Why

We want a norm on the vector space of continuous functions.

Definition

Consider a function from a closed real interval to the real numbers. The *absolute supremum* of the function is the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

Prop. 1. *The functional mapping $f \in C[a,b]$ to its absolute supremum is a norm.*

Proof. Let R denote the set of real numbers. Define $\phi : C[a,b] \to R$ by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a,b]\}.$$

1. $|f(x)| \geq 0$ for all $x \in [a,b]$, so $\phi(f) \geq 0$.
2. If $\phi(f) = 0$ then $|f(x)| \leq 0$ for all x and so $f(x) = 0$ for all $x \in [a,b]$. If $f = 0$, then $|f(x)| = 0$ for all $x \in [a,b]$
3. For all α real, $|\alpha f(x)| = |\alpha|\,|f(x)|$. so $\phi(\alpha f) = |\alpha|\,\phi(f)$
4. For all $f, g \in C[a,b]$, and $x \in [a,b]$, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ by the triangle inequality for absolute value. Thus,

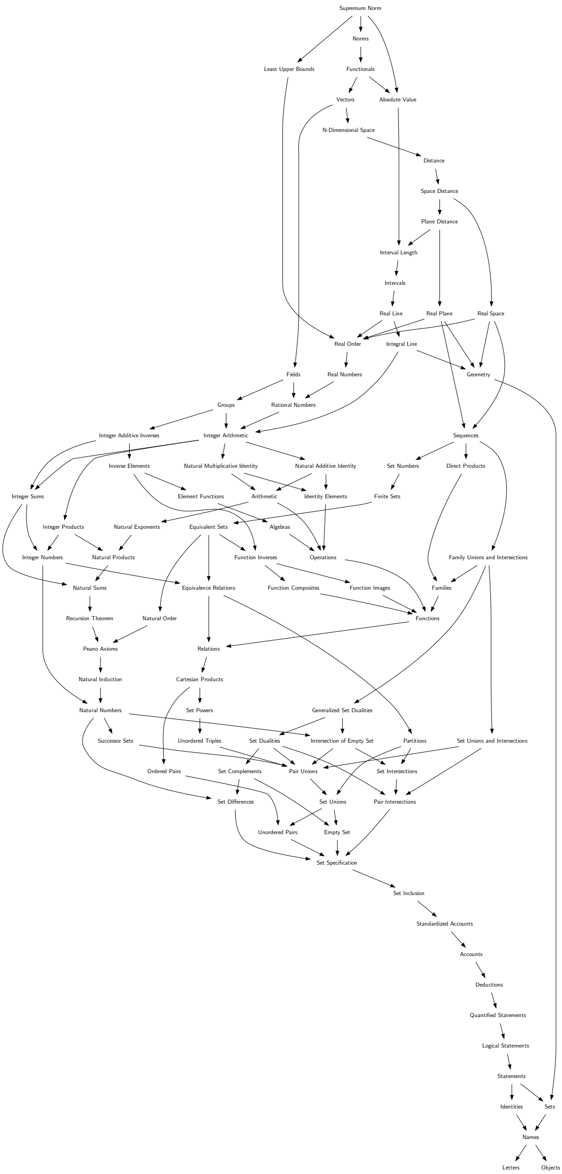
$$\begin{aligned}\phi(f+g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a,b]\} \\ &\leq \sup\{|f(x)| \mid x \in [a,b]\} + \sup\{|g(x)| \mid x \in [a,b]\} \\ &= \phi(f) + \phi(g)\end{aligned}$$

$\square$

We call the functional ϕ defined above the *supremum norm*.

Notation

Let $f \in C[a, b]$. We denote the supremum norm of f by $|f|_{\text{sup}}$.


Supremum Norm
Norms
Least Upper Bounds
Functionals
Vectors
Absolute Value
N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Line
Real Plane
Real Space
Real Order
Integral Line
Fields
Real Numbers
Geometry
Groups
Rational Numbers
Integer Additive Inverses
Integer Arithmetic
Sequences
Inverse Elements
Natural Multiplicative Identity
Natural Additive Identity
Set Numbers
Direct Products
Integer Sums
Element Functions
Arithmetic
Identity Elements
Finite Sets
Integer Products
Natural Exponents
Equivalent Sets
Algebras
Integer Numbers
Natural Products
Function Inverses
Operations
Family Unions and Intersections
Natural Sums
Equivalence Relations
Function Composites
Function Images
Families
Recursion Theorem
Natural Order
Functions
Peano Axioms
Relations
Natural Induction
Cartesian Products
Natural Numbers
Set Powers
Generalized Set Dualities
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects