Why

We want to measure the size of vectors in $\mathbf{R}^n$.[1]

Definition

The *norm* (*Euclidean norm*) of $x \in \mathbf{R}^n$ is

$$\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Notation

We denote the norm of x by $\|x\|$. In other words, $\|\cdot\| : \mathbf{R}^n \to \mathbf{R}$ is a function from vectors to real numbers. The notation follows the notation of absolute value, the *magnitude* of a real number, and the double verticals remind us that x is a vector. A warning: some authors write $|x|$ for the norm of x when it is understood that $x \in \mathbf{R}^n$.

We understand the norm of x by comparison with the distance function $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. On one hand, the norm of x is $d(x, 0)$. So $\|x\|$ measures the length of the vector x from the origin 0. On the other hand, $d(x, y) = \|x - y\|$. So $\|x - y\|$ measures the distance between x and y.

Properties

The norm has several important properties

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[1]Future editions may expand. They will certainly include images of the plane.

1. $\|\alpha x\| = |\alpha| \|x\|$, called *(absolute) homogeneity*,

2. $\|x + y\| \le \|x\| + \|y\|$, called the *triangle inequality*,

3. $\|x\| \ge 0$, called *non-negativity*, and

4. $\|x\| = 0 \longleftrightarrow x = 0$, called *definiteness*.

Visualization

Real Norm

N-Dimensional Space

Absolute Value

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Line   Real Plane   Real Space

Integral Line   Real Order

Rational Order   Real Numbers   Geometry

Rational Numbers

Integer Arithmetic

Integer Products   Natural Multiplicative Identity   Natural Additive Identity   Sequences

Arithmetic   Identity Elements   Set Numbers

Natural Powers   Operations   Finite Sets   Direct Products

Equivalent Sets

Integer Sums   Natural Products   Integer Order   Comparisons   Function Inverses   Family Unions and Intersections

Integer Numbers   Natural Sums   Orders   Function Composites   Function Images   Families

Equivalence Relations   Recursion Theorem   Natural Order   Converse Relations   Functions

Peano Axioms   Relations

Natural Induction   Set Products

Natural Numbers   Set Powers   Generalized Set Dualities

Partitions   Successor Sets   Intersection of Empty Set   Unordered Triples   Set Dualities   Set Unions and Intersections

Set Intersections   Pair Unions   Ordered Pairs   Set Complements

Set Differences   Set Unions   Pair Intersections

Unordered Pairs   Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects