Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Given a distribution $p : \Omega \to \mathbf{R}$, the *probability of an event* $A \subset \Omega$ is $\sum_{a \in A} p(a)$, the sum of probabilities of its outcomes.

Notation

Define define $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ by

$$\mathbf{P}(A) = \textstyle\sum_{a \in A} p(a).$$

We call $\mathbf{P}$ the *event probability function* (or *probability measure*) of (or induced by) p.

Example: die

Define $p : \{1, \ldots, 6\} \to \mathbf{R}$ by $p(\omega) = 1/6$ for $\omega = 1, \ldots, 6$. Define the event $E = \{2, 4, 6\}$. Then

$$\mathbf{P}(E) = \textstyle\sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Basic properties

Notice that for all $A \subset \Omega$, (i) $\mathbf{P}(A) \geq 0$. In particular, (ii) $\mathbf{P}(\Omega) = 1$ (and $\mathbf{P}(\varnothing) = 0$). For all $A, B \subset \Omega$, $\mathbf{P}(A \cup B) =$

$\mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. In particular, if $A \cap B = \varnothing$, (iii) $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

Conversely, suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satifies (i), (ii), (iii). These three conditions are sometimes called the *axioms of probability* (for finite sets). Define $p : \Omega \to \mathbf{R}$ by

$$p(\omega) = f(\{\omega\}).$$

In case f satisfies the axioms, p is a probability distribution (nonnegative and sums to one). For this reason we call f satisfying (i)-(iii) an *event probability function* (or *probability measure*). In the case that we think of a probability event function $\mathbf{P}$ as induced by a distribution p, we write $\mathbf{P}_p$.

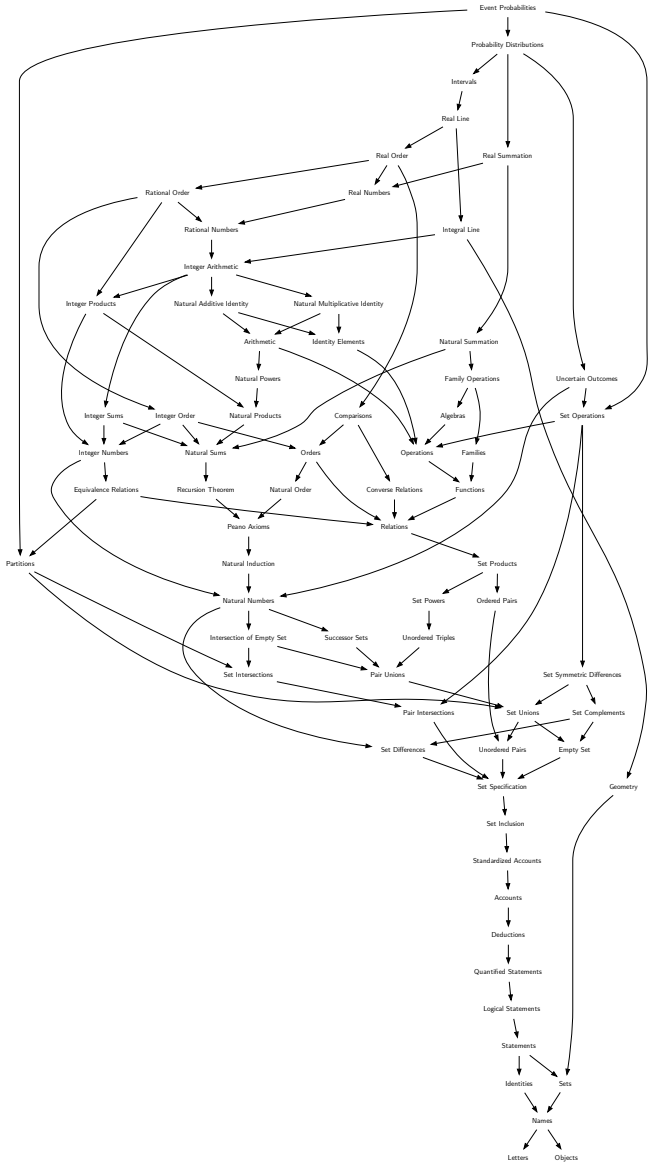
We conclude that p and $\mathbf{P}$ are two perspectives. We can think of elementary events (outcomes) and define their probabilities individually in a way that they sum to one and are nonnegative. Or we can think of the compound events, and define their probabilities in a way consistent with (i)-(iii).

Probability by cases

Let $\mathbf{P}$ be a probability event function. Suppose $A_1, \ldots, A_n$ partition Ω. Then for an $B \subset \Omega$,

$$\mathbf{P}(B) = \textstyle\sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

Some authors call this the *law of total probability*.


Event Probabilities
Probability Distributions
Intervals
Real Line
Real Order
Real Summation
Rational Order
Real Numbers
Rational Numbers
Integral Line
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Arithmetic
Identity Elements
Natural Summation
Natural Powers
Family Operations
Uncertain Outcomes
Integer Sums
Integer Order
Natural Products
Comparisons
Algebras
Set Operations
Integer Numbers
Natural Sums
Orders
Operations
Families
Equivalence Relations
Recursion Theorem
Natural Order
Converse Relations
Functions
Peano Axioms
Relations
Partitions
Natural Induction
Set Products
Natural Numbers
Set Powers
Ordered Pairs
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Intersections
Pair Unions
Set Symmetric Differences
Pair Intersections
Set Unions
Set Complements
Set Differences
Unordered Pairs
Empty Set
Set Specification
Geometry
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects