



## Why

We discuss a decomposition using eigenvalues and eigenvectors.<sup>1</sup>

## Defining result

An *eigenvalue decomposition* of a matrix  $A \in \mathbf{R}^{n \times n}$  is an ordered pair  $(X, \Lambda)$  in which  $X$  is invertible,  $\Lambda$  is diagonal, and  $A = X\Lambda X^{-1}$ .

In this case,  $AX = X\Lambda$ , in other words,

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

in which  $x_i$  is the  $i$ th column of  $X$  and  $\lambda_i$  is the  $i$ th diagonal element of  $\Lambda$ . We have  $Ax_i = \lambda_i x_i$  for  $i = 1, \dots, n$ . In other words, the  $i$ th column of  $X$  is an eigenvector of  $A$  and the  $j$ th entry of  $\Lambda$  is the corresponding eigenvalue.

If  $X$  is orthonormal, so that  $X^{-1} = X^\top$ , then we can interpret such a decomposition as a change of basis to *eigenvector coordinates*. If  $Ax = b$ , and  $A = X\Lambda X^{-1}$  then  $(X^{-1}b) = \Lambda(X^{-1}x)$ . Here,  $X^{-1}x$  expands  $x$  in the basis of columns of  $X$ . So to compute  $Ax$ , we first expand into the basis of columns of  $X$ , scale by  $\Lambda$ , and then interpret the result as the coefficients of a linear combination of the columns of  $X$ .

In this case that  $A = X\Lambda X^\top$  for an eigenvalue decomposition  $(X, \Lambda)$  of  $A$ , we can also write

$$A = X\Lambda X^\top = \sum_{i=1}^n \Lambda_{ii} x_i x_i^\top.$$

**Proposition 1.** *Every real symmetric matrix has an eigenvalue decomposition  $(X, \Lambda)$  in which  $X$  is orthonormal.*<sup>2</sup>

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<sup>1</sup>Future editions will expand.

<sup>2</sup>In future editions, this may be the motivating result for the definition of eigenvalues.

