Why

We discuss a decomposition using eigenvalues and eigenvectors.[1]

Defining result

An *eigenvalue decomposition* of a matrix $A \in \mathbf{R}^{n \times n}$ is an ordered pair (X, Λ) in which X is invertible, Λ is diagonal, and $A = X\Lambda X^{-1}$.

In this case, $AX = X\Lambda$, in other words,

$$
\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}
\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}
=
\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}
\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.
$$

in which x_i is the ith column of X and λ_i is the ith diagonal element of Λ. We have $Ax_i = \lambda_i x_i$ for $i = 1, \dots, n$. In other words, the ith column of X is an eigenvector of A and the jth entry of Λ is the corresponding eigenvalue.

If X is orthonormal, so that $X^{-1} = X^\top$, then we can interpret such a decomposition as a change of basis to *eigenvector coordinates*. If $Ax = b$, and $A = X\Lambda X^{-1}$ then $(X^{-1}b) = \Lambda(X^{-1}x)$. Here, $X^{-1}x$ expands x is the basis of columns of X. So to compute Ax , we first expand into the basis of columns of X, scale by Λ, and then interpret the result as the coefficients of a linear combination of the columns of X.

In this case that $A = X\Lambda X^\top$ for an eigenvalue decomposition (X, Λ) of A, we can also write

$$
A = X\Lambda X^\top = \sum_{i=1}^{n} \Lambda_{ii} x_i x_i^\top.
$$

Proposition 1. *Every real symmetric matrix has an eigenvalue decomposition (X, Λ) in which X is orthonormal.*[2]

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[1] Future editions will expand.

[2] In future editions, this may be the motivating result for the definition of eigenvalues.

Eigenvalue Decomposition

Orthogonal Matrices

Matrix Transpose

Eigenvalues and Eigenvectors · Orthonormal Set of Vectors

Matrices · Inner Product Norms

Matrix Determinants · Norms · Vectors as Matrices · Inner Products · Orthonormal Set of Real Vectors

Real Matrices · Functionals · Real Inner Product

Linear Equations · Vectors · Real Norm

Linear Functions · Real Vectors · Space Norm

Plane Vector Sums and Differences · Space Vectors · n-Dimensional Space · Plane Inner Product · Plane Norm

Arrays · Distance · Plane Vectors · Absolute Value · Real Square Roots · Right Triangle Sides Relation

Monic Polynomials · Space Distance · Real Squares

Plane Distance · Real Arithmetic

Permutations · Interval Length · Real Multiplicative Inverse

Intervals · Real Line · Real Products · Natural Square Roots

Polynomials · Groups · Fields · Real Summation · Real Sums · Rational Arithmetic · Rational Multiplicative Inverse · Irrational Line · Square Numbers

Real Plane · Real Addition Inverses · Geometry

Lists · Rings · Set Numbers · Real Numbers · Rational Sum · Rational Order · Rational Products · Rational Numbers

Finite Sets

Direct Products · Integer Arithmetic · Integer Products

Inverse Elements · Equivalent Sets · Natural Summation · Natural Multiplication Identity · Natural Addition Identity · Identity Elements · Arithmetic · Natural Equations · Integer Order · Integer Sums

Family Unions and Intersections · Function Inverses · Family Operations · Comparisons · Element Functions · Operations · Equations · Order · Natural Powers · Integer Numbers

Families · Function Composites · Function Images · Natural Order · Equivalence Relations · Natural Products

Functions · Converse Relations · Natural Sums

Relations · Natural Numbers

Recursion Theorem

Peano Axioms

Set Products

Natural Induction

Generalized Set Duality · Set Powers

Natural Numbers

Set Unions and Intersections · Set Dualities · Unordered Triples · Intersection of Empty Set · Successor Sets · Partitions · Set Symmetric Differences

Ordered Pairs · Pair Unions · Set Intersection · Set Complements

Pair Intersection · Unordered Pairs · Empty Set · Set Differences

Set Unions

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects