



POSITIVE SEMIDEFINITE MATRIX ORDER

Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering \geq on \mathbf{S}^d defined by

$$A \geq B \iff A - B \geq 0 \iff A - B \in \mathbf{S}_+^d.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \iff A - B > 0 \iff A - B \in \mathbf{S}_{++}^d.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$\begin{aligned} \alpha A &\geq 0 && \text{for all } \alpha > 0, A \geq 0, \\ A + B &\geq 0 && \text{for all } A, B \geq 0, \\ A \geq B \text{ and } B \geq A &\longrightarrow A = B && \text{for all } A, B \in \mathbf{S}^d, \\ \lim_{n \rightarrow \infty} A_n = A &\longrightarrow A \geq 0 && \text{for all sequences } (A_n)_n \text{ in } \mathbf{S}_+^d. \end{aligned}$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that \geq is antisymmetric. Moreover,

$$\begin{aligned} A &\geq A && \text{for all } A \in \mathbf{S}^d, \text{ and} \\ A \geq B \text{ and } B \geq C &\longrightarrow A \geq C && \text{for all } A, B, C \in \mathbf{S}^d. \end{aligned}$$

In other words, \geq is also reflexive and transitive. In other words, \geq is a partial order (see **Partial Orders**).¹

For $d = 1$, \geq reduces to the familiar total order of the real line (see **Real Order**). The converse perspective is to see the positive semidefinite order as an extension of the order on \mathbf{R} to the space \mathbf{S}^d . Of course, the key difference is that two matrices may not be comparable. The order is *partial*.

For example, the matrices $A, B \in \mathbf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.

¹Future editions will include more formal accounts.

