

AREA

Why

We list some principles which our intuition of area in planar geometry satisfies.

Common notions

We take two common notions; these are analagous to those we developed for length.

- 1. The area of the whole is the sum of the area of the parts; the additivity principle.
- 2. If one whole contains another, the first's area at least as large as the second's area; the *containment principle*.

Again, the task is to make precise the use of "whole," "parts," and "contains." We start with rectangles.

Definition

The area of an rectangle is the sum of the lengths of its sides.

Two rectangles are *non-overlapping* if their intersection is a single point or empty. The *area* of the union of two non-overlapping intervals is the sum of their areas.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A countably simple subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets

of the real numbers countably simple? Answer: no. So, what can we say?

A cover of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

Cuts

If a, b are real numbers and a < b, then we cut an interval with a and b as its endpoints by selecting c such that a < c and c < b. We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the cut pieces.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

All sets

Proposition 1. Not all subsets of real numbers are simple.

Exhibit: R is not finite.

Proposition 2. Not all subsets of real numbers are countably simple.

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

