## Why

We want to solve linear equations. Our approach is to "eliminate" variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

## Two-variable example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$3x_1 + 2x_2 = 10, \text{ and}$$

$$6x_1 + 5x_2 = 20.$$

We can list the coefficients in a two-dimensional array $A = (3, 2; 6, 5)$ and $b = (10, 20)$. We can eliminate $x_1$ from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$3x_1 + 2x_2 = 10 \text{ and}$$

$$x_2 = 0.$$

The key insight is that this system has the *same solution set*. We call the process of moving between these two systems a *row reduction*.

## Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate $x_1$ (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear system $S_1 = (A^1, b^1)$ in which

$$
A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and } b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}
$$

The trick is that, since $A'_{22} \neq 0$, we can take the same route to eliminate $x_2$, to obtain the system $S_2 = (A^2, b^2)$ in which

$$
A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \text{and } b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}
$$

Likewise for $x_3$, we obtain $S_3 = (A^3, b^3)$ in which

$$
A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and } b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.
$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one, $(A^3, b^3)$, is easy to solve. We solve it by *back substitution*. First, since $2x_4 = 3$, we find $x_4 = 3/2$. Second, since $2x_3 + 2x_4 = -1$, we find $x_3 = -2$. Similarly we find $x_2 = 1/2$ and $x_3 = 5/4$.

**Definition**

Let $S = (A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system. The lower *row reduction* of $S$ for index $i$ with $A_{ii} \neq 0$ (or the *i*-row reduction) is the linear system $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$ if $i < s \leq m$ and $A_{st}$ otherwise. We say that the system $(A, b)$ is *ordinarily reducible*.

Let $a^k, \tilde{a}^k \in \mathbf{R}^n$ denote the $k$th row of $A$ and $\tilde{A}$, respectively. Then if $k \neq i$, $\tilde{a}^k = a^k - \alpha_k a^i$ where $\alpha_k = A_{kj}/A_{ij}$. In other words, a row $k$ of the matrix $\tilde{A}$ is obtained by subtracting a multiple of the $i$th row of matrix $A$ from row $k$ of matrix $A$. We are "reducing" the rows of $A$.

**Proposition 1.** *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to $(C, d)$. Then $x \in \mathbf{R}^n$ is a solution of $(A, b)$ if and only if it is a solution of $(C, d)$.*[1]

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = ( \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} ).$$

We then subtract three times row 2 from row 3 and four times

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[1] Future editions will include an account.

row 2 from row 4 to obtain

$$S_2 = \left( \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

Finally, we subtract two times row 3 from row 4 to obtain $S_4$, which we can write as

$$2x_1 + x_2 + x_3 = 1,$$
$$x_2 + x_3 + x_4 = 0,$$
$$2x_3 + 2x_4 = -1, \quad \text{and}$$
$$2x_4 = 3.$$

We can now back-substitute to find $x_4 = {}^3\!/\!_2$, $x_3 = -2$, $x_2 = {}^1\!/\!_2$ and $x_1 = {}^5\!/\!_4$. Proposition **??** says that this is the only solution of $S$, as well.

Linear System Row Reductions

Linear Equation Solutions

Linear Equations

Linear Combinations

Arrays

Vectors

N-Dimensional Space

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Space   Real Plane   Real Line

Sequences   Real Order   Integral Line   Real Summation

Direct Products   Geometry   Fields   Real Numbers

Set Numbers   Groups   Rational Numbers

Finite Sets   Integer Additive Inverse   Integer Arithmetic

Inverse Elements   Natural Multiplicative Identity   Natural Additive Identity   Integer Sums   Natural Summation

Family Unions and Intersections   Element Functions   Family Operations   Identity Elements   Equivalent Sets   Arithmetic   Natural Exponents   Integer Products   Natural Equations

Algebras   Function Inverse   Integer Numbers   Natural Order   Natural Products   Equation

Families   Operations   Function Composites   Function Images

Functions   Equivalence Relations   Natural Sums

Relations   Recursion Theorem

Cartesian Products   Peano Axioms

Natural Induction

Generalized Set Dualities   Set Powers   Natural Numbers

Set Unions and Intersections   Partitions   Intersection of Empty Set   Set Dualities   Unordered Triples   Successor Sets

Set Intersections   Pair Unions   Set Complements   Ordered Pair

Pair Intersections   Set Unions   Set Differences

Empty Set   Unordered Pairs

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets   Identities

Names

Letters   Objects