Why

How should we modify probabilities, given that we know some aspect of the outcome (i.e., that some event has occurred)?

Definition

Suppose Ω is a set of outcomes and $\mathbf{P} : \mathcal{P}(\Omega) \to \mathbf{R}$ is a finite probability measure. Suppose $A, B \subset \Omega$ with $\mathbf{P}(B) \neq 0$. The *conditional probability of A given B* is the ratio of the probability of $A \cap B$ to the probability of B.

Notation

In a slightly slippery but universally standard notation, we denote the conditional probability of A given B by $\mathbf{P}(A \mid B)$. In other words, we define

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

for all $A, B \subset \Omega$, whenever $\mathbf{P}(B) \neq 0$.

For example, we can express the law of total probability (see Event Probabilities) as

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{P}(B \mid A_i),$$

where $A_1, \ldots, A_n$ partition Ω and $B \subset \Omega$ with $\mathbf{P}(B) > 0$.

Conditional probability measure

It happens that $\mathbf{P}(\cdot \mid B)$ is itself a probability measure on Ω. We therefore refer to $\mathbf{P}(\cdot \mid B)$ as a *conditional probability measure*.

To see this, suppose $B \subset \Omega$ and $\mathbf{P}(B) > 0$. Then (i) $\mathbf{P}(A \mid B) \geq 0$ for all $A \subset \Omega$, since $\mathbf{P}(A \cap B) \geq 0$. Moreover, (ii)

$$\mathbf{P}(\Omega \mid B) = \mathbf{P}(\Omega \cap B)/\mathbf{P}(B) = \mathbf{P}(B)/\mathbf{P}(B) = 1$$

Similarly, $\mathbf{P}(\varnothing \mid B) = \mathbf{P}(\varnothing \cap B)/\mathbf{P}(B) = 0/\mathbf{P}(B) = 0$.

Finally, (iii) if $A \cap C = \varnothing$, then

$$
\begin{aligned}
\mathbf{P}(A \cap C \mid B) &= \mathbf{P}((A \cap C) \cap B)/\mathbf{P}(B) \\
&= \mathbf{P}((A \cap B) \cap (C \cap B))/\mathbf{P}(B) \\
&\stackrel{(a)}{=} (\mathbf{P}(A \cap B) + \mathbf{P}(C \cap B))/\mathbf{P}(B) \\
&= \mathbf{P}(A \cap B)/\mathbf{P}(B) + P(C \cap B)/\mathbf{P}(B) \\
&= \mathbf{P}(A \mid B) + P(C \mid B).
\end{aligned}
$$

where (a) follows since $A \cap B$ and $C \cap B$ are disjoint.

Induced conditional distribution

Therefore, we expect there to also correspond a new distribution on the set of outcomes. For $\mathbf{P}_p$, define $q : \Omega \to \mathbf{R}$ by

$$
q(\omega) = \begin{cases} \frac{p(\omega)}{\mathbf{P}(B)} & \text{if } \omega \in B \\ 0 & \text{otherwise.} \end{cases}
$$

In this case $\mathbf{P}_q(A) = \mathbf{P}_p(A \mid B)$. We call q the *conditional distribution* induced by *conditioning on* the event B.

Finite intersections

As a simple repeated application of our definition, suppose $A_1, \ldots, A_n \subset \Omega$. Then

$$
\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1) \cdots \mathbf{P}(A_n \mid A_1 \cap \cdots \cap A_{n-1})
$$

Many authors call this the *chain rule*. The order of the A_i is inconsequential.

Conditional Event Probabilities

Event Probabilities

Outcome Probabilities

Real Functions

Intervals

Real Line

Real Order — Real Summation — Size of Direct Product

Rational Order — Real Numbers — Number of Set Products — Natural Summation

Rational Numbers — Number of Disjoint Unions — Integral Line — Finite Set Examples — Lists

Integer Arithmetic — Set Number — Direct Products

Integer Products — Natural Multiplicative Identity — Natural Additive Identity — Finite Sets

Natural Powers — Arithmetic — Identity Elements — Equivalent Sets

Integer Sums — Integer Order — Comparisons — Family Operations — Function Inverse — Uncertain Events — Family Unions and Intersections

Natural Products — Integer Numbers — Orders — Operations — Families — Function Images — Function Composites

Natural Sums — Natural Order — Converse Relations — Functions — Equivalence Relations

Recursion Theorem — Relations

Peano Axioms — Ordered Pair Projections — Uncertain Outcomes

Natural Induction — Set Products

Natural Numbers — Set Powers — Generalized Set Dualities

Set Symmetric Differences — Successor Sets — Unordered Triples — Set Dualities — Intersection of Empty Set — Partitions — Set Unions and Intersections

Set Complements — Pair Unions — Set Intersections — Ordered Pairs

Set Differences — Set Unions — Empty Set — Unordered Pairs — Pair Intersections

Set Specification — Geometry

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities — Sets

Names

Letters — Objects