Almost Everywhere

Why

We treat properties failing on a set of measure zero as though they occur everyhwere; especially in discussions of convergence.

Definition

A subset of the base set of a measure space is *negligible* if there exists a measurable set with measure zero containing the subset. Negligible sets need not be measurable.

A property holds *almost everywhere* with respect to a measure on a measure space if the set of elements of the base set on which the property does not hold is negligible.

If the property holds everywhere, it holds almost everywhere. In this sense we call the almost everywhere sense "weaker" than the everywhere sense.

Notation

Let $(X, \mathcal{A}, \mu)$ be a measure space. A set $N \subset X$ is negligible if there exists $A \in \mathcal{A}$ with $N \subset A$ and $\mu(N) = 0$.

We abbreviate almost everywhere as "a.e.," read "almost everywhere". We say that a property "holds a.e." If the measure μ is not clear from context, we say that the property holds almost everywhere $[\mu]$ or μ-a.e., read "mu almost everywhere."

Examples

Let $(X, \mathcal{A}, \mu)$ be a measure space and let R be the real numbers.

Function Comparisons

Let $f, g : X \to R$ be two functions on X, not necessarily measurable. Then $f = g$ almost everywhere if the set of points at which the functions disagree is μ-negligible. Similarly, $f \geq g$ almost everywhere if the set of points where f is less than g is μ-negligible. If f and g are $\mathcal{A}$-measurable, then the sets

$$\{x \in X \mid f(x) \neq g(x)\} \text{ and } \{x \in X \mid f(x) < g(x)\}$$

are measurable; but they need not be measurable otherwise.

Function Limits

Let $f_n : X \to R$ for each natural number n and let $f : X \to R$ be a function. The sequence $(f_n)_n$ *converges to* f *almost everywhere* if

$$\left\{x \in X \;\middle|\; \lim_n f_n \text{ does not exist, or } f(x) \neq \lim_n f_n\right\}$$

is μ-negligible. In this case, we write " $f = \lim_n f_n$ almost everywhere."