Why

Can we represent the function associating a linear system with its row reduction by matrix multiplication?

Main observation

The following proposition affirmatively answers the question.

Proposition 1. *Let* $(A \in \mathbf{R}^{m\times m}, b \in \mathbf{R}^m)$ *be a linear system with* $A_{kk} \neq 0$ *and* (C, d) *the* k*th reduction of* (A, b)*. Then there exists a matrix* $L \in \mathbf{R}^{m\times m}$ *so that* $C = LA$ *and* $d = Lb$*.*

Proof. Define $L \in \mathbf{R}^{m\times m}$ by $L_{st} = 1$ if $s = t$, $-A_{sj}/A_{ij}$ if $k < s \leq m$ and zero otherwise. $\square$

For this reason, we call L in Proposition 1 a *row reducer matrix* or *row reducing matrix* or *row reducer*. The row reducer matrix for the kth reduction of (A, b) has the form

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & A_{ik}/A_{kk} & 1 & & \\ & & \vdots & & \ddots & \\ & & A_{mk}/A_{kk} & & & 1 \end{bmatrix}$$

So the following is immediate

Prop. 2. *Row reducing matrices are unit lower triangular.*

Example

For example, the $(1,1)$-reduction of (A,b) in which

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 6 & 7 & 9 & 5 \\ 8 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

is the linear system

$$A' = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \text{ and } b' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The row reducer is $L \in R^{4\times 4}$ defined by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that $A' = LA$ and $b' = Lb$, and clearly L is unit lower triangular.

Row Reducer Matrices
Triangular Matrices
Linear System Row Reductions
Diagonal Matrices
Matrices
Linear Equation Solutions
Real Matrices
Vectors
Linear Equations
Real Vectors
Linear Functions
Space Vectors
Plane Vector Sums and Differences
N-Dimensional Space
Plane Vectors
Distance
Space Distance
Arrays
Plane Distance
Interval Length
Real Space
Intervals
Real Plane
Real Line
Integral Line
Real Order
Real Summation
Geometry
Fields
Real Numbers
Rational Order
Rings
Rational Numbers
Integer Arithmetic
Lists
Integer Products
Set Number
Natural Multiplicative Identity
Natural Additive Identity
Direct Products
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Natural Summation
Natural Powers
Function Inverses
Family Operations
Comparisons
Natural Equations
Natural Products
Integer Order
Integer Sums
Family Unions and Intersections
Families
Function Composites
Function Images
Operations
Equations
Orders
Natural Sums
Integer Numbers
Functions
Converse Relations
Natural Order
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Intersection of Empty Set
Unordered Triples
Partitions
Successor Sets
Set Dualities
Set Symmetric Differences
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects