



# Norms

## 1 Why

We want to measure the size of an element in a vector space.

## 2 Definition

A *norm* is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisfies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms.

A *norm space* is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

### 2.1 Notation

Let  $(X, F)$  be a vector space where  $F$  is the field of real numbers or the field of complex numbers. Let  $R$  denote the set of real numbers. Let  $f : X \rightarrow R$ . The functional  $f$  is a norm if

1.  $f(v) \geq 0$  for all  $x \in V$
2.  $f(v) = 0$  if and only if  $x = 0 \in X$ .
3.  $f(\alpha x) = |\alpha|f(x)$  for all  $\alpha \in F, x \in X$

4.  $f(x + y) \leq f(x) + f(y)$  for all  $x, y \in X$ .

In this case, for  $x \in X$ , we denote  $f(x)$  by  $|x|$ , read aloud “norm x”. The notation follows the notation of absolute value as a norm. When we wish to distinguish the norm from the absolute value function, we may write  $\|x\|$ . In some cases, we go further, and for a norm indexed by some parameter  $\alpha$  or set  $A$  we write  $\|x\|_\alpha$  or  $\|x\|_A$ .