Why

If we stack two rectangles, with equal base lengths but different heights, on top of each other, the additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property, as it ought to.

Result

Prop. 1. *The simple non-negative integral operator is additive.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathrm{SF}_+(X)$ denote the non-negative real-valued simple functions on X. Define $s : \mathrm{SF}_+(X) \to [0, \infty]$ by $s(f) = \int f d\mu$ for $f \in \mathrm{SF}_+ X$.

In this notation, we want to show that $s(f+g) = s(f)+s(g)$ for all $f, g \in \mathrm{SF}_+(X)$. Toward this end, let $f, g \in \mathrm{SF}_+(X)$ with the simple partitions:

$$\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n \subset \mathcal{A} \quad \text{and} \quad \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset [0, \infty].$$

We consider the refinement of the two partitions. TODO: this is why you don't do the unique maximal partition business. $\{A_i \cap B_j\}_{i,j=1}^{i=m,j=n}$.

First, let $\alpha \in (0, \infty)$. Then $\alpha f \in \mathrm{SF}_+(X)$, with the simple partition $\{A_n\} \subset \mathcal{A}$ and $\{\alpha a_n\} \subset [0, \infty]$.

$$s(\alpha f) = \sum_{i=1}^n \alpha a_n \mu(A_i) = \alpha \sum_{i=1}^n a_n \mu(A_i) = \alpha s(f).$$

If $\alpha = 0$, then αf is uniformly zero; it is the non-negative simple with partition $\{X\}$ and $\{0\}$. Regardless of the measure of X, this non-negative simple function is zero Recall that we define $0 \cdot \infty = \infty \cdot 0 = 0$. $\qquad\square$

Simple Integral Additivity

Simple Integrals

Measurable Functions

Measures

Sigma Algebras

Extended Real Numbers

Real Series

Real Limits

Absolute Value

Simple Functions

Interval Length

Real Functions

Intervals

Real Line

Real Sequences

Real Order

Real Numbers

Rational Order

Integral Line

Rational Numbers

Integer Arithmetic

Subset Algebra

Lists

Natural Additive Identity

Natural Multiplicative Identity

Integer Products

Cardinality

Set Numbers

Finite Sets

Identity Elements

Arithmetic

Direct Products

Operations

Equivalent Sets

Natural Powers

Family Unions and Intersections

Function Inverses

Comparisons

Integer Order

Integer Sums

Natural Products

Families

Function Images

Function Composites

Orders

Integer Numbers

Natural Sums

Functions

Converse Relations

Equivalence Relations

Natural Order

Recursion Theorem

Relations

Peano Axioms

Subset Systems

Set Products

Characteristic Functions

Natural Induction

Ordered Pairs

Generalized Set Dualities

Set Powers

Natural Numbers

Set Unions and Intersections

Set Dualities

Unordered Triples

Successor Sets

Intersection of Empty Set

Partitions

Set Symmetric Differences

Pair Unions

Set Intersections

Set Complements

Pair Intersections

Set Unions

Set Differences

Unordered Pairs

Empty Set

Set Specification

Geometry

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects