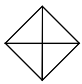


Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Suppose p is a distribution on Ω. For any event $A \subset \Omega$, we call the value $\sum_{a \in A} p(a)$ the *event probability*. We refer to the probability *of* A. The probability of A is the sum of the probabilities of its outcomes.

Notation

We can define a function $\mathsf{P} : \mathcal{P}(\Omega) \to \mathsf{R}$ by $\mathsf{P}(A) = \sum_{a \in A} p(a)$. We call P *the event probability function* (or *the probability measure*) induced by p. Since P depends on the sample space Ω and the distribution p, we occasionally denote this dependence by $\mathsf{P}_{\Omega,p}$ or P_p.

Many authors associate an event $A \subset \Omega$ with a function $\pi : \Omega \to \{0, 1\}$ so that $A = \{\omega \in \Omega \mid \pi(\omega) = 1\}$. In this context, it is common to write $P[\pi]$ for $\mathsf{P}[\pi^{-1}(1)]$ which is $\mathsf{P}(A)$.

Example: die

Define $p : \{1, \dots, 6\} \to \mathsf{R}$ by $p(\omega) = 1/6$ for $\omega = 1, \dots, 6$. Define the event $E = \{2, 4, 6\}$. Then

$$\mathsf{P}(E) = \textstyle\sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of P

As a result of the conditions on p, P satisfies

1. $\mathsf{P}(A) \geq 0$ for all $A \subset \Omega$;

2. $\mathsf{P}(\Omega) = 1$ (and $\mathsf{P}(\varnothing) = 0$);

3. $\mathbf{P}(A) + \mathbf{P}(B)$ for all $A, B \subset \Omega$ and $A \cap B = \varnothing$. This statement follows from the more general identity

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

for $A, B \subset \Omega$, by using $\mathbf{P}(\varnothing) = 0$ of (2) above.

These three conditions are sometimes called the *axioms of probability for finite sets*. Do all such $\mathbf{P}$ satisfying (1)-(3) have a corresponding underlying probability distribution?

In other words, suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satisfies (1)-(3). Define $q : \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an *event probability function* (or a *(finite) probability measure*).

Other basic consequences

Probability by cases

Let $\mathbf{P}$ be a probability event function. Suppose $A_1, \ldots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$\mathbf{P}(B) = \textstyle\sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

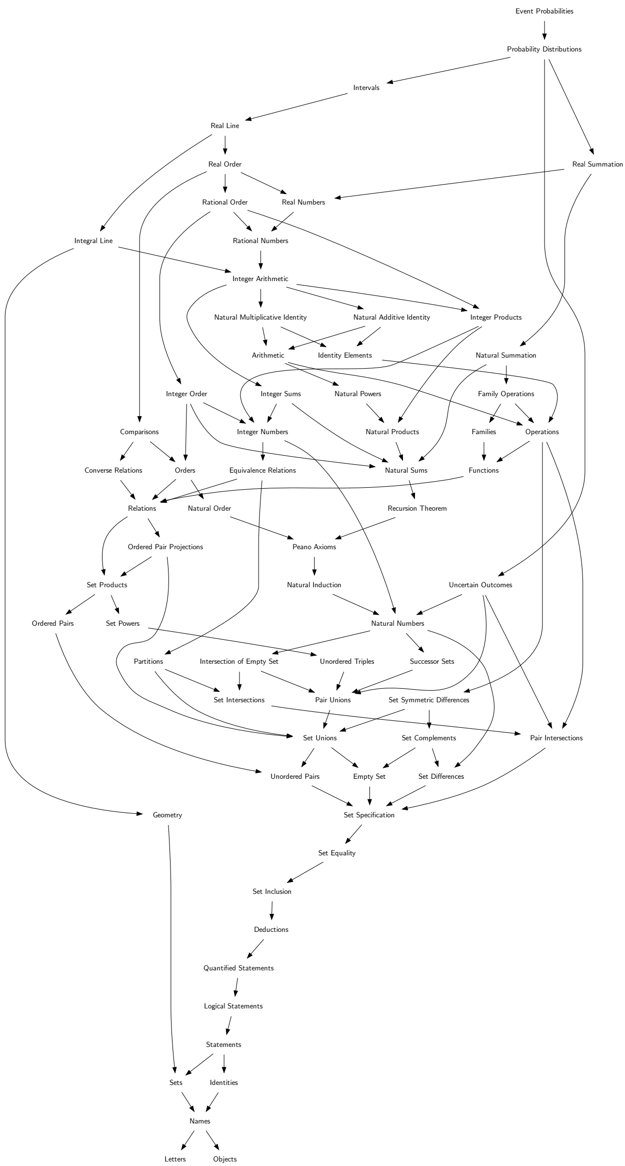
Some authors call this the *law of total probability.*

Monotonicity

If $A \subseteq B$, then $\mathbf{P}(A) \leq P(B)$. This is easy to see by splitting B into $A \cap B$ and $B - A$, and applying (1) and (3).

Subadditivity

For $A, B \subset \Omega$, $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$. This is easy to see from the more general identity in (3) above. This is sometimes referred to as a *union bound*, in reference to *bounding* the quantity $\mathbf{P}(A \cup B)$.


Event Probabilities
Probability Distributions
Intervals
Real Line
Real Order
Real Summation
Rational Order
Real Numbers
Integral Line
Rational Numbers
Integer Arithmetic
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Arithmetic
Identity Elements
Natural Summation
Integer Order
Integer Sums
Natural Powers
Family Operations
Comparisons
Integer Numbers
Natural Products
Families
Operations
Converse Relations
Orders
Equivalence Relations
Natural Sums
Functions
Relations
Natural Order
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Set Products
Natural Induction
Uncertain Outcomes
Ordered Pairs
Set Powers
Natural Numbers
Partitions
Intersection of Empty Set
Unordered Triples
Successor Sets
Set Intersections
Pair Unions
Set Symmetric Differences
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Empty Set
Set Differences
Geometry
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects