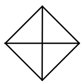


Why

We want to talk about sequences of real numbers which, as we go further and further in the sequence, get closer and closer to some fixed real number.

Definition

We need to talk about "where" a sequence is "approaching."

A *limit* of a sequence $(x_n)_{n \in \mathbf{N}}$ of real numbers is a real number x_0 with the property that, for any interval centered at x_0, we can find a final part of the sequence wholly contained in that interval; no matter how small the interval. By making the interval small, we capture to intuition that the sequence is "close" to its limit.

In other words, you propose a limit for a sequence. To test this proposal, I specify some small positive real number. Then we look for a final part of the sequence which is wholly contained in the interval whose width is twice the small positive number. If we can always find the final part, no matter how small the positive number I specified, then the proposed limit is true.

Existence

Some sequences have no limit. Consider the sequence which alternates between $+1$ and -1. To show that the limit does not exist, we argue indirectly. We take any real number and test it with the interval length one. No matter which real number we have selected, $+1$ and -1 are a distance two apart, and so can not both be contained in an interval of width one.

In the case that there exists a limit for a sequence, we say that the sequence *converges to* its limit (or we say that it *converges*, or call it *convergent* or *converging*).

Uniqueness

If a sequence has a limit, it has only one limit. So, from here on, we will speak of *the limit* of the sequence.

To see this uniqueness, suppose that two real numbers satisfy the limiting property. We now argue indirectly: suppose also that they are not equal. Denote the distance between them by x. Then ask for final parts in intervals of width $x/2$ for both limits.

Approximation

We use limits to speak about the terminating behavior of infinite processes. We think about the sequence as approximating the limit. The sequence may never actually take the value of its limit, so the limit need not be in the set of terms of the sequence. But the idea, of course, is that the set of terms, especially those "far out" in the sequence, are close to the limit.

The definition, moreover, ensures that the sequence will get arbitrarily close. We can operationalize this property, by taking the first element of that final part after which all elements are close to the limit. This element of the sequence approximates the limit value well.

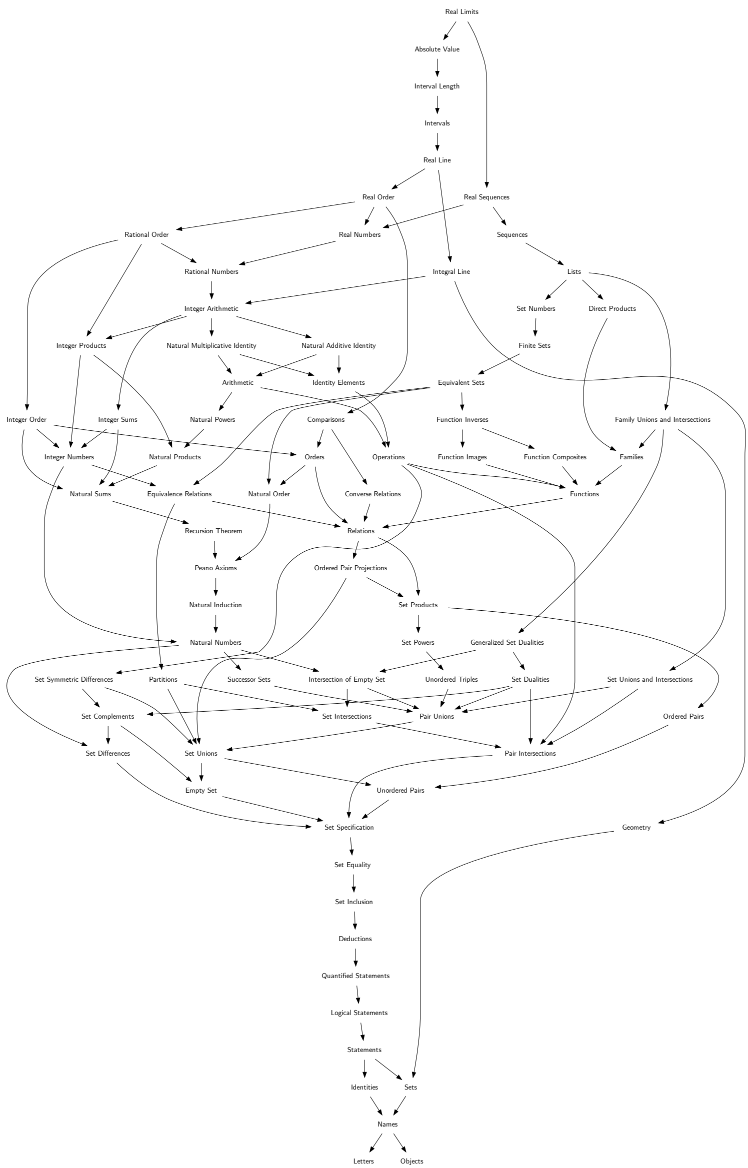
Notation

Let $(a_n)_n$ be a sequence of real numbers. Let a be a real number. We denote that a is the limit of $(a_n)_n$ by

$$a = \lim_{n \to \infty} a_n.$$

The abbreviation "lim" is from the Latin word *limes*, which means boundary, or limit.

We read this statement aloud as "a is the limit of a sub n." The above statement asserts two facts: (1) the sequence $(a_n)_n$ has a limit and (2) the limit is the real number a. We sometimes abbreviate this by writing $a = \lim_n a_n$.


Real Limits
Absolute Value
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Rational Order
Real Numbers
Sequences
Rational Numbers
Integral Line
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Integer Order
Integer Sums
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Images
Function Composites
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Geometry
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects