

Measure Vector Space

1 Why

If both signed measures are finite, then their difference is always well-defined. Is the difference a finite signed measure?

2 Preliminary Result

Proposition 1. A linear combination of finite signed measures is a finite signed measure.

Proof. Let (X, \mathcal{A}) be a measurable space. Let μ and ν be finite signed measures. Let R denote the real numbers. Then $(\alpha\mu)(\varnothing) = \alpha \cdot \mu(\varnothing) = \alpha \cdot 0 = 0$. Also for $(A_n)_n \subset \mathcal{A}$ disjoint,

$$(\alpha\mu)(\cup A_n) = \alpha\mu(\cup A_n) = \alpha\sum_{n=1}^{\infty}\mu(A_n)$$
$$= \sum_{n=1}^{\infty}\alpha\mu(A_n) = (\alpha\mu)(A_n)$$

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Similarly, $(\mu + \nu)(\varnothing) = \mu(\varnothing) + \nu(\varnothing) = 0$. And, for $(A_n)_n \subset \mathcal{A}$ disjoint,

$$(\mu + \nu)(\cup A_n) = \mu(\cup A_n) + \nu(\cup A_n) = \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \nu(A_n)$$
$$= \sum_{n=1}^{\infty} \mu(A_n) + \nu(A_n) = \sum_{n=1}^{\infty} (\mu + \nu)(A_n)$$

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3 Main Result

Proposition 2. The set of finite signed measures is a vector space.

Proof. Use the previous proposition. Observe that the function $\mu \equiv 0$ is a measure. and $\nu + \mu = \nu$ for all measures ν .

3.1 Notation

Denote the set of real numbers by R. We denote the vector space of signed measures on measurable space (X, A) by by M(X, A, R).