



## SIGNED SET DECOMPOSITION

### Why

Are all signed measures the difference of two positive measures?

Suppose we could partition the base set into two sets, one containing all the sets with positive measure and one containing all sets with negative measure. We could “restrict” the measure to the former and it would be positive, and we could “restrict” it to the latter and it would be negative.

Any measurable set could be partitioned into a piece in the former and a piece in the latter, and so its signed measure could be written as a sum of measures of these pieces.

### Definition

By “positive” and “negative” we mean “non-negative” and “non-positive.” Let  $(X, \mathcal{A})$  be a measurable space. Let  $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$  be a signed measure.

A *positive set* is a measurable set with the property that each of its subsets have non-negative measure under  $\mu$ . A *negative set* is a measurable set with the property that each of its subsets have non-positive measure under  $\mu$ .

A *signed-set decomposition* of  $X$  under  $\mu$  is a partition of  $X$  into a positive and a negative set. Some authors call it a *Hahn decomposition*.

### Notation

Denote by  $P$  a positive and by  $N$  a negative set. When we say “let  $(P, N)$  be a signed-set decomposition of  $X$  under  $\mu$ ”, we mean that

$P$  is the positive set and  $N$  is the negative set.

### **Motivating Implication**

Does such a decomposition always exist? Is it unique? We are motivated to find answers by the following observation.

Suppose there was a signed-set decomposition of  $(X, \mathcal{A})$  under  $\mu$ ; denote it by  $(P, N)$ . Then  $\mu(A \cap P) \geq 0$  and  $\mu(A \cap N) \leq 0$  for all  $A \in \mathcal{A}$ .

Define  $\mu_1 = \mu(A \cap P)$  and  $\mu_2 = -\mu(A \cap N)$ , then  $\mu_1$  and  $\mu_2$  are finite measures. Moreover,  $\mu = \mu_1 - \mu_2$ . Thus, if we had a signed-set decomposition we could write  $\mu$  as the difference of two measures.

