



Measure Vector Space

1 Why

If both signed measures are finite, then their difference is always well-defined. Is the difference a finite signed measure?

2 Preliminary Result

Proposition 1. *A linear combination of finite signed measures is a finite signed measure.*

Proof. Let (X, \mathcal{A}) be a measurable space. Let μ and ν be finite signed measures. Let R denote the real numbers. Then $(\alpha\mu)(\emptyset) = \alpha \cdot \mu(\emptyset) = \alpha \cdot 0 = 0$. Also for $(A_n)_n \subset \mathcal{A}$ disjoint,

$$\begin{aligned} (\alpha\mu)(\cup A_n) &= \alpha\mu(\cup A_n) = \alpha \sum_{n=1}^{\infty} \mu(A_n) \\ &= \sum_{n=1}^{\infty} \alpha\mu(A_n) = (\alpha\mu)(A_n) \end{aligned}$$

Similarly, $(\mu + \nu)(\emptyset) = \mu(\emptyset) + \nu(\emptyset) = 0$. And, for $(A_n)_n \subset \mathcal{A}$ disjoint,

$$\begin{aligned} (\mu + \nu)(\cup A_n) &= \mu(\cup A_n) + \nu(\cup A_n) = \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \nu(A_n) \\ &= \sum_{n=1}^{\infty} \mu(A_n) + \nu(A_n) = \sum_{n=1}^{\infty} (\mu + \nu)(A_n) \end{aligned}$$

□

3 Main Result

Proposition 2. *The set of finite signed measures is a vector space.*

Proof. Use the previous proposition. Observe that the function $\mu \equiv 0$ is a measure. and $\nu + \mu = \nu$ for all measures ν . \square

3.1 Notation

Denote the set of real numbers by R . We denote the vector space of signed measures on measurable space (X, \mathcal{A}) by $M(X, \mathcal{A}, R)$.