Why

Definition

Let X be a (nonempty) set and k a field. Let $F \subset (X \to k)$ and let $\langle \cdot, \cdot \rangle : F \times F \to k$ be an inner product so that $(F, \langle \cdot, \cdot \rangle)$ is a complete inner product space.

A *reproducing kernel* of $(F, \langle \cdot, \cdot \rangle)$ is a map $R : X \times X \to k$ satisfying (1) for every $y \in X$ the function $R(\cdot, y) : X \to k$ is an element of F and (2) for every $f \in F$, at every $y \in X$, $f(y) = \langle f, R(\cdot, y) \rangle$ (the *reproducing property*).

R is called a "reproducing" kernel because of the following implication of the reproducing property. Notice that $R(\cdot, y) \in F$. For this reason,

Properties

If a reproducing kernel exists, it is unique.

Let X be nonempty (index) set. For example, X may be $\{1, 2, \ldots, N\}$, Z, $[0, 1]$, R^d, $\{x \in \mathsf{R}^3 \mid \|x\| \leq 1\}$ (the unit sphere), or $\{x \in \mathsf{R}^3 \mid \alpha \leq \|x\| \leq \beta\}$ (the atmosphere, or volume between two concentric spheres).

A symmetric, real-valued function $k : X \times X \to \mathsf{R}$ of two variables is said to be *positive semidefinite* if for any $n \in \mathsf{N}$, for any real $a_1, \ldots, a_n \in \mathsf{R}$ and $x_1, \ldots, x_n \in X$, we have

$$\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) \geq 0,$$

and *positive definite* if the above holds with "$>$".[1]

Positive semidefinite kernels are useful for the following constructive reason:

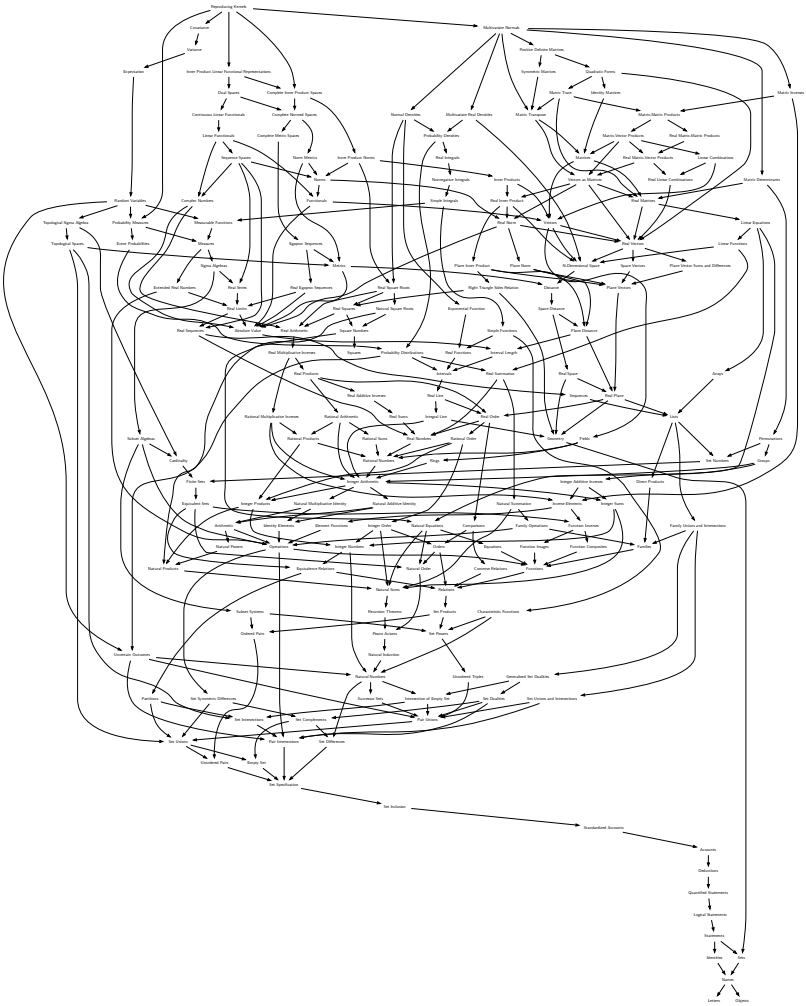
Proposition 1. *Let $X \neq \varnothing$ be a set. If $k : X \times X \to \mathsf{R}$ is positive semidefinite, then there exists a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ and a family of zero-mean normal real-valued random variables $\{f_x : \Omega \to \mathsf{R}\}_{x \in X}$ with covariance function k, that is,*

$$\mathsf{E} f(a) f(b) = k(a, b), \qquad \textit{for all } a, b \in X.^2$$

This result is known by the names *Kolmogorov extension theorem*, *Kolmogorov existence theorem*, *Kolmogorov consistency theorem* and *Daniell-Kolmogorov theorem.*

[1]Some authors use the term "positive definite" for our term positive semidefinite and the term "*strictly positive definite*" for our term positive definite.

[2]Future editions will prove this result.


Reproducing Kernels
Multivariate Normals
Commutator
Vectorizer
Permutation Matrices
Symmetric Matrices
Quadratic Forms
Expectation
Inner Product Linear Functional Representations
Matrix Trace
Matrix Transposition
Matrix Inverses
Real Squares
Complex Inner Product Spaces
Matrix Product
Continuous Linear Functionals
Complete Normed Spaces
Normal Densities
Multivariate Real Densities
Matrix Determinants
Linear Functionals
Complete Metric Spaces
Probability Densities
Matrix Vector Products
Real Matrix Products
Sequence Spaces
Metric Spaces
Inner Product Norms
Real Integrals
Matrix Addition
Real Linear Combinations
Linear Combinations
Measurable Functions
Measurable Spaces
Real Inner Product
Matrix as Matrix
Real Matrices
Random Variables
Complete Densities
Functionals
Simple Integrals
Real Space
Vectors
Vector Spaces
Topological Vector Algebra
Probability Measures
Measurable Functions
Real Vectors
Linear Functions
Topological Spaces
Event Probabilities
Measures
Signature Sequences
Matrices
Plane Inner Product
Plane Norm
R-Dimensional Spaces
Space Metrics
Plane Vector Norm and Differences
Signed Algebras
Extended Real Numbers
Real Series
Real Squares Sequences
Real Square Roots
Right Triangle Area Relations
Distance
Plane Metrics
Real Limits
Real Numbers
Natural Square Roots
Exponential Function
Space Distance
Real Sums
Natural Addition
Real Differences
Power Relations
Simple Functions
Plane Distance
Real Multiplication Inverses
Squares
Probability Distributions
Real Exponents
Natural Logs
Real Products
Real Integrals
Real Numbers
Real Sums
Sequences
Real Plane
Integers
Real Addition Inverses
Real Line
Rational Multiplication Inverses
Natural Numbers
Real Powers
Integral Line
Real Line
Lines
Subset Algebras
Natural Products
Natural Series
Real Reciprocals
Rational Order
Planes
Fields
Permutations
Cardinals
Rings
Set Numbers
Sets
Integer Sets
Integer Addition
Even Products
Integer Addition Inverses
Exponentiation Sets
Integer Products
Natural Multiplication Identity
Natural Addition Identity
Natural Cardinalities
Integer Exponents
Integer Sums
Integer Multiplication
Natural Functions
Integer Order
Natural Numbers
Composition
Family Intersections
Function Inverses
Family Unions and Intersections
Natural Powers
Integers
Integer Numbers
Order
Equations
Function Images
Function Compositions
Families
Natural Products
Equivalence Relations
Natural Sets
Natural Order
Function Relations
Functions
Natural Sets
Predicates
Subset Systems
Recursive Functions
Set Products
Characteristic Functions
Ordered Pairs
Partial orders
Set Powers
Natural Subsets
Ordered Outcomes
Natural Subsets
Unordered Triples
Generalized Set Doubles
Partitions
Set Symmetric Differences
Nonempty Sets
Intersection of Empty Set
Set Doubles
Set Unions and Intersections
Set Intersections
Set Complements
Pair Unions
Set Unions
Pair Intersections
Set Differences
Unordered Pairs
Empty Set
Set Symbolization
Set Inclusion
Standardized Accounts
Accounts
Definitions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Numbers
Lemmas
Theorems