Why

Are all signed measures the difference of two positive measures?

Suppose we could partition the base set into two sets, one containing all the sets with positive measure and one containing all sets with negative measure. We could restrict the measure to the former and it would be positive, and we could restrict it to the latter and it would be negative.

Any measurable set could be partitioned into a piece in the former and a piece in the latter, and so its signed measure could be written as a sum of measures of these pieces.

Definition

By "positive" and "negative" we mean "non-negative" and "non-positive." Let $(X, \mathcal{A})$ be a measurable space. Let $\mu : \mathcal{A} \to [-\infty, \infty]$ be a signed measure.

A *positive set* is a measurable set with the property that each of its subsets have non-negative measure under μ. A *negative set* is a measurable set with the property that each of its subsets have non-positive measure under μ.

A *signed-set decomposition* of X under μ is a partition of X into a positive and a negative set. Some authors call it a *Hahn decomposition*.

Notation

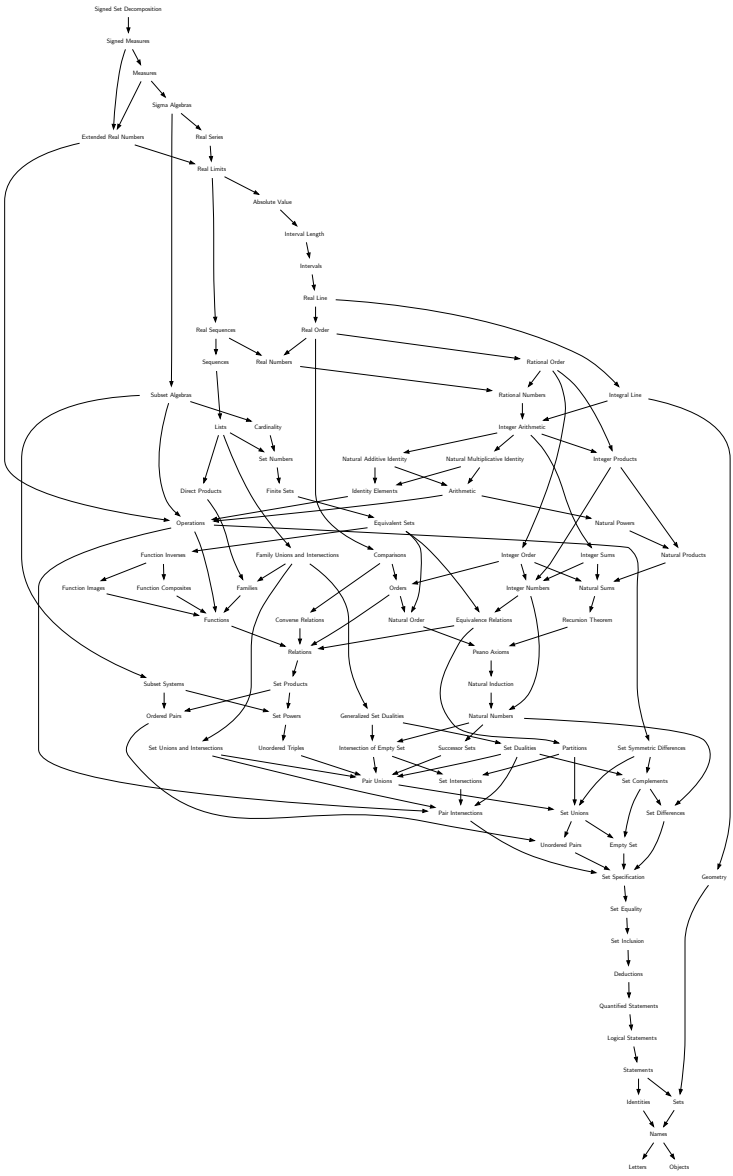
Denote by P a positive and by N a negative set. When we say "let (P, N) be a signed-set decomposition of X under μ", we mean that P is the positive set and N is the negative set.

Motivating implication

Does such a decomposition always exist? Is it unique? We are motivated to find answers by the following observation.

Suppose there was a signed-set decomposition of $(X, \mathcal{A})$ under μ; denote it by (P, N). Then $\mu(A \cap P) \geq 0$ and $\mu(A \cap N) \leq 0$ for all $A \in \mathcal{A}$.

Define $\mu_1 = \mu(A \cap P)$ and $\mu_2 = -\mu(A \cap P)$, then μ_1 and μ_2 are finite measures. Moreover, $\mu = \mu_1 - \mu_2$. Thus, if we had a signed-set decomposition we could write μ as the difference of two measures.


Signed Set Decomposition
Signed Measures
Measures
Sigma Algebras
Extended Real Numbers
Real Series
Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Sequences
Real Order
Sequences
Real Numbers
Rational Order
Subset Algebras
Rational Numbers
Integral Line
Lists
Cardinality
Integer Arithmetic
Set Numbers
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Direct Products
Finite Sets
Identity Elements
Arithmetic
Operations
Equivalent Sets
Natural Powers
Function Inverses
Family Unions and Intersections
Comparisons
Integer Order
Integer Sums
Natural Products
Function Images
Function Composites
Families
Orders
Integer Numbers
Natural Sums
Functions
Converse Relations
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Set Powers
Generalized Set Dualities
Natural Numbers
Set Unions and Intersections
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Dualities
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Geometry
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
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Objects