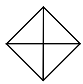

Why

We want to select a normal density which summarizes well a dataset.

Formulation

Let $D = (x^1, \dots, x^n)$ be a dataset in $\mathbf{R}$. We want to select a density from among normal densities, which require specifying a mean and covariance.

Following the principle of maximum likelihood, we want to solve

$$\begin{array}{rl} \textbf{find} & \mu, \sigma \in \mathbf{R} \\ \textbf{to maximize} & \displaystyle\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x^k - \mu}{\sigma}\right)^2\right) \\ \textbf{subject to} & \sigma > 0 \end{array}$$

We call a solution to the above problem a *maximum likelihood normal density* with respect to the dataset.

Solution

Proposition 1. *Let $(x^1, \dots, x^n)$ be a dataset in $\mathbf{R}$. Let f be a normal density with mean*

$$\frac{1}{n}\sum_{k=1}^{n} x^k$$

and covariance

$$\frac{1}{n}\sum_{k=1}^{n}\left(x^k - \frac{1}{n}\sum_{k=1}^{n} x^k\right)^2.$$

Then f is a maximum likelihood normal density.

Proof. Every normal density has two parameters: the mean and the covariance. If the likelihood of one normal is less than or equal to the likelihood of another, then so also with their log likelihoods. Let f be a normal density with parameter μ and σ^2. We express the log likelihood of f by

$$\sum_{k=1}^{n}\left(\frac{1}{2\sigma^2}(x^k-\mu)^2-\frac{1}{2}\log 2\pi\sigma^2\right)$$

The partial derivative of the log likelihood with respect to the mean $(\partial_\mu \ell) : \mathbf{R}^2 \to \mathbf{R}$ is

$$(\partial_\mu \ell)(\mu, \sigma^2) = -\sum_{k=1}^{n}\frac{1}{\sigma^2}(x-\mu)$$

and with respect to the covariance $(\partial_{\sigma^2}\ell) : \mathbf{R}^2 \to \mathbf{R}$ is

$$(\partial_{\sigma^2}\ell)(\mu, \sigma^2) = \left(\frac{-1}{2(\sigma^2)^2}\sum_{k=1}^{n}(x^k-\mu)^2\right)-\frac{1}{2\sigma^2}$$

We are interested in finding $\mu_0 \in \mathbf{R}$ and $\sigma_0^2 > 0$, at which $\partial_\mu \ell(\mu_0, \sigma_0^2) = 0$ and $\partial_{\sigma^2}\ell(\mu_0, \sigma_0^2) = 0$. So we have two equations. First, notice that $\partial_\mu \ell$ is zero if an only if its first argument (the mean) is $\frac{1}{n}\sum_{k=1}^{n} x^k$. Second, notice that for all μ, σ^2, $\partial_{\sigma^2}\ell$ is zero if and only if

$$\sigma^2 = \sum_{k=1}^{n}(x^k-\mu)^2.$$

So the pair

$$\left(\frac{1}{n}\sum_{k=1}^{k} x^k, \frac{1}{n}\sum_{k=1}^{n}(x_k - \frac{1}{n}\sum_{k=1}^{n} x^k)^2\right)$$

is a stationary point of ℓ. $\square$


Maximum Likelihood Normals
Normal Densities
Maximum Likelihood Densities
Probability Densities
Exponential Function
Real Integrals
Nonnegative Integrals
Simple Integrals
Partial Derivatives
Measurable Functions
Multivariate Functions
Differentiable Functions
Simple Functions
Measures
N-Dimensional Space
Real Functions
Sigma Algebras
Distance
Real Square Roots
Real Series
Extended Real Numbers
Space Distance
Real Squares
Real Limits
Plane Distance
Real Arithmetic
Absolute Value
Maximum Likelihood Distributions
Interval Length
Real Multiplicative Inverses
Probability Distributions
Natural Square Roots
Real Space
Intervals
Real Products
Square Numbers
Real Plane
Real Line
Real Additive Inverses
Real Sequences
Squares
Integral Line
Real Order
Real Sums
Real Summation
Rational Arithmetic
Rational Multiplicative Inverses
Sequences
Subset Algebras
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Lists
Cardinality
Optimizers
Rational Numbers
Set Numbers
Integer Arithmetic
Direct Products
Finite Sets
Comparisons
Integer Order
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Natural Summation
Inverse Elements
Equivalent Sets
Natural Equations
Orders
Integer Sums
Arithmetic
Identity Elements
Family Operations
Element Functions
Function Inverses
Family Unions and Intersections
Natural Order
Equations
Natural Powers
Integer Numbers
Operations
Function Composites
Function Images
Families
Natural Products
Converse Relations
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Set Products
Subset Systems
Characteristic Functions
Peano Axioms
Ordered Pairs
Set Powers
Natural Induction
Uncertain Outcomes
Natural Numbers
Unordered Triples
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects