Why

We are frequently interested in finding minimizers of real functions.[1]

Definition

An *optimization problem* is a pair $(\mathcal{X}, f : \mathcal{X} \to \mathbf{R})$ in which $\mathcal{X}$ is a nonempty set called the *constraint set* and f is called the *objective* (or *cost function*).

If $\mathcal{X}$ is finite we call the optimization problem *discrete*. If $\mathcal{X} \subset \mathbf{R}^d$ we call the optimization problem *continuous*.

We refer to all elements of the constraint set as *feasible*. We refer to an element $x \in \mathcal{X}$ of the constraint set as *optimal* if $f(x) = \inf_{z \in \mathcal{X}} f(z)$. We also refer to optimal elements as *solutions* of the optimization problem.

It is common for f and $\mathcal{X}$ to depend on some other, known, given objects. In this case, these objects are often called *parameters* or *problem data*.

Notation

We often write optimization problems as

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{X}. \end{aligned}$$

In this case we call x the *decision variable*.

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[1] Future editions will modify and expand.

Extended reals

It is common to let $f : \mathcal{X} \to \bar{\mathsf{R}}$, and allow there to exist $x \in \mathcal{X}$ for which $f(x) = \infty$. This is a trick to embed further constraints in the objective.

Maximization

If we have some function $g : \mathcal{X} \to \bar{\mathsf{R}}$ that we wish to maximize, we can always convert it to an optimization problem by defining $f : \mathcal{X} \to \bar{\mathsf{R}}$ by $f(x) = -g(x)$. In this case g is often called a *reward*, *utility* or *profit*.

Solvers

Let $\mathcal{P} = \{(X_a, f_a : X_a \to \bar{\mathsf{R}})\}_{a \in A}$ be a family of optimization problems. Define $\mathcal{X} = \cup_a X_a$. A *solver* (or *solution method* or *solution algorithm*) for $\mathcal{P}$ is a function $S : A \to \mathcal{S}$ such that S_a is a solution of the problem (X_a, f_a).

Loosely speaking, the difficulty of "solving" the optimization problem $(\mathcal{X}, f)$ depends on the properties of $\mathcal{X}$ and f and the problem "size". For example, when $\mathcal{X} \subset \mathsf{R}^d$ the difficulty is related to the "dimension" d of $x \in \mathcal{X}$.

Optimization Problems

Optimizers Greatest Lower Bounds N-Dimensional Space

Total Orders

Distance

Space Distance

Plane Distance Real Functions

Interval Length Extended Real Numbers

Partial Orders

Intervals Real Limits

Real Space Real Plane Real Line Real Order Integral Line Real Sequences

Geometry Real Numbers

Sequences Rational Numbers

Direct Products Set Numbers Integer Arithmetic

Finite Sets Natural Multiplicative Identity Natural Additive Identity

Identity Elements Arithmetic Set Operations

Equivalent Sets Operations Natural Exponents Integer Products

Family Unions and Intersections Function Inverse Natural Products Integer Sums

Families Function Composite Function Images

Functions Natural Sums Integer Numbers

Natural Order Equivalence Relations Recursion Theorem

Relations Peano Axioms

Cartesian Products Natural Induction

Generalized Set Dualities Set Powers Natural Numbers

Set Unions and Intersections Set Dualities Unordered Triples Successor Sets Intersection of Empty Set Partitions Set Symmetric Differences

Ordered Pairs Pair Unions Set Intersections Set Complements

Pair Intersections Set Unions Set Differences

Unordered Pairs Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities Sets

Names

Letters Objects