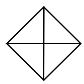


Why

How should we modify probabilities, given that we know some aspect of the outcomes (i.e., that some event has occurred).

Definition

Let $\mathsf{P} : \mathcal{P}(\Omega) \to \mathsf{R}$ be a finite probability measure. Let $A, B \subset \Omega$ and $\mathsf{P}(B) \neq 0$. The *conditional probability* of A *given* B is fraction of the probability of $A \cap B$ over the probability of B.

Notation

In a slightly slippery but universally standard notation, we denote the conditional probability of A given B by $\mathsf{P}(A \mid B)$. In other words, we define

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)},$$

for all $A, B \subset \Omega$, whenever $\mathsf{P}(B) \neq 0$.

For example, we can express the law of total probability (see Event Probabilities) as

$$\mathsf{P}(B) = \textstyle\sum_{i=1}^{n} \mathsf{P}(A_i)\mathsf{P}(B \mid A_i),$$

where $A_1, \dots, A_n$ partition Ω and $B \subset \Omega$ with $\mathsf{P}(B) > 0$.

Conditional probability measure

Suppose $B \subset \Omega$ and $\mathsf{P}(B) > 0$. Then (i) $\mathsf{P}(A \mid B) \geq 0$ for all $A \subset \Omega$, since $\mathsf{P}(A \cap B) \geq 0$. Moreover, (ii)

$$\mathsf{P}(\Omega \mid B) = \mathsf{P}(\Omega \cap B)/\mathsf{P}(B) = \mathsf{P}(B)/\mathsf{P}(B) = 1$$

Similarly, $\mathsf{P}(\varnothing \mid B) = \mathsf{P}(\varnothing \cap B)/\mathsf{P}(B) = 0/\mathsf{P}(B) = 0$.

Finally, (iii) if $A \cap C = \varnothing$, then

$$\begin{aligned}
\mathsf{P}(A \cap C \mid B) &= \mathsf{P}((A \cap C) \cap B)/\mathsf{P}(B) \\
&= \mathsf{P}((A \cap B) \cap (C \cap B)/\mathsf{P}(B) \\
&\overset{(a)}{=} (\mathsf{P}(A \cap B) + \mathsf{P}(C \cap B))/\mathsf{P}(B) \\
&= \mathsf{P}(A \cap B)/\mathsf{P}(B) + P(C \cap B)/\mathsf{P}(B) \\
&= \mathsf{P}(A \mid B) + P(C \mid B).
\end{aligned}$$

where (a) follows since $A \cap B$ and $C \cap B$ are disjoint.

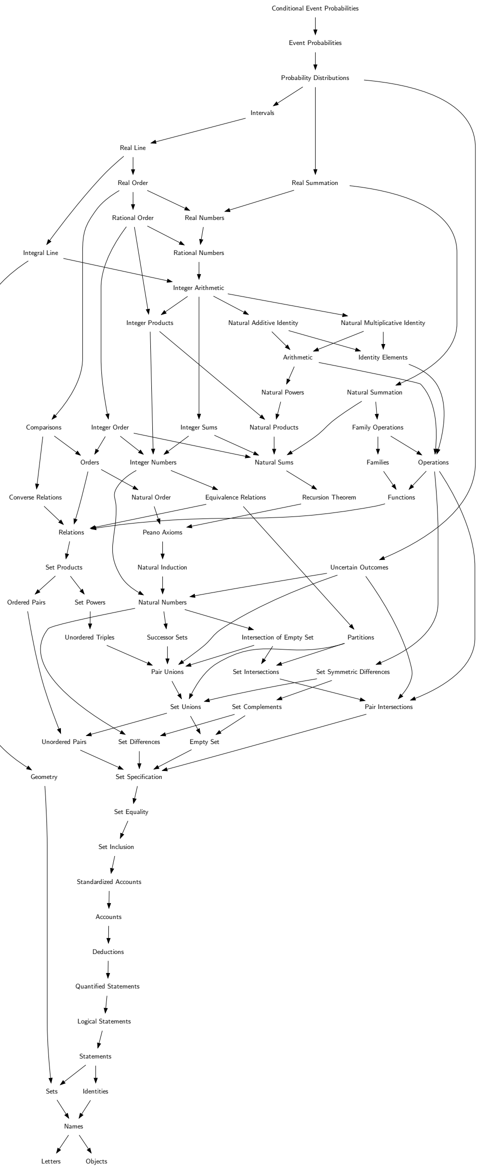
Together, (i)-(iii) mean that $\mathsf{P}(\cdot \mid B)$ is itself a probability measure on Ω. We therefore refer to $\mathsf{P}(\cdot \mid B)$ as a *conditional probability measure.*

Induced conditional distribution

Therefore, we expect there to also correspond a new distribution on the set of outcomes. For P_p, define $q : \Omega \to \mathbf{R}$ by

$$q(\omega) = \begin{cases} \frac{p(\omega)}{\mathsf{P}(B)} & \text{if } \omega \in B \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\mathsf{P}_q(A) = \mathsf{P}_p(A \mid B)$. We call q the *conditional distribution* induced by *conditioning on* the event B.



Conditional Event Probabilities
Event Probabilities
Probability Distributions
Intervals
Real Line
Real Order
Real Summation
Rational Order
Real Numbers
Integral Line
Rational Numbers
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Arithmetic
Identity Elements
Natural Powers
Natural Summation
Comparisons
Integer Order
Integer Sums
Natural Products
Family Operations
Orders
Integer Numbers
Natural Sums
Families
Operations
Converse Relations
Natural Order
Equivalence Relations
Recursion Theorem
Functions
Relations
Peano Axioms
Set Products
Natural Induction
Uncertain Outcomes
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Pair Unions
Set Intersections
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Set Differences
Empty Set
Geometry
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects