



### Why

We want to talk about a set with a prescribed quantitative degree of closeness (or distance) between its elements.

### Definition

The correspondences which serve as a degree of closeness, or measure of distance, must satisfy our previously developed (see Distance) notion of distance.

A function on ordered pairs which does not depend on the order of the elements so considered is *symmetric*. A function into the real numbers which takes only nonnegative values is *nonnegative*. A repeated pair is an ordered pair of the same element twice. A function which satisfies a triangle inequality for any three elements is *triangularly transitive*.

A *metric* (or *distance function*) is a function on ordered pairs of elements of a set which is symmetric, non-negative, zero only on repeated pairs, and triangularly transitive. A *metric space* is an ordered pair whose first coordinate is a nonempty set and whose second coordinate is a metric.

In a metric space, we say that one pair of objects is *closer* together if the metric of the first pair is smaller than the metric value of the second pair.

Notice that a set can be made into different metric spaces by using different metrics.

## Notation

Let  $A$  be a set. We commonly denote a metric by the letter  $d$ , as a mnemonic for “distance.” Let  $d : A \times A \rightarrow \mathbf{R}$ . Then  $d$  is a metric if:

1. it is non-negative, which we tend to denote by

$$d(a, b) \geq 0, \quad \forall a, b \in A.$$

2. it is 0 only on repeated pairs, which we tend to denote by

$$d(a, b) = 0 \iff a = b, \quad \forall a, b \in A.$$

3. it is symmetric, which we tend to denote by:

$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$

4. it is triangularly transitive, which we tend to denote by

$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of  $A$  with  $d$  by  $(A, d)$ .

## Examples

$\mathbf{R}$  with the absolute value distance is a metric space. As is  $\mathbf{R}^2$  and  $\mathbf{R}^3$  with the Euclidean distance.  $\mathbf{R}^n$  with Euclidean metric is an example of a metric space for which the objects ( $n$ -dimensional tuples of real numbers) are impossible to visualize.

