



Why

Vectors can be identified with matrices of width 1.

Canonical identification

We identify \mathbf{R}^n with $\mathbf{R}^{n \times 1}$ in the obvious way. For this reason, we call $x \in \mathbf{R}^{n \times 1}$ (meaning $x \in \mathbf{R}^n$) a *column vector*.

For the reasons that we identify \mathbf{R}^n with $\mathbf{R}^{n \times 1}$, we write the vector $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

We could as easily also identify \mathbf{R}^n with $\mathbf{R}^{1 \times n}$. We avoid this convention. However, by analogy with the language “column vector,” we refer to the *matrix* $y \in \mathbf{R}^{1 \times n}$ as a *row vector*.

Matrix transpose

We frequently move from $\mathbf{R}^{n \times 1}$ and $\mathbf{R}^{1 \times n}$. If $a \in \mathbf{R}^{n \times 1}$, we denote $b \in \mathbf{R}^{1 \times n}$ defined by $b_i = a_i$ by a^\top .

More generally, given a matrix $A \in \mathbf{R}^{m \times n}$, we denote the matrix $B \in \mathbf{R}^{m \times n}$ defined by $B_{ij} = A_{ji}$ by A^\top . Notice that the entries of i and j have swapped. We call the matrix B the *transpose* of A , and similarly call a^\top the *transpose* of the vector a . Clearly, $(A^\top)^\top = A$, which includes $(a^\top)^\top = a$.

Reals as vectors

There is a similar, and similarly obvious, identification of scalars $a \in \mathbf{R}$ with the 1-vectors \mathbf{R}^1 (and so with the 1 by 1 matrices $\mathbf{R}^{1 \times 1}$). Given our definition of matrix-vector products, if we identify $a \in \mathbf{R}$ with $A \in \mathbf{R}^{1 \times 1}$ where $A_{11} = a$, then $Ax = ax$.

Familiar concepts, new notation

These identifications and the notation of transposition give allow us to write several familiar concepts in a compact notation. We write the norm $\|x\|$ as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^\top x}.$$

We write the inner product as

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^\top y.$$

We express the symmetry of the inner product by $x^\top y = y^\top x$.

