Why

Can we characterize positive (semi-)definite matrices in terms of their eigenvalues?

Main Result

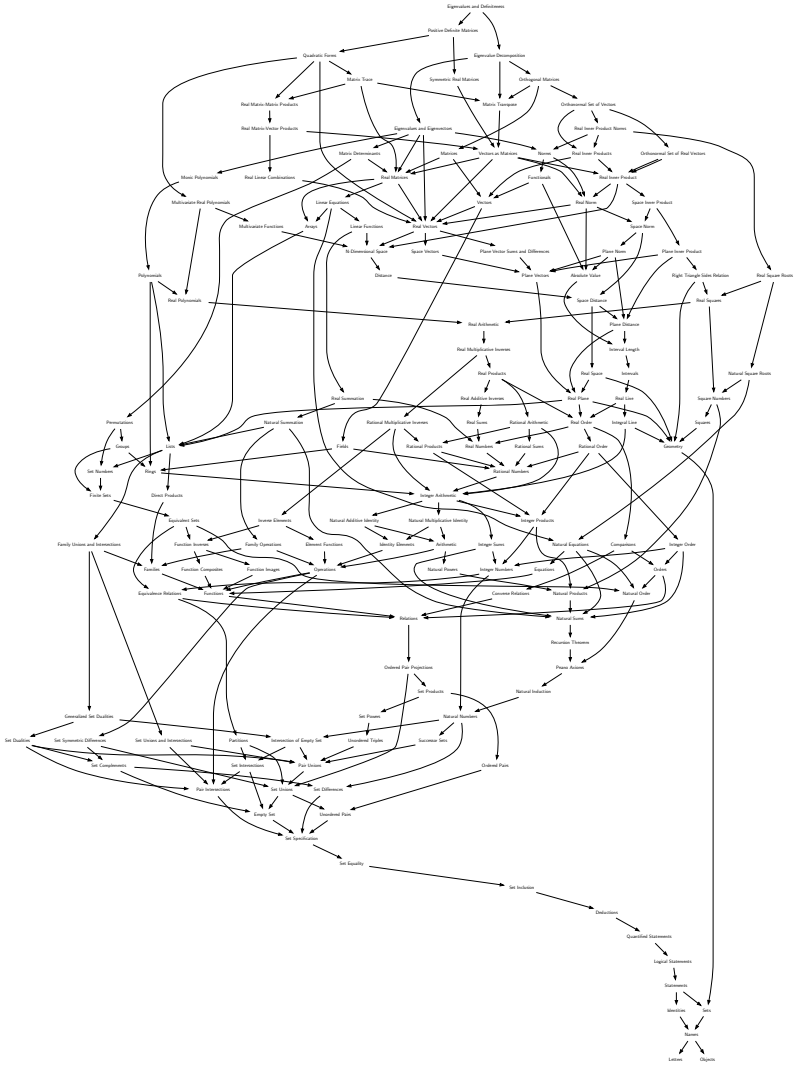
Using eigenvalue decompositions, we can answer in the affirmative.

Proposition 1. *Suppose $A \in \mathbf{S}^d$ has smallest eigenvalue $\lambda_{\min}(A)$. Then*

$$\begin{aligned} A \in \mathbf{S}^d_+ \quad &\longleftrightarrow \quad \lambda_{\min}(A) \geq 0 \\ &\longleftrightarrow \quad \operatorname{tr} AB \geq 0 \text{ for all } B \in \mathbf{S}^d_+. \end{aligned}$$

and

$$\begin{aligned} A \in \mathbf{S}^d_{++} \quad &\longleftrightarrow \quad \lambda_{\min}(A) > 0 \\ &\longleftrightarrow \quad \operatorname{tr} AB > 0 \text{ for all nonzero } B \in \mathbf{S}^d_{++}. \end{aligned}$$


Eigenvalues and Definiteness
Positive Definite Matrices
Quadratic Forms
Eigenvalue Decomposition
Matrix Trace
Symmetric Real Matrices
Orthogonal Matrices
Real Matrix-Matrix Products
Matrix Transpose
Orthonormal Set of Vectors
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Set Powers
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Set Dualities
Set Symmetric Differences
Set Unions and Intersections
Partitions
Intersection of Empty Set
Disordered Triples
Successor Sets
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Pair Intersections
Set Unions
Set Differences
Empty Set
Disordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
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Sets
Nouns
Lexicon
Objects