Definition

Suppose V is a subspace over **F**. A vector space U over **F** is a *subspace* (or *linear subspace, vector subspace*) of V if $U \subset V$ and vector addition and scalar multiplication defined for U agree with those defined for V. In other words, a subspace is a subset of a vector space which is closed under vector addition and scalar multiplication.

For example, the entire set of vectors is a subspace. As a second example, the set consisting only of the zero vector is a subspace; we call this the *zero subspace*. These two subspaces are the *trivial subspaces*. A *nontrivial subspace* is a subspace that is not trivial.

Notation

Let $(V, \mathbf{F})$ be a vector space. Let $U \subset V$ with

$$\alpha u + \beta v \in U$$

for all $\alpha, \beta \in \mathbf{F}$ and $u, v \in U$. Then U is a subspace of $(V, \mathbf{F})$.

Characterization

Proposition 1. *Suppose V is a vector space over a field $\mathbf{F}$ and $U \subset V$. U is a subspace if and only if U satisifes*

1. *$0 \in U$ (contains additive identity)*

2. *$u + w \in U$ for all $u, w \in U$ (closed under addition)*

3. *$\alpha u \in U$ for all $\alpha \in \mathbf{F}$ and $u \in U$ (closed under scalar addition)*

Properties

Proposition 2. *The intersection of a family of subspaces is a subspace.*

Proposition 3. *There exists a family of subspaces whose union is not a subspace;*

Remark 1. *In other words: the union of a family subspaces need not be a subspace.*

Proposition 4. *A subspace must contain the zero vector; in other words, every subspace is nonempty.*

Vector Subspaces

Real Subspaces

Real Vectors Span

Vectors

Real Linear Combinations

Real Vector

Plane Vector Sums and Differences · Space Vectors · N-Dimensional Space

Plane Vectors

Distance

Space Distance

Plane Distance

Interval Length

Intervals · Real Space

Real Plane · Real Line

Real Order · Integral Line

Fields · Real Numbers · Rational Order · Geometry

Lists

Rings · Rational Numbers

Direct Products · Set Numbers

Natural Fractions · Integer Arithmetic

Finite Sets

Natural Additive Identity · Natural Multiplicative Identity · Integer Products

Equivalent Sets

Natural Arithmetic · Identity Elements

Family Unions and Intersections · Function Inverses · Comparisons · Natural Powers · Integer Sums · Integer Order

Families · Function Composites · Function Images · Operations · Orders · Natural Products · Integer Numbers

Functions · Converse Relations · Equivalence Relations · Natural Order · Natural Sums

Relations

Recursion Theorem

Ordered Pair Projections

Peano Axioms

Set Products

Generalized Set Dualities · Set Powers

Natural Induction

Set Unions and Intersections · Set Dualities · Unordered Triples · Successor Sets · Intersection of Empty Set · Set Partitions · Set Symmetric Differences

Natural Numbers

Ordered Pairs · Pair Unions · Set Intersections · Set Complements

Pair Intersections · Set Unions · Set Differences

Unordered Pairs · Empty Set

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities · Sets

Names

Letters · Objects