



# Equivalence Relations

## 1 Why

We want to handle at once all elements which are indistinguishable or equivalent in some aspect.

## 2 Definition

Let  $R$  be a relation on  $A$ .  $R$  is an **equivalence relation** if it is reflexive, symmetric, and transitive.

For an element  $a \in A$ , we call the set of elements in relation  $R$  to  $a$  the **equivalence class** of  $a$ . The key observation, recorded and proven below, is that the equivalence classes partition the set  $A$ . A frequent technique is to define an appropriate equivalence relation on a large set  $A$  and then to work with the set of equivalence classes of  $A$ .

We call the set of equivalence classes the **quotient set** of  $A$  under  $R$ . An equally good name is the divided set of  $A$  under  $R$ , but this terminology is not standard. The language in both cases reminds us that  $\sim$  partitions the set  $A$  into equivalence classes.

### 2.1 Notation

If  $R$  is an equivalence relation on a set  $A$ , we use the symbol  $\sim$ . When alone,  $\sim$  is read aloud as “sim,” but we still read  $a \sim b$  aloud as “a equivalent to b.” We denote the quotient set of  $A$  under  $\sim$  by  $A/\sim$ , read aloud as “A quotient sim”.

## 2.2 Results

## 3 Order Relations

Here we survey a two other special relation on a set. Let  $R$  a relation on the non-empty set  $A$ . We call  $R$  **anti-symmetric** if for two nonequal elements  $a, b \in A$ ,  $(a, b) \in R \implies (b, a) \notin R$ . If  $R$  is reflexive, transitive, and anti-symmetric then we call  $R$  a **partial order** on  $A$ .

A **partially ordered set** is a set together with a partial order. The language partial is meant to suggest that two elements need not be comparable. For example, suppose  $R$  is  $\{(a, a) \mid a \in A\}$ ; we may justifiably call this no order at all and call  $A$  totally unordered, but it is a partial order by our definition.

Often we want all elements of the set  $A$  to be comparable. We call  $R$  **connexive** if for all  $a, b \in A$ ,  $(a, b) \in R$  or  $(b, a) \in R$ . If  $R$  is a partial order and connexive, we call it a **total order**.

A **totally ordered set** is a set together with a total order. The language is a faithful guide: we can compare any two elements. Still, we prefer one word to three, and so we will use the shorter term **chain** for a totally ordered set; other terms include **simply ordered set** and **linearly ordered set**.

### 3.1 Notation

We denote total and partial orders on a set  $A$  by  $\preceq$ . We read  $\preceq$  aloud as “precedes or equal to” and so read  $a \preceq b$  aloud as “a precedes or is equal to b.” If  $a \preceq b$  but  $a \neq b$ , we write  $a \prec b$ , read aloud as “a precedes b.”