

Definition

The *covariance* between two random variables which are each integrable and whose product is integrable is the expectation of their product less the product of their expectation.

Notation

Let f and g be two integrable random variables with fg integrable. Denote the covariance of f with g by $\mathbf{cov}(f,g)$. We defined it:

$$\mathbf{cov}(f,g) = \mathbf{E}(fg) - \mathbf{E}(f)\mathbf{E}(g).$$

Properties

Prop. 1. Covariance is symmetric and billinear. ¹

Prop. 2. The covariance of a random variable with itself is its variance.

Proof. Let f be a square-integrable real-valued random variable, then

$$\operatorname{cov}(f,f) = \operatorname{E}(ff) - \operatorname{E}(f)\operatorname{E}(f) = \operatorname{E}(f^2) - (\operatorname{E}(f))^2 = \operatorname{var}(f).$$

Prop. 3. The variance of a sum of integrable real-valued random variables whose pairwise products are integrable is the double sum of the pairwise covariances.

Proof. Let f_1, \ldots, f_n be integrable random variables with $f_i f_j$ integrable for all $i, j = 1, \ldots, n$. Using the billinearity,

$$\begin{aligned} \operatorname{var}\!\left(\sum_{i=1}^n f_i\right) &= \operatorname{cov}\!\left(\sum_{i=1}^n f_i, \sum_{i=1}^n f_i\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \operatorname{cov}(f_i, f_j) \end{aligned}$$

¹Future editions will include an account.

