



Definition

The *trace* of a square real matrix is the sum of its diagonal entries.

Notation

We denote the function which associates a matrix with its trace by $\text{tr} : \mathbf{R}^{n \times n} \rightarrow \mathbf{R}$. The trace of $A \in \mathbf{R}^{n \times n}$ is

$$\text{tr } A = \sum_{i=1}^n A_{ii}.$$

Properties

Prop. 1. *The trace is a linear function on the vector space of $n \times n$ real matrices.*

Proof. Let $A, B \in \mathbf{R}^{n \times n}$ and $\alpha, \beta \in \mathbf{R}$. Define $C = \alpha A + \beta B$. Then $C_{ii} = \alpha A_{ii} + \beta B_{ii}$. So

$$\begin{aligned} \text{tr } C &= \sum_{i=1}^n C_{ii} = \sum_{i=1}^n \alpha A_{ii} + \beta B_{ii} \\ &= \alpha \sum_{i=1}^n A_{ii} + \beta \sum_{i=1}^n B_{ii} \\ &= \alpha \text{tr } A + \beta \text{tr } B. \end{aligned}$$

□

Prop. 2. *Let $A, B \in \mathbf{R}^{n \times n}$. Then*

$$\text{tr } (AB) = \text{tr } (BA).$$

In other words, “matrices commute under the trace operator.”

Prop. 3. *Let $A \in \mathbf{R}^{n \times n}$. Then $\text{tr } A = \text{tr } A^\top$.*

