



## Why

Many functions of interest are additive and homogenous.

## Definition

A transformation is *linear* (a *linear transformation*, *linear map*) if the result of a linear combination of the two vectors is the linear combination of the results of the vectors (using the same coefficients). The transformation is linear *with respect to* the field of the two vector spaces.

We use the term transformation (see **Transformations**) to emphasize and remind that the function is defined on a *vector* space. Of course,  $\mathbf{R}$  is a vector space and so a function  $f : \mathbf{R} \rightarrow \mathbf{R}$  may be linear. The linear maps from  $\mathbf{R}$  to  $\mathbf{R}$  are the *linear functions* (see **Real Linear Functions**).

## Notation

Let  $(V, \mathbf{F})$  and  $(W, \mathbf{F})$  be two vector spaces over the same field. Suppose  $T : V \rightarrow W$ .  $T$  is linear means

$$T(\alpha u + \beta v) = \beta T(u) + \alpha T(v) \quad \text{for all } \alpha, \beta \in F \text{ and } u, v \in V.$$

As usual, the condition that  $T$  is linear condition is equivalent to the two conditions:

1.  $T(u + v) = f(u) + T(v)$  for all  $u, v \in V$ , and
2.  $f(\lambda u) = \lambda f(u)$  for all  $\lambda \in \mathbf{F}$  and  $u \in V$ .

If  $T$  satisfies (1), we call  $T$  *additive* (has the property of *additivity*). If  $T$  satisfies (2) we call  $T$  *homogeneous* (has the property of *homogeneity*).

For linear maps, it is common to denote  $T(v)$  by  $Tv$ ; notice that we have dropped the usual parentheses.

We denote the set of all linear maps by  $\mathcal{L}(V, W)$ . It is understood when using this notation that  $V$  and  $W$  are vector spaces with respect to the same field  $\mathbf{F}$ .

## Examples

Throughout, we consider vector spaces  $V$  and  $W$  over some fixed field  $\mathbf{F}$ .

*Constant zero map.* The map  $T \in \mathcal{L}(V, W)$  defined by

$$T(v) = 0 \in W \quad \text{for all } v \in V$$

is called the *zero map* (or *zero transformation*). It is common to overload the symbol 0 so that  $0 \in \mathcal{L}(V, W)$  denotes the zero map. In other words, the map 0 is defined by

$$0v = 0$$

Some care is required to interpret this equation. The 0 on the left hand side refers to a function, from  $V$  to  $W$ . The 0 on the right hand side is the additive identity in  $W$ . Usually context disambiguates the overloaded notation.

*The identity map.* The map  $T \in \mathcal{L}(V, V)$  defined by

$$Tv = v \quad \text{for all } v \in V$$

is called the *identity map* (or *identity transformation*). It is common to denote this map by  $I$ .

*Differentiation of polynomials* Suppose  $P$  is the set of all polynomials with coefficients in  $\mathbf{R}$ . (Some authors denote this set by  $\mathcal{P}(\mathbf{R})$ . Recall that every  $p \in \mathcal{P}(\mathbf{R})$  is differentiable and  $p' \in \mathcal{P}(\mathbf{R})$ . The map  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$  defined by

$$Tp = p'$$

is linear. To see this, recall  $(f + g)' = f' + g'$  and  $(\lambda f)' = \lambda f'$  whenever  $f, g$  are differentiable and  $\lambda \in \mathbf{R}$  (see Derivative of Sums and Derivatives of Scalar Multiples).

*Integration of polynomials* As in the previous paragraph,  $\mathcal{P}(\mathbf{R})$  denotes the vector space of polynomials with coefficients in  $\mathbf{R}$ . The map  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$  defined by

$$Tp = \int_{[0,1]} p$$

is linear To see this, recall that  $\int(f+g) = \int f + \int g$  and  $\int \lambda f = \lambda \int f$  whenever  $f, g$  are differentiable and  $\lambda \in \mathbf{R}$  (see Real Integral Additivity and Real Integral Homogeneity).

*Multiplication by a quadratic.* As in the previous paragraph,  $\mathcal{P}(\mathbf{R})$  denotes the vector space of polynomials with coefficients in  $\mathbf{R}$ . The map  $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$  defined by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbf{R}, p \in \mathcal{P}(\mathbf{R})$$

is linear. (Prove this).

*Sequence backward shift.* Denote the space of infinite sequences in a field  $\mathbf{F}$  by  $\mathbf{F}^{\mathbf{N}}$  as usual. Define  $T \in \mathcal{L}(\mathbf{F}^{\mathbf{N}}, \mathbf{F}^{\mathbf{N}})$  by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

$T$  is called the *backward shift operator*.

*From real space to the real plane.* Define  $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$  by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

*From  $\mathbf{F}^n$  to  $\mathbf{F}^m$ .* Generalizing the previous example, suppose  $m$  and  $n$  are natural numbers, and let  $A_{i,j} \in \mathbf{F}$  for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . Define  $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$  by

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

(It happens that every linear map from  $\mathbf{F}^n$  to  $\mathbf{F}^m$  has this form.)

*A counterexample:* <sup>1</sup>Notice  $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ . True,  $\cos$  is not homogenous. that  $\cos 2x = 2\cos(x)\cos(x) - \sin(x)\sin(x)$  and But this does not hold for all reals:  $\cos \lambda x \neq \lambda \cos(x)$ .

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<sup>1</sup>Need to add a sheet for trigonometric functions.



