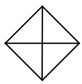

Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering $\geq$ on S^d defined by

$$A \geq B \quad \longleftrightarrow \quad A - B \geq 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_+.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \quad \longleftrightarrow \quad A - B > 0 \quad \longleftrightarrow \quad A - B \in \mathsf{S}^d_{++}.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$\begin{aligned} \alpha A \geq 0 &\quad \text{for all } \delta > 0, A \geq 0, \\ A + B \geq 0 &\quad \text{for all } A, B \geq 0, \\ A \geq B \text{ and } B \geq A \longrightarrow A = B &\quad \text{for all } A, B \in \mathsf{S}^d, \\ \lim_{n \to \infty} A_n = A \longrightarrow A \geq 0 &\quad \text{for all sequences } (A_n)_n \text{ in } \mathsf{S}^d_+. \end{aligned}$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that $\geq$ is antisymmetric. Moreover,

$$A \geq A \quad \text{for all } A \in \mathsf{S}^d, \text{ and}$$

$$A \geq B \text{ and } B \geq C \longrightarrow A \geq C \text{ for all } A, B, C \in \mathsf{S}^d.$$

In other words, $\geq$ is also reflexive and transitive. In other words, $\geq$ is a partial order (see Orders).[1]

[1] Future editions will include more formal accounts.

For $d = 1$, $\geq$ reduces to the familiar total order of the real line (see Real Order). The converse perspective is to see the positive semidefinite order as an extension of the order on $\mathbf{R}$ to the space $\mathbf{S}^d$. Of course, the key difference is that two matrices may not be comparable. The order is *partial.*

For example, the matrices $A, B \in \mathbf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.


Real Positive Semidefinite Matrix Order
Real Positive Semidefinite Matrix Cone
Real Convex Cones
Real Convex Sets
Topological Interiors
Real Halfspaces
Real Hyperplanes
Subspace Orthogonal Complements
Subspace Dimensions
Vector Space Dimensions
Real Positive Semidefinite Matrices
Quadratic Forms
Affine Set Dimensions
Vector Space Bases
Real Matrix Space
Matrix Trace
Affinely Independent Vectors
Span
Orthogonal Complements
Linearly Dependent Vectors
Symmetric Matrices
Real Matrix-Matrix Products
Real Affine Hulls
Real Affine Sets and Subspaces
Real Subspace Dimensions
Vector Subspaces
Symmetric Real Matrices
Matrix Transpose
Real Matrix-Vector Products
Cones
Real Affine Sets
Real Vector Bases
Real Subspaces
Linear Combinations
Real Inner Products
Vectors as Matrices
N-Dimensional Line Segments
Topological Closures
N-Dimensional Half-Lines
Real Affine Combinations
Independent Set of Real Vectors
Real Vectors Span
Vectors
Real Inner Product
Real Matrices
Real Cones
Topological Neighborhoods
N-Dimensional Lines
Real Translates
Real Linear Combinations
Space Inner Product
Real Norm
Multivariate Real Polynomials
Real Linear Equations
Real Set Closures
Topologies
Space Norm
Real Vectors
Multivariate Functions
Linear Functions
Arrays
Real Limits
Metrics
N-Dimensional Space
Plane Inner Product
Plane Norm
Plane Vector Sums and Differences
Space Vectors
Distance
Absolute Value
Right Triangle Sides Relation
Plane Vectors
Space Distance
Real Squares
Real Polynomials
Plane Distance
Real Arithmetic
Interval Length
Real Multiplicative Inverses
Intervals
Real Space
Real Products
Real Line
Square Numbers
Real Plane
Real Additive Inverses
Real Sequences
Integral Line
Squares
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Real Summation
Fields
Sequences
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Natural Summation
Rings
Rational Numbers
Lists
Integer Arithmetic
Natural Fractions
Set Numbers
Direct Products
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Natural Arithmetic
Inverse Elements
Equivalent Sets
Natural Powers
Integer Order
Integer Sums
Element Functions
Comparisons
Natural Equations
Function Inverses
Family Operations
Family Unions and Intersections
Natural Products
Integer Numbers
Operations
Orders
Equations
Function Composites
Function Images
Families
Natural Sums
Converse Relations
Natural Order
Equivalence Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Set Partitions
Set Unions and Intersections
Set Complements
Ordered Pairs
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects