Why

For each value of a random variable's codomain, the set of outcomes corresponding to that value is the inverse image of the random variable. We can speak of the probability that a random variable takes a value then, by assigning it the probability of the set of outcomes corresponding to that value.

Definition

Let $p : \Omega \to \mathbf{R}$ be a probability distribution with corresponding probability measure $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$. Suppose $x : \Omega \to V$ is an outcome variable. The *probability* $x = a$, for $a \in \Omega$, is

$$\mathbf{P}(\{\omega \in \Omega \mid x(\omega) = a\}).$$

From the definition of $\mathbf{P}$, we express the above as

$$\textstyle\sum_{\omega \in \Omega \mid x(\omega) = a} p(\omega).$$

We refer to the probability of the *event* that $x = a$.

Notation

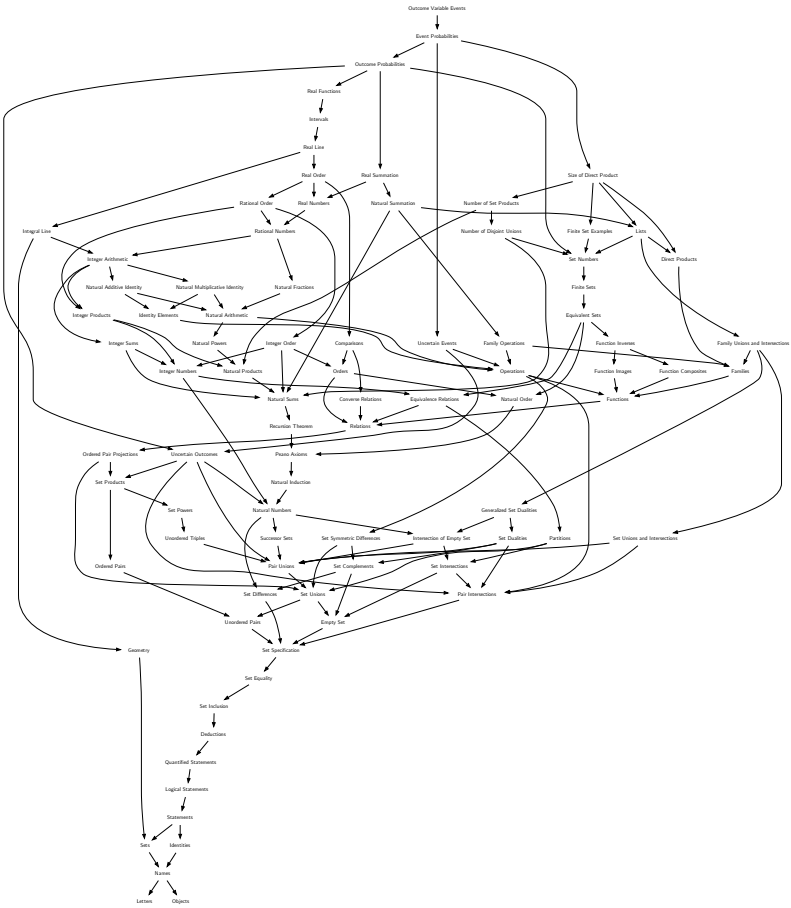
We denote the probability that $x = a$ by $\mathbf{P}[x = a]$. Our square brackets deviate from the slightly slippery but universally standard notation $\mathbf{P}(x = a)$. We prefer the square brackets, since $x = a$ is not itself an argument to $\mathbf{P}$, but shorthand for the set $\{\omega \in \Omega \mid x(\omega) = a\}$.

There are many similar notations. For example, $\mathbf{P}[x \in C]$ means $\mathbf{P}(\{x \in \Omega \mid x(\omega) \in C\})$. In particular, if $x : \Omega \to \mathbf{R}$, $\mathbf{P}[x \geq a]$ means $\mathbf{P}(\{\omega \in \Omega \mid x(\omega) \geq a\})$. Since the *event* that $x = a$ is the inverse image of $\{a\}$ under x, we also use the notations $\mathbf{P}(x^{-1}(a))$ and $\mathbf{P}(x^{-1}(C))$.

Example: sum of two dice

Define $\Omega = \{1, \ldots, 6\}^2$ and define $p : \Omega \to \mathbf{R}$ with $p(\omega) = 1/36$ for each $\omega \in \Omega$. Define $x : \Omega \to \mathbf{N}$ by $x(\omega_1, \omega_2) = \omega_1 + \omega_2$. Then

$$\mathbf{P}[x = 4] = p((2, 2)) + p(1, 3) + p(3, 1) = 1/12.$$


Outcome Variable Events
Event Probabilities
Outcome Probabilities
Real Functions
Intervals
Real Line
Real Order
Real Summation
Size of Direct Product
Rational Order
Real Numbers
Natural Summation
Number of Set Products
Integral Line
Rational Numbers
Number of Disjoint Unions
Finite Set Examples
Lists
Integer Arithmetic
Set Numbers
Direct Products
Natural Additive Identity
Natural Multiplication Identity
Natural Fractions
Finite Sets
Integer Products
Identity Elements
Natural Arithmetic
Equivalent Sets
Integer Sums
Natural Powers
Integer Order
Comparisons
Uncertain Events
Family Operations
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Images
Function Composites
Families
Natural Sums
Converse Relations
Equivalence Relations
Natural Order
Functions
Recursion Theorem
Relations
Ordered Pair Projections
Uncertain Outcomes
Peano Axioms
Set Products
Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Unordered Triples
Successor Sets
Set Symmetric Differences
Intersection of Empty Set
Set Dualities
Partitions
Set Unions and Intersections
Ordered Pairs
Pair Unions
Set Complements
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
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Statements
Sets
Identities
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Objects