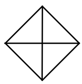

Why

Definition

Let X be a (nonempty) set and k a field. Let $F \subset (X \to k)$ and let $\langle \cdot, \cdot \rangle : F \times F \to k$ be an inner product so that $(F, \langle \cdot, \cdot \rangle)$ is a complete inner product space.

A *reproducing kernel* of $(F, \langle \cdot, \cdot \rangle)$ is a map $R : X \times X \to k$ satisfying (1) for every $y \in X$ the function $R(\cdot, y) : X \to k$ is an element of F and (2) for every $f \in F$, at every $y \in X$, $f(y) = \langle f, R(\cdot, y) \rangle$ (the *reproducing property*).

R is called a "reproducing" kernel because of the following implication of the reproducing property. Notice that $R(\cdot, y) \in F$. For this reason,

Properties

If a reproducing kernel exists, it is unique.

Let X be nonempty (index) set. For example, X may be $\{1, 2, \ldots, N\}$, $\mathbf{Z}$, $[0, 1]$, $\mathbf{R}^d$, $\{x \in \mathbf{R}^3 \mid \|x\| \leq 1\}$ (the unit sphere), or $\{x \in \mathbf{R}^3 \mid \alpha \leq \|x\| \leq \beta\}$ (the atmosphere, or volume between two concentric spheres).

A symmetric, real-valued function $k : X \times X \to \mathbf{R}$ of two variables is said to be *positive semidefinite* if for any $n \in \mathbf{N}$, for any real $a_1, \ldots, a_n \in \mathbf{R}$ and $x_1, \ldots, x_n \in X$, we have

$$\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) \geq 0,$$

and *positive definite* if the above holds with "$>$".[1]

Positive semidefinite kernels are useful for the following constructive reason:

Proposition 1. *Let $X \neq \varnothing$ be a set. If $k : X \times X \to \mathbf{R}$ is positive semidefinite, then there exists a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and a family of zero-mean normal real-valued random variables $\{f_x : \Omega \to \mathbf{R}\}_{x \in X}$ with covariance function k, that is,*

$$\mathbf{E} f(a) f(b) = k(a, b), \qquad \textit{for all } a, b \in X.^2$$

This result is known by the names *Kolmogorov extension theorem*, *Kolmogorov existence theorem*, *Kolmogorov consistency theorem* and *Daniell-Kolmogorov theorem.*

[1]Some authors use the term "positive definite" for our term positive semidefinite and the term "*strictly positive definite*" for our term positive definite.

[2]Future editions will prove this result.