



Why

We want to discuss real-valued random functions whose family of random variables have simple densities.¹

Definition

A *normal random function* is a real-valued random function whose family of real-valued random variables has the property that any subfamily is jointly normal.

For this reason, we call the family of random variables (or stochastic process) corresponding to the random function a *gaussian process* or *normal process*.

Notation

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and A a set. Let $x : \Omega \rightarrow (A \rightarrow \mathbf{R})$ be a random function with family $y : A \rightarrow (\Omega \rightarrow \mathbf{R})$.

The random function x is a normal if, for all $a^1, \dots, a^m \in A$, $(y(a^1), \dots, y(a^m))$ is jointly normal.

Mean and covariance function

Proposition 1. *Let $x : \Omega \rightarrow (A \rightarrow \mathbf{R})$ be a normal random function with family $X : A \rightarrow (\Omega \rightarrow \mathbf{R})$. There exists unique functions $m : A \rightarrow \mathbf{R}$ and $k : A \times A \rightarrow \mathbf{R}$ so that the mean of the random variable X_a is $m(a)$ for all A and the covariance of*

¹Future editions will expand.

the random variables X_a and $X_{a'}$ is $k(a, a')$ for all $a, a' \in A$.²

For this reason, we call m the *mean function* and k the *covariance function* of the random function.

Conversely, let $m : A \rightarrow \mathbf{R}$ and $k : A \times A \rightarrow \mathbf{R}$. Then if k satisfies the property that for all a^1, \dots, a^m , the $m \times m$ matrix

$$\begin{pmatrix} k(a^1, a^1) & \cdots & k(a^1, a^m) \\ \vdots & \ddots & \vdots \\ k(a^m, a^1) & \cdots & k(a^m, a^m) \end{pmatrix}$$

is positive semidefinite, then we can construct a Gaussian process with mean function m and covariance function k .³ For this reason, we call k with such a property *positive semidefinite* or a *covariance function*. Notice, of course, that k is symmetric. The matrix above is sometimes called the *Gram matrix* for k and a^1, \dots, a^m .

Example

Let $A = \{1, \dots, n\}$ and let $K \in \mathbf{R}^{n \times n}$ be symmetric positive semidefinite. Define $m : A \rightarrow \mathbf{R}$ to be $m \equiv 0$ (the constant zero function) and $k(i, j) = K_{ij}$. Then the normal random function $x : \Omega \rightarrow (A \rightarrow \mathbf{R})$ with mean m and covariance k is in

²Future editions may include an account.

³Some authors belabor this point because of the natural inclination to want to specify an *inverse* covariance function, which need not satisfy the consistency property. The consistency property ensures that any marginal of a subfamily's density is the density of that further subfamily. Future editions may expand.

one to one correspondence with the gaussian random vectors with mean zero.

