



Why

We can characterize the dependence of two events in terms of the rank of a particular matrix.

Definition

Given a probability measure $\mathbf{P} : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$ on the finite set Ω and two events $A, B \subset \Omega$, the *joint probability matrix* of A and B is the matrix

$$M = \begin{bmatrix} \mathbf{P}(A \cap B) & \mathbf{P}(A \cap C_{\Omega}(B)) \\ \mathbf{P}(C_{\Omega}(A) \cap B) & \mathbf{P}(C_{\Omega}(A) \cap C_{\Omega}(B)) \end{bmatrix}.$$

Characterization of independence

If A and B are independent, then so are A and $C_{\Omega}(B)$, B and $C_{\Omega}(A)$, and $C_{\Omega}(A)$ and $C_{\Omega}(B)$. In other words,

$$M = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_{\Omega}(A)) \end{bmatrix} \begin{bmatrix} \mathbf{P}(B) & \mathbf{P}(C_{\Omega}(B)) \end{bmatrix}.$$

In this case, we see that $\text{rank}(M) = 1$.

Conversely, suppose $\text{rank}(M) = 1$. Then, using the law of total probability, each row is a multiple of

$$M1 = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_{\Omega}(A)) \end{bmatrix}.$$

In particular, we have $\mathbf{P}(A \cap B) = \alpha \mathbf{P}(A)$ and $\mathbf{P}(C_{\Omega}(A) \cap B) = \alpha \mathbf{P}(C_{\Omega}(A))$. So

$$\mathbf{P}(A \cap B) + \mathbf{P}(C_{\Omega}(A) \cap B) = \alpha(\mathbf{P}(A) + \mathbf{P}(C_{\Omega}(A))),$$

from which we deduce $\alpha = \mathbf{P}(B)$. Likewise, the multiplier for the second column of M is $\mathbf{P}(C_{\Omega}(B))$. In other words, A and B are independent. We conclude that A and B are independent if and only if $\text{rank}(M) = 1$.

