



Why

We characterize chordal graphs using vertex separators, and vice versa.¹

Main Result

Proposition 1 (Chordal Graphs and Vertex Separators). *An undirected graph is chordal if and only if all minimal vertex separators are complete.*

Proof. Let $G = (V, E)$ be an undirected graph.

First, suppose that all minimal vertex separators of G are complete. Let c be a cycle of length greater than 3. Let v, w be nonconsecutive vertices in c . If v and w are adjacent in G , then $\{v, w\} \in E$ is a chord. If v and w are nonadjacent, then vw -separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any vw -separator T . Split the cycle into the path from v to w , call it p_1 and the path from w to v , call it p_2 . T must include an interior point of p_1 , call it u_1 , otherwise v and w are connected. Similarly, T must include an interior point of p_2 , call it u_2 . u_1 and u_2 are not consecutive in c , since they are distinct from x and y .

Let S be a minimal vw -separator. Let $s, t \in S$ be two non-consecutive vertices in the cycle different from v and w . By assumption S is complete, so s and t are adjacent in G .

Second, let $G = (V, E)$ be a chordal graph. Let S be a minimal vw -separator. Let C_v and C_w be the connected components

¹Future editions will expand and may include graphics.

containing v and w of the subgraph induced by $V - S$.

If $|S| = 1$, then S is complete. Otherwise, let $x, y \in S$ be distinct. We want to show $\{x, y\} \in E$. The key insight is that x is adjacent to vertices in C_v and C_w . If there were no such vertex, $S - \{x\}$ would be a vw -separator and S would not be minimal. Similarly with y . Also, $|C_v|, |C_w| \geq 1$.

With these observations, there exists a path from x to y through C_v . Let $p_v = (x, v_1, \dots, v_k, y)$ be a path of shortest length with at least one interior vertex (so $k \geq 1$) from x to y using interior vertices in $v_1, \dots, v_k \in C_v$. Let $p_w = (y, w_1, \dots, w_l, x)$ be a path of shortest length with at least one interior vertex (so $l \geq 1$) from y to x using interior vertices $w_1, \dots, w_l \in C_w$. Use p_v and p_w to define the cycle $c = (x, v_1, \dots, v_k, y, w_1, \dots, w_l, x)$ which has length at least four. G is chordal, so c has a chord.

We argue that the chord of c is $\{x, y\}$. Since C_w and C_v are different connected components (whose vertices are not included in S), there are no edges $\{v_i, w_j\}$ for $i = 1, \dots, k$ and $j = 1, \dots, l$. Since p_v and p_w are paths of shortest length, they have no chords. In particular, there is no edge $\{v_i, v_j\}$ for $|i - j| > 1$ or $\{v_i, x\}$ for $i = 1, \dots, k$. Similarly, there is no edge $\{w_i, w_j\}$ for $|i - j| > 1$ or $\{w_i, y\}$ for $i = 1, \dots, l$. The only remaining pair is $\{x, y\}$, and so it must be the chord. \square

