



## Why

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## Result

This result is called sometimes called the *probability inverse transform*.

**Proposition 1.** *Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space and let  $X : \Omega \rightarrow \mathbf{R}$  be a random variable with cumulative distribution function  $F_X : \mathbf{R} \rightarrow [0, 1]$ . Suppose  $F_X^{-1} : [0, 1] \rightarrow \mathbf{R}$  exists, then  $Y = F_X^{-1} \circ X$  is a random variable with cumulative distribution function  $F_Y : [0, 1] \rightarrow [0, 1]$  satisfying  $F_Y(y) = y$ .*

**Remark 1.** *The conclusion is equivalent to the following:  $Y$  has a density and that density is the the standard uniform density (see *Uniform Densities*).*

*Proof.* Express  $F_Y(\gamma) = \mathbf{P}[Y \leq \gamma] = \mathbf{P}(Y^{-1}([0, \gamma]))$  Notice

$$\begin{aligned} Y^{-1}([0, \gamma]) &= \{\omega \in \Omega \mid Y(\omega) \leq \gamma\} \\ &= \{\omega \in \Omega \mid F_X(X(\omega)) \leq \gamma\} \\ &= \{\omega \in \Omega \mid X(\omega) \leq F_X^{-1}(\gamma)\}. = X^{-1}(\dots)^2 \end{aligned}$$

□

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<sup>1</sup>Future editions will include.

**Remark 2.** *Using different notation the above can be expressed succinctly as*

$$\begin{aligned} F_Y(\gamma) &= \mathbf{P}[Y \leq \gamma] = \mathbf{P}[F_X \circ X \leq \gamma] \\ &= \mathbf{P}[X \leq F_X^{-1}(\gamma)] = F_X(F_X^{-1}(\gamma)) = \gamma. \end{aligned}$$

Future editions will discuss *inverse transform sampling*.



