

#### MONOTONIC MATRIX FUNCTIONS

# Why

Since we have a partial order on the set of positive semidefinite matrices, we can study which familiar functions are have order-preserving or order-reversing properties.

## **Norms**

It would be nice if the matrix norm induced by the matrix scalar produce (see Matrix Scalar Product) was an isotonic function. In other words, if  $A, B \in \mathbf{S}^d$  satisfy  $A \geq B$ , does  $||A|| \geq ||B||$ ?

Since  $||A||^2 = \operatorname{tr} A^2$ , we should study the trace first..

## **Trace**

**Proposition 1.** Let  $f: S^d \to R$  defined by  $f(A) = \operatorname{tr} A$ .

In other words, the function f is the restriction of the trace function onto the set of symmetric matrices.

**Proposition 2.** Let  $B \in \mathbf{S}^d$  Let  $f_B : \mathbf{S}^d \to \mathbf{R}$  defined by  $f(A) = \operatorname{tr} AB$ .

#### Inversion

**Proposition 3.** Let  $A \in \mathbf{S}_{++}^d$ . Then the map  $f : \mathbf{S}_{++}^d \to \mathbf{S}_{++}^d$  satisfying  $f(A) = A^{-1}$  is an isotonic function mapping the (open) positive definite cone into itself.<sup>1</sup>

 $<sup>^1\</sup>mathrm{Future}$  editions will include a proof.

