Affine MMSE Predictors

We want to find A and b to minimize

$$\mathbf{E}\,|Ax+b-y|^2.$$

Proof. We can express $\mathbf{E}(|Ax+b-y|^2)$ as $\mathbf{E}((Ax+b-y)^\top(Ax+b-y))$

$$\begin{array}{cccccc} + & \mathbf{tr}(A\,\mathbf{E}(xx^\top)A^\top) & + & \mathbf{E}(x)^\top A^\top b & - & \mathbf{tr}(A^\top\,\mathbf{E}(yx^\top)) \\ + & b^\top A\,\mathbf{E}(x) & + & b^\top b & - & b^\top\,\mathbf{E}(y) \\ - & \mathbf{tr}(A\,\mathbf{E}(xy^\top)) & - & \mathbf{E}(y)^\top b & + & \mathbf{E}(yy^\top) \end{array}$$

The gradients with respect to b are

$$\begin{array}{cccccc} + & 0 & + & A\,\mathbf{E}(x) & - & 0 \\ + & A\,\mathbf{E}(x) & + & 2b & - & \mathbf{E}(y) \\ - & 0 & - & \mathbf{E}(y) & + & 0 \end{array}$$

so $2A\,\mathbf{E}(x)+2b-2\,\mathbf{E}(y)$. The gradients with respect to A are

$$\begin{array}{cccccc} + & \mathbf{E}(xx^\top)A^\top + \mathbf{E}(xx^\top)^\top A^\top & + & \mathbf{E}(x)b^\top & - & \mathbf{E}(yx^\top)^\top \\ + & \mathbf{E}(x)b^\top & + & 0 & - & 0 \\ - & \mathbf{E}(xy^\top) & - & 0 & + & 0 \end{array}$$

so $2\,\mathbf{E}(xx^\top)A^\top + 2\,\mathbf{E}(x)b^\top - 2\,\mathbf{E}(xy^\top)$. We want A and b solutions to

$$A\,\mathbf{E}(x)+b-\mathbf{E}(y)=0$$

$$\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)b^\top - \mathbf{E}(xy^\top) = 0$$

so first get $b = \mathbf{E}(y) - A\,\mathbf{E}(x)$. Then express

$$\begin{aligned}
\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)(\mathbf{E}(y) - A\,\mathbf{E}(x))^\top - \mathbf{E}(xy^\top) &= 0. \\
\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)\,\mathbf{E}(y)^\top - \mathbf{E}(x)\,\mathbf{E}(x)^\top A^\top - \mathbf{E}(xy^\top) &= 0. \\
(\mathbf{E}(xx^\top) - \mathbf{E}(x)\,\mathbf{E}(x)^\top)A^\top &= \mathbf{E}(xy^\top) - \mathbf{E}(x)\,\mathbf{E}(y)^\top. \\
\mathbf{cov}(x,x)A^\top &= \mathbf{cov}(x,y).
\end{aligned}$$

So $A^\top = \mathbf{cov}(x,x)^{-1}\,\mathbf{cov}(x,y)$ means $A = \mathbf{cov}(y,x)\,\mathbf{cov}(x,x)^{-1}$ is a solution. Then $b = \mathbf{E}(y) - \mathbf{cov}(y,x)\,\mathbf{cov}(x,x)^{-1}\,\mathbf{E}(x)$. So to summarize, the estimator $\phi(x) = Ax + b$ is

$$\mathbf{cov}(y,x)\,(\mathbf{cov}\,x,x)^{-1}\,x + \mathbf{E}(y) - \mathbf{cov}(y,x)\,\mathbf{cov}(x,x)^{-1}\,\mathbf{E}(x)$$

or

$$\mathbf{E}(y) + \mathbf{cov}(y,x)\,(\mathbf{cov}\,x,x)^{-1}\,(x - \mathbf{E}(x))$$

$\square$