Why

Can we order the cone of positive semidefinite matrices?

Definition

The *positive semidefinite matrix order* (or *Loewner order*) is a partial ordering $\geq$ on $\mathbf{S}^d$ defined by

$$A \geq B \quad \longleftrightarrow \quad A - B \geq 0 \quad \longleftrightarrow \quad A - B \in \mathbf{S}^d_+.$$

We define the partial order $>$ on symmetric matrices by

$$A > B \quad \longleftrightarrow \quad A - B > 0 \quad \longleftrightarrow \quad A - B \in \mathbf{S}^d_{++}.$$

Properties

Each of the following results from the geometric properties of the positive semidefinite cone:

$$\begin{aligned} \alpha A \geq 0 \quad & \text{for all } \delta > 0, A \geq 0, \\ A + B \geq 0 \quad & \text{for all } A, B \geq 0, \\ A \geq B \text{ and } B \geq A \longrightarrow A = B \quad & \text{for all } A, B \in \mathbf{S}^d, \\ \lim_{n\to\infty} A_n = A \longrightarrow A \geq 0 \quad & \text{for all sequences } (A_n)_n \text{ in } \mathbf{S}^d_+. \end{aligned}$$

Partial Order

$A \geq B$ and $B \geq A$ giving $A = B$ means that $\geq$ is antisymmetric. Moreover,

$$A \geq A \quad \& \text{ for all } A \in \mathbf{S}^d, \text{ and}$$

$$A \geq B \text{ and } B \geq C \longrightarrow A \geq C \& \text{ for all } A, B, C \in \mathbf{S}^d.$$

In other words, $\geq$ is also reflexive and transitive. In other words, $\geq$ is a partial order (see Partial Orders).[1]

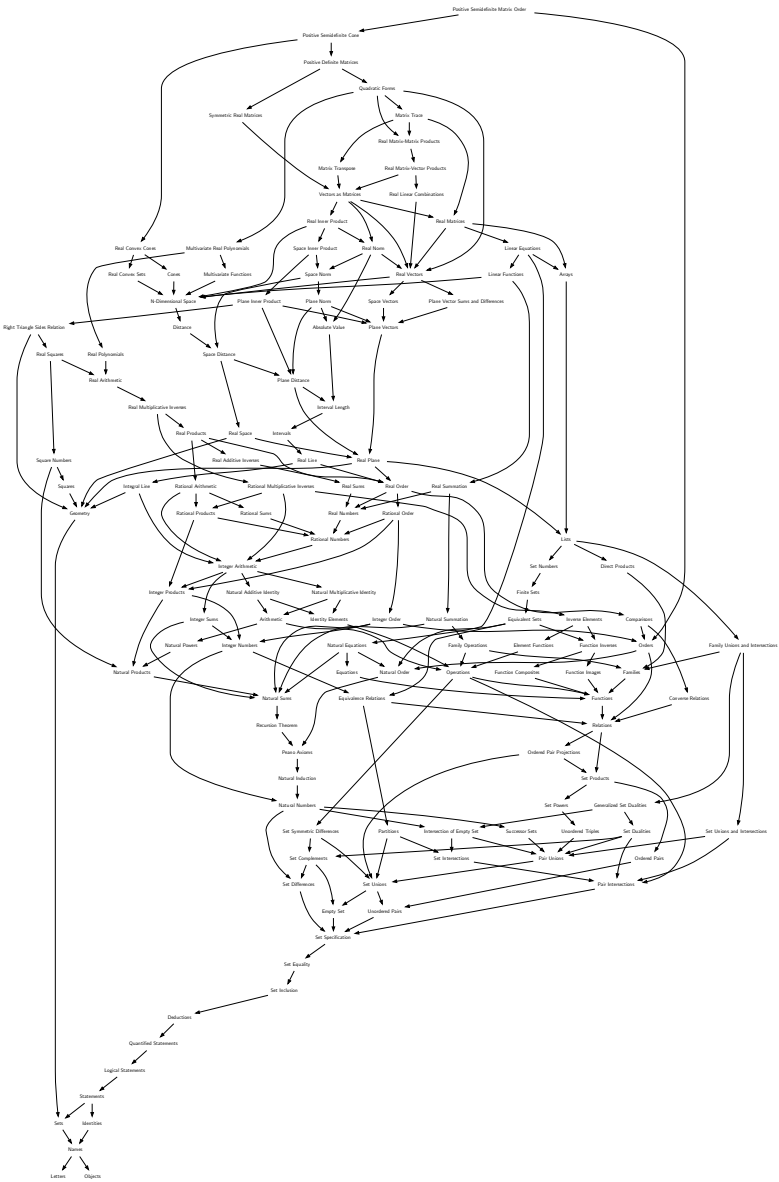
[1] Future editions will include more formal accounts.

For $d = 1$, $\geq$ reduces to the familiar total order of the real line(see Real Order). The converse perspective is to see the positive semidefinite order as an extension of the order on $\mathbf{R}$ to the space $\mathbf{S}^d$. Of course, the key difference is that two matrices may not be comparable. The order is *partial*.

For example, the matrices $A, B \in \mathbf{S}^2$ defined by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not comparable. Neither $A \geq B$ nor $B \geq A$ holds.


Positive Semidefinite Matrix Order
Positive Semidefinite Cone
Positive Definite Matrices
Quadratic Forms
Symmetric Real Matrices
Matrix Trace
Real Matrix-Matrix Products
Matrix Transpose
Real Matrix-Vector Products
Vectors as Matrices
Real Linear Combinations
Real Inner Product
Real Matrices
Real Convex Cones
Multivariate Real Polynomials
Space Inner Product
Real Norm
Linear Equations
Real Convex Sets
Cones
Multivariate Functions
Space Norm
Real Vectors
Linear Functions
Arrays
N-Dimensional Space
Plane Inner Product
Plane Norm
Space Vectors
Plane Vector Sums and Differences
Right Triangle Sides Relation
Distance
Absolute Value
Plane Vectors
Real Squares
Real Polynomials
Space Distance
Real Arithmetic
Plane Distance
Real Multiplicative Inverses
Interval Length
Real Products
Real Space
Intervals
Square Numbers
Real Additive Inverses
Real Line
Real Plane
Squares
Integral Line
Rational Arithmetic
Rational Multiplicative Inverses
Real Sums
Real Order
Real Summation
Geometry
Rational Products
Rational Sums
Real Numbers
Rational Order
Rational Numbers
Lists
Set Numbers
Direct Products
Integer Arithmetic
Finite Sets
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Integer Sums
Arithmetic
Strictly Ordered
Integer Order
Natural Summation
Equivalent Sets
Inverse Elements
Comparisons
Natural Powers
Integer Numbers
Natural Equations
Family Operations
Element Functions
Function Inverses
Orders
Family Unions and Intersections
Natural Products
Equations
Natural Order
Composition
Function Compositions
Function Images
Functions
Natural Sums
Equivalence Relations
Functions
Converse Relations
Relations
Recursion Theorem
Peano Axioms
Ordered Pair Properties
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects