



## Definition

The *trace* of a square real matrix is the sum of its diagonal entries.

## Notation

We denote the function which associates a matrix with its trace by  $\text{tr} : \mathbf{R}^{n \times n} \rightarrow \mathbf{R}$ . The trace of  $A \in \mathbf{R}^{n \times n}$  is

$$\text{tr } A = \sum_{i=1}^n A_{ii}.$$

## Properties

**Proposition 1.** *The trace is a linear function on the vector space of  $n \times n$  real matrices.*

*Proof.* Let  $A, B \in \mathbf{R}^{n \times n}$  and  $\alpha, \beta \in \mathbf{R}$ . Define  $C = \alpha A + \beta B$ . Then  $C_{ii} = \alpha A_{ii} + \beta B_{ii}$ . So

$$\begin{aligned} \text{tr } C &= \sum_{i=1}^n C_{ii} = \sum_{i=1}^n \alpha A_{ii} + \beta B_{ii} \\ &= \alpha \sum_{i=1}^n A_{ii} + \beta \sum_{i=1}^n B_{ii} \\ &= \alpha \text{tr } A + \beta \text{tr } B. \end{aligned}$$

□

**Proposition 2.** *Let  $A, B \in \mathbf{R}^{n \times n}$ . Then*

$$\text{tr } (AB) = \text{tr } (BA).$$

In other words, “matrices commute under the trace operator.”

**Proposition 3.** *Let  $A \in \mathbf{R}^{n \times n}$ . Then  $\text{tr } A = \text{tr } A^\top$ .*



