Why

What is the optimal tree approximator of a multivariate normal density?

Result

Prop. 1. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a normal density with mean $\mu \in \mathbf{R}^d$ and covariance $\Sigma \in \mathbf{S}^d_{++}$. The normal density $f^*_T : \mathbf{R}^d \to \mathbf{R}$ with mean μ and precision matrix P defined by*

- $P_{11} = \Sigma_{11}^{-1} + \sum_{\mathbf{pa}\, j=1} \Sigma_{j1}^2 \Sigma_{11}^{-2} \Sigma_{j|1}^{-1}$
- *for* $i = 2, \ldots, d$, $P_{ii} = \Sigma_{i|\mathbf{pa}\, i}^{-1} + \sum_{\mathbf{pa}\, j=i} \Sigma_{ji}^2 \Sigma_{ii}^{-2} \Sigma_{j|i}^{-1}$
- $i, j = 1, \ldots d$ *and* $i = \mathbf{pa}\, j$, $P_{ij} = P_{ji} = -\Sigma_{ji} \Sigma_{jj}^{-1} \Sigma_{j|i}^{-1}$

where $\mathbf{pa}\, i$ is the parent of i in an optimal approximator tree T $(i = 2, \ldots, n)$ is an optimal tree approximator of g.

Proof. Using Proposition 1 of Best Tree Density Approximators, express an optimal tree approximator of g by

$$(1/c) \exp\left(-\frac{1}{2} \left(\Sigma_{11}^{-1} \bar{x}_1^2 + \sum_{i \neq 1} \left(\bar{x}_i - \Sigma_{i,\mathbf{pa}\, i} \Sigma_{\mathbf{pa}\, i,\mathbf{pa}\, i}^{-1} \bar{x}_{\mathbf{pa}\, i} \right)^2 \Sigma_{i|\mathbf{pa}\, i}^{-1} \right) \right)$$

where $\bar{x}_i = x_i - \mu_i$ and $c = \sqrt{(2\pi)^d \Sigma_{11} \prod_{i \neq 1} \Sigma_{i|\mathbf{pa}\, i}}$.

Second, express the quadratic in the exponential as

$$\Sigma_{11}^{-1} \bar{x}_1^2 + \sum_{i \neq 1} \Big[\Sigma_{i|\mathbf{pa}\, i}^{-1} \bar{x}_i^2 - 2 \Sigma_{i,\mathbf{pa}\, i} \Sigma_{\mathbf{pa}\, i,\mathbf{pa}\, i}^{-1} \Sigma_{i|\mathbf{pa}\, i}^{-1} \bar{x}_i \bar{x}_{\mathbf{pa}\, i} + \Sigma_{i,\mathbf{pa}\, i}^2 \Sigma_{\mathbf{pa}\, i,\mathbf{pa}\, i}^{-2} \Sigma_{i|\mathbf{pa}\, i}^{-1} \bar{x}_{\mathbf{pa}}^2$$

With P defined as earlier, we can express the above as $\bar{x}^\top P \bar{x}$.

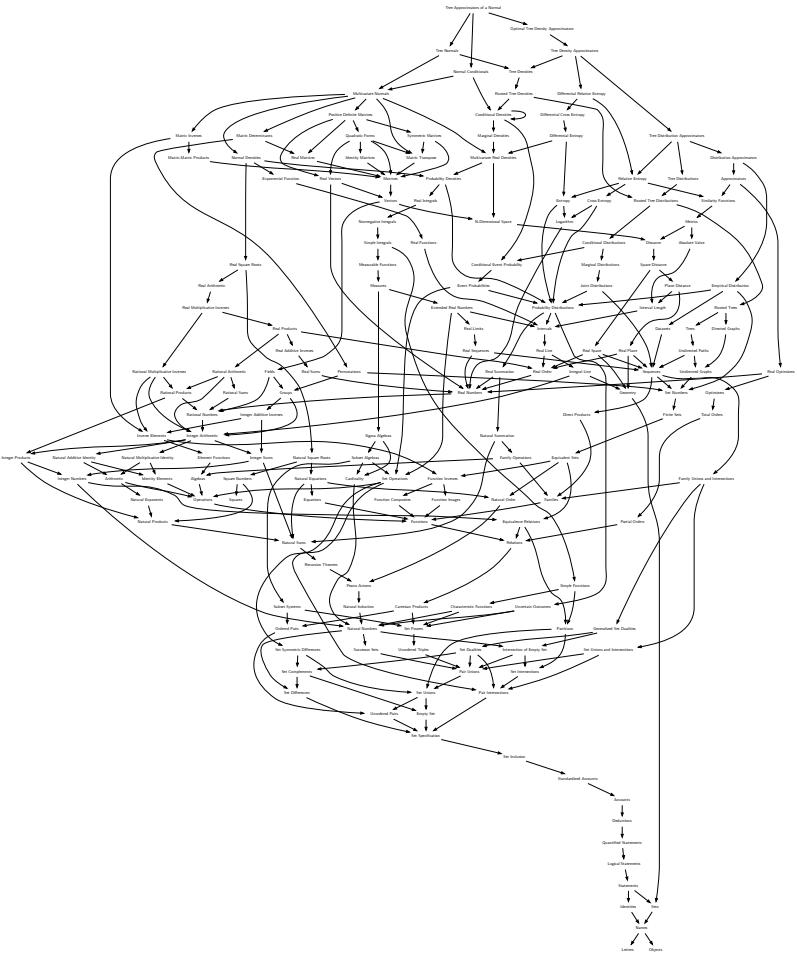
Third, note that c is $\sqrt{(2\pi)^d \,\mathbf{det}\, P^{-1}}$ since f_T^* is a density and so integrates to one.

$\square$

Notice that f_T^* is a tree normal density.

Empirical Normal

In particular, notice that we can approximate the empirical normal density of a dataset with a density that factors according to a tree.


Tree Approximation of a Normal
Optimal Tree-Density Approximation
Tree Models
Tree-Density Approximation
Normal Coordinands
Tree-Densities
Multivariate Normal
Rooted Tree-Densities
Differential Relative Entropy
Positive Definite Matrices
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Real Line
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Set Products
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Set Symmetric Differences
Successor Sets
Greatest Digits
Set Qualities
Intersection of Empty Sets
Set Unions and Intersections
Set Complements
Set Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Symmetries
Set Inclusion
Standardized Assessors
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Quantified Statements
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Sums
Limits
Objects