



## Why

If a directed graph has no cycles, then it has a nice property.<sup>1</sup>

## Definition

Directed and acyclic graphs (or *directed acyclic graphs*, *DAGs*) have partial ordering property on vertices. We call a vertex  $s$  an *ancestor* of a vertex  $u$  if there is a directed path from  $s$  to  $u$ .

## Partial Order

We call a vertex  $s$  an *ancestor* of a vertex  $t$  if there is a directed path from  $s$  to  $t$ . The relation  $R$  defined by  $(s, t) \in R$  if  $s$  is an ancestor of  $t$  is a partial order.

Clearly, every subgraph induced on a directed acyclic graph is a directed acyclic graph.

**Proposition 1.** *Let  $(V, E)$  be a directed acyclic graph. Then there exists a vertex  $v \in V$  which is a source and a vertex  $w \in V$  which is a sink.*

*Proof.* There exists a directed path of maximum length. It must start at a source and end at a sink.<sup>2</sup> □

## Topological numbering

We can choose a total ordering that is consistent with the partial order of ancestry.

A *topological numbering* (or *topological sort*, *topological ordering*) of a directed graph  $(V, E)$  is a numbering  $\sigma : \{1, \dots, |V|\} \rightarrow V$  satisfying

$$(v, w) \in E \longrightarrow \sigma^{-1}(v) < \sigma^{-1}(w).$$

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<sup>1</sup>Future editions will expand this vague introduction.

<sup>2</sup>Future editions will expand.

<sup>3</sup>Future editions will further explain this concept.

**Proposition 2.** *There exists a topological sort for every acyclic graph.*

*Proof.* Let  $(V, F)$  be a directed acyclic graph. There exists a source vertex,  $v_1$ . Set  $\sigma(1) = v_1$ . Take the subgraph induced by  $V - \{v_1\}$ . It is directed acyclic, and so has a source vertex,  $v_2$ . Set  $\sigma(2) = v_2$ . Continue in this way.<sup>4</sup> □

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<sup>4</sup>Future editions will clarify and expand.



