



### Definition

Let  $X$  be a (nonempty) set and  $k$  a field. Let  $F \subset (X \rightarrow k)$  and let  $\langle \cdot, \cdot \rangle : F \times F \rightarrow k$  be an inner product so that  $(F, \langle \cdot, \cdot \rangle)$  is a complete inner product space.

A *reproducing kernel* of  $(F, \langle \cdot, \cdot \rangle)$  is a map  $R : X \times X \rightarrow k$  satisfying

- (1) for every  $y \in X$  the function  $R(\cdot, y) : X \rightarrow k$  is an element of  $F$  and
- (2) for every  $f \in F$ , at every  $y \in X$ ,  $f(y) = \langle f, R(\cdot, y) \rangle$  (the *reproducing property*).

$R$  is called a “reproducing” kernel because of the following implication of the reproducing property. Notice that  $R(\cdot, y) \in F$ . For this reason,

### Properties

If a reproducing kernel exists, it is unique.

### Separate sheet

Let  $X$  be nonempty (index) set. For example,  $X$  may be  $\{1, 2, \dots, N\}$ ,  $\mathbf{Z}$ ,  $[0, 1]$ ,  $\mathbf{R}^d$ ,  $\{x \in \mathbf{R}^3 \mid \|x\| \leq 1\}$  (the unit sphere), or  $\{x \in \mathbf{R}^3 \mid \alpha \leq \|x\| \leq \beta\}$  (the atmosphere, or volume between two concentric spheres).

A symmetric, real-valued function  $k : X \times X \rightarrow \mathbf{R}$  of two variables is said to be *positive semidefinite* if for any  $n \in \mathbf{N}$ , for any real  $a_1, \dots, a_n \in \mathbf{R}$  and  $x_1, \dots, x_n \in X$ , we have

$$\sum_{i,j=1}^n a_i a_j k(x_i, x_j) \geq 0,$$

and *positive definite* if the above holds with “ $>$ ”.<sup>1</sup>

Positive semidefinite kernels are useful for the following constructive reason:

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<sup>1</sup>Some authors use the term “positive definite” for our term positive semidefinite and the term “*strictly positive definite*” for our term “positive definite.”

**Proposition 1.** *Let  $X \neq \emptyset$  be a set. If  $k : X \times X \rightarrow \mathbf{R}$  is positive semidefinite, then there exists a probability space  $(\Omega, \mathbf{CA}, \mathbf{P})$  and a family of zero-mean normal real-valued random variables  $\{f_x : \Omega \rightarrow \mathbf{R}\}_{x \in X}$  with covariance function  $k$ , that is,*

$$\mathbf{E}f(a)f(b) = k(a, b), \quad \text{for all } a, b \in X.$$

This result is known by the names *Kolmogorov extension theorem*, *Kolmogorov existence theorem*, *Kolmogorov consistency theorem* and *Daniell-Kolmogorov theorem*.



