Why

We extend integrability to complex functions.[1]

Definition

A *measurable* complex-valued function on a measurable space is one whose real and imaginary parts are both measurable. An *integrable* complex-valued function on a measurable space is one whose real and imaginary parts are both integrable.

The *integral* of a integrable complex-valued function on a measurable space is the complex number whose real part is the integral of the real part of the function and whose imaginary part is the integral imaginary part of the function.

Notation

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let C denote the set of complex numbers. Let $f : X \to C$ be a function. f is measurable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable. f is integrable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ is integrable. If f is integrable, we denote its integral by $\int f d\mu$. We have defined it by:

$$\int f d\mu = \int \mathrm{Re}(f) d\mu + \int \mathrm{Im}(f) d\mu.$$

Results

Proposition 1. *A linear combination of two integrable complex-valued functions is an integrable complex-valued function.*

Proposition 2. *The integral is a linear operator on the vector space of integrable complex-valued functions.*

Proposition 3. *The absolute value of the integral of a complex-valued function is smaller than the integral of the absolute value of the function.*

[1] Future editions will modify this motivation.

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbf{C}$ integrable. There exists $\alpha \in C$ with $|\alpha| = 1$ such that

$$|\int f d\mu| = \alpha \int f d\mu.$$

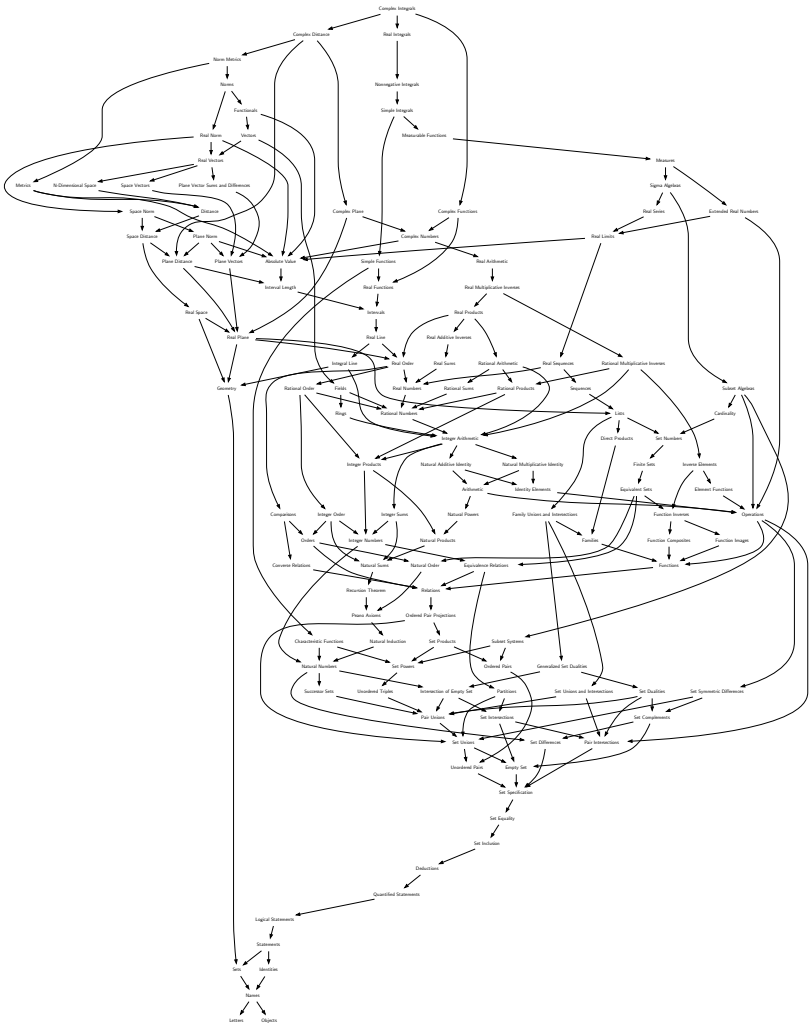
Since the integral is homogenous,

$$|\int f d\mu| = \int \alpha f d\mu \quad = \int \mathrm{Re}(\alpha f) d\mu + i \int \mathrm{Im}(\alpha f) d\mu.$$

Since $\mathbf{C} mod \int f d\mu$ is real, $\int \mathrm{Im}(\alpha f) d\mu = 0$, so

$$|\int f d\mu| = \int \mathrm{Re}(\alpha f) d\mu \quad \leq \int |\alpha f| d\mu = \int |f| d\mu,$$

since $\mathrm{Re}(z) \leq |z|$ for all complex numbers z and $|\alpha| = 1$. $\square$


Complex Integrals
Complex Distance
Real Integrals
Norm Metrics
Norms
Nonnegative Integrals
Functionals
Simple Integrals
Real Norm
Vectors
Measurable Functions
Real Vectors
Measure
Metrics
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Sigma Algebras
Space Norm
Distance
Complex Plane
Complex Functions
Real Series
Extended Real Numbers
Space Distance
Plane Norm
Complex Numbers
Real Limits
Plane Distance
Plane Vectors
Absolute Value
Simple Functions
Real Arithmetic
Interval Length
Real Functions
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Integral Line
Real Order
Real Sums
Rational Arithmetic
Real Sequences
Rational Multiplicative Inverses
Geometry
Rational Order
Fields
Real Numbers
Rational Sums
Rational Products
Sequences
Subset Algebras
Rings
Rational Numbers
Zero
Cardinality
Integer Arithmetic
Direct Products
Set Numbers
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Inverse Elements
Arithmetic
Identity Elements
Equivalent Sets
Element Functions
Comparisons
Integer Order
Integer Sums
Natural Powers
Family Unions and Intersections
Function Inverses
Operations
Order
Integer Numbers
Natural Products
Families
Function Composites
Function Images
Converse Relations
Natural Sums
Natural Order
Equivalence Relations
Functions
Recursion Theorems
Relations
Ordered Pair Projections
Peano Axioms
Characteristic Functions
Natural Induction
Set Products
Subset Systems
Natural Numbers
Set Powers
Ordered Pairs
Generalized Set Dualities
Successor Sets
Unordered Triples
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Dualities
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects