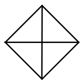


Why

We want to solve linear equations. Our approach is to "eliminate" variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

Two-variable example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$3x_1 + 2x_2 = 10, \text{ and}$$
$$6x_1 + 5x_2 = 20.$$

We can list the coefficients in a two-dimensional array $A = (3, 2; 6, 5)$ and $b = (10, 20)$. We can eliminate x_1 from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$3x_1 + 2x_2 = 10 \text{ and}$$
$$x_2 = 0.$$

The key insight is that this system has the *same solution set.* We call the process of moving between these two systems a *row reduction.*

Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate x_1 (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear system $S_1 = (A^1, b^1)$ in which

$$A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \text{ and } b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The trick is that, since $A'_{22} \neq 0$, we can take the same route to eliminate x_2, to obtain the system $S_2 = (A^2, b^2)$ in which

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \text{ and } b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Likewise for x_3, we obtain $S_3 = (A^3, b^3)$ in which

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one, (A^3, b^3), is easy to solve. We solve it by *back substitution.* First, since $2x_4 = 3$, we find $x_4 = 3/2$. Second, since $2x_3 + 2x_4 = -1$, we find $x_3 = -2$. Similarly we find $x_2 = 1/2$ and $x_3 = 5/4$.

Definition

Let $S = (A \in \mathbf{R}^{m\times n}, b \in \mathbf{R}^n)$ be a linear system. The lower *row reduction* of S for index i with $A_{ii} \neq 0$ (or the i-row reduction) is the linear system $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$ if $i < s \leq m$ and A_{st} otherwise. We say that the system (A, b) is *ordinarily reducible.*

Let $a^k, \tilde{a}^k \in \mathbf{R}^n$ denote the kth row of A and $\tilde{A}$, respectively. Then if $k \neq i$, $\tilde{a}^k = a^k - \alpha_k a^i$ where $\alpha_k = A_{kj}/A_{ij}$. In other words, a row k of the matrix $\tilde{A}$ is obtained by subtracting a multiple of the ith row of matrix A from row k of matrix A. We are "reducing" the rows of A.

Proposition 1. *Let* $(A \in \mathbf{R}^{m\times n}, b \in \mathbf{R}^n)$ *be a linear system which row reduces to* (C, d)*. Then* $x \in \mathbf{R}^n$ *is a solution of* (A, b) *if and only if it is a solution of* (C, d)*.*[1]

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = (\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}).$$

We then subtract three times row 2 from row 3 and four times

[1]Future editions will include an account.

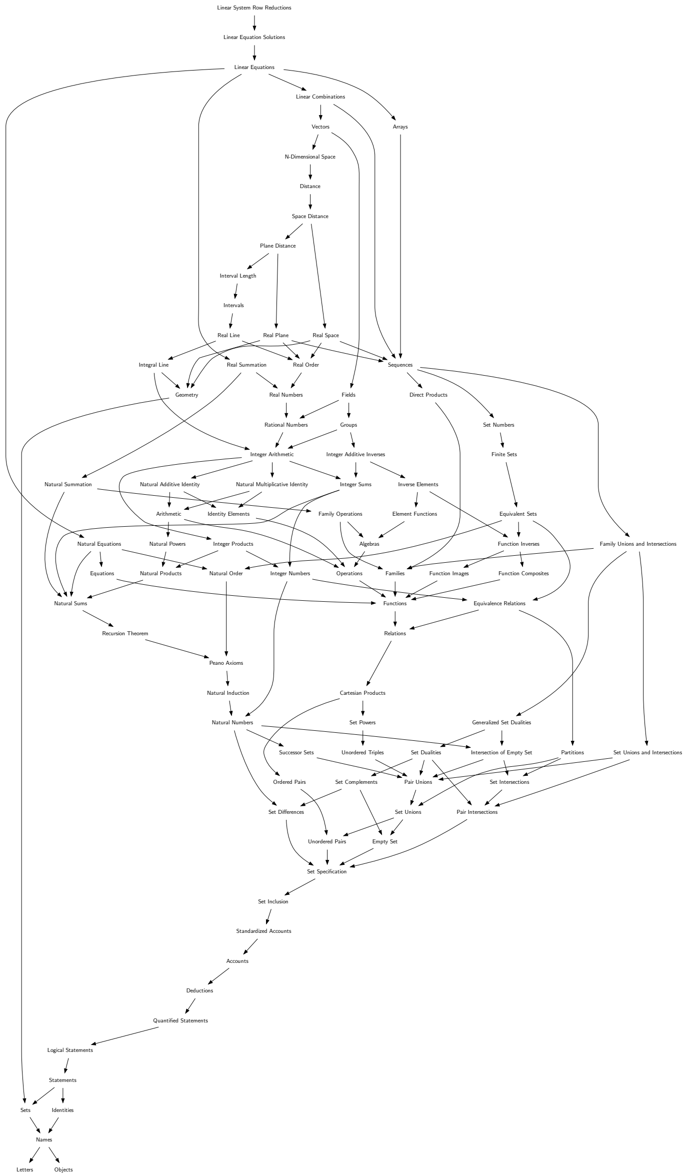
row 2 from row 4 to obtain

$$S_2 = (\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}).$$

Finally, we subtract two times row 3 from row 4 to obtain S_4, which we can write as

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ x_2 + x_3 + x_4 &= 0, \\ 2x_3 + 2x_4 &= -1, \quad \text{and} \\ 2x_4 &= 3. \end{aligned}$$

We can now back-substitute to find $x_4 = {}^3\!/_2$, $x_3 = -2$, $x_2 = {}^1\!/_2$ and $x_1 = {}^5\!/_4$. Proposition **??** says that this is the only solution of S, as well.


Linear System Row Reductions
Linear Equation Solutions
Linear Equations
Linear Combinations
Vectors
Arrays
N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Line
Real Plane
Real Space
Integral Line
Real Summation
Real Order
Sequences
Geometry
Real Numbers
Fields
Direct Products
Rational Numbers
Groups
Set Numbers
Integer Arithmetic
Integer Additive Inverses
Finite Sets
Natural Summation
Natural Additive Identity
Natural Multiplicative Identity
Integer Sums
Inverse Elements
Arithmetic
Identity Elements
Family Operations
Element Functions
Equivalent Sets
Natural Equations
Natural Powers
Integer Products
Algebras
Function Inverses
Family Unions and Intersections
Equations
Natural Products
Natural Order
Integer Numbers
Operations
Families
Function Images
Function Composites
Natural Sums
Functions
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Natural Induction
Cartesian Products
Natural Numbers
Set Powers
Generalized Set Dualities
Successor Sets
Unordered Triples
Set Dualities
Intersection of Empty Set
Partitions
Set Unions and Intersections
Ordered Pairs
Set Complements
Pair Unions
Set Intersections
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects