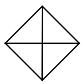


Definition

We want to estimate a random variable $x : \Omega \to \mathbf{R}^n$ from a random variable $y : \Omega \to \mathbf{R}^n$ using an estimator $\phi : \mathbf{R}^m \to \mathbf{R}^n$ which is affine.[1] In other words, $\phi(\xi) = A\xi + b$ for some $A \in \mathbf{R}^{n\times m}$ and $b \in \mathbf{R}^n$. We will use the mean squared error cost.

We want to find A and b to minimize

$$\mathbf{E}\|Ax + b - y\|^2.$$

Proof. Express $\mathbf{E}(\|Ax + b - y\|^2)$ as $\mathbf{E}((Ax + b - y)^\top (Ax + b - y))$

$$\begin{array}{cccccc}
+ & \operatorname{tr}(A\mathbf{E}(xx^\top)A^\top) & + & \mathbf{E}(x)^\top A^\top b & - & \operatorname{tr}(A^\top \mathbf{E}(yx^\top)) \\
+ & b^\top A\mathbf{E}(x) & + & b^\top b & - & b^\top \mathbf{E}(y) \\
- & \operatorname{tr}(A\mathbf{E}(xy^\top)) & - & \mathbf{E}(y)^\top b & + & \mathbf{E}(yy^\top)
\end{array}$$

The gradients with respect to b are

$$\begin{array}{cccccc}
+ & 0 & + & A\mathbf{E}(x) & - & 0 \\
+ & A\mathbf{E}(x) & + & 2b & - & \mathbf{E}(y) \\
- & 0 & - & \mathbf{E}(y) & + & 0
\end{array}$$

so $2A\mathbf{E}(x) + 2b - 2\mathbf{E}(y)$. The gradients with respect to A are

$$\begin{array}{cccccc}
+ & \mathbf{E}(xx^\top)A^\top + \mathbf{E}(xx^\top)^\top A^\top & + & \mathbf{E}(x)b^\top & - & \mathbf{E}(yx^\top)^\top \\
+ & \mathbf{E}(x)b^\top & + & 0 & - & 0 \\
- & \mathbf{E}(xy^\top) & - & 0 & + & 0
\end{array}$$

so $2\mathbf{E}(xx^\top)A^\top + 2\mathbf{E}(x)b^\top - 2\mathbf{E}(xy^\top)$. We want A and b solutions to

$$A\mathbf{E}(x) + b - \mathbf{E}(y) = 0$$
$$\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)b^\top - \mathbf{E}(xy^\top) = 0$$

so first get $b = \mathbf{E}(y) - A\mathbf{E}(x)$. Then express

$$\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)(\mathbf{E}(y) - A\mathbf{E}(x))^\top - \mathbf{E}(xy^\top) = 0.$$
$$\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)\mathbf{E}(y)^\top - \mathbf{E}(x)\mathbf{E}(x)^\top A^\top - \mathbf{E}(xy^\top) = 0.$$
$$(\mathbf{E}(xx^\top) - \mathbf{E}(x)\mathbf{E}(x)^\top)A^\top = \mathbf{E}(xy^\top) - \mathbf{E}(x)\mathbf{E}(y)^\top.$$
$$\operatorname{cov}(x,x)A^\top = \operatorname{cov}(x,y).$$

[1] Actually, the development flips this. Future editions will correct.

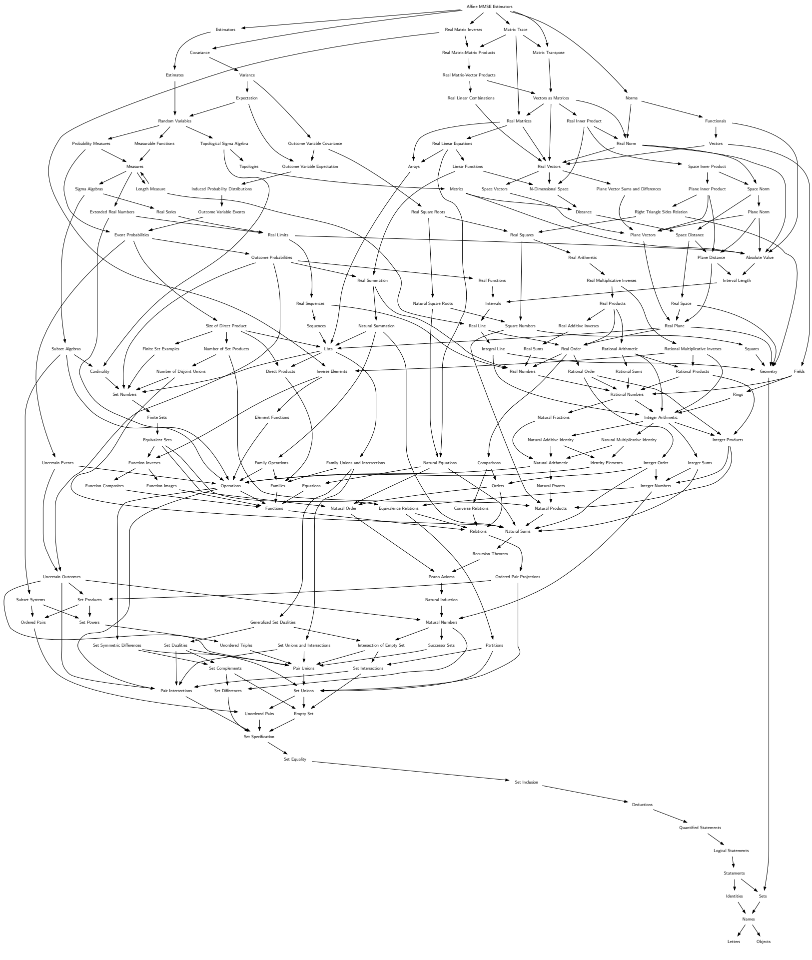
So $A^\top = (\operatorname{cov}(x,x))^{-1}\operatorname{cov}(x,y)$ means $A = \operatorname{cov}(y,x)(\operatorname{cov}(x,x))^{-1}$ is a solution. Then $b = \mathbf{E}(y) - \operatorname{cov}(y,x)\operatorname{cov}(x,x)^{-1}\mathbf{E}(x)$. So to summarize, the estimator $\phi(x) = Ax + b$ is

$$\operatorname{cov}(y,x)(\operatorname{cov} x,x)^{-1}x + \mathbf{E}(y) - \operatorname{cov}(y,x)(\operatorname{cov}(x,x))^{-1}\mathbf{E}(x)$$

or

$$\mathbf{E}(y) + \operatorname{cov}(y,x)(\operatorname{cov} x,x)^{-1}(x - \mathbf{E}(x))$$

$\square$


Affine MMSE Estimators
Estimators
Real Matrix Inverses
Matrix Trace
Covariance
Real Matrix-Matrix Products
Matrix Transpose
Estimates
Variance
Real Matrix-Vector Products
Expectation
Real Linear Combinations
Vectors as Matrices
Norms
Random Variables
Real Matrices
Real Inner Product
Functionals
Probability Measures
Measurable Functions
Topological Sigma Algebra
Outcome Variable Covariance
Real Linear Equations
Real Norm
Vectors
Measures
Topologies
Outcome Variable Expectation
Arrays
Linear Functions
Real Vectors
Space Inner Product
Sigma Algebras
Length Measure
Induced Probability Distributions
Metrics
Space Vectors
N-Dimensional Space
Plane Vector Sums and Differences
Plane Inner Product
Space Norm
Extended Real Numbers
Real Series
Outcome Variable Events
Real Square Roots
Distance
Right Triangle Sides Relation
Plane Norm
Event Probabilities
Real Limits
Real Squares
Plane Vectors
Space Distance
Outcome Probabilities
Real Arithmetic
Plane Distance
Absolute Value
Real Summation
Real Functions
Real Multiplicative Inverses
Interval Length
Real Sequences
Natural Square Roots
Intervals
Real Products
Real Space
Size of Direct Product
Sequences
Natural Summation
Real Line
Square Numbers
Real Additive Inverses
Real Plane
Subset Algebras
Finite Set Examples
Number of Set Products
Lists
Integral Line
Real Sums
Real Order
Rational Arithmetic
Rational Multiplicative Inverses
Squares
Cardinality
Number of Disjoint Unions
Direct Products
Inverse Elements
Real Numbers
Rational Order
Rational Sums
Rational Products
Geometry
Fields
Set Numbers
Rational Numbers
Rings
Finite Sets
Element Functions
Natural Fractions
Integer Arithmetic
Equivalent Sets
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Uncertain Events
Function Inverses
Family Operations
Family Unions and Intersections
Natural Equations
Comparisons
Natural Arithmetic
Identity Elements
Integer Order
Integer Sums
Function Composites
Function Images
Operations
Families
Equations
Orders
Natural Powers
Integer Numbers
Functions
Natural Order
Equivalence Relations
Converse Relations
Natural Products
Relations
Natural Sums
Recursion Theorem
Uncertain Outcomes
Peano Axioms
Ordered Pair Projections
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Set Powers
Generalized Set Dualities
Natural Numbers
Set Symmetric Differences
Set Dualities
Unordered Triples
Set Unions and Intersections
Intersection of Empty Set
Successor Sets
Partitions
Set Complements
Pair Unions
Set Intersections
Pair Intersections
Set Differences
Set Unions
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects