



# Optimal Tree Distribution Approximators

## 1 Why

Which is the optimal tree to use for tree distribution approximation?

## 2 Definition

We want to choose a tree whose corresponding approximator for the given distribution achieves minimum relative entropy with the given distribution among all tree distribution approximators. We call such a distribution an *optimal* tree approximator of the given distribution. We call a tree according to which an optimal tree approximator factors and *optimal* approximator tree.

## 3 Result

**Proposition 1.** *Let  $A_1, \dots, A_n$  be finite non-empty sets. Define  $A = \prod_{i=1}^n A_i$ . Let  $q : A \rightarrow [0, 1]$  a distribution. A tree  $T$  on  $\{1, \dots, n\}$  is an optimal approximator tree if and only if it is a maximal spanning tree of the mutual information graph of  $q$ .*

*Proof.* First, denote the optimal tree distribution approximator of  $q$  for tree  $T$  by  $p_T^*$ . Express

$$p_T^* = q_1 \prod_{i \neq 1} q_{i|\mathbf{pa}_i}$$

Second, express  $d(q, p) = H(q, p) - H(q)$ . Since  $H(q)$  does not depend on  $T$ ,  $p_T^*$  is a minimizer (w.r.t.  $T$ ) of  $d(q, p_T^*)$  if and only if it is a minimizer of  $H(q, p_T^*)$ .

Third, express the cross entropy of  $p_T^*$  relative to  $q$  as

$$\begin{aligned}
H(q, p_T^*) &= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) \log q_{i|\mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i, \mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) - \log q_{\mathbf{pa}_i}(a_{\mathbf{pa}_i})) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) (\log q_{i, \mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) - \log q_{\mathbf{pa}_i}(a_{\mathbf{pa}_i}) - \log q_i(a_i) + \log q_i(a_i)) \\
&= \sum_{i=1}^n H(q_i) - \sum_{i \neq 1} I(q_i, q_{\mathbf{pa}_i}) \\
&= \sum_{i=1}^n H(q_i) - \sum_{\{i, j\} \in T} I(q_i, q_j)
\end{aligned}$$

where  $\mathbf{pa}_i$  denotes the parent of vertex  $i$  in  $T$  rooted at vertex 1 ( $i = 2, \dots, n$ ). For  $i = 1, \dots, n$ ,  $H(q_i)$  does not depend on the choice of tree. Therefore selecting a tree to minimize the second term in the final expression above is equivalent to choosing a maximal spanning tree from the weighted graph with mutual information edge weights; namely, the mutual information graph of  $q$ .

□

Proposition 1 says that to we should first select a maximum spanning tree of the mutual information graph of the distribution we are approximating. Then, we should pick the best approximator to  $q$  which factors according to that tree.