Signed Measures

1 Why

Can we view the set of measures as a vector space?

Not quite: the difference of two measures may take negative values on some set. This functional will be countably additive, however, and so behaves similar to a measure.

2 Definition

An extended-real-valued function on a sigma algebra is **countably additive** if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually. The limit of the partial sums must exist irregardless of the summand order.

A **signed measure** is an extended-real-valued function on a sigma algebra that is (1) zero on the empty set and (2) countably additive. We call the result of the function applied to a set in the sigma algebra the **signed measure** (or when no ambiguity arises, the **measure**) of the set.

A **finite** signed measure is one for which the measure of every set is finite. This condition is equivalent to the base set having finite measure (see below).

When speaking of a measure, which is non-negative, in contrast to a signed measure, we will call it a **positive measure**.

2.1 Notation

Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to [-\infty, \infty]$. Then μ is a signed measure if

1. $\mu(\varnothing) = 0$ and
2. $\mu(\cup_i A_i) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k)$ for all disjoint $\{A_n\}_n$.

3 Finiteness

For the difference of two (signed) measures to be well-defined, we need one or the other to be finite. Otherwise we may subtract ∞ from ∞. We can characterize this condition, that the measure of every set is finite, by the condition that the base set is. This result parallels the definition of finiteness for positive measures.

Proposition 1. *A signed measure is finite if and only if it is finite on the base set.*

Proof. Let $(X, \mathcal{A})$ be a measurable space. Let $\mu : \mathcal{A} \to [-\infty, \infty]$ be a signed measure. $(\Rightarrow)$ If μ is finite, then $\mu(X)$ is finite since

$X \in \mathcal{A}$. ($\Leftarrow$) Next, suppose $\mu(X)$ is finite. Let $A \in \mathcal{A}$. Then $X = A \cup (X - A)$, with these sets disjoint, so by countable additivity of μ, $\mu(X) = \mu(A) + \mu(X - A)$. Since $\mu(X)$ finite, $\mu(A)$ and $\mu(X - A)$ are both finite. $\square$

Proposition 2. *A signed measure never takes both positive infinity and negative infinity.*

Proof. Let $(X, \mathcal{A})$ be a measurable space. Let $\mu : \mathcal{A} \to [-\infty, \infty]$ be a signed measure. First, suppose $\mu(X)$ is finite, Then by Proposition 1 μ is finite for each $A \in \mathcal{A}$.

Suppose $\mu(X) = \infty$. Let $A \in \mathcal{A}$. As before, $\mu(X) = \mu(A) + \mu(X - A)$. Since $\mu(X) = +\infty$, then both of $\mu(A)$ and $\mu(X - A)$ must be either finite or $+\infty$. Argue similarly for $\mu(X) = -\infty$. $\square$