



Why

We want to talk about how knowledge of one aspect of an outcome can give us knowledge about another aspect.

Definition

Two events $A, B \subset \Omega$ are *independent* under a probability measure $\mathbf{P} : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$ if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B).$$

In other words, they are independent if the probability of their intersection is the product of their respective probabilities. Otherwise, we call A and B *dependent*.

In the case that $\mathbf{P}(B) \neq 0$, then $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$ is equivalent to $\mathbf{P}(A \mid B) = \mathbf{P}(A)$, which more clearly expresses the intuition captured by the definition. Roughly speaking, we interpret this second expression as encoding the fact that the occurrence of event B does not change the probability—intuitively, the “credibility”—of the event A .

Example: two dice

Define $\Omega = \{(\omega_1, \omega_2) \mid \omega_i \in \{1, \dots, 6\}\}$, and interpret $\omega \in \Omega$ as corresponding to pips face up after rolling two dice. Define $p : \Omega \rightarrow \mathbf{R}$ by $p(\omega) = 1/36$.

Two events are $A = \{\omega \in \Omega \mid \omega_1 + \omega_2 > 5\}$, “the sum is greater than 5”, and $B = \{\omega \in \Omega \mid \omega_1 > 3\}$, “the number of pips on the first die is greater than 3”. Then $\mathbf{P}(A) = 26/36$. Also, $\mathbf{P}(A \mid B) = 17/16$. So, these events are dependent. Roughly speaking, we say that knowing B tells us something about A . In this case, we say that it “makes A more probable.”

In the language used to describe the events, we say that knowledge that the number of pips on the first die is greater than three makes it more probable that the sum of the number of pips on each die is greater than 5.

Basic implications

Since $\mathbf{P}(\cdot \mid B)$ is a probability measure, and the events A and $C_\Omega(A)$ partition Ω , we have

$$\mathbf{P}(A \mid B) + \mathbf{P}(C_\Omega(A) \mid B) = 1.$$

From which we deduce $\mathbf{P}(C_\Omega(A) \mid B) = 1 - \mathbf{P}(A) = \mathbf{P}(C_\Omega(A))$. Which is equivalent to $\mathbf{P}(C_\Omega(A) \cap B) = \mathbf{P}(C_\Omega(A))\mathbf{P}(B)$. In other words, B and $C_\Omega(A)$ are independent events. Similarly, A and $C_\Omega(B)$ are independent events. Since $(\Omega - A) \cap (\Omega - B) = \Omega - (A \cup B)$, we have

$$\mathbf{P}(A \cup B) + \mathbf{P}(C_\Omega(A) \cap C_\Omega(B)) = 1.$$

Since $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$, we obtain

$$\mathbf{P}(C_\Omega(A) \cap C_\Omega(B)) = 1 - \mathbf{P}(A) - \mathbf{P}(B) + \mathbf{P}(A \cap B).$$

We can express the right hand side as $(1 - \mathbf{P}(A))(1 - \mathbf{P}(B))$ or $\mathbf{P}(C_\Omega(A))\mathbf{P}(C_\Omega(B))$. In other words, $C_\Omega(A)$ and $C_\Omega(B)$ are independent.

