Iterated Integrals

1 Why

We integrate over a product space by integrating one coordinate at a time. TODO: WHY?

2 Result

Proposition 1. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measurable spaces. Let $f : X \times Y \to [-\infty, \infty]$ be $\mathcal{A} \times \mathcal{B}$-measurable and $\mu \times \nu$-integrable. Then*

1. *For μ-almost every x in X the section f_x is ν-integrable and for ν-almost every y in Y the section f^y is μ-integrable,*

2. *the functions I_f and J_f defined by*

$$I_f(x) = \begin{cases} \int_Y f_x d\nu & \text{if } f_x \text{ is } \nu\text{-integrable,} \\ 0 & \text{otherwise} \end{cases}$$

and

$$J_f(y) = \begin{cases} \int_X f^y d\mu & \text{if } f^y \text{ is } \mu\text{-integrable,} \\ 0 & \text{otherwise} \end{cases}$$

belong to $\mathcal{L}(X, \mathcal{A}, \mu, R)$ and $\mathcal{L}(Y, \mathcal{B}, \nu, R)$ respectively, and

3. *the relation*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X I_f d\mu = \int_Y J_f d\nu$$

holds.

The above is called *Fubini's Theorem*