Why

We generalize the algebraic structure of addition and multiplication over the integers.[1]

Definition

A *ring* (or *ring with identity*) (A, f, g) is a set A and two operations on A satisfying the following set of conditions.

(A) (i) f is associative. (ii) f is *commutative*, (iii) A has an identity for f (i.e., is $e \in A$ with $f(a, e) = f(e, a) = a$ for all $a \in A$ (iv) A has inverse elements for f (i.e., for any $a \in A$, there is $\tilde{a}$ satisfying $f(a, \tilde{a}) = f(\tilde{a}, b) = e$)

(B) (i) g is associative; (ii) A has an idenity element for g (i.e., there is $\tilde{e} \in A$ satisfying $g(a, \tilde{e}) = g(\tilde{e}, a) = a$)

(C) (i) g left distributes: $g(f(a, b), c) = f(g(a, c), g(b, c))$; (ii) g right distributes: $g(c, f(a, b)) = f(g(c, a), g(c, b))$.

Conditions (A) concern f, conditions (B) concern g, and conditions (C) relate the two. Define $\psi : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by $\psi(a, b) = a + b$ and $\pi : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by $\pi(a, b) = a \cdot b$. We have defined a ring so that $(\mathbf{Z}, \psi, \pi)$ is one. The element referred to in (A.2) is $0 \in \mathbf{Z}$, so we refer to this element in any ring as the *additive identity*. That referred to (A.3) is $1 \in \mathbf{Z}$, so we refer to this element in any ring as the *multipliciative identity*. We

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[1]Future editions will likely modify this sheet, and give a genetic treatment involving the solution of polynomial equations by Galois.

refer to the elements mentioned in (A.4) as *additive inverses*. We call to f *ring addition* and g *ring multiplication*. Although integer products are commutative, we have not required this aspect (future editions will elaborate).

Notation

Our notation furthers this analogy with **Z**. We denote the ring addition by $+$ and ring multiplication by $\cdot$. Moreover, we denote the ring's additive identity by 0 and the ring's multiplicative identity by 1. Finally, we denote the additive inverse of $a \in A$ by $-a$.

These notational conventions make the condtions (A), (B), (C) familiar relations among integers. (A) (1) $a + (b + c) = (a + b) + c$; (2) $a + b = b + a$; (3) $a + 0 = 0 + a = a$; (4) $a + (-a) = 0$. (B) (1) $a(bc) = (ab)c$; (2) $1a = a1 = a$. (C) (1) $(a + b)c = ac + bc$; (2) $(c(a + b) = ca + cb$.

Immediate consequences

We need not require that $0x = 0$, because we can deduce it:

$$0x + x = (0 + 1)x = 1x = x.$$

Similarly, $(-a)b = -(ab)$ since

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

Other familiar relations among the integers, e.g. $(-a)(-b) = ab$, may be deduced.

Rings

Integer Arithmetic

Natural Additive Identity   Natural Multiplicative Identity

Identity Elements   Arithmetic

Natural Powers   Integer Products

Natural Products   Integer Sums

Operations   Natural Sums   Integer Numbers

Functions   Equivalence Relations   Recursion Theorem

Relations   Peano Axioms

Set Products   Natural Induction

Ordered Pairs   Set Powers   Natural Numbers

Partitions   Intersection of Empty Set   Unordered Triples   Successor Sets

Set Intersections   Pair Unions   Set Symmetric Differences

Pair Intersections   Set Unions   Set Complements

Unordered Pairs   Empty Set   Set Differences

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects