



## Measures

### 1 Why

We want to generalize the notion of length, area, volume beyond the Lebesgue measure on the product spaces of real numbers.

### 2 Definition

An extended-real-valued non-negative function on an algebra is *finitely additive* if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a sigma algebra is *countably additive* if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A *finitely additive measure* is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A *countably additive measure* is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call countably additive measures *measures*, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countable

additive.

In the context of measure, we call a countably unital subset algebra a *measurable space*. We call the distinguished sets *measurable sets*. A *measure space* is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of the measurable space.

### 3 Notation

Let  $A$  a set. Let  $\mathcal{A}$  a sigma algebra on  $A$ . The pair  $(A, \mathcal{A})$  is a measurable space.

Let  $\mu : \mathcal{A} \rightarrow [0, \infty]$  a measure; thus: (a)  $\mu(\emptyset) = 0$  and (b) for disjoint  $\{A_n\} \subset \mathcal{A}$ ,  $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ . The triple  $(A, \mathcal{A}, \mu)$  is a measure space.

We use  $\mu$  since it is a mnemonic for “measure”. We often also use  $\nu$  to denote measures, since it is after  $\mu$  in the Greek alphabet, and  $\lambda$ , since it is before  $\mu$  in the Greek alphabet.

### 4 Examples

**Example 1.** Let  $(A, \mathcal{A})$  a measurable space. Let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  such that  $\mu(A)$  is  $|A|$  if  $A$  is finite and  $\mu(A)$  is  $+\infty$  otherwise. Then  $\mu$  is a measure. We call  $\mu$  the counting measure.

**Example 2.** Let  $(A, \mathcal{A})$  measurable. Fix  $a \in A$ . Let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  such that  $\mu(A)$  is 1 if  $a \in A$  and  $\mu(A)$  is 0 otherwise. Then  $\mu$  is a measure. We call  $\mu$  the point mass concentrated at  $a$ .

**Example 3.** Let  $R$  denote the real numbers. The Lebesgue measure on

the measurable space  $(R, \mathcal{B}(R))$  is a measure.

**Example 4.** Let  $N$  be the natural numbers. Let  $\mathcal{A}$  the finite co-finite algebra on  $N$ . Let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  be such that  $\mu(A)$  is 1 if  $A$  is infinite or 0 otherwise. Then  $\mu$  is a finitely additive measure. However it is impossible to extend  $\mu$  to be a countably additive measure. Observe that if  $A_n = \{n\}$  the  $\mu(\cup_n A_n) = 1$  but  $\sum_n \mu(A_n) = 0$ .

**Example 5.** Let  $(A, \mathcal{A})$  a measurable space. Let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  be 0 if  $A = \emptyset$  and  $\mu(A)$  is  $+\infty$  otherwise. Then  $\mu$  is a measure.

**Example 6.** Let  $A$  be set with at least two elements ( $|A| \geq 2$ ). Let  $\mathcal{A} = A^*$ . Let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  such that  $\mu(A)$  is 0 if  $A = \emptyset$  and  $\mu(A) = 1$  otherwise. Then  $\mu$  is not a measure, nor is  $\mu$  finitely additive.

*Proof.* Let  $B, C \in \mathcal{A}$ ,  $B \cap C = \emptyset$  then using finite additivity we obtain a contradiction  $1 = \mu(B \cup C) = \mu(B) + \mu(C) = 2$ .  $\square$

## 5 Why

We want to generalize the notions of length, area, and volume.

## 6 Definition

A *measurable space* is a sigma algebra. We call the distinguished subsets the *measurable sets*.

A *measure* on a measurable space is a function from the sigma algebra to the positive extended reals. A *measure space* is a measurable space and a measure.

## 6.1 Notation

## 6.2 Properties

**Proposition 7.** *Let  $(A, \mathcal{A})$  be a measurable space and  $m : \mathcal{A} \rightarrow [0, \infty]$  be a measure.*

*If  $B \subset C \subset A$ , then  $m(B) \leq m(C)$ . We call this property the of measures monotonicity of measure.*

**Proposition 8.** *For a measure space  $(A, \mathcal{A}, m)$ .*

*If  $B \subset C \subset A$ , then  $m(B) \leq m(C)$ .*

*We call this property the monotonicity of measure.*

**Proposition 9.** *For a measure space  $(A, \mathcal{A}, m)$ .*

*If  $\{A_n\} \subset \mathcal{A}$  a countable family, then  $m(\cup A_n) \leq \sum_i m(A_i)$ .*

*We this property the sub-additivity of measure.*

**Proposition 10.** *For a measure space  $(A, \mathcal{A}, m)$ .*

*If  $\{A_n\} \subset \mathcal{A}$  a countable family, then  $m(\cup A_n) \leq \sum_i m(A_i)$ .*

*We this property the sub-additivity of measure.*

**Proposition 11.** *For a measure space  $(A, \mathcal{A}, m)$ .*

$$m(\cup_{n=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} m(A_i)$$

**Proposition 12.** *For a measure space  $(A, \mathcal{A}, m)$ .*

$$m(\cap_{n=1}^{\infty} A_i) = \lim_{n \rightarrow \infty} m(A_i)$$

## 6.3 Examples

**Example 13.** *counting measure*