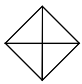


Definition

An optimization problem $(\mathcal{X}, f)$ is called *linear* (a *linear optimization problem*) if $\mathcal{X} \subset \mathbf{R}^n$ is a polyhedron and $f : \mathbf{R}^n \to \bar{\mathbf{R}}$ is a linear function.

Problem data

Recall that f is linear means there exists $c \in \mathbf{R}^n$ such that

$$f(x) = c^\top x \quad \text{for all } x \in \mathbf{R}^n$$

Also, $\mathcal{X}$ polyhedral means there exists $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^d$ such that

$$\mathcal{X} = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

For this reason, the *problem data* (A, b, c) is sufficient to specify a linear optimization problem. Recall that $Ax \leq b$ means element-wise inequality (i.e., that the inequality holds in each component).

Task

Given data $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}^n$, we want to find $x \in \mathbf{R}^d$ to

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

We either want $x^\star \in \mathbf{R}^n$ so that $Ax^\star \leq b$ and $c^\top x^\star \leq c^\top x$ for all $x \in \mathbf{R}^n$, or we want to know that $\{x \mid Ax \leq b\} = \varnothing$, or we want to know that for all $\alpha \in \mathbf{R}$, there is an $x \in \mathbf{R}^n$ satisfying $Ax \leq b$ and $c^\top x \leq \alpha$. This problem is regularly called *linear programming* (a *linear program*). Many authors define this problem with the goal as maximization: of course, minimizing $c^\top x$ is equivalent to (has the same set of optimal solutions) maximizing $-c^\top x$.

Notation

In the context of linear optimization, $c^\top x$ is often abbreviated cx. A linear program is sometimes abbreviated $\min\{cx \mid Ax \leq b\}$ (here the matrices and vectors are assumed to be conforming). As usual, $x \in \mathbf{R}^n$ is called *feasible* (a *feasible solution*) if $Ax \leq b$. $x^\star \in \mathbf{R}^n$ is called *optimal* (an *optimal solution*, *optimum solution*) if $c^\top x^\star \leq c^\top x$ for all $x \in \mathbf{R}^n$. We sometimes denote the rows of a by $\bar{a}_i^\top \in \mathbf{R}^n$ for $i = 1, \ldots, m$, i.e.,

$$A = \begin{bmatrix} - & \bar{a}_1^\top & - \\ & \vdots & \\ - & \bar{a}_m^\top & - \end{bmatrix} \in \mathbf{R}^{m \times n}$$

and refer to the inequality $\bar{a}_i^\top x \leq b_i$ as an *inequality constraint* for $i = 1, \ldots, m$. The set or the expression $Ax \leq b$ are both sometimes called the *inequality constraints* of the problem.

The problem (A, b, c) is *infeasible* if the polyhedron

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

is empty. In symbols, if $P = \varnothing$. The problem is *unbounded* if for all $\alpha \in \mathbf{R}$, there exists $x \in P$ with $c^\top x < \alpha$.

Here's an infeasible instance. Define

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \end{bmatrix}$$

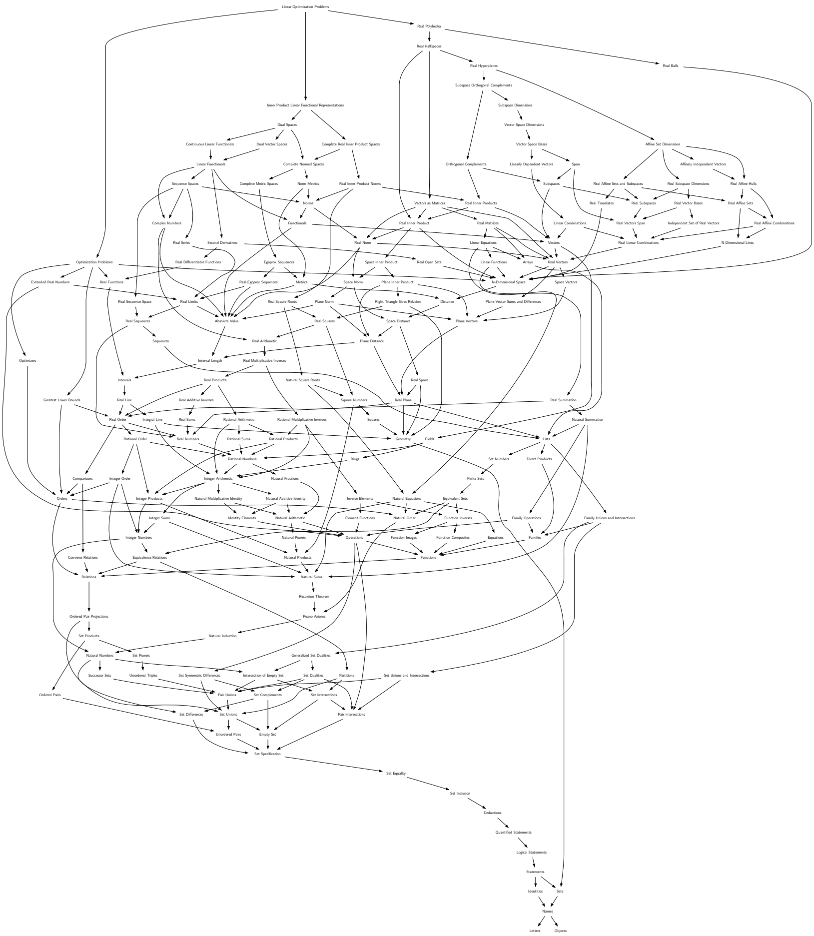
We want $x \in \mathbf{R}^1$ so that $x \leq -1$ and $x \geq 1$. There is no such x.

Here's an unbounded instance: drop the second inequality constraint. Then we are interested in finding x_1 to minimize x_1 subject to $x_1 \leq -1$. Given $\alpha \in \mathbf{R}$, if $\alpha > 0$ pick $x_1 = -1$, else pick 2α. This problem is unbounded.

Here's a simple example with an optimal solution. Modify $b = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$. Now we want to find x_1 so that

$$x_1 \leq 1 \quad \text{and } x_1 \geq 0$$

and we minimize x_1 Clearly $x_1 = 0$ is an optimal solution. Indeed, it is the unique optimal solution in this case.


Linear Optimization Problems
Real Polyhedra
Real Halfspaces
Real Hyperplanes
Real Balls
Subspace Orthogonal Complements
Inner Product Linear Functional Representations
Subspace Dimensions
Dual Spaces
Vector Space Dimensions
Continuous Linear Functionals
Dual Vector Spaces
Complete Real Inner Product Spaces
Vector Space Bases
Affine Set Dimensions
Linear Functionals
Complete Normed Spaces
Orthogonal Complements
Linearly Dependent Vectors
Span
Affinely Independent Vectors
Sequence Spaces
Complete Metric Spaces
Norm Metrics
Real Inner Product Norms
Subspaces
Real Affine Sets and Subspaces
Real Subspace Dimensions
Real Affine Hulls
Norms
Real Inner Products
Real Translates
Real Subspaces
Real Vector Bases
Real Affine Sets
Vectors as Matrices
Complex Numbers
Functionals
Real Inner Product
Real Matrices
Linear Combinations
Real Vectors Span
Independent Set of Real Vectors
Real Affine Combinations
Real Series
Second Derivatives
Real Norm
Linear Equations
Vectors
Real Linear Combinations
N-Dimensional Lines
Optimization Problems
Real Differentiable Functions
Egoprox Sequences
Space Inner Product
Real Open Sets
Linear Functions
Arrays
Real Vectors
Extended Real Numbers
Real Functions
Real Egoprox Sequences
Metrics
Space Norm
Plane Inner Product
N-Dimensional Space
Space Vectors
Real Sequence Space
Real Limits
Real Square Roots
Plane Norm
Right Triangle Sides Relation
Distance
Plane Vector Sums and Differences
Real Sequences
Absolute Value
Real Squares
Space Distance
Plane Vectors
Sequences
Real Arithmetic
Plane Distance
Optimizers
Interval Length
Real Multiplicative Inverses
Intervals
Real Products
Natural Square Roots
Real Space
Greatest Lower Bounds
Real Line
Real Additive Inverses
Square Numbers
Real Plane
Real Summation
Real Order
Integral Line
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Squares
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Rational Sums
Rational Products
Geometry
Fields
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Rational Numbers
Rings
Set Numbers
Direct Products
Comparisons
Integer Order
Integer Arithmetic
Natural Fractions
Finite Sets
Orders
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Inverse Elements
Natural Equations
Equivalent Sets
Integer Sums
Identity Elements
Natural Arithmetic
Element Functions
Natural Order
Function Inverses
Family Operations
Family Unions and Intersections
Integer Numbers
Natural Powers
Operations
Function Images
Function Composites
Equations
Families
Converse Relations
Equivalence Relations
Natural Products
Functions
Relations
Natural Sums
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Natural Numbers
Set Powers
Generalized Set Dualities
Successor Sets
Unordered Triples
Set Symmetric Differences
Intersection of Empty Set
Set Dualities
Partitions
Set Unions and Intersections
Ordered Pairs
Pair Unions
Set Complements
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects