Why

We want a notion of reversing functions.

Definition

Reversing functions does not make sense if the function is not one-to-one. Let $f : X \to Y$. If x_1 goes to y and x_2 goes to y (i.e., $f(x_1) = f(x_2) = y$), then what should y go to. One answer is that we should have a function which gives all the domain values which could lead to y. This is the inverse image (see Function Images) $f^{-1}(\{y\})$. Nor does reversing funcitons make sense if f is not onto. If there does not exist $x \in X$ so that $y = f(x)$, then $f^{-1}(\{y\}) = \varnothing$.

In the case, however, that the function is one-to-one and onto, then each element of the domain corresponds to one and only one element of the codomain and vice versa. In this case, for all $y \in Y$, $f^{-1}(\{y\})$ is a singleton $\{x\}$ where $f(x) = y$. In this case, we define a fucntion $g : Y \to X$ so that $g(y) = x$ if and only if $f(x) = y$.

In general, if we have two functions, where the codomain of the first is the domain of the second, and the codomain of the second is the domain of the first, we call them *inverse functions* if the composition of the second with the first is the identity function on the first's domain and the composition of the first with the second is the identity function on the second's domain (see Functions and Function Composites).

In this case we say that the second function is an *inverse* of the second, and vice versa. When an inverse exists, it is unique,[1] so we refer to *the inverse* of a function. We call the first function *invertible*. Other names for an invertible function include *bijection*.

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[1]Future editions will prove this assertion and all unproven propositions herein.

Let A a non-empty set. We denote the identity function on A by id_A, read aloud as "identity on A." id_A maps A onto A. Let A, B be non-empty sets. Let $f : A \to B$ and $g : B \to A$ be functions. f and g are inverse functions if $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

The Inverse

Proposition 1 (Uniqueness). *Let $f : A \to B$, $g : B \to A$, and $h : B \to A$. If g and h are both inverse functions of f, then $g = h$.*

Proposition 2 (Existence). *If a function is one-to-one and onto, it has an inverse; and conversely.*[2]

ßComposites and Inveres

Let $f : X \to Y$ and $g : Y \to Z$. Then g^{-1} maps $\mathcal{P}(Z)$ to $\mathcal{P}(Y)$ and f^{-1} maps $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Then the following is immediate

Proposition 3. $(gf)^{-1} = f^{-1}g^{-1}$

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[2]A proof will appear in future editions.

Function Inverses

Function Composites   Function Images

Functions

Relations

Set Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions

Set Unions

Unordered Pairs   Empty Set

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects