



Definition

A subset of \mathbf{R}^n is *affine* (or an *affine variety*, *linear variety*, or *flat*) if it contains the lines through each of its points.

Some authors use the term *affine subspace*, but we avoid this terminology because affine sets are not subspaces. In fact, they are translated subspaces.

Examples

The empty set is trivially an affine set. The entire set of points in n -dimensional space is an affine set. Any singleton is an affine set.

Our definition is meant to capture the intuitive idea of an infinite, uncurved set, like a line or a plane in space.

Notation

As usual, let $L(x, y)$ denote the line between $x, y \in \mathbf{R}^n$. The set $M \subset \mathbf{R}^n$ is affine if $L(a, b) \subset M$ for all $a, b \in M$. For any $x, y \in M$ and $\lambda \in \mathbf{R}$,

$$(1 - \lambda)x + \lambda y \in M.$$

Translations

The set of affine sets is closed under translation.

Proposition 1. *Let $M \subset \mathbf{R}^n$ affine. Then $M + a$ is affine for any $a \in \mathbf{R}^n$.*

Proof. Suppose $x, y \in M + a$. There exist $\tilde{x}, \tilde{y} \in M$ with $x = \tilde{x} + a$ and $y = \tilde{y} + a$. For any $\lambda \in \mathbf{R}$,

$$\begin{aligned} (1 - \lambda)x + \lambda y &= (1 - \lambda)(\tilde{x} + a) + \lambda(\tilde{y} + a) \\ &= ((1 - \lambda)\tilde{x} + \lambda\tilde{y}) + a. \end{aligned}$$

□

An affine set M is *parallel* to an affine set L if there exists $a \in \mathbf{R}^n$ so that $M = L + a$. The relation “ M is parallel to L ” on the set of affine sets is an equivalence relation. This notion of parallelism is more restrictive than the natural one, in which we may speak of a line being parallel to a plane. We must speak of a line which is parallel to another line within the given plane.

Properties

The set of affine sets is closed under intersection.

Proposition 2. *If \mathcal{M} is a family of affine sets, then $\bigcap \mathcal{M}$ is affine.*

