Image Measures

1 Why

A measurable function from a first measure space to a second measurable space induces a measure on the latter.

2 Definition

Consider two measurable spaces and a measurable function between them. The *image measure* of a measure on the first space *under* the measurable function is the measure on the second space which assigns to each measurable set the measure of the inverse image of that measurable set.

We say that the function *induces* the image measure on the codomain. Alternatively, we say that we *push forward* the measure to the codomain, and so call the image measure a *push forward measure*.

2.1 Notation

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. Let $f : X \to Y$ be a measurable function. Let $\mu : \mathcal{A} \to [0, \infty]$ be a measure. We denote the image measure of μ under f by $\mu \circ f^{-1}$, for the reason that it

$$\mu \circ f^{-1}(B) = \mu(f^{-1}(B))$$

for every $B \in \mathcal{B}$.

3 Change of Variables

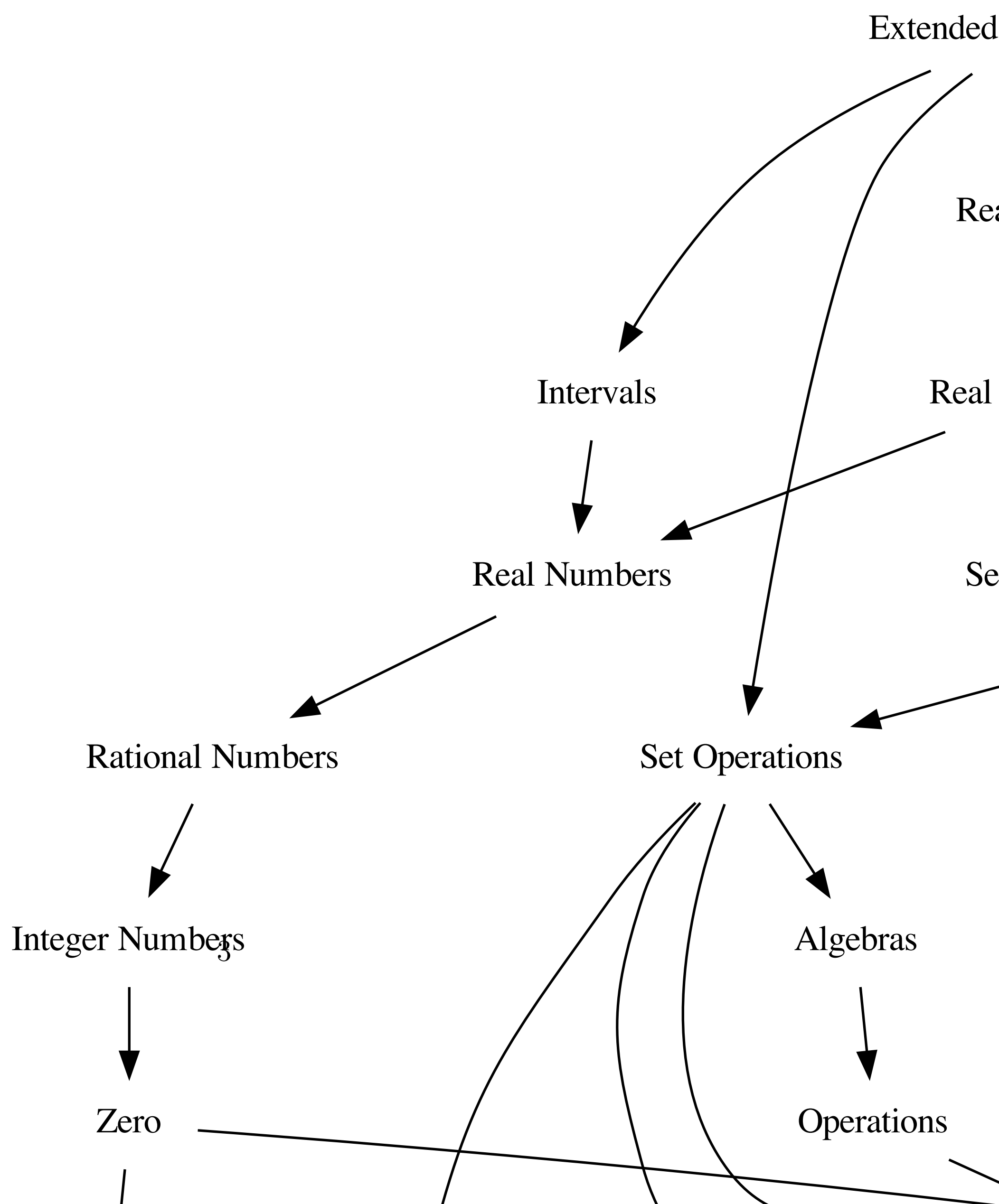
The main property we would like to hold is that integration on the new measure space is the same as integration on the old.

Proposition 1. *Let* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *be measurable spaces and let* R *denote the real numbers. Let* $f : X \to Y$ *be a measurable function and let* $\mu : \mathcal{A} \to [0, \infty]$ *be a measure.*

Then $g : Y \to R$ *is integrable with respect to* $\mu \circ f^{-1}$ *if and only if* $g \circ f$ *is integrable with respect to* μ*. In this case,*

$$\int g d(\mu \circ f^{-1}) = \int g \circ f d\mu.$$

Proof. □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Intervals
Real
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