



LINEAR SYSTEM ROW REDUCTIONS

Why

We want to solve linear equations. Our approach is to “eliminate” variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

Two-variable example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$3x_1 + 2x_2 = 10, \text{ and}$$

$$6x_1 + 5x_2 = 20.$$

We can list the coefficients in a two-dimensional array $A = (3, 2; 6, 5)$ and $b = (10, 20)$. We can eliminate x_1 from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$3x_1 + 2x_2 = 10 \text{ and}$$

$$x_2 = 0.$$

The key insight is that this system has the *same solution set*. We call the process of moving between these two systems a *row reduction*.

Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate x_1 (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear system $S_1 = (A^1, b^1)$ in which

$$A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and} \quad b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The trick is that, since $A'_{22} \neq 0$, we can take the same route to eliminate x_2 , to obtain the system $S_2 = (A^2, b^2)$ in which

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \text{and} \quad b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Likewise for x_3 , we obtain $S_3 = (A^3, b^3)$ in which

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one, (A^3, b^3) , is easy to solve. We solve it by *back substitution*. First, since $2x_4 = 3$, we find $x_4 = 3/2$. Second, since $2x_3 + 2x_4 = -1$, we find $x_3 = -2$. Similarly we find $x_2 = 1/2$ and $x_1 = 5/4$.

Definition

Let $S = (A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system. The lower *row reduction* of S for index i with $A_{ii} \neq 0$ (or the i -row reduction) is

the linear system $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$ if $i < s \leq m$ and A_{st} otherwise. We say that the system (A, b) is *ordinarily reducible*.

Let $a^k, \tilde{a}^k \in \mathbf{R}^n$ denote the k th row of A and \tilde{A} , respectively. Then if $k \neq i$, $\tilde{a}^k = a^k - \alpha_k a^i$ where $\alpha_k = A_{kj}/A_{ij}$. In other words, a row k of the matrix \tilde{A} is obtained by subtracting a multiple of the i th row of matrix A from row k of matrix A . We are “reducing” the rows of A .

Proposition 1. *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to (C, d) . Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d) .¹*

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = \left(\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

We then subtract three times row 2 from row 3 and four times row 2 from row 4 to obtain

$$S_2 = \left(\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

Finally, we subtract two times row 3 from row 4 to obtain S_4 , which

¹Future editions will include an account.

we can write as

$$2x_1 + x_2 + x_3 = 1,$$

$$x_2 + x_3 + x_4 = 0,$$

$$2x_3 + 2x_4 = -1, \quad \text{and}$$

$$2x_4 = 3.$$

We can now back-substitute to find $x_4 = 3/2$, $x_3 = -2$, $x_2 = 1/2$ and $x_1 = 5/4$. Proposition ?? says that this is the only solution of S , as well.

