



### Defining result

**Proposition 1.** *Let  $(X, \mathcal{A})$  be a measurable space. Let  $\mu$  and  $\nu$  be finite measures with  $\nu \ll \mu$ . There exists  $g : X \rightarrow [0, \infty)$  such that*

$$\nu(A) = \int_A g \, d\mu$$

*for all  $A \in \mathcal{A}$ . The function  $g$  is  $\mu$ -almost everywhere unique.*

*Proof.* Define

$$\mathcal{F} = \left\{ f : X \rightarrow [0, \infty) \mid f \text{ measurable and } \int_A f \, d\mu \leq \nu(A) \right\}.$$

The function  $f \equiv 0$  is in  $\mathcal{F}$ , since it is a measurable simple function whose integral over every measurable set is zero. If  $f_1$  and  $f_2$  are in  $\mathcal{F}$ , then  $f_1 \vee f_2$  is in  $\mathcal{F}$ . To check, let  $A \in \mathcal{A}$ , and define the sets  $A_1 = \{x \in A \mid f_1(x) > f_2(x)\}$  and  $A_2 = \{x \in A \mid f_1(x) \leq f_2(x)\}$ .  $A_1$  and  $A_2$  partition  $A$ , so

$$\begin{aligned} \int_A f_1 \vee f_2 &= \int_{A_1} f_1 \vee f_2 + \int_{A_2} f_1 \vee f_2 \\ &= \int_{A_1} f_1 + \int_{A_2} f_2 \\ &\leq \nu(A_1) + \nu(A_2) \end{aligned}$$

Since  $A_1$  and  $A_2$  partition  $A$ ,

$$\nu(A_1) + \nu(A_2) = \nu(A_1 \cup A_2) = \nu(A).$$

Select a sequence of functions  $(f_n)_n$  in  $\mathcal{F}$  so that

$$\lim_n \int f_n = \sup \left\{ \int f \mid f \in \mathcal{F} \right\}.$$

Toward ensuring the sequence is increasing, define  $g_1 = f_1$ ,  $g_2 = g_1 \vee f_2$ , and  $g_n = g_{n-1} \vee f_n$  for  $n \geq 3$ . Using the observation in the previous paragraph,  $g_n \in \mathcal{F}$  for each  $n$ .

Let  $g$  be the pointwise limit of the  $(g_n)_n$ . The monotone convergence of integrals shows

$$\int_A g = \lim_n \int_A g_n.$$

for each  $A \in \mathcal{A}$ . Since  $\int_A g_n \leq \nu(A)$ , so too is the limit and thus so too is  $\int_A g$ . Thus,  $g \in \mathcal{F}$ . By construction, for  $A = X$ ,  $\int g = \sup\{\int f \mid f \in \mathcal{F}\}$ . We have constructed an element of  $\mathcal{F}$  attaining the supremum.

We know that the integral of  $g$  on  $A$  with respect to  $\mu$  is bounded above by  $\nu(A)$ . We want the gap to be zero. Regardless of the gap, the function  $\nu_0 : \mathcal{A} \rightarrow [0, \infty)$  defined by

$$\nu_0(A) = \nu(A) - \int (g, A, \mu),$$

for each  $A \in \mathcal{A}$  is a positive measure. If  $\nu_0$  is identically zero, then there is no gap.

Suppose there is a gap: then there exists a measurable set with strictly positive measure under  $\nu_0$ . Since the base set contains this set, and measures are monotone, the base set must have strictly positive measure. Since  $\mu$  is finite, there exists a natural number  $n$  so that

$$\nu_0(X) > \frac{1}{n}\mu(X).$$

Define a new measure  $\nu_1 = \nu_0 - \frac{1}{n}\mu$ . Denote a signed-set decomposition of  $\nu_1$  by  $(P, N)$ . Then  $\nu_1(A \cap P) \geq 0$ , or equivalently,

$$\nu_0(A \cap P) - \frac{1}{n}\mu(A \cap P) \geq 0,$$

for all  $A$ , and so

$$\begin{aligned} \nu(A) &= \nu_0(A) + \int (g, A, \mu) \\ &\geq \nu_0(A \cap P) + \int (g, A, \mu). \end{aligned}$$

□

Many authorities refer to this result as the *Radon-Nikodym theorem*.

## Notation

