



## Why

We characterize chordal graphs using vertex separators, and vice versa.<sup>1</sup>

## Main Result

**Proposition 1** (Chordal Graphs and Vertex Separators). *An undirected graph is chordal if and only if all minimal vertex separators are complete.*

*Proof.* Let  $G = (V, E)$  be an undirected graph. First, suppose that all minimal vertex separators of  $G$  are complete. Let  $c$  be a cycle of length greater than 3. Let  $v, w$  be nonconsecutive vertices in  $c$ . If  $v$  and  $w$  are adjacent in  $G$ , then  $\{v, w\} \in E$  is a chord. If  $v$  and  $w$  are nonadjacent, then  $vw$ -separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any  $vw$ -separator  $T$ . Split the cycle into the path from  $v$  to  $w$ , call it  $p_1$  and the path from  $w$  to  $v$ , call it  $p_2$ .  $T$  must include an interior point of  $p_1$ , call it  $u_1$ , otherwise  $v$  and  $w$  are connected. Similarly,  $T$  must include an interior point of  $p_2$ , call it  $u_2$ .  $u_1$  and  $u_2$  are not consecutive in  $c$ , since they are distinct from  $x$  and  $y$ .

Let  $S$  be a minimal  $vw$ -separator. Let  $s, t \in S$  be two non-consecutive vertices in the cycle different from  $v$  and  $w$ . By assumption  $S$  is complete, so  $s$  and  $t$  are adjacent in  $G$ .

Second, let  $G = (V, E)$  be a chordal graph. Let  $S$  be a minimal  $vw$ -separator. Let  $C_v$  and  $C_w$  be the connected components containing  $v$  and  $w$  of the subgraph induced by  $V - S$ .

If  $\nu m S = 1$ , then  $S$  is complete. Otherwise, let  $x, y \in S$  be distinct. We want to show  $\{x, y\} \in E$ . The key insight is that  $x$  is adjacent to vertices in  $C_v$  and  $C_w$ . If there were no such vertex,  $S - \{x\}$  would be a  $vw$ -separator and  $S$  would not be minimal. Similarly with  $y$ . Also,  $\nu m C_v, \nu m C_w \geq 1$ .

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<sup>1</sup>Future editions will expand and may include graphics.

With these observations, there exists a path from  $x$  to  $y$  through  $C_v$ . Let  $p_v = (x, v_1, \dots, v_k, y)$  be a path of shortest length with at least one interior vertex (so  $k \geq 1$ ) from  $x$  to  $y$  using interior vertices in  $v_1, \dots, v_k \in C_v$ . Let  $p_w = (y, w_1, \dots, w_l, x)$  be a path of shortest length with at least one interior vertex (so  $l \geq 1$ ) from  $y$  to  $x$  using interior vertices  $w_1, \dots, w_l \in C_w$ . Use  $p_v$  and  $p_w$  to define the cycle  $c = (x, v_1, \dots, v_k, y, w_1, \dots, w_l, x)$  which has length at least four.  $G$  is chordal, so  $c$  has a chord.

We argue that the chord of  $c$  is  $\{x, y\}$ . Since  $C_w$  and  $C_v$  are different connected components (whose vertices are not included in  $S$ ), there are no edges  $\{v_i, w_j\}$  for  $i = 1, \dots, k$  and  $j = 1, \dots, l$ . Since  $p_v$  and  $p_w$  are paths of shortest length, they have no chords. In particular, there is no edge  $\{v_i, v_j\}$  for  $|i - j| > 1$  or  $\{v_i, x\}$  for  $i = 1, \dots, k$ . Similarly, there is no edge  $\{w_i, w_j\}$  for  $|i - j| > 1$  or  $\{w_i, y\}$  for  $i = 1, \dots, l$ . The only remaining pair is  $\{x, y\}$ , and so it must be the chord.  $\square$

