Why

For general measure theory, we need an algebra of sets closed under countable unions; we define such an object.[1]

Definition

A *countably summable subset algebra* is a subset algebra for which (1) the base set is distinguished (2) the complement of a distinguished set is distinguished (3) the union of a sequence of distinguished sets is distinguished.

The name is justified, as each countably summable subset algebra is a subset algebra, because the union of $A_1, \ldots, A_n$ coincides with the union of $A_1, \cdots, A_n, A_n, A_n \cdots$.

We call the set of distinguished sets a *sigma algebra* (or *sigma field*) *on* the base set. This language is justified (as for a regular subset algebra) by the closure properties of the sigma algebra under the usual set operations. We sometimes write are σ-algebra and σ-field.

A *sub-σ-algebra* (*sub-sigma-algebra*) is a subset of a sigma algebra which is itself a sigma algebra.

Notation

We often denote a sigma algebra by $\mathcal{A}$ or $\mathcal{F}$; the former is a mnemonic for "algebra" and the second is a mnemonic for "field". The calligraphic typeface is meant as a reminder that the object so denoted is a set *of sets*.

A common pattern is also to use the calligraphic font of whichever letter is being used for the base set. Thus, if A is a set, then we might choose to denote a sigma algebra on A by $\mathcal{A}$.

[1] Future editions will make no reference to measure theory. The entire development will follow the genetic approach, and so roughly follow the historical development for handling integration.

Often, instead of saying "let $(A, \mathcal{A})$ be a countably summable subset algebra" we say instead "let $\mathcal{A}$ be a sigma algebra on A." Since the largest element of the sigma algebra is the base set, we can also say (without ambiguity): "let $\mathcal{A}$ be a sigma algebra." The base set is $\cup\mathcal{A}$.

Examples

Example 1. *For any set A, 2^A is a sigma algebra.*

Example 2. *For any set A, $\{A, \varnothing\}$ is a sigma algebra.*

Example 3. *Let A be an infinite set. Let $\mathcal{A}$ the collection of finite subsets of A. $\mathcal{A}$ is not a sigma algebra.*

Example 4. *Let A be an infinite set. Let $\mathcal{A}$ be the collection subsets of A such that the set or its complement is finite. $\mathcal{A}$ is not a sigma algebra.*

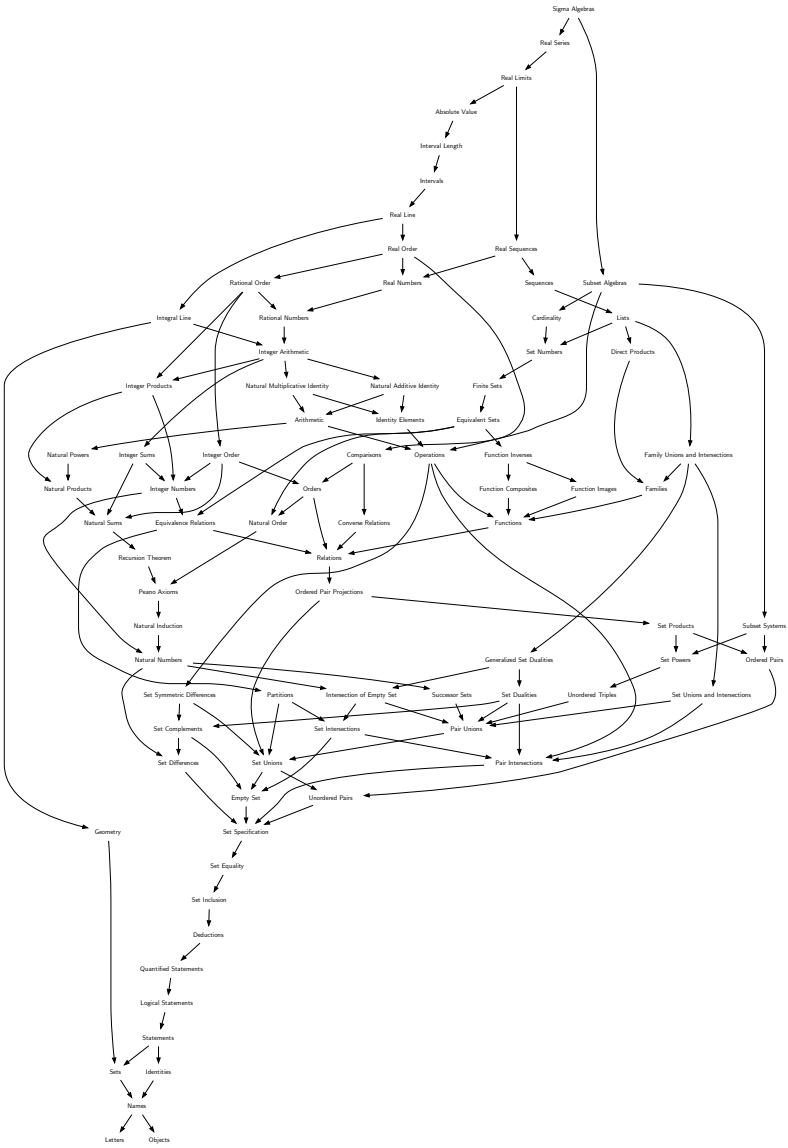
Proposition 1. *The intersection of a family of sigma algebras is a sigma algebra.*

Example 5. *For any infinite set A, let $\mathcal{A}$ be the set*

$$\{B \subset A \mid |B| \leq \aleph_0 \vee |C_A(B)| \leq \aleph_0\}.$$

$\mathcal{A}$ is an algebra; the countable/co-countable algebra.[2]

[2]Future editions will clean up and modify these examples.


Sigma Algebras
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Rational Order
Real Numbers
Sequences
Subset Algebras
Integral Line
Rational Numbers
Cardinality
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Operations
Function Inverses
Family Unions and Intersections
Natural Products
Integer Numbers
Orders
Function Composites
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Set Products
Subset Systems
Natural Induction
Natural Numbers
Generalized Set Dualities
Set Powers
Ordered Pairs
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects