Why

Some reasonable statements do not hold *for all* elements of the base set of a measurable space. Even so, these statements often hold *broadly*, in the sense that the measure of the set on which they fail is zero. This idea allows one to handle statements which fail on a set of measure zero as though they hold everywhere. This approach is useful in discussions of convergence and integration.

Definition

For this sheet, suppose $(X, \mathcal{A}, \mu)$ is a measure space. Call a set $N \subset X$ *negligible* if there exists a measurable set $A \in \mathcal{A}$ with $\mu(A) = 0$ and $N \subset A$. In english, there is a measurable set containing N that has measure zero.

The qualification "if there exists a measurable set..." enables one to speak of *nonmeasurable* negligible sets. Negligible sets need not be measurable.

Given the measure μ, a statements holds *almost everywhere* with respect to μ if the set of elements on which the statement fails is negligible. In symbols, there exists $A \in \mathcal{A}$ with $\mu(A) = 0$ and

$$\{x \in X \mid \neg s(x)\} \subset A$$

A statement which holds "everywhere" holds "almost everywhere" also. With this in mind, we call the almost everywhere sense "weaker" than the everywhere sense.

Notation

We abbreviate almost everywhere as "a.e.," read "almost everywhere". We say that a statement "holds a.e." If the measure μ is not clear from context, we say that the property holds almost everywhere $[\mu]$ or μ-a.e., read "mu almost everywhere."

Function comparisons

Let $f, g : X \to \mathbf{R}$, not necessarily measurable. Then $f = g$ almost everywhere if the set of points at which the functions disagree is μ-negligible. Similarly, $f \geq g$ almost everywhere if the set of points where f is less than g is μ-negligible.

If f and g were both in fact $\mathcal{A}$-measurable, then the sets

$$\{x \in X \mid f(x) \neq g(x)\} \text{ and } \{x \in X \mid f(x) < g(x)\}$$

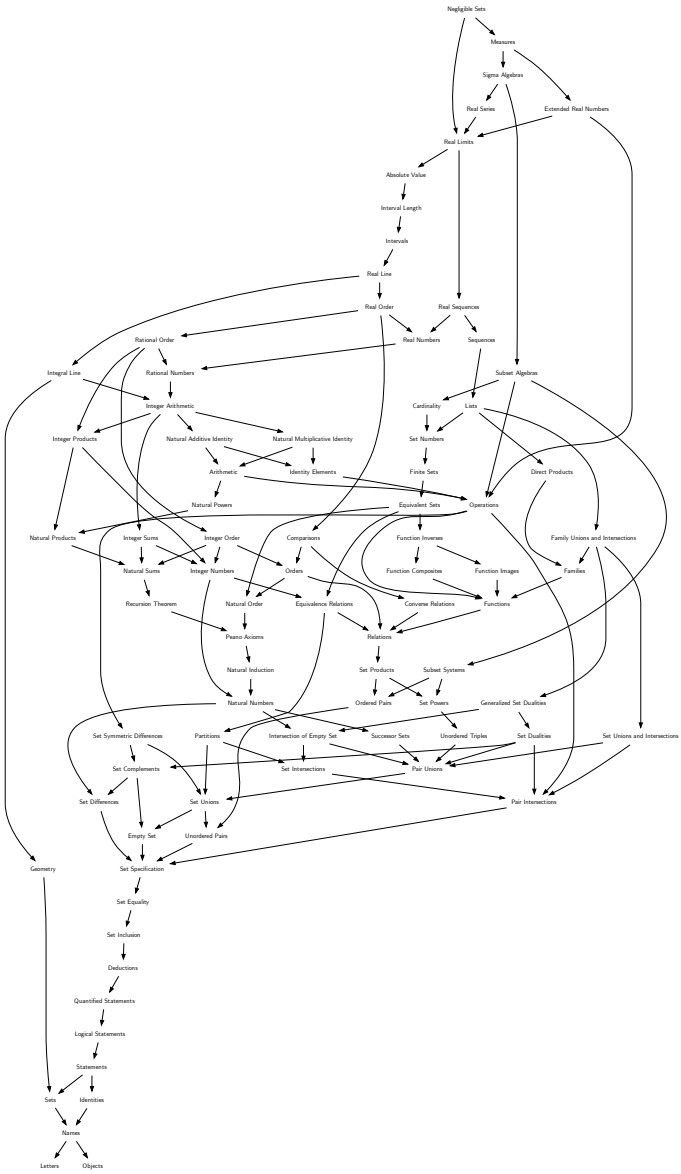
would be measurable also. But in general, these sets need not be measurable, and so this is one simple example to justify our including non-measurable sets in the definition of neglible sets, as mentioned above.

Function limits

Let $f_n : X \to R$ for each natural number n and let $f : X \to R$ be a function. The sequence $(f_n)_n$ *converges to* f *almost everywhere* if

$$\left\{x \in X \;\middle|\; \lim_n f_n(x) \text{ does not exist, or } f(x) \neq \lim_n f_n(x)\right\}$$

is μ-negligible. In this case, we write "$f = \lim_n f_n$ a.e."


Negligible Sets
Measures
Sigma Algebras
Real Series
Extended Real Numbers
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Real Numbers
Sequences
Rational Order
Integral Line
Rational Numbers
Subset Algebras
Cardinality
Lists
Integer Arithmetic
Set Numbers
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Arithmetic
Identity Elements
Finite Sets
Direct Products
Natural Powers
Equivalent Sets
Operations
Natural Products
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Natural Sums
Integer Numbers
Orders
Function Compositos
Function Images
Families
Recursion Theorem
Natural Order
Equivalence Relations
Converse Relations
Functions
Peano Axioms
Relations
Natural Induction
Set Products
Subset Systems
Natural Numbers
Ordered Pairs
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects