

REPRODUCING KERNELS

Why

Definition

Let X be a (nonempty) set and k a field. Let $F \subset (X \to k)$ and let $\langle \cdot, \cdot \rangle : F \times F \to k$ be an inner product so that $(F, \langle \cdot, \cdot \rangle)$ is a complete inner product space.

A reproducing kernel of $(F, \langle \cdot, \cdot \rangle)$ is a map $R: X \times X \to k$ satisfying (1) for every $y \in X$ the function $R(\cdot, y): X \to k$ is an element of F and (2) for every $f \in F$, at every $y \in X$, $f(y) = \langle f, R(\cdot, y) \rangle$ (the reproducing property).

R is called a "reproducing" kernel because of the following implication of the reproducing property. Notice that $R(\cdot, y) \in F$. For this reason,

Properties

If a reproducing kernel exists, it is unique.

Let X be nonempty (index) set. For example, X may be $\{1, 2, ..., N\}$, **Z**, [0,1], \mathbf{R}^d , $\{x \in \mathbf{R}^3 \mid \|x\| \le 1\}$ (the unit sphere), or $\{x \in \mathbf{R}^3 \mid \alpha \le \|x\| \le \beta\}$ (the atmosphere, or volume between two concentric spheres).

A symmetric, real-valued function $k: X \times X \to \mathbf{R}$ of two variables is said to be *positive semidefinite* if for any $n \in \mathbf{N}$, for any real $a_1, \ldots, a_n \in \mathbf{R}$ and $x_1, \ldots, x_n \in X$, we have

$$\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) \ge 0,$$

and positive definite if the above holds with ">".1"

Positive semidefinite kernels are useful for the following constructive reason:

Proposition 1. Let $X \neq \emptyset$ be a set. If $k: X \times X \to \mathsf{R}$ is positive semidefinite, then there exists a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ and a family of zero-mean normal real-valued random variables $\{f_x: \Omega \to \mathsf{R}\}_{x \in X}$ with covariance function k, that is,

$$\mathsf{E} f(a) f(b) = k(a,b), \quad \text{for all } a,b \in X.^2$$

This result is known by the names Kolmogorov extension theorem, Kolmogorov existence theorem, Kolmogorov consistency theorem and Daniell-Kolmogorov theorem.

¹Some authors use the term "positive definite" for our term positive semidefinite and the term "strictly positive definite" for our term positive definite.

²Future editions will prove this result.

