

REAL LENGTH IMPOSSIBLE

Why

Given a subset of the real line, what is its length?

Background

Let $a, b \in R$ with $a \leq b$. The *length* of the closed interval of the real numbers [a, b] is b - a. The length is non-negative.

A family $\{A_{\alpha}\}_{{\alpha}\in I}$ is disjoint if for ${\alpha},{\beta}\in I, {\alpha}\neq {\beta}$, then $A_{\alpha}\cap A_{\beta}=\varnothing$. A set A can be partioned into a family if there exists a disjoint family whose union is A. A set $A\subset R$ is simple if it can be partitioned into a countable family whose members are closed intervals. The above discussion suggests that we should define the length of a simple set as the sum of the lengths of sets which parition it.

The above discussion suggests that if we wish to define a function length : $2^R \to R \cup \{-\infty, \infty\}$, we should ask that (1) length $(A) \ge 0$, (2) length([a,b]) = b - a, (3) for disjoint closed intervals $\{A_n\}_{n \in \mathbb{N}}$, length $(A_i) = \sum_i \text{length}(A_i)$, and (4) for all $A \subset R$ and $a \in R$, length(A + x) = length(A).

Converse

Define the equivalence relation \sim on R by by $x \sim y$ if $x \sim y \in Q$

Notation

Let A be a set and $A \subset \mathcal{P}(A)$. We denote the subset algebra of A and A by (A, A), read aloud as "A, script A."

Properties

Prop. 1. For any set A, 2^A is a sigma algebra.

Prop. 2. The intersection of a family of sigma algebras is a sigma algebra.

Generation

Prop. 3. Let A a set and \mathcal{B} a set of subsets. There is a unique smallest sigma algebra (A, \mathcal{A}) with $\mathcal{B} \subset \mathcal{A}$.

We call the unique smallest sigma algebra containing B the generated sigma algebra of B.

