



Simple Integral Additivity

1 Why

If we stack two rectangles, with equal base lengths but different heights, on top of each other, the additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property, as it ought to.

2 Result

Proposition 1. *The simple non-negative integral operator is additive.*

Proof. Let (X, \mathcal{A}, μ) be a measure space. Let $\mathcal{SF}_+(X)$ denote the non-negative real-valued simple functions on X . Define $s : \mathcal{SF}_+(X) \rightarrow [0, \infty]$ by $s(f) = \int f d\mu$ for $f \in \mathcal{SF}_+(X)$.

In this notation, we want to show that $s(f + g) = s(f) + s(g)$ for all $f, g \in \mathcal{SF}_+(X)$. Toward this end, let $f, g \in \mathcal{SF}_+(X)$ with the simple partitions:

$$\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n \subset \mathcal{A} \quad \text{and} \quad \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset [0, \infty].$$

We consider the refinement of the two partitions. TODO: this is why you don't do the unique maximal partition business. $\{A_i \cap B_j\}_{i,j=1}^{i=m, j=n}$.

First, let $\alpha \in (0, \infty)$. Then $\alpha f \in \mathcal{SF}_+(X)$, with the simple partition $\{A_n\} \subset \mathcal{A}$ and $\{\alpha a_n\} \subset [0, \infty]$.

$$s(\alpha f) = \sum_{i=1}^n \alpha a_n \mu(A_i) = \alpha \sum_{i=1}^n a_n \mu(A_i) = \alpha s(f).$$

If $\alpha = 0$, then αf is uniformly zero; it is the non-negative simple with partition $\{X\}$ and $\{0\}$. Regardless of the measure of X , this non-negative simple function is zero. Recall that we define $0 \cdot \infty = \infty \cdot 0 = 0$. \square