



Supremum Norm

1 Why

We want a norm on the vector space of continuous functions.

2 Definition

Consider a function from a closed real interval to the real numbers. The *absolute supremum* of the function is the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

Proposition 1. *The functional mapping $f \in C[a, b]$ to its absolute supremum is a norm.*

Proof. Let R denote the set of real numbers. Define $\phi : C[a, b] \rightarrow R$ by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a, b]\}.$$

1. $|f(x)| \geq 0$ for all $x \in [a, b]$, so $\phi(f) \geq 0$.
2. If $\phi(f) = 0$ then $|f(x)| \leq 0$ for all x and so $f(x) = 0$ for all $x \in [a, b]$. If $f = 0$, then $|f(x)| = 0$ for all $x \in [a, b]$
3. For all α real, $|\alpha f(x)| = |\alpha||f(x)|$. so $\phi(\alpha f) = |\alpha|\phi(f)$

4. For all $f, g \in C[a, b]$, and $x \in [a, b]$, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ by the triangle inequality for absolute value. Thus,

$$\begin{aligned}\phi(f + g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a, b]\} \\ &\leq \sup\{|f(x)| \mid x \in [a, b]\} + \sup\{|g(x)| \mid x \in [a, b]\} \\ &= \phi(f) + \phi(g)\end{aligned}$$

□

We call the functional ϕ defined above the *supremum norm*.

2.1 Notation

Let $f \in C[a, b]$. We denote the supremum norm of f by $|f|_{\text{sup}}$.