Why

We want to model uncertain outcomes in dynamical systems.[1]

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_{T-1}$ be sets. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Let $\mathcal{W}_0, \dots, \mathcal{W}_T$. Let $w_t : \Omega \to \mathcal{W}_t$ for $t = 0, \dots, T$ be random variables. For $t = 0, \dots, T-1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathcal{X}_{t+1}$.

We call the sequence

$$\mathcal{D} = ((\mathcal{X}_t)_{t=0}^{T}), (\mathcal{U}_t)_{t=0}^{T-1}, (w_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1})$$

a *stochastic discrete-time dynamical system*. We call w_t the *noise* variables.

Problem

Let $x_0 : \Omega \to \mathcal{X}_0$ be a random variable. Define $x_1 : \Omega \to \mathcal{X}_1$, $\dots$, $x_T : \Omega \to \mathcal{X}_t$ by

$$x_{t+1} = f_t(x_t, u_t, w_t)$$

for $t = 0, \dots, T-1$. Roughly speaking, the state transition functions are nondeterministic. In other words, it is uncertain which state we will arrive in given our current state and action. The choice u_t only determines the distribution of x_{t+1}. Here x_0 is (still) called the *initial state* and is a random variable, usually assumed independent of the w_t.

Let $g_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathbf{R} \cup \{\infty\}$ for $t = 0, \dots, T-1$ and $g_T : \mathcal{X}_T \times \mathcal{W}_T \to \mathbf{R} \cup \{\infty\}$. We call $(x_0, \mathcal{D}, (g_t)_{t=0}^{T})$ a *stochastic dynamic*

[1] Future editions will expand.

optimization problem. As with dynamic optimization problems, we call g_t the *stage cost function* and g_T the *terminal cost function.* It is common for these to not depend on w_T (in other words, to be deterministic). It is also common for these to take infinite values to encode constraints.

As before, a stochastic dynamic optimization problem is just an optimization problem. Define $U = \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_{T-1}$ and let $u \in U$. Define $C : \Omega \to \mathbf{R}$ by

$$C = \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T).$$

We call C the *total cost* for actions u. It is a random variable.

Define $J : U \to \mathbf{R} \cup \{\infty\}$ by

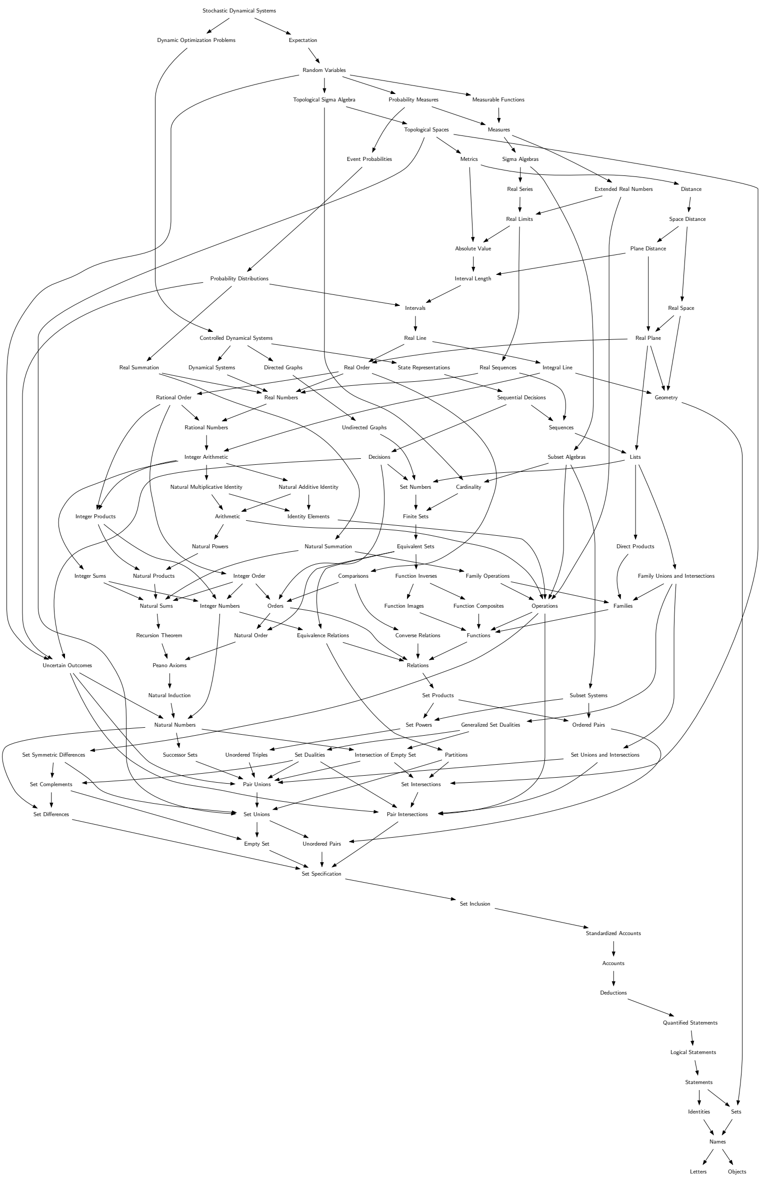
$$J = \mathbf{E}\left(\sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T) \right).$$

$J(u)$ is the *expected total cost* for inputs u.

The optimization problem is (U, J). In other words, the objective is the mean total stage cost plus the terminal cost.

Other terminology

Stochastic dynamic optimization problems are frequently called *stochastic control problems.*


Stochastic Dynamical Systems
Dynamic Optimization Problems
Expectation
Random Variables
Topological Sigma Algebra
Probability Measures
Measurable Functions
Topological Spaces
Measures
Event Probabilities
Metrics
Sigma Algebras
Real Series
Extended Real Numbers
Distance
Real Limits
Space Distance
Absolute Value
Plane Distance
Probability Distributions
Interval Length
Intervals
Real Space
Controlled Dynamical Systems
Real Line
Real Plane
Real Summation
Dynamical Systems
Directed Graphs
Real Order
State Representations
Real Sequences
Integral Line
Rational Order
Real Numbers
Sequential Decisions
Geometry
Rational Numbers
Undirected Graphs
Sequences
Integer Arithmetic
Decisions
Subset Algebras
Lists
Natural Multiplicative Identity
Natural Additive Identity
Set Numbers
Cardinality
Integer Products
Arithmetic
Identity Elements
Finite Sets
Natural Powers
Natural Summation
Equivalent Sets
Direct Products
Integer Sums
Natural Products
Integer Order
Comparisons
Function Inverses
Family Operations
Family Unions and Intersections
Natural Sums
Integer Numbers
Orders
Function Images
Function Composites
Operations
Families
Recursion Theorem
Natural Order
Equivalence Relations
Converse Relations
Functions
Uncertain Outcomes
Peano Axioms
Relations
Natural Induction
Set Products
Subset Systems
Natural Numbers
Set Powers
Generalized Set Dualities
Ordered Pairs
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects