



## Why

Take a vector space, and consider the set of continuous linear functionals on that space. Given a suitable norm, this space is a complete normed space.

## Defining result

**Proposition 1.** *Let  $(V, \|\cdot\|)$  be a normed space. The set  $V^*$  of all continuous linear functionals on  $V$  is a complete normed space with respect to pointwise algebraic operations and norm  $\|\cdot\|_* : V \rightarrow \mathbf{R}$  defined by*

$$\|F\|_* = \sup_{x \in V, \|x\| \leq 1} |F(x)|.$$

*Proof.* We argue (1)  $V^*$  is a vector space, (2)  $\|\cdot\|_*$  is a norm, and (3)  $(V, \|\cdot\|_*)$  is complete.<sup>1</sup> □

We call  $(V^*, \|\cdot\|_*)$  the *dual space* (or *Banach dual of  $V$* ). Notice that  $(V^*, \|\cdot\|_*)$  is complete regardless of whether the original normed space  $(V, \|\cdot\|)$  is complete.

## Basic dual norm property

Notice that the dual norm satisfies a familiar property.

**Proposition 2.** *For any vector  $x$  in a normed space  $(V, \|\cdot\|)$  and any continuous linear functional  $F$  on  $E$ ,*

$$|F(x)| \leq \|F\|_* \|x\|.$$

*Proof.* If  $x = 0$ , then  $\|x\| = 0$  and  $F(x) = 0$  ( $F$  is linear). Otherwise,  $x/\|x\|$  is a unit vector and so

$$\|F\|_* \geq |F(x/\|x\|)| = \frac{|F(x)|}{\|x\|}.$$

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<sup>1</sup>Future editions will include an account.

where the inequality is from the definition of  $\|\cdot\|_*$  (as a supremum) and the equality follows from the linearity of  $F$ .  $\square$

