## Why

When does the technique of row reductions prevail?

## Multivariable row reductions

Let $S = (A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be a linear system with $A_{kk} \neq 0$. The *kth row reduction* of $S$ is the linear system $(C, d)$ with $C_{st} = A_{st} - (A_{sk}/A_{kk})A_{kt}$ if $i < s \leq m$ and $C_{st} = A_{st}$ otherwise.

The idea, as in the example in Linear System Row Reductions, is to eliminate variable $k$ from equations $k+1, \dots, m$. We are taking the $k$th column of $A$ from

$$\begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ A_{k+1,k} \\ \vdots \\ A_{mk} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{1k} \\ \vdots \\ A_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We interpret the $i$th row reduction as *subtracting equations* of the system or *reducing rows* of the array $A$. If $a^i, c^i \in \mathbf{R}^n$ denote the $i$th rows of $A$ and $C$, $c^i = a^i - (A_{ik}/A_{kk})a^k$ for $k < i \leq m$, In other words, we obtain the $i$th row of matrix $C$ by subtracting a multiple of the $k$th row of matrix $A$ from the $i$th row of matrix $A$, for $k < i \leq m$. The following is an immediate consequence of real arithmetic.

**Proposition 1.** *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system* *which row reduces to* $(C, d)$. *Then* $x \in \mathbf{R}^n$ *is a solution of* $(A, b)$ *if and only if it is a solution of* $(C, d)$.

### Ordinary reductions

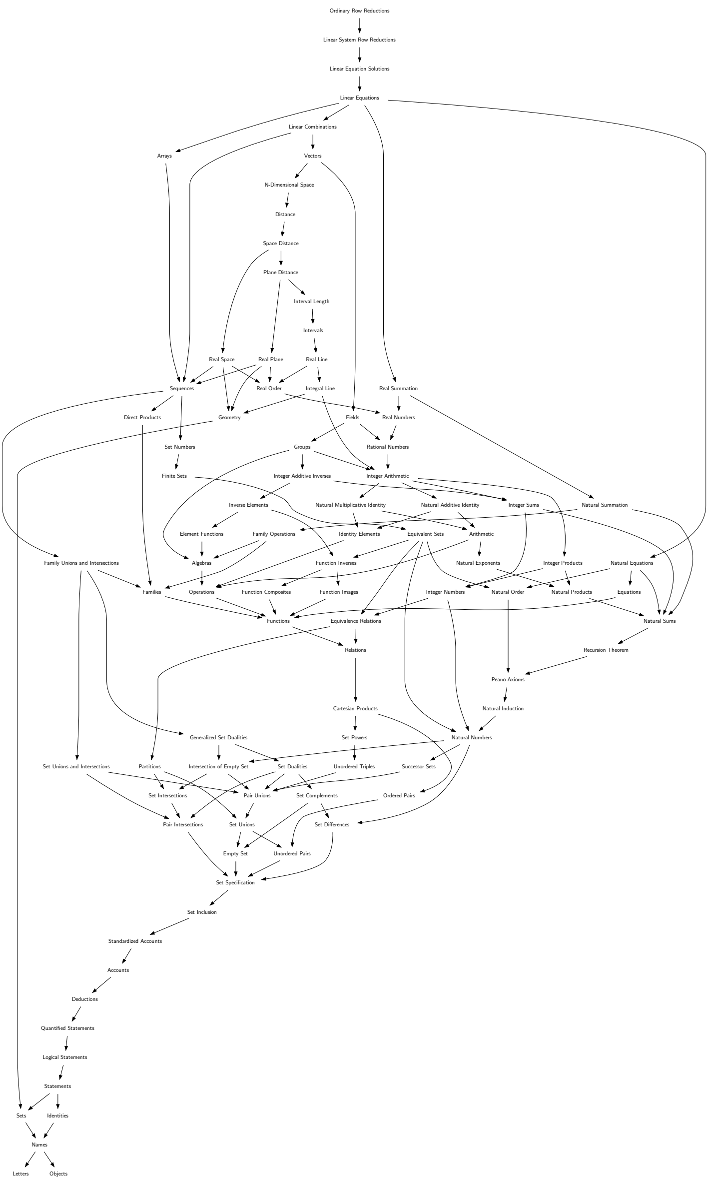
We call the system $S$ *ordinarily reducible* if there exists a sequence of systems $S_1, \ldots, S_{m-1}$ so that $S_1$ is the 11-reduction of $S$ and $S_i$ is the $ii$-reduction of $S_{i-1}$ for $i = 1, \ldots, n-1$. In this case, we call $S_{n-1}$ the *final ordinary reduction* (or just *ordinary reduction*) of $S$. The following is an immediate consequence of Proposition 1.

**Proposition 2.** *Let* $S'$ *be the (final) ordinary reduction of* $S$. *Then* $S$ *and* $S'$ *have equivalent solution sets.*

The idea is that a system is ordinarily reducible if we can take row reductions in sequence an end up with a system that is easy to back-substitute and solve. The difficulty is that this need not be the case. For example, consider the following obvious difficulty. The system $(A, b)$ in which

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is not ordinarily reducible, but clearly solvable.


Ordinary Row Reductions
Linear System Row Reductions
Linear Equation Solutions
Linear Equations
Linear Combinations
Arrays
Vectors
N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Plane
Real Line
Sequences
Real Order
Integral Line
Real Summation
Direct Products
Geometry
Fields
Real Numbers
Set Numbers
Groups
Rational Numbers
Finite Sets
Integer Additive Inverses
Integer Arithmetic
Inverse Elements
Natural Multiplicative Identity
Natural Additive Identity
Integer Sums
Natural Summation
Element Functions
Family Operations
Identity Elements
Equivalent Sets
Arithmetic
Family Unions and Intersections
Algebras
Function Inverses
Natural Exponents
Integer Products
Natural Equations
Families
Operations
Function Composites
Function Images
Integer Numbers
Natural Order
Natural Products
Equations
Functions
Equivalence Relations
Natural Sums
Relations
Recursion Theorem
Peano Axioms
Cartesian Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Partitions
Intersection of Empty Set
Set Dualities
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Complements
Ordered Pairs
Pair Intersections
Set Unions
Set Differences
Empty Set
Unordered Pairs
Set Specification
Set Inclusion
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Deductions
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