Why

When does the technique of row reductions prevail?

Multivariable row reductions

Let $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$. The *kth row reduction* of S is the linear system (C, d) with $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$ if $i < s \leq m$ and $C_{st} = A_{st}$ otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable k from equations $k + 1, \ldots, m$. We are taking the kth column of A from

$$
\begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ A_{k+1,k} \\ \vdots \\ A_{mk} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

We interpret the ith row reduction as *subtracting equations* of the system or *reducing rows* of the array A. If $a^i, c^i \in \mathbf{R}^n$ denote the ith rows of A and C, $c^i = a^i - (A_{ik}/A_{kk})a^k$ for $k < i \leq m$, In other words, we obtain the ith row of matrix C by subtracting a multiple of the kth row of matrix A from the ith row of matrix A, for $k < i \leq m$. The following is an immediate consequence of real arithmetic.

Proposition 1. *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to (C, d). Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d).*

Ordinary reductions

We call the system S *ordinarily reducible* if there exists a sequence of systems $S_1, \ldots, S_{m-1}$ so that S_1 is the 11-reduction of S and S_i is the ii-reduction of S_{i-1} for $i = 1, \ldots, n - 1$. In this case, we call S_{n-1} the

final ordinary reduction (or just *ordinary reduction*) of S. The following is an immediate consequence of Proposition 1.

Proposition 2. *Let S' be the (final) ordinary reduction of S. Then S and S' have equivalent solution sets.*

This process of constructing the ordinary reduction is called *Gauss elimination* or *Gaussian elimination*. We call the kkth entry of system S_{k-1} the *pivot*. In an ordinarily reducible system, the pivots are nonzero. The idea is that a system is ordinarily reducible if we can take row reductions in sequence and end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system (A, b) in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.

Ordinary Row Reductions

Linear System Row Reductions

Linear Equation Solutions

Linear Equations

Linear Functions

Arrays

N-Dimensional Space

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Space

Real Plane

Real Line

Real Summation

Real Order

Integral Line

Natural Summation

Real Numbers

Rational Order

Geometry

Lists

Rational Numbers

Direct Products

Set Numbers

Integer Arithmetic

Natural Fractions

Finite Sets

Natural Multiplicative Identity

Natural Additive Identity

Integer Products

Equivalent Sets

Identity Elements

Natural Arithmetic

Family Unions and Intersections

Family Operations

Function Inverses

Comparisons

Natural Equations

Integer Order

Integer Sums

Natural Powers

Families

Function Composites

Function Images

Operations

Equations

Orders

Integer Numbers

Natural Products

Functions

Converse Relations

Equivalence Relations

Natural Order

Natural Sums

Relations

Ordered Pair Projections

Recursion Theorem

Peano Axioms

Set Products

Generalized Set Dualities

Set Powers

Natural Induction

Set Unions and Intersections

Set Dualities

Unordered Triples

Successor Sets

Intersection of Empty Set

Partitions

Set Symmetric Differences

Natural Numbers

Ordered Pairs

Pair Unions

Set Intersections

Set Complements

Pair Intersections

Set Unions

Set Difference

Unordered Pairs

Empty Set

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects