



FUNCTIONS

Why

We want a notion for a correspondence between two sets.

Definition

A *function* f from a set X to a set Y is a relation (see *Relations*) whose domain is X and whose range is a subset of Y such that for each $x \in X$, there exists a unique $y \in Y$ so that $(x, y) \in f$.

We call the unique $y \in Y$ the *result* of the function *at* the *argument* x . We call Y the *codomain*. If the range is Y we say that f is a function from X *onto* Y (or f is *surjective*). If distinct elements of X are mapped to distinct elements of Y , we say that the function is *one-to-one* (or f is *injective*).

We say that the function *maps* elements from the domain to the codomain. Since the word function and the verb “maps” connote activity, some authors refer to the concept that we have defined as a function as the *graph* of a function—namely, the set of ordered pairs which that function produces—and leave the concept of function undefined.

Notation

Let X and Y denote sets. We denote a function named f whose domain is X and whose codomain is Y by $f : X \rightarrow Y$. We read the notation aloud as “ f from X to Y ”. We denote the set

of all functions from X to Y (which is a subset of $(X \times Y)^*$) by Y^X . A less standard but equally good notation is $X \rightarrow Y$, read aloud as “ A to B ”. Using the notations introduced so far, we denote that $f \in (A \rightarrow B)$ by $f : A \rightarrow B$. We tend to denote function by lower case latin letters, especially f , g , and h . f is a mnemonic for function and g and h are nearby.

Let $f : A \rightarrow B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud “f of a.” We sometimes drop the parentheses, and write the result as f_a , read aloud as “f sub a.” Let $g : A \times B \rightarrow C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as “g of a and b”. We read g_{ab} aloud as “g sub a b.”

Examples

If $X \subset Y$, the function $\{(x, y) \in X \times Y \mid x = y\}$ is the *inclusion function* of X into Y . We often introduce such a function as “the function from X to Y defined by $f(x) = y$ ”. We mean by this that f is a function and that we are specifying the appropriate ordered pairs using the statement, called *argument-value notation*. The inclusion function of X into X is called the *identity function* of X . If we view the identity function as a relation on X , it is the relation of equality on X .

The functions $f : (X \times Y) \rightarrow X$ defined by $f(x, y) = x$ is the *pair projection* of $X \times Y$ onto X . Similarly $g : (X \times Y) \rightarrow Y$ defined by $g(x, y) = y$ is the pair projection of $X \times Y$ onto Y . The identity function is one-to-one and onto, the inclusion

functions are one-to-one but not always onto, and the pair projections are usually not one-to-one.

