

Why

We want to sum infinitely many real numbers.

Definition

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence in **R**. Define $(s_n)_{n\in\mathbb{N}}$ by

$$s_n = \sum_{k=1}^n a_k.$$

We call s_n the *nth partial sum* of (x_k) . In other words, the first partial sum s_1 is a_1 , the second partial sum s_2 is $a_1 + a_2$, the third partial sum s_3 is $a_1 + a_2 + a_3$ and so on. We call (s_n) the sequence of partial sums or series of (a_k) . If the series converges, then we say that (a_k) is summable. Clearly not every series is summable: consider, for example, $a_k = 1$ for all k. It has the divergent series $(1, 2, 3, 4, 5, \ldots)$.

Notation

If the sequence is summable, then there exists a unique $s \in \mathbf{R}$ (the limit), which we denote

$$s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{k=1}^n a_k.$$

We read these relations aloud as "s is the limit as n goes to infinity of s n" and "s is the limit as n goes to infinity of the sum of a k from k equals 1 to n." We often avoid referencing s_n by abbreviating the above with

$$\sum_{k=1}^{\infty} a_k = s.$$

We read this notation aloud as "the sum from 1 to infinity of a k is s." The notation is subtle, and requires justification by the algebra of series.

¹Future editions will include such justification.

Convergence

For a series to converge, intuition suggests that the additional terms added should be getting smaller and smaller. Indeed:

Proposition 1. Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers. If (a_k) is summable then a_k converges to $0.^2$

The converse of this theorem has immediate relevance as a preliminary test for determining whether a series converges.

Proposition 2. If (a_k) does not converge or converges to $a_0 \neq 0$, then it is not summable.

²Future editions will include an account.

