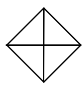


Why

Historically considered a fountainhead for all of mathematics.

Discussion

So far we know that ω is the unique smallest successor set. In other words, we know that $0 \in \omega$, $n \in \omega \longrightarrow n^+ \in \omega$ and that if these two properties hold of some $S \subset \omega$, then $S = \omega$. We can add two important statements to this list. First, that 0 has no successor. I.e., $n^+ \neq 0$ for all $n \in \omega$. Second, that if two numbers have the same successor, then they are the same number I.e., $n^+ = m^+ \longrightarrow n = m$

These five properties were historically considered the fountainhead of all of mathematics. One by the name of Peano used them to show the elementary properties of arithmetic. They are:

1. $0 \in \omega$.
2. $n \in \omega \longrightarrow n^+ \in \omega$ for all $n \in \omega$.
3. If S is a successor set contained in ω, then $S = \omega$.
4. $n^+ \neq 0$ for all $n \in \omega$
5. $n^+ = m^+ \longrightarrow n = m$ for all $n, m \in \omega$.

These are collectively known as the *Peano axioms*. Recall that the third statement in this list is the *principle of mathematical induction.*

Statements

Here are the statements.[1]

Proposition 1 (Peano's First Axiom). $0 \in \omega$.

Proposition 2 (Peano's Second Axiom). $n \in \omega \longrightarrow n^+ \in \omega$.

Proposition 3 (Peano's Third Axiom). *Suppose* $S \subset \omega$, $0 \in S$, *and* $(n \in S \longrightarrow n^+ \in S$. *Then* $S = \omega$.

Proposition 4 (Peano's Fourth Axiom). $n^+ \neq 0$ *for all* $n \in \omega$.

The last one uses the following two useful facts.

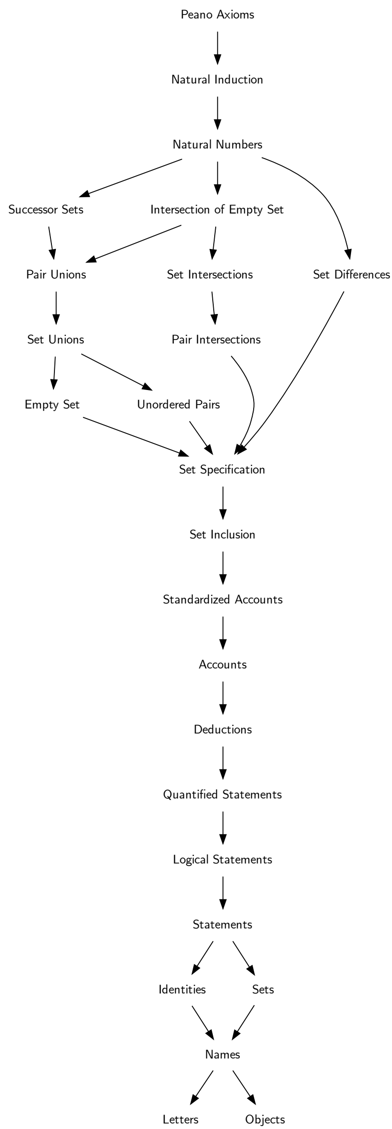
Proposition 5. $x \in n \longrightarrow n \not\subset x$.

Proposition 6. $(x \in y \wedge y \in n) \longrightarrow x \in n$

This latter proposition is sometimes described by saying that n is a *transitive set*. This notion of transitivity is not the same as that described in Relations. Using these one can show:

Proposition 7 (Peano's Fifth Axiom). *Suppose* $n, m \in \omega$ *with* $n^+ = m^+$. *Then* $n = m$.

[1] Accounts of all of these will appear in future editions.


Peano Axioms
Natural Induction
Natural Numbers
Successor Sets
Intersection of Empty Set
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects