Why

How big can a quadratic form be? How small?

Result

Proposition 1. *Suppose $A \in \mathbf{S}^n$ has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_n x^\top x \leq x^\top A x \leq \lambda_1 x^\top x,$$

for all $x \in \mathbf{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix $Q \in \mathbf{R}^{n \times n}$ with $A = Q \Lambda Q^\top$. Express

$$x^\top A x = x^\top Q \Lambda Q^\top x = (Q^\top x)^\top \Lambda (Q^\top x)$$

$$= \sum_{i=1}^n \lambda_i (q_i^\top x)^2$$

$$= \lambda_1 \sum_{i=1}^n (q_i^\top x) = \lambda_1 \|Q^\top x\|^2 = \lambda_1 \|x\|^2.$$

Similarly,

$$x^\top A x = \sum_{i=1}^n \lambda_i (q_i^\top x)^2$$

$$\geq \lambda_n \sum_{i=1}^n (q_i^\top x) = \lambda_n \|Q^\top x\|^2 = \lambda_n \|x\|^2.$$

$\square$

Notation

For this reason, it is common to order the eigenvalues of $A \in \mathbf{S}^n$ by magnitude with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. λ_1 is sometimes denoted $\lambda_{\max}$ and λ_n is sometimes denoted $\lambda_{\min}$.

Optimization implication

If $z = \alpha x$, then $z^\top A z = \alpha^2 x^\top A x$. Consider finding $x \in \mathbf{R}^n$ to maximize

$$\text{maximize} \quad x^\top A x$$
$$\text{subject to} \quad \|x\| = 1.$$

Since the objective is $x^\top A x \leq \lambda_1$ for all $x \in \mathbf{R}^n$ with $\|x\| = 1$, a solution of this problem is the eigenvector $q_1 \in \mathbf{R}^n$ corresponding to λ_1. In other words, these inequalities are tight.

Quadratic Form Inequalities

Quadratic Forms

Matrix Trace — Identity Matrices

Real Matrix-Matrix Products — Matrix Transpose — Symmetric Real Matrix Eigenvalues

Real Matrix-Vector Products — Symmetric Real Matrices — Eigenvalues and Eigenvectors

Vectors as Matrices — Real Inner Product — Real Matrices — Matrices — Matrix Determinants — Norms — Functionals

Real Linear Combination — Real Norm — Space Inner Product — Multivariate Polynomials — Linear Equation — Vectors

Real Vectors — Space Norm — Multivariate Functions — Linear Functions — Arrays

Space Vectors — N-Dimensional Space — Plane Vector Sums and Differences — Plane Norm — Plane Inner Product

Distance — Plane Vectors — Absolute Value — Right Triangle Sides Relation

Space Distance — Real Square — Real Polynomials — Monic Polynomials

Plane Distance — Real Arithmetic

Interval Length — Real Multiplicative Inverse

Real Space — Real Products

Real Plane — Square Numbers — Real Line — Intervals — Real Additive Inverse — Polynomials

Squares — Integer Line — Real Order — Rational Arithmetic — Real Sums — Rational Multiplicative Inverse — Real Summation — Lists — Permutations

Geometry — Rational Order — Rational Products — Rational Sum — Real Numbers — Fields — Set Numbers — Groups

Rational Numbers — Rings — Finite Sets

Integer Arithmetic — Direct Products

Integer Products — Natural Multiplicative Identity — Natural Additive Identity

Arithmetic — Identity Elements

Natural Powers — Natural Summation — Inverse Elements — Equivalent Sets — Family Unions and Intersections

Comparisons — Integer Order — Integer Sums — Natural Products — Natural Equations — Family Operations — Element Functions — Function Inverses

Order — Integer Numbers — Natural Sums — Equations — Operations — Function Image — Function Composites — Families

Coarsene Relation — Recursion Theorem — Natural Order — Equivalence Relation — Functions

Relation — Peano Axioms

Set Products — Natural Induction

Natural Numbers — Generalized Set Dualities

Set Powers — Set Symmetric Difference — Unordered Triple — Successor Sets — Set Dualities — Intersection of Empty Set — Partitions — Set Unions and Intersections

Ordered Pairs — Set Complements — Pair Unions — Set Intersections

Set Differences — Set Unions — Pair Intersections

Unordered Pairs — Empty Set

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deduction

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects