



Why

Given a subset of the real line, what is its length?

Background

Let $a, b \in \mathbb{R}$ with $a \leq b$. The *length* of the closed interval of the real numbers $[a, b]$ is $b - a$. The length is non-negative.

A family $\{A_\alpha\}_{\alpha \in I}$ is *disjoint* if for $\alpha, \beta \in I$, $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \emptyset$. A set A can be *partitioned* into a family if there exists a disjoint family whose union is A . A set $A \subset \mathbb{R}$ is *simple* if it can be partitioned into a countable family whose members are closed intervals. The above discussion suggests that we should define the length of a simple set as the sum of the lengths of sets which partition it.

The above discussion suggests that if we wish to define a function $\text{length} : 2^{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, we should ask that (1) $\text{length}(A) \geq 0$, (2) $\text{length}([a, b]) = b - a$, (3) for disjoint closed intervals $\{A_n\}_{n \in \mathbb{N}}$, $\text{length}(\bigcup_i A_i) = \sum_i \text{length}(A_i)$, and (4) for all $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, $\text{length}(A + a) = \text{length}(A)$.

Converse

Define the equivalence relation \sim on \mathbb{R} by $x \sim y$ if $x - y \in \mathbb{Q}$

Notation

Let A be a set and $\mathcal{A} \subset \mathcal{P}(A)$. We denote the subset algebra of A and \mathcal{A} by (A, \mathcal{A}) , read aloud as “ A , script \mathcal{A} .”

Properties

Prop. 1. *For any set A , 2^A is a sigma algebra.*

Prop. 2. *The intersection of a family of sigma algebras is a sigma algebra.*

Generation

Prop. 3. *Let A a set and \mathcal{B} a set of subsets. There is a unique smallest sigma algebra (A, \mathcal{A}) with $\mathcal{B} \subset \mathcal{A}$.*

We call the unique smallest sigma algebra containing B the *generated sigma algebra* of B .

