Why

Can we represent the function associating a linear system with its row reduction by matrix multiplication?

Main observation

The following proposition affirmatively answers the question.

Proposition 1. *Let $(A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$ and (C, d) the kth reduction of (A, b). Then there exists a matrix $L \in \mathbf{R}^{m \times m}$ so that $C = LA$ and $d = Lb$.*

Proof. Define $L \in \mathbf{R}^{m \times m}$ by $L_{st} = 1$ if $s = t$, $-A_{sj}/A_{ij}$ if $k < s \leq m$ and zero otherwise. $\square$

For this reason, we call L in Proposition 1 a *row reducer matrix* or *row reducing matrix* or *row reducer*. The row reducer matrix for the kth reduction of (A, b) has the form

$$
L_k = \begin{bmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & A_{ik}/A_{kk} & 1 & & & \\
& & \vdots & & \ddots & & \\
& & A_{mk}/A_{kk} & & & 1 &
\end{bmatrix}
$$

So the following is immediate

Prop. 2. *Row reducing matrices are unit lower triangular.*

Example

For example, the $(1,1)$-reduction of (A, b) in which

$$
A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 6 & 7 & 9 & 5 \\ 8 & 7 & 9 & 8 \end{bmatrix} \quad \text{and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.
$$

is the linear system

$$
A' = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and } b' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.
$$

The row reducer is $L \in R^{4 \times 4}$ defined by

$$
L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.
$$

One can check that $A' = LA$ and $b' = Lb$, and clearly L is unit lower triangular.

Row Reducer Matrices

Triangular Matrices

Linear System Row Reductions

Diagonal Matrices

Matrices

Linear Equation Solutions

Real Matrices

Vectors

Linear Equations

Real Vectors

Arrays

Linear Functions

Plane Vector Sums and Differences

Space Vectors

N-Dimensional Space

Plane Vector

Distance

Space Distance

Plane Distance

Interval Length

Real Space

Intervals

Real Plane

Real Line

Integral Line

Real Order

Real Summation

Geometry

Fields

Real Numbers

Rational Order

Lists

Rings

Rational Numbers

Integer Arithmetic

Set Numbers

Natural Multiplicative Identity

Natural Additive Identity

Integer Products

Direct Products

Finite Sets

Identity Elements

Arithmetic

Natural Summation

Family Unions and Intersections

Family Operations

Function Inverses

Equivalent Sets

Natural Powers

Comparisons

Integer Sums

Integer Order

Natural Products

Natural Equations

Families

Operations

Function Composites

Function Images

Integer Numbers

Orders

Natural Sums

Equations

Functions

Equivalence Relations

Converse Relations

Natural Order

Recursion Theorem

Relations

Peano Axioms

Set Products

Natural Induction

Generalized Set Dualities

Set Powers

Natural Numbers

Set Unions and Intersections

Set Symmetric Differences

Set Dualities

Unordered Triples

Successor Sets

Intersection of Empty Set

Partitions

Set Complements

Pair Unions

Set Intersections

Ordered Pairs

Pair Intersections

Set Unions

Set Differences

Empty Set

Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects