



## Why

We want to model uncertain outcomes in dynamical systems.<sup>1</sup>

## Definition

Let  $\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_T$  and  $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_{T-1}$  be sets. Let  $(\Omega, \mathcal{A}, \mathbf{P})$  be a probability space. Let  $\mathcal{W}_0, \dots, \mathcal{W}_T$ . Let  $w_t : \Omega \rightarrow \mathcal{W}_t$  for  $t = 0, \dots, T$  be random variables. For  $t = 0, \dots, T - 1$ , let  $f_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \rightarrow \mathcal{X}_{t+1}$ .

We call the sequence

$$\mathcal{D} = ((\mathcal{X}_t)_{t=0}^T, (\mathcal{U}_t)_{t=0}^{T-1}, (w_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1})$$

a *stochastic discrete-time dynamical system*. We call  $w_t$  the *noise* variables.

## Problem

Let  $x_0 : \Omega \rightarrow \mathcal{X}_0$  be a random variable. Define  $x_1 : \Omega \rightarrow \mathcal{X}_1, \dots, x_T : \Omega \rightarrow \mathcal{X}_T$  by

$$x_{t+1} = f_t(x_t, u_t, w_t)$$

for  $t = 0, \dots, T - 1$ . Roughly speaking, the state transition functions are nondeterministic. In other words, it is uncertain which state we will arrive in given our current state and action. The choice  $u_t$  only determines the distribution of  $x_{t+1}$ . Here

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<sup>1</sup>Future editions will expand.

$x_0$  is (still) called the *initial state* and is a random variable, usually assumed independent of the  $w_t$ .

Let  $g_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \rightarrow \mathbf{R} \cup \{\infty\}$  for  $t = 0, \dots, T-1$  and  $g_T : \mathcal{X}_T \times \mathcal{W}_T \rightarrow \mathbf{R} \cup \{\infty\}$ . We call  $(x_0, \mathcal{D}, (g_t)_{t=0}^T)$  a *stochastic dynamic optimization problem*. As with dynamic optimization problems, we call  $g_t$  the *stage cost function* and  $g_T$  the *terminal cost function*. It is common for these to not depend on  $w_T$  (in other words, to be deterministic). It is also common for these to take infinite values to encode constraints.

As before, a stochastic dynamic optimization problem is just an optimization problem. Define  $U = \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_{T-1}$  and let  $u \in U$ . Define  $C : \Omega \rightarrow \mathbf{R}$  by

$$C = \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T).$$

We call  $C$  the *total cost* for actions  $u$ . It is a random variable.

Define  $J : U \rightarrow \mathbf{R} \cup \{\infty\}$  by

$$J = \mathbf{E} \left( \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T) \right).$$

$J(u)$  is the *expected total cost* for inputs  $u$ .

The optimization problem is  $(U, J)$ . In other words, the objective is the mean total stage cost plus the terminal cost.

### Other terminology

Stochastic dynamic optimization problems are frequently called *stochastic control problems*.

