



Tree-Structured Distributions

1 Why

2 Definition

2.1 Rooted Definition

Consider a distribution on a direct product of n sets. The distribution has n marginals over each of its components, and $n(n-1)$ conditionals of each component given another. Consider a rooted tree over n vertices.

The distribution **factors according to the rooted tree** if the probability for every outcome can be written as a product of the marginal probability of the outcome at the root with the produce of the probabilities of the conditionals corresponding to the edges in the tree.

2.1.1 Notation

Let p be a distribution on a d -dimensional space. p **factors according to the rooted tree** on $\{1, \dots, d\}$ rooted at a vertex k if it can be written as a product of p_k and the conditionals of

$p_{i|j}$ for $i, j = 1, \dots, d$ and $i \neq j$ and $i \neq k$ where j is the parent of i in the rooted tree.

2.2 Defining Result

Proposition 1. *If a distribution factors according to a tree rooted at a vertex it factors according to that tree rooted at any vertex.*

2.3 Undirected Definition

A distribution p **factors according to the tree** T if it factors according to the T rooted at any vertex.

3 Existence and Uniqueness

Trees are not a property of distributions, since there is no one-to-one correspondence, as demonstrated by the following propositions.

3.1 Existence

A distribution p need not factor according to a tree.

3.2 Uniqueness

A distribution p may factor according to multiple trees.