

PROBABILISTIC DATASET MODELS

Why

Let
$$X = \{a, b\}$$
 and $Y = \{0, 1\}$. Define $f: X \to Y$ by $f \equiv 0$.

The dataset (a, 0) is consistent with f. So are the datasets ((a, 0), (a, 0)) and ((a, 0), (a, 0), (a, 0)). Unfortunately, these datasets are "bad" in the sense that we do not see the value associated with b. Each dataset is consistent with the (functional) relations $\{(a, 0), (b, 0)\}$ and $\{(a, 0), (b, 1)\}$.

In other words, a dataset may be incomplete. In spite of this limitation, we want to discuss an inductor's performance on consistent datasets. One route is to put a measure on the set of training sets and only consider high-measure subsets.

Definition

Let (X, \mathcal{X}) and (Y, \mathcal{Y}) be measurable spaces and R be a relation on $X \times Y$. Let $\mu : \mathcal{X} \times \mathcal{Y} \to [0, 1]$ be a probability measure so that $\mathcal{D} = (X \times Y, \mathcal{X} \times \mathcal{Y}, \mu)$ is a probability space.

If $\mu(B) = 0$ for all $B \subset C_{X \times Y}(R)$, then we call \mathcal{D} a probabilistic dataset model for R. In other words, μ gives zero measure to any set of points not in the relation. Equivalently, $\mu(R) = 1$; or we observe a pair in R almost surely.

If R is functional, then we call \mathcal{D} a supervised probabilistic dataset model. In this case, since there is a functional relation between X and Y, we call the marginal measure $\mu_X : \mathcal{X} \to [0,1]$ the data-generating distribution or underlying distribution since $\mu(A) = \mu_X(\{x \in X \mid (x,y) \in A)\}$). In this case we call the (functional) relation R the correct labeling function. Many authors refer to a supervised probabilistic data model as the statistical learning (theory) framework.

Probable datasets

For datasets of size n, we use the product measure $((X \times Y)^n, (\mathcal{X} \times \mathcal{Y})^n, \mu^n)$. We interpret this measure as modeling independent and iden-

tically distributed elements of R.

For $\delta \in (0,1)$, $\mathcal{S} \subset (X \times Y)^n$ is $1 - \delta$ -probable if $\mu^n(S) \geq 1 - \delta$. We call δ the confidence parameter. If δ is small, we interpret \mathcal{S} as a set of "reasonable" datasets.

