



## PIVOTED ROW REDUCTIONS

### Why

We want to modify ordinary row reduction to handle the case in which a pivot is zero by selecting another suitable pivot.

### Example

Let  $A \in \mathbf{R}^{5 \times 5}$ . If  $A_{11} \neq 0$ , we may subtract multiples of row 1 from row  $2, \dots, 5$  to eliminate variable  $x_1$  from those equations. If  $A$  reduces to  $C \in \mathbf{R}^{5 \times 5}$  and  $C_{22} \neq 0$ , then step 2 moves from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & C_{22} & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \text{ to } \begin{bmatrix} \times & \times & \times & \times & \times \\ & C_{22} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \end{bmatrix}.$$

What if  $C_{22} = 0$ ? In this case suppose we pick a different row. For example, if  $C_{42} \neq 0$  we can move from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & C_{42} & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} \text{ to } \begin{bmatrix} \times & \times & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \\ & C_{42} & \times & \times & \times \\ & \mathbf{0} & \times & \times & \times \end{bmatrix}.$$

Alternatively, we could introduce zeros in column 3 rather than column 2. For example, if we pick the pivot  $C_{43}$  we move from

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & C_{43} & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \text{ to } \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \mathbf{0} & \times & \times \\ \times & \times & \mathbf{0} & \times & \times \\ \times & C_{43} & \times & \times & \times \\ \times & \times & \mathbf{0} & \times & \times \end{bmatrix}.$$

We can choose any nonzero entry in  $C_{k:m,k:m}$  as the pivot.

Suppose we pick pivot  $C_{st} \neq 0$  for  $k \leq s, t \leq m$ . Define  $\tilde{C}$  by swapping row  $s$  of  $C$  with row  $k$  of  $C$  and column  $t$  of  $C$

with column  $k$  of  $C$ . Then  $\tilde{C}_{kk} = C_{st} \neq 0$  and there exists an ordinary row reduction for  $\tilde{C}$ . We call this reduction of  $(\tilde{C}, \tilde{d})$  a *pivoted row reduction* of  $C$  or the *st-reduction* of  $C$ .

If all remaining pivots are zero, then there is no viable pivot. In this case, at least one variable is free and we do not have a unique solution. For convenience, in this case, we still call the system an *st-reduction* of itself.

### Definition

At step  $k$  of ordinary elimination, multiples of row  $k$  are subtracted from rows  $k + 1, \dots, m$  to introduce zeros in entry  $k$  of the rows. If we denote the matrix at the beginning of that step by  $X$ , then row  $k$  of  $X$ , column  $k$  of  $X$  and especially the pivot  $X_{kk}$  play a role. Ordinarily, we subtract from every entry in the submatrix  $X_{k+1:m,k:m}$  the product of a number in row  $k$  and a number in column  $k$ , divided by the pivot  $X_{kk}$ . Generally, however, we can choose as pivot any nonzero entry of  $X_{k:m,k:m}$ .

An  $m$ -variable system  $(A, b)$  is *pivot reducible* (or *reducible*) if there exists a sequence of systems  $S_1, \dots, S_{m-1}$  so that  $S_1$  is a reduction of  $(A, b)$  and  $S_i$  is a reduction of  $S_{i-1}$  for  $i = 1, \dots, m - 1$ . We call  $S_{m-1}$  the *final reduction* (or *reduction*) of  $(A, b)$ . An immediate consequence of our definition is

**Proposition 1.** *All systems are reducible.*

