The Bourbaki Project

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Contents

The Bourbaki Project is a collection of documents describing mathematical concepts, terminology, results and notation. Each document is named and labeled with the names of those documents that should be read before it by the unaquainted reader.

Aims. Our primary aim is to understand and develop the mathematical concepts ourselves. Besides this, we have two other goals. First, to provide useful exposition to teach the concepts to an unaquainted reader. And second, to serve as a reference for further work.

Sheets. We call these documents *sheets*. They are only ever two-pages long, and sometimes shorter. They can be printed on the front and back of a single sheet of a paper, hence the name sheet. The decision to cap at two pages is arbitrary. But our experience suggests it is convenient.

Needs. We call the sheets that should be read before a particular sheet X the *needs* of X. For example, the sheet *Relations* needs the sheet *Ordered Pairs*. The reason, in this case, is that the concept of a relation is discussed using the concept of an ordered pair of objects. And since the phrase "ordered pair of objects" makes sense only if we know what is meant by object (discussed in the sheet *Objects*), the sheet *Relations* needs the sheet *Objects* also. The reader unaquainted with ordered pairs and objects must read (at least) these two sheets before the sheet on relations. So needs order the sheets to be read.

The needs of a sheet are naturally ordered by their respective needs. Suppose X needs both Y and Z, and Y in turn needs Z. In this case, Z ought to be read first, Y second, and X last. We ensure that such an ordering always exists by enforcing the following constraint: if a sheet X needs a sheet Y, then Y can not need X or any sheet that needs X.

For convenience, the needs listed on a page are minimal. That is to say, for sheet X we only list the sheets which are in the needs of X and not needed by any other sheet in the needs of X. If X, Y and Z are as before, then we only list Y as X's needs because Z is implicit (through Y). The sheets and their needs are probably best explored by browsing the project; the index is a reasonable starting point.

Caveats. There are two caveats. First, Bourbaki gives only one path to concepts. Bourbaki is like a map: the landmarks are concepts. Walking is reading. And you must walk along the trails specified by the needs. The point is that the Bourbaki way of structuring the concepts is just one way, and there are many ways, since there are equivalent concepts, alternate proofs, and so on. The second caveat is a wink: these sheets are fiction. They contain only ideas. We have done our best to eliminate all false statements. But very little is said about fitting these puzzle pieces to reality.

Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *script* is a collection of written marks or symbols called *letters*. In *phonetic* languages, the letters correspond to sounds. A *written word* is a succession of letters. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into *written words* which are meant to denote the *spoken words* of the English language. The written words on this page are several letters one after the other. For example, the word "word" is composed of the letters "w", "o", "r", "d".

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. We call these the *lower case latin letters.*

$$\begin{array}{ccccccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} & \mathrm{i} \\ \mathrm{j} & \mathrm{k} & \mathrm{l} & \mathrm{m} & \mathrm{n} & \mathrm{o} & \mathrm{p} & \mathrm{q} & \mathrm{r} \\ \mathrm{s} & \mathrm{t} & \mathrm{u} & \mathrm{v} & \mathrm{w} & \mathrm{x} & \mathrm{y} & \mathrm{z} & \end{array}$$

And we call these the *upper case latin letters.*

$$\begin{array}{ccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H} & \mathrm{I} \\ \mathrm{J} & \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{O} & \mathrm{P} & \mathrm{Q} & \mathrm{R} \\ \mathrm{S} & \mathrm{T} & \mathrm{U} & \mathrm{V} & \mathrm{W} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \end{array}$$

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We will also use the following symbols. We call these the *Arabic numerals.*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Other symbols

We will also use the following symbols We call thes the *logical symbols.*

$$(\quad) \quad \vee \quad \wedge \quad \neg \quad \forall \quad \exists \quad \implies \quad \iff \quad = \quad \in$$

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an accent $'$ or subscripts or superscripts. When we write the symbol, we say that it *denotes* the object. We call the symbols the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{\text{A}}$ or $\boxed{\text{A'}}$. The box works well to group in the accent, and also clarifies that $\boxed{\text{A}}\boxed{\text{A}}$ is different from $\boxed{\text{AA}}$. But experience shows that the boxes are mostly unnecessary.

We indicate a name for an object with italics. Instead of $\boxed{\text{A'}}$ we use A'. Experience shows that this subtelty is enough for clarity, and it agrees with traditional and modern practice.

No repetitions

We will also agree that we will never use the same name to refer to two different objects. It is in the nature of things—and of names in particular—that we can not do this without confusion.

Names are objects

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can references themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the several marks on the page, all of which are meant to look similar, which ar meant to denote the object, are uses of the name.

Placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is beset understood by example.

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

Symmetry

"A equals B" means the same as "B equals A". This is because the identity of the object is not changed by the order in which the names are given.

This fact is called the *symmetry of identity*. It is obvious. Not subtle in the slightest. We can switch the spots of A and B and say the same thing. There are two ways to say the same thing.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself.

This fact is called the *reflexivity of identity*. It too is obvious. And not subtle. We can always declare that the same symbol denotes the same object. We agreed upon this in *Names*.

Why

We want to talk about none, one, or several objects considered as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set*. We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

The objects in a set may be other sets. In other words, an element of a set may be another set. This may be subtle at first glance, but becomes familiar with experience.

We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object belongs to the set. In other words, the object is a member or an element of the set. We say “The object denoted by a belongs to the set denoted by A”.

Asymmetry

Notice that belonging is not symmetric. Saying “the object denoted by a belongs to the set denoted by A” does not mean the same as “the set denoted by A belongs to the object denoted by a” In fact, the latter sentence is nonsensical unless the object a is also a set.

Nontransitive

Let a denote an object and let A denote a set and B denote a set. If the object denoted by a is *a part of* a the set denoted by A, and the set denoted by A is *a part of* the set denoted by B, then usual English usage would suggest that a is a *a part of* the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing. The relation of belonging is not the same. We do not allow this with sets. If a thing is an element of a thing, that second thing may be an element of the third thing, but this does not mean that the.

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of Latin letters: a, b, c, . . . , x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Why

We want to write about objects belonging to sets.

Definition

Let A denote a set; in other words, an intangible object which has some objects as members. Let a denote an object. Recall that if two names refer to the same object, the names are equal. Similarly, if the object denoted by a is an element of the set denoted by A, then we say that the former name belongs to the latter name. We write that the name a belongs to the name A by $a \in A$.

We read this sequence of symbols aloud as "a in A." The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means "belongs" in ancient greek. Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of". Of course, we must take care. The first name is not an element on the second name. Rather, the object denoted by the first name is an element of the set (object) denoted by the second name.

We tend to denote sets by upper case latin letters: for example, A, B, and C. To aid our memory, we tend to use the lower case form of the letter for an element of the set. For example, let A and B denote nonempty sets. We tend to denote by a an object which is an element of A. And similarly, we tend to denote by b an object which is an element of B

Why

We want symbols to represent identity and belonging.

Definition

In the English language, *nouns* are words that name people, places and things. In these sheets, *names* (see *Names*) serve the role of nouns. In the English language, *verbs* are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense. A *statement* is several symbols.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings. As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object in the set of the sheet *Identity*. We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And if we read it aloud, we would read a as "the object denoted by a", then $=$ as "is", then b as "the object denoted by b". Altogether then, "the object denoted by a is the object denoted by b." We might box these three symbols $\boxed{a = b}$ to make clear that they are meant to be read together, but experience shows that (as with English sentences and words) we do not need boxes.

The symbol $=$ is (appropriately) a symmetric symbol. If we flip it left and right, it is the same symbol. This reflects the symmetry of the English sentences represented. $A = B$ means the same as $B = A$.

Belonging

As a second example, consider the symbol $\in$. Let a denote an object and let A denote a set. Let us suppose that the object denoted by a belongs to the set denoted by A. We agree that $\in$ means "belongs to" in the sense of "is an element of" or "is a member of" as given in *Sets*. Then we write $a \in A$. We read these symbols as "the object denoted by a belongs to the set denoted by A".[1]

The symbol $\in$ is not symmetric. If we flip it left and right it looks different. And as we discussed in *Sets*, $a \in A$ does not the mean the same as $A \in a$.

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[1]The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for the ancient Greek word $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means, roughly, "belongs". Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of".

Why

We want symbols for "and", "or", "not", and "implies".

Definition

In *Statements* we discussed that nouns are names and that we will only use the present tense of the verbs "is" and "belongs". We had statements like $a = b$ (identity) and $a \in A$ (belonging).

We call $=$ and $\in$ *relational symbols*. They say how the objects denoted by a pair of placeholder names relate to each other in the sense of being or belonging. We call $\_ = \_$ and $\_ \in \_$ *simple statements*. They denote simple sentences "the object denoted by _ is the object denoted by _" and "the object denoted by _ belongs to the set denoted by _". We want want to assert that one make more complicated statements.

Conjunction

Consider the symbol $\land$. We will agree that it means "and". If we want to make two simple statements like $a = b$ and $a \in A$ at once, we write write $(a = b) \land (a \in A)$. The symbol $\land$ is symmetric, reflecting the fact that a statement like $(a \in A) \land (a = b)$ means the same as $(a = b) \land (a \in A)$.

Disjunction

Consider the symbol $\lor$. We will agree that it means "or" in the sense of either one, the other, or both. If we want to say that possibly only one, but at least one of the simple statements like $a = b$ and $a \in A$, we write write $(a = b) \vee (a \in A)$. The symbol $\vee$ is symmetric, reflecting the fact that a statement like $(a \in A) \vee (a = b)$ means the same as $(a = b) \vee (a \in A)$.

Negation

Consider the simple $\neg$. We will agree that it means "not", in the sense of negating whatever it follows. If we want to say the opposite of a simple statement like $a = b$ we will write $\neg(a = b)$. We read it aloud as "not a is b". Of course, the more desirable english expression is "a is not b". Similarly, $\neg(a \in A)$ we read as "not, the object denoted by a belongs to the set denoted by A". Again, the more desirable english expression is something like "the object denoted by a does not belong to the set A" For these reasons, we introduce two new symbols $\neq$ and $\notin$. $a \neq b$ means $\neg(a = b)$ and $a \notin A$ means $\neg(a \in A)$.

Implication

Consider the symbol $\implies$. We will agree that it means "implies". For example $(a \in A) \implies (a \in B)$ means "the object denoted by a belongs to the object denoted by A implies the obejcted denoted by a belongs to the set denoted by B" It is the same as $(\neg(a \in A)) \vee (a \in B)$. In other words, if $a \in A$, then always $a \in B$. The symbol $\implies$ is not symmetric, since implication is not symmetric.

Why

We want to make conclusions.

Definition

Suppose we have a list of logical statements. We want to write down o

Why

We want symbols for talking about some object of a set or all objets of a set.

Definition

Existential Quantifier

Consider the symbol $\exists$. We agree that it means "there exists an object". We write $\exists a \in A$. We read this as "there exists an object in the set denoted by A, and denote that very same object by a". We call $\exists$ the existential quantifier.

Universal Quantifier

Consider the symbol $\forall$. We agree that it means "for every object". We write $\forall a \in A$. We read this as "for every object in the set denoted by A, and denote one such object by a".

Why

We want to succinctly and clearly make several statements about objects and sets. We want to track the names we use, taking care to avoid using the same name twice.

Definition

An *account* is a list of naming, logical, and quantified statements. We use the words "let _ denote an _" to introduce a name as a placeholder for a thing, and we use the symbols $\_ = \_$ and $\_ \in \_$ to denote statements of identity and belonging. In other words, we have three sentence kinds to record.

1. **Names.** State we are using a name.
2. **Identity.** We want to make statements of identity.
3. **Belonging.** We want to make statements of belonging.

Our main purpose is to keep a list names and logical statements about them and then deductions. We want to group our usage of names. In the English language we use paragraphs or sections to do so. In these sheets, we will use *accounts*, which will be a list of statements, each of which is labeled by an Arabic numeral (see *Letters*).

Experience suggests that we start with an example. Suppose we want to summarize the following english language description of some names and objects.

> Denote an object by a. Also, denote the same object by b Also, denote a set by A. Also, the object denoted by a is an element of the set denoted by A. Also denote an object by c. Also c is the same object as b.

In our usual manner of speaking, we drop the word "also". In these sheets, we translate each of the sentences into our symbols. For names we use, we write `name` in that font followed by the name. For logical statements we "have", we write `have` followed by the logical statement. So we write:

Account 1. First Example

1	`name`	a	
2	`name`	b	
3	`have`	$a = b$	
4	`name`	A	
5	`have`	$a \in A$	
6	`name`	c	
7	`have`	$c = b$	
8	`thus`	$a = c$	`by 3,7`

Why

We want to do our best to have only one way to write accounts.

Discussion

Consider the account.

Account 2. First Example

1	name	a
2	name	b
3	have	$a = b$
4	name	c
5	have	$c = b$
6	thus	$a = c$ by 3,5

We standardize it:

Account 3. Standardized First Example

1-3	name	a, b, c
4	have	$a = b$
5	have	$c = b$
6	thus	$a = c$ by 4,5,IdentityAxioms:1

Why

When are two sets the same?

Definition

Given sets A and B, if $A = B$ then every element of A is an element of B and every element of B is an element of A.

Account 4. Joint Membership

1-3	`name`	A, B, x	
4	`have`	$A = B$	
5	`have`	$x \in B$	
6	`thus`	$x \in A$	`by 4,5`

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Is $A = B$ true? We define it to be so. Two sets are *equal* if and only if every element of one is an element of the other. In other words, two sets are the same if they have the same elements. This statement is sometimes called the *axiom of extension*. Roughly speaking, if we refer to the elements of a set as its *extension*, then we have declared that if we know the extension then we know the set. A set is determined by its extension.

This definition gives us a way to argue that $A = B$ from the properties of the elements of A and B. It may not be obvious that the sets are the same. We first argue that each element of A is an element of B and then argue that each element of B is an element of A. With these two implications, we use the axiom of extension to conclude that the sets are the same.

The logical statement is: $(((\forall x)(x \in A \implies x \in B) \land (\forall x)(x \in B \implies x \in A))) \implies (A = B)$ Here is an example of applying that:

Account 5. Extension

1-2	`name`	A, B	
3	`have`	$(\forall x)((x \in A) \implies (x \in B))$	
4	`have`	$(\forall x)((x \in B) \implies (x \in A))$	
5	`thus`	$A = B$	`by 3,4`

A Contrast

We can compare the axiom of extension for sets and their elements with an analogous statement for human beings and their ancestors.

On the one hand, if two human beings are equal then they have the same ancestors. The ancestors being the person's parents, grandparents, greatgrandparents, and so on. This direction, same human implies same ancestors, is the analogue of the "only if" part of the axiom of extension. It is true. On the other hand, if two human beings have the same set of ancestors, they need not be the same human. This direction, same ancestors implies same human, is the analogue of the "if" part of the axiom of extension. It is false. For example, siblings have the same ancestors but are different people.

We conclude that the axiom of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Why

We want language for all of the elements of a first set being the elements of a second set.

Definition

Denote a set by A and a set by B. If every element of the set denoted by A is an element of the set denoted by B, then we say that the set denoted by A is a *subset* of the set denoted by B. We say that the set denoted by A is *included* in the set denoted by B. We say that the set denoted by B is a *superset* of the set denoted by A or that the set denoted by B *includes* the set denoted by A. A set includes and is included in itself.

If the sets denoted by A and B are identical, then each contains the other. If $A = B$, then the set denoted by A includes the set denoted by B and the set denoted by B includes the set denoted by A. The axiom of extension asserts the converse also holds. If the set denoted by A includes the set denoted by B and the set denoted by$B$ includes the set denoted by A, then A and B denote the same set. In other words, if the set denoted by A is a subset of the set denoted by B and the set denoted by B a subset of the set denoted by A, then $A = B$.

The empty set is a subset of every other set.

Account 6. Empty Set Inclusion

1-2	`name`	$A, \varnothing$	
3	`have`	$\neg((\exists x)(x \in \varnothing))$	
4	`thus`	$(\forall x)((x \in \varnothing) \implies (x \in A))$	`by 3`
5	`i.e.`	$\varnothing \subset A$	`by 4`

Suppose toward contradiction that A were a set which did not include the empty set. Then there would exist an element in the empty set which is not in A. But then the empty set would not be empty. We call the empty set and A *improper subsets* of A. All other subsets we call *proper subsets*. In other words, B is an improper subset of A if and only if A includes B, $B \neq A$ and $B \neq \varnothing$.

Notation

Given two sets A and B, we denote that A is included in B by $A \subset B$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A".

In this notation, we express the axiom of extension

$$A = B \Leftrightarrow (A \supset B) \land (A \subset B).$$

The notation $A \subset B$ is a concise symbolism for the sentence "every element of A is an element of B." Or for the alternative notation $a \in A \implies a \in B$.

Properties

Given a set A, $A \subset A$. Like equality, we say that inclusion is *reflexive*. Given sets A and B, if $A \subset B$ and $B \subset C$ then $A \subset C$. Like equality, we say that inclusion is *transitive*. If $A \subset B$ and $B \subset A$, then $A = B$ (by the axiom of extension). Unlike equality, which is symmetric, we say that inclusion is *antisymmetric*.

Comparison with belonging

Given a set A inclusion is reflexive. $A \subset A$ is always true. Is $A \in A$ ever true? Also, inclusion is transitive. Whereas belonging is not.

Why

If there is a set, there is an empty set. Are there many such sets? How do they (or it) relate to other sets?

Empty Set

An immediate consequence of the axiom of extension is that there is a unique set that is empty.

Account 7.

1-2	`name`	A, B		
3	`have`	$\neg((\exists a)(a \in A))$		
4	`have`	$\neg((\exists b)(b \in B))$		
5	`thus`	$(\forall x)(x \in A \implies b \in A)$	`by`	`3`
6	`thus`	$(\forall x)(x \in B \implies b \in B)$	`by`	`4`
7	`thus`	$A = B$	`by`	`5,6`

Definition

First, we assume there exists a set. As a consequence, there exists a set which contains no elements at all. We use the axiom of specification with a condition that is always false, and so selects no elements.

As a result of the axiom of extension, this set with no elements is unique. We call this empty set *the empty set.*

Notation

We denote the empty set by $\varnothing$.

Why

We want to construct new sets out of old ones. So, can we always construct subsets?

Definition

We will say that we can. Let A denote a set. Let s denote a statement in which the symbol x and A appear unbound.

We assert that there is a set, denote it by A' for which belonging is equivalent to the statement denoted by s. It is a consequence of the axiom of extension that this set is unique. This assertion is sometimes called the *axiom of specification* is this assertion. We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence.

All the basic principles of set theory other than the axiom of extension assert that we can construct new sets out of old ones in reasonable ways.

For example:

Account 8. Example Specification

1 `name` A,y

2 `thus` $(\exists A')((x \in A') \iff (x \neq y)))$ `by Axiom:Specifica`

Notation

Let A be a set. Let $S(a)$ be a sentence. We use the notation

$$\{a \in A \mid \ S(a)\ \}$$

to denote the subset of A specified by S. We read the symbol $|$ aloud as "such that." We read the whole notation aloud as "a in A such that..."

We call the notation *set-builder notation*. Set-builder notation avoids enumerating elements. This notation is really indispensable for sets which have many members, too many to reasonably write down.

Example

For example, let a, b, c, d be distinct objects. Let $A = \{a, b, c, d\}$. Then $\{x \in A \mid x \neq a\}$ is the set $\{b, c, d\}$

Now let B be an arbitrary set. The set $\{b \in B \mid b \neq b\}$ specifies the empty set. Since the statement $b \neq b$ is false for all objects b.

Why

Are there enough sets to ensure that every set is an element of some set? What of one set and another set — is there a set that they both belong to?

Definition

We will say that there is. For one set and another set, there exists a set that they both belong to. We refer to this as the *axiom of pairing*.

If there exists a set that contains both the sets we began with, then there exists a set which contains them and nothing else. First, use the axiom of pairing to obtain a set containing both sets, and then use the axiom of specification with a sentence that is true only if the element considered is one of the sets we began with. As a result of the axiom of extension, there can be only one set with this property. We call this set a *pair* or an *unordered pair*.

Notation

Let a and b be objects. We denote the set which contains a and b as elements and nothing else by $\{a, b\}$.

The pair of a with itself is the set $\{a, a\}$ which we will denote by $\{a\}$.

Why

We want to consider the elements of two sets together at one. Does a set exist which contains all elements which appear in either of one set or another?

Definition

We say yes. For every set of sets there exists a sets which contains all the elements that belong to at least one set of the given collection. We refer to this as the *axiom of unions*. If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the axiom of unions may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification to form the set which contains only those elements which are appear in at least one of any of the sets. As a result of the axiom of extension, this set is unique. We call it the *union* of the set of sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\cup \mathcal{A}$.

Simple Facts

Proposition 1. $\cup \varnothing = \varnothing$

Proposition **2.** $\cup\{A\} = A$

Why

We want to consider the elements which are shared between two sets. Does a set exist which contains all the elements whcih appear in both of one set and other.

Definition

Yes. The *intersection* of one set with another is the set obtained by specifying the elements of the former set which are members of the latter set. The intersection is symmetric. The intersection of one set with another is the same as the latter set with the former.

Why

We want to consider the elements of a set and another set which are in either, but not in both.

Definition

The *symmetric difference* of a set with another set is the union of the difference between the latter set and the former set and the difference between teh former and the latter.

Why

We want to consider the elements of one set which are not contained in another set. Does such a set exist?

Definition

Yes: use the axiom of specification on the first set with the condition that the element not be in the second set. The axiom of extension guarantees uniqueness. And so we call this set the *relative complement* of the latter set in the first set. We also call it the *difference* between the former and the latter.

Notation

Let A and B be sets. We denote the difference of A with B by $A - B$. We express $A - B$ as

$$\{a \in A \mid x \notin B\}.$$

Why

We want to consider the subsets of a given set. Does a set exist which contains all the subsets.

Definition

We say yes.

We call this set the *power set*. It includes the set itself and the empty set.

Notation

We denote the power set of A by A^*, read aloud as "powerset of A." $A \in A^*$ and $\varnothing \in A^*$. However, $A \subset A^*$ is false.

Example

Let a, b, c be distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in A^*$. As always, $\varnothing \in A^*$ and $A \in A^*$ as well. In this case, we can list the elements (which are sets) of the power set:

$$A^* = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$

Why

We speak of an ordered pair of objects: one selected from a first set and one selected from a second set.

Definition

Let A and B be nonempty sets. Let $a \in A$ and $b \in B$. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is a and the *second coordinate* is b.

The *product* of A and B is the set of all ordered pairs. This set is also called the *cartesian product.* If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b). We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Taste

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second "coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life. Plus, suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Equality

Proposition 3. *$(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.*

Proof. TODO $\square$

Why

How can we relate the elements of two sets?

Definition

A *relation* between two nonempty sets is a subset of their cross product. A relation on a single set is a subset of the cross product of it with itself.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation. The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation.

Notation

Let A and B be two nonempty sets. A relation on A and B is a subset of $A \times B$. Let C be a nonempty set. A relation on a C is a subset of $C \times C$.

Let $a \in A$ and $b \in B$. The ordered pair (a, b) may or may not be in a relation on A and B. Also notice that if $A \neq B$, then (b, a) is not a member of the product $A \times B$, and therefore not in any relation on A and B. If $A = B$, however, it may be that (b, a) is in the relation.

Notation

Let A and B be nonempty sets with $a \in A$ and $b \in B$. Since relations are sets, we can use upper case Latin letters. Let R be a relation on A and B. We denote that $(a, b) \in R$ by aRb, read aloud as "a in relation R to b."

When $A = B$, we tend to use other symbols instead of letters. For example, $\sim$, $=$, $<$, $\leq$, $\prec$, and $\preceq$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *antisymmetric*antisymmetric if two different objects are related only in one order, and never both. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related.

Notation

Let R be a relation on a non-empty set A. R is reflexive if

$$(a, a) \in R$$

for all $a \in A$. R is transitive if

$$(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$$

for all $a, b, c \in A$. R is symmetric if

$$(a, b) \in R \implies (b, a) \in R$$

for all $a, b \in A$. R is anti-symmetric if

$$(a, b) \in R \implies (b, a) \notin R$$

for all $a, b \in A$.

Why

We want a notion for a correspondence between two sets.

Definition

A *functional* relation on two sets relates each element of the first set with a unique element of the second set. A *function* is a functional relation.

The *domain* of the function is the first set and *codomain* of the function is the second set. The function *maps* elements *from* the domain *to* the codomain. We call the codomain element associated with the domain element the *result* of *applying* the function to the domain element.

Notation

Let A and B be sets. If A is the domain and B the codomain, we denote the set of functions from A to B by $A \to B$, read aloud as "A to B".

We denote functions by lower case latin letters, especially f, g, and h. The letter f is a mnemonic for function; g and h follow f in the Latin alphabet. We denote that $f \in (A \to B)$ by $f : A \to B$, read aloud as "f from A to B".

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a."

Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

The set $\{(a, f(a)) \in A \times B \mid a \in A\}$ of ordered pairs is the *graph* of f.

Why

We consider the set of results of a set of domain elements.

Definition

The *image* of a set of domain elements under a function is the set of their results. Though the set of domain elements may include several distinct elements, the image may still be a singleton, since the function may map all of elements to the same result.

The *range* of a function is the image of its domain. The range includes all possible results of the function. If the range does not include some element of the codomain, then the function maps no domain elements to that codomain element.

Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for every $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. Following this notation for function images, we denote the range of f by $f(A)$.

Why

We want to "combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations to *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\,g\,b$ instead. We call this *infix notation*.

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Why

We want to consider the elements of two sets together at once, and other sets created from two sets.

Definitions

Let A and B be two sets.

The *union* of A with B is the set whose elements are in either A *or* B *or* both. The key word in the definition is *or*.

The *intersection* of A with B is the set whose elements are in both A *and* B. The keyword in the definition is *and*.

Viewed as operations, both union and intersection commute; this property justifies the language "with." The intersection is a subset of A, of B, and of the union of A with B.

The *symmetric difference* of A and B is the set whose elements are in the union but not in the intersection. The symmetric difference commutes because both union and intersection commute; this property justifies the language "and." The symmetric difference is a subset of the union.

Let C be a set containing A. The *complement* of A in C is the symmetric difference of A and C. Since $A \subset C$, the union is C and the intersection is A. So the complement is the "left-over" elements of B after removing the elements of A.

We call these four operations *set-algebraic operations*.

Notation

Let A, B be sets. We denote the union of A with B by $A \cup B$, read aloud as "A union B." $\cup$ is a stylized U. We denote the intersection of A with B by $A \cap B$, read aloud as "A intersect B." We denote the symmetric difference of A and B by $A+B$, read aloud as "A symdiff B." "Delta" is a mnemonic for difference.

Let C be a set containing A. We denote the complement of A in C by $C - A$, read aloud as "C minus A."

Results

Proposition 4. *For all sets A and B the operations $\cup$, $\cap$, and $+$ commute.*

Proposition 5. *Let S a set. For all sets $A, B \subset S$,*

$$\text{(1)} \quad S - (A \cup B) = (S - A) \cap (S - B)$$
$$\text{(2)} \quad S - (A \cap B) = (S - A) \cup (S - B).$$

Proposition 6. *Let S a set. For all sets $A, B \subset S$,*

$$A + B = (A \cup B) \cap C_S(A \cap B)$$

$\boxed{TODO: notation}$

Why

We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Why

Halmos: "There are occasions when the range of a functionis deemed to be more important than the function itself. When that is the case, both the terminology and the notation undergo radical alterations." It is useful to have some language and notation for talking about a set of sets.

Definition

A *family* of sets is a set of sets. Experience shows that it is useful to have these associated with the elements of a well-known second set.

An *indexed family of sets* is a function from one set to the power set of a second set. We call the first set the *index set*. We call the second set the *base set*. The range of the indexed family of sets is, of course, a family.

Notation

Let A and I be be a non-empty sets. We use I as a mnemonic for "index" set. Let $a : I \to A^*$ be a family. For $i \in I$, we follow the function notation and denote the result of applying a to i by a_i.

We denote the range of the family by family of a_α indexed with I by $\{a_\alpha\}_{\alpha \in I}$, which is short-hand for set-builder notation. We read this notation "a sub-alpha, alpha in I."

Why

Family set operations are common. TODO: this works for infinite stuff too

Definition

We define the set whose elements are the objects which are contained in at least one family member the *family union*. We define the set whose elements are the objects which are contained in all of the family members the *family intersection*.

Notation

We denote the family union by $\cup_{\alpha \in I} A_\alpha$. We read this notation as "union over alpha in I of A sub-alpha." We denote family intersection by $\cap_{\alpha \in I} A_\alpha$. We read this notation as "intersection over alpha in I of A sub-alpha."

Results

Proposition **7.** *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \{i, j\}$ then*

$$\cup_{\alpha \in I} A_\alpha = A_i \cup A_j$$

and

$$\cap_{\alpha \in I} A_\alpha = A_i \cap A_j.$$

Proposition **8.** *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S, if $I = \varnothing$, then*

$$\cup_{\alpha \in I} A_\alpha = \varnothing$$

and

$$\cap_{\alpha \in I} A_\alpha = S.$$

PROPOSITION **9.** *For an indexed family $\{A_\alpha\}_{\alpha \in I}$ in S.*

$$C_S(\cup_{\alpha \in I} A_\alpha) = \cap_{\alpha \in I} C_S(A_\alpha)$$

and

$$C_S(\cap_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I} C_S(A_\alpha).$$

Why

We want to generalize operations beyond two objects.

Operations

The *pairwise extension* of a commutative operation is the function from finite families of the ground set to the ground set obtained by applying the operation pairwise to elements. TODO: this is not a function if the operation is not commutative.

The *ordered pairwise extension* of an operation is the function from finite families ground set to the ground set obtained by applying the operation pairwise to elements in order.

Notation

Let $(A, +)$ be an algebra and $\{A_i\}_{i=1}^{n}$ a finite family of elements of A. We denote the pairwise extension by

$$\mathop{+}_{i=1}^{n} A_i$$

Why

We want a notion for applying two functions one after the other. We apply a first function then a second function.

Definition

Consider two functions for which the codomain of the first function is the domain of the second function.

The *composite* or *composition* of the second function with the first function is the function which associates each element in the first's domain with the element in the second's codomain that the second function associates with the result of the first function.

The idea is that we take an element in the first domain. We apply the first function to it. We obtain an element in the first's codomain. This result is an element of the second's domain. We apply the second function to this result. We obtain an element in the second's codomain. The composition of the second function with the first is the function so constructed.

Notation

Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$. We denote the composition of g with f by $g \circ f$ read aloud as "g composed with f." To make clear the domain and comdomain, we denote the composition $g \circ f : A \to C$.

In previously introduced notation, $g \circ f$ satisfies

$$(g \circ f)(a) = g(f(a))$$

for all $a \in A$.

Why

We want a notion of reversing functions.

Definition

An *identity function* is a relation on a set which is functional and reflexive. It associates each element in the set with itself. There is only one identity function associated to each set.

Consider two functions for which the codomain of the first function is the domain of the second function and the codomain of the second function is the domain of the first function. These functions are *inverse functions* if the composition of the second with the first is the identity function on the first's domain and the composition of the first with the second is the identity function on the second's domain.

In this case we say that the second function is an *inverse* of the second, and vice versa. When an inverse exists, it is unique, so we refer to the *inverse* of a function. We call the first function *invertible*. Other names for an invertible function include *bijection*.

Notation

Let A a non-empty set. We denote the identity function on A by id_A, read aloud as "identity on A." id_A maps A onto A.

Let A, B be non-empty sets. Let $f : A \to B$ and $g : B \to A$ be functions. f and g are inverse functions if $g \circ f = \mathrm{id}_A$ and

$f \circ g = \mathrm{id}_B$.

The Inverse

We discuss existence and uniqueness of an inverse.

PROPOSITION **10.** *Let $f : A \to B$, $g : B \to A$, and $h : B \to A$.*

If g and h are both inverse functions of f, then $g = h$.

Proof. $\square$

PROPOSITION **11.** *If a function is one-to-one and onto, it has an inverse.*

Proof. $\square$

Inverse Images

The *inverse image* of a codomain set under a function is the set of elements which map to elements of the codomain set under the function. We denote the inverse image of $D \subset B$ by $f^{-1}(D)$, read aloud as "f inverse D."

Why

We want to define the natural numbers. TODO: better why

Definition

The *successor* of a set is the union of the set with the singleton whose element is the set. This definition holds for any set, but is of interest only for the sets which will be defined in this sheet.

These sets are the following (and their successors): *One* is the successor of the empty set. *Two* is the successor of one. *Three* is the successor of two. *Four* is the successor of three. And so on; using the English language in the usual manner.

Can this be carried on and on? We will say yes. We will say that there exists a set which contains one and contains the successor of each of its elements. So, this set contains one. Since it contains one, it contains two. Since it contains two, it contains three. And so on. We call this assertion the *axiom of infinity.*

A set is a *successor set* if it contains one and if it contains the successor of each of its elements. In these words, the axiom of infinity asserts the existence of a successor set. We want this set to be unique. So we have a successor set. By the axiom of specification, the intersection of all the successor sets included in this first successor set exists. Moreover, this intersection is a successor set. Even more, this intersection is unique. For this, take a second successor set. Its intersection with the first successor set is contained in the first successor set. Thus, this intersection of two sets is one of the successor sets contained in the first set, and so, is contained in the intersection of all such sets. So then, that first intersection is contained in second intersection of two sets, which is, of course, contained in the second successor set. In other words, we start with a successor set. Use it to construct a succesor set contained in it, in such a way that every other successor set also contains this successor set so constructed. The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique.

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4.

We denote the set of natural numbers by N, a mnemonic for natural. We often denote elements of N by n, a mnemonic for number, or m, a letter close to n.

Why

We represent rectangles by functions.

Definition

The *characteristic function* of a subset of some base set is the function from the base set to the real numbers which maps elements contained in the subset to value one and maps all other elements to zero. The range of the funtion is the set consisting of the real numbers one and zero.

If the base set is the real numbers and the subset is an interval, then the characteristic function is a rectangle with height one and the width of the interval.

Notation

Let A be a non-empty set and $B \subset A$. We denote the characteristic function of B in A by $\chi_B : A \to R$. The Greek letter χ is a mnemonic for "characteristic".

The subscript indicates the set on which the function is one. In other words, for all $B \subset A$, $\chi_B^{-1}(\{1\}) = B$.

If B is an interval and α is a real number then $\alpha\chi_B$ is a rectangle with height α.

Why

Definition

integer numbers integers

TODO

Why

We generalize the algebraic structure of addition over the integers.

Definition

A *group* is an algebra with: (1) an associative operation, (2) an identity element, and (3) an inverse for each element. We call the operation of the algebra *group addition*. A *commutative group* is a group whose operation commutes.

Notation

TODO

Why

Definition

Why

We generalize the algebraic structure of addition and multiplication over the rationals.

Definition

A *field* is two algebras over the same ground set with: (1) both algebras are commutative groups (2) the operation of the second algebra distributes over the operation of the first algebra.

We call the operation of the first algebra *field addition*. We call the operation of the second algebra *field multiplication*.

Notation

We denote an arbitrary field by F, a mnemonic for "field."

TODO

Real Numbers

Why

Definition

Why

We want to find roots of negative numbers

Definition

A *complex number* is an ordered pair of real numbers. The *real part* of a complex number is its first coordinate. The *imaginary part* of a complex number is its second coordinate.

Notation

Let z be a complex number. We denote the real part of z by $\mathsf{Re}(z)$, read "real of z," and the imaginary part by $\mathsf{Im}(z)$, read "imaginary of z." So if $z = (a, b)$, then $\mathsf{Re}(z) = a$ and $\mathsf{Im}(z) = b$.

Why

We want a notion of distance between elements of the real line.

Definition

We define a function mapping a real number to its length from zero.

Notation

We denote the absolute value of a real number $a \in \mathsf{R}$ by $|a|$. Thus $|\cdot| : \mathsf{R} \to \mathsf{R}$ can be viewed as a real-valued function on the real numbers which is nonnegative.

Why

We name and denote subsets of the set of real numbers which correspond to segments of a line.

Definition

Take two real numbers, with the first less than the second.

An *interval* is one of four sets:

1. the set of real numbers larger than the first number and smaller than the second; we call the interval *open*.

2. the set of real numbers larger than or equal to the first number and smaller than or equal to the second number; we call the interval *closed*.

3. the set of real numbers larger than the first number and smaller than or equal to the second; we call the interval *open on the left* and *closed on the right*.

4. the set of real numbers larger than or equal to the first number and smaller than the second; we call the interval *closed on the left* and *open on the right*.

If an interval is neither open nor closed we call it *half-open* or *half-closed*

We call the two numbers the *endpoints* of the interval. An open interval does not contain its endpoints. A closed interval contains its endpoints. A half-open/half-closed interval contains only one of its endpoints. We say that the endpoints *delimit* the interval.

Notation

Let a, b be two real numbers which satisfy the relation $a < b$.

We denote the open interval from a to b by (a, b). This notation, although standard, is the same as that for ordered pairs; no confusion arises with adequate context.

We denote the closed interval from a to b by $[a, b]$. We record the fact $(a, b) \subset [a, b]$ in our new notation.

We denote the half-open interval from a to b, closed on the right, by $(a, b]$ and the half-open interval from a to b, closed on the left, by $[a, b)$.

Why

We want to define the length of a subset of real numbers.

Notions

We take two common notions:

1. The length of a whole is the sum of the lengths of its parts; the *additivity principle*.

2. The length of a whole is the at least the length of any whole it contains the *containment principle*.

The task is to make precise the use of "whole,", "parts," and "contains." We start with intervals.

Definition

By whole we mean set. By part we mean an element of a partition. By contains we mean set containment.

The *length* of an interval is the difference of its endpoints: the larger minus the smaller.

Two intervals are *non-overlapping* if their intersection is a single point or empty. The *length* of the union of two non-overlapping intervals is the sum of their lengths.

A *simple* subset of the real numbers is a finite union of non-overlapping intervals. The length of a simple subset is the sum of the lengths of its family.

A *countably simple* subset of the real numbers is a countable union of non-overlapping intervals. The length of a countably simple subset is the limit of the sum of the lengths of its family; as we have defined it, length is positive, so this series is either bounded and increasing and so converges, or is infinite, and so converges to $+\infty$.

At this point, we must confront the obvious question: are all subsets of the real numbers countably simple? Answer: no. So, what can we say?

A *cover* of a set A of real numbers is a family whose union is a contains A. Since a cover always contains the set A, it's length, which we understand, must be larger (containment principles) than A. So what if we declare that the length of an arbitrary set A be the greatest lower bound of the lengths of all sequences of intervals covering A. Will this work?

Cuts

If a, b are real numbers and $a < b$, then we *cut* an interval with a and b as its endpoints by selecting c such that $a < c$ and $c < b$. We obtain two intervals, one with endpoints a, c and one with endpoints c, b; we call these two the *cut pieces*.

Given an interval, the length of the interval is the sum of any two cut pieces, because the pieces are non-overlapping.

All sets

PROPOSITION **12.** *Not all subsets of real numbers are simple.*

Exhibit: R is not finite.

PROPOSITION **13.** *Not all subsets of real numbers are countably simple.*

Exhibit: the rationals.

Here's the great insight: approximate a set by a countable family of intervals.

Notation

Why

We want to talk about the "distance" between objects in a set.

Common Notions

Our inspiration is the notion of distance in the plane of geometry. The objects are points and the distance between them is the length of the line segment joining them. We note a few properties of this notion of distance:

1. The distance between any two distinct objects is not zero.
2. The distance between any two objects does not depend on the order in which we consider them.
3. The distance between two objects is no larger than the sum of the distances of each with any third object

The first observation is natural: if two points are not the same, then they are some distance apart. In other words, the line segment between them has length.

The second observation is natural: the line segment connecting two points does not depend on the order specifying the points. This observation justifies the word "between." If it were not the case, then we should use different words, and be careful to speak of the distance "from" a first point "to" a second point.

The third property is a non-obvious property of distance in the plane. It says, in other words, that the length of any side of a triangle is no larger than the sum of the lengths of the two other sides. With experience in geometry, the observation may become natural. But it does not seem to be superficially so.

A more muddled but superficially natural justification for our concern with third observation is that it says something about the transitivity of closeness. Two objects are close if their distance is small. Small is a relative concept, and needs some standard of comparison. Let us fix two points, take the distance between them, and call it a unit. We call two objects close with respect to our unit if their distance is less than a unit.

In this language, the third observation says that if we know two objects are each half of a unit distance from a third object, then the two objects are close (their distance is less than a unit). We might call this third object the reference object. Here, then, is the usefulness of the third property: we can infer closeness of two objects if we know their distance to a reference object.

Why

Sometimes "distance" as used in the English language refers to an asymmetric concept. This apparent paradox further illuminates the symmetry property.

Apparent Paradox

Distance in the plane is symmetric: the distance from one point to another does not depend on the order of the points so considered. We took this observation as a definiting property of our abstract notion of distance. The meaning, strength, and limitation of this property is clarified by considering an asymmetric case.

Contrast walking up a hill with walking down it. The "distance" between these two points, the top of the hill and a point on its base, may not be symmetric with respect to the time taken or the effort involved. Experience suggests that it will take longer to walk up the hill than to walk down it. A superficial justification may include reference to the some notion of uphill walking requiring more effort.

If we were going to model the top and base of the hill as points in space, however, the distance between them is the same: it is symmetric. It is even the same if we take into account that some specific path, a trail say, must be followed.

If planning a backpacking trip, such symmetry appears foolish. The distance between two locations must not be considered symmetric. Going up the mountain takes longer than going down. It may justify, in the English phrase, “going around, ather than going over.”

Why

We want to talk about a set with a prescribed quantitative degree of closeness (or distance) between its elements.

Definition

The correspondences which serve as a degree of closeness, or measure of distance, must satisify our notions of distances previously developed.

A function on ordered pairs which does not depend on the order of the elements so considered is *symmetric*. A function into the real numbers which takes only non-negative values is *non-negative*. A repeated pair is an ordered pair of the same element twice. A function which satisfies a triangle inequality for any three elements is *triangularly transitive*.

A *metric* (or *distance function*) is a function on ordered pairs of elements of a set which is symmetric, non-negative, zero only on repeated pairs, and triangularly transitive. A *metric space* is an ordered pair: a nonempty set with a metric on the set.

In a metric space, we say that one pair of objects is *closer* together if the metric of the first pair is smaller than the metric value of the second pair.

Notice that a set can be made into different metric spaces by using different metrics.

Notation

Let A be a set and let R be the set of real numbers. We commonly denote a metric by the letter d, as a mnemonic for "distance." Let $d : A \times A \to R$. Then d is a metric if:

1. it is non-negative, which we tend to denote by
$$d(a, b) \geq 0, \quad \forall a, b \in A.$$
2. it is 0 only on repeated pairs, which we tend to denote by
$$d(a, b) = 0 \Leftrightarrow a = b, \quad \forall a, b \in A.$$
3. it is symmetric, which we tend to denote by:
$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$
4. it is triangularly transitive, which we tend to denote by
$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of A with d by (A, d).

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