

OUTCOME VARIABLE PROBABILITIES

Why

Given a probability measure on the events of a set of outcomes, we can discuss probabilities of outcome variables.

Definition

Let $p: \Omega \to \mathbf{R}$ be a probability distribution with corresponding probability measure $\mathbf{P}: \mathcal{P}(\Omega) \to \mathbf{R}$. Suppose $x: \Omega \to V$ is an outcome variable. The *probability* x = a, for $a \in \Omega$, is

$$P(\{\omega \in \Omega \mid x(\omega) = a\}).$$

From the definition of **P**, we express the above as

$$\sum_{\omega \in \Omega \mid x(\omega) = a} p(\omega).$$

We refer to the probability of the *event* that x = a.

Notation

We denote the probability that x=a by $\mathbf{P}[x=a]$. Our square brackets deviate from the slightly slippery but universally standard notation $\mathbf{P}(x=a)$. We prefer the square brackets, since x=a is not itself an argument to \mathbf{P} , but shorthand for $(\{\omega \in \Omega \mid x(\omega) = a\})$.

Accordingly, there are many similar notations. For example if $V = \mathbf{N}$, $\mathbf{P}[x \geq a]$ is $\mathbf{P}(\{\omega \in \Omega \mid x(\omega) \geq a\})$. Or, $\mathbf{P}[x \in C]$ means $\mathbf{P}(\{x \in \Omega \mid x(\omega) \in C\})$. Since the *event* that x = a is the inverse image of $\{a\}$ under x, we also use the notation $x^{-1}(a)$. Or generally, $x^{-1}(C)$.

Example: sum of two dice

Define $\Omega = \{1, \ldots, 6\}^2$ and define $p : \Omega \to \mathbb{R}$ with $p(\omega) = 1/36$ for each $\omega \in \Omega$. Define $x : \Omega \to \mathbb{N}$ by $x(\omega_1, \omega_2) = \omega_1 + \omega_2$. Then $\mathbf{P}[x = 4] = p((2, 2)) + p(1, 3) + p(3, 1) = 1/12$.

Induced probability

For $x: \Omega \to V$, the events $x^{-1}(a)$ for $a \in V$ partition Ω . Define $q: V \to \mathbf{R}$ by

$$q(a) = \mathbf{P}[x = a].$$

Since
$$\bigcup_{a \in V} x^{-1}(a) = \Omega$$
, $\sum_{a \in A} q(a) = 1$.

We call q the induced distribution (or induced probability mass function) of the random variable x. It is common to denote it by p_x . Thus we can think of V as a set of outcomes, which we call the outcomes induced by x.

If $x: \Omega \to V$ is a random variable and $f: V \to U$, then if we define $y: \Omega \to V$ so that $y \equiv f(x)$, y is a random variable with induced distribution $p_y: \Omega \to \mathbb{R}$ satisfying

$$p_y(b) = \sum_{a \in V | y(a) = b} p_x(a).$$

As a matter of practical computation, we can evaluate probabilities having to do with the outcome variable x using p_x instead of p. For example with x as in the example above, $\mathbf{P}(x=4 \text{ or } x=5)=p_x(4)+p_x(5)$, rather than $\sum_{\omega\in\Omega|x(\omega)=4 \text{ or } x(\omega)=5}p(\omega)$.

