Why

We want to talk about making decisions over time.[1]

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{T-1}$ be sets. For $t = 0, \ldots,$ $T - 1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \to \mathcal{X}_{t+1}$. We call the sequence

$$\left((\mathcal{X}_t)_{t=0}^T, (\mathcal{U}_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1} \right)$$

a *deterministic discrete-time dynamical system*. We call the index t the *epoch*, the *stage* or the *period*.

Let $x_0 \in \mathcal{X}_0$. Let $u_0 \in \mathcal{U}_0, \ldots, u_{T-1} \in \mathcal{U}_{T-1}$. Define a state sequence $x_1 \in \mathcal{X}_1, \ldots, x_T \in \mathcal{X}_T$ by

$$x_{t+1} = f_t(x_t, u_t).$$

In this case we call x_0 the *initial state*. We call the x_t the *states*. We call the u_t a sequence of *inputs* (or *actions*, *decisions*, *choices*, or *controls*). We call f_t the *transition function* or *dynamics function*.

We call T the *horizon*. In the case that we have an infinite sequence of state sets, input sets, and dynamics, then we refer to a *infinite-horizon* dynamical system. To use language in contrast with this case, we refer to the dynamical system when T is finite as a *finite-horizon* dynamical system.

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[1]Future editions will modify, and may develop dynamic systems via the genetic approach by appealing to their classical use in Newtonian physics for modeling celestial mechanics.

State

The current action u_t affects future states x_s for $s > t$, but not the current or past states. The current state x_t depends on the initial state x_0 and the sequence of past actions $u_0, \ldots, u_{t-1}$. So the state is a "link" between the past and the future. Given x_t and $u_t, \ldots, u_{s-1}$, for $s > t$, we can compute x_s. In other words, the prior actions $u_0, \ldots, u_{t-1}$ are not relevant.

This nonrelevancy of prior actions and prior states simplifies the sequential decision problem (see Sequential Decisions).

Variations

The dynamical system is called *time-invariant* if $\mathcal{X}_t$, $\mathcal{U}_t$ and f_t do not depend on t. A simple variation is that $\mathcal{U}_t$ depends on x_t.[2]

Examples

Finite dynamical system

A dynamical system is finite if the state and action sets are finite. For example, $\mathcal{X} = \{1, \ldots, n\}$ and $\mathcal{U} = \{1, \ldots, m\}$. Then $f_t : \mathcal{X} \times \mathcal{U} \to \mathcal{U}$ is called a *transition map*.

Or else, let (V, E) be a directed graph, then $\mathcal{X} = V$, $\mathcal{U}_{x_t} = \{(u, v) \in E \mid u = x_t\}$ and $f_t(x_t, u_t) = v$ when $u_t = (x_t, v)$ is a dynamical system. Roughly this system models "moving" on the graph.

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[2]Future editions will say more here.

Discrete-time linear dynamical system

Let $\mathcal{X} = \mathbf{R}^n$ and $\mathcal{U} = \mathbf{R}^m$. Define $f_t : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ by

$$x_{t+1} = f_t(x_t, u_t) = A_t x_t + B_t u_t + c_t$$

for $y = 1, \ldots, T - 1$.[3]

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[3]This very special form of dynamics arises in many applications. Future editions will say more.

Dynamical Systems

Directed Graphs

N-Dimensional Space

State Representations

Undirected Graphs

Distance

Space Distance

Plane Distance

Interval Length

Intervals

Real Line

Real Plane

Real Space

Integral Line

Real Order

Geometry

Real Numbers

Rational Numbers

Sequential Decisions

Integer Arithmetic

Sequences

Decisions

Natural Multiplicative Identity

Natural Additive Identity

Set Numbers

Total Orders

Arithmetic

Identity Elements

Finite Sets

Direct Products

Integer Products

Natural Powers

Operations

Equivalent Sets

Family Unions and Intersections

Integer Sums

Natural Products

Function Inverses

Integer Numbers

Natural Sums

Function Images

Function Composites

Families

Equivalence Relations

Recursion Theorem

Natural Order

Functions

Partial Orders

Peano Axioms

Relations

Natural Induction

Cartesian Products

Uncertain Outcomes

Generalized Set Dualities

Natural Numbers

Set Powers

Partitions

Set Dualities

Intersection of Empty Set

Successor Sets

Unordered Triples

Set Unions and Intersections

Ordered Pairs

Set Complements

Set Intersections

Pair Unions

Set Differences

Set Unions

Pair Intersections

Unordered Pairs

Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets

Identities

Names

Letters

Objects