Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering $\geq$ on $\mathbf{S}^d$ defined by

$$A \geq B \quad \longleftrightarrow \quad A - B \geq 0 \quad \longleftrightarrow \quad A - B \in \mathbf{S}^d_+.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \quad \longleftrightarrow \quad A - B > 0 \quad \longleftrightarrow \quad A - B \in \mathbf{S}^d_{++}.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$\begin{aligned} \alpha A \geq 0 \quad & \text{for all } \delta > 0, A \geq 0, \\ A + B \geq 0 \quad & \text{for all } A, B \geq 0, \\ A \geq B \text{ and } B \geq A \longrightarrow A = B \quad & \text{for all } A, B \in \mathbf{S}^d, \\ \lim_{n\to\infty} A_n = A \longrightarrow A \geq 0 \quad & \text{for all sequences } (A_n)_n \text{ in } \mathbf{S}^d_+. \end{aligned}$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that $\geq$ is antisymmetric. Moreover,

$$A \geq A \quad \& \text{ for all } A \in \mathbf{S}^d, \text{ and}$$

$$A \geq B \text{ and } B \geq C \longrightarrow A \geq C \& \text{ for all } A, B, C \in \mathbf{S}^d.$$

In other words, $\geq$ is also reflexive and transitive. In other words, $\geq$ is a partial order (see Partial Orders).[1]

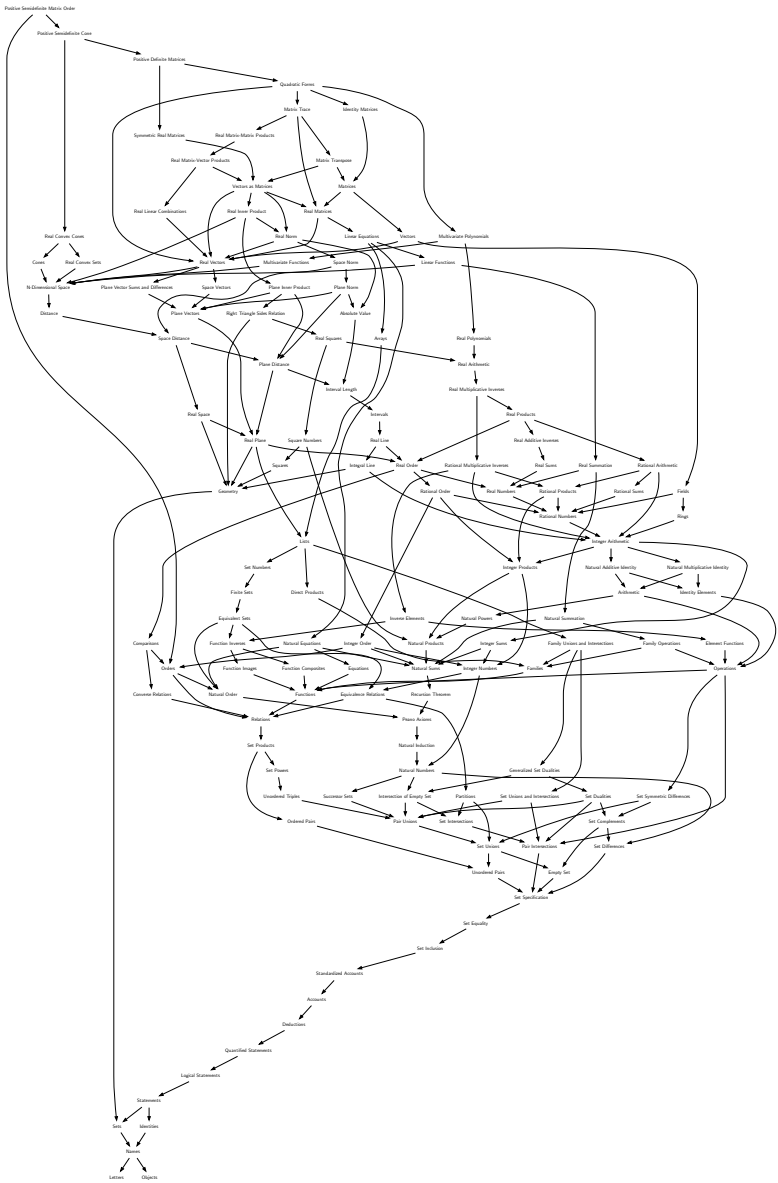
[1] Future editions will include more formal accounts.

For $d = 1$, $\geq$ reduces to the familiar total order of the real line(see Real Order). The converse perspective is to see the positive semidefinite order as an extension of the order on $\mathbf{R}$ to the space $\mathbf{S}^d$. Of course, the key difference is that two matrices may not be comparable. The order is *partial*.

For example, the matrices $A, B \in \mathbf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.


Positive Semidefinite Matrix Order
Positive Semidefinite Cone
Positive Definite Matrices
Quadratic Forms
Matrix Trace
Identity Matrices
Symmetric Real Matrices
Real Matrix-Matrix Products
Matrix Transpose
Real Matrix-Vector Products
Vectors as Matrices
Matrices
Real Linear Combinations
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Real Cones Cones
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Plane Vector Sums and Differences
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Real Space
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Rational Multiplicative Inverses
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Family Unions and Intersections
Family Operations
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Generalized Set Doubles
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Successor Sets
Intersection of Empty Set
Functions
Set Unions and Intersections
Set Doubles
Set Symmetric Differences
Ordered Pairs
Pair Unions
Set Intersections
Set Complements
Set Unions
Pair Intersections
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
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Statements
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Objects