

## PERMUTATION MATRICES

## Why

Can permuting the rows or columns of a matrix be represented by matrix multiplication?

### Definition

Let  $\sigma: \{1, 2, ..., n\} \to \{1, 2, ..., n\}$  be a permutation of n. The permutation matrix of  $\sigma$  is the matrix P defined by  $P_{ij} = 1$  if  $\sigma(i) = j$  and 0 otherwise. This is sometimes called the column representation (in contrast to the row representation, in which  $P_{ij} = 1$  if  $\sigma(j) = i$ .

Let  $A \in \mathbf{R}^{n \times n}$ . Then pre-multiplying A by P permutes the rows of A. In other words PA has the same rows as A but permuted according to  $\sigma$ . Similarly, post-multiplying by P permutes the columns of A. In other words, AP has the same columns as A but permuted according to  $\sigma$ . Clearly, we can also speak of permuting the components of a vector.

# Composition

Let  $\pi, \sigma \in S_n$  with corresponding permutation matrices  $P_{\sigma}$  and  $P_{\pi}$ . Then  $P_{\pi}P_{\sigma}A$  has the same rows as A but permuted by  $\pi\sigma$ . Likewise,  $AP_{\pi}P_{\sigma}$  has the same columns as A but permuted by  $\pi\sigma$ . Clearly, the identity permutation on  $\{1, 2, \ldots, n\}$  is the identity  $I \in \mathbf{R}^{n \times n}$ .

#### Inverses

It is clear from the definition that  $P_{\sigma}^{-1} = P_{\sigma^{-1}}$  and so if P is a permutation matrix then  $P^{-1}$  is  $P^{\top}$ .

