Why

We want a norm on the vector space of continuous functions.

Definition

Consider a function from a closed real interval to the real numbers. The *absolute supremum* of the function is the supremum of the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

Proposition 1. *The functional mapping $f \in C[a, b]$ to its absolute supremum is a norm.*

Proof. Let R denote the set of real numbers. Define $\phi : C[a, b] \to R$ by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a, b]\}.$$

1. $|f(x)| \geq 0$ for all $x \in [a, b]$, so $\phi(f) \geq 0$.

2. If $\phi(f) = 0$ then $|f(x)| \leq 0$ for all x and so $f(x) = 0$ for all $x \in [a, b]$. If $f = 0$, then $|f(x)| = 0$ for all $x \in [a, b]$

3. For all α real, $|\alpha f(x)| = |\alpha||f(x)|$. So $\phi(\alpha f) = |\alpha|\phi(f)$

4. For all $f, g \in C[a, b]$, and $x \in [a, b]$, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ by the triangle inequality for absolute value. Thus,

$$\begin{aligned}
\phi(f + g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a, b]\} \\
&\leq \sup\{|f(x)| \mid x \in [a, b]\} + \sup\{|g(x)| \mid x \in [a, b]\} \\
&= \phi(f) + \phi(g)
\end{aligned}$$

$\square$

We call the functional ϕ defined above the *supremum norm*.

Notation

Let $f \in C[a, b]$. We denote the supremum norm of f by $\|f\|_{\sup}$.

Supremum Norm

Norms

Least Upper Bounds

Functionals

Vectors

Real Norm

Space Norm

Real Vector

Plane Vector Sums and Differences

Space Vector

Plane Norm

N-Dimensional Space

Absolute Value

Plane Vectors

Distance

Space Distance

Plane Distance

Interval Length

Real Space

Intervals

Real Line

Real Plane

Real Order

Integral Line

Geometry

Rational Order

Real Numbers

Fields

Lists

Rational Numbers

Rings

Integer Arithmetic

Set Numbers

Direct Products

Integer Products

Natural Multiplicative Identity

Natural Additive Identity

Finite Sets

Arithmetic

Identity Elements

Equivalent Sets

Comparisons

Integer Order

Integer Sums

Natural Powers

Operations

Function Inverses

Family Unions and Intersections

Order

Integer Numbers

Natural Products

Equivalence Relations

Function Composes

Function Images

Families

Converse Relations

Relations

Natural Sums

Natural Order

Functions

Ordered Pair Projections

Recursion Theorem

Peano Axioms

Set Products

Natural Induction

Set Powers

Natural Numbers

Generalized Set Dualities

Unordered Triples

Successor Sets

Set Symmetric Differences

Intersection of Empty Set

Set Dualities

Partitions

Set Unions and Intersections

Ordered Pairs

Pair Unions

Set Complements

Set Intersections

Set Differences

Pair Intersections

Set Unions

Unordered Pairs

Empty Set

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects