Joint Probability Matrices

Why

We can characterize the dependence of two events in terms of the rank of a particular matrix.

Definition

Given a probability measure $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ on the finite set Ω and two events $A, B \subset \Omega$, the *joint probability matrix* of A and B is the matrix

$$M = \begin{bmatrix} \mathbf{P}(A \cap B) & \mathbf{P}(A \cap C_\Omega(B)) \\ \mathbf{P}(C_\Omega(A) \cap B) & \mathbf{P}(C_\Omega(A) \cap C_\Omega(B)) \end{bmatrix}.$$

Characterization of independence

If A and B are independent, then so are A and $C_\Omega(B)$, B and $C_\Omega(A)$, and $C_\Omega(A)$ and $C_\Omega(B)$. In other words,

$$M = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_\Omega(A)) \end{bmatrix} \begin{bmatrix} \mathbf{P}(B) & \mathbf{P}(C_\Omega(B)) \end{bmatrix}.$$

In this case, we see that $\mathsf{rank}(M) = 1$.

Conversely, suppose $\mathsf{rank}(M) = 1$. Then, using the law of total probabiliy, each row is a multiple of

$$M1 = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_\Omega(A)) \end{bmatrix}.$$

In particular, we have $\mathbf{P}(A \cap B) = \alpha \mathbf{P}(A)$ and $\mathbf{P}(C_\Omega(A) \cap B) = \alpha P(C_\Omega(A))$. So

$$\mathbf{P}(A \cap B) + \mathbf{P}(C_\Omega(A) \cap B) = \alpha(\mathbf{P}(B) + \mathbf{P}(C_\Omega(A))),$$

from which we deduce $\alpha = \mathbf{P}(B)$ Likewise, the multiplier for the second column of M is $\mathbf{P}(C_\Omega(B))$. In other words, A and B are independent. We conclude that A and B are independent if and only if $\mathsf{rank}(M) = 1$.

Joint Probability Matrices

Dependent Events

Real Matrix Rank

Conditional Event Probabilities

Real Matrices

Real Subspaces

Linear Equations

Linear Functions

Real Vectors

N-Dimensional Space

Space Vectors

Plane Vector Sums and Differences

Event Probabilities

Distance

Plane Vectors

Space Distance

Plane Distance

Interval Length

Probability Distributions

Real Summation

Intervals

Real Space

Real Line

Real Plane

Real Order

Integral Line

Real Numbers

Rational Order

Geometry

Rational Numbers

Arrays

Integer Arithmetic

Lists

Integer Products

Natural Additive Identity

Natural Multiplicative Identity

Set Numbers

Arithmetic

Identity Elements

Finite Sets

Direct Products

Natural Summation

Natural Powers

Equivalent Sets

Function Inverses

Natural Equations

Natural Products

Integer Sums

Integer Order

Comparisons

Family Operations

Function Composites

Function Images

Family Unions and Intersections

Equations

Natural Sums

Integer Numbers

Orders

Operations

Families

Recursion Theorem

Natural Order

Converse Relations

Equivalence Relations

Functions

Peano Axioms

Relations

Uncertain Outcomes

Natural Induction

Set Products

Natural Numbers

Set Powers

Generalized Set Dualities

Partitions

Set Unions and Intersections

Set Symmetric Differences

Successor Sets

Unordered Triples

Set Dualities

Intersection of Empty Set

Ordered Pairs

Set Complements

Pair Unions

Set Intersections

Set Differences

Set Unions

Pair Intersections

Empty Set

Unordered Pairs

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects