The Bourbaki Project

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CONTENTS

Why

We want to communicate and remember.

Discussion

A *language* is a conventional correspondence of sounds to affections of mind. We deliberately leave the definition of *affections* vague. A *spoken word* is a succession of sounds. By using these sounds, our mind can communicate with other minds.

A *symbol* is a written mark. A *script* is a collection symbols called *letters*. In *phonetic* languages the letters correspond to sounds and rules for composing these letters into successions called written words. This succession of letters corresponds to a succession of sounds and so a written word corresponds to a spoken word. By making marks, we communicate with other minds—including our own—in the future.

To write this sheet, we use Latin letters arranged into written words which are meant to denote the spoken words of the English language. The written words on this page are several letters one after the other. For example, the word "word" is composed of the letters "w", "o", "r", "d".

These endeavors are at once obvious and remarkable. They are obvious by their prevalence, and remarkable by their success. We do not long forget the difficulty in communicating affections of the mind, however, and this leads us to be very particular about how we communicate throughout these sheets.

Latin letters

We will start by officially introducing the letters of the Latin language. These come in two kinds, or cases. The *lower case latin letters.*

a b c d e f g h i
j k l m n o p q r
s t u v w x y z

And the *upper case latin letters.*

A B C D E F G H I
J K L M N O P Q R
S T U V W X Y Z

So, A is the upper case of a, and a the lower case of A. Similarly with b and B, with c and C, and all the rest.

Arabic numerals

We also use the *Arabic numerals.*

0 1 2 3 4 5 6 7 8 9

Other symbols

We also use the following symbols.

′ ( ) { } ∨ ∧ ¬ ∀ ∃ ⟶ ⟷ = ∈ →

Letters (**??**) does not immediately need any sheet.

Letters (**??**) is immediately needed by:

Names (**??**)

Letters (**??**) gives the following terms.

language
affections
spoken word
symbol
script
letters
phonetic
lower case latin letters
upper case latin letters
Arabic numerals

Letters

Why

We want to talk and write about things.

Definition

We use the word *object* with its usual sense in the English language. Objects that we can touch we call *tangible*. Otherwise, we say that the object is *intangible*.

Examples

We pick up a pebble for an example of a tangible object. The pebble is an object. We can hold and touch it. And because we can touch it, the pebble is tangible.

We consider the color of the pebble as an example of an intangible object. The color is an object also, even though we can not hold it or touch it. Because we can not touch it, the color is intangible. These sheets discuss other intangible objects and little else besides.

Objects (**??**) does not immediately need any sheet.

Objects (**??**) is immediately needed by:

Definitions (**??**)

Names (**??**)

Objects (**??**) gives the following terms.

object
tangible
intangible

Objects

Why

We (still) want to talk and write about things.

Names

As we use sounds to speak about objects, we use symbols to write about objects. In these sheets, we will mostly use the upper and lower case latin letters to denote objects. We sometimes also use an *accent* ′ or subscripts or superscripts. When we write the symbols we say that the composite symbol formed *denotes* the object. We call it the *name* of the object.

Since we use these same symbols for spoken words of the English language, we want to distinguish names from words. One idea is to box our names, and agree that everything in a box is a name, and that a name always denotes the object. For example, $\boxed{A}$ or $\boxed{A'}$ or $\boxed{A_0}$. The box works well to group the symbols and clarifies that $\boxed{A\,A}$ is different from $\boxed{AA}$. But experience shows that we need not use boxes.

We indicate a name for an object with italics. Instead of $\boxed{A'}$ we use A', instead of $\boxed{A_0}$ we use A_0. Experience shows that this subtlety is enough for clarity and it agrees with traditional and modern practice. Other examples include A'', A''', A'''', B, C, D, E, F, f, f' f_a.

No repetitions

We never use the same name to refer to two different objects. Using the same name for two different objects causes confusion. We make clear when we reuse symbols to mean different objects. We tend to introduce the names used at the beginning of a paragraph or section.

Names are objects

There is an odd aspect in these considerations. A may denote itself, that particular mark on the page. There is no helping it. As soon as we use some symbols to identify any object, these symbols can reference themselves.

An interpretation of this peculiarity is that names are objects. In other words, the name is an abstract object, it is that which we use to refer to another object. It is the thing pointing to another object. And the marks on the page which are meant to look similar are the several uses of a name.

Names as placeholders

We frequently use a name as a *placeholder*. In this case, we will say "let A denote an object". By this we mean that A is a name for an object, but we do not know what that object is. This is frequently useful when the arguments we will make do not depend upon the particular object considered. This practice is also old. Experience shows it is effective. As usual, it is best understood by example.

Names (**??**) immediately needs:

Letters (**??**)
Objects (**??**)

Names (**??**) is immediately needed by:

Identities (**??**)
Sets (**??**)

Names (**??**) gives the following terms.

accent
denotes
name
assertion
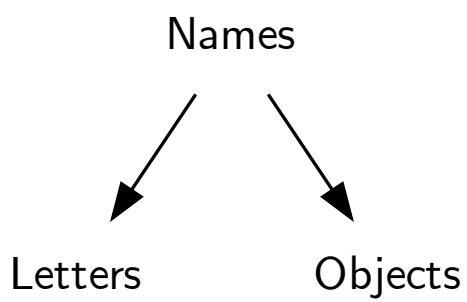*names*
accent
letter
terms
relations
placeholder

Names

Letters　　Objects

Why

We can give the same object two different names.

Definition

An object *is* itself. If the object denoted by one name is the same as the object denoted by a second name, then we say that the two names are *equal*. The object associated with a *name* is the *identity* of the name.

Let A denote an object and let B denote an object. Here we are using A and B as placeholders. They are names for objects, but we do not know—or care—which objects. We say "A equals B" as a shorthand for "the object denoted by A is the same as the object denoted by B". In other words, A and B are two names for the same object.

Symmetry

Let A denote an object and let B denote an object. "A equals B" means the same as "B equals A". The identity of the names is not dependent on the order in which the names are given. We call this the *symmetry of identity*. It means we can switch the spots of A and B and say the same thing. In other words, there are two ways to make the statement.

Reflexivity

Let A denote an object. Since every object is the same as itself, the object denoted by A is the same as the object denoted by A. We say "A equals A". In other words, every name equals itself. This fact is called the *reflexivity of identity*. A name is equal to itself because an object is itself.

Identities (**??**) immediately needs:

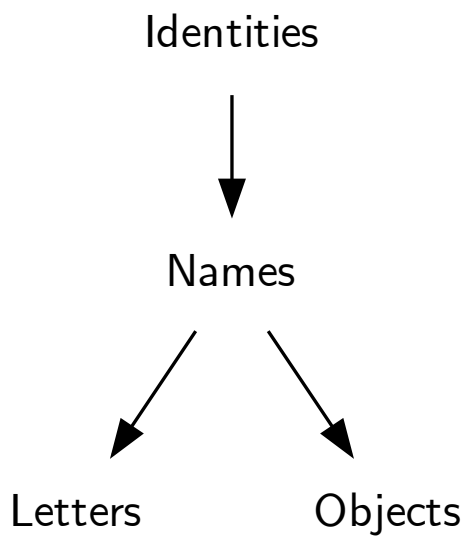Names (**??**)

Identities (**??**) is immediately needed by:

Equation Solutions (**??**)
Statements (**??**)

Identities (**??**) gives the following terms.

is
equation
indeterminate
is
equal
name
identity
symmetry of identity
reflexivity of identity
reflexive
symmetric
transitive
equals
reflexive
symmetric
transitive

Identities

↓

Names

↓ ↓

Letters   Objects

Why

We want to talk about none, one, or several objects considered together, as an aggregate.

Definition

When we think of several objects considered as an intangible whole, or group, we call the intangible object which is the group a *set*. We say that these objects *belong* to the set. They are the set's *members* or *elements*. They are *in* the set.

Principle 1 (Existence of Sets). *Intangible groups exist.*

A set may have other sets as its members. This is subtle but becomes familiar. We call a set which contains no objects *empty*. Otherwise we call a set *nonempty*.

Denoting a set

Let A denote a set. Then A is a name for an object. That object is a set. So A is a name for an object which is a grouping of other objects.

Belonging

Let a denote an object and A denote a set. So we are using the names a and A as placeholders for some object and some set, we do not particularly know which. Suppose though, that whatever this object and set are, it is the case that the object

belongs to the set. In other words, the object is a member or an element of the set. We say "The object denoted by a belongs to the set denoted by A".

Not symmetric

Notice that belonging is not symmetric. Saying "the object denoted by a belongs to the set denoted by A" does not mean the same as "the set denoted by A belongs to the object denoted by a" In fact, the latter sentence is nonsensical unless the object denoted by a is also a set.

Not transitive

Let a denote an object and let A and B both denote sets. If the object denoted by a is "a part of" a the set denoted by A, and the set denoted by A is "a part of" the set denoted by B, then usual English usage would suggest that a is "a part of" the set denoted by B. In other words, if a thing is a part of a second thing, and the second thing is part of a third thing, then the first thing is often said to be a part of the third thing. The relation of belonging is not quite this. If a thing is an element of a thing, that second thing may be an element of the third thing, but this does not mean that the the first thing is an element of the third thing.

Sets (**??**) immediately needs:
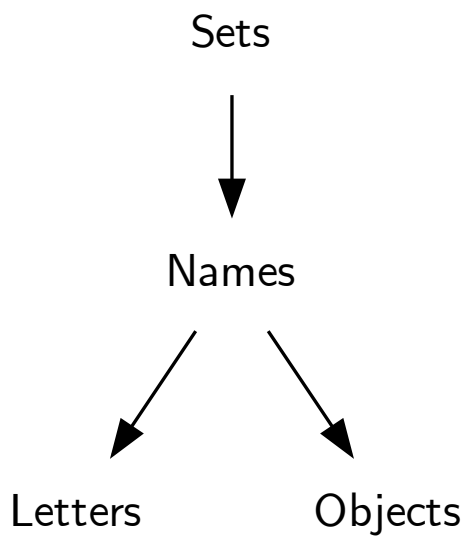
Names (**??**)

Sets (**??**) is immediately needed by:

Set Examples (**??**)
Statements (**??**)

Sets (**??**) gives the following terms.

set
belong
members
elements
in
empty
nonempty

Sets

Names

Letters   Objects

Why

We give some examples of objects and sets.

Examples

For familiar examples, let us start with some tangible objects. Find, or call to mind, a deck of playing cards.

First, consider the set of all the cards. This set contains fifty-two elements. Second, consider the set of cards whose suit is hearts. This set contains thirteen elements: the ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, and king of hearts. Third, consider the set of twos. This set contains four elements: the two of clubs, the two of spades, the two of hearts, and the two of diamonds.

We can imagine many more sets of cards. If we are holding a deck, each of these can be made tangible: we can touch the elements of the set. But the set itself is always abstract: we can not touch it. It is the idea of the group as distinct from any individual member.

Moreover, the elements of a set need not be tangible. First, consider the set consisting of the suits of the playing card: hearts, diamonds, spades, and clubs. This set has four elements. Each element is a suit, whatever that is.

Second, consider the set consisting of the card types. This set has thirteen elements: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. The subtlety here is that

this set is different than the set of hearts, namely those thirteen cards which are hearts. However these sets are similar: they both have thirteen elements, and there is a natural correspondence between their elements: the ace of hearts with the type ace, the two of hearts with the type two, and so on.

Of course, sets need have nothing to do with playing cards. For example, consider the set of seasons: autumn, winter, spring, and summer. This set has four elements. For another example, consider the set of lower case latin letters (introduced in Letters): a, b, c, ..., x, y, z. This set has twenty-six elements. Finally, consider a pack of wolves, or a bunch of grapes, or a flock of pigeons.

Set Examples (**??**) immediately needs:

 Sets (**??**)

Set Examples (**??**) is not immediately needed by any sheet.

Set Examples (**??**) gives no terms.

Set Examples

↓

Sets

↓

Names

Letters   Objects

Why

We want symbols to represent identity and belonging.

Definition

In the English language, nouns are words that name people, places and things. In these sheets, names (see Names) serve the role of nouns. In the English language, verbs are words which talk about actions or relations. In these sheets, we use the verbs "is" and "belongs" for the objects discussed. And we exclusively use the present tense.

Experience shows that we can avoid the English language and use symbols for verbs. By doing this, we introduce odd new shapes and forms to which we can give specific meanings. As we use italics for names to remind us that the symbol is denoting a possibly intangible arbitrary object, we use new symbols for verbs to remind us that we are using particular verbs, in a particular sense, with a particular tense. A *statement* is a succession of symbols.

Identity

As an example, consider the symbol $=$. Let a denote an object and b denote an object. Let us suppose that these two objects are the same object (see Identities). We agree that $=$ means "is" in this sense. Then we write $a = b$. It's an odd series of symbols, but a series of symbols nonetheless. And if we read it

aloud, we would read a as "the object denoted by a", then $=$ as "is", then b as "the object denoted by b". Altogether then, "the object denoted by a is the object denoted by b." We might box these three symbols $\boxed{a = b}$ to make clear that they are meant to be read together, but experience shows that (as with English sentences and words) we do not need boxes.

The symbol $=$ is (appropriately) a symmetric symbol. If we flip it left and right, it is the same symbol. This reflects the symmetry of the English sentences represented (see Identities). The symbols $a = b$ mean the same as the symbols $b = a$.

Belonging

As a second example, consider the symbol $\in$. Let a denote an object and let A denote a set. We agree that $\in$ means "belongs to" in the sense of "is an element of" or "is a member of" (see Sets). Then we write $a \in A$. We read these symbols as "the object denoted by a belongs to the set denoted by A".[1]

The symbol $\in$ is not symmetric. If we flip it left and right it looks different. This reflects that $a \in A$ does not the mean the same as $A \in a$ (see Sets). As with english words, the order of symbols is significant. The word "word" is not the same as the word "drow". Our symbolism for belonging reflects the concept's lack of symmetry.

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[1] The symbol $\in$ is a stylized lower case Greek letter ε, which is a mnemonic for the ancient Greek word $\grave{\varepsilon}\sigma\tau\acute{\iota}$ which means, roughly, "belongs". Since in English, ε is read aloud "ehp-sih-lawn," $\in$ is also a mnemonic for "element of".

Statements (**??**) immediately needs:

Identities (**??**)
Sets (**??**)

Statements (**??**) is immediately needed by:

Logical Statements (**??**)

Statements (**??**) gives the following terms.

statement
relational symbol
name symbol
relational symbol
name symbol
relational symbols
terminal
assertion
membership assertion
identity assertion
primitive sentence
logical form
sentence
belongs to
member

Statements

Identities   Sets

Names

Letters   Objects

Why
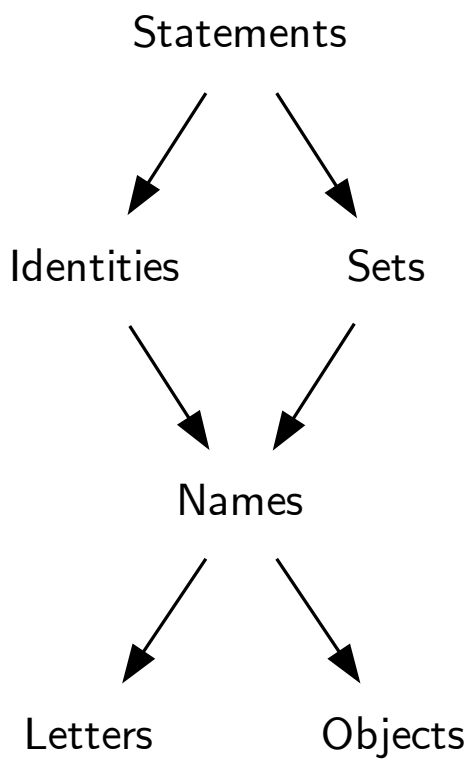
We want symbols for "and", "or", "not", and "implies".[2]

Overview

We call $=$ and $\in$ *relational symbols*. They say how the objects denoted by a pair of placeholder names relate to each other in the sense of being or belonging. We call _ $=$ _ and _ $\in$ _ *simple statements*. They denote simple sentences "the object denoted by _ is the object denoted by _" and "the object denoted by _ belongs to the set denoted by _". The symbols introduced here are *logical symbols* and statements using them are *logical statements*.

Conjunction

Consider the symbol $\wedge$. We will agree that it means "and". If we want to make two simple statements like $a = b$ and $a \in A$ at once, we write write $(a = b) \wedge (a \in A)$. The symbol $\wedge$ is symmetric, reflecting the fact that a statement like $(a \in A) \wedge (a = b)$ means the same as $(a = b) \wedge (a \in A)$.

Disjunction

Consider the symbol $\vee$. We will agree that it means "or" in the sense of either one, the other, or both. If we want to say that

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[2]This sheet does not explain logic. In the next edition there will be several more sheets serving this function.

at least one of the simple statements like $a = b$ and $a \in A$, we write write $(a = b) \vee (a \in A)$. The symbol $\vee$ is symmetric, reflecting the fact that a statement like $(a \in A) \vee (a = b)$ means the same as $(a = b) \vee (a \in A)$.

Negation

Consider the symbol $\neg$. We will agree that it means "not". We will use it to say that one object "is not" another object and one object "does not belong to" another object. If we want to say the opposite of a simple statement like $a = b$ we will write $\neg(a = b)$. We read it aloud as "not a is b" or (the more desirable) "a is not b". Similarly, $\neg(a \in A)$ we read as "not, the object denoted by a belongs to the set denoted by A". Again, the more desirable english expression is something like "the object denoted by a does not belong to the set A" For these reasons, we introduce two new symbols $\neq$ and $\notin$. $a \neq b$ means $\neg(a = b)$ and $a \notin A$ means $\neg(a \in A)$.

Implication

Consider the symbol $\longrightarrow$. We will agree that it means "implies". For example $(a \in A) \longrightarrow (a \in B)$ means "the object denoted by a belongs to the object denoted by A implies the object denoted by a belongs to the set denoted by B" It is the same as $(\neg(a \in A)) \vee (a \in B)$. In other words, if $a \in A$, then always $a \in B$. The symbol $\longrightarrow$ is not symmetric, since implication is not symmetric. The symbol $\longleftrightarrow$ means "if and only if".

Logical Statements (**??**) immediately needs:

Statements (**??**)

Logical Statements (**??**) is immediately needed by:

Deductions (**??**)
Quantified Statements (**??**)

Logical Statements (**??**) gives the following terms.

relational symbols
simple statements
logical symbols
logical statements

Logical Statements

Statements

Identities   Sets

Names
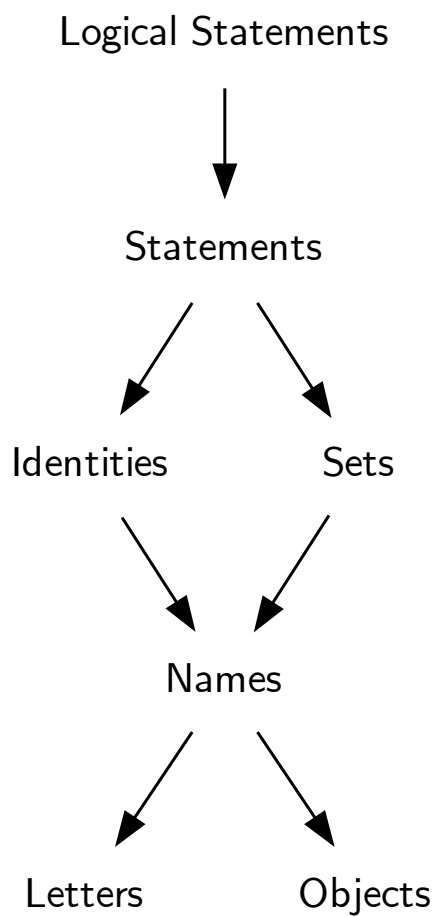
Letters   Objects

Why

We want to make conclusions.

Definition

Suppose we have a list of logical statements. We want to write down o

Deductions (**??**) immediately needs:

 Logical Statements (**??**)

Deductions (**??**) is immediately needed by:

 Accounts (**??**)

Deductions (**??**) gives no terms.

Deductions

$\downarrow$

Logical Statements

$\downarrow$

Statements

Identities   Sets

Names

Letters   Objects

Why
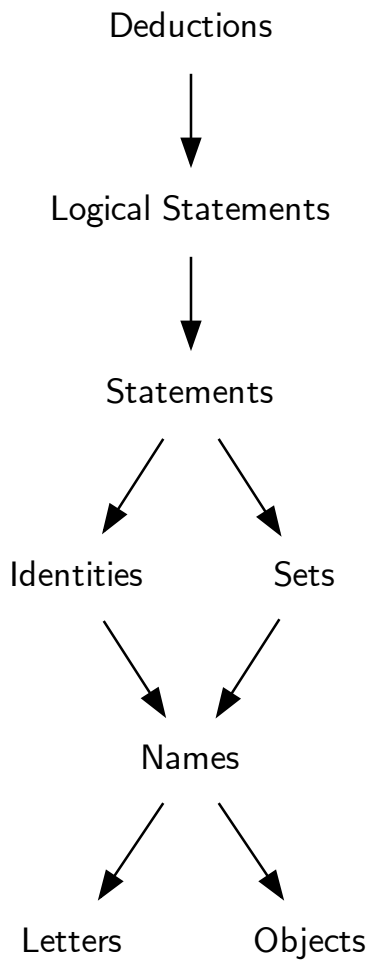
We want symbols for talking about the existence of objects and for making statements which hold for all objects.[3]

Definition

If we say there exists an object that is blue, we mean the same as if we say that not every object is not blue. If we say that every object is blue, we mean the same as if we say there does not exist an object that is not blue. In other words, "there exists an object so that _" is the same as "not every object is not _". Or, "every object is _" is the same as "there does not exist an object that is _".

When we assert something of every object we also assert the nonexistence of the contrary of that assertion. And likewise when we assert that an object exists with some conditions, we assert that not every object exists without that condition.

The content of our assertions will be logical statements (see Logical Statements) and when we want to make them for all objects or for no object we will use the following symbols. The symbols introduced here are *quantifier symbols* and statements using them are *quantified statements*.

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[3]This sheet does not explain quantifiers. In the next edition there will be several more sheets serving this function

Existential Quantifier

Consider the symbol $\exists$. We agree that it means "there exists an object". We write $(\exists x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\exists x)(x \in A)$ to mean "there exists an object in the set denoted by A" We call $\exists$ the *existential quantifier* symbol.

Universal Quantifier

Consider the symbol $\forall$. We agree that it means "for every object". We write $(\forall x)(\_)$ and then substitute any logical statement which uses the name x for $\_$. For example, we write $(\forall x)((x \in A) \longrightarrow (x \in B)$ to mean, "every object which is in the set denoted by A is in the set denoted by B". We call $\forall$ the *universal quantifier* symbol.

Binding

When we have a name following a $\forall$ or $\exists$ we say that the name is *bound*. If a name is bound, then the statement uses it in one sense but not in another. The name is only used in that single statement. Regular names in statements we call *unbound*

Negations

The statement $\neg(\forall x)(\_)$ is the same as $(\exists x)(\neg(\_))$ and $\neg(\exists x)(\_)$ is the same as $(\forall x)(\neg(\_))$.

Quantified Statements (**??**) immediately needs:

Logical Statements (**??**)

Quantified Statements (**??**) is immediately needed by:

Accounts (**??**)

Quantified Statements (**??**) gives the following terms.

quantifier symbols
quantified statements
existential quantifier
universal quantifier
bound
unbound

Quantified Statements

$\downarrow$

Logical Statements

$\downarrow$

Statements

Identities   Sets

Names

Letters   Objects

Why

We want to succinctly and clearly make several statements about objects and sets. We want to track the names we use, taking care to avoid using the same name twice.

Definition

An *account*[4] is a list of naming, logical, and quantified statements. We use the words "let _ denote an _" to introduce a name as a placeholder for a thing, and we use the symbols _ = _ and _ ∈ _ to denote statements of identity and belonging. In other words, we have three sentence kinds to record.

1. **Names.** State we are using a name.

2. **Identity.** We want to make statements of identity.

3. **Belonging.** We want to make statements of belonging.

Our main purpose is to keep a list names, of quantified, logical and simple statments about them, and then statements we can deduce from these. In particular we want to group our name usage. In the English language we use paragraphs or sections to do so. In these sheets, we will use accounts. We will list the statements and label each with Arabic numerals (see Letters). which will be a list of statements, each of which is labeled by an Arabic numeral (see Letters).

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[4]This sheet will be expanded in future editions.

Experience suggests that we start with an example. Suppose we want to summarize the following english language description of some names and objects.

> Denote an object by a. Also, denote the same object by b Also, denote a set by A. Also, the object denoted by a is an element of the set denoted by A. Also denote an object by c. Also c is the same object as b.

In our usual manner of speaking, we drop the word "also". In these sheets, we translate each of the sentences into our symbols. For names we use, we write `name` in that font followed by the name. For logical statements we `have`, we write `have` followed by the logical statement. For deductions we write `thus` followed by the conclusion and then `by` followed by the Arabic numerals of the premisses. So we write:

Account 1. First Example

1	`name`	a
2	`name`	b
3	`have`	$a = b$
4	`name`	A
5	`have`	$a \in A$
6	`name`	c
7	`have`	$c = b$
8	`thus`	$a = c$ `by` ??,??

Accounts (**??**) immediately needs:

Deductions (**??**)
Quantified Statements (**??**)

Accounts (**??**) is immediately needed by:

Standardized Accounts (**??**)

Accounts (**??**) gives the following terms.

account

Accounts

Deductions   Quantified Statements

Logical Statements

Statements

Identities   Sets
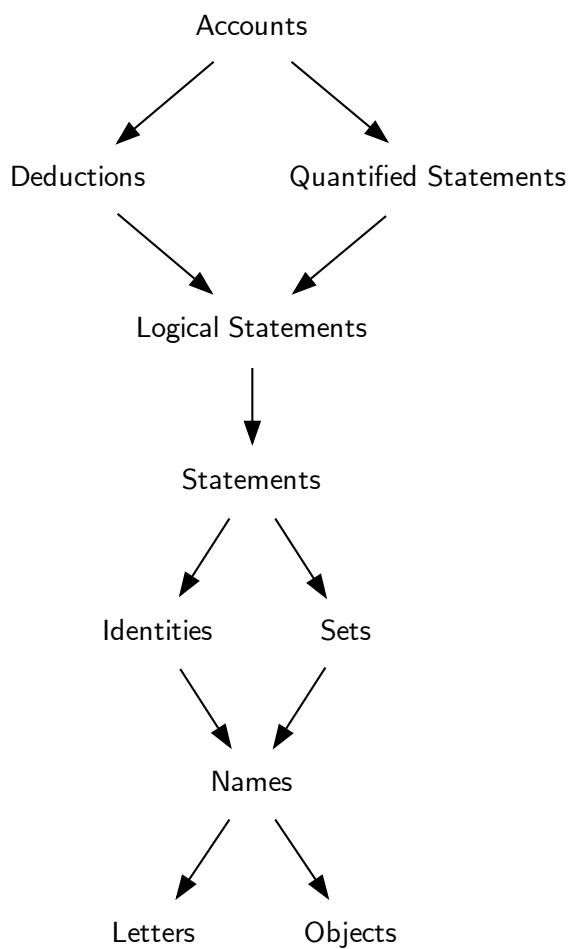
Names

Letters   Objects

Why

We want to do our best to have only one way to write accounts.

Definition

A *standard account*[5] lists all names, then lists all premisses, then lists all conclusions.

Example

Consider the account.

Account 2. First Example

1	name	a
2	name	b
3	have	$a = b$
4	name	c
5	have	$c = b$
6	thus	$a = c$ by ??,??

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[5]This sheet will be expanded in future editions.

Account 3. Standardized First Example

1	name	a
2	name	b
3	have	$a = b$
4	name	c
5	have	$c = b$
6	thus	$a = c$ by ??,??

We can abbreviate the names:

Account 4. Abbreviated First Example

1-3	name	a, b, c
4	have	$a = b$
5	have	$c = b$
6	thus	$a = c$ by ??,??,IdentityAxioms:1

Standardized Accounts (**??**) immediately needs:

Accounts (**??**)

Standardized Accounts (**??**) is immediately needed by:

Set Inclusion (**??**)

Standardized Accounts (**??**) gives the following terms.

standard account

Standardized Accounts

↓

Accounts

Deductions ← → Quantified Statements

↓ ↓

Logical Statements

↓

Statements

Identities ← → Sets

↓ ↓

Names
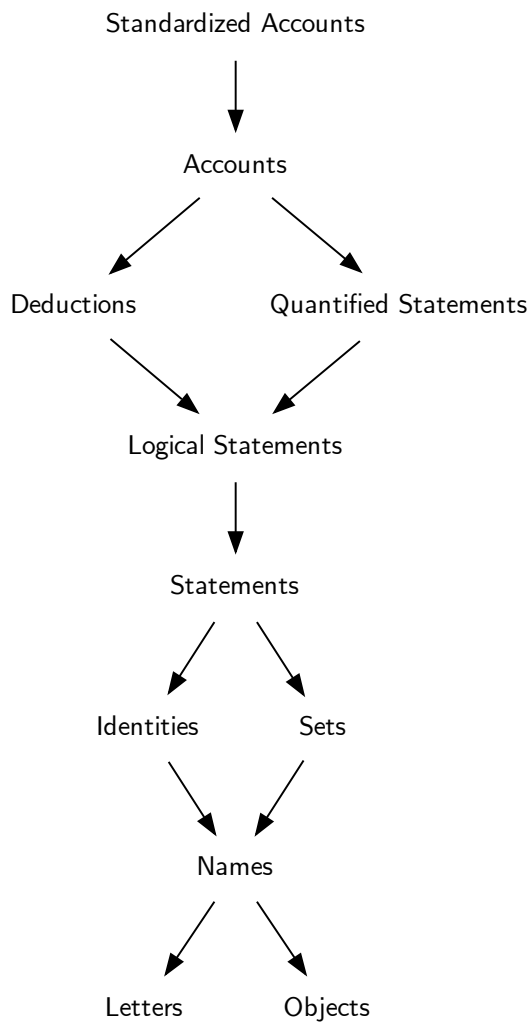
Letters ← → Objects

All definitions are *nominal.* They are made to give us language and to save space. They are labeled with arabic numerals to enable to us to reference them throughout the project.

They are given set apart from the text with the bold word **Definition #.** and then often with naming or sumamry text between parantheses. The definition body statement is in italics.

Definitions (**??**) immediately needs:

Objects (**??**)

Definitions (**??**) is immediately needed by:

Set Inclusion (**??**)

Definitions (**??**) gives the following terms.

nominal

Definitions

↓

Objects

Why

We want language for all of the elements of a first set being the elements of a second set.

Definition

Denote a set by A and a set by B.

Definition 1 (Subsets). If every element of the set denoted by A is an element of the set denoted by B, then we say that the set denoted by A is a *subset* of the set denoted by B.

We say that the set denoted by A is *included* in the set denoted by B. We say that the set denoted by B is a *superset* of the set denoted by A or that the set denoted by B *includes* the set denoted by A.

Every set is included in and includes itself.

Notation

Let A denote a set and B denote a set. We denote that the set A is included in the set B by $A \subset B$. In other words, $A \subset B$ means $(\forall x)((x \in A) \longrightarrow (x \in B))$. We read the notation $A \subset B$ aloud as "A is included in B" or "A subset B". Or we write $B \supset A$, and read it aloud "B includes A" or "B superset A". $B \supset A$ also means $(\forall x)((x \in A) \longrightarrow (x \in B))$.

Properties

There are some properties that our intuition suggests inclusion should have. First, every set should include itself. We describe this fact by saying that inclusion is *reflexive.*

Proposition 1 (Reflexive)**.** *Every set is included in itself*

Proof. (1) `name` A; (2) `have` $(\forall x)(x \in A \longrightarrow x \in A)$; (3) `thus` $A \subset A$ `by` Definition **??**. $\square$

Next, we expect that if one set is included in another, This fact is described by saying that inclusion is *transitive*

Proposition 2 (Transitive)**.** *If a one set is included in another, and the latter in yet another, then the former is included in the last.*

Proof. (1) `name` A, B, C; (2) `have` $A \subset B$ (3) `have` $B \subset C$ (4) `thus` $A \subset C$ `by` modus ponens. $\square$

Equality ($=$) shares these two properties. Let A denote an object. Then $A = A$. Let B and C also denote objects. If $A = B$ and $B = C$, then $A = C$. Of course, inclusion is not symmetric.. Belonging ($\in$) may be, but need not be reflexive and transitive.

Set Inclusion (**??**) immediately needs:

Definitions (**??**)
Standardized Accounts (**??**)

Set Inclusion (**??**) is immediately needed by:

Set Equality (**??**)
Set Specification (**??**)

Set Inclusion (**??**) gives the following terms.

subset
included
superset
includes
improper subsets
proper subsets
reflexive
transitive

Set Inclusion

Standardized Accounts

Definitions

Accounts

Deductions

Quantified Statements

Logical Statements

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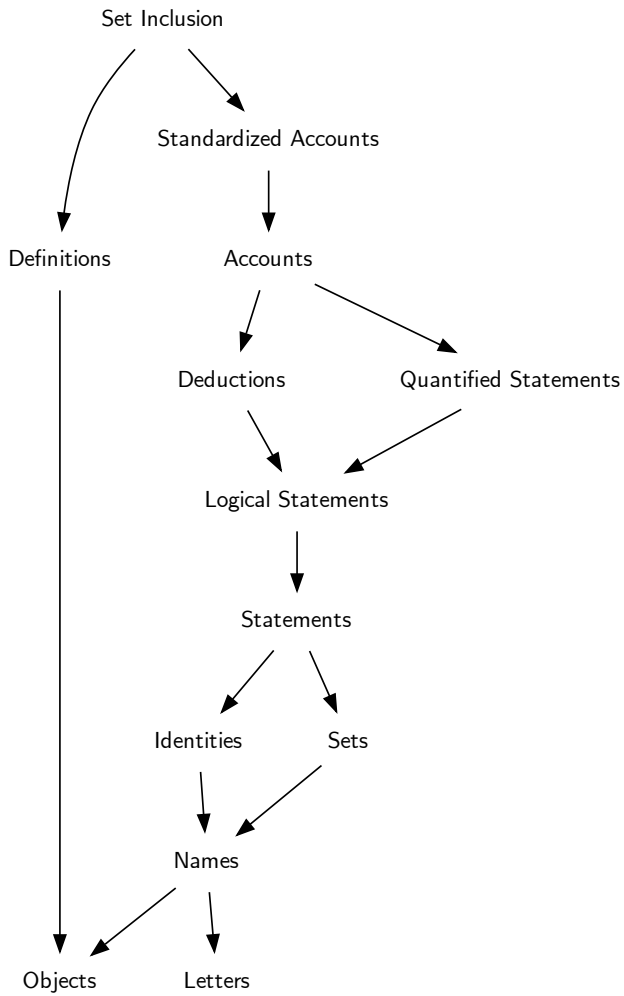
Why

When are two sets the same?

Definition

Let A and B denote sets. If $A = B$ then every element of A is an element of B and every element of B is an element of A. In other words, $(A = B) \longrightarrow ((A \subset B) \wedge (B \subset A))$.

What of the converse? Suppose every element of A is an element of B and every element of B is an element of A. Then $A = B$? We define it to be so. Sets are determined by their members.

Principle 2 (Extension). *Sets are the same if every member of one is a member of the other and vice versa.*

In other words, two sets are identical if and only if every element of one is an element of the other. This principle is sometimes called the *principle of extension*. We refer to the elements of a set as its *extension*. Roughly speaking, we have declared that if we know the extension then we know the set. A set is determined by its extension.

Deductive principle

We can use this definition to deduce $A = B$ if we first deduce $A \subset B$ and $B \subset A$. With these two implications, we use the principle of extension to conclude that the sets are the same.

In other words, $(A = B) \longleftrightarrow ((A \subset B) \wedge (B \subset A))$. We also describe this fact by saying that inclusion ($\subset$) is *antisymmetric*.

Belonging and sets compared with ancestry and humans

Compare the principle of extension for identifying sets from their elements with an analogous principle for identifying people from their ancestors.

We can consider a person's ancestors. Namely, the person's parents, grandparents, great grandparents and so on. It is clear that if we label the same human with two names A and B, then A and B have the same ancestors. In other words, same human implies same ancestors. This is the analog of "if two sets are equal they have the same members".

On the other hand, if we have two people denoted by A and B, and we know that A has the same ancestors as B, we can not conclude that A and B denote the same human. For example, siblings have the same ancestors but are different people. This direction, same ancestors implies same human, is the analogue of "if they have the same elements, two sets are the same". It is false for humans and ancestors, but we define it to be true for sets and members.

The principle of extension is more than a statement about equality. It is also a statement about our notion of belonging, of what it means to be an element of a set, and what a set is.

Set Equality (**??**) immediately needs:

Set Inclusion (**??**)

Set Equality (**??**) is not immediately needed by any sheet.

Set Equality (**??**) gives the following terms.

principle of extension
extension
antisymmetric

Set Equality

↓

Set Inclusion

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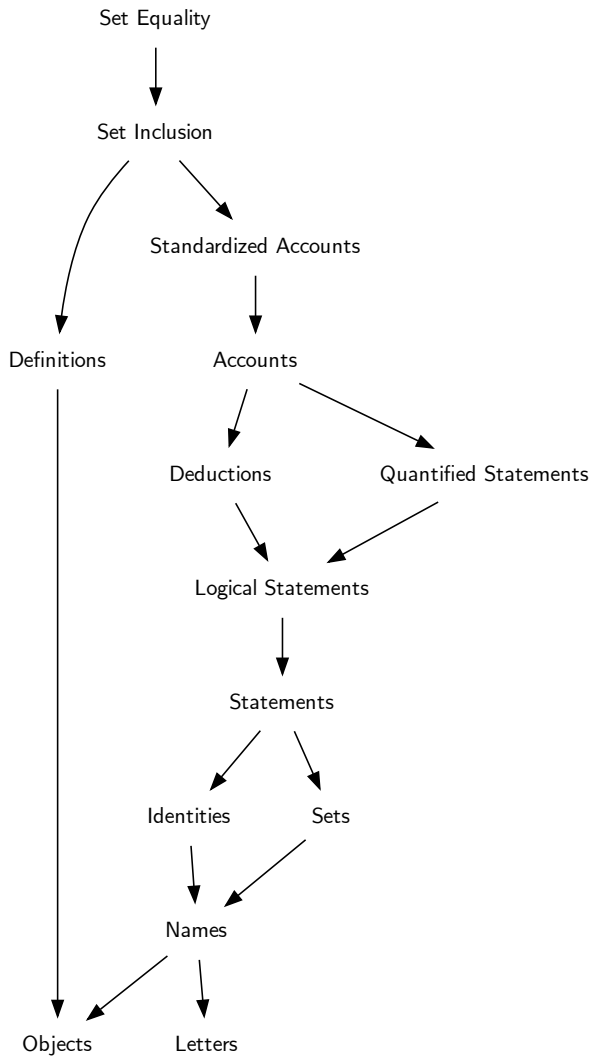
Why

We want to construct new sets out of old ones. So, can we always construct subsets?

Definition

We will say that we can. More specifically, if we have a set and some statement which may be true or false for the elements of that set, a set exists containing all and only the elements for which the statement is true.

Roughly speaking, the principle is like this. We have a set which contains some objects. Suppose the set of playing cards in a usual deck exists. We are taking as a principle that the set of all fives exists, so does the set of all fours, as does the set of all hearts, and the set of all face cards. Roughly, the corresponding statements are "it is a five", "it is a four", "it is a heart", and "it is a face card".

Principle 3 (Specification)**.** *For any statement and any set, there is a subset whose elements satisfy the statement.*

We call this the *principle of specification.* We call the second set (obtained from the first) the set obtained by *specifying* elements according to the sentence. The principle of extension (see Set Equality) says that this set is unique. All basic principles about sets (other than the principle of extension, see Set Equality) assert that we can construct new sets out of old ones in reasonable ways.

Notation

Let A denote a set. Let s denote a statement in which the symbol x and A appear unbound. We assert that there is a set, denote it by B, for which belonging is equivalent to membership in A and s. In other words,

$$(\forall x)((x \in B) \longleftrightarrow ((x \in A) \land s(x))).$$

We denote B by $\{x \in A \mid s(x)\}$. We read the symbol $\mid$ aloud as "such that." We read the whole notation aloud as "a in A such that..." We call it *set-builder notation*.

Nothing contains everything

As an example of the principle of specification and important consequence, consider the statement $x \notin x$. Using this statement and the principle of specification, we can prove that there is not set which contains every thing.

Proposition 3. *No set contains all sets.*[6]

Proof. Suppose there exists a set, denote it A which contains all sets. In other words, suppose $(\exists A)(\forall x)(x \in A)$. Use the principle of specification to construct $B = \{x \in A \mid x \notin x\}$. So $(\forall x)(x \in B \longleftrightarrow (x \in A \land x \notin x))$ In particular, $(B \in B \longleftrightarrow (B \in A \land B \notin B))$. So $B \notin A$. $\qquad\square$

---

[6]We might call such a set, if we admitted its existence, a *universe of discourse* or *universal set*. With the principle of specification, a "principle of a universal set" would give a contradiction (called *Russell's paradox*).

Set Specification (**??**) immediately needs:

  Set Inclusion (**??**)

Set Specification (**??**) is immediately needed by:

  Empty Set (**??**)
  Pair Intersections (**??**)
  Set Differences (**??**)
  Unordered Pairs (**??**)

Set Specification (**??**) gives the following terms.

set-builder notation
principle of specification
specifying
universe of discourse
universal set
Russell's paradox

Set Specification

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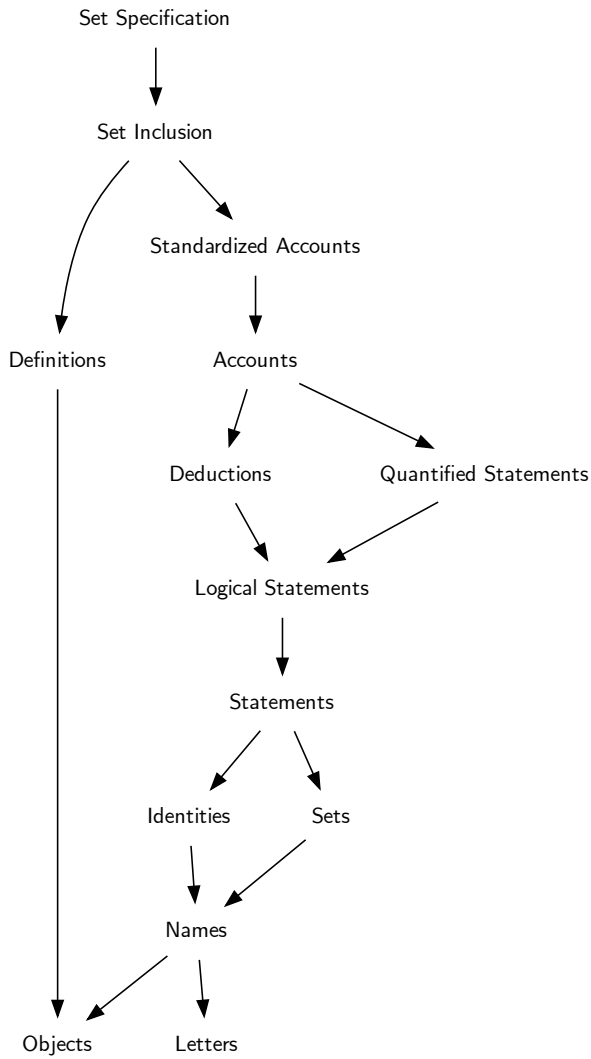
Why

Can a set have no elements?

Definition

Sure. A set exists by the principle of existence (see Sets); denote it by A. Specify elements (see Set Specification) of any set that exists using the universally false statement $x \neq x$. We denote that set by $\{x \in A \mid x \neq x\}$. It has no elements. In other words, $(\forall x)(x \notin A)$. The principle of extension (see Set Equality) says that the set obtained is unique (contradiction).[7]

Definition 2 (Empty Set). We call the unique set with no elements *the empty set*.

Notation

We denote the empty set by $\varnothing$. In other words, in all future accounts (see Accounts), there are two implicit lines. First, "`name` $\varnothing$" and second "`have` $(\forall x)(x \notin \varnothing)$".

Properties

It is immediate from our definition of the empty set and of the definition of inclusion (see Set Inclusion) that the empty set is included in every set (including itself).

Proposition 4. $(\forall A)(\varnothing \subset A)$

---

[7]This account will be expanded in the next edition.

Proof. Suppose toward contradiction that $\varnothing \not\subset A$. Then there exists $y \in \varnothing$ such that $y \notin A$. But this is impossible, since $(\forall x)(x \notin \varnothing)$. $\qquad\square$

Empty Set (**??**) immediately needs:

Set Specification (**??**)

Empty Set (**??**) is immediately needed by:

Set Complements (**??**)
Set Unions (**??**)

Empty Set (**??**) gives the following terms.

the empty set
empty set

Empty Set

Set Specification

Set Inclusion

Standardized Accounts

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Objects   Letters
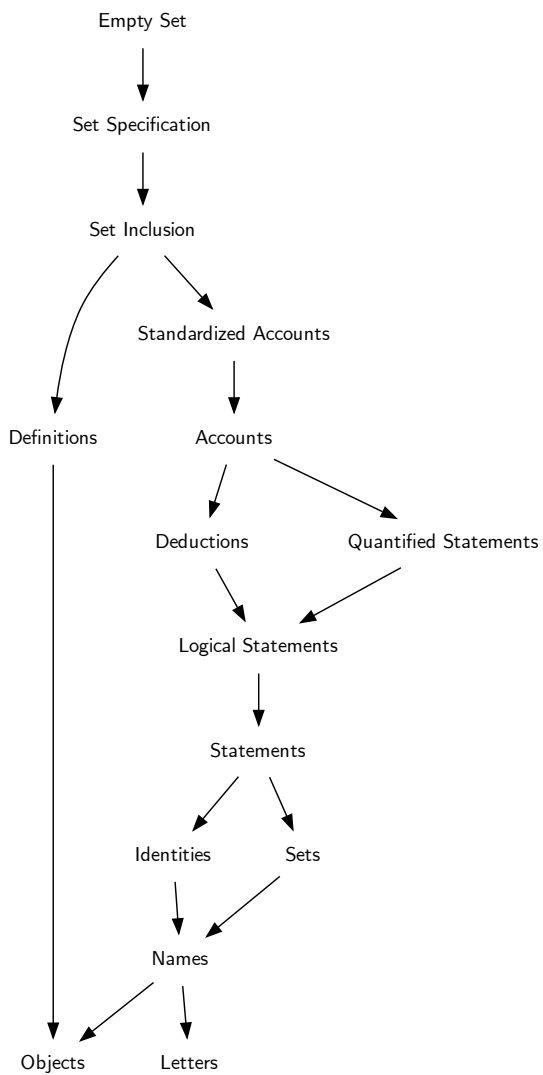
Why

Can we always make a set out of two objects?

Definition

We say yes.

Principle 4 (Pairing). *Given two objects, there exists a set containing them.*

We refer to this as the *principle of pairing*. Denote one object by a and the other by b. This principle gives us the existence of a set that contains the objects. The principle of specification (see Set Specification) gives use the subset for the statement "$x = a \lor x = b$". The principle of extension (see Set Equality) says this set is unique. We call this set a *pair* or an *unordered pair*.

If the object denoted by a is the object denoted by b, then we call the pair the *singleton* of the object denoted by a. Every element of the singleton of the object denoted by a is a.

In other words, the principle of pairing says that every object is an element of some set. That set may be the singleton, or it may be the pair with any other object. We can construct several sets using this principle: the singleton of the object denoted by a, the singleton of the singleton of the object denoted by a, the singleton of the singleton of the singleton of the object denoted by a, and so on.

Notation

We denote the set which contains a and b as elements and nothing else by $\{a, b\}$. The pair of a with itself is the set $\{a, a\}$ is the singleton of a. We denote it by $\{x\}$. The principle of pairing also says that $\{\{a\}\}$ exists and $\{\{\{a\}\}\}$ exists, as well as $\{a, \{a\}\}$.

Note well that $a \neq \{a\}$. a denotes the object a. $\{a\}$ denotes the set whose only element is a. In other words $(\forall x)(x \in \{a\} \longleftrightarrow x = a)$. The moral is that a sack with a potato is not the same thing as a potato.

Unordered Pairs (**??**) immediately needs:

Set Specification (**??**)

Unordered Pairs (**??**) is immediately needed by:

Graphs (**??**)
Ordered Pairs (**??**)
Set Unions (**??**)

Unordered Pairs (**??**) gives the following terms.

principle of pairing
pair
unordered pair
singleton

Unordered Pairs

↓

Set Specification

↓

Set Inclusion

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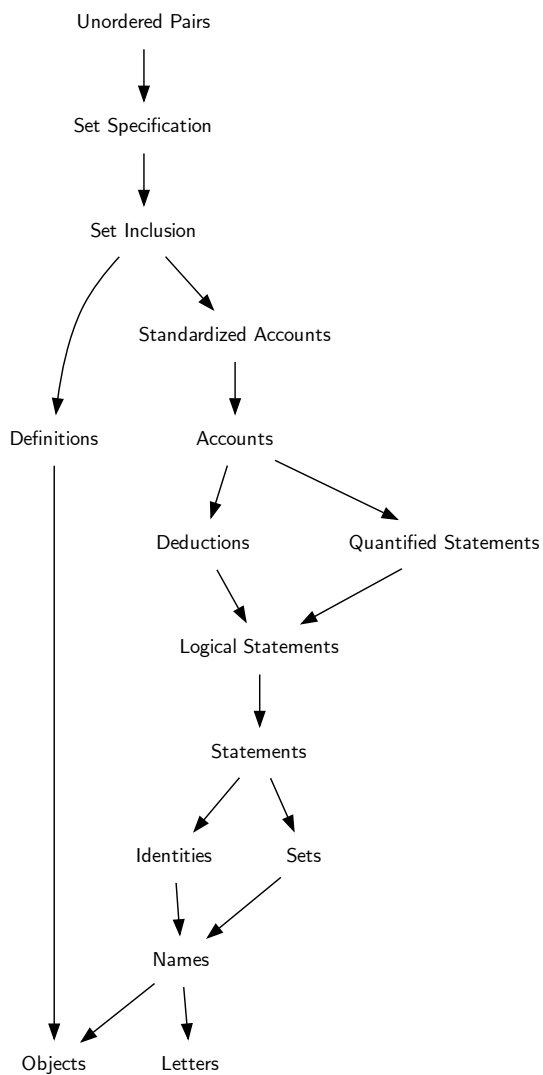
Why

Can we combine sets?

Definition

We say yes. For example, if we have a first set denoted A and a second set denoted B, then we want a third set including all the elements of the set denoted by A and the elements of the set denoted by B. If an object appears in the set denoted by A and in the set denoted by B, it appears in the new set. If an object appears in one set but not the other, it appears in the new set. Indeed, if we have a set of sets, the same should hold.

Principle 5 (Union). *Given a set of sets, there exists a set which contains all elements which belong to any of the sets.*

We call this the *principle of union.* If we have one set and another, the axiom of unions says that there exists a set which contains all the elements that belong to at least one of the former or the latter.

The set guaranteed by the principle of union may contain more elements than just those which are elements of a member of the the given set of sets. No matter: apply the axiom of specification (see Set Specification) to form the set which contains only those elements which are appear in at least one of any of the sets. The set is unique by the principle of extension. We call that unique set *the union* of the sets.

Notation

Let $\mathcal{A}$ be a set of sets. We denote the union of $\mathcal{A}$ by $\bigcup \mathcal{A}$. So

$$(\forall x)((x \in (\bigcup \mathcal{A})) \longleftrightarrow (\exists A)((A \in \mathcal{A}) \wedge x \in A)).$$

Simple Facts

It is reasonable for the union of the empty set to be empty and for the union of the singleton of a set to be itself.

PROPOSITION **1.** $\bigcup \varnothing = \varnothing$

Proof. Immediate[8] $\hfill\square$

PROPOSITION **2.** $\bigcup \{A\} = A$

Proof. Immediate[9] $\hfill\square$

---

[8]Future editions will include the account.
[9]Future editions will include the account.

Set Unions (**??**) immediately needs:

Empty Set (**??**)
Unordered Pairs (**??**)

Set Unions (**??**) is immediately needed by:

Graph Complements (**??**)
Ordered Pair Projections (**??**)
Pair Unions (**??**)
Partitions (**??**)
Set Operations (**??**)
Set Symmetric Differences (**??**)
Trees (**??**)

Set Unions (**??**) gives the following terms.

principle of union
the union

Set Unions

Empty Set   Unordered Pairs

Set Specification

Set Inclusion

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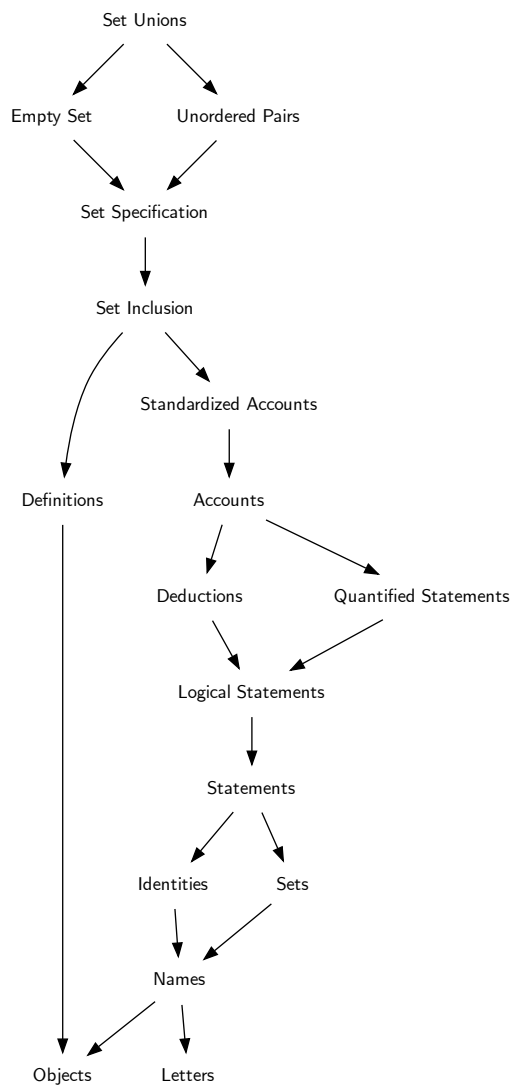
Why

We often unite the elements of one set with another.

Discussion

Let A and B denote sets. We call $\cup\{A, b\}$ the *pair union* of A and B. We denote the union of the pair $\{A, B\}$ by $A \cup B$. Clearly the pair union does not depend on the order of A and B. In other words, $A \cup B = B \cup A$.

Facts

Here are some basic facts about unions of a pair of sets.[10] Let A and B denote sets.

Proposition 5 (Identity Element). $A \cup \varnothing = A$

Proposition 6 (Commutativity). $A \cup B = B \cup A$

Proposition 7 (Associativity). $(A \cup B) \cup C = A \cup (B \cup C)$

Proposition 8 (Idempotence). $A \cup A = A.$

Proposition 9. $A \subset B \longleftrightarrow A \cup B = B$

---

[10]Proofs will appear in the next edition.

Pair Unions (**??**) immediately needs:

Set Unions (**??**)

Pair Unions (**??**) is immediately needed by:

Intersection of Empty Set (**??**)
Set Decompositions (**??**)
Set Dualities (**??**)
Set Unions and Intersections (**??**)
Successor Sets (**??**)
Unordered Triples (**??**)

Pair Unions (**??**) gives the following terms.

pair union

Pair Unions

↓

Set Unions

↙ ↘

Empty Set   Unordered Pairs

↘ ↙

Set Specification

↓

Set Inclusion

↙ ↘

Definitions   Standardized Accounts

↓

Accounts

↙ ↘

Deductions   Quantified Statements

↘ ↙

Logical Statements

↓

Statements

↙ ↘

Identities   Sets

↘ ↙

Names

↙ ↘

Objects   Letters
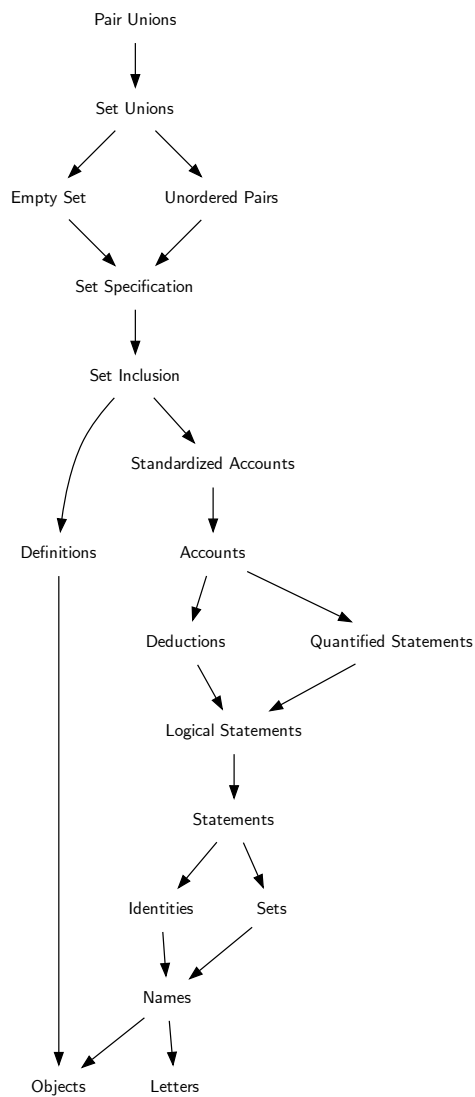
Why

$$\{a\} \cup \{b\} = \{a, b\}$$

Definition

Let a, b and c denote objects. From the associativity of pair unions (see Pair Unions), we have

$$(\{a\} \cup \{b\}) \cup \{b\} = \{a\} \cup (\{b\} \cup \{c\}).$$

So we will drop the parentheses, and write $\{a\} \cup \{b\} \cup \{c\}$. We call such a set the *unordered triple* of a, b and c. The unordered triple of a, b and c is the set containing these elements and no others.

Notation

Such sets are so commonplace that we denote the unordered triple of a, b and c by $\{a, b, c\}$.

Quadruples

Let d denote an object. Again, the associativity of pair unions allows us to drop the parentheses from

$$((((\{a\} \cup \{b\}) \cup \{c\}) \cup \{d\})).$$

We can therefore write $\{a\} \cup \{b\} \cup \{c\} \cup \{d\}$ without ambiguity. We call this set the *unordered quadruple*. As before, the unordered quadruple contains of a, b, c and d contains a, b, c, and d and nothing besides these.

Notation

We denote the unordered quadruple of the objected denoted by a, b, c and d, denote this set by $\{a, b, c, d\}$.

The case of several named objects

In a similar way we speak of *unordered pentuples*, *unordered sextuples*, *unordered septuples* and so on. If we have several objects named, we denote the set containing these objects be writing their names in between the left brace $\{$ and right brace$\}$, separating the names by commas. For example, if we A, b, x and Y and z denote objects, then we denote the set containing these elements by

$$\{A, b, x, Y, z\}.$$

Unordered Triples (**??**) immediately needs:

Pair Unions (**??**)

Unordered Triples (**??**) is immediately needed by:

Counts (**??**)
Ordering Sets (**??**)
Set Powers (**??**)

Unordered Triples (**??**) gives the following terms.

unordered triple
unordered quadruple
unordered pentuples
unordered sextuples
unordered septuples

Unordered Triples

↓

Pair Unions

↓

Set Unions

Empty Set   Unordered Pairs

Set Specification

↓

Set Inclusion

Standardized Accounts

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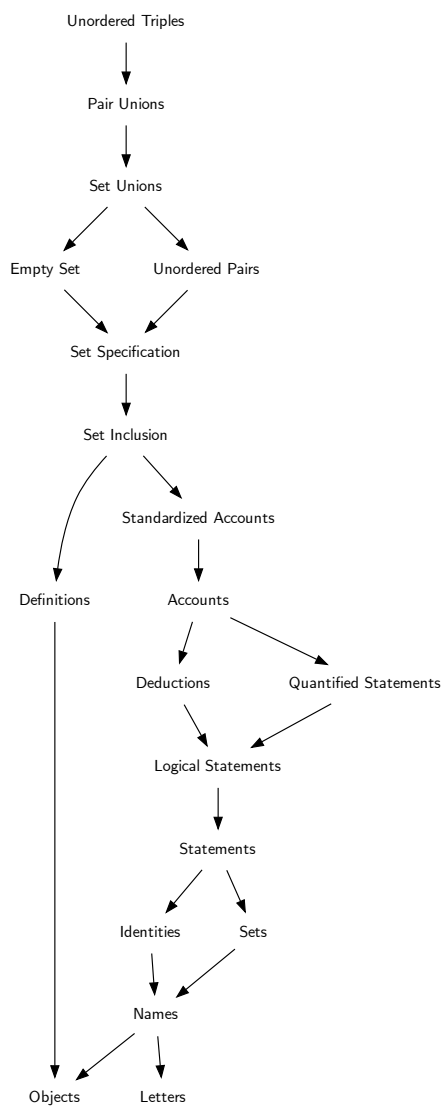
Why

Does a set exist containing the elements shared between two sets? How might we construct such a set?

Definition

Let A and B denote sets. Consider the set $\{x \in A \mid x \in B\}$. This set exists by the principle of specification (see Set Specification). Moreover $(y \in \{x \in A \mid x \in B\}) \longleftrightarrow (y \in A \land y \in B)$. In other words, $\{x \in A \mid x \in B\}$ contains all the elements of A that are also elements of B.

We can also consider $\{x \in B \mid x \in A\}$, in which we have swapped the positions of A and B. Similarly, the set exists by the principle of specification (see Set Specification) and again $y \in \{x \in B \mid x \in A\} \longleftrightarrow (y \in B \land y \in B)$. Of course, $y \in A \land y \in B$ means the same as[11] $y \in B \land y \in A$ and so by the principle of extension (see Set Equality)

$$\{x \in A \mid x \in B\} = \{x \in B \mid x \in A\}.$$

We call this set the *pair intersection* of the set denoted by A with the set denoted by B.

Notation

We denote the intersection fo the set denoted by A with the set denoted by B by $A \cap B$. We read this notation aloud as "A intersect B".

---

[11]Future editions will name and cite this rule.

Basic Properties

All the following results are immediate.[12]

Proposition 10. $A \cap \varnothing = \varnothing$

Proposition 11 (Commutativity)**.** $A \cap B = B \cap A$

Proposition 12 (Associativity)**.** $(A \cap B) \cap C = A \cap (B \cap C)$

Proposition 13. $A \cap A = A$

Proposition 14. $(A \subset B) \longleftrightarrow (A \cap B = A)$.

---

[12]Proofs of these results will appear in the next edition.

Pair Intersections (**??**) immediately needs:

Set Specification (**??**)

Pair Intersections (**??**) is immediately needed by:

Set Decompositions (**??**)
Set Dualities (**??**)
Set Intersections (**??**)
Set Unions and Intersections (**??**)
Successor Sets (**??**)

Pair Intersections (**??**) gives the following terms.

pair intersection

Pair Intersections

Set Specification

Set Inclusion

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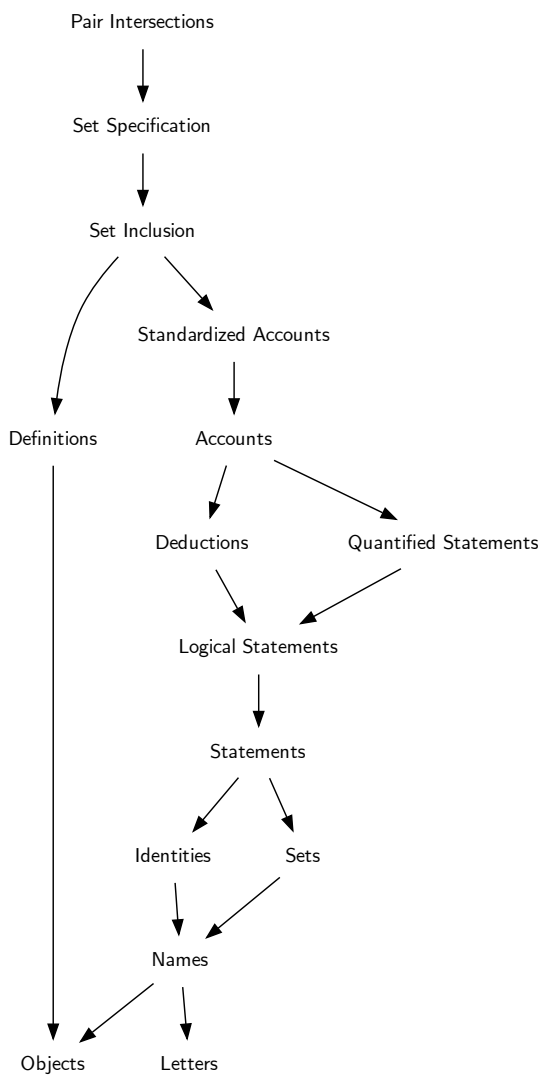
Why

We can consider intersections of more than two sets.

Definition

Let $\mathcal{A}$ denote a set of sets. In other words, every element of $\mathcal{A}$ is a set. And suppose that $\mathcal{A}$ has at least one set (i.e., $\mathcal{A} \neq \varnothing$). Let C denote a set such that $C \in \mathcal{A}$. Then consider the set,

$$\{x \in C \mid (\forall A)(A \in \mathcal{A} \longrightarrow x \in A)\}.$$

This set exists by the principle of specification (see Set Specification). Moreover, the set does not depend on which set we picked. So the dependence on C does not matter. It is unique by the axiom of extension (see Set Equality). This set is called the *intersection* of $\mathcal{A}$.

Notation

We denote the intersection of $\mathcal{A}$ by $\bigcap \mathcal{A}$.

Equivalence with pair intersections

As desired, the the set denoted by $\mathcal{A}$ is a pair (see Unordered Pairs) of sets, the pair intersection (see Pair Intersections) coincides with intersection as we have defined it in this sheet.[13]

Proposition 15. $\bigcap\{A, B\} = A \cap B$

---

[13]A full account of the proof will appear in future editions.

Set Intersections (??) immediately needs:

  Pair Intersections (??)

Set Intersections (??) is immediately needed by:

  Intersection of Empty Set (??)
  Natural Numbers (??)
  Partitions (??)
  Powers and Intersections (??)
  Set Operations (??)

Set Intersections (??) gives the following terms.

intersection

Set Intersections

↓

Pair Intersections

↓

Set Specification

↓

Set Inclusion

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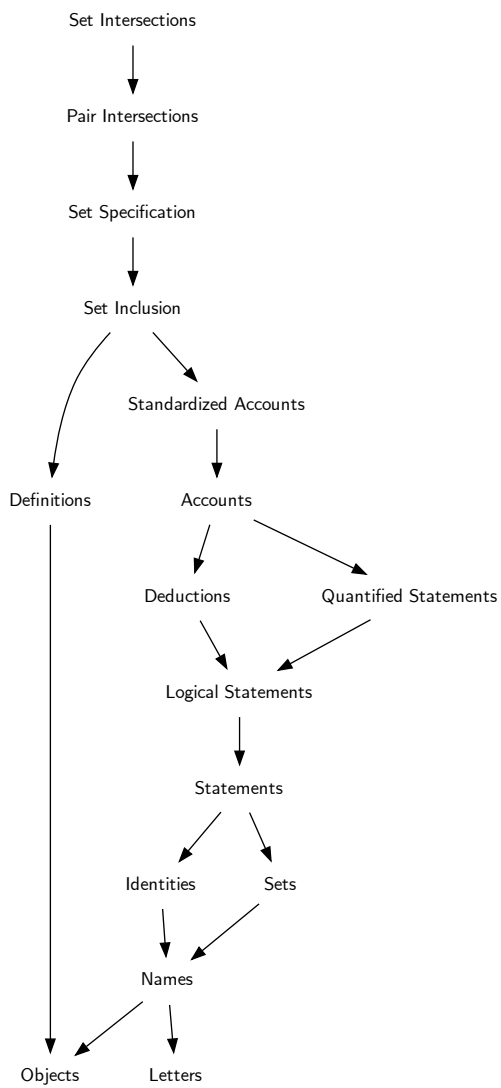
Why

We only define set intersections for nonempty sets of sets.
Why?

Discussion

Which objects are specified by the sentence $(\forall x \in \varnothing)(x \in X)$?
Well, since no objects fail to satisfy the statement,[14] the sentence specifies all objects. So in other words, the condition we used to define set intersections (Set Intersections) specifies the "set of everything". In order to maintain other more desirable set principles like selection, we have said that such a set does not exist (see Set Specification).

If, however, all sets under consideration are subsets of one paticular set—denote it E—then we can define intersections as follows. Let $\mathcal{C}$ be a possibly nonempty collection of sets

$$\bigcap \mathcal{C} = \{X \in E\}(\forall X \in \mathcal{C})(x \in X).$$

This definition agrees with that given in Set Intersections. In particular, it is the intersection of the set $\mathcal{C} \cup \{E\}$

Another definition

This begs the following question. Why not define intersections by selecting from the union. Let $\mathcal{A}$ be a possibly nonempty

---

[14]Future editions will offer an account of this.

set of sets. Then define:

$$\bigcap \mathcal{A} = \{x \in \bigcup \mathcal{A} \mid (\forall A \in \mathcal{A})(x \in A)\}.$$

If $\mathcal{A}$ is empty, so is $\bigcup \mathcal{A}$ and then there are no elements in the set to select from so $\bigcap \mathcal{A}$ is empty. This does not agree with the previous definitions for the empty set, but does for all other sets of sets.

For these reasons, the intersection of the empty set is a delicate thing.[15]

---

[15]Future editions are likely to expand on, and perhaps mention why we prefer the former definition.

Intersection of Empty Set (??) immediately needs:

Pair Unions (??)
Set Intersections (??)

Intersection of Empty Set (??) is immediately needed by:

Generalized Set Dualities (??)

Intersection of Empty Set (??) gives no terms.

Intersection of Empty Set

Pair Unions
Set Intersections

Set Unions
Pair Intersections

Empty Set
Unordered Pairs

Set Specification

Set Inclusion

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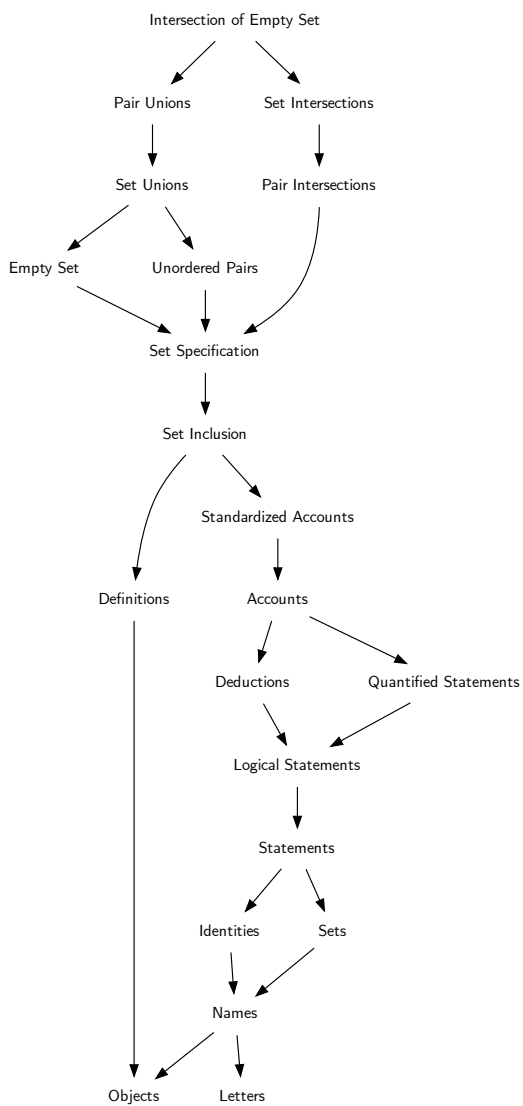
Why

We study how intersection and union interact.

Results

The following are easy results.[16] They are known as the *distributive laws.*

Proposition 16. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proposition 17. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

---

[16]The accounts will appear in future editions.

Set Unions and Intersections (??) immediately needs:

Pair Intersections (??)
Pair Unions (??)

Set Unions and Intersections (??) is immediately needed by:

Family Unions and Intersections (??)

Set Unions and Intersections (??) gives the following terms.

distributive laws

Set Unions and Intersections

Pair Unions

Pair Intersections

Set Unions

Empty Set

Unordered Pairs

Set Specification

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Why
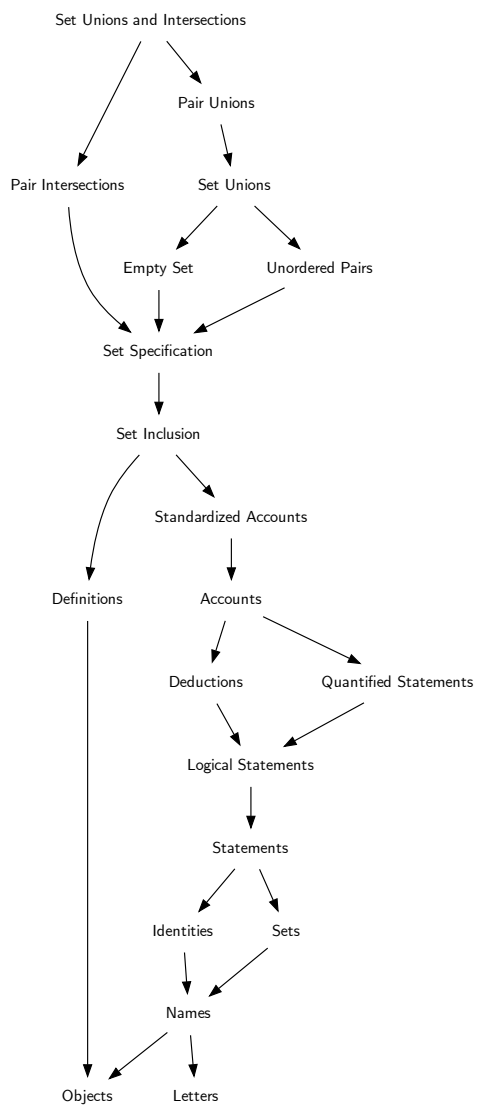
We want to consider the elements of one set which are not contained in another set.

Definition

Let A and B denote sets. The *difference* between A and B is the set $\{x \in A \mid x \notin B\}$. It is not necessary that $B \subset A$.

Notation

We denote the difference between A and B by $A - B$.

Properties

The following are straightforward.[17]

Proposition 18. $A - \varnothing = A$

Proposition 19. $A - A = \varnothing$

---

[17]Accounts will appear in future editions.

Set Differences (**??**) immediately needs:

Set Specification (**??**)

Set Differences (**??**) is immediately needed by:

Set Complements (**??**)

Set Differences (**??**) gives the following terms.

difference

Set Differences

↓

Set Specification

↓

Set Inclusion

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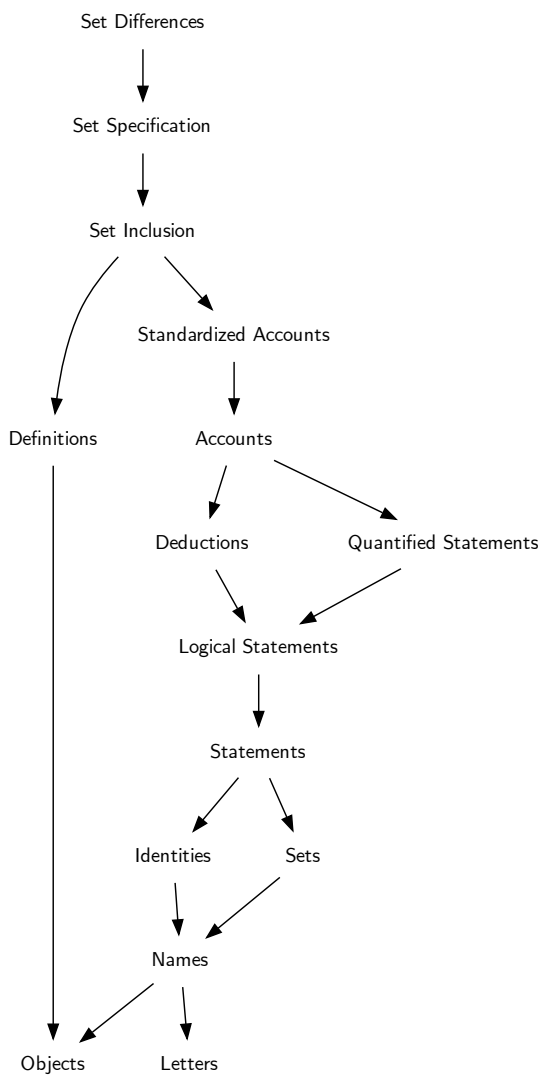
Why

It is often the case in considering set differences that all sets considered are subsets of one set.

Definition

Let A and B denote sets. In many cases, we take the difference between a set and one contained in it. In other words, we assume that $B \subset A$. In this case, we often take complements relative to the same set A. So we do not refer to it, and instead refer to the relative complement of B in A as the *complement* of B.

Notation

Let A denote a set, and let B denote a set for which $B \subset A$. We denote the relative complement of B in A by $C_A(B)$. When we need not mention the set A, and instead speak of the complement of B without qualification, we denote this complement by $C(B)$.

Complement of a complement

One nice property of a complement when $B \subset A$ is:

Proposition 20. $(B \subset A) \longleftrightarrow (C_A(C_A(B)) = B)$

Basic Facts

Let E denote a set and let A and B denote sets satisfying $A, B \subset E$. Then take all complements with respect to E. Here are some immediate consequences of the definition of complements.[18]

Proposition 21. $C(C(A)) = A$

Proposition 22. $C(\varnothing) = E$

Proposition 23. $C(E) = \varnothing$

Proposition 24. $A \subset B \longleftrightarrow C(B) \subset C(A)$

---

[18]Proofs will appear in future editions.

Set Complements (**??**) immediately needs:

  Empty Set (**??**)
  Set Differences (**??**)

Set Complements (**??**) is immediately needed by:

  Graph Complements (**??**)
  Set Decompositions (**??**)
  Set Dualities (**??**)
  Set Symmetric Differences (**??**)

Set Complements (**??**) gives the following terms.

  complement

Set Complements

Empty Set   Set Differences

Set Specification

Set Inclusion

Standardized Accounts

Definitions   Accounts

Deductions   Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Objects   Letters
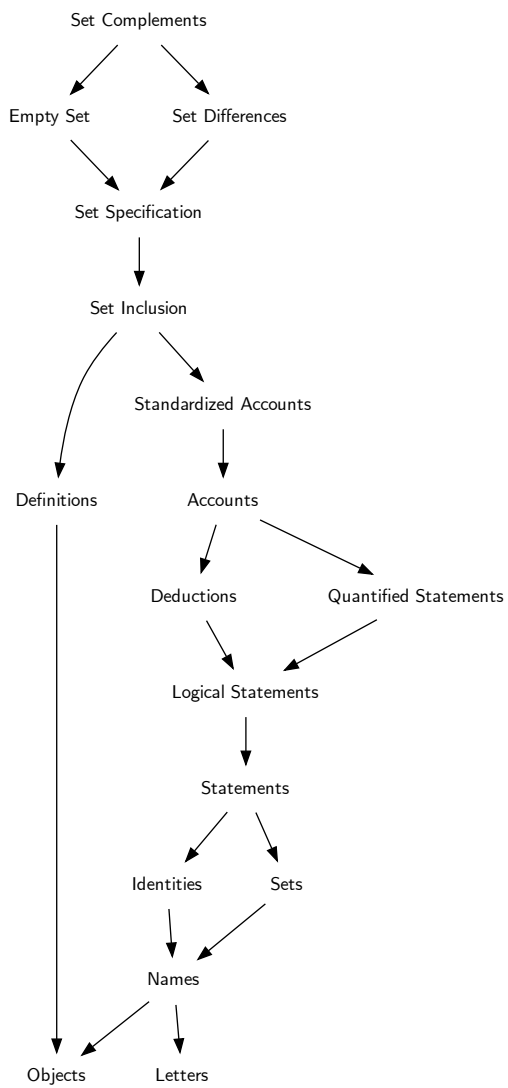
Why

Let E denote a set and let A denote a set with $A \subset E$. A and $C(A)$ as breaking E into two pieces which do not overlap.

Discussion for complements

To make this precise, let us say that by "breaking E into two pieces" we mean that these two pieces are all of E. In other words, every element of E is contained either in A or $C(A)$. We use the language of set unions (Pair Unions).

Proposition 25 (Breaking). $A \cup C(A) = E$

Next, let us say that "do not overlap" means that no elemenet of A is an element of $C(A)$ and vice versa. We use the language of set interserctions (see Pair Intersections).

Proposition 26 (Non-overlapping). $A \cap C(A) = \varnothing$

Definition

We call a pair $\{A, B\}$ a *decomposition* of E if $A \cap B = \varnothing$ and $A \cup B = E$. If $A \cap B$ we say that $\{A, B\}$ are *disjoint*. If we have a set of sets $\mathcal{A}$ satisfying $(A \in \mathcal{A} \wedge B \in \mathcal{A}) \longrightarrow (A \cap B = \varnothing)$ then we call $\mathcal{A}$ *pairwise disjoint*.

Set Decompositions (**??**) immediately needs:

Pair Intersections (**??**)
Pair Unions (**??**)
Set Complements (**??**)

Set Decompositions (**??**) is immediately needed by:

Set Exercises (**??**)

Set Decompositions (**??**) gives the following terms.

decomposition
disjoint
pairwise disjoint

Set Decompositions

Pair Intersections → Pair Unions

Set Unions → Set Complements

Unordered Pairs → Empty Set → Set Differences

Set Specification

Set Inclusion

Standardized Accounts

Definitions → Accounts

Deductions → Quantified Statements

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Identities → Sets

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Objects → Letters
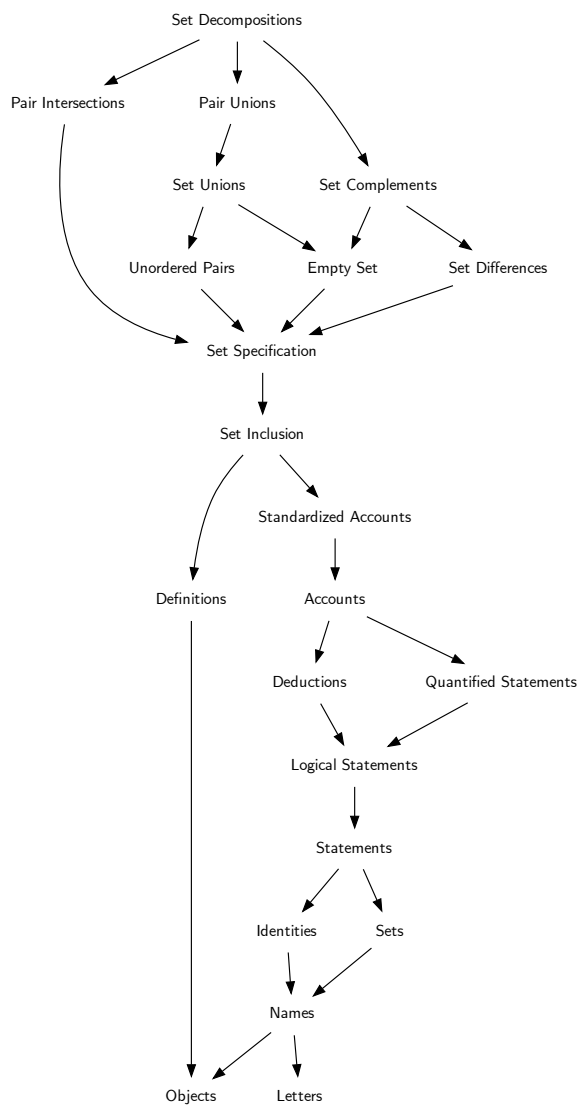
Why

We divide a set into disjoint subsets whose union is the whole set. In this way we can handle each subset of the main set individually, and so handle the entire set piece by piece.

Definition

A *partition* of a set X is a set of pairwise disjoint (see Set Decompositions) subsets of X whose union is X. We call the elements ofa partition the *pieces* of the partition. When speaking of a partition, we commonly call the set of sets *mutually exclusive* (by which we mean that they are pairwise disjoint) and *collectively exhaustive* (by which we mean that their union is full set).

Notation

Let X be a set and $\mathcal{C}$ be a set of subsets of X. $\mathcal{C}$ is a partition of X means $(\forall A)(\forall B)((A \in \mathcal{C} \wedge A \in \mathcal{C}) \longrightarrow A \cap B = \varnothing)$ and $\bigcup \mathcal{C} = X$.

Partitions (**??**) immediately needs:

Set Intersections (**??**)
Set Unions (**??**)

Partitions (**??**) is immediately needed by:

Equivalence Relations (**??**)
Simple Functions (**??**)
Split Graphs (**??**)
Total Probability (**??**)

Partitions (**??**) gives the following terms.

partition
pieces
mutually exclusive
collectively exhaustive

Partitions

Set Intersections   Set Unions

Pair Intersections   Empty Set   Unordered Pairs

Set Specification

Set Inclusion

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Names

Objects   Letters
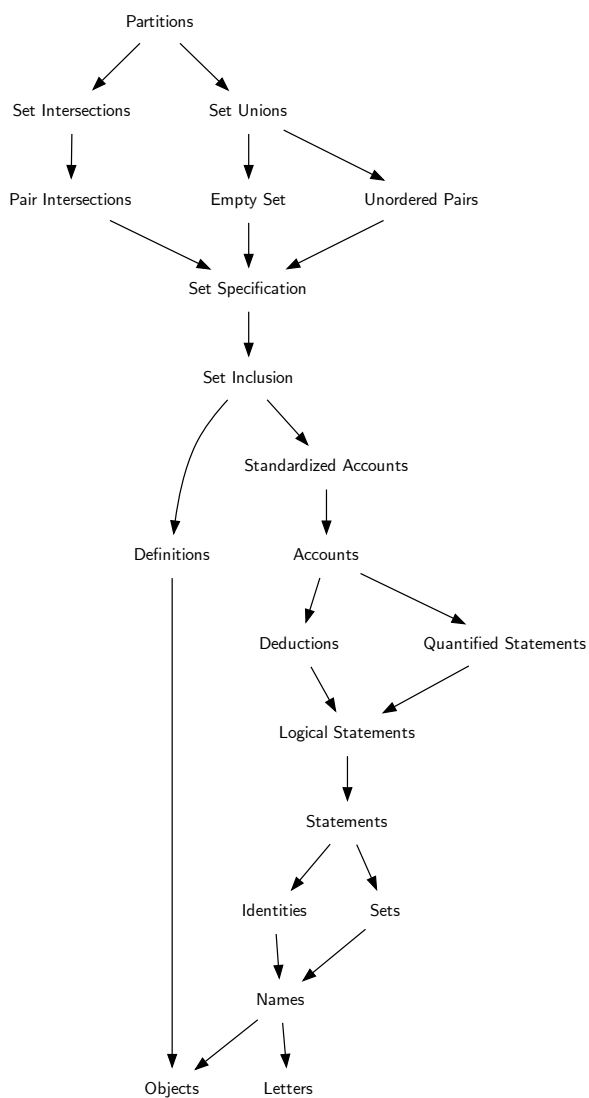
Why

How does taking complements relate to forming unions and intersections.

Complements of unions or intersections

Let E denote a set. Let A and B denote sets and $A, B \subset E$. All complements are taken with respect to E. The following are known as *DeMorgan's Laws*.[19]

Proposition 27. $C(A \cup B) = C(A) \cap C(B)$

Proposition 28. $C(A \cap B) = C(A) \cup C(B)$

Principle of duality

As a result of DeMorgan's Laws[20] and basic facts about complements (see Set Complements) theorems having to do with sets come in pairs. In other words, given an inclusion or identity relation involving complements, unions and intersections of some set (above E) if we replace all sets by their complemnets, swap unions and intersections, and flip all inclusions we obtain another result. This is called the *principle of duality for sets*.

---

[19]Proofs will appear in a future edition.

[20]A future edition will change the name to remove the reference to DeMorgan in accordance with the project's policy.

Set Dualities (**??**) immediately needs:

Pair Intersections (**??**)
Pair Unions (**??**)
Set Complements (**??**)

Set Dualities (**??**) is immediately needed by:

Generalized Set Dualities (**??**)
Set Exercises (**??**)

Set Dualities (**??**) gives the following terms.

DeMorgan's Laws
principle of duality for sets

Set Dualities

Pair Intersections

Pair Unions

Set Unions

Set Complements

Unordered Pairs

Empty Set

Set Differences

Set Specification

Set Inclusion

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Identities

Sets

Names

Objects
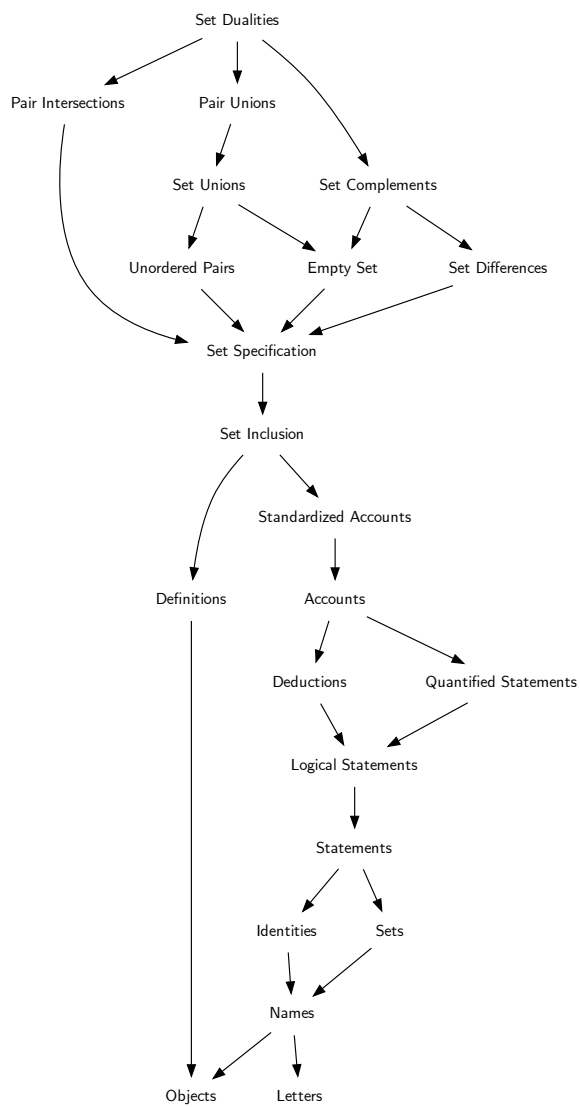
Letters

Why

Here are some exercises on sets.[21]

Exercise 1. *Let A, B, C denote sets. Show $((A \cap B) \cup C = A \cap (B \cup C)) \longleftrightarrow (C \subset A)$ Observe that the condition does not involve B.*

Exercise 2.
$$A - B = A \cap B'.$$

Exercise 3.
$$A \subset B \text{ if and only if} A - B = \varnothing.$$

Exercise 4.
$$A - (A - B) = A \cap B.$$

Exercise 5.
$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

Exercise 6.
$$(A \cap B) \subset ((A \cap C) \cup (A \cap C')).$$

Exercise 7.
$$((A \cup C) \cap (B \cup C')) \subset (A \cup B).$$

---

[21]Future editions will give the hypotheses more clearly.

Set Exercises (**??**) immediately needs:

Set Decompositions (**??**)
Set Dualities (**??**)

Set Exercises (**??**) is not immediately needed by any sheet.

Set Exercises (**??**) gives no terms.

Set Exercises

Set Decompositions → Set Dualities

Pair Unions → Set Complements → Pair Intersections

Set Unions → Set Differences

Unordered Pairs → Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Definitions → Accounts

Deductions → Quantified Statements

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Identities → Sets

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Objects → Letters

Why
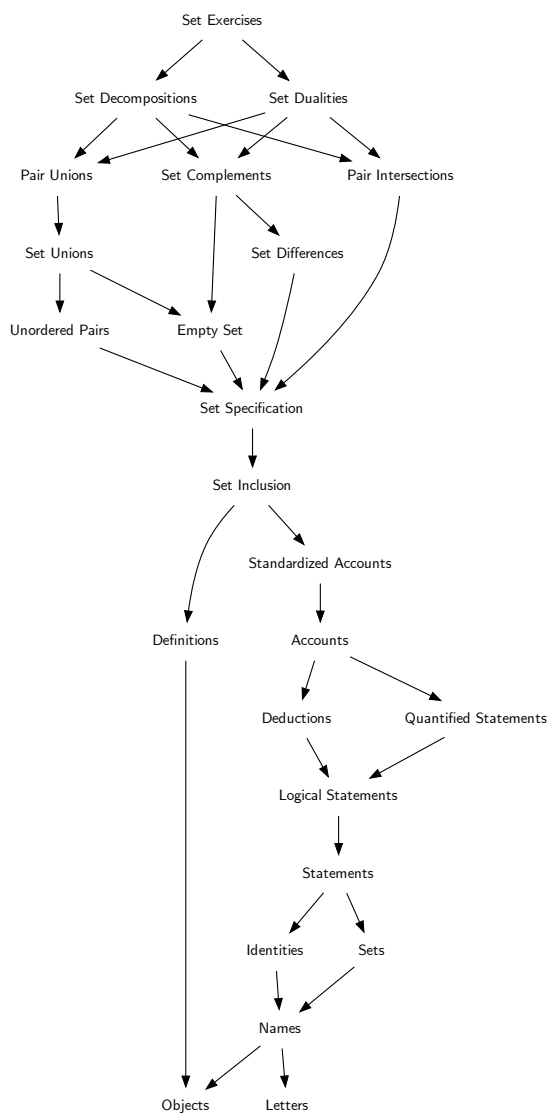
We want to consider the no-overlapping elements of a pair of sets.

Definition

In other words, we want to consider the set of elements which is one or the other but not in both. The *symmetric difference* of a set with another set is the union of the difference between the latter set and the former set and the difference between the former and the latter. The symmetric differences is also called the *Boolean sum* of A and B[22]

Notation

Let A and B denote sets. We denote the symmetric difference by $A + B$.
$$A + B = (A - B) \cup (B - A)$$

Properties

Here are some immediate properties of symmetric differences.[23]

Proposition 29 (Commutative)**.** $A + B = B + A$.

Proposition 30 (Associative)**.** $(A + B) + C = A + (B + C)$.

---

[22]Future editions will likely remove or modify this term in accordance with the project's policy on using names.

[23]Future editions will have more detailed (but obvious) hypotheses stated.

Proposition 31 (Identity). $(A + \varnothing) = A$

Proposition 32 (Inverse). $(A + A) = \varnothing$

Set Symmetric Differences (**??**) immediately needs:

Set Complements (**??**)
Set Unions (**??**)

Set Symmetric Differences (**??**) is immediately needed by:

Set Operations (**??**)

Set Symmetric Differences (**??**) gives the following terms.

symmetric difference
Boolean sum

Set Symmetric Differences

Set Complements   Set Unions

Set Differences   Empty Set   Unordered Pairs

Set Specification

Set Inclusion

Standardized Accounts

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SET POWERS

Why
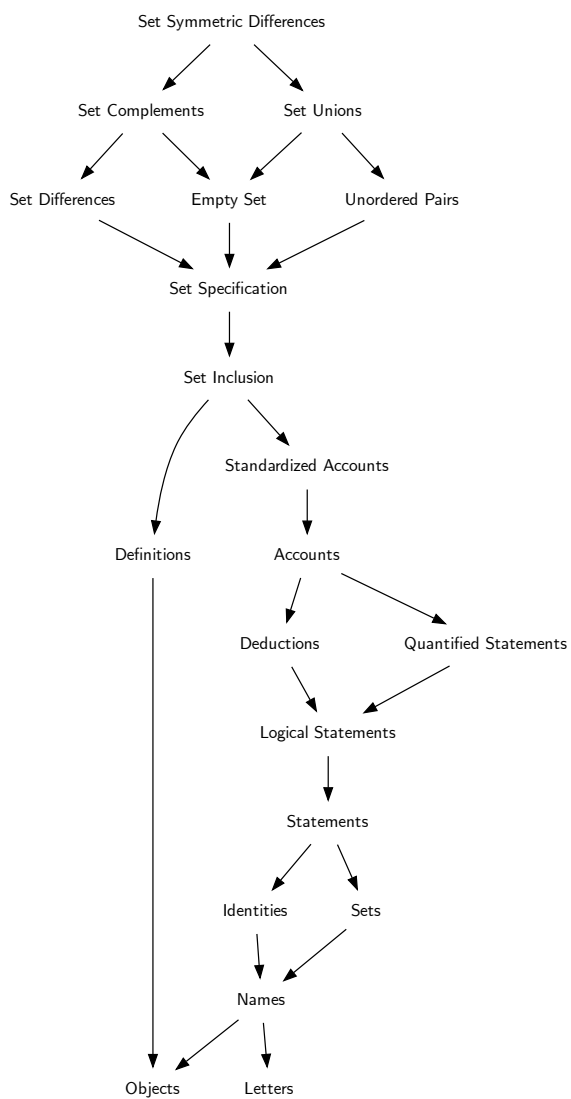
We want to consider all the subsets of a given set.

Definition

We do not yet have a principle stating that such a set exists, but our intuition suggests that it does.

Principle 6 (Powers). *For every set, there exists a set of its subsets.*

We call the existence of this set the *principles of powers* and we call the set the *power set.*[24] As usual, the principle of extension gives uniqueness (see Set Equality). The power set of a set includes the set itself and the empty set.

Notation

Let A denote a set. We denote the power set of A by A^*, read aloud as "powerset of A." $A \in A^*$ and $\varnothing \in A^*$. However, $A \subset A^*$ is false.

Examples

Let a, b, c denote distinct objects. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $B \subset A$. In other notation, $B \in A^*$. We can walk through examples of power sets.

---

[24]This terminology is standard, but unfortunate. Future editions may change these terms.

Empty Set

Proposition 33. $\varnothing^* = \{\varnothing\}$

Singletons

Proposition 34. $\{a\}^* = \{\varnothing, \{a\}\}$

Pairs

Proposition 35. $\{a, b\}^* = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

Triples

Proposition 36. $\{a, b, c\}^* = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b,$

Properties

We can guess the following easy properties.[25]

Proposition 37. $\varnothing \in A^*$

Proposition 38. $A \in A^*$

We call A and $\varnothing$ the *improper* subsets of A. All other subset we call *proper*.

Basic Fact

Proposition 39. $E \subset F \longrightarrow E^* \subset F^*$

---

[25]Future editions will expand this account.

Set Powers (**??**) immediately needs:

  Unordered Triples (**??**)

Set Powers (**??**) is immediately needed by:

  Cartesian Products (**??**)
  Characteristic Functions (**??**)
  Powers and Intersections (**??**)
  Powers and Unions (**??**)
  Probability Events (**??**)
  Real Length Impossible (**??**)
  Subset Systems (**??**)

Set Powers (**??**) gives the following terms.

principles of powers
power set
improper
proper

Set Powers

↓

Unordered Triples

↓

Pair Unions

↓

Set Unions

↓ ↓

Empty Set   Unordered Pairs

↓ ↓

Set Specification

↓

Set Inclusion

↓ ↓

Definitions   Standardized Accounts

↓

Accounts

↓ ↓

Deductions   Quantified Statements

↓ ↓

Logical Statements

↓

Statements

↓ ↓

Identities   Sets

↓ ↓

Names

↓ ↓

Objects   Letters

Why
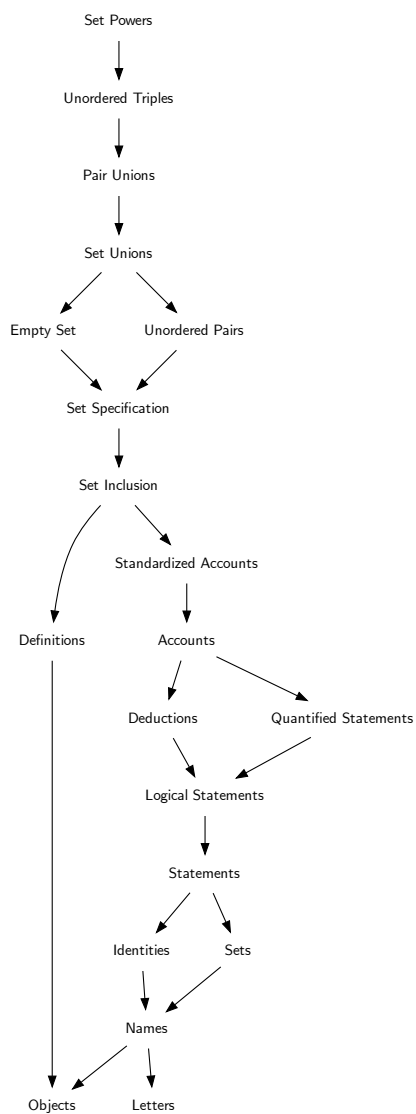
How does the power set relate to an intersection?

Notation Preliminaries

First, if we have a set of sets—denote it $\mathcal{C}$—and all members are subsets of a fixed set—denote it E—then the set of sets is a subset of E^*. In this case, we can write

$$\bigcap \{X \in E^* \mid x \in \mathcal{C}\}$$

Which is a sort of justification for the notation

$$\bigcap_{X \in \mathcal{C}} X.$$

Basic Properties

Here are some basic interactions between the powerset and intersections.[26]

Proposition 40. $A^* \cap F^* = (A \cap F)^*$

Proposition 41. $\bigcap_{X \in \mathcal{A}} A^* = \left(\bigcap_{X \in \mathcal{A}} A\right)^*$

Proposition 42. $\bigcap_{X \in E^*} X = \varnothing$

---

[26]Future editions will expand on these propositions and provide accounts of them.

Powers and Intersections (**??**) immediately needs:

Set Intersections (**??**)
Set Powers (**??**)

Powers and Intersections (**??**) is not immediately needed by any sheet.

Powers and Intersections (**??**) gives no terms.

Powers and Intersections

Set Intersections

Set Powers

Unordered Triples

Pair Intersections

Pair Unions

Set Unions

Empty Set

Unordered Pairs

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Why
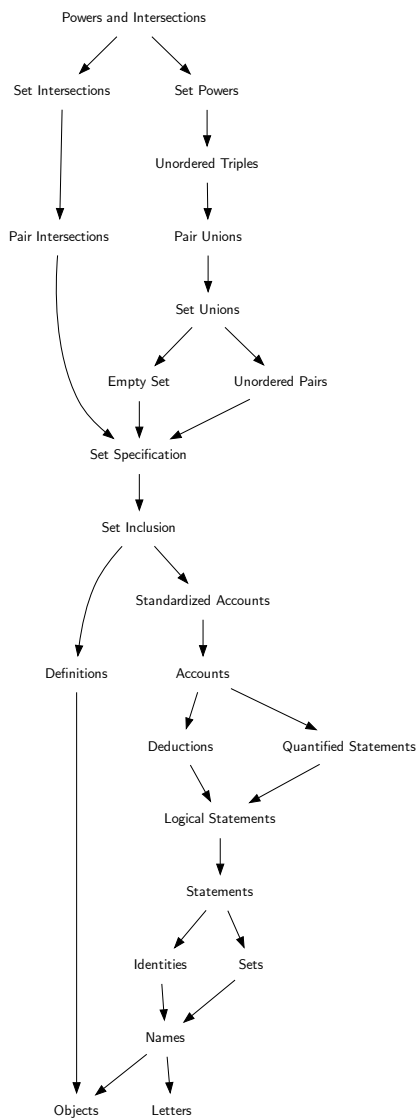
How does the power set relate to a union?

Notation Preliminaries

Let E denote a set. Let $\mathcal{A}$ denote a set of subsets of the set denoted by E. We define $\bigcup_{A \in \mathcal{A}} A$ to mean $\cap \mathcal{A}$.

Basic Properties

Here are some basic interactions between the powerset and unions.[27]

Proposition 43. $E^* \cup F^* \subset (E \cup F)^*$

Proposition 44. $\bigcup_{X \in \mathcal{C}} X^* \subset (\bigcup_{X \in \mathcal{C}} X)^*$

Proposition 45. $E = \bigcup E^*$

Proposition 46. $(\bigcup E)^* \supset E$.

Typically $E \neq (\bigcup E)^*$, in which case E is a proper subset.

---

[27]Future editions will expand on these propositions and provide accounts of them.

Powers and Unions (**??**) immediately needs:

Set Powers (**??**)

Powers and Unions (**??**) is not immediately needed by any sheet.

Powers and Unions (**??**) gives no terms.

Powers and Unions

↓

Set Powers

↓

Unordered Triples

↓

Pair Unions

↓

Set Unions

Empty Set   Unordered Pairs

Set Specification

↓

Set Inclusion

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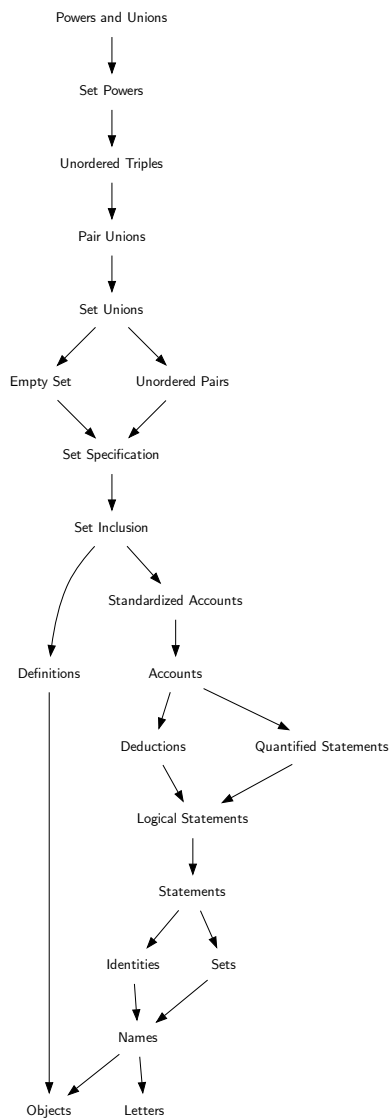
Why

If all sets considered in a union or intersection are subsets of a fixed set, then the union and intersection of any set of sets is well defined. We can then derive generalized version of DeMorgan's laws.[28]

New Notation

Let E denote a set. Let $\mathcal{A}$ denote a set of subsets of E. Then define

$$\bigcup_{A \in \mathcal{A}} A := \bigcup \mathcal{A}, \quad \bigcap_{A \in \mathcal{A}} A := \bigcap \mathcal{A}.$$

In this case we have

Proposition 47. $C(\cup_{A \in \mathcal{A}} A) = \cap_{A \in A} C(A).$

Proposition 48. $C(\cap_{A \in \mathcal{A}} A) = \cup_{A \in A} C(A).$

---

[28]In future editions, this sheet may not exist.

Generalized Set Dualities (??) immediately needs:

 Intersection of Empty Set (??)
 Set Dualities (??)

Generalized Set Dualities (??) is immediately needed by:

 Family Unions and Intersections (??)

Generalized Set Dualities (??) gives no terms.

Generalized Set Dualities

Set Dualities
Intersection of Empty Set

Pair Unions
Set Intersections

Set Complements
Set Unions
Pair Intersections

Set Differences
Empty Set
Unordered Pairs

Set Specification

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Why

We want to arrange the elements of a set in an order using only the concept of sets.

Discussion

What does this mean? Well, we often arrange objects in orders. For example, the letters of this page are arranged into words. Take two such words: 'note' and 'tone'. If letters are objects, what are words?

A first guess is that words seem like groups of letters, and sets seem like groups, and so a word is a set of letters. So, the word 'note' is the set {'n', 'o', 't', 'e'}, and then word 'tone' is the set {'t', 'o', 'n', 'e'}. The rub, of course, is that these are the same set.

The trick is that a word is not just the set of letters, it is that set in some order. Since 'tone' and 'note' have the same letters, they have the same set of letters. The question is whether there is a way of saying what a word is in terms of letters by using sets in such a way that the set corresponding to 'tone' is distinguishable from the set corresponding to 'note'.

The way we read English offers a hint. When reading 'tone' we scan from left to right seeing 't', then 'to', then 'ton' then 'tone'. Suppose that for each spot in the ordering of the letters, we consider those leters that appear at or before the spot. In other words, we can consider the sets {'t'},

{'t', 'o'}, {'t', 'o', 'n'}, {'t', 'o', 'n', 'e'}. Let us say that 'tone' corresponds to the set of these sets, denoted by $\mathcal{C}$,

$$\mathcal{C} = \{\{\text{'n'}, o, t\}, \{n, o, t, e\}, \{t\}, \{o, t\}\}.$$

Given $\mathcal{C}$, can we recover 'tone' (intead of 'note')? Sure. First, look for a set contained in all the others. The singleton {'t'} is the only one. So the first letter is 't'. Next look for a set distinct from "t''" which is contained in all the rest. The pair {'o', 't'} is the only one. Since we already have 't', the next letter is 'o'. We do the same twice more, getting 'n' and 'e', in that order.

There is a certain peculiarity in all these considerations. Every time we write down a set, we write the names (see Names) of the elements in some order. Indeed, whenever we speak of objects, we must say their names in some order. But of course, no matter how we denote or speak of the set, the concept of set has no concept of ordering.

Generally

Let a, b, c and d denote objects, no two of which are the same (i.e., $a \neq b$, $b \neq c$, etc.). Suppose we want to consider the elements of the quadruple $\{a, b, c, d\}$ in the order c, b, d, a. We include in the set all objects that that occur at or before that position. For the order c, b, d, a of the objects in the set $\{a, b, c, d\}$ we use $\{c\}$, $\{c, b\}$, $\{c, b, d\}$ and $\{c, b, d, a\}$.

Ordering Sets (**??**) immediately needs:

Unordered Triples (**??**)

Ordering Sets (**??**) is not immediately needed by any sheet.

Ordering Sets (**??**) gives no terms.

Ordering Sets

↓

Unordered Triples

↓

Pair Unions

↓

Set Unions

↓   ↓

Empty Set   Unordered Pairs

↓   ↓

Set Specification

↓

Set Inclusion

↓   ↓

Standardized Accounts

↓

Definitions   Accounts

↓   ↓

Deductions   Quantified Statements

↓   ↓

Logical Statements

↓

Statements

↓   ↓

Identities   Sets

↓   ↓

Names

↓   ↓

Objects   Letters
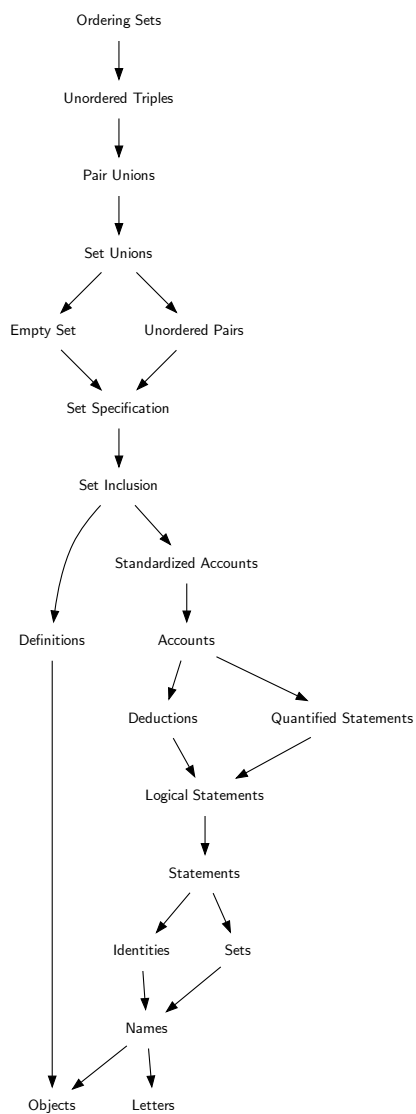
Why

We want to order two objects.

Definition

Let a and b denote objects. The *ordered pair* of a and b is the set $\{\{a\}, \{a, b\}\}$. The *first coordinate* of $\{\{a\}, \{a, b\}\}$ is the object denoted by a and the *second coordinate* is the object denoted by b.

Notation

We denote the ordered pair $\{\{a\}, \{a, b\}\}$ by (a, b).

Equality

Our intuition of two objects in order dictates that if we have the same objects in the same order then we have the same ordered pair. Conversely, if we have two identical ordered pairs, they must consist of the same objects in the same location. In other words, two ordered pairs should be equal if and only if they consist of the same objects in the same order. Our definition agrees with this intuition. Indeed,

Proposition 49. $(((a, b) = (x, y)) \longleftrightarrow (a = x \land b = y))$[29]

---

[29]The proof of this proposition will be found in future editions.

Ordered Pairs (**??**) immediately needs:

Unordered Pairs (**??**)

Ordered Pairs (**??**) is immediately needed by:

Cartesian Products (**??**)
Counts (**??**)
Ordered Pair Pathologies (**??**)
Product Sections (**??**)
Subset Systems (**??**)

Ordered Pairs (**??**) gives the following terms.

ordered pair
first coordinate
second coordinate

Ordered Pairs

↓

Unordered Pairs

↓

Set Specification

↓

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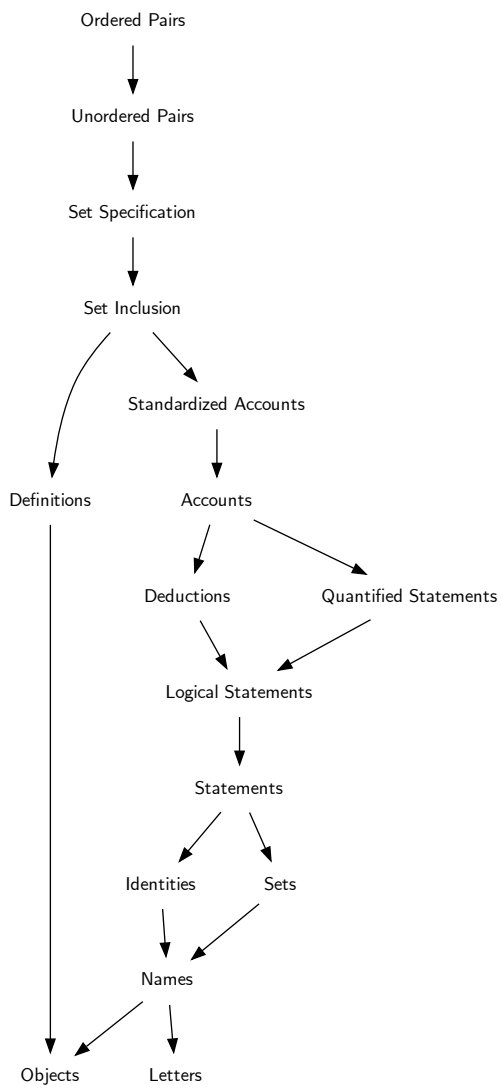
Why

Why define ordered pairs in terms of sets? Why not make them their own type of object?

Pathologies

Notice that $a \notin (a, b)$ and similarly $b \notin (a, b)$. These facts led us to use the terms first and second "coordinate" above rather than element. Neither a nor b is an element of the ordered pair (a, b). On the other hand, it is true that $\{a\} \in (a, b)$ and $\{a, b\} \in (a, b)$. These facts are odd. Should they bother us?

We chose to define ordered pairs in terms of sets so that we could reuse notions about a particular type of object (sets) that we had already developed. We chose what we may call conceptual simplicty (reusing notions from sets) over defining a new type of object (the ordered pair) with its own primitive properties. Taking the former path, rather than the latter is a matter of taste, really, and not a logical consequence of the nature of things.

The argument for our taste is as follows. We already know about sets, so let's use them, and let's forget cases like $\{a, b\} \in (a, b)$ (called by some authors "pathologies"). It does not bother us that our construction admits many true (but irrelevant) statements. Such is the case in life.

Suppose we did choose to make the object (a, b) primitive. Sure, we would avoid oddities like $\{a\} \in (a, b)$. And we might

even get statements like $a \in (a, b)$ to be true. But to do so we would have to define the meaning of $\in$ for the case in which the right hand object is an "ordered pair". Our current route avoids introducing any new concepts, and simply names a construction in our current concepts.

Ordered Pair Pathologies (??) immediately needs:

Ordered Pairs (??)

Ordered Pair Pathologies (??) is not immediately needed by any sheet.

Ordered Pair Pathologies (??) gives no terms.

Ordered Pair Pathologies

↓

Ordered Pairs

↓

Unordered Pairs

↓

Set Specification

↓

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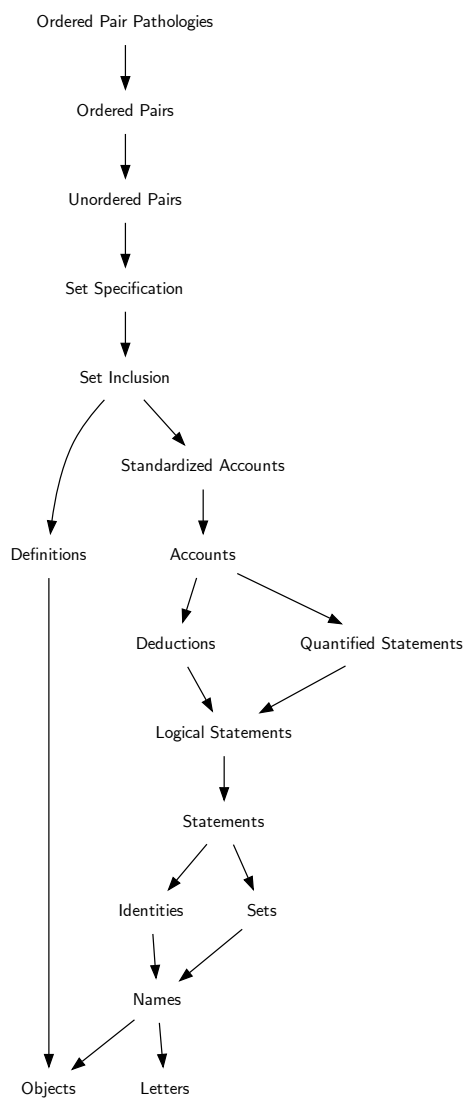
Why

Does a set exist of all the ordered pairs of elements from an ordered pair of sets?

Definition

Let A and B denote sets. Ordered pairs are sets of singletons and pairs. So to construct the set of all ordered pairs taken from two sets, we want to specify the elements of a set which contains all singletons $\{a\}$ and pairs $\{a, b\}$ for $a \in A$, $b \in B$.

Notice that $a \in A$ and $b \in A$ mean $a, b \in (A \cup B)$. In other words, $\{a\} \subset A$ and $\{b\} \subset B$ and $\{a\}, \{b\} \subset (A \cup B)$. In particular, $\{a\} \in (A \cup B)^*$. Similarly, $\{a, b\} \in (A \cup B)^*$. And so $\{\{a\}, \{a, b\}\} \in (A \cup B)^{**}$.

We define the set of "all ordered pairs" from A and B by specifying the appropriate pairs of this set.[30]

$$\{(a, b) \in (A \cup B)^{**} \mid a \in A \wedge b \in B\}$$

We name this set the *product* of the set denoted by A and the set denoted by B is the set of all ordered pairs. This set is also called the *cartesian product*.[31] If $A \neq B$, the ordering causes the product of A and B to differ from the product of B with A. If $A = B$, however, the symmetry holds.

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[30]The specific statement used here requires some translation. A discussion of this and the full statement will appear in a future edition.

[31]This is the current name of the sheet, but may change in future editions, in accordance with the project policy on using names.

Notation

We denote the product of A with B by $A \times B$, read aloud as "A cross B." In this notation, if $A \neq B$, then $A \times B \neq B \times A$.

Cartesian Products (**??**) immediately needs:

Ordered Pairs (**??**)
Set Powers (**??**)

Cartesian Products (**??**) is immediately needed by:

Ordered Pair Projections (**??**)
Relations (**??**)

Cartesian Products (**??**) gives the following terms.

product
cartesian product

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

Pair Unions

Set Unions

Unordered Pairs

Empty Set

Set Specification

Set Inclusion

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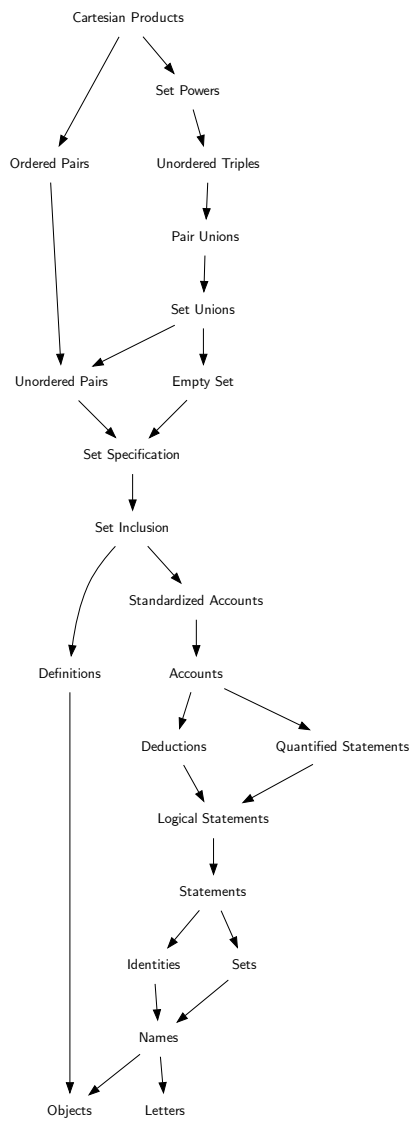
Why

The product of two sets is a (sub)set of ordered pairs. Is every set or ordered pairs a subset of a product of two sets?

Result

The answer is easily seen to be yes. Let R denote a set or ordered pairs. So for $x \in R$, $x = \{\{a\}, \{a, b\}\}$. First consider $\bigcup R$. Then $\{a\} \in \bigcup R$ and $\{a, b\} \in \bigcup R$. Next consder $\bigcup \bigcup R$. Then $a, b \in \bigcup \bigcup R$. So if we want to sets—denote them by A and B—so that $R \subset A \times B$, we can take both A and B to be the set $\bigcup \bigcup R$.

We often want to shrink the sets A and B to only include the relevant members. In other words, we specify the elements of $\bigcup \bigcup R$ which are actually a first coordinate or second coordinate for some ordered pair in the set R. In other words, we define $A' = \{a \in A \mid (\exists b)((a, b) \in R)\}$ and likewise $B' = \{b \in B \mid (\exists a)((a, b) \in R)\}$. We call A' the *projection of R onto the first coordinate* and B' the *projection of R onto the second coordinate.*

Ordered Pair Projections (**??**) immediately needs:

Cartesian Products (**??**)
Set Unions (**??**)

Ordered Pair Projections (**??**) is not immediately needed by any sheet.

Ordered Pair Projections (**??**) gives the following terms.

projection of R onto the first coordinate
projection of R onto the second coordinate.

Ordered Pair Projections

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions

Set Unions

Unordered Pairs   Empty Set

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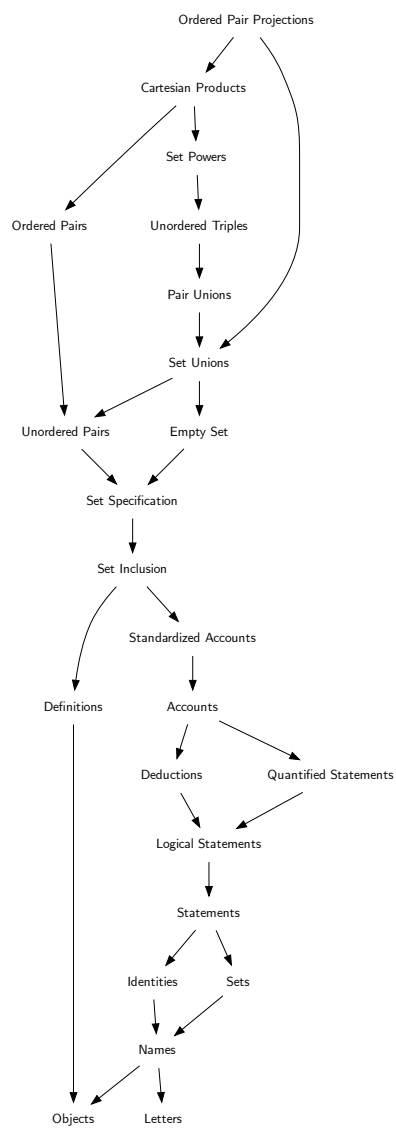
Why

How can we relate the elements of two sets?

Definition

A *relation* is a set of ordered pairs (see Ordered Pairs). So if an object z is an element of a relation, there exists two other objects x, y so that $z = (x, y)$.

The *domain* of a relation is the set of all elements which appear as the first coordinate of some ordered pair of the relation (the projection onto the first coordinate, see Ordered Pair Projections) The *range* of a relation is the set of all elements which appear as the second coordinate of some ordered pair of the relation (the projection onto the second coordinate).

When the domain of a relation R is a subset of X and the range is a subset of Y, we say R is *from X to Y* or *between X and Y*. If $X = Y$, then R speak of a relation *in* or *on X*.

Notation

If R is a relation, we express that $(x, y) \in R$ by writing $x \, R \, y$, which we read as "x is in relation R to y". We denote the domain of R by dom R and the range of R by ran R.

Examples

For an uninteresting relation, consider the empty set. In the empty (set) relation, no object is related to any other. Both

the domain and range of $\varnothing$ are $\varnothing$. For another simple relation, consider the product of any two sets X and Y. In $X \times Y$, all objects are related. The domain is X and the range is Y.

For a more interesting example, consider the set

$$R := \{(x, y) \in X \times X \mid x = y\}.$$

This relation is the relation of equality (see Identities) between two objects. Here $x \, R \, y \longleftrightarrow x = y$. dom $R = $ ran $R = X$. Another similar example is if we consider the set X and X^*, and the relation

$$R := \{(x, y) \in X \times X^* \mid x \in y\}.$$

This relation is the relation of belonging (see Sets). Here $x \, R \, y \longleftrightarrow x \in y$. Here dom $R = X$ and ran $R = X^*$.

Properties

Often relations are defined over a single set, and there are a few useful properties to distinguish.

A relation is *reflexive* if every element is related to itself. A relation is *symmetric* if two objects are related regardless of their order. A relation is *transitive* if a first element is related to a second element and the second element is related to the third element, then the first and third element are related. Equality is reflexive, symmetric and transitive whereas belonging is neither. Exercise: what is inclusion?

Relations (**??**) immediately needs:

Cartesian Products (**??**)

Relations (**??**) is immediately needed by:

Converse Relations (**??**)
Equivalence Relations (**??**)
Functions (**??**)
Partial Orders (**??**)
Relation Composites (**??**)

Relations (**??**) gives the following terms.

relation
domain
range
from X to Y
between
in
on
reflexive
symmetric
antisymmetric
transitive

Relations

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

Pair Unions

Set Unions

Unordered Pairs

Empty Set

Set Specification

Set Inclusion

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EQUIVALENCE RELATIONS
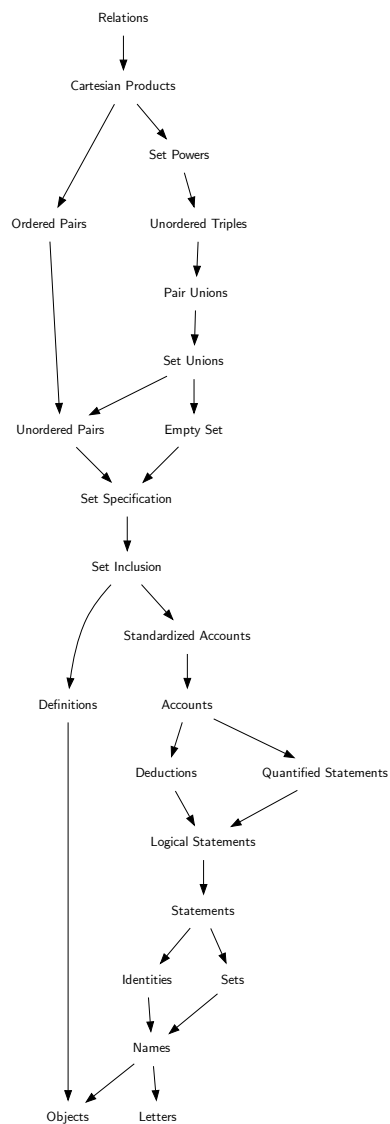
Why

We want to handle at once all the objects of a set which are indistinguishable or equivalent in some aspect.

Definition

An *equivalence relation* on a set X is a reflexive, symmetric, and transitive relation on X (see Relations). The smallest equivalence relation in a set X is the relation of equality in X. The largest equivalence relation in a set X is $X \times X$.

Equivalence relations are useful because they partition (see Partitions) the set. If R is an equivalence relation on X, the *equivalence class* of an object $x \in X$ is the set $\{y \in X \mid x \mathrel{R} y\}$. We call the set of equivalence classes the *quotient set* of the set under the relation. An equally good name is the divided set of the set under the relation, but this terminology is not standard. The language in both cases reminds us that the relation partitions the set into equivalence classes.

If $\mathcal{C}$ is a partition of X, we can define a relation R on X for which $x \mathrel{R} y \longleftrightarrow (\exists A \in \mathcal{C})(x \in A \wedge y \in A)$. In other words, if x and y are in the same piece (see Partitions) of $\mathcal{C}$.

The key result is that every equivalence relation partitions the set and every partition of the set is an equivalence relation. Moreover, if we start with an equivalence relation, look for the partition, and then get the relation defined by the partition, we end up with the relation we started with. Likewise, if we

start with a partition relation, get the equivalence relation, and then get the partition defined by the relation, we end up with the partition we started with. Before stating and proving this result, we give some notation.

Notation

Let R denote an equivalence relation on a set denoted by X. We denote the equivalence class of $x \in X$ by x/R. We denote the set of equivalence classes of R by X/R, read aloud as "X modulo R" or "x mod R".

Main Results

The proofs of these results are straightforward.[32]

Proposition 50. *$X/\mathcal{C}$ is an equivalence relation*

Proposition 51. *X/R is a a partition.*

Proposition 52. *If R is an equivalence relation on X, then $X/(X/R) = R$*

Proposition 53. *If $\mathcal{C}$ is a partition of X, then $X/(X/\mathcal{C}) = \mathcal{C}$.*

These last two propositions make clear the rationale for the notation. Of course, we have not yet had occasion to speak of numbers, much less of division.

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[32]Nonetheless, the full accounts will appear in future editions.

Equivalence Relations (**??**) immediately needs:

Partitions (**??**)
Relations (**??**)

Equivalence Relations (**??**) is immediately needed by:

Canonical Maps (**??**)
Equivalent Sets (**??**)
Inverses of Composite Relations (**??**)
Matrix Similarity (**??**)

Equivalence Relations (**??**) gives the following terms.

quotient set
equivalence class
equivalence relation
equivalence class

Equivalence Relations

Relations

Cartesian Products

Set Powers

Unordered Triples   Ordered Pairs

Partitions   Pair Unions

Set Intersections   Set Unions

Pair Intersections   Empty Set   Unordered Pairs

Set Specification

Set Inclusion

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Why

We want a notion for a correspondence between two sets.

Definition

A *function f from* a set X *to* a set Y is a relation (see Relations) whose domain is X and whose range is a subset of Y such that for each $x \in X$, there exists a unique $y \in Y$ so that $(x, y) \in f$.

We call the unique $y \in Y$ the *result* of the function *at* the *argument* x. We call Y the *codomain.* If the range is Y we say that f is a function from X *onto* Y (or f is *surjective*). If distinct elements of X are mapped to distinct elements of Y, we say that the function is *one-to-one* (or f is *injective*).

We say that the function *maps* elements from the domain to the codomain. Since the word function and the verb "maps" connote activity, some authors refer to the concept that we have defined as a function as the *graph* of a function—namely, the set of ordered pairs which that function produces—and leave the concept of function undefined.

Notation

Let X and Y denote sets. We denote a function named f whose domain is X and whose codomain is Y by $f : X \to Y$. We read the notation aloud as "f from X to Y". We denote the set of all functions from X to Y (which is a subset of $(X \times Y)^*$) by Y^X. A less standard but equally good notation is $X \to Y$,

read aloud as "A to B". Using the notations introduced so far, we denote that $f \in (A \to B)$ by $f : A \to B$. We tend to denote function by lower case latin letters, especially f, g, and h. f is a mnemonic for function and g and h are nearby.

Let $f : A \to B$. For each element $a \in A$, we denote the result of applying f to a by $f(a)$, read aloud "f of a." We sometimes drop the parentheses, and write the result as f_a, read aloud as "f sub a." Let $g : A \times B \to C$. We often write $g(a, b)$ or g_{ab} instead of $g((a, b))$. We read $g(a, b)$ aloud as "g of a and b". We read g_{ab} aloud as "g sub a b."

Examples

If $X \subset Y$, the function $\{(x, y) \in X \times Y \mid x = y\}$ is the *inclusion function* of X into Y. We often introduce such a function as "the function from X to Y defined by $f(x) = y$". We mean by this that f is a function and that we are specifying the appropraite ordered pairs using the statement, called *argument-value notation.* The inclusion function of X into X is called the *identity function* of X. If we view the identity function as a relation on X, it is the relation of equality on X.

The functions $f : (X \times Y) \to X$ defined by $f(x, y) = x$ is the *pair projection* of $X \times Y$ ono X. Similarly $g : (X \times Y) \to Y$ defined by $g(x, y) = y$ is the pair projection of $X \times Y$ onto Y. The identity function is one-to-one and onto, the inclusion functions are one-to-one but not always onto, and the pair projections are usually not one-to-one.

Functions (**??**) immediately needs:

  Relations (**??**)

Functions (**??**) is immediately needed by:

  Canonical Maps (**??**)
  Categories (**??**)
  Families (**??**)
  Function Composites (**??**)
  Function Images (**??**)
  Function Restrictions and Extensions (**??**)
  Operations (**??**)
  Quasiconcave Functions (**??**)

Functions (**??**) gives the following terms.

function
from
to
result
at
argument
codomain
onto
surjective
one-to-one
injective
maps
graph
inclusion function
argument-value notation
identity function
pair projection

Functions

↓

Relations

↓

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions

Set Unions

Unordered Pairs   Empty Set

Set Specification

↓

Set Inclusion

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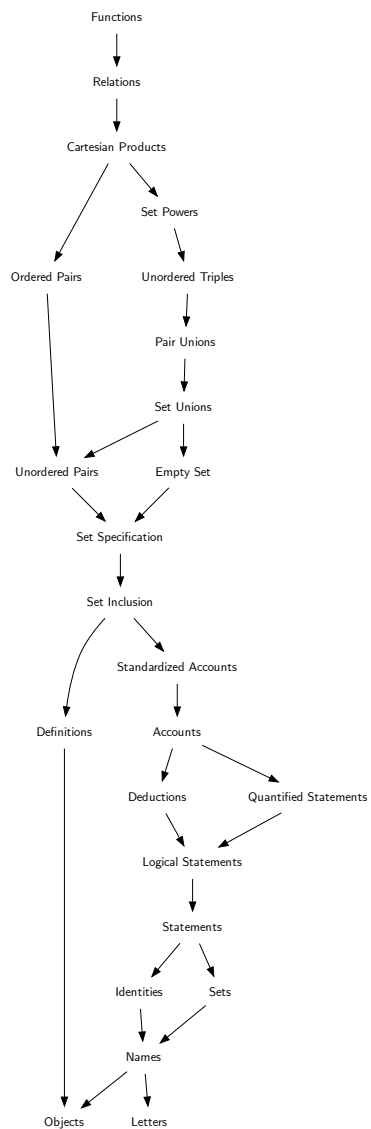
Why

The relationship between the inclusion map and the identity map is characteristic of making small functions out of large ones.

Definition

Let $X \subset Y$ and $f : Y \to Z$. There is a natural function $g : X \to Z$, namely the one defined by $g(x) = f(x)$ for all $x \in X$. We call g the *restriction* of f to X. We call f an *extension* of g to Y. Clearly, there may be more than one extension of a function

Notation

We denote the restriction of $f : Y \to Z$ to the set $X \subset Y$ by $f|X$.

Example

A simple example is the that the inclusion mapping from X to Y with $X \subset Y$ is a restriction of the identity map on X

Function Restrictions and Extensions (**??**) immediately needs:

   Functions (**??**)

Function Restrictions and Extensions (**??**) is not immediately needed by any sheet.

Function Restrictions and Extensions (**??**) gives the following terms.

restriction
extension

Function Restrictions and Extensions

↓

Functions

↓

Relations

↓

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

↓

Pair Unions

↓

Set Unions

Unordered Pairs

Empty Set

↓

Set Specification

↓

Set Inclusion

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Logical Statements

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Why

We consider the set of results of a set of domain elements.

Definition

The *image* of a set of domain elements under a function is the set of their results. Though the set of domain elements may include several distinct elements, the image may still be a singleton, since the function may map all of elements to the same result.

Using this language, the range (see Functions) of a function is the image of its domain. The range includes all possible results of the function. If the range does not include some element of the codomain, then the function maps no domain elements to that codomain element.

Notation

Let $f : A \to B$. We denote the image of $C \subset A$ by $f(C)$, read aloud as "f of C." This notation is overloaded: for every $c \in C$, $f(c) \in A$, whereas $f(C) \subset A$. Read aloud, the two are indistinguishable, so we must be careful to specify whether we mean an element c or a set C. Following this notation for function images, we denote the range of f by $f(A)$. In this notation, we can record that f maps X onto Y by $f(X) = Y$.

Notational ambiguity

The notation $f(A)$ is can be ambiguous in the case that A is both an element and a set of elements of the domain of f. For example, consider $f : \{\{a\}, \{b\}, \{a, b\}\} \to X$. Then $f(\{a, b\})$ is ambiguous. We will avoid this ambiguity by making clear which we mean in particular cases.

Inverse Images

Similarly to how we can define $f : X^* \to Y^*$ for $A \subset X$

$$f(A) = \{y \in Y \mid (\exists x)(x \in a \land y = f(x))\},$$

we can define $f^{-1} : Y^* \to X^*$ for $B \subset X$

$$f^{-1}(B) = \{x \in X \mid (\exists y)(y \in B \land y = f(x))\}.$$

In other words, $f^{-1}(B)$ is the set of all elements of the domain which give the elements in B of the range. We call $f^{-1}(B)$ the *inverse image* of B.

Connections

Here are some connections.[33]

Proposition 54. *Let* $f : X \to Y$ *and* $B \subset Y$. $f(f^{-1}(B)) \subset B$. *If* f *is onto, then* $f(f^{-1}(B)) \subset B$.

Proposition 55. *Let* $f : X \to Y$ *and* $A \subset X$. $A \subset f^{-1}(f(A))$. *If* f *is one-to-one, then* $A = f^{-1}(f(A))$.

---

[33]The proofs are straightfoward, and will appear in future editions.

Function Images (**??**) immediately needs:

Functions (**??**)

Function Images (**??**) is immediately needed by:

Function Inverses (**??**)

Function Images (**??**) gives the following terms.

image
inverse image

Function Images

↓

Functions

↓

Relations

↓

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

↓

Pair Unions

↓

Set Unions

Unordered Pairs   Empty Set

Set Specification

↓

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Logical Statements

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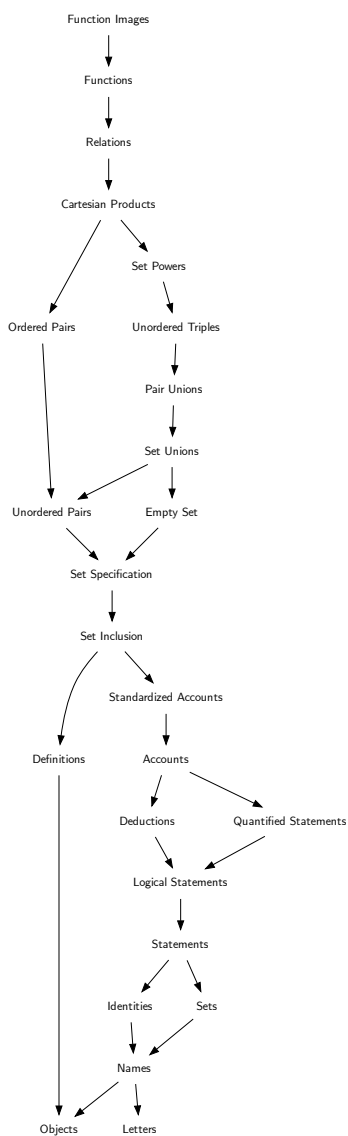
Why

How do equivalence classes and functions relate

Definition

We can associate to each element of a set its equivalence class under an equivalence relation. Let X denote a set and R an equivalence relation. We call the function $f : X \to X/R$ defined by $f(x) = x/R$ the *canonical map* from X to X/R.

Conversely, if f is an arbitrary function from X onto Y, we can naturally define an equivalence relation R in X so that for $a, b \in X$, $a\,R\,b \longleftrightarrow f(a) = f(b)$ f was onto, so for each $y \in Y$, there exists an $x \in X$ with $f(x) = y$. Now let $g : Y \to X/R$ be defined by $g(y) = x/R$. The values of g are the subset X which are mapped to the same value under f. Moreover, the function g is one-to-one.

Canonical Maps (**??**) immediately needs:

Equivalence Relations (**??**)
Functions (**??**)

Canonical Maps (**??**) is not immediately needed by any sheet.

Canonical Maps (**??**) gives the following terms.

canonical map

Canonical Maps

Equivalence Relations → Functions

Relations

Cartesian Products

Set Powers

Unordered Triples → Ordered Pairs

Partitions → Pair Unions

Set Intersections → Set Unions

Pair Intersections → Empty Set → Unordered Pairs

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Objects → Letters

Why

We often use functions to keep track of several objects by the objects of some well-known set with which they correspond. In this case, we use specific language and notation.

Definition

Let I and X denote sets. A *family* is a function from I to X. We call an element of I an *index* and we call I the *index set*. Of course, the letter I was picked here to be a mnemonic for "index". We call the range of the family the *indexed set* and we call the value of the family at an index i a *term* of the family at i or the *ith term* of the family.

Experience shows that it is useful to discuss sets using indices, especially when discussing a set of sets. If the values of the family are sets, we speak of a *family of sets*. Indeed, we often speak of a *family of* whatever object the values of the function are. So for instance, a family of subsets of X is understood to be a function from some index set into X^*.

Notation

Let $x : I \to X$ be a family. We denote the ith term of x by x_i. We sometimes denote the family by $\{x_i\}_{i \in I}$.

Families (**??**) immediately needs:

Functions (**??**)

Families (**??**) is immediately needed by:

Direct Products (**??**)
Family Operations (**??**)
Family Unions and Intersections (**??**)
Natural Families (**??**)

Families (**??**) gives the following terms.

family of sets
ordered family
family
index
index set
indexed set
term
ith term
family of sets
family of

Families

Functions

Relations

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions

Set Unions

Unordered Pairs   Empty Set

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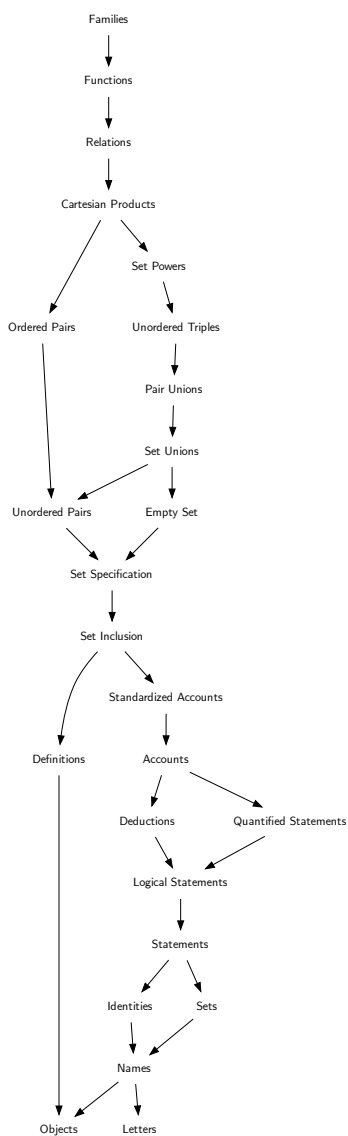
Why

We can use families to think about unions and intersections.

Family Unions

Let $A : I \to X^*$ be a family of subsets. We refer to the union (see Set Unions) of the range (see Relations) of the family the *family union*. We denote it $\cup_{i \in I} A_i$.

Proposition 56. $(x \in \cup_{i \in I} A_i) \longleftrightarrow (\exists i)(x \in A_i)$

If $I = \{a, b\}$ is a pair with $a \neq b$, then $\cup_{i \in I} = A_a \cup A_b$.

There is no loss of generality in considering family unions. Every set of sets is a family: consider the identity function from the set of sets to itself.

We can also show generalized associative and commutative law[34] for unions.

Proposition 57. *Let $\{I_j\}$ be a family of sets and define $K = \cup_j I_j$. Then $\cup_{k \in K} A_k = \cup_{j \in J}(\cup_{i \in I_j} A_i)$.*[35]

Family Intersection

If we have a nonempty family of subsets $A : I \to X^*$, we call the intersection (see Set Intersections) of the range of the family the *family intersection*. We denote it $\cap_{i \in I} A_i$.

---

[34]The commutative law will appear in future editions.

[35]An account will appear in future editions.

Proposition 58. $x \in \cap_{i \in I} A_i \longleftrightarrow (\forall i)(x \in A_i)$

Similarly we can derive associative and commutative laws for intersection[36]. They can be derived as for unions, or from the facts of unions using generalized DeMorgan's laws (see Generalized SSet Dualities.

Connections

The following are easy[37]

Let $\{A_i\}$ be a family of subsets of X and let $B \subset X$.

Proposition 59. $B \cap \bigcup_i A_i = \bigcup_i (B \cap A_i)$

Proposition 60. $B \cup \bigcap_i A_i = \bigcap_i (B \cup A_i)$

Let $\{A_i\}$ and $\{B_j\}$ be families of sets.[38]

Proposition 61. $(\bigcup_i A_i) \cap (\bigcup_j B_j) = \bigcup_{i,j} (A_i \cap B_j)$

Proposition 62. $(\bigcap_i A_u) \cup (\bigcap_j B_j) = \bigcap_{i,j} (A_i \cup B_j)$.

Proposition 63. $\cap_i X_i \subset X_j \subset \cup_i X_i$ *for each* j.

---

[36]Statements of these will be given in future editions.

[37]Accounts will appear in future editions.

[38]An account of the notation used and the proofs will appear in future editions.

Family Unions and Intersections (**??**) immediately needs:

Families (**??**)

Generalized Set Dualities (**??**)

Set Unions and Intersections (**??**)

Family Unions and Intersections (**??**) is immediately needed by:

Family Products and Unions (**??**)

Inverses Unions Intersections and Complements (**??**)

Family Unions and Intersections (**??**) gives the following terms.

family union
family intersection

Family Unions and Intersections

Families

Functions

Relations

Cartesian Products

Generalized Set Dualities

Set Powers

Set Dualities   Intersection of Empty Set   Set Unions and Intersections   Unordered Triples

Set Intersections   Pair Unions   Ordered Pairs

Set Complements   Pair Intersections   Set Unions

Set Differences   Empty Set   Unordered Pairs

Set Specification

Set Inclusion

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Why

We can generalize the product of two sets to a product of a family of sets.

Discussion for pairs

Let A and B be sets. There is a natural correspondence between $A \times B$ (see Cartesian Products) and a particular set of families. The particular set of families $z : \{i, j\} \to (A \cup B)$ with $z_i \in A$ and $z_b \in B$. The family z corresponds with the pair (z_i, z_i). The pair (a, b) corresponds to the family $z : \{i, j\} \in (A \cup B)$ defined by $z(i) = a$ and $z(j) = b$. In other words, we can think about ordered pairs as special families. The generalization of Cartesian products to families generalizes the notion for families.

Direct Products

Let X be a set. Let $A : I \to X$ be a family of subsets of X. The *direct product* or *family Cartesian product* of A is the set of all families $a : I \to X$ which satisfy $a_i \in A_i$ for every $i \in I$. A function on a product is called a *function of several variables* and, in particular, a function on the product $X \times Y$ is called a *funciton of two variables*.

Notation

We denote the product of the family $\{A_i\}$ by

$$\prod_{i \in I} A_i$$

We read this notation as "product over i in I of A sub-i."

Projections

The word "projection" is used in two senses with families. Let I be a set, and let $\{A_i\}$ be a family of sets. Define $A = \prod_{i \in I} A_i$. Then if $J \subset I$, there is a natural correspondence between the elements of X and those of $\prod_{j \in J} A_j$. To each element $x \in X$, we restrict x to J and this is an element of $\prod_{j \in J} A_j$. The correspondence is called the *projection* of X onto $\prod_{i \in J} X_i$ Also, the value of x at j is called the *projection of x onto index j* or the *j-coordinate* of x.

Direct Products (**??**) immediately needs:

Families (**??**)

Direct Products (**??**) is immediately needed by:

Datasets (**??**)
Family Products and Unions (**??**)
Joint Distributions (**??**)
Marginal Distributions (**??**)
Product Metrics (**??**)
Product Sigma Algebras (**??**)
Random Variable Independence (**??**)
Sequences (**??**)
Undirected Paths (**??**)

Direct Products (**??**) gives the following terms.

direct product
n-tuples
sequences
family Cartesian product
function of several variables
funciton of two variables.
consecutive
projection
projection of x onto index j
j-coordinate

Direct Products

↓

Families

↓

Functions

↓

Relations

↓

Cartesian Products

↓

Set Powers

Ordered Pairs   Unordered Triples

↓

Pair Unions

↓

Set Unions

Unordered Pairs   Empty Set

↓

Set Specification

↓

Set Inclusion

Standardized Accounts

Definitions   Accounts

Deductions   Quantified Statements

↓

Logical Statements

↓

Statements

Identities   Sets

↓

Names

↓

Objects   Letters
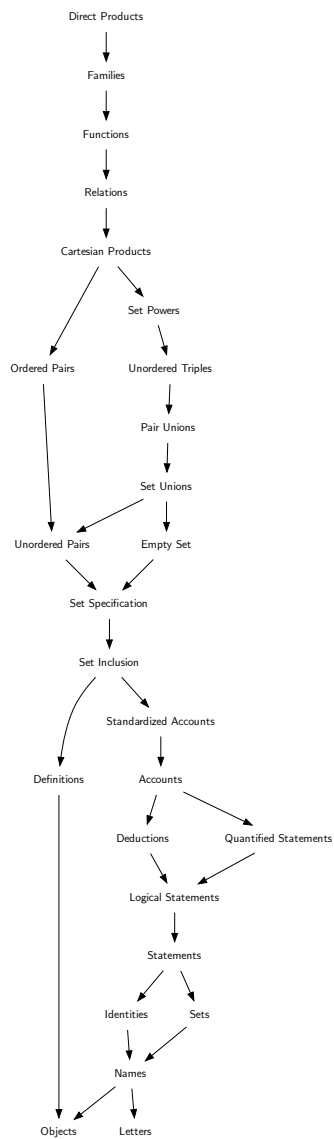
Why

We study how family unions and direct products interact.

Result

The following is easy.[39]

Proposition 64. $(\cup_i A_i) \times (\cup_j B_j) = \cup_{i,j}(A_i \times B_j)$.

---

[39]An account will appear in future editions.

Family Products and Unions (**??**) immediately needs:

Direct Products (**??**)
Family Unions and Intersections (**??**)

Family Products and Unions (**??**) is not immediately needed by any sheet.

Family Products and Unions (**??**) gives no terms.

Family Products and Unions

Family Unions and Intersections → Direct Products

Direct Products → Families

Families → Functions

Functions → Relations

Relations → Cartesian Products

Cartesian Products → Set Powers

Generalized Set Dualities

Set Unions and Intersections

Intersection of Empty Set

Unordered Triples

Set Dualities

Ordered Pairs

Set Intersections

Pair Unions

Pair Intersections

Set Complements

Set Unions

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Empty Set

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Why

We want a notion for applying two functions one after the other. We apply a first function then a second function.

Definition

Consider two functions. And suppose the range of the first is a subset of the domain of the second. In other words, every value of the first is in the domain (and so can be used as an argument) for the second.

The *composite* or *composition* of the second function with the first function is the function which associates each element in the first's domain with the element in the second's codomain that the second function associates with the result of the first function.

The idea is that we take an element in the first domain. We apply the first function to it. We obtain an element in the first's codomain. This result is an element of the second's domain. We apply the second function to this result. We obtain an element in the second's codomain. The composition of the second function with the first is the function so constructed. Of course the order of composition is important.

Notation

Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$. We denote the composition of g with f by $g \circ f$ read aloud as "g

composed with f." To make clear the domain and comdomain, we denote the composition $g \circ f : A \to C$. $g \circ f$ is defined by

$$(g \circ f)(a) = g(f(a)).$$

for all $a \in A$. Sometimes the notation gf is used for $g \circ f$.

Basic Properties

Function composiiton is associative but not commutative.[40] Indeed, even if $f \circ g$ is defined, $g \circ f$ may not be.

Proposition 65 (Associative). *Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to U$ Then $(f \circ g) \circ h = f \circ (g \circ h)$*[41]

---

[40]Future editions will include a counterexample.
[41]The proof is straightforward. Future editions will include it.

Function Composites (**??**) immediately needs:

Functions (**??**)

Function Composites (**??**) is immediately needed by:

Function Inverses (**??**)
Sequences (**??**)

Function Composites (**??**) gives the following terms.

composite
composition

Function Composites

↓

Functions

↓

Relations

↓

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

Pair Unions

Set Unions

Unordered Pairs

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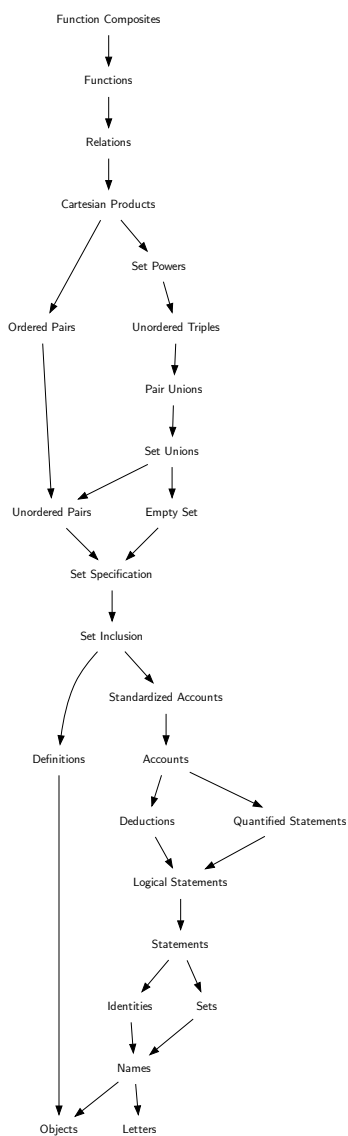
Why

We want a notion of reversing functions.

Definition

Reversing functions does not make sense if the function is not one-to-one. Let $f : X \to Y$. If x_1 goes to y and x_2 goes to y (i.e., $f(x_1) = f(x_2) = y$), then what should y go to. One answer is that we should have a function which gives all the domain values which could lead to y. This is the inverse image (see Function Images) $f^{-1}(\{y\})$. Nor does reversing funcitons make sense if f is not onto. If there does not exist $x \in X$ so that $y = f(x)$, then $f^{-1}(\{y\}) = \varnothing$.

In the case, however, that the function is one-to-one and onto, then each element of the domain corresponds to one and only one element of the codomain and vice versa. In this case, for all $y \in Y$, $f^{-1}(\{y\})$ is a singleton $\{x\}$ where $f(x) = y$. In this case, we define a fucntion $g : Y \to X$ so that $g(y) = x$ if and only if $f(x) = y$.

In general, if we have two functions, where the codomain of the first is the domain of the second, and the codomain of the second is the domain of the first, we call them *inverse functions* if the composition of the second with the first is the identity function on the first's domain and the composition of the first with the second is the identity function on the second's domain (see Functions and Function Composites).

In this case we say that the second function is an *inverse* of the second, and vice versa. When an inverse exists, it is unique,[42] so we refer to *the inverse* of a function. We call the first function *invertible*. Other names for an invertible function include *bijection*.

Notation

Let A a non-empty set. We denote the identity function on A by id_A, read aloud as "identity on A." id_A maps A onto A. Let A, B be non-empty sets. Let $f : A \to B$ and $g : B \to A$ be functions. f and g are inverse functions if $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

The Inverse

PROPOSITION **3** (Uniqueness). *Let $f : A \to B$, $g : B \to A$, and $h : B \to A$. If g and h are both inverse functions of f, then $g = h$.*

PROPOSITION **4** (Existence). *If a function is one-to-one and onto, it has an inverse; and conversely.*

Composites and Inveres

Let $f : X \to Y$ and $g : Y \to Z$. Then g^{-1} maps Z^* to Y^* and f^{-1} maps Y^* to X^*. Then the following is immediate

Proposition 66. $(gf)^{-1} = f^{-1}g^{-1}$

---

[42]Future editions will prove this assertion and all unproven propositions herein.

Function Inverses (**??**) immediately needs:

Function Composites (**??**)
Function Images (**??**)

Function Inverses (**??**) is immediately needed by:

Equivalent Sets (**??**)
Isometries (**??**)
Permutations (**??**)

Function Inverses (**??**) gives the following terms.

inverse functions
inverse
the inverse
invertible
bijection

Function Inverses

Function Composites → Functions ← Function Images

Functions

Relations

Cartesian Products

Set Powers

Ordered Pairs Unordered Triples

Pair Unions

Set Unions

Unordered Pairs Empty Set

Set Specification

Set Inclusion

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Why

The inverse of a function interacts nicely with family unions, family intersections and complements.

Results

Let $f : X \to Y$. Throughout this sheet, let $f^{-1} : Y^* \to X^*$. And take $\{B_i\}$ to be a family of subsets of Y.[43]

Proposition 67. $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$

Proposition 68. $f^{-1}(\cup_i B_i) = \cap_i f^{-1}(B_i)$

Proposition 69. $f^{-1}(Y - B) = X - f^{-1}(B)$

Properties for Function Image

Notice that $f(\cup_i A_i) = \cup_i f(A_i)$ but not for interesctions. Nor is there a similar correspondence for complements. There are some relations, which we list below.[44]

Proposition 70. $f(A \cap B) = f(A) \cap f(B)$ *if and only if f is one-to-one.*

Proposition 71. *For all $A \subset X$, $f(X - A) = Y - f(A)$ if and only if f is one-to-one.*

Proposition 72. *For all $A \subset X$, $Y - f(A) \subset f(X - A)$ if and only if f is onto.*

---

[43]The proofs of the following will appear in future editions.

[44]Accounts of these facts will appear in future editions.

Inverses Unions Intersections and Complements (??) immediately needs:

   Family Unions and Intersections (??)

Inverses Unions Intersections and Complements (??) is not immediately needed by any sheet.

Inverses Unions Intersections and Complements (??) gives no terms.

Inverses Unions Intersections and Complements

Family Unions and Intersections

Families

Functions

Relations

Cartesian Products

Generalized Set Dualities

Set Powers

Set Dualities

Intersection of Empty Set

Set Unions and Intersections

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Set Intersections

Pair Unions

Ordered Pairs

Set Complements

Pair Intersections

Set Unions

Set Differences

Empty Set

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Why

If x is related to y and y to z, then x and z are related.

Definition

Let R be a relation from X to Y and S a relation from Y to Z. The *composite relation* from X to Z contains the pair $(x, z) \in (X \times Z)$ if and only if there exists a $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in S$. This composite relation is sometimes called the *relative product*.

Notation

We denote the composite relation of R and S by $R \circ S$ or RS.

Example

Let X be the set of people and let R be the relation in X "is a brother of" and S be the relation in X "is a father of". Then RS is the relation "is an uncle of".

Properties

Composition of relation is associative but not commutative.[45]

---

[45] A fuler account will appear in future editions.

Relation Composites (**??**) immediately needs:

Relations (**??**)

Relation Composites (**??**) is immediately needed by:

Inverses of Composite Relations (**??**)

Relation Composites (**??**) gives the following terms.

composite relation
relative product

Relation Composites

↓

Relations

↓

Cartesian Products

↓ ↘

Set Powers

Ordered Pairs   Unordered Triples

↓

Pair Unions

↓

Set Unions

↓ ↘

Unordered Pairs   Empty Set

↓

Set Specification

↓

Set Inclusion

↓ ↘

Standardized Accounts

Definitions   Accounts

↓ ↘

Deductions   Quantified Statements

↓

Logical Statements

↓

Statements

↓ ↘

Identities   Sets

↓

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↓ ↘

Objects   Letters

Why
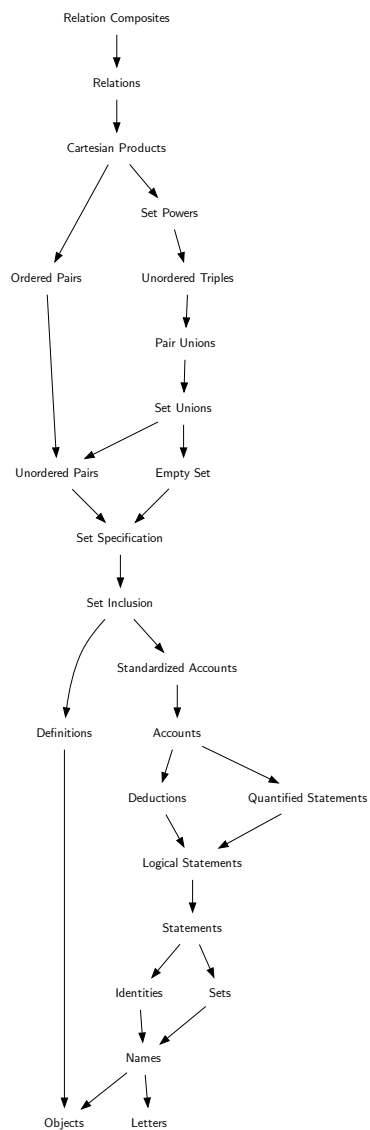
If x is related to y, the y is related to x, but how?

Definition

If R is a relation between X and Y, then the *converse* or *inverse* relation of R is a relation on Y and X relating $y \in Y$ to $x \in X$ if and only if $x \, R \, y$. If $R = R^{-1}$ then R is symmetric.

Notation

We denote the converse relation of R by R^{-1}.

Example

Let X be the set of people and let R be a relation in X. If R is "is a father of", then R^{-1} is "is a son of". If R is "is a mother of", then R^{-1} is "is a daughter of". If R is "is a brother of", then R^{-1} is "is a brother of". The relation "is a brother of" is symmetric.

Converse Relations (**??**) immediately needs:

Relations (**??**)

Converse Relations (**??**) is immediately needed by:

Inverses of Composite Relations (**??**)

Converse Relations (**??**) gives the following terms.

converse
inverse

Converse Relations

↓

Relations

↓

Cartesian Products

↓ ↘

Set Powers

↓ ↓

Ordered Pairs   Unordered Triples

↓ ↓

Pair Unions

↓

Set Unions

↓ ↘ ↓

Unordered Pairs   Empty Set

↓ ↓

Set Specification

↓

Set Inclusion

↓ ↘

Standardized Accounts

↓ ↓

Definitions   Accounts

↓ ↘

Deductions   Quantified Statements

↓ ↓

Logical Statements

↓

Statements

↓ ↘

Identities   Sets

↓ ↓

Names

↓ ↓

Objects   Letters
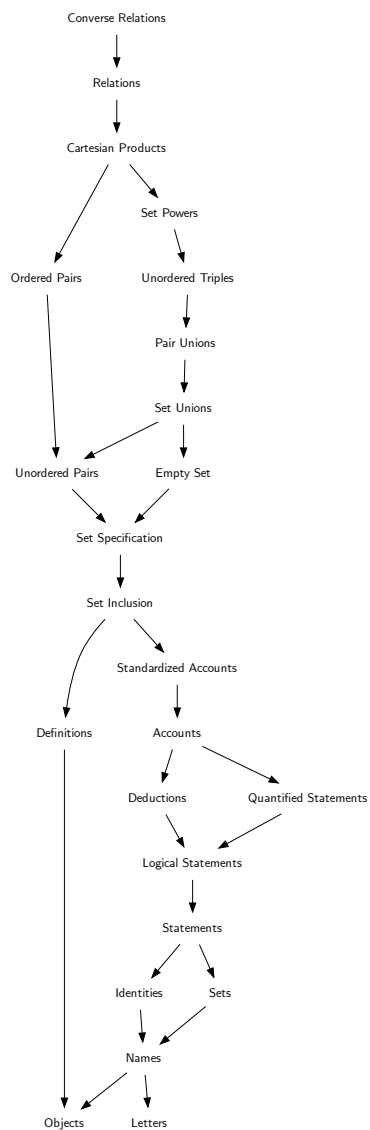
Why

How do inverse and converse relations interact.

Results

Let R be a relation between X and Y and let S be a relation between Y and Z.

Proposition 73. $(RS)^{-1} = S^{-1}R^{-1}$

Identity Relations

Recall that I is the identity relation on X if $x\,I\,y$ if and only if $x = y$.

Proposition 74. *Let R be a relation on X. Let I be the identity relation on X. Then $RI = IR = R$.*

One would like $RR^{-1} \supset I$, $R^{-1}R \supset I$. The father of the son is the father and the son of the father is the son. But the empty relation violates these claims.

Relation Properties

Proposition 75. *R is symmetric if and only if $R \subset R^{-1}$*

Proposition 76. *R is reflextive if and only if $I \subset R$*

Proposition 77. *R is transitive if and only if $RR \subset R$.*

Inverses of Composite Relations (??) immediately needs:

Converse Relations (??)
Equivalence Relations (??)
Relation Composites (??)

Inverses of Composite Relations (??) is not immediately needed by any sheet.

Inverses of Composite Relations (??) gives no terms.

Inverses of Composite Relations

Converse Relations   Relation Composites   Equivalence Relations

Relations

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions   Partitions

Set Unions   Set Intersections

Unordered Pairs   Empty Set   Pair Intersections

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Why

We want numbers to count with.[46]

Definition

The *successor* of a set is the set which is the union of the set with the singleton of the set. In other words, the successor of a set A is $A \cup \{A\}$. This definition has sense for any set, but is of interest only for those particular sets introduced here.

These sets are the following (and their successors): We call the empty set *zero*.[47] We call the successor of the empty set *one*. In other words, one is $\varnothing \cup \{\varnothing\} = \{\varnothing\}$. We call the successor of one *two*. In other words, two is $\{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$. Likewise, the successor of two we call *three* and the successor of three we call *four*. And we continue as usual,[48] using the English language in the typical way.

A set is a *successor set* if it contains zero and if it contains the successor of each of its elements.

Notation

Let x be a set. We denote the successor of x by x^+. We defined it by

$$x^+ := x \cup \{x\}$$

---

[46]Future editions will expand on this sheet with a more justified why.

[47]In future editions, zero may be a separate sheet.

[48]Future editions will assume less in the introduction of natural numbers.

We denote one by 1. We denote two by 2. We denote three by 3. We denote four by 4. So

$$0 = \varnothing$$
$$1 = 0^+ = \{0\}$$
$$2 = 1^+ = \{0, 1\}$$
$$3 = 2^+ = \{0, 1, 2\}$$
$$4 = 3^+ = \{0, 1, 2, 3\}$$

Successor Sets (**??**) immediately needs:

Pair Intersections (**??**)
Pair Unions (**??**)

Successor Sets (**??**) is immediately needed by:

Natural Numbers (**??**)

Successor Sets (**??**) gives the following terms.

successor
zero
one
two
three
four
successor set

Successor Sets

Pair Unions

Pair Intersections

Set Unions

Empty Set

Unordered Pairs

Set Specification

Set Inclusion

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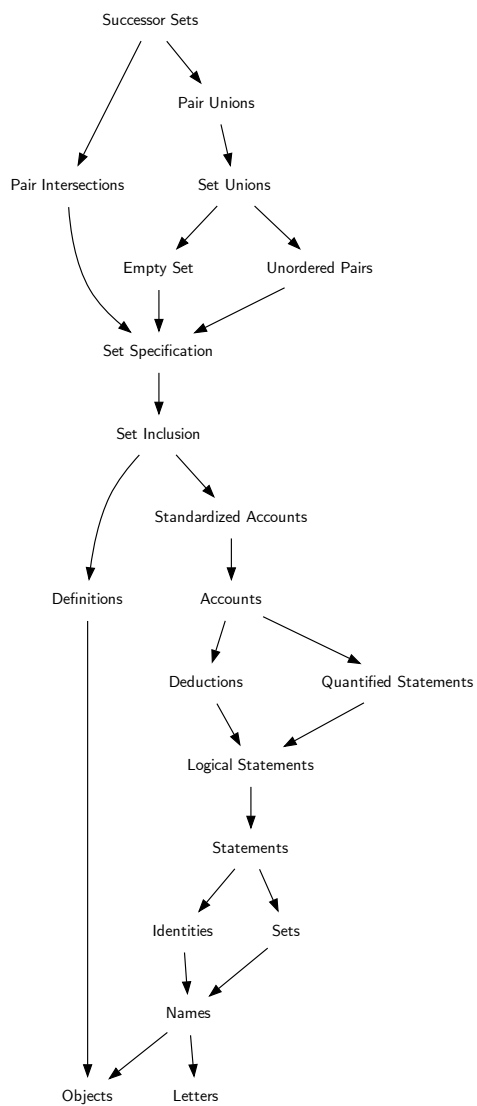Letters

Why

Does a set exist which contains zero, and on and two, and three, and all the rest?

Definition

In Successor Sets, we said "and we continue as usual using the English language..." in our definition of zero, and one and two and three. Can this really be carried on and on? We will say yes. We will say that there exists a set which contains zero and contains the successor of each of its elements.

Principle 7 (Natural Numbers). *A set which contains 0 and contains the successor each of its elements exists.*

This principle is sometimes also called the *principle of infinity.*

We want this set to be unique. The principle says one successor set exists, but not that it is unique. To see that it is unique, notice that the intersection of a nonempty family of successor sets is a successor set.[49] Consider the intersection of the family of all successor sets. The intersection is nonempty by the principle of infinity (see Intersection of Empty Set for this subtlety). The axiom of extension guarantees that this intersection, which is a successor set contained in every other successor set, is unique. We summarize:

---

[49]This account will be expanded in future editions.

Proposition 78. *There exists a unique successor smallest successor set.*

A *natural number* or *number* or *natural* is an element of this minimal successor set. The *set of natural numbers* or *natural numbers* or *naturals* or *numbers* is the minimal successor set.

Notation

We denote the set which exists by Proposition **??** by ω. [50] We denote the set of natural numbers without 0 by N, a mnemonic for natural. In other words $\mathsf{N} = \omega - \{0\}$. We often denote elements of ω or N by n, a mnemonic for number, or m, a letter close to n.

---

[50]We use this notation to follow many authorities on the subject, and to meet the exigencies of time in producing this first edition. Future editions are likely to rework the treatment.

Natural Numbers (??) immediately needs:

  Set Intersections (??)
  Successor Sets (??)

Natural Numbers (??) is immediately needed by:

  Cardinality (??)
  Characteristic Functions (??)
  Coins (??)
  Dice (??)
  Equivalent Sets (??)
  Integer Numbers (??)
  Natural Families (??)
  Natural Induction (??)
  Natural Order (??)
  Natural Products (??)
  Natural Sums (??)
  Peano Axioms (??)
  Sequences (??)

Natural Numbers (??) gives the following terms.

principle of infinity
natural number
number
natural
set of natural numbers
natural numbers
naturals
numbers
zero
natural numbers with zero
addition

Natural Numbers

Set Intersections

Successor Sets

Pair Intersections

Pair Unions

Set Unions

Empty Set

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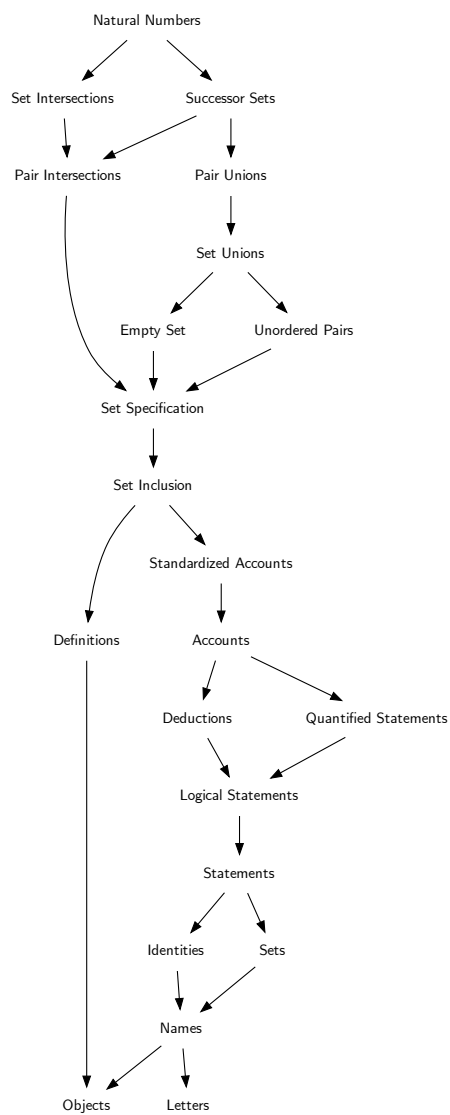
Why

We introduce language for the steps of an infinite process.

Definition

Let A be a non-empty set. A *sequence in A* is a function from the natural numbers to the set. The *nth term* of a sequence is the result of the nth natural number; it is an element of the set.

Notation

Let A be a non-empty set. $a : \mathsf{N} \to A.$ is a sequence in A. $a(n)$ is the nth term. We also denote a by $(a_n)_n$ and $a(n)$ by a_n.

Subsequences

A *subindex* is a monotonically increasing function from and to the natural numbers. Roughly, it selects some ordered infinite subset of natural numbers. A *subsequence* of a first sequence is any second sequence which is the composition of the first sequence with a subindex.

Notation

Let $i : N \to N$ such that $n < m \longrightarrow i(n) < i(m)$. Then i is a subindex. Let $b = a \circ i$. Then b is a subsequence of a. We denote it by $\{b_{i(n)}\}_n$ and the nth term by $b_{i(n)}$.

TODO: integrate, from direct products

If I is the set of natural numbers we denote the direct product by

$$\prod_{i=1}^{\infty} A_i.$$

We denote an element of $\prod_{i=1}^{\infty} A_i$ by (a_i) with the understanding that $a_i \in A_i$ for all $i = 1, 2, 3, \ldots$. If $A_i = A$ for all $i = 1, 2, 3, \ldots$, then (a_i) is a sequence in A.

Sequences (??) immediately needs:

Direct Products (??)
Function Composites (??)
Natural Numbers (??)

Sequences (??) is immediately needed by:

Almost Everywhere (??)
Central Limit Theorem (??)
Egoprox Sequences (??)
Monotone Algebras (??)
Monotone Classes (??)
Monotone Sequences (??)
Nets (??)
Random Variable Sigma Algebra (??)
Real Continuity (??)
Real Integral Series Convergence (??)
Real Sequences (??)
Tail Sigma Algebra (??)

Sequences (??) gives the following terms.

sequence in
nth term
subindex
subsequence

Sequences

Direct Products

Families   Function Composites

Functions

Relations

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions   Natural Numbers

Set Unions   Set Intersections

Unordered Pairs   Empty Set   Pair Intersections

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Why

Historically considered a fountainhead for all of mathematics.

Discussion

Our discussion of sets has suceeded in making Peano's axioms provable propositions.

1. $0 \in \omega$.

2. $n \in \omega \longrightarrow n^+ \in \omega$.

3. Suppose $S \subset \omega$, $0 \in S$, and $(n \in S \longrightarrow n^+ \in S$. Then $S = \omega$.

Proposition 79 (Peano's First Axiom). $0 \in \omega$.

Proposition 80 (Peano's Second Axiom). $n \in \omega \longrightarrow n^+ \in \omega$.

Proposition 81 (Peano's Third Axiom). *Suppose $S \subset \omega$, $0 \in S$, and $(n \in S \longrightarrow n^+ \in S$. Then $S = \omega$.*

Peano's third axiom is called the *principle of mathematical induction*.

Proposition 82 (Peano's Fourth Axiom). $n^+ \neq 0$ *for all $n \in \omega$.*

Proposition 83 (Peano's Fifth Axiom). *Suppose $n, m \in \omega$ with $n^+ = m^+$. Then $n = m$.*
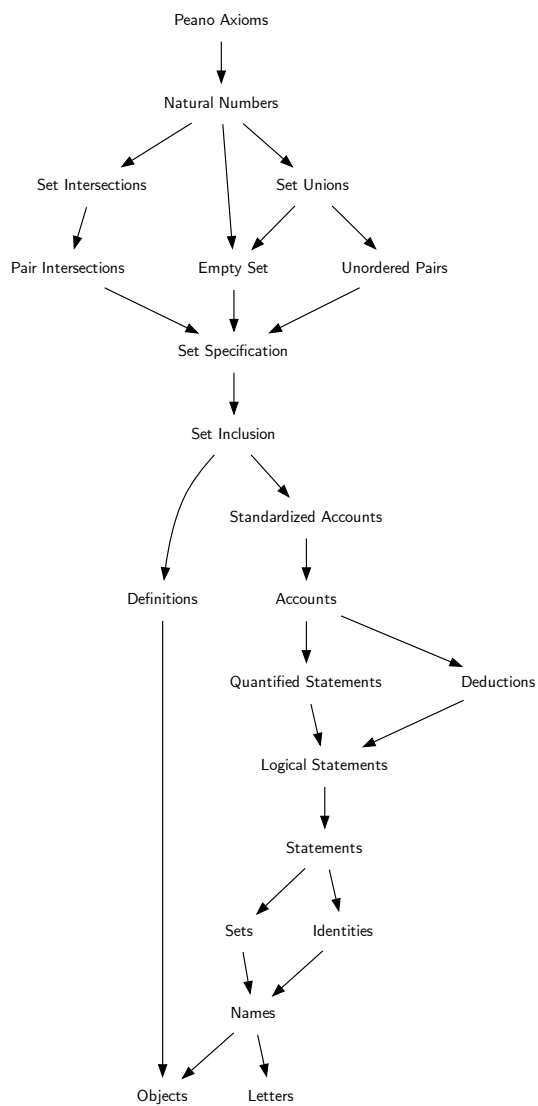
Peano Axioms (**??**) immediately needs:

Natural Numbers (**??**)

Peano Axioms (**??**) is not immediately needed by any sheet.

Peano Axioms (**??**) gives the following terms.

principle of mathematical induction

Peano Axioms

↓

Natural Numbers

↓ ↓ ↓

Set Intersections   Set Unions

↓ ↓ ↓ ↓

Pair Intersections   Empty Set   Unordered Pairs

↓ ↓ ↓

Set Specification

↓

Set Inclusion

↓ ↓

Definitions   Standardized Accounts

↓

Accounts

↓ ↓

Quantified Statements   Deductions

↓ ↓

Logical Statements

↓

Statements

↓ ↓

Sets   Identities

↓ ↓

Names

↓ ↓

Objects   Letters

Why

We want to "combine" elements of a set.

Definition

Let A be a non-empty set. An *operation* on A is a function from ordered pairs of elements of the set to the same set. Operations to *combine* elements. We *operate* on ordered pairs.

Notation

Let A be a set and $g : A \times A \to A$. We tend to forego the notation $g(a, b)$ and write $a\,g\,b$ instead. We call this *infix notation*.

Using lower case latin letters for elements and for operators confuses, so we tend to use special symbols for operations. For example, $+$, $-$, $\cdot$, $\circ$, and $\star$.

Let A be a non-empty set and $+ : A \times A \to A$ be an operation on A. According to the above paragraph, we tend to write $a + b$ for the result of applying $+$ to (a, b).

Operations (??) immediately needs:

  Functions (??)

Operations (??) is immediately needed by:

  Algebras (??)
  Associative Operations (??)
  Commutative Operations (??)
  Identity Elements (??)

Operations (??) gives the following terms.

operation
combine
operate
infix notation

Operations

↓

Functions

↓

Relations

↓

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

↓

Pair Unions

↓

Set Unions

Unordered Pairs

Empty Set

↓

Set Specification

↓

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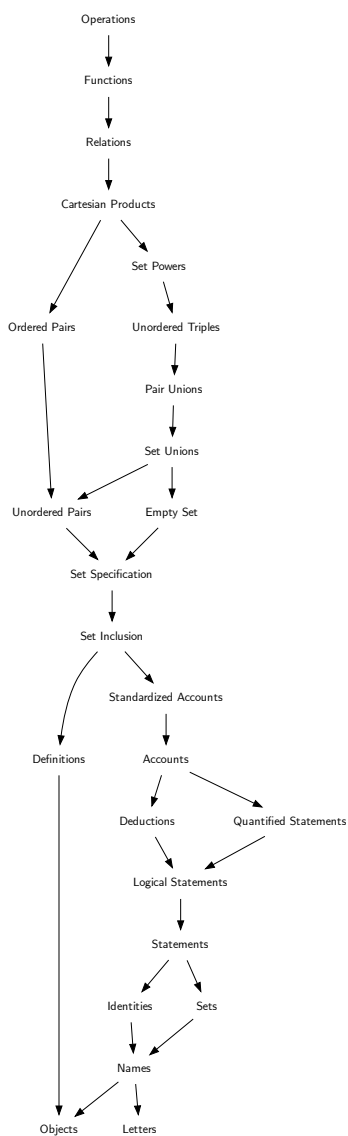
Why

We want to generalize operations beyond two objects.

Operations

The *pairwise extension* of a commutative operation is the function from finite families of the ground set to the ground set obtained by applying the operation pairwise to elements. TODO: this is not a function if the operation is not commutative.

The *ordered pairwise extension* of an operation is the function from finite families ground set to the ground set obtained by applying the operation pairwise to elements in order.

Notation

Let $(A, +)$ be an algebra and $\{A_i\}_{i=1}^n$ a finite family of elements of A. We denote the pairwise extension by

$$\overset{n}{\underset{i=1}{+}} A_i$$

Family Operations (**??**) immediately needs:

Algebras (**??**)
Families (**??**)

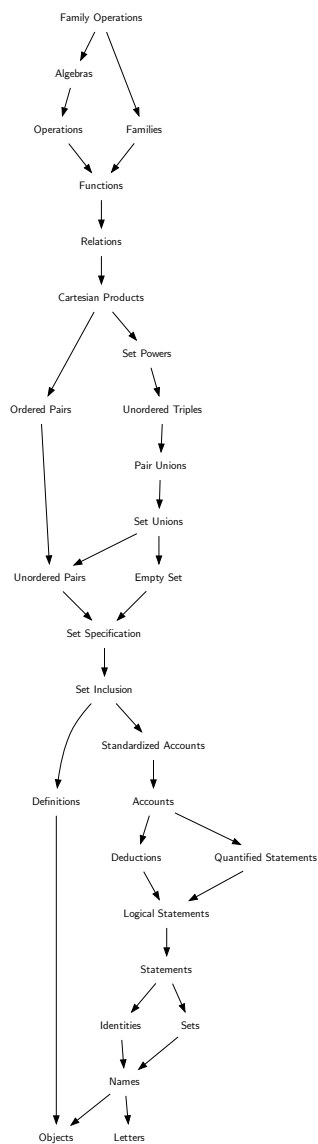Family Operations (**??**) is immediately needed by:

Natural Summation (**??**)
Summation (**??**)

Family Operations (**??**) gives the following terms.

pairwise extension
ordered pairwise extension

Family Operations

Algebras

Operations   Families

Functions

Relations

Cartesian Products

Set Powers

Ordered Pairs   Unordered Triples

Pair Unions

Set Unions

Unordered Pairs   Empty Set

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Why

We want to consider the elements of two sets together at once, and other sets created from two sets.

Definitions

Let A and B be two sets.

The *union* of A with B is the set whose elements are in either A *or* B *or* both. The key word in the definition is *or*.

The *intersection* of A with B is the set whose elements are in both A *and* B. The keyword in the definition is *and*.

Viewed as operations, both union and intersection commute; this property justifies the language "with." The intersection is a subset of A, of B, and of the union of A with B.

The *symmetric difference* of A and B is the set whose elements are in the union but not in the intersection. The symmetric difference commutes because both union and intersection commute; this property justifies the language "and." The symmetric difference is a subset of the union.

Let C be a set containing A. The *complement* of A in C is the symmetric difference of A and C. Since $A \subset C$, the union is C and the intersection is A. So the complement is the "left-over" elements of B after removing the elements of A.

We call these four operations *set-algebraic operations*.

Notation

Let A, B be sets. We denote the union of A with B by $A \cup B$, read aloud as "A union B." $\cup$ is a stylized U. We denote the intersection of A with B by $A \cap B$, read aloud as "A intersect B." We denote the symmetric difference of A and B by $A + B$, read aloud as "A symdiff B." "Delta" is a mnemonic for difference.

Let C be a set containing A. We denote the complement of A in C by $C - A$, read aloud as "C minus A."

Results

PROPOSITION **5.** *For all sets A and B the operations $\cup$, $\cap$, and $+$ commute.*

PROPOSITION **6.** *Let S a set. For all sets $A, B \subset S$,*

$$(1) \quad S - (A \cup B) = (S - A) \cap (S - B)$$
$$(2) \quad S - (A \cap B) = (S - A) \cup (S - B).$$

PROPOSITION **7.** *Let S a set. For all sets $A, B \subset S$,*

$$A + B = (A \cup B) \cap C_S(A \cap B)$$

$\boxed{TODO : notation}$

Exercises

If A, B, X, and Y are sets, then

1. $(A \cup B) \times X) = (A \times X) \cup (B \times X)$,

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2. $(A \cap B) \times (A \cap Y) = (A \times X) \cap (B \times Y)$,

3. $(A - B) \times X = (A \times X) - (B \times X)$.

If either $A = \varnothing$ or $B = \varnothing$, then $A \times B = \varnothing$, and conversely. If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$, and (provided $A \times B \neq \varnothing$) conversely.

Set Operations (??) immediately needs:

Algebras (??)
Set Intersections (??)
Set Symmetric Differences (??)
Set Unions (??)

Set Operations (??) is immediately needed by:

Convex Sets (??)
Event Probabilities (??)
Extended Real Numbers (??)
Generated Sigma Algebra (??)
Monotone Classes (??)
Pointwise vs Measure Limits (??)
Real Length Impossible (??)
Subset Algebras (??)
Tail Sigma Algebra (??)
Topological Spaces (??)

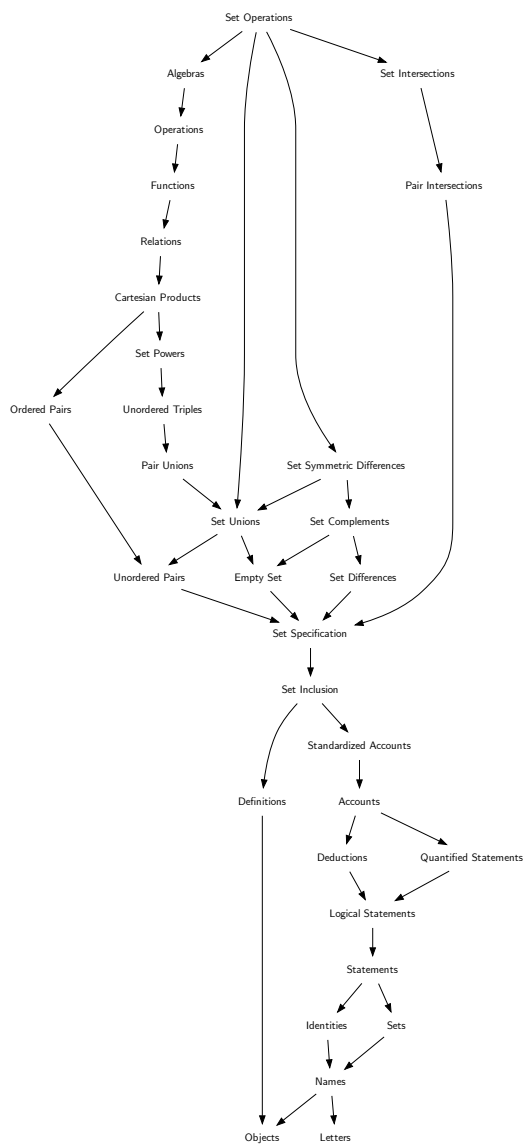Set Operations (??) gives the following terms.

intersection
symmetric difference
complement
set-algebraic operations
union

Set Operations

Algebras

Operations

Functions

Relations

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

Pair Unions

Set Symmetric Differences

Set Intersections

Pair Intersections

Set Unions

Set Complements

Unordered Pairs

Empty Set

Set Differences

Set Specification

Set Inclusion

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We name a set together with an operation.

Definition

An *algebra* is an ordered pair whose first element is a non-empty set and whose second element is an operation on that set. The *ground set* of the algebra is the set on which the operation is defined.

Notation

Let A be a non-empty set and let $+ : A \times A \to A$ be an operation on A. As usual, we denote the ordered pair by $(A, +)$.

Algebras (**??**) immediately needs:

Operations (**??**)

Algebras (**??**) is immediately needed by:
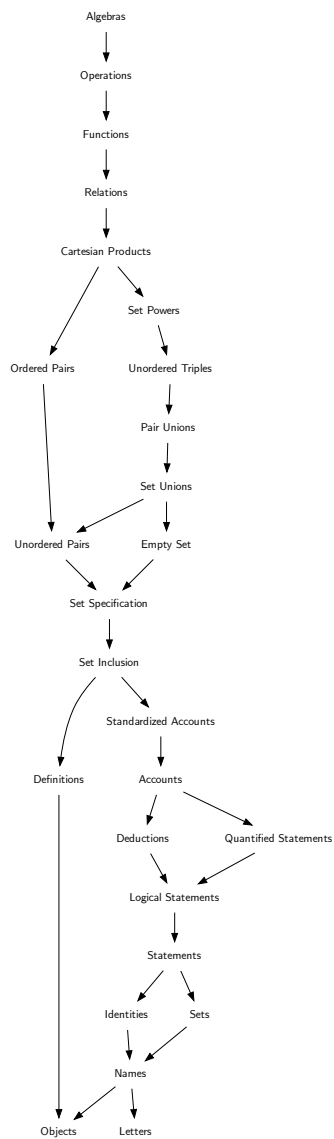
Element Functions (**??**)
Family Operations (**??**)
Groups (**??**)
Set Operations (**??**)

Algebras (**??**) gives the following terms.

algebra
ground set

Algebras

↓

Operations

↓

Functions

↓

Relations

↓

Cartesian Products

Set Powers

Ordered Pairs

Unordered Triples

Pair Unions

Set Unions

Unordered Pairs

Empty Set

Set Specification

↓

Set Inclusion

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↓

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Why

We want to talk about the size of a set.

Definition

Two sets are *equivalent* if there exists a bijection between them.

PROPOSITION **8.** *Set equivalence in the sense defined above is an equivalence relation in the power set of a set.*

PROPOSITION **9.** *Every proper subset of a natural number is equivalent to some smaller natural number.*

Proof. TODO induction □

TODO: smaller defined?

PROPOSITION **10.** *A set can be equivalent to a proper subset of itself.*

Halmos' example here is not a bijection, though...

PROPOSITION **11.** *If n is a natural number, then n is not equivalent ot a proper subset of itself.*

PROPOSITION **12.** *A set can be equivalent to at most one natural number.*

PROPOSITION **13.** *The set of natural numbers is infinite.*

PROPOSITION **14.** *A finite set is never equivalent to a proper subset of itself.*

PROPOSITION **15.** *Every subset of a finite set is finite.*

PROPOSITION **16.** *Every subset of a natural number is equivalent to a natural number.*

Equivalent Sets (**??**) immediately needs:

Equivalence Relations (**??**)
Function Inverses (**??**)
Natural Numbers (**??**)

Equivalent Sets (**??**) is immediately needed by:

Finite Sets (**??**)

Equivalent Sets (**??**) gives the following terms.

equivalent