



## Why

A simple (albeit indirect) way to obtain a sigma algebra, is to start with some sets, and then to add all the sets needed to make the starting set closed under the various operations.

## Definition

The *generated sigma algebra* for a set of subsets is the smallest sigma algebra containing the set of subsets. We must prove the existence and uniqueness of this sigma algebra.

**Proposition 1.** *The intersection of a non-empty set of sigma algebras over the same set is a sigma algebra.*

*Proof.* Given a family of sigma algebras  $\{(A, \mathcal{A}_\alpha)\}_{\alpha \in I}$  over some set, define  $\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ .

1. For all  $\alpha \in I$ ,  $A \in \mathcal{A}_\alpha$ , thus  $A \in \mathcal{A}$ ; condition (a).
2. For all  $B \in \mathcal{A}$ , for all  $\alpha \in I$ ,  $B \in \mathcal{A}_\alpha$ . Thus, for all  $\alpha \in I$ ,  $C_A(B) \in \mathcal{A}_\alpha$ . And so  $C_A(B) \in \mathcal{A}$ ; condition (b).
3. For all sequences  $\{B_n\} \subset \mathcal{A}$ ,  $\{B_n\} \subset \mathcal{A}_\alpha$  for all  $\alpha$ . Thus  $\bigcup_n B_n \in \mathcal{A}_\alpha$  for all  $\alpha$  and so  $\bigcup_n B_n \in \mathcal{A}$ ; condition (c).

□

On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

**Proposition 2.** *If  $A$  is a set and  $\mathcal{A} \subset 2^A$ , then there is a unique smallest sigma algebra containing  $\mathcal{A}$ .*

*Proof.* We know of one sigma algebra containing  $\mathcal{A}$ : the power set of  $A$ . Thus, the set of sigma algebras containing  $\mathcal{A}$  is not empty. Proposition ?? implies the intersection of all such sigma algebras (containing

$\mathcal{A}$ ) is a sigma algebra. The intersection contains  $\mathcal{A}$ , and is contained in all other sigma algebras with this property, so is a smallest sigma algebra containing  $\mathcal{A}$ . If  $\mathcal{B}, \mathcal{C}$  were two smallest sigma algebras, then  $\mathcal{B} \subset \mathcal{C}$  and  $\mathcal{C} \subset \mathcal{B}$ , but then  $\mathcal{B} = \mathcal{C}$ ; thus the smallest sigma algebra is unique.  $\square$

### **Notation**

Let  $A$  be a set and  $\mathcal{A} \subset \mathcal{P}(A)$ . We denote the sigma algebra generated by  $\mathcal{A}$  by  $\sigma(\mathcal{A})$ .

