Why

Can a set have no elements?

Definition

Sure. A set exists by the principle of existence (see Sets); denote it by A. Specify elements (see Set Specification) of any set that exists using the universally false statement $x \neq x$. We denote that set by $\{x \in A \mid x \neq x\}$. It has no elements. In other words, $(\forall x)(x \notin A)$. The principle of extension (see Set Equality) says that the set obtained is unique (contradiction).[1] We call the unique set with no elements the *empty set*. If a set is not the empty set, we call it *nonempty*.

Notation

We denote the empty set by $\varnothing$. In other words, in all future accounts (see Accounts), there are two implicit lines. First, "`name` $\varnothing$" and second "`have` $(\forall x)(x \notin \varnothing)$".

Properties

It is immediate from our definition of the empty set and of the definition of inclusion (see Set Inclusion) that the empty set is included in every set (including itself).

Proposition 1. $(\forall A)(\varnothing \subset A)$

Proof. Suppose toward contradiction that $\varnothing \not\subset A$. Then there exists $y \in \varnothing$ such that $y \notin A$. But this is impossible, since $(\forall x)(x \notin \varnothing)$. $\square$

[1] This account will be expanded in the next edition.

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\begin{tikzcd}
\text{Empty Set} \arrow[d] \\
\text{Set Specification} \arrow[d] \\
\text{Set Equality} \arrow[d] \\
\text{Set Inclusion} \arrow[d] \\
\text{Deductions} \arrow[d] \\
\text{Quantified Statements} \arrow[d] \\
\text{Logical Statements} \arrow[d] \\
\text{Statements} \arrow[dl] \arrow[dr] & \\
\text{Identities} \arrow[dr] & \text{Sets} \arrow[dl] \\
& \text{Names} \arrow[dl] \arrow[dr] & \\
\text{Letters} & & \text{Objects}
\end{tikzcd}
$$