Why

How does convergence pointwise (or almost everywhere pointwise) relate to convergence in measure?

Results

PROPOSITION **1.** *There exists a measure space and a sequence of measurable real-valued functions on that space converging everywhere (and so almost everywhere) but not converging in measure.*

PROPOSITION **2.** *There exists a measure space and a sequence of measurable real-valued functions on that space converging in measure but not converging almost everywhere (nor everywhere).*

PROPOSITION **3.** *On finite measure spaces, all sequences of measurable real-valued functions converging almost everywhere converge in measure.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of measurable functions on X such that $f_n \longrightarrow f$ almost everywhere. Let $\varepsilon > 0$.

For each $x \in X$, if $|f_n(x) - f(x)| > \varepsilon$ for infinitely many n, then $f_n(x) \not\longrightarrow f(x)$. Let A be the set of such x. and let $B = \{x \in X \mid f_n(x) \not\longrightarrow f(x)\}$. A is a subset of B. The measure of B is zero since $f_n \longrightarrow f$. Use the the monotonicity

of measure to conclude. $\mu(A) \leq \mu(B) = 0$. Since $\mu(A) \geq 0$, $\mu(A) = 0$.

For natural k, let E_k be the $\{x \in X \mid |f_k(x) - f(x)| > \varepsilon\}$. Then $x \in A$ means that for every natural n, there exists a $k \geq n$ such that $x \in E_k$. In particular, for every n, x is in $\cup_{k=n}^{\infty} E_k$; denote this set by B_n. If x is in B_n for every n, then $x \in \cap_{n=1}^{\infty} B_n$. So we can write

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} B_n.$$

The sequence of sets $(B_n)_n$ is decreasing. So since μ is finite,

$$\lim_{n \to \infty} \mu(B_n) = \mu(A) = 0.$$

For every n, the set B_n contains $\{x \in X \mid |f_n(x) - f(x)| > \varepsilon\}$, namely E_n, the first set in the union. So then $\mu(E_n) \leq \mu(B_n)$ by monotonicity and so

$$0 \leq \lim_{n \to \infty} \mu(E_n) \leq \lim_{n \to \infty} \mu(B_n) = 0,$$

and we conclude $\lim_n E_n = 0$. Since ε was arbitrary, we conclude $f_n \longrightarrow f$ in measure.

$\square$

PROPOSITION 4. *On any measure space, for a sequence of measurable real-valued functions converging in measure to a measurable real-valued limit function, there exists a subsequence convergeng to the limit function almost everywhere.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_n$ be a sequence of measurable functions on X such that $f_n \longrightarrow f$ in measure.

There exists n_1 so that

$$\mu(\{x \in X \mid |f_{n_1}(x) - f(x)| > 1\}) < \frac{1}{2}.$$

Can find $n_2 > n_1$ so that

$$\mu\left(\left\{x \in X \;\middle|\; |f_{n_2}(x) - f(x)| > \frac{1}{2}\right\}\right) < \frac{1}{4}.$$

We can inductively find a sequence $\{n_k\}_k$ so that:

$$\mu\left(\left\{x \in x \;\middle|\; |f_{n_k}(x) - f(x)| > \frac{1}{k}\right\}\right) \leq \frac{1}{2^k}.$$

$\square$

Pointwise vs Measure Limits

Real Limits

Real Sequences   Absolute Value

Sequences   Real Numbers   Natural Differences   Length

Set Operations   Function Composites   Direct Products   Rational Numbers

Algebras   Natural Families   Natural Order   Integer Numbers

Operations   Families   Zero   Natural Sums

Set Symmetric Differences   Functions   Natural Numbers   Equation Solutions

Set Unions   Relations   Set Intersections

Set Complements   Ordered Pairs   Power Set

Unordered Pairs

Set Specification

Set Inclusion

Equations

Empty Set

Set Equality

Standardized Accounts   Identity

Accounts

Deductions   Quantified Statements

Logical Statements

Statements

Identities   Sets

Objects