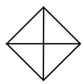


Why

It is believable that $1/2, 1/4, 1/8, \dots$ has a convergent series. And likewise with $1/3, 1/9, 1/27, \dots$ What of $a_k = x^k$ for $x \in \mathbf{R}$.

Definition

Let $x \in \mathbf{R}$. The *geometric series* of x is the series of the sequence (a_k) defined by $a_k = x^k$.

Characterization of convergence

Does the geometric series of x converge? In other words, does (s_n) defined by $s_n = \sum_{k=1}^{n} x^k$ have a limit.

For $x = 1$ and $x = -1$, we have seen (see Real Series) that the series diverges. However for the cases $x = 1/2$ and $x = 1/3$ the geometric series converges.

Proposition 1. *If $|x| < 1$, then the geometric series of x converges and*

$$\lim_{n\to\infty} \sum_{k=1}^{n} x^k = \frac{x}{1-x}$$

If $|x| \geq 1$ then the geometric series of x diverges.

Proof. Define $s_n = \sum_{k=1}^{n} x^k$. Then

$$\begin{aligned} x \cdot s_n &= x \cdot (x^1 + x^2 + \cdots + x^n) \\ &= x^2 + x^3 + \cdots + x^{n+1} \\ &= s_n - x + x^{n+1}. \end{aligned}$$

From which we deduce, $s_n(1-x) = x(1-x^n)$. If $x \neq 1$, then

$$s_n = \frac{1}{1-x}(1-x^n)$$

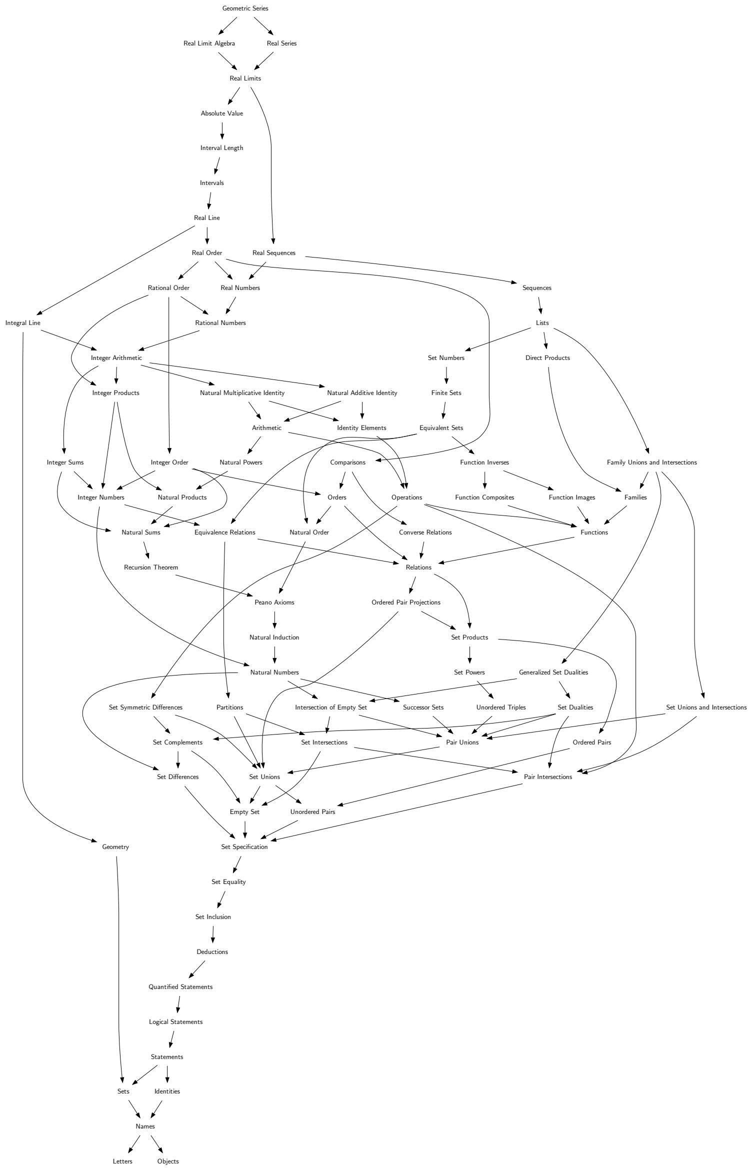
If $|x| < 1$, then using the algebra of limits (see Real Limit Algebra) we deduce

$$\lim_{n\to\infty} \frac{1}{1-x}(1-x^n) = \frac{1}{1-x}(1-0) = \frac{1}{1-x},$$

since $\lim_{k\to\infty} x^k = 0$ for $|x| < 1$.

If $x = 1$ or $x = -1$, then we have seen that (s_n) diverges.[1] $\square$

[1] Future editions will include the trivial account about the case $|x| > 1$.


Geometric Series
Real Limit Algebra
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Rational Order
Real Numbers
Sequences
Integral Line
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Integer Sums
Integer Order
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects