## Why

We want to model uncertain outcomes in dynamical systems.[1]

## Definition

Let $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_T$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{T-1}$ be sets. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Let $\mathcal{W}_0, \ldots, \mathcal{W}_T$. Let $w_t : \Omega \to \mathcal{W}_t$ for $t = 0, \ldots, T$ be random variables. For $t = 0, \ldots, T-1$, let $f_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathcal{X}_{t+1}$.

We call the sequence

$$\mathcal{D} = ((\mathcal{X}_t)_{t=0}^{T}), (\mathcal{U}_t)_{t=0}^{T-1}, (w_t)_{t=0}^{T-1}, (f_t)_{t=1}^{T-1})$$

a *stochastic discrete-time dynamical system.* We call $w_t$ the *noise* variables.

## Problem

Let $x_0 : \Omega \to \mathcal{X}_0$ be a random variable. Define $x_1 : \Omega \to \mathcal{X}_1, \ldots, x_T : \Omega \to \mathcal{X}_t$ by

$$x_{t+1} = f_t(x_t, u_t, w_t)$$

for $t = 0, \ldots, T-1$. Roughly speaking, the state transition functions are nondeterministic. In other words, it is uncertain which state we will arrive in given our current state and action. The choice $u_t$ only determines the distribution of $x_{t+1}$. Here $x_0$ is (still) called the *initial state* and is a random variable, usually assumed independent of the $w_t$.

Let $g_t : \mathcal{X}_t \times \mathcal{U}_t \times \mathcal{W}_t \to \mathbf{R} \cup \{\infty\}$ for $t = 0, \ldots, T-1$ and $g_T : \mathcal{X}_T \times \mathcal{W}_T \to \mathbf{R} \cup \{\infty\}$. We call $(x_0, \mathcal{D}, (g_t)_{t=0}^{T})$ a *stochastic dynamic optimization problem.* As with dynamic optimization problems, we call $g_t$ the *stage cost function* and $g_T$ the *terminal cost function.* It is common for these to not depend on $w_T$ (in other words, to be deterministic). It is also common for these to take infinite values to encode constraints.

[1] Future editions will expand.

As before, a stochastic dynamic optimization problem is just an optimization problem. Define $U = \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_{T-1}$ and let $u \in U$. Define $C : \Omega \to \mathbf{R}$ by

$$C = \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T).$$

We call $C$ the *total cost* for actions $u$. It is a random variable.

Define $J : U \to \mathbf{R} \cup \{\infty\}$ by

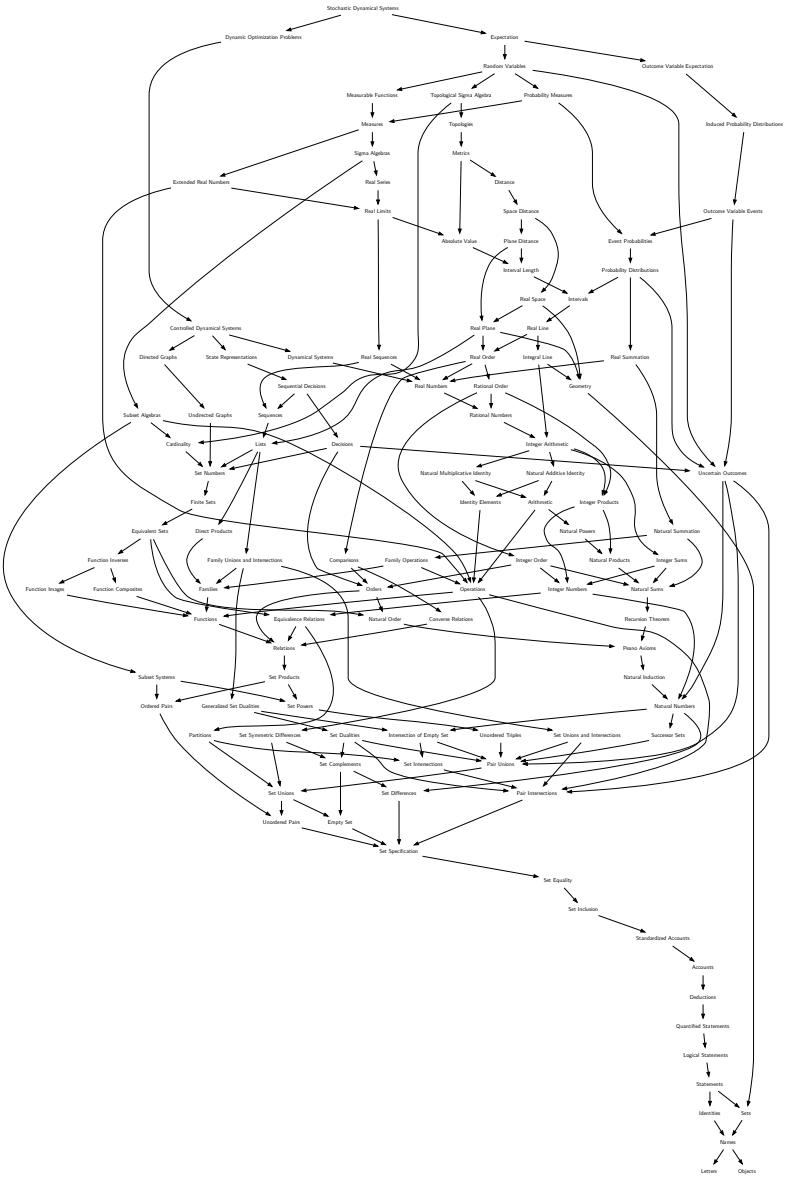
$$J = \mathbf{E}(\sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T, w_T)).$$

$J(u)$ is the *expected total cost* for inputs $u$.

The optimization problem is $(U, J)$. In other words, the objective is the mean total stage cost plus the terminal cost.

### Other terminology

Stochastic dynamic optimization problems are frequently called *stochastic control problems.*


Stochastic Dynamical Systems
Dynamic Optimization Problems
Expectation
Random Variables
Outcome Variable Expectation
Measurable Functions
Topological Sigma Algebra
Probability Measure
Induced Probability Distributions
Measures
Topologies
Sigma Algebras
Metrics
Extended Real Numbers
Real Series
Distances
Real Limits
Space Distance
Outcome Variable Events
Absolute Value
Plane Distance
Event Probabilities
Interval Length
Probability Distributions
Real Space
Intervals
Controlled Dynamical Systems
Real Plane
Real Line
Directed Graphs
State Representations
Dynamical Systems
Real Sequences
Real Order
Integral Line
Real Summation
Sequential Decisions
Real Numbers
Rational Order
Geometry
Subset Algebras
Undirected Graphs
Sequences
Rational Numbers
Cardinality
Lists
Decisions
Integer Arithmetic
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Uncertain Outcomes
Finite Sets
Identity Elements
Arithmetic
Integer Products
Equivalent Sets
Direct Products
Natural Powers
Natural Summation
Function Inverses
Family Unions and Intersections
Comparisons
Family Operations
Integer Order
Natural Products
Integer Sums
Function Images
Function Composites
Families
Orders
Operations
Integer Numbers
Natural Sums
Functions
Equivalence Relations
Natural Order
Converse Relations
Recursion Theorem
Relations
Peano Axioms
Subset Systems
Set Products
Natural Induction
Ordered Pairs
Generalized Set Dualities
Set Powers
Natural Numbers
Partitions
Set Symmetric Differences
Set Dualities
Intersection of Empty Set
Unordered Triples
Set Unions and Intersections
Successor Sets
Set Complements
Set Intersections
Pair Unions
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects