Metrics

1 Why

We want to talk about the "closeness" of objects in a set.

2 Definition

Our common sense notions will be the following: (a) close objects are not distant (b) distance is a positive quantity (a) any two distinct objects are some discernible distance apart (b) closeness is transitive: if one object is close to a second object, and the second is close to the third, then the first is close to the third.

Two objects are close if they are not distant. We name a class of functions which capture the essence of distance from the plane of analytic geometry.

A **metric** on a set is a function on ordered pairs of elements of the set which is symmetric, non-negative, definite, and triangularly transitive. A **metric space** is an ordered pair: a set with a metric on the set.

That the function is symmetric is natural: the distance of two objects should not depend on their order.

In a metric space, we say that one pair of objects is **closer** together if the metric of the first pair is smaller than the metric value of the second pair.

Thee properties of a metric are satisfied by the absolue value on the real numbers and by the usual distance in two-dimensional real plane.

2.1 Notation

Let A be a set and let R be the set of real numbers. We commonly denote a metric by the letter d, as a mnemonic for "distance." Let $d : A \times A \to R$. Then d is a metric if:

1. it is non-negative, which we tend to denote by
$$d(a, b) \geq 0, \quad \forall a, b \in A.$$
2. it is definite, which we tend to denote by
$$d(a, b) = 0 \Leftrightarrow a = b, \quad \forall a, b \in A.$$
3. it is symmetric, which we tend to denote by:
$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$
4. it is triangularly transitive, which we tend to denote by
$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of A with d by (A, d).