



Why

We explore matrix-vector multiplication.

Definition

Given a matrix $A \in \mathbf{R}^{m \times n}$ and a vector $x \in \mathbf{R}^n$, the *product* of A with x is the vector $y \in \mathbf{R}^m$ defined by

$$y_i = \sum_{j=1}^n A_{ij}x_j, \quad i = 1, \dots, m.$$

Notation

We denote the product of A with x by Ax . With which we concisely write the system linear equations (A, b) as $b = Ax$.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function $x \mapsto Ax$. In other words, we are interested in the existence of an inverse element of A . The geometric interpretation is that A transforms the vector x .

Conversely, we can view x as transforming (acting on) A . Let $a^j \in \mathbf{R}^m$ denote the j th column of A , then we can write the product

$$y = Ax = \sum_{j=1}^n x_j a^j$$

In other words, y is linear combination of the columns of A .

Properties

We call the function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ defined by $f(x) = Ax$ the *matrix multiplication function* (or *matrix-vector product function*) associated with A . f satisfies the following two important properties:

1. $A(x + y) = Ax + Ay$
2. $A(\alpha x) = \alpha Ax$.

We call such a function f *linear*. In other words, the matrix multiplication function is linear. Conversely, if $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is linear, there exists a matrix inducing g .

Proposition 1. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be linear. Then there exists a unique $A \in \mathbf{R}^{m \times n}$ satisfying $f(x) = Ax$ for all $x \in \mathbf{R}^n$.*

Proof. Evaluate f at the standard unit vectors e_i . The i th component of e_i is 1 and all other components are 0. \square

