Why

A simple way to obtain a sigma algebra, is to ask it to obtain some sets, and then to ask it to contain all the sets it needs to fulfill the properties.

Definition

The *generated sigma algebra* for a set of subsets is the smallest sigma algbera containing the set of subsets. We must prove the existence and uniqueness of this sigma algebra.

Proposition **1.** *The intersection of a non-empty set of sigma algebras on the same base set is a sigma algebra.*

Proof. Let $\{(A, \mathcal{A}_\alpha\}_{\alpha \in I}$ a familiy of sigma algebras on the same base set. Define $\mathcal{A}$ as $\cap_{\alpha \in I} \mathcal{A}_\alpha$.

1. For all $\alpha \in I$, $A \in \mathcal{A}_\alpha$, thus $A \in \mathcal{A}$; condition (a).
2. For all $B \in \mathcal{A}$, for all $\alpha \in I$, $B \in \mathcal{A}_\alpha$. Thus, for all $\alpha \in I$, $C_A(B) \in \mathcal{A}_\alpha$. And so $C_A(B) \in \mathcal{A}$; condition (b).
3. For all sequences $\{B_n\} \subset \mathcal{A}$, $\{B_n\} \subset \mathcal{A}_\alpha$ for all α. Thus $\cup_n B_n \in \mathcal{A}_\alpha$ for all α and so $\cup_n B_n \in \mathcal{A}$; condition (c).

$\square$

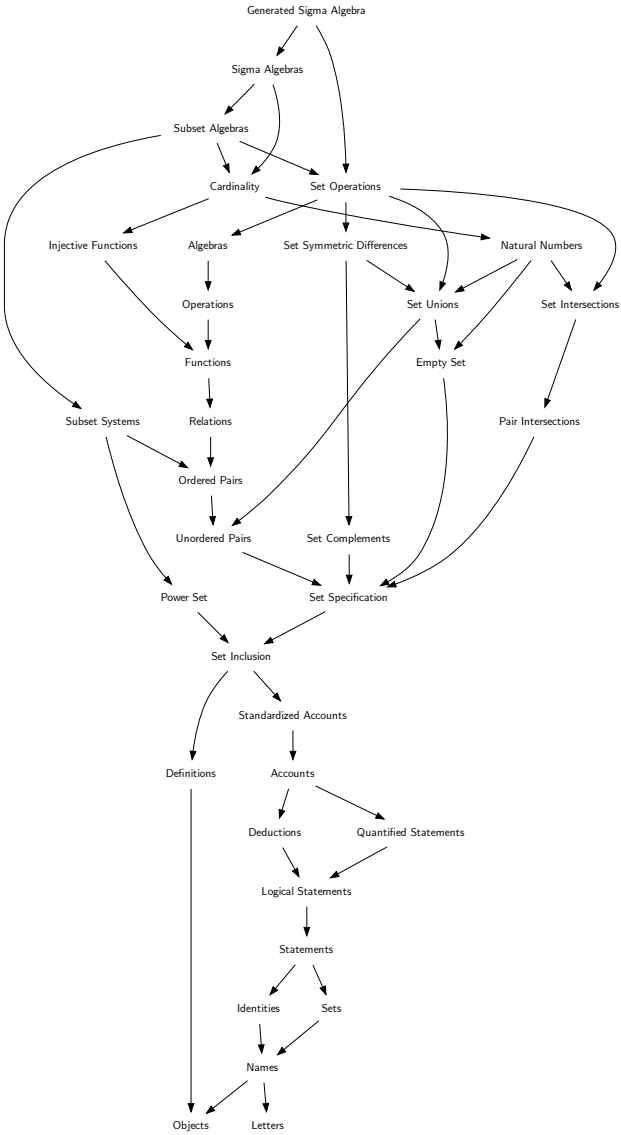
On the other hand, the union of a set of sigma algebras can fail to be a sigma algebra.

Proposition **2.** *If A is a set and $\mathcal{A} \subset 2^A$, then there is a unique a smallest sigma algebra containing $\mathcal{A}$.*

Proof. We know of one sigma algebra containing $\mathcal{A}$: the power set of A. Thus, the set of sigma algebras containing $\mathcal{A}$ is not empty. Proposition 1 implies the intersection of all such sigma algebras (containing $\mathcal{A}$) is a sigma algebra. The intersection contains $\mathcal{A}$, and is contained in all other sigma algebras with this property, so is a smallest sigma algebra containing $\mathcal{A}$. If $\mathcal{B}, \mathcal{C}$ were two smallest sigma algebras, then $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{C} \subset \mathcal{B}$, but then $\mathcal{B} = \mathcal{C}$; thus the smallest sigma algebra is unique. $\square$

Notation

Let A be a set and $\mathcal{A} \subset 2^A$. We denote the sigma algebra generated by $\mathcal{A}$ by $\sigma(\mathcal{A})$.


Generated Sigma Algebra
Sigma Algebras
Subset Algebras
Cardinality
Set Operations
Injective Functions
Algebras
Set Symmetric Differences
Natural Numbers
Operations
Set Unions
Set Intersections
Functions
Empty Set
Subset Systems
Relations
Pair Intersections
Ordered Pairs
Unordered Pairs
Set Complements
Power Set
Set Specification
Set Inclusion
Standardized Accounts
Definitions
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Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Objects
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