



# Supremum Norm

## 1 Why

We want a norm on the vector space of continuous functions.

## 2 Definition

Consider a function from a closed real interval to the real numbers. The *absolute supremum* of the function is the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

**Proposition 1.** *The functional mapping  $f \in C[a, b]$  to its absolute supremum is a norm.*

*Proof.* Let  $R$  denote the set of real numbers. Define  $\phi : C[a, b] \rightarrow R$  by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a, b]\}.$$

1.  $|f(x)| \geq 0$  for all  $x \in [a, b]$ , so  $\phi(f) \geq 0$ .
2. If  $\phi(f) = 0$  then  $|f(x)| \leq 0$  for all  $x$  and so  $f(x) = 0$  for all  $x \in [a, b]$ . If  $f = 0$ , then  $|f(x)| = 0$  for all  $x \in [a, b]$

3. For all  $\alpha$  real,  $|\alpha f(x)| = |\alpha||f(x)|$ . so  $\phi(\alpha f) = |\alpha|\phi(f)$
4. For all  $f, g \in C[a, b]$ , and  $x \in [a, b]$ ,  $|f(x)+g(x)| \leq |f(x)| + |g(x)|$  by the triangle inequality for absolute value. Thus,
- $$\begin{aligned}\phi(f + g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a, b]\} \\ &\leq \sup\{|f(x)| \mid x \in [a, b]\} + \sup\{|g(x)| \mid x \in [a, b]\} \\ &= \phi(f) + \phi(g)\end{aligned}$$

□

We call the functional  $\phi$  defined above the *supremum norm*.

## 2.1 Notation

Let  $f \in C[a, b]$ . We denote the supremum norm of  $f$  by  $|f|_{\text{sup}}$ .