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Definition

We want to estimate a random variable $x : \Omega \rightarrow \mathbf{R}^n$ from a random variable $y : \Omega \rightarrow \mathbf{R}^n$ using an estimator $\phi : \mathbf{R}^m \rightarrow \mathbf{R}^n$ which is affine.² In other words, $\phi(\xi) = A\xi + b$ for some $A \in \mathbf{R}^{n \times m}$ and $b \in \mathbf{R}^n$. We will use the mean squared error cost.

We want to find A and b to minimize

$$\mathbf{E}\|Ax + b - y\|^2.$$

Proof. Express $\mathbf{E}(\|Ax + b - y\|^2)$ as $\mathbf{E}((Ax + b - y)^\top (Ax + b - y))$

$$\begin{aligned} &+ \operatorname{tr}(A\mathbf{E}(xx^\top)A^\top) + \mathbf{E}(x)^\top A^\top b - \operatorname{tr}(A^\top \mathbf{E}(yx^\top)) \\ &+ b^\top A\mathbf{E}(x) + b^\top b - b^\top \mathbf{E}(y) \\ &- \operatorname{tr}(A\mathbf{E}(xy^\top)) - \mathbf{E}(y)^\top b + \mathbf{E}(yy^\top) \end{aligned}$$

The gradients with respect to b are

$$\begin{aligned} &+ 0 + A\mathbf{E}(x) - 0 \\ &+ A\mathbf{E}(x) + 2b - \mathbf{E}(y) \\ &- 0 - \mathbf{E}(y) + 0 \end{aligned}$$

so $2A\mathbf{E}(x) + 2b - 2\mathbf{E}(y)$. The gradients with respect to A are

$$\begin{aligned} &+ \mathbf{E}(xx^\top)A^\top + \mathbf{E}(xx^\top)^\top A^\top + \mathbf{E}(x)b^\top - \mathbf{E}(yx^\top)^\top \\ &+ \mathbf{E}(x)b^\top + 0 - 0 \\ &- \mathbf{E}(xy^\top) - 0 + 0 \end{aligned}$$

so $2\mathbf{E}(xx^\top)A^\top + 2\mathbf{E}(x)b^\top - 2\mathbf{E}(xy^\top)$. We want A and b solutions to

$$\begin{aligned} A\mathbf{E}(x) + b - \mathbf{E}(y) &= 0 \\ \mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)b^\top - \mathbf{E}(xy^\top) &= 0 \end{aligned}$$

¹Future editions will include an account.

²Actually, the development flips this. Future editions will correct.

so first get $b = \mathbf{E}(y) - A\mathbf{E}(x)$. Then express

$$\begin{aligned}\mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)(\mathbf{E}(y) - A\mathbf{E}(x))^\top - \mathbf{E}(xy^\top) &= 0. \\ \mathbf{E}(xx^\top)A^\top + \mathbf{E}(x)\mathbf{E}(y)^\top - \mathbf{E}(x)\mathbf{E}(x)^\top A^\top - \mathbf{E}(xy^\top) &= 0. \\ (\mathbf{E}(xx^\top) - \mathbf{E}(x)\mathbf{E}(x)^\top)A^\top &= \mathbf{E}(xy^\top) - \mathbf{E}(x)\mathbf{E}(y)^\top. \\ \text{cov}(x, x)A^\top &= \text{cov}(x, y).\end{aligned}$$

So $A^\top = \text{cov}(x, x)^{-1} \text{cov}(x, y)$ means $A = \text{cov}(y, x) \text{cov}(x, x)^{-1}$ is a solution. Then $b = \mathbf{E}(y) - \text{cov}(y, x) \text{cov}(x, x)^{-1} \mathbf{E}(x)$. So to summarize, the estimator $\phi(x) = Ax + b$ is

$$\text{cov}(y, x) (\text{cov } x, x)^{-1} x + \mathbf{E}(y) - \text{cov}(y, x) \text{cov}(x, x)^{-1} \mathbf{E}(x)$$

or

$$\mathbf{E}(y) + \text{cov}(y, x) (\text{cov } x, x)^{-1} (x - \mathbf{E}(x))$$

□

