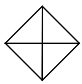


Why

We want to select a normal density which summarizes well a dataset.

Formulation

Let $D = (x^1, \dots, x^n)$ be a dataset in $\mathbf{R}$. We want to select a density from among normal densities, which require specifying a mean and covariance.

Following the principle of maximum likelihood, we want to solve

$$\begin{array}{rl} \textbf{find} & \mu, \sigma \in \mathbf{R} \\ \textbf{to maximize} & \displaystyle\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x^k - \mu}{\sigma}\right)^2\right) \\ \textbf{subject to} & \sigma > 0 \end{array}$$

We call a solution to the above problem a *maximum likelihood normal density* with respect to the dataset.

Solution

Prop. 1. *Let $(x^1, \dots, x^n)$ be a dataset in $\mathbf{R}$. Let f be a normal density with mean*

$$\frac{1}{n}\sum_{k=1}^{n} x^k$$

and covariance

$$\frac{1}{n}\sum_{k=1}^{n}\left(x^k - \frac{1}{n}\sum_{k=1}^{n} x^k\right)^2.$$

Then f is a maximum likelihood normal density.

Proof. Every normal density has two parameters: the mean and the covariance. If the likelihood of one normal is less than or equal to the likelihood of another, then so is are their log likelihoods. Let f be a normal density with parameter μ and σ^2. We express the log likelihood of f by

$$\sum_{k=1}^{n}\left(\frac{1}{2\sigma^2}(x^k - \mu)^2 - \frac{1}{2}\log 2\pi\sigma^2\right)$$

The partial derivative of the log likelihood with respect to the mean $(\partial_\mu \ell) : \mathbf{R}^2 \to \mathbf{R}$ is

$$(\partial_\mu \ell)(\mu, \sigma^2) = -\sum_{k=1}^{n} \frac{1}{\sigma^2}(x - \mu)$$

and with respect to the covariance $(\partial_{\sigma^2} \ell) : \mathbf{R}^2 \to \mathbf{R}$ is

$$(\partial_{\sigma^2} \ell)(\mu, \sigma^2) = \left(\frac{-1}{2(\sigma^2)^2} \sum_{k=1}^{n} (x^k - \mu)^2 \right) - \frac{1}{2\sigma^2}$$

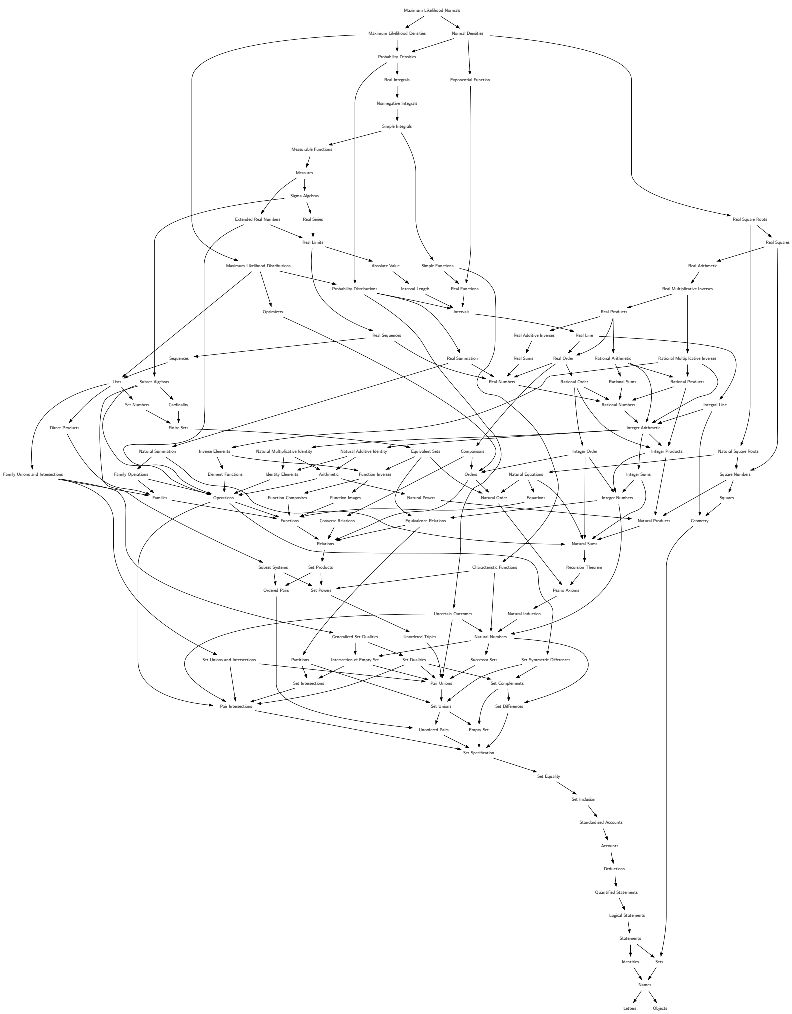
We are interested in finding $\mu_0 \in \mathbf{R}$ and $\sigma_0^2 > 0$, at which $\partial_\mu \ell(\mu_0, \sigma_0^2) = 0$ and $\partial_{\sigma^2} \ell(\mu_0, \sigma_0^2) = 0$. So we have two equations. First, notice that $\partial_\mu \ell$ is zero if an only if its first argument (the mean) is $\frac{1}{n}\sum_{k=1}^{n} x^k$. Second, notice that for all μ, σ^2, $\partial_{\sigma^2} \ell$ is zero if and only if

$$\sigma^2 = \sum_{k=1}^{n} (x^k - \mu)^2.$$

So the pair

$$\left(\frac{1}{n} \sum_{k=1}^{k} x^k, \frac{1}{n} \sum_{k=1}^{n} (x_k - \frac{1}{n} \sum_{k=1}^{n} x^k)^2 \right)$$

is a stationary point of ℓ. $\square$


Maximum Likelihood Normals
Maximum Likelihood Densities
Normal Densities
Probability Densities
Real Integrals
Exponential Function
Nonnegative Integrals
Simple Integrals
Measurable Functions
Measures
Sigma Algebras
Extended Real Numbers
Real Series
Real Square Roots
Real Limits
Real Squares
Maximum Likelihood Distributions
Absolute Value
Simple Functions
Real Arithmetic
Probability Distributions
Interval Length
Real Functions
Real Multiplicative Inverses
Optimizers
Intervals
Real Products
Real Sequences
Real Additive Inverses
Real Line
Sequences
Real Summation
Real Sums
Real Order
Rational Arithmetic
Rational Multiplicative Inverses
Lists
Subset Algebras
Real Numbers
Rational Order
Rational Sums
Rational Products
Set Numbers
Cardinality
Rational Numbers
Integral Line
Direct Products
Finite Sets
Integer Arithmetic
Natural Summation
Inverse Elements
Natural Multiplicative Identity
Natural Additive Identity
Equivalent Sets
Comparisons
Integer Order
Integer Products
Natural Square Roots
Family Unions and Intersections
Family Operations
Element Functions
Identity Elements
Arithmetic
Function Inverses
Orders
Natural Equations
Integer Sums
Square Numbers
Families
Operations
Function Composites
Function Images
Natural Powers
Natural Order
Equations
Integer Numbers
Squares
Functions
Converse Relations
Equivalence Relations
Natural Products
Geometry
Relations
Natural Sums
Subset Systems
Set Products
Characteristic Functions
Recursion Theorem
Ordered Pairs
Set Powers
Peano Axioms
Uncertain Outcomes
Natural Induction
Generalized Set Dualities
Unordered Triples
Natural Numbers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Dualities
Successor Sets
Set Symmetric Differences
Set Intersections
Pair Unions
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects