



Why

If a directed graph has no cycles, then it has a nice property.¹

Definition

Directed and acyclic graphs (or *directed acyclic graphs*, *DAGs*) have partial ordering property on vertices. We call a vertex s an *ancestor* of a vertex u if there is a directed path from s to u ; c.f. .

Partial Order

We call a vertex s an *ancestor* of a vertex t if there is a directed path from s to t . The relation R defined by $(s, t) \in R$ if s is an ancestor of t is a partial order.

Clearly, every subgraph induced on a directed acyclic graph is a directed acyclic graph.

Proposition 1. *Let (V, E) be a directed acyclic graph. Then there exists a vertex $v \in V$ which is a source and a vertex $w \in V$ which is a sink.*

Proof. There exists a directed path of maximum length. It must start at a source and end at a sink.² □

Topological numbering

We can choose a total ordering that is consistent with the partial order of ancestry.

A *topological numbering* (or *topological sort*, *topological ordering*) of a directed graph (V, E) is a numbering $\sigma : \{1, \dots, |V|\} \rightarrow V$ satisfying

$$(v, w) \in E \longrightarrow \sigma^{-1}(v) < \sigma^{-1}(w).$$

¹Future editions will expand this vague introduction.

²Future editions will expand.

³Future editions will further explain this concept.

Proposition 2. *There exists a topological sort for every acyclic graph.*

Proof. Let (V, F) be a directed acyclic graph. There exists a source vertex, v_1 . Set $\sigma(1) = v_1$. Take the subgraph induced by $V - \{v_1\}$. It is directed acyclic, and so has a source vertex, v_2 . Set $\sigma(2) = v_2$. Continue in this way.⁴ □

⁴Future editions will clarify and expand.

