



# Real Integrals

## 1 Why

We define the area under an extended real function.

## 2 Definition

The *positive part* of an extended-real-valued function is the function mapping each element to the maximum of the function's result and zero. The *negative part* of an extended-real-valued function is the function mapping each element to the maximum of the negative of function's result and zero.

We decompose an extended-real-valued function as the difference of its positive part and its negative part. Both the positive and negative parts are non-negative extended-real-valued functions.

Consider a measure space. An *integrable* function is a measurable extended real function for which the non-negative integral of the positive part and the non-negative integral of the negative part of the function are finite.

The *integral* of an integrable function is the difference of the non-negative integral of the positive part and the non-negative integral of the negative part.

If one but not both of the parts of the function are finite, we say that the integral *exists* and again define it as before. In this way we avoid arithmetic between two infinities.

## 2.1 Notation

Let  $A$  a non-empty set. Let  $g : A \rightarrow [-\infty, \infty]$ . We denote the positive part of  $g$  by  $g^+$  and the negative part of  $g$  by  $g^-$ :

$$g^+(x) = \max\{g(x), 0\} \quad \text{and} \quad g^-(x) = \max\{-g(x), 0\}.$$

Moreover, we decompose  $g$  as  $g = g^+ - g^-$ . We observed that  $g^+(x) \geq 0$  and  $g^-(x) \geq 0$  for all  $x \in X$ .

Let  $(X, \mathcal{A}, \mu)$  be a measure space. Let  $f : X \rightarrow [-\infty, +\infty]$  measurable and one of  $\int f^+ d\mu$  or  $\int f^- d\mu$  is finite (if both are finite,  $f$  is integrable).

We denote the integral of  $f$  with respect to the measure  $\mu$  by  $\int f d\mu$ . We defined:

$$\int f d\mu = \left( \int f^+ d\mu \right) - \left( \int f^- d\mu \right).$$