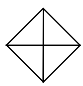


Why

We expect measure to have the common sense properties we stated when trying to define a notion of length for the real line.

Monotonicity

An extended-real-valued function on an alebra is *monotone* if, given a first distinguished set contained in a distinguished second set, the result of the first is no greater than the result of the second.

Proposition 1. *All measures are monotone.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $A, B \in \mathcal{A}$ and $A \subset B$. Then $B = A \cup (B - A)$, a disjoint union. So

$$\mu(B) = \mu(A \cup (B - A)) = \mu(A) + \mu(B - A),$$

by the additivity of μ. Since $\mu(B - A) \geq 0$, we conclude $\mu(A) \leq \mu(B)$. □

Proposition 2. *If $A \subset B$ and B finite, then $\mu(B - A) = \mu(B) - \mu(A)$.*[1]

Subadditivity

Monotonicity along with additivity of measures give us one other convenient property: subadditivity.

An extended-real-valued function on an algebra is *subadditive* if, given a sequence of distinguished sets, the result of union of the sequence is no greater than the limit of the partial sums of the results on each element of the sequence.

[1] Future editions will contain a proof.

Proposition 3. *All measures are subadditive.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space.

Let $\{A_n\} \subset \mathcal{A}$. Define $\{B_n\} \subset \mathcal{A}$ with $B_n := A_n - \cup_{i=1}^{n-1} A_i$. Then $\cup_n A_n = \cup_n B_n$, $\{B_n\}$ is a disjoint sequence, and $B_n \subset A_n$ for each n. So

$$\mu(\cup_n A_n) = \mu(\cup_n B_n) = \sum_{i=1}^{\infty} \mu(B_n) \leq \sum_{i=1}^{\infty} \mu(A_n),$$

by additivity and then montonicity of measure. □

The inequality involved in subadditivity is sometimes called Boole's inequality *or* Bonferroni's inequalities *or the* union bound*; each of these terms is most common with discussion* Probability Measures*.*

Limits

Measures also behave well under limits.

An extended-real-valued function on an algebra *resolves under increasing limits* if the result of the union of an increasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets. An extended-real-valued function on an algebra *resolves under decreasing limits* if the result of the intersection of a decreasing sequence of distinguished sets coincides with the limit of the sequence of results on the individual sets.

Proposition 4. *All measures resolve under increasing limits.*

Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\}$ be an increasing sequence in $\mathcal{A}$. Then we want to show: $\mu(\cup_n A_n) = \lim_{n \to \infty} \mu(A_n)$.

Define $\{B_n\}$ such that $B_n := A_n - \cup_{i=1}^{n-1} A_i$. Then $\{B_n\}$ is disjoint, $A_n = \cup_{i=1}^{n} B_i$ for each n, $\cup_n A_n = \cup_n B_n$, and $\mu(\cup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \mu(B_i)$, by additivity. So

$$\begin{aligned}
\mu(\cup_n A_n) &= \mu(\cup_n B_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) \\
&= \lim_{n\to\infty} \mu(\cup_{i=1}^{n} B_i) \\
&= \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

□

Proposition 5. *Measures resolve under decreasing limits if there is a finite set in the decreasing sequence.*

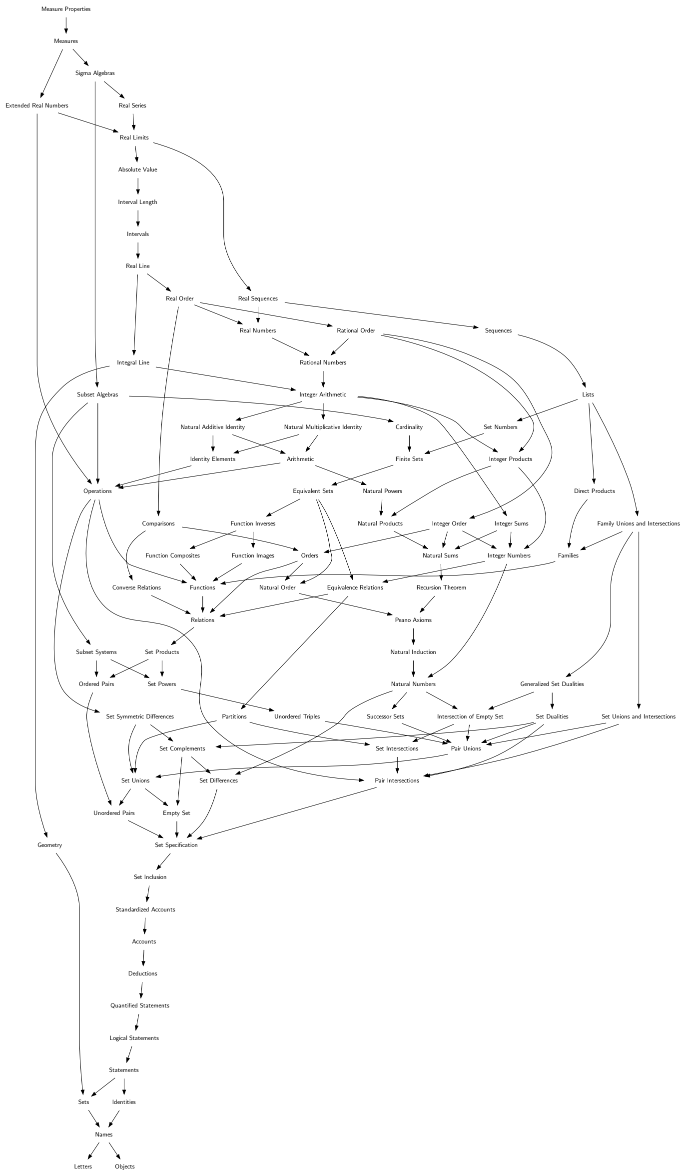
Proof. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let $\{A_n\}$ be a decreasing sequence in $\mathcal{A}$ with one element finite. Then we want to show: $\mu(\cap_n A_n) = \lim_{n\to\infty} \mu(A_n)$.

On one hand, let n_0 be the index of the first finite element of the sequence. Then for all $n \geq n_0$, the sequence is finite because of the monotonicity of measure. Denote this decreasing finite subsequence of sets by $\{B_n\}$. Then $\cap_n A_n = \cap_n B_n$ and $\lim_n A_n = \lim_n B_n$.

On the other hand, the sequence $\{B_1 - B_n\}$ is an increasing sequence in $\mathcal{A}$. Also $\cap_n B_n = B_1 - \cup_n (B_1 - B_n)$. So

$$\begin{aligned}
\mu(\cap_n B_n) &= \mu(B_1 - \cup_n (B_1 - B_n)) \\
&= \mu(B_1) - \mu(\cup_n (B_1 - B_n)) \\
&= \mu(B_1) - \lim_n \mu(B_1 - B_n) \\
&= \mu(B_1) - \left(\lim_n \mu(B_1) - \mu(B_n)\right) \\
&= \lim_n B_n.
\end{aligned}$$

$\square$


Measure Properties
Measures
Sigma Algebras
Extended Real Numbers
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Real Numbers
Rational Order
Sequences
Integral Line
Rational Numbers
Subset Algebras
Integer Arithmetic
Lists
Natural Additive Identity
Natural Multiplicative Identity
Cardinality
Set Numbers
Identity Elements
Arithmetic
Finite Sets
Integer Products
Operations
Equivalent Sets
Natural Powers
Direct Products
Comparisons
Function Inverses
Natural Products
Integer Order
Integer Sums
Family Unions and Intersections
Function Composites
Function Images
Orders
Natural Sums
Integer Numbers
Families
Converse Relations
Functions
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Unordered Triples
Successor Sets
Intersection of Empty Set
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects