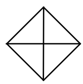


Why

We explore matrix-vector multiplication.

Definition

Given a matrix $A \in \mathbf{R}^{m \times n}$ and a vector $x \in \mathbf{R}^n$, the *product* of A *with* x is the vector $y \in \mathbf{R}^m$ defined by

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \quad i = 1, \dots, m.$$

Notation

We denote the product of A with x by Ax. With which we concisely write the system of linear equations (A, b) as $b = Ax$.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function $x \mapsto Ax$. In other words, we are interested in the existence of an inverse element of A. The geometric interpretation is that A transforms the vector x.

Conversely, we can view x as transforming (acting on) A. Let $a^j \in \mathbf{R}^m$ denote the jth column of A, then

$$Ax = \sum_{j=1}^{n} x_j a^j.$$

In other words, y is linear combination of the columns of A.

Properties

We call the function $f : \mathbf{R}^n \to \mathbf{R}^m$ defined by $f(x) = Ax$ the *matrix multiplication function* (or *matrix-vector product function*) associated with A. f is satisfies the following two important properties:

1. $A(x + y) = Ax + Ay$

2. $A(\alpha x) = \alpha Ax$.

We call such a function f *linear*. In other words, the matrix multiplication function is linear. Conversely, if $g : \mathbf{R}^n \to \mathbf{R}^m$ is linear, there exists a matrix inducing g.

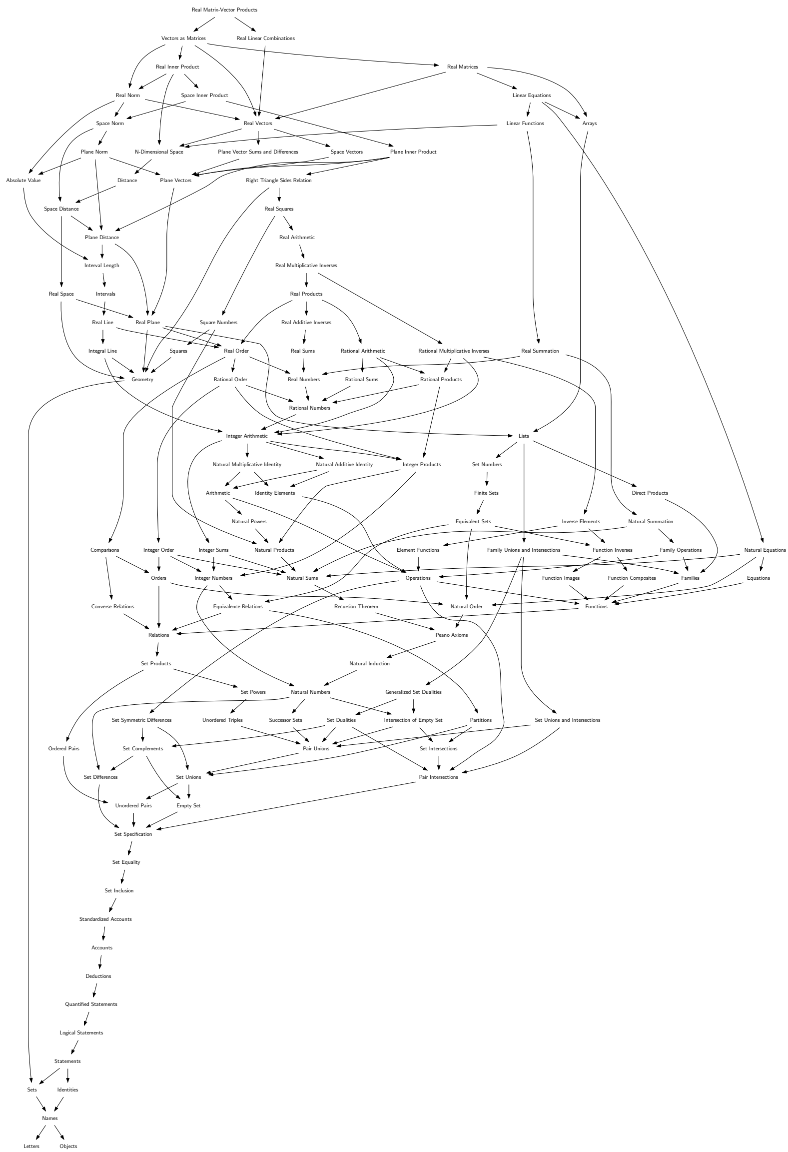
Proposition 1. *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be linear. Then there exists a unique $A \in \mathbf{R}^{m\times n}$ satisfying $f(x) = Ax$ for all $x \in \mathbf{R}^n$.*

Proof. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. $\square$

Moreover, this matrix representation of f is unique.

Proposition 2. *If $A, B \in \mathbf{R}^{m\times n}$ are two matrices so that $f(x) = Ax = Bx$, then $A = B$.*

Proof. We have $Ax - Bx = 0$ so $(A - B)x = 0$ for every x. In particular $y^\top(A-B)x = 0$ for every $x \in \mathbf{R}^n, y \in \mathbf{R}^m$. In particular, $e_i^\top(A-b)e_j = 0$. Conclusion: $A_{ij} - B_{ij} = 0$, and conclude that $A_{ij} = B_{ij}$. Thus, $A = B$. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. $\square$


Real Matrix-Vector Products
Vectors as Matrices
Real Linear Combinations
Real Inner Product
Real Matrices
Real Norm
Space Inner Product
Linear Equations
Space Norm
Real Vectors
Linear Functions
Arrays
Plane Norm
N-Dimensional Space
Plane Vector Sums and Differences
Space Vectors
Plane Inner Product
Absolute Value
Distance
Plane Vectors
Right Triangle Sides Relation
Space Distance
Real Squares
Plane Distance
Real Arithmetic
Interval Length
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Line
Real Plane
Square Numbers
Real Additive Inverses
Integral Line
Squares
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Real Summation
Geometry
Rational Order
Real Numbers
Rational Sums
Rational Products
Rational Numbers
Integer Arithmetic
Lists
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Set Numbers
Arithmetic
Identity Elements
Finite Sets
Direct Products
Natural Powers
Equivalent Sets
Inverse Elements
Natural Summation
Comparisons
Integer Order
Integer Sums
Natural Products
Element Functions
Family Unions and Intersections
Function Inverses
Family Operations
Natural Equations
Orders
Integer Numbers
Natural Sums
Operations
Function Images
Function Composites
Families
Equations
Converse Relations
Equivalence Relations
Recursion Theorem
Natural Order
Functions
Relations
Peano Axioms
Set Products
Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Unordered Triples
Successor Sets
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects