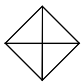


Why

We explore matrix-vector multiplication.

Definition

Given a matrix $A \in \mathbf{R}^{m \times n}$ and a vector $x \in \mathbf{R}^n$, the *product* of A *with* x is the vector $y \in \mathbf{R}^m$ defined by

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \quad i = 1, \dots, m.$$

Notation

We denote the product of A with x by Ax. With which we concisely write the system of linear equations (A, b) as $b = Ax$.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function $x \mapsto Ax$. In other words, we are interested in the existence of an inverse element of A. The geometric interpretation is that A transforms the vector x.

Conversely, we can view x as transforming (acting on) A. Let $a^j \in \mathbf{R}^m$ denote the jth column of A, then

$$Ax = \sum_{j=1}^{n} x_j a^j.$$

In other words, y is linear combination of the columns of A.

Properties

We call the function $f : \mathbf{R}^n \to \mathbf{R}^m$ defined by $f(x) = Ax$ the *matrix multiplication function* (or *matrix-vector product function*) associated with A. f is satisfies the following two important properties:

1. $A(x + y) = Ax + Ay$

2. $A(\alpha x) = \alpha Ax$.

We call such a function f *linear*. In other words, the matrix multiplication function is linear. Conversely, if $g : \mathbf{R}^n \to \mathbf{R}^m$ is linear, there exists a matrix inducing g.

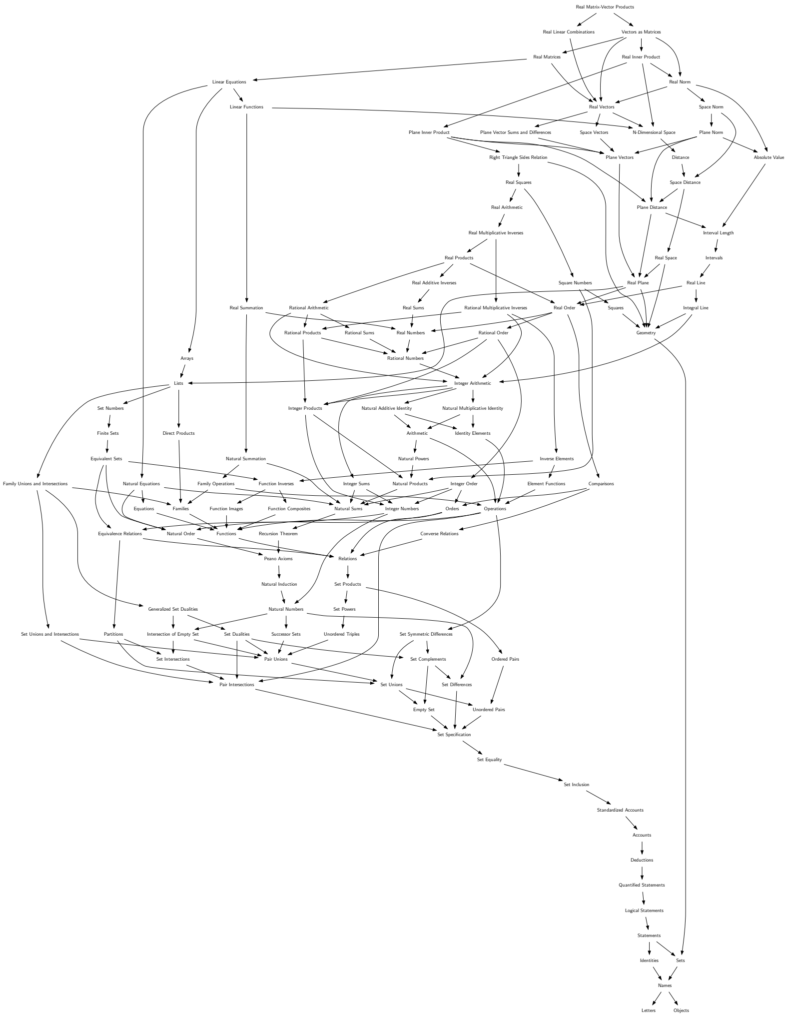
Proposition 1. *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be linear. Then there exists a unique $A \in \mathbf{R}^{m\times n}$ satisfying $f(x) = Ax$ for all $x \in \mathbf{R}^n$.*

Proof. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. $\square$

Moreover, this matrix representation of f is unique.

Proposition 2. *If $A, B \in \mathbf{R}^{m\times n}$ are two matrices so that $f(x) = Ax = Bx$, then $A = B$.*

Proof. We have $Ax - Bx = 0$ so $(A - B)x = 0$ for every x. In particular $y^\top(A-B)x = 0$ for every $x \in \mathbf{R}^n, y \in \mathbf{R}^m$. In particular, $e_i^\top(A-b)e_j = 0$. Conclusion: $A_{ij} - B_{ij} = 0$, and conclude that $A_{ij} = B_{ij}$. Thus, $A = B$. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. $\square$


Real Matrix-Vector Products
Real Linear Combinations
Vectors as Matrices
Real Matrices
Real Inner Product
Linear Equations
Real Norm
Linear Functions
Real Vectors
Space Norm
Plane Inner Product
Plane Vector Sums and Differences
Space Vectors
N-Dimensional Space
Plane Norm
Right Triangle Sides Relation
Plane Vectors
Distance
Absolute Value
Real Squares
Space Distance
Real Arithmetic
Plane Distance
Real Multiplicative Inverses
Interval Length
Real Products
Real Space
Intervals
Real Additive Inverses
Square Numbers
Real Plane
Real Line
Real Summation
Rational Arithmetic
Real Sums
Rational Multiplicative Inverses
Real Order
Squares
Integral Line
Rational Products
Rational Sums
Real Numbers
Rational Order
Geometry
Arrays
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Direct Products
Arithmetic
Identity Elements
Equivalent Sets
Natural Summation
Natural Powers
Inverse Elements
Family Unions and Intersections
Natural Equations
Family Operations
Function Inverses
Integer Sums
Natural Products
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Element Functions
Comparisons
Equations
Families
Function Images
Function Composites
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Recursion Theorem
Converse Relations
Peano Axioms
Relations
Natural Induction
Set Products
Generalized Set Dualities
Natural Numbers
Set Powers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Dualities
Successor Sets
Unordered Triples
Set Symmetric Differences
Set Intersections
Pair Unions
Set Complements
Ordered Pairs
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects