying the operation pairwise to elements in order.

Notation

Let $(A, +)$ be an algebra and $\{A_i\}_{i=1}^{n}$ a finite family of elements of A. We denote the pairwise extension by

$$\mathop{+}_{i=1}^{n} A_i$$

Family Operations (43) immediately needs:

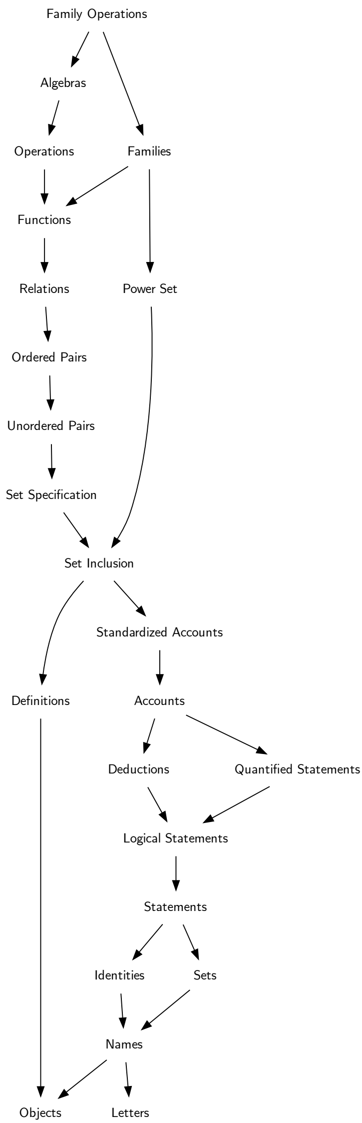
Algebras (41)
Families (42)

Family Operations (43) is immediately needed by:

Family Set Operations (44)
Natural Summation (**??**)
Summation (**??**)

Family Operations (43) gives the following terms.

pairwise extension
ordered pairwise extension


Family Operations
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Why

Family set operations are common. TODO: this works for infinite stuff too

Definition

We define the set whose elements are the objects which are contained in at least one family member the *family union*. We define the set whose elements are the objects which are contained in all of the family members the *family intersection*.

Notation

We denote the family union by $\cup_{\alpha \in I} A_\alpha$. We read this notation as "union over alpha in I of A sub-alpha." We denote family intersection by $\cap_{\alpha \in I} A_\alpha$. We read this notation as "intersection over alpha in I of A sub-alpha."

Results

Proposition **7.** *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \{i, j\}$ then*

$$\cup_{\alpha \in I} A_\alpha = A_i \cup A_j$$

and

$$\cap_{\alpha \in I} A_\alpha = A_i \cap A_j.$$

Proposition **8.** *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \varnothing$, then*

$$\cup_{\alpha \in I} A_\alpha = \varnothing$$

and

$$\cap_{\alpha \in I} A_\alpha = S.$$

Proposition **9.** *For an indexed family* $\{A_\alpha\}_{\alpha \in I}$ *in* S.

$$C_S(\cup_{\alpha \in I} A_\alpha) = \cap_{\alpha \in I} C_S(A_\alpha)$$

and

$$C_S(\cap_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I} C_S(A_\alpha).$$

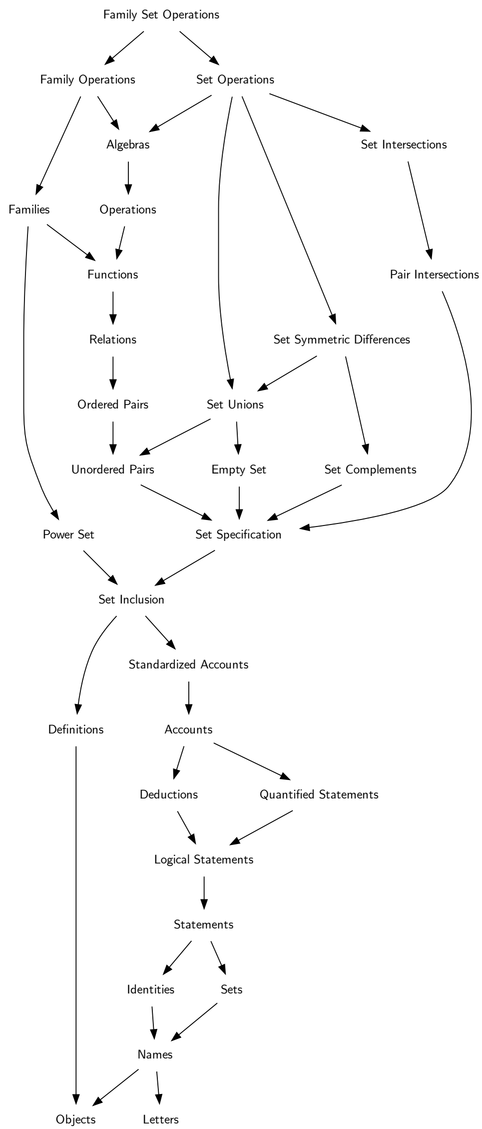
Family Set Operations (44) immediately needs:

Family Operations (43)
Set Operations (40)

Family Set Operations (44) is not immediately needed by any sheet.

Family Set Operations (44) gives the following terms.

family union
family intersection


Family Set Operations
Family Operations
Set Operations
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Why

We want a notion for applying two functions one after the other. We apply a first function then a second function.

Definition

Consider two functions for which the codomain of the first function is the domain of the second function.

The *composite* or *composition* of the second function with the first function is the function which associates each element in the first's domain with the element in the second's codomain that the second function associates with the result of the first function.

The idea is that we take an element in the first domain. We apply the first function to it. We obtain an element in the first's codomain. This result is an element of the second's domain. We apply the second function to this result. We obtain an element in the second's codomain. The composition of the second function with the first is the function so constructed.

Notation

Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$. We denote the composition of g with f by $g \circ f$ read aloud as "g composed with f." To make clear the domain and comdomain, we denote the composition $g \circ f : A \to C$.

In previously introduced notation, $g \circ f$ satisfies

$$(g \circ f)(a) = g(f(a))$$

for all $a \in A$.

Function Composites (45) immediately needs:

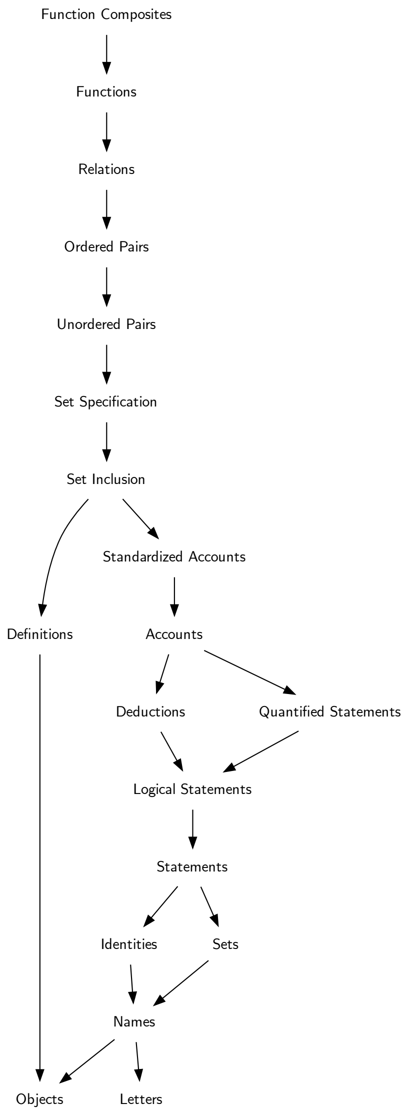
Functions (34)

Function Composites (45) is immediately needed by:

Function Inverses (48)
Sequences (52)

Function Composites (45) gives the following terms.

composite
composition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Why

An element of the codomain may be the result of several elements of the domain. This overlapping, using an element of the codomain more than once, is a regular occurrence.

Definition

If a function is a unique correspondence in that every domain element has a different result, we call it *one-to-one*. This language is meant to suggest that each element of the domain corresponds to one and exactly one element of the codomain, and vice versa.

We also call such functions *injective*.

Injective Functions (46) immediately needs:

Functions (34)

Injective Functions (46) is immediately needed by:

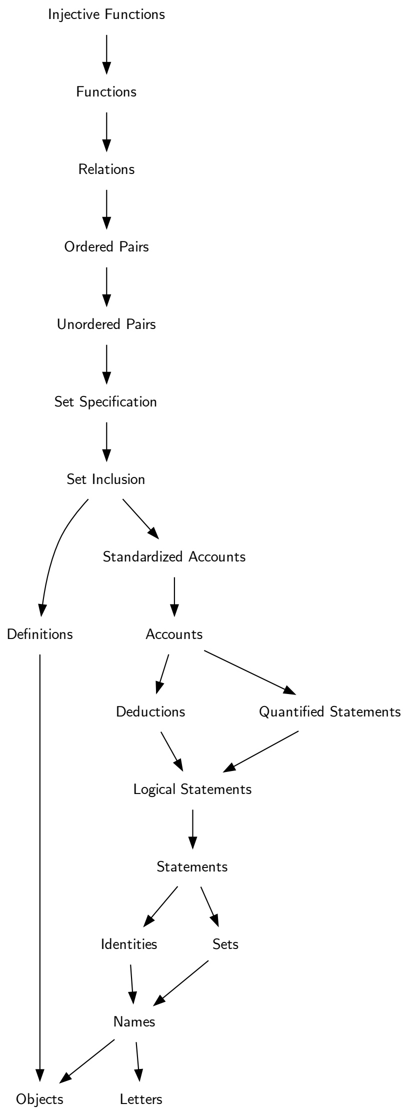
Cardinality (**??**)

Function Inverses (48)

Injective Functions (46) gives the following terms.

one-to-one

injective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Why

The range need not equal the codomain; though it, like every other image, is a subset of the codomain.

Definition

The function maps to domain *on* to the codomain if the range and codomain are equal; in this case we call the function *onto*. This language suggests that every element of the codomain is used by the function. It means that for each element of the codomain, we can find an element of the domain whose result is that first element of the codomain.

We also call the function *surjective*.

Notation

Let $f : A \to B$. Using prior notation, we can state that f is onto by writing $f(A) = B$.

Surjective Functions (47) immediately needs:

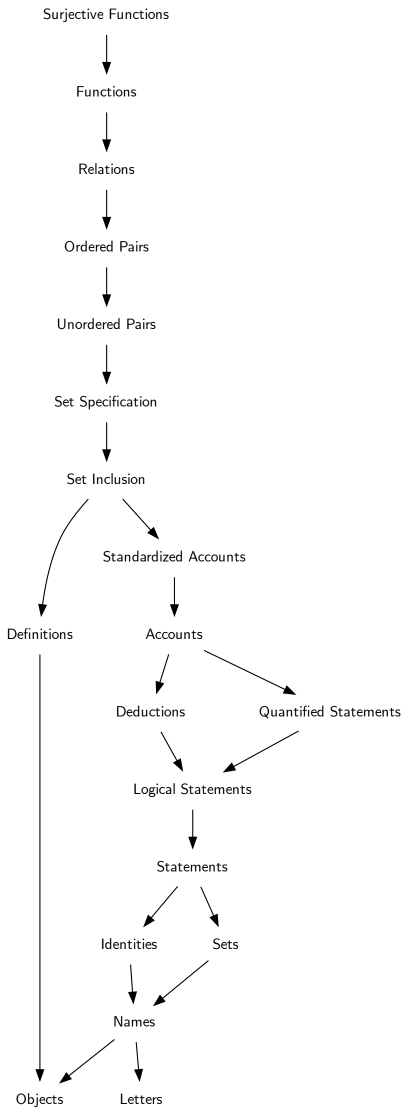
Functions (34)

Surjective Functions (47) is immediately needed by:

Function Inverses (48)

Surjective Functions (47) gives the following terms.

on
onto
surjective


Surjective Functions
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Why

We want a notion of reversing functions.

Definition

An *identity function* is a relation on a set which is functional and reflexive. It associates each element in the set with itself. There is only one identity function associated to each set.

Consider two functions for which the codomain of the first function is the domain of the second function and the codomain of the second function is the domain of the first function. These functions are *inverse functions* if the composition of the second with the first is the identity function on the first's domain and the composition of the first with the second is the identity function on the second's domain.

In this case we say that the second function is an *inverse* of the second, and vice versa. When an inverse exists, it is unique, so we refer to the *inverse* of a function. We call the first function *invertible.* Other names for an invertible function include *bijection.*

Notation

Let A a non-empty set. We denote the identity function on A by id_A, read aloud as "identity on A." id_A maps A onto A.

Let A, B be non-empty sets. Let $f : A \to B$ and $g : B \to A$ be functions. f and g are inverse functions if $g \circ f = \mathrm{id}_A$ and

$f \circ g = \mathrm{id}_B$.

The Inverse

We discuss existence and uniqueness of an inverse.

PROPOSITION **10.** *Let $f : A \to B$, $g : B \to A$, and $h : B \to A$.*

If g and h are both inverse functions of f, then $g = h$.

Proof. □

PROPOSITION **11.** *If a function is one-to-one and onto, it has an inverse.*

Proof. □

Inverse Images

The *inverse image* of a codomain set under a function is the set of elements which map to elements of the codomain set under the function. We denote the inverse image of $D \subset B$ by $f^{-1}(D)$, read aloud as "f inverse D."

Function Inverses (48) immediately needs:

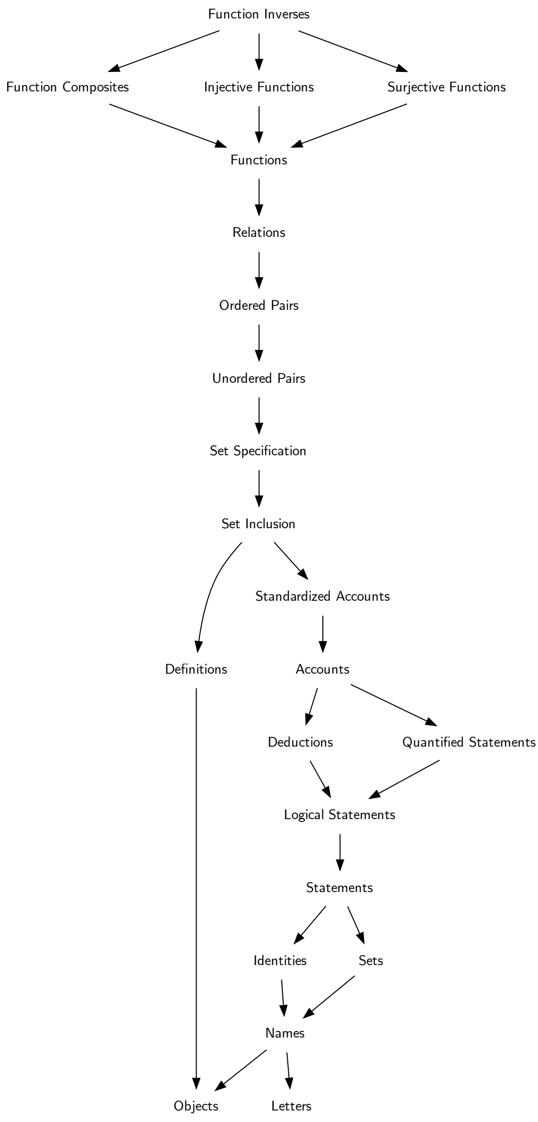
Function Composites (45)
Injective Functions (46)
Surjective Functions (47)

Function Inverses (48) is immediately needed by:

Equivalent Sets (49)
Isometries (**??**)
Permutations (**??**)

Function Inverses (48) gives the following terms.

identity function
inverse functions
inverse
inverse
inverse image
invertible
bijection


Function Inverses
Function Composites
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Why

We want to talk about the size of a set.

Definition

Two sets are *equivalent* if there exists a bijection between them.

Proposition **12.** *Set equivalence in the sense defined above is an equivalence relation in the power set of a set.*

Proposition **13.** *Every proper subset of a natural number is equivalent to some smaller natural number.*

Proof. TODO induction □

TODO: smaller defined?

Proposition **14.** *A set can be equivalent to a proper subset of itself.*

Halmos' example here is not a bijection, though...

Proposition **15.** *If n is a natural number, then n is not equivalent ot a proper subset of itself.*

Proposition **16.** *A set can be equivalent to at most one natural number.*

Proposition **17.** *The set of natural numbers is infinite.*

Proposition **18.** *A finite set is never equivalent to a proper subset of itself.*

Proposition **19.** *Every subset of a finite set is finite.*

Proposition **20.** *Every subset of a natural number is equivalent to a natural number.*

Equivalent Sets (49) immediately needs:

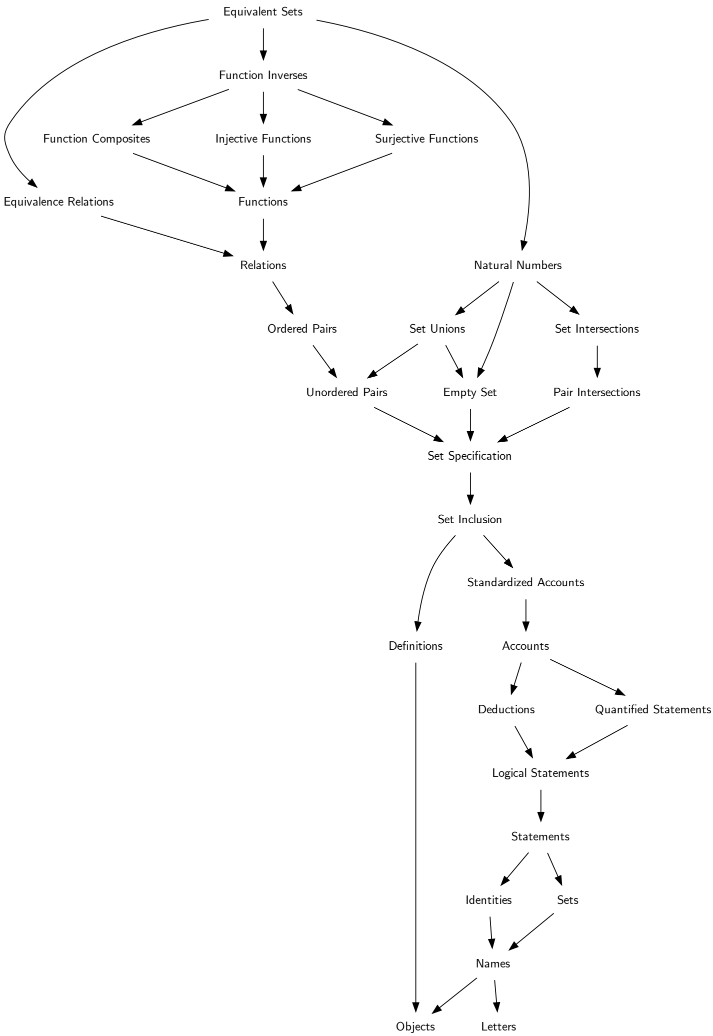
Equivalence Relations (**??**)
Function Inverses (48)
Natural Numbers (50)

Equivalent Sets (49) is immediately needed by:

Finite Sets (**??**)

Equivalent Sets (49) gives the following terms.

equivalent



Equivalent Sets
Function Inverses
Function Composites
Injective Functions
Surjective Functions
Equivalence Relations
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Why

We want to define the natural numbers. TODO: better why

Definition

The *successor* of a set is the union of the set with the singleton whose element is the set. This definition holds for any set, but is of interest only for the sets which will be defined in this sheet.

These sets are the following (and their successors): *One* is the successor of the empty set. *Two* is the successor of one. *Three* is the successor of two. *Four* is the successor of three. And so on; using the English language in the usual manner.

Can this be carried on and on? We will say yes. We will say that there exists a set which contains one and contains the successor of each of its elements. So, this set contains one. Since it contains one, it contains two. Since it contains two, it contains three. And so on. We call this assertion the *axiom of infinity*.

A set is a *successor set* if it contains one and if it contains the successor of each of its elements. In these words, the axiom of infinity asserts the existence of a successor set. We want this set to be unique. So we have a successor set. By the axiom of specification, the intersection of all the successor sets included in this first successor set exists. Moreover, this intersection is a successor set. Even more, this intersection is unique. For this, take a second successor set. Its intersection with the first successor set is contained in the first successor set. Thus, this intersection of two sets is one of the successor sets contained in the first set, and so, is contained in the intersection of all such sets. So then, that first intersection is contained in second intersection of two sets, which is, of course, contained in the second successor set. In other words, we start with a successor set. Use it to construct a succesor set contained in it, in such a way that every other successor set also contains this successor set so constructed. The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique.

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4.

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n.

Natural Numbers (50) immediately needs:

Empty Set (17)
Set Intersections (23)
Set Unions (19)

Natural Numbers (50) is immediately needed by:

Cardinality (**??**)
Coins (**??**)
Dice (**??**)
Equivalent Sets (49)
Natural Families (**??**)
Natural Induction (**??**)
Natural Order (**??**)
Natural Products (**??**)
Natural Sums (**??**)
Zero (54)

Natural Numbers (50) gives the following terms.

successor

One

Two

Three

Four

axiom of infinity

successor set

natural number

number

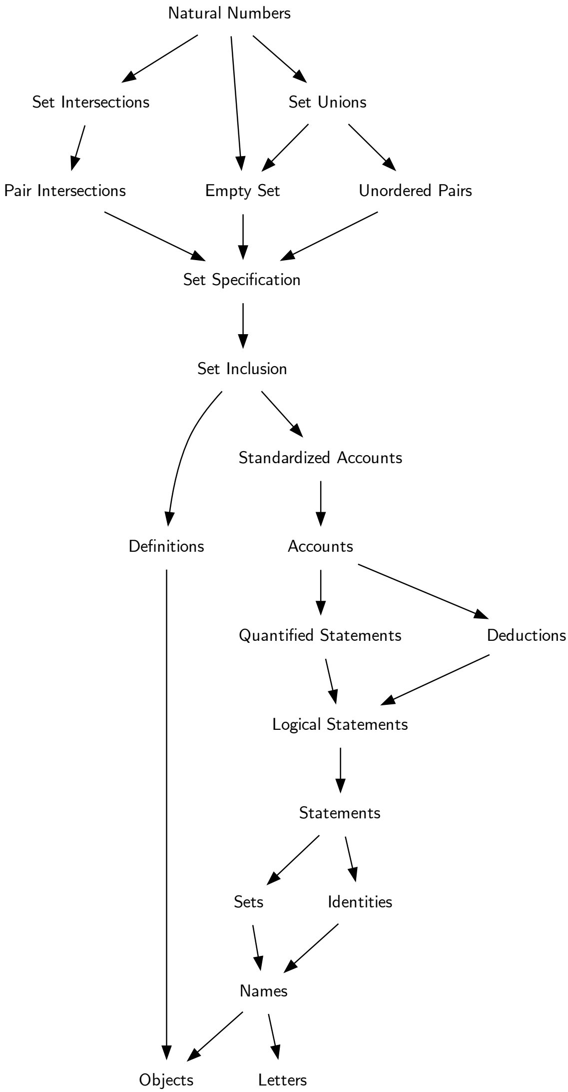
natural

set of natural numbers

natural numbers

naturals

numbers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Why

We generalize the idea of a product of two sets to a product of n sets.

Direct Products

The *direct product* of a natural family is the set of ordered sequences of elements from each set in the family. We call the elements of the direct product n*-tuples*. We call the ith element in an n-tuple the ith coordinate. This language is meant to follow that used in defining ordered pairs. Two coordinates in a sequence are *consecutive* if their natural difference is 1.

Notation

Let $A_1, \dots, A_n$ be a natural family of sets. We denote its direct product by

$$\prod_{i=1}^{n} A_i.$$

We read this notation as "product over alpha in I of A sub-alpha." We denote an element of $\prod_{i=1}^{n} A_i$ by $(a_1, a_2, \dots, a_n)$ with the understanding that $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$.

If $A_i = A$ for $i = 1, \dots, n$, then we denote $\prod_{i=1}^{n} A_i$ by A^n.

Direct Products (51) immediately needs:

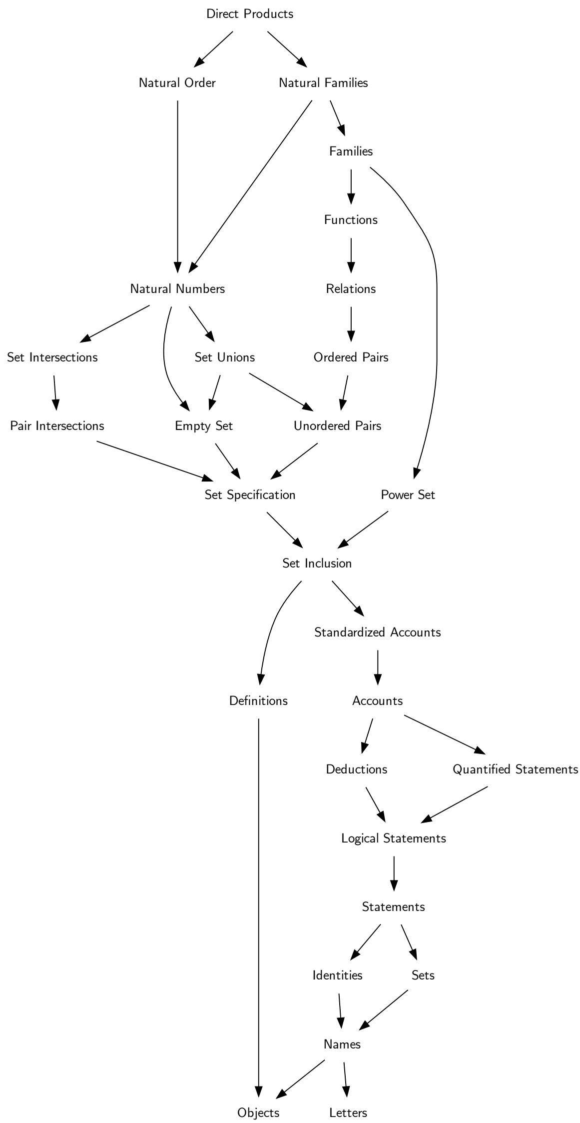
Natural Families (??)
Natural Order (??)

Direct Products (51) is immediately needed by:

Datasets (??)
Joint Distributions (??)
Marginal Distributions (??)
Product Metrics (??)
Product Sigma Algebras (??)
Random Variable Independence (??)
Sequences (52)
Undirected Paths (??)

Direct Products (51) gives the following terms.

direct product
n-tuples
sequences
consecutive


Direct Products
Natural Order
Natural Families
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Why

We introduce language for the steps of an infinite process.

Definition

Let A be a non-empty set. A *sequence in A* is a function from the natural numbers to the set. The *nth term* of a sequence is the result of the nth natural number; it is an element of the set.

Notation

Let A be a non-empty set. $a : \mathsf{N} \to A$. is a sequence in A. $a(n)$ is the nth term. We also denote a by $(a_n)_n$ and $a(n)$ by a_n.

Subsequences

A *subindex* is a monotonically increasing function from and to the natural numbers. Roughly, it selects some ordered infinite subset of natural numbers. A *subsequence* of a first sequence is any second sequence which is the composition of the first sequence with a subindex.

Notation

Let $i : N \to N$ such that $n < m \longrightarrow i(n) < i(m)$. Then i is a subindex. Let $b = a \circ i$. Then b is a subsequence of a. We denote it by $\{b_{i(n)}\}_n$ and the nth term by $b_{i(n)}$.

TODO: integrate, from direct products

If I is the set of natural numbers we denote the direct product by

$$\prod_{i=1}^{\infty} A_i.$$

We denote an element of $\prod_{i=1}^{\infty} A_i$ by (a_i) with the understanding that $a_i \in A_i$ for all $i = 1, 2, 3, \ldots$. If $A_i = A$ for all $i = 1, 2, 3, \ldots$, then (a_i) is a sequence in A.

Sequences (52) immediately needs:

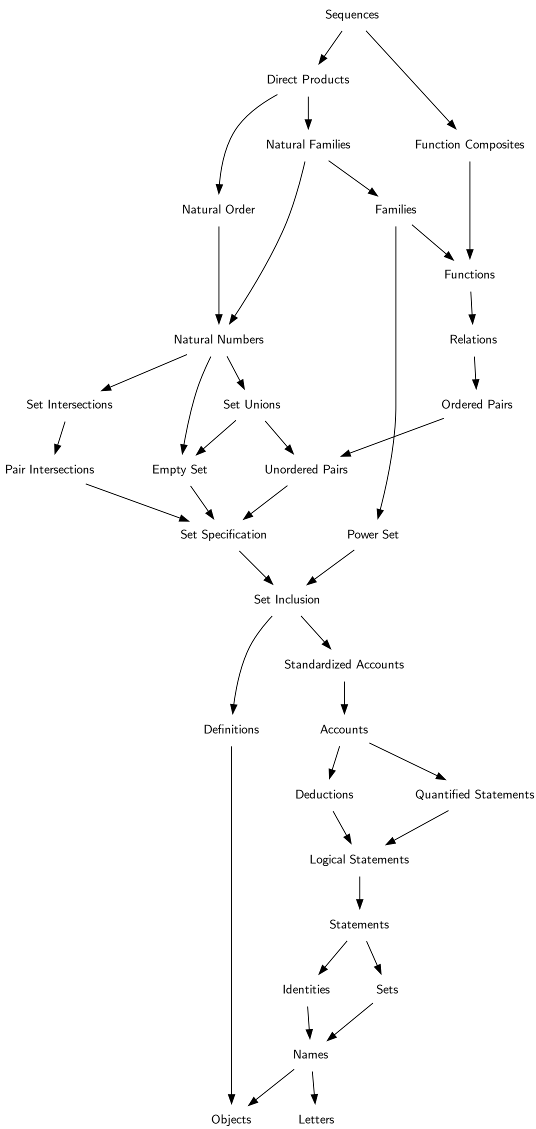
Direct Products (51)
Function Composites (45)

Sequences (52) is immediately needed by:

Almost Everywhere (**??**)
Central Limit Theorem (**??**)
Egoprox Sequences (**??**)
Monotone Algebras (**??**)
Monotone Classes (**??**)
Monotone Sequences (**??**)
Nets (**??**)
Random Variable Sigma Algebra (**??**)
Real Continuity (**??**)
Real Integral Series Convergence (**??**)
Real Sequences (**??**)
Tail Sigma Algebra (**??**)

Sequences (52) gives the following terms.

sequence in
nth term
subindex
subsequence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Direct Products
Natural Families
Function Composites
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Why

We represent rectangles by functions.

Definition

The *characteristic function* of a subset of some base set is the function from the base set to the real numbers which maps elements contained in the subset to value one and maps all other elements to zero. The range of the funtion is the set consisting of the real numbers one and zero.

If the base set is the real numbers and the subset is an interval, then the characteristic function is a rectangle with height one and the width of the interval.

Notation

Let A be a non-empty set and $B \subset A$. We denote the characteristic function of B in A by $\chi_B : A \to R$. The Greek letter χ is a mnemonic for "characteristic".

The subscript indicates the set on which the function is one. In other words, for all $B \subset A$, $\chi_B^{-1}(\{1\}) = B$.

If B is an interval and α is a real number then $\alpha\chi_B$ is a rectangle with height α.

Characteristic Functions (53) immediately needs:

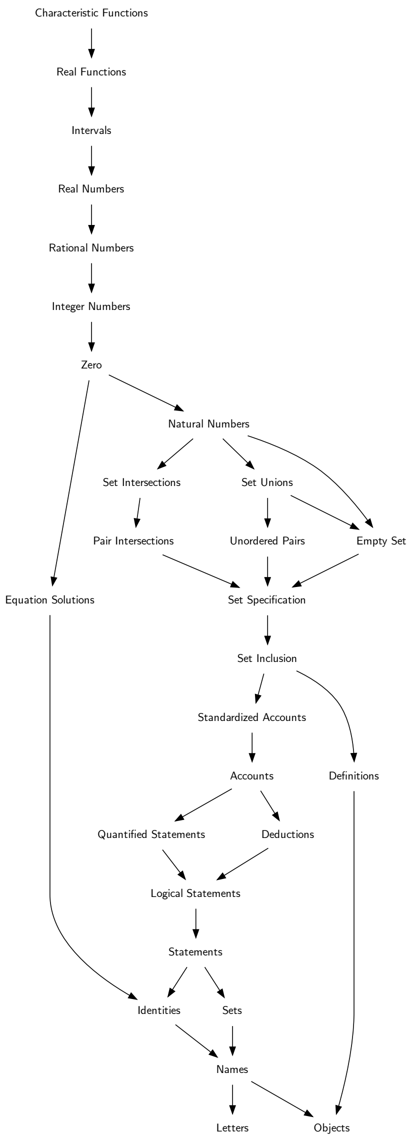
Real Functions (**??**)

Characteristic Functions (53) is immediately needed by:

Simple Functions (**??**)

Characteristic Functions (53) gives the following terms.

characteristic function
characteristic function
indicator function


Characteristic Functions
Real Functions
Intervals
Real Numbers
Rational Numbers
Integer Numbers
Zero
Natural Numbers
Set Intersections
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Equation Solutions
Set Specification
Set Inclusion
Standardized Accounts
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Why

If I am holding three pebbles, and I have three in my left hand, how many might I have in my right hand? None, of course!

In the notation we have developed we find solutions of

$$3 + n = 3,$$

where n is a natural number. Unfortunately, any natural number added to three is a number different than three. So there is no natural number such that this equation holds.

How can we expand our algebra so that we can express this new, but common, situation in our language?

Definition

We consider a superset of the natural numbers with one additional element. We call this new elemenet *zero* and we call this new set the *natural numbers with zero*.

Extending Arithmetic

We extend the algebra on the natural numbers to an algebra on the natural numbers with zero.

We define an extension of addition, also called *addition*. Given two natural numbers, we define the result as the sum. Given a natural number and zero, we define the result to be the natural number, and vice versa, so that addition still commutes. As a function, addition by zero is the identity.

We extend multiplication. Given two natural numbers, we define their product as before. Given a natural number and zero, we define the product to be the zero, and vice versa, so that addition still commutes. As a function, mutliplication by zero is constant.

A new solution

We have extended our algebra. We can search for solutions of:

$$3 + n = 3$$

where n is a natural number or zero. Zero is a solution (by design)! Since no natural number solves, zero is the only solution.

Notation

We denote zero by 0. For every natural number n, we defined

$$n + 0 = 0 + n = n \quad \text{and} \quad n \cdot 0 = 0 \cdot n = 0$$

Zero (54) immediately needs:

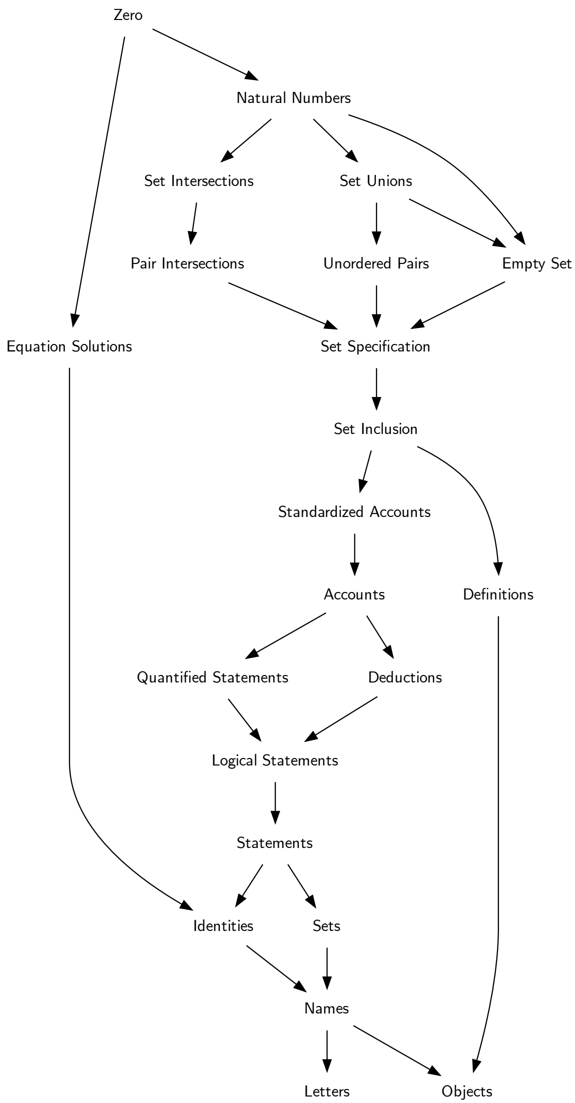
Equation Solutions (**??**)
Natural Numbers (50)

Zero (54) is immediately needed by:

Integer Numbers (55)

Zero (54) gives the following terms.

zero
natural numbers with zero
addition


Zero
Natural Numbers
Set Intersections
Set Unions
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Equation Solutions
Set Specification
Set Inclusion
Standardized Accounts
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Quantified Statements
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Why

Definition

integer numbers *integers*

TODO

Integer Numbers (55) immediately needs:

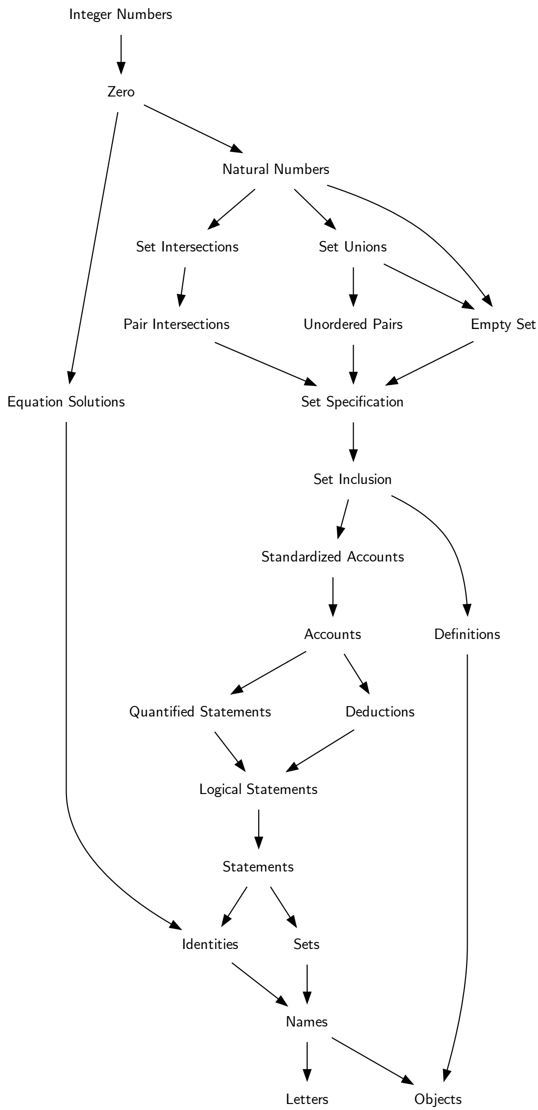
Zero (54)

Integer Numbers (55) is immediately needed by:

Groups (56)
Rational Numbers (57)
Rings (**??**)

Integer Numbers (55) gives the following terms.

integer numbers
integers


Integer Numbers
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Natural Numbers
Set Intersections
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Unordered Pairs
Empty Set
Equation Solutions
Set Specification
Set Inclusion
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Accounts
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Why

We generalize the algebraic structure of addition over the integers.

Definition

A *group* is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra *group addition*. A *commutative group* is a group whose operation commutes.

Notation

TODO

Groups (56) immediately needs:

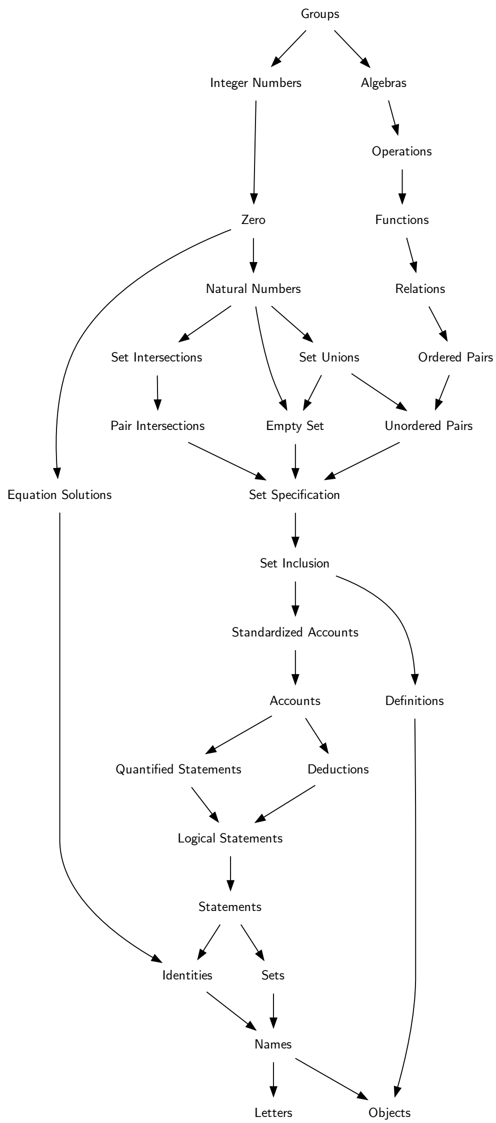
Algebras (41)
Integer Numbers (55)

Groups (56) is immediately needed by:

Fields (58)
Homomorphism (**??**)

Groups (56) gives the following terms.

group
group addition
commutative group


Groups
Integer Numbers
Algebras
Operations
Zero
Functions
Natural Numbers
Relations
Set Intersections
Set Unions
Ordered Pairs
Pair Intersections
Empty Set
Unordered Pairs
Equation Solutions
Set Specification
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Why

Definition

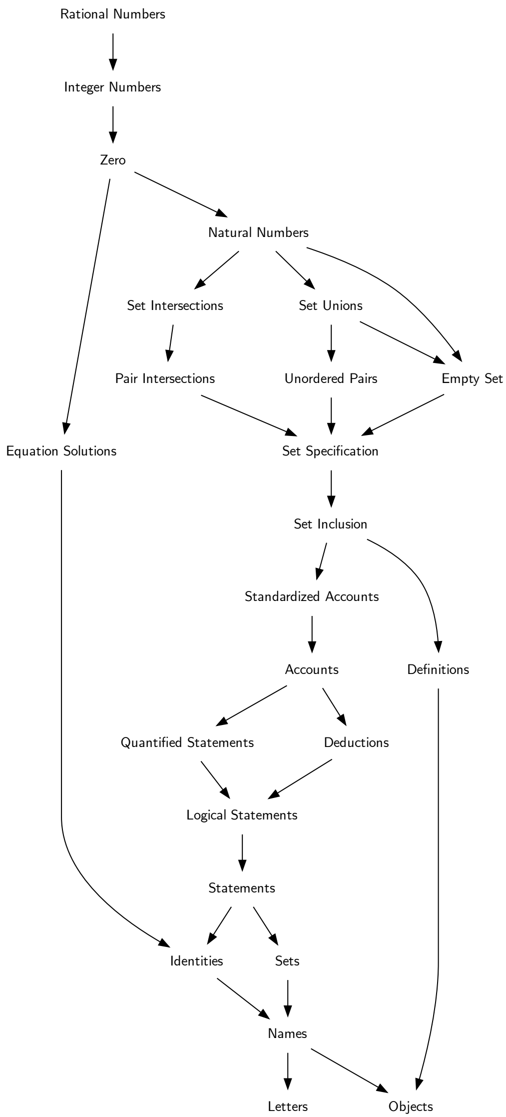
Rational Numbers (57) immediately needs:

Integer Numbers (55)

Rational Numbers (57) is immediately needed by:

Fields (58)
Real Numbers (59)

Rational Numbers (57) gives no terms.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Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *field addition*. We call the operation of the second algebra *field multiplication*.

Notation

We denote an arbitrary field by F, a mnemonic for "field."

TODO

Fields (58) immediately needs:

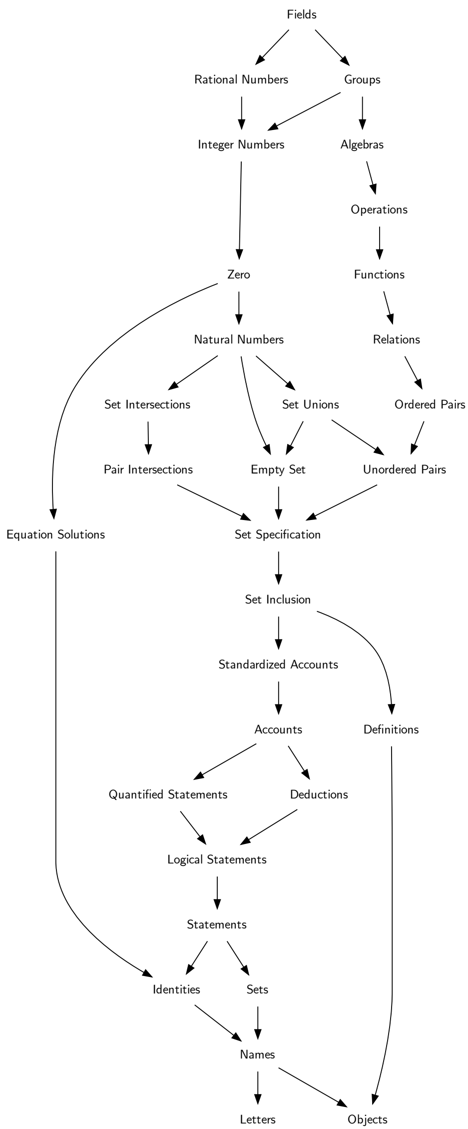
Groups (56)
Rational Numbers (57)

Fields (58) is immediately needed by:

Vectors (**??**)

Fields (58) gives the following terms.

field
field addition
field multiplication


Fields
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Why

Definition

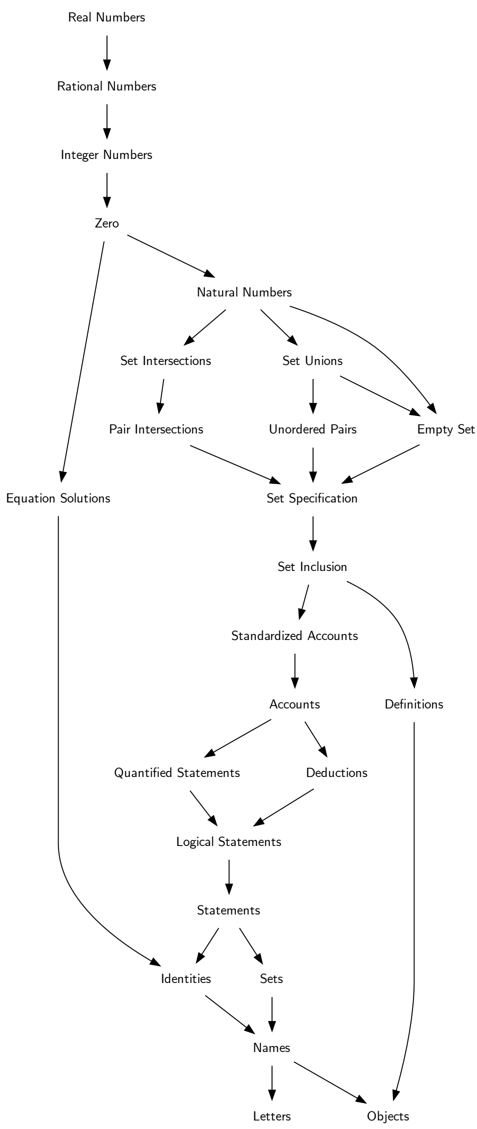
Real Numbers (59) immediately needs:

Rational Numbers (57)

Real Numbers (59) is immediately needed by:

Absolute Value (61)
Complex Numbers (60)
Intervals (62)
Logarithm (??)
Loss Functions (??)
Metrics (66)
N-Dimensional Space (??)
Quadratic Forms (??)
Real Continuity (??)
Real Length Impossible (??)
Real Optimizers (??)
Real Sequences (??)
Real Square Roots (??)
Real Summation (??)
Real Vectors (??)
Supremum (??)
Weighted Graphs (??)

Real Numbers (59) gives no terms.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Why

We want to find roots of negative numbers

Definition

A *complex number* is an ordered pair of real numbers. The *real part* of a complex number is its first coordinate. The *imaginary part* of a complex number is its second coordinate.

Notation

Let z be a complex number. We denote the real part of z by $\mathsf{Re}(z)$, read "real of z," and the imaginary part by $\mathsf{Im}(z)$, read "imaginary of z." So if $z = (a, b)$, then $\mathsf{Re}(z) = a$ and $\mathsf{Im}(z) = b$.

Complex Numbers (60) immediately needs:

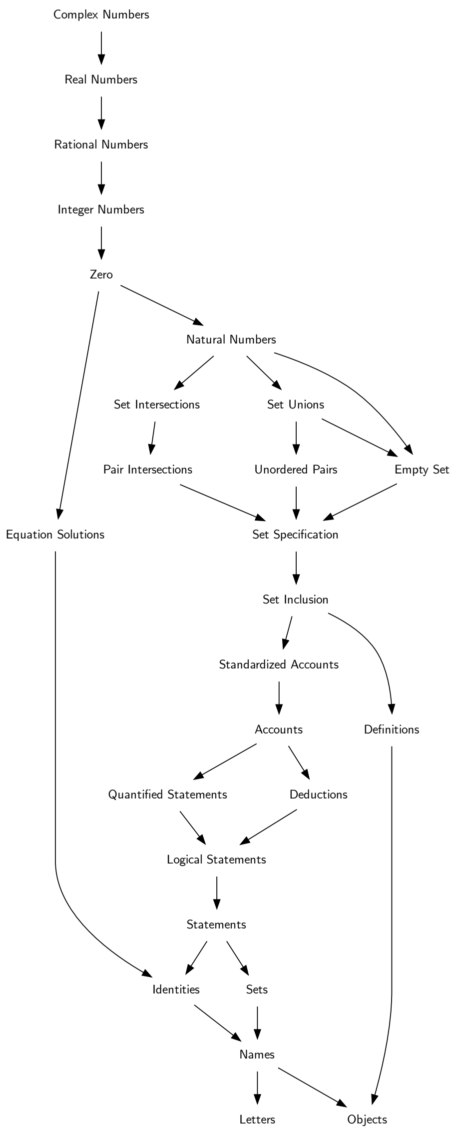
Real Numbers (59)

Complex Numbers (60) is immediately needed by:

Complex Inner Products (**??**)
Complex Integrals (**??**)

Complex Numbers (60) gives the following terms.

complex number
real part
imaginary part


Complex Numbers
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Why

We want a notion of distance between elements of the real line.

Definition

We define a function mapping a real number to its length from zero.

Notation

We denote the absolute value of a real number $a \in \mathsf{R}$ by $|a|$. Thus $|\cdot| : \mathsf{R} \to \mathsf{R}$ can be viewed as a real-valued function on the real numbers which is nonnegative.

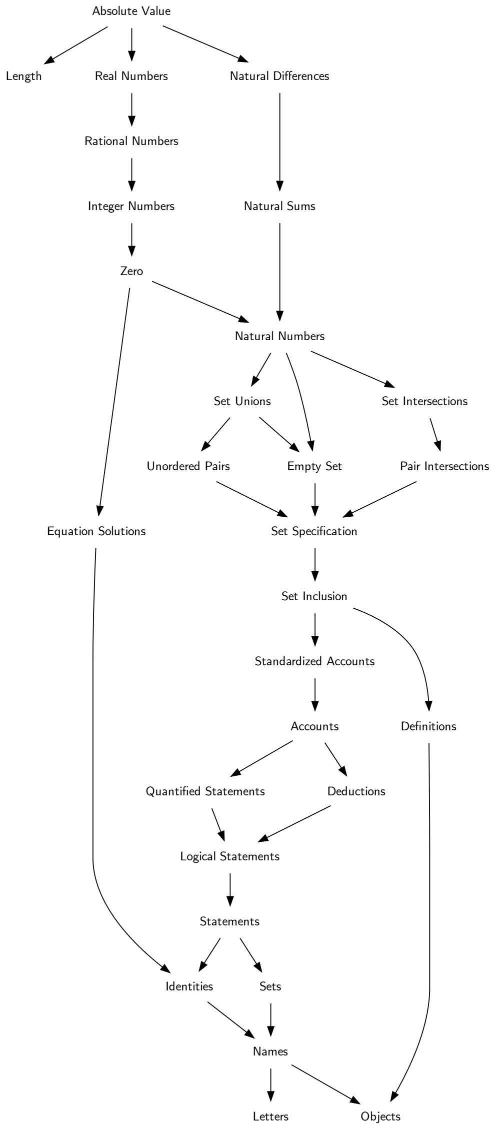
Absolute Value (61) immediately needs:

Length (??)
Natural Differences (??)
Real Numbers (59)

Absolute Value (61) is immediately needed by:

Chordal Graphs (??)
Complex Integrals (??)
Convergence In Measure (??)
Convergence In Probability (??)
Expectation Deviation Upper Bound (??)
Functionals (??)
Integrable Function Spaces (??)
Metric Space Examples (??)
Pointwise vs Measure Limits (??)
Real Continuity (??)
Real Convergence (??)
Real Integral Monotone Convergence (??)
Singular Measures (??)
Supremum Norm (??)
Variation Measure (??)

Absolute Value (61) gives no terms.


Absolute Value
Length
Real Numbers
Natural Differences
Rational Numbers
Integer Numbers
Natural Sums
Zero
Natural Numbers
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Pair Intersections
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Why

We name and denote subsets of the set of real numbers which correspond to segments of a line.

Definition

Take two real numbers, with the first less than the second.

An *interval* is one of four sets:

1. the set of real numbers larger than the first number and smaller than the second; we call the interval *open*.
2. the set of real numbers larger than or equal to the first number and smaller than or equal to the second number; we call the interval *closed*.
3. the set of real numbers larger than the first number and smaller than or equal to the second; we call the interval *open on the left* and *closed on the right*.
4. the set of real numbers larger than or equal to the first number and smaller than the second; we call the interval *closed on the left* and *open on the right*.

If an interval is neither open nor closed we call it *half-open* or *half-closed*

We call the two numbers the *endpoints* of the interval. An open interval does not contain its endpoints. A closed interval contains its endpoints. A half-open/half-closed interval contains only one of its endpoints. We say that the endpoints *delimit* the interval.

Notation

Let a, b be two real numbers which satisfy the relation $a < b$.

We denote the open interval from a to b by (a, b). This notation, although standard, is the same as that for ordered pairs; no confusion arises with adequate context.

We denote the closed interval from a to b by $[a, b]$. We record the fact $(a, b) \subset [a, b]$ in our new notation.

We denote the half-open interval from a to b, closed on the right, by $(a, b]$ and the half-open interval from a to b, closed on the left, by $[a, b)$.

Intervals (62) immediately needs:

Real Numbers (59)

Intervals (62) is immediately needed by:

Convex Sets (**??**)

Extended Real Numbers (**??**)

Interval Graphs (**??**)

Interval Length (**??**)

Interval Partitions (**??**)

Probability Distributions (**??**)

Product Sections (**??**)

Real Functions (**??**)

Uniform Densities (**??**)

Intervals (62) gives the following terms.

interval

open

closed

open on the left

closed on the right

closed on the left

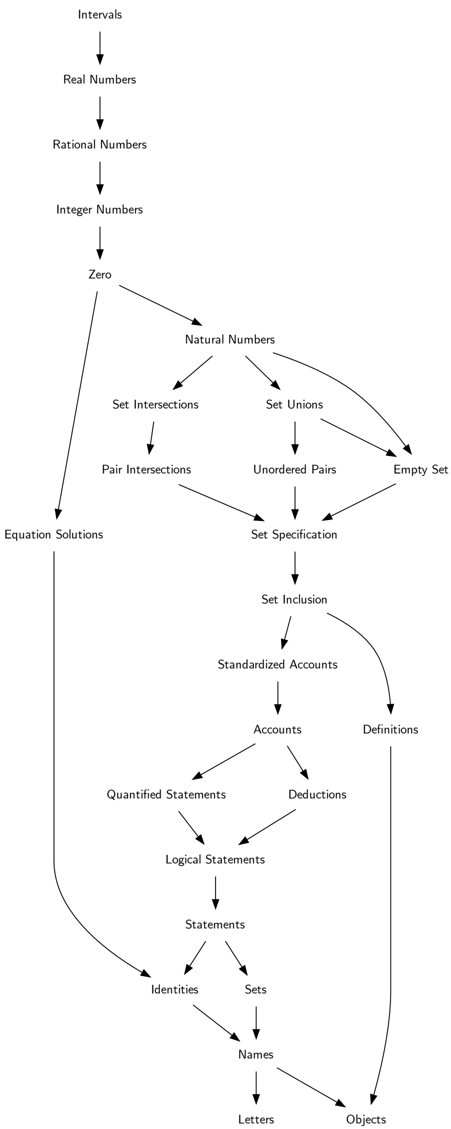
open on the right

half-open

half-closed

endpoints

delimit


Intervals
Real Numbers
Rational Numbers
Integer Numbers
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Why

We want to define the length of a subset of real numbers.

Notions

We take two common notions:

1. The length of a whole is the sum of the lengths of its parts; the *additivity principle*.

2. The length of a whole is the at least the length of any whole it contains the *containment principle*.

The task is to make precise the use of "whole,", "parts," and "contains." We start with intervals.

Definition

By whole we mean set. By part we mean an element of a partition. By contains we mean set containment.

The *length* of an interval is the difference of its endpoints: the larger minus the smaller.

Two intervals are *non-overlapping* if their intersection is a single point or empty. The *length* of the union of two non-overlapping intervals is the sum of their lengths.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A *countably simple* subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A *cover* of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

Cuts

If a, b are real numbers and $a < b$, then we *cut* an interval with a and b as its endpoints by selecting c such that $a < c$ and $c < b$. We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the *cut pieces*.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

All sets

PROPOSITION **21.** *Not all subsets of real numbers are simple.*

Exhibit: R is not finite.

PROPOSITION **22.** *Not all subsets of real numbers are countably simple.*

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

Notation

Length Common Notions (63) does not immediately need any sheet.

Length Common Notions (63) is immediately needed by:

Distance (64)

Length Common Notions (63) gives the following terms.

additivity principle

containment principle

length

non-overlapping

length

simple

countably simple

cover

cut

cut pieces

Length Common Notions

Why

We want to talk about the "distance" between objects in a set.

Common Notions

Our inspiration is the notion of distance in the plane of geometry. The objects are points and the distance between them is the length of the line segment joining them. We note a few properties of this notion of distance:

1. The distance between any two distinct objects is not zero.
2. The distance between any two objects does not depend on the order in which we consider them.
3. The distance between two objects is no larger than the sum of the distances of each with any third object

The first observation is natural: if two points are not the same, then they are some distance apart. In other words, the line segment between them has length.

The second observation is natural: the line segment connecting two points does not depend on the order specifying the points. This observation justifies the word "between." If it were not the case, then we should use different words, and be careful to speak of the distance "from" a first point "to" a second point.

The third property is a non-obvious property of distance in the plane. It says, in other words, that the length of any side of a triangle is no larger than the sum of the lengths of the two other sides. With experience in geometry, the observation may become natural. But it does not seem to be superficially so.

A more muddled but superficially natural justification for our concern with third observation is that it says something about the transitivity of closeness. Two objects are close if their distance is small. Small is a relative concept, and needs some standard of comparison. Let us fix two points, take the distance between them, and call it a unit. We call two objects close with respect to our unit if their distance is less than a unit.

In this language, the third observation says that if we know two objects are each half of a unit distance from a third object, then the two objects are close (their distance is less than a unit). We might call this third object the reference object. Here, then, is the usefulness of the third property: we can infer closeness of two objects if we know their distance to a reference object.

Distance (64) immediately needs:

Length Common Notions (63)

Distance (64) is immediately needed by:

Distance Asymmetry (65)
Metrics (66)

Distance (64) gives no terms.

Distance

$\downarrow$

Length Common Notions

Why

Sometimes "distance" as used in the English language refers to an asymmetric concept. This apparent paradox further illuminates the symmetry property.

Apparent Paradox

Distance in the plane is symmetric: the distance from one point to another does not depend on the order of the points so considered. We took this observation as a definiting property of our abstract notion of distance. The meaning, strength, and limitation of this property is clarified by considering an asymmetric case.

Contrast walking up a hill with walking down it. The "distance" between these two points, the top of the hill and a point on its base, may not be symmetric with respect to the time taken or the effort involved. Experience suggests that it will take longer to walk up the hill than to walk down it. A superficial justification may include reference to the some notion of uphill walking requiring more effort.

If we were going to model the top and base of the hill as points in space, however, the distance between them is the same: it is symmetric. It is even the same if we take into account that some specific path, a trail say, must be followed.

If planning a backpacking trip, such symmetry appears foolish. The distance between two locations must not be considered symmetric. Going up the mountain takes longer than going down. It may justify, in the English phrase, “going around, ather than going over.”

Distance Asymmetry (65) immediately needs:

Distance (64)

Distance Asymmetry (65) is not immediately needed by any sheet.

Distance Asymmetry (65) gives no terms.

$$
\begin{tikzcd}
\text{Distance Asymmetry} \arrow[d] \\
\text{Distance} \arrow[d] \\
\text{Length Common Notions}
\end{tikzcd}
$$

Why

We want to talk about a set with a prescribed quantitative degree of closeness (or distance) between its elements.

Definition

The correspondences which serve as a degree of closeness, or measure of distance, must satisify our notions of distances previously developed.

A function on ordered pairs which does not depend on the order of the elements so considered is *symmetric*. A function into the real numbers which takes only non-negative values is *non-negative*. A repeated pair is an ordered pair of the same element twice. A function which satisfies a triangle inequality for any three elements is *triangularly transitive*.

A *metric* (or *distance function*) is a function on ordered pairs of elements of a set which is symmetric, non-negative, zero only on repeated pairs, and triangularly transitive. A *metric space* is an ordered pair: a nonempty set with a metric on the set.

In a metric space, we say that one pair of objects is *closer* together if the metric of the first pair is smaller than the metric value of the second pair.

Notice that a set can be made into different metric spaces by using different metrics.

Notation

Let A be a set and let R be the set of real numbers. We commonly denote a metric by the letter d, as a mnemonic for "distance." Let $d : A \times A \to R$. Then d is a metric if:

1. it is non-negative, which we tend to denote by

$$d(a, b) \geq 0, \quad \forall a, b \in A.$$

2. it is 0 only on repeated pairs, which we tend to denote by

$$d(a, b) = 0 \Leftrightarrow a = b, \quad \forall a, b \in A.$$

3. it is symmetric, which we tend to denote by:

$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$

4. it is triangularly transitive, which we tend to denote by

$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of A with d by (A, d).

Metrics (66) immediately needs:

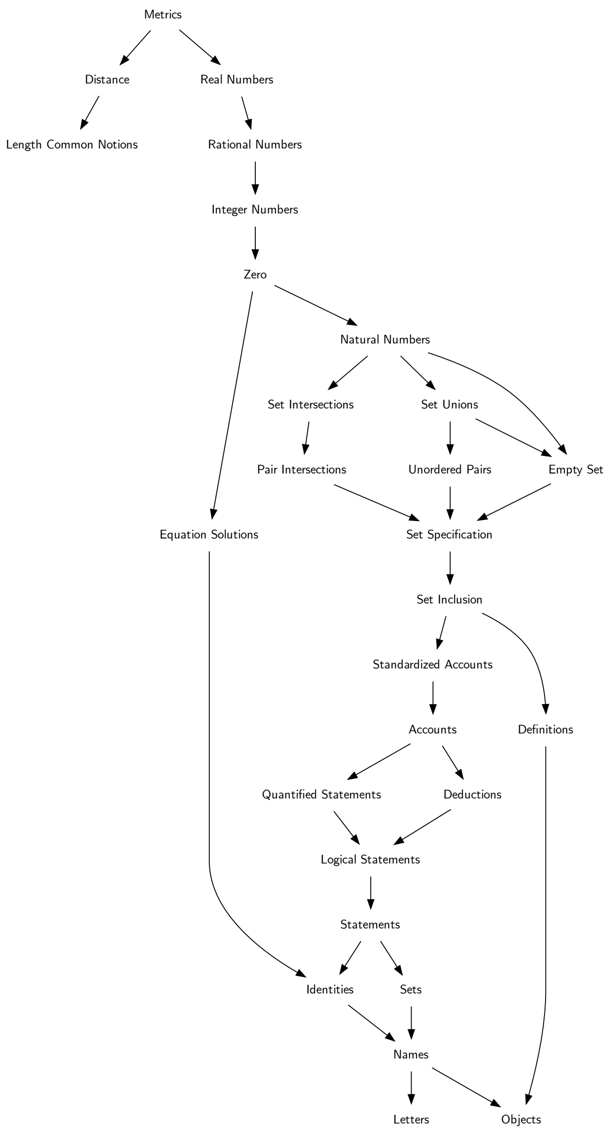
Distance (64)
Real Numbers (59)

Metrics (66) is immediately needed by:

Egoprox Sequences (**??**)
Isometries (**??**)
Metric Balls (**??**)
Metric Continuity (**??**)
Metric Convergence (**??**)
Metric Space Examples (**??**)
Metric Space Functions (**??**)
Norm Metrics (**??**)
Product Metrics (**??**)
Similarity Functions (**??**)
Topological Spaces (**??**)

Metrics (66) gives the following terms.

symmetric
non-negative
triangularly transitive
metric
distance function
metric space
closer


Metrics
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Real Numbers
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Why

We want to visualize relations.

Definition

A *graph* is a set and a relation on the set. The graph is *undirected* if the relation is symmetric; otherwise the graph is *directed*.

A *vertex* of the graph is an element of the set. The set is called the *vertex set*. An *edge* of the graph is an element of the relation. The relation is called the *edge set*.

If the graph is directed, we call the first element of an edge the *parent* of the second element. We call the second element of an edge the *child* of the first element. So we can discuss the set of parents or set of children of a particular vertex (these sets may be empty).

A graph is *complete* if it has all possible edges.

Notation

We denote the vertex set by V, a mnemonic for vertex. We denote the edge set by E, a mnemonic for edge. We denote a graph by (V, E). If the vertex set is assumed, or if every vertex appears in E we can unambiguously refer to the graph by E.

Visualization

We visualize a graph by drawing a point for each vertex. If the pair of two vertices u and v is an edge we draw a line segment whose endpoints are the points corresponding to the vertices.

Paths

A path in a graph is a sequence of vertices with the property that consecutive vertices are related. A path *cycles* if a vertex appears more than once. A path is *finite* if the sequence is finite. A *loop* is a finite path that cycles once. A finite path from vertex u to vertex v is a path starting with u and ending with v. The *length* of a finite path is the length of the sequence.

Properties

A graph is *connected* if there is a path between every pair of vertices. A graph is *acyclic* if none of its paths cycle.

Graphs (67) immediately needs:

Unordered Pairs (18)

Graphs (67) is immediately needed by:

Conditional Dependency Graph (**??**)
Graph Cliques (**??**)
Graph Complements (**??**)
Subgraphs (**??**)
Trees (68)
Weighted Graphs (**??**)

Graphs (67) gives the following terms.

graph

undirected

directed

vertex

vertex set

edge

edge set

bipartite

cycles

finite

loop

length

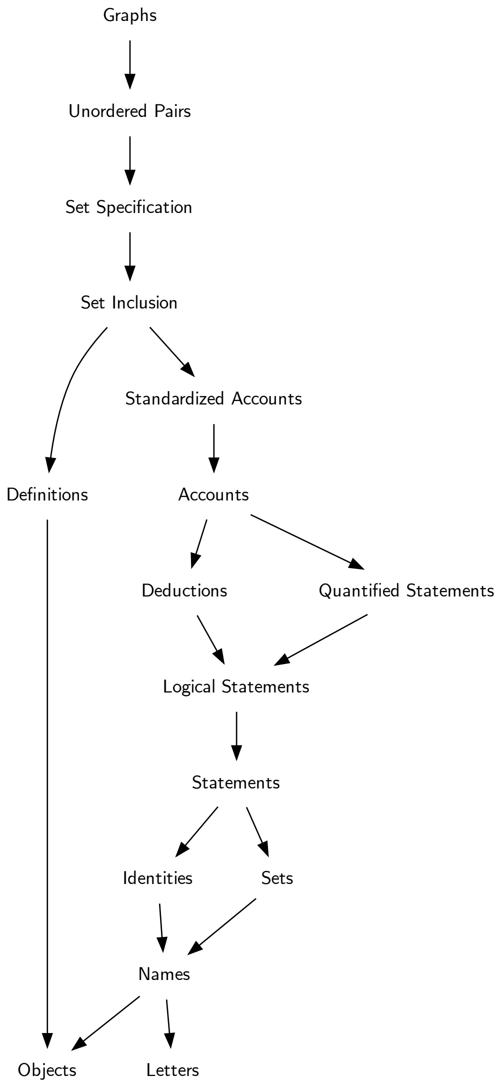
connected

acyclic

parent

child

complete


Graphs
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Set Specification
Set Inclusion
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TODO: make this not jsut the edge set...

Why

Tree branches split and do not recombine. We formalize this property in the language of graphs.

Definition

A *tree* is a connected and acyclic undirected graph. A *forest* is an undirected graph if it does not contain any cycles.

Notation

Let $T = (V, E)$ be a tree, a mnemonic for "tree."

Properties

PROPOSITION **23.** *A unique path exists between any two vertices of a tree.*

Proof. The path exists because a tree is connected. The path is unique, since were it not, we could create a cycle by combining these paths. $\square$

Trees (68) immediately needs:

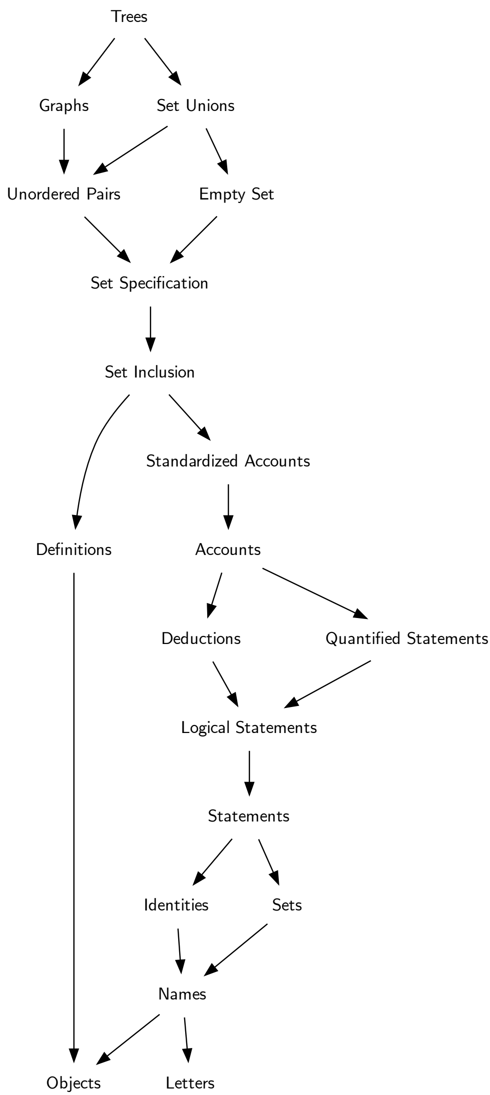
Graphs (67)
Set Unions (19)

Trees (68) is immediately needed by:

Rooted Trees (**??**)
Spanning Trees (**??**)

Trees (68) gives the following terms.

tree
forest


Trees
Graphs
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Definitions
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Objects
Letters

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