



Why

We want to capture the useful properties of the standard basis vectors.

Definition

A set of vectors $\{v_1, \dots, v_k\}$ is *independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0.$$

Notice that independence is a property of a set of vectors, not of any vector in particular. Another way of saying this is that no vector can be represented as a linear combination of another.

Unique representation

Suppose v_1, \dots, v_k are independent and we have

$$x = \sum_{i=1}^k \alpha_i v_i \quad \text{and} \quad x = \sum_{i=1}^k \beta_i v_i.$$

Then

$$0 = x - x = \sum_{i=1}^n (\alpha_i - \beta_i) v_k.$$

Using the definition of independence, we conclude $\alpha_i - \beta_i = 0$ for $i = 1, \dots, k$. Consequently, $\alpha_i = \beta_i$. In other words, if x can be represented as a linear combination of the vectors v_1, \dots, v_k , that representation is *unique*. We have shown that independence implies uniqueness? What of the converse?

We show that lack of independence gives a lack of uniqueness. Suppose there exists $\alpha_1, \dots, \alpha_k$, not all zero, so that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0.$$

In particular, suppose $\alpha_i \neq 0$. Then we have

$$v_i = (1/\alpha_i) \sum_{j \neq i} \alpha_j v_j.$$

Suppose x can be written as a linear combination of v_1, \dots, v_k . In other words, there are β_1, \dots, β_k so that

$$x = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

