



Why

What of lower upper triangular factorizations when the system is symmetric?

Definition

A *symmetric lower upper triangular factorization* of A is a pair of matrices ($L \in \mathbf{R}^{m \times m}, U \in \mathbf{R}^{m \times m}$) where L is unit lower triangular, U is upper triangular and $A = LU$.

$$A = LL^\top.$$

Other terminology includes *lower upper triangular factorization*, *LU decomposition*, *LU factorization*. Define $R = L^\top$, then

$$A = R^\top R.$$

Let $A \in \mathbf{R}^{n \times n}$. Then the ordinary row reduction of A is a matrix U which is upper triangular. A *lower upper triangular decomposition* A is a pair of matrices (L, L^\top) where $L \in \mathbf{R}^{n \times n}$ is lower triangular, has nonnegative real diagonal entries, and satisfies

$$A = LL^\top.$$

Other terminology includes *lower upper triangular factorization*, *LU decomposition*, *LU factorization*. Define $R = L^\top$, then

$$A = R^\top R.$$

Basic properties

Proposition 1. *Let $A \in \mathbf{R}^{m \times m}$ be positive definite. Then there exists unique lower triangular matrix $L \in \mathbf{R}^{n \times n}$ so that*

$$A = LL^\top.$$

So, in the case that A is positive definite, a lower upper triangular decomposition exists and is unique. Therefore we refer to it as *the upper lower triangular decomposition* of A . It is also known (universally) as the *Cholesky decomposition* or *Cholesky factorization* of A .

Proposition 2. *If A is positive semidefinite, there exists a permutation matrix P for which there is a unique L so that*

$$P^\top AP = LL^\top.$$

This second proposition says that there is a unique Cholesky decomposition for a particular permutation (or pivoting) of A .

Unitriangular form

A *lower diagonal upper decomposition* (or *lower diagonal upper factorization*) of a matrix A a sequence (L, D, L^\top) where $L \in \mathbf{R}^{n \times n}$ is unit lower triangular, $D \in \mathbf{R}^{n \times n}$ is diagonal with real nonnegative entries and

$$A = LDL^\top.$$

Other terminology includes *LDL decomposition*, *LDL factorization*, *LDU factorization*, *LDU decomposition*.

If $(L \in \mathbf{R}^{n \times n}, D \in \mathbf{R}^{n \times n}, L^\top)$ is a LDU decomposition of $A \in \mathbf{R}^{n \times n}$, then $(LD^{1/2}A = LDL^\top$ then $(\tilde{L}D^{1/2}, D^{1/2}L^\top)$ is a LU decomposition. Conversely, if (B, B^\top) is a LU decomposition and S is the diagonal matrix satisfying $S_{ii} = B_{ii}$ for $i = 1, \dots, n$, then $(BS^{-1}, S^2, S^{-1}B^\top)$ is a LDU decomposition of A .

