Optimal Tree Distribution Approximators

1 Why

Which is the optimal tree to use for tree distribution approximation?

2 Definition

We want to choose a tree whose corresponding approximator for the given distribution achieves minimum relative entropy with the given distribution among all tree distribution approximators. We call such a distribution an *optimal* tree approximator of the given distribution. We call a tree according to which an optimal tree approximator factors and *optimal* approximator tree.

3 Result

Proposition 1. *Let $A_1, \dots, A_n$ be finite non-empty sets. Define $A = \prod_{i=1}^n A_i$. Let $q : A \to [0, 1]$ a distribution. A tree T on $\{1, \dots, n\}$ is an optimal approximator tree if and only if it is a maximal spanning tree of the mutual information graph of q.*

Proof. First, denote the optimal approximator of q for tree T by p_T^*. Recall

$$p_T^* = q_1 \prod_{i \neq 1} q_{i|\mathsf{pa}_i}$$

Second, recall $d(q, p) = H(q, p) - H(q)$. Since $H(q)$ does not depend on p, p is a minimizer of $d(q, p)$ if and only if it is a minimizer of $H(q, p)$.

Third, express the cross entropy of p_T^* relative to q as

$$
\begin{aligned}
H(q, p_T^*) &= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a) \log q_{i|pai}(a_i, a_{\mathbf{pa}_i}) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a)(\log q_{i,\mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) - \log q_{\mathbf{pa}_i}(a_{\mathbf{pa}_i})) \\
&= H(q_1) - \sum_{i \neq 1} \sum_{a \in A} q(a)(\log q_{i,\mathbf{pa}_i}(a_i, a_{\mathbf{pa}_i}) - \log q_{\mathbf{pa}_i}(a_{\mathbf{pa}_i}) - \log q_i(a_i) + \log q_i(a_i)) \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{i \neq 1} I(q_i, q_{\mathbf{pa}_i}) \\
&= \sum_{i=1}^{n} H(q_i) - \sum_{\{i,j\} \in T} I(q_i, q_j)
\end{aligned}
$$

where $\mathbf{pa}_i$ denotes the parent of vertex i in T $(i = 2, \ldots, n)$. $H(q_i)$ does not depend on the choice of tree. Choosing a tree to minimize the second term in the final expression above is equivalent to choosing a maximal spanning tree from the weighted graph with mutual information edge weights; namely, the mutual information graph of q.

$\square$