



Why

We can view the set of sequences as vector spaces and give them norms.

Bounded sequences

Let $\mathbf{C}^{\mathbf{N}}$ denote the set of complex-valued sequences. Define $\ell^\infty \subset \mathbf{C}^{\mathbf{N}}$ to be the set of all *bounded sequences*. That is,

$$\ell^\infty = \{x \in \mathbf{C}^{\mathbf{N}} \mid \exists M \in \mathbf{R} \text{ with } |x_i| < M \text{ for all } i\}.$$

Then ℓ^∞ with componentwise addition and scalar multiplication is a vector space.

Exercise 1. Define $\|\cdot\|_\infty : \ell^\infty \rightarrow \mathbf{R}$ by

$$\|x\|_\infty = \sup_{n \in \mathbf{N}} |x_n|.$$

Then $\|\cdot\|_\infty$ is a norm on ℓ^∞ .

Absolutely summable sequences

Let $\ell^1 \subset \mathbf{C}^{\mathbf{N}}$ denote the set of all *absolutely summable sequences*. In other words, for $x \in \mathbf{C}^{\mathbf{N}}$, $x \in \ell^1$ if

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

Then ℓ^1 is a vector space with componentwise addition and scalar multiplication.

Exercise 2. Define $\|\cdot\|_1 : \ell^1 \rightarrow \mathbf{R}$ by

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

Then $\|\cdot\|_1$ is a norm on ℓ^1 .

