Why

We want to talk about influencing natural phenomena.[1]

Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{T-1}$ be sets. For $t = 0, \ldots, T-1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \to \mathcal{X}_{t+1}$. We call the sequence

$$\left((\mathcal{X}_t)_{t=0}^{T}), (\mathcal{U}_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1}\right)$$

a *controlled kdeterministic discrete-time dynamical system*. We call the index t the *epoch*, the *stage* or the *period*.

Let $x_0 \in \mathcal{X}_0$. Let $u_0 \in \mathcal{U}_0, \ldots, u_{T-1} \in \mathcal{U}_{T-1}$. Define a state sequence $x_1 \in \mathcal{X}_1, \ldots, x_T \in \mathcal{X}_T$ by

$$x_{t+1} = f_t(x_t, u_t).$$

In this case we call x_0 the *initial state*. We call the x_t the *states*. We call the u_t a sequence of *inputs* (or *actions*, *decisions*, *choices*, or *controls*). We call f_t the *transition function* or *dynamics function*.

We call T the *horizon*. In the case that we have an infinite sequence of state sets, input sets, and dynamics, then we refer to a *infinite-horizon* dynamical system. To use language in contrast with this case, we refer to the dynamical system when T is finite as a *finite-horizon* dynamical system.

State

The current action u_t affects future states x_s for $s > t$, but not the current or past states. The current state x_t depends on the initial state x_0 and the sequence of past actions $u_0, \ldots, u_{t-1}$. So the state is a "link" between the

[1] Future editions will modify, and may restore former editions language: "We want to talk about making decisions over time." Though this language may also be used in a sheet on finite controlled dynamical systems.

past and the future. Given x_t and $u_t, \ldots, u_{s-1}$, for $s > t$, we can compute x_s. In other words, the prior actions $u_0, \ldots, u_{t-1}$ are not relevant.

This nonrelevancy of prior actions and prior states simplifies the sequential decision problem (see Sequential Decisions).

Variations

The dynamical system is called *time-invariant* if $\mathcal{X}_t$, $\mathcal{U}_t$ and f_t do not depend on t. A simple variation is that $\mathcal{U}_t$ depends on x_t.[2]

Examples

Finite dynamical system

A dynamical system is finite if the state and action sets are finite. For example, $\mathcal{X} = \{1, \ldots, n\}$ and $\mathcal{U} = \{1, \ldots, m\}$. Then $f_t : \mathcal{X} \times \mathcal{U} \to \mathcal{U}$ is called a *transition map*.

Or else, let (V, E) be a directed graph, then $\mathcal{X} = V$, $\mathcal{U}_{x_t} = \{(u, v) \in E \mid u = x_t\}$ and $f_t(x_t, u_t) = v$ when $u_t = (x_t, v)$ is a dynamical system. Roughly this system models "moving" on the graph.

Discrete-time linear dynamical system

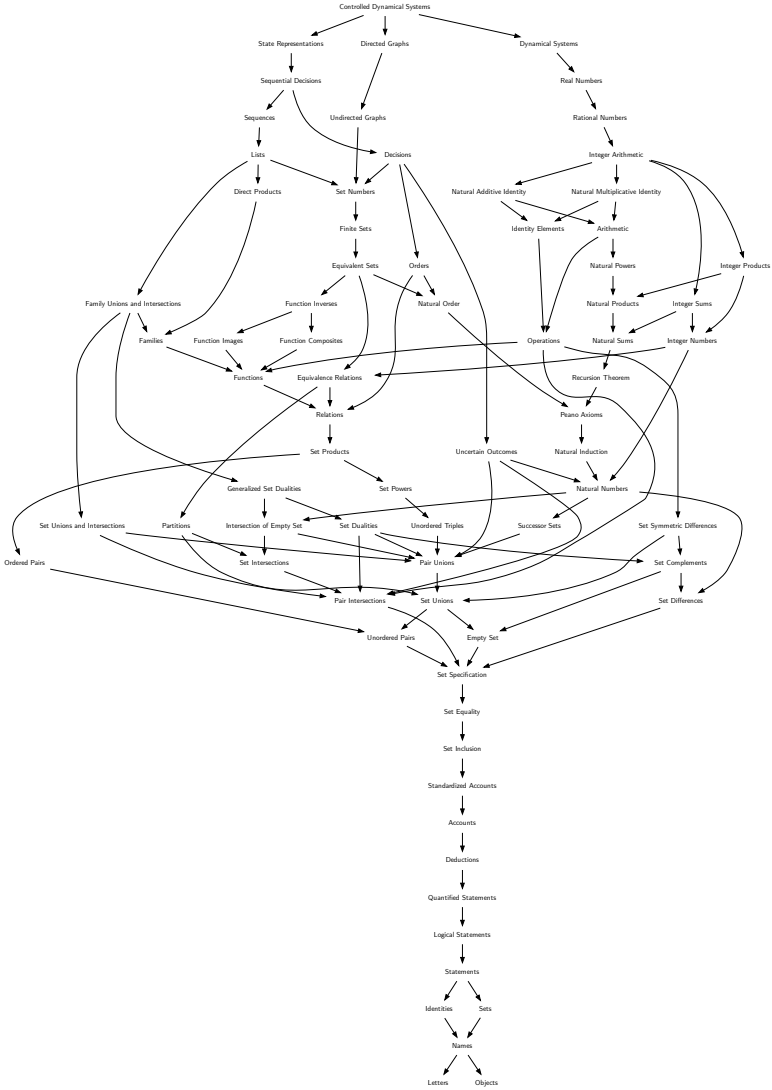
Let $\mathcal{X} = \mathbf{R}^n$ and $\mathcal{U} = \mathbf{R}^m$. Define $f_t : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ by

$$x_{t+1} = f_t(x_t, u_t) = A_t x_t + B_t u_t + c_t$$

for $y = 1, \ldots, T - 1$.[3]

[2] Future editions will say more here.

[3] This very special form of dynamics arises in many applications. Future editions will say more.


Controlled Dynamical Systems
State Representations
Directed Graphs
Dynamical Systems
Sequential Decisions
Real Numbers
Sequences
Undirected Graphs
Rational Numbers
Lists
Decisions
Integer Arithmetic
Direct Products
Set Numbers
Natural Additive Identity
Natural Multiplicative Identity
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Orders
Natural Powers
Integer Products
Family Unions and Intersections
Function Inverses
Natural Order
Natural Products
Integer Sums
Families
Function Images
Function Composites
Operations
Natural Sums
Integer Numbers
Functions
Equivalence Relations
Recursion Theorem
Relations
Peano Axioms
Set Products
Uncertain Outcomes
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Dualities
Unordered Triples
Successor Sets
Set Symmetric Differences
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Pair Intersections
Set Unions
Set Differences
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects