



## Why

We want to sum infinitely many real numbers.

## Definition

Let  $(a_k)_{k \in \mathbf{N}}$  be a sequence in  $\mathbf{R}$ . Define  $(s_n)_{n \in \mathbf{N}}$  by

$$s_n = \sum_{k=1}^n a_k.$$

We call  $s_n$  the *n*th partial sum of  $(x_k)$ . In other words, the first partial sum  $s_1$  is  $a_1$ , the second partial sum  $s_2$  is  $a_1 + a_2$ , the third partial sum  $s_3$  is  $a_1 + a_2 + a_3$  and so on.

We call  $(s_n)$  the *sequence of partial sums* or *series* of  $(a_k)$ . If the *series* converges, then we say that  $(a_k)$  is *summable*. Clearly not every series is summable: consider, for example,  $a_k = 1$  for all  $k$ . It has the divergent series  $(1, 2, 3, 4, 5, \dots)$ .

## Notation

If the sequence is summable, then there exists a unique  $s \in \mathbf{R}$  (the limit), which we denote

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k. \quad (1)$$

We read these relations aloud as “ $s$  is the limit as  $n$  goes to infinity of  $s_n$ ” and “ $s$  is the limit as  $n$  goes to infinity of the sum of  $a_k$  from  $k$  equals 1 to  $n$ .” We often avoid referencing

$s_n$  by abbreviating Equation (1) by

$$\sum_{k=1}^{\infty} a_k = s.$$

We read this notation aloud as “the sum from 1 to infinity of a k is s.” The notation is subtle, and requires justification by the algebra of series.<sup>1</sup>

## Convergence

For a series to converge, intuition suggests that the additional terms added should be getting smaller and smaller. Indeed:

**Proposition 1.** *Let  $(a_k)_{k \in \mathbf{N}}$  be a sequence of real numbers. If  $(a_k)$  is summable then  $a_k$  converges to 0.*<sup>2</sup>

The converse of this theorem has immediate relevance as a preliminary test for determining whether a series converges.

**Proposition 2.** *If  $(a_k)$  does not converge or converges to  $a_0 \neq 0$ , then it is not summable.*

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<sup>1</sup>Future editions will include such justification.

<sup>2</sup>Future editions will include an account.

