Why

We discuss a decomposition using eigenvalues and eigenvectors.[1]

Defining result

An *eigenvalue decomposition* of a matrix $A \in \mathbf{R}^{n \times n}$ is an ordered pair (X, Λ) in which X is invertible, Λ is diagonal, and $A = X\Lambda X^{-1}$.

In this case, $AX = X\Lambda$, in other words,

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

in which x_i is the ith column of X and λ_i is the ith diagonal element of Λ. We have $Ax_i = \lambda_i x_i$ for $i = 1, \ldots, n$. In other words, the ith column of X is an eigenvector of A and the jth entry of Λ is the corresponding eigenvalue. If X is orthonormal, so that $X^{-1} = X^\top$, then we can interpret such a decomposition as a change of basis to *eigenvector coordinates*. If $Ax = b$, and $A = X\Lambda X^{-1}$ then $(X^{-1}b) = \Lambda(X^{-1}x)$. Here, $X^{-1}x$ expands x is the basis of columns of X. So to compute Ax , we first expand into the basis of columns of X, scale by Λ, and then interpret the result as the coefficients of a linear combination of the columns of X.

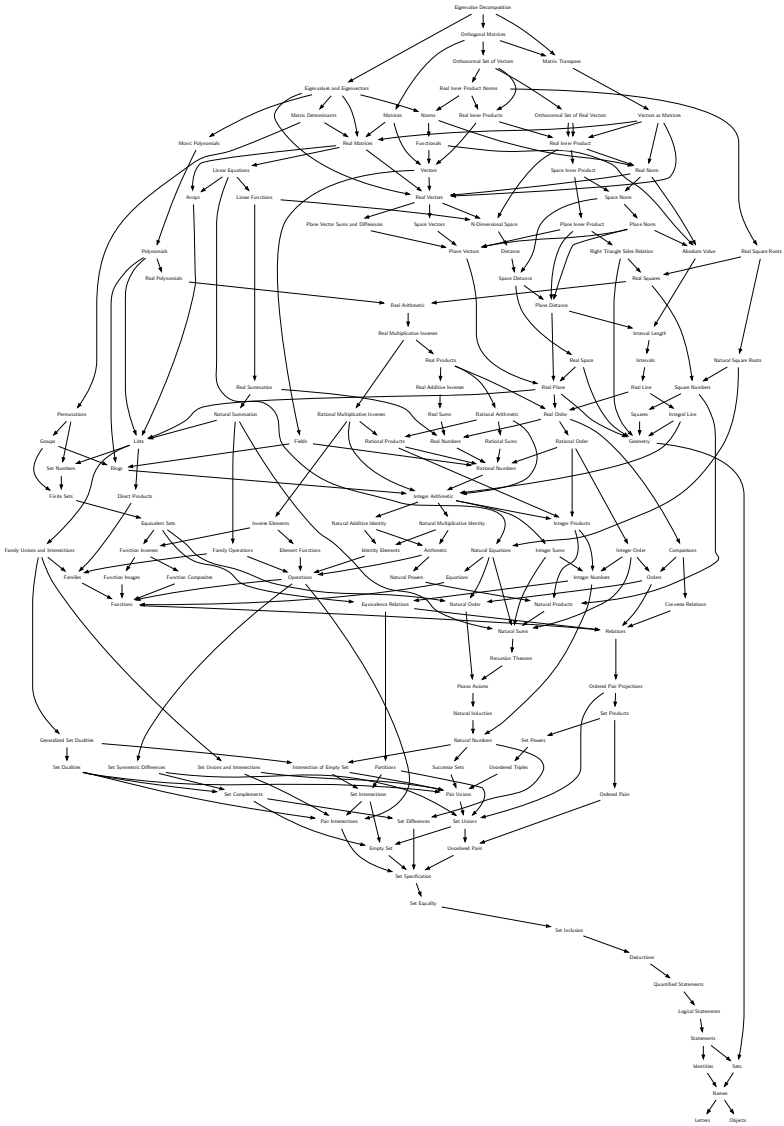
In this case that $A = X\Lambda X^\top$ for an eigenvalue decomposition (X, Λ) of A, we can also write

$$A = X\Lambda X^\top = \sum_{i=1}^{n} \Lambda_{ii} x_i x_i^\top.$$

Proposition 1. *Every real symmetric matrix has an eigenvalue decomposition (X, Λ) in which X is orthonormal.*[2]

[1] Future editions will expand.

[2] In future editions, this may be the motivating result for the definition of eigenvalues.


Eigenvalue Decomposition
Orthogonal Matrices
Orthonormal Set of Vectors
Matrix Transpose
Eigenvalues and Eigenvectors
Real Inner Product Norms
Matrix Determinants
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Family Unions and Intersections
Function Inverses
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Intersection of Empty Set
Partitions
Successor Sets
Unordered Triples
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Pair Intersections
Set Differences
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Unordered Pairs
Set Specification
Set Equality
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