



Why

Consider bounding the measure where two functions differ. We would look at their absolute value, and the measure of where this is greater than zero. The absolute value of their difference is a non-negative measurable random variable.

Result

We bound the measure that a non-negative measurable real-valued function exceeds some value by its integral.

Prop. 1. *Let (X, \mathcal{A}, μ) be a measure space. Let $g : X \rightarrow [0, \infty]$ be measurable. Then for all $t > 0$,*

$$\mu(\{x \in X \mid g(x) \geq t\}) \leq \frac{\int g d\mu}{t}.$$

Proof. Let $A = \{x \in X \mid g(x) \geq t\}$. Define $h : X \rightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} 1 & \text{if } g(x) \geq t \\ 0 & \text{otherwise.} \end{cases}$$

First, $\mu(A) = \int h d\mu$. Second, $h \leq g$. So:

$$\mu(\{x \in X \mid g(x) \geq t\}) = \int h d\mu \leq \int g/t d\mu = \frac{\int g d\mu}{t}.$$

□

The above is also called *Markov's Inequality*

