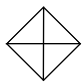


Why

Many functions of interest are additive and homogenous.

Definition

A transformation is *linear* (a *linear transformation*, *linear map*) if the result of a linear combination of the two vectors is the linear combination of the results of the vectors (using the same coefficients). The transformation is linear *with respect to* the field of the two vector spaces.

We use the term transformation (see Transformations) for emphasis and reminder that the function is defined on a vector space. Of course, R is a vector space and so a function $f : \mathsf{R} \to \mathsf{R}$ may be linear. The linear maps from R to R are the the *linear functions* (see Real Linear Functions)
.

Often authors will use the word *operator* for linear functions. It seems, generally, that this term is commonly reserved for the case in which the vector space discussed is a function space (or, at least, infinite dimensional).

Notation

Let (V, F) and (W, F) be two vector spaces over the same field. Suppose $T : V \to W$. T is linear means

$$T(\alpha u + \beta v) = \beta T(u) + \alpha T(v) \quad \text{for all } \alpha, \beta \in F \text{ and } u, v \in V.$$

As usual, the condition that T is linear condition is equivalent to the two conditions:

1. $T(u + v) = f(u) + T(v)$ for all $u, v \in V$, and
2. $f(\lambda u) = \lambda f(u)$ for all $\lambda \in \mathsf{F}$ and $u \in V$.

If T satisfies (1), we call T *additive* (has the property of *additivity*) and if it satisfies (2) we call T T *homogeneous* (has the property of *homogeneity*).

For linear maps, it is common to denote $T(v)$ by Tv; notice that we have dropped the usual parentheses.

We denote the set of all linear maps by $\mathcal{L}(V, W)$. It is understood when using this notation that V and W are vector spaces with respect to the same field $\mathbf{F}$.

Examples

Throughout, we consider vector spaces V and W over some fixed field $\mathbf{F}$.

Constant zero map. The map $T \in \mathcal{L}(V, W)$ defined by

$$T(v) = 0 \in W \quad \text{for all } v \in V$$

is called the *zero map* (or *zero transformation*). It is common to overload the symbol 0 so that $0 \in \mathcal{L}(V, W)$ denotes the map zero map. In other words, the map 0 is defined by

$$0v = 0$$

Some care is required to interpret this equation. The 0 on the left hand side refers to a function, from V to W. The 0 on the right hand side is the additive identity in W. Usually context disambiguates the overloaded notation.

The identity map. The map $T \in \mathcal{L}(V, V)$ defined by

$$Tv = v \quad \text{for all } v \in V$$

is called the *identity map* (or *identity transformation*) It is common to denote this map by I.

Differentiation of polynomials Suppose P is the set of all polynomials with coefficients in $\mathbf{R}$. (Some authors denote this set by $\mathcal{P}(\mathbf{R})$. Recall that every $p \in \mathcal{P}(\mathbf{R})$ is differentiable and $p' \in \mathcal{P}(\mathbf{R})$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ defined by

$$Tp = p'$$

is linear. To see this, recall $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda f'$ whenever f, g are differentiable and $\lambda \in \mathbf{R}$ (see Derivative of Sumsand Derivatives of Scalar Multiples) .

Integration of polynomials As in the previous paragraph, $\mathcal{P}(\mathbf{R})$ denotes the vector space of polynomials with coefficients in $\mathbf{R}$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$ defined by

$$Tp = \int_{[0,1]} p$$

is linear To see this, recall that $\int(f+g) = \int f + \int g$ and $\int \lambda f = \lambda \int f$ whenever f, g are differentiable and $\lambda \in \mathbf{R}$ (see Real Integral Additivityand Real Integral Homogeneity.

Multipliciation by a quadratic. As in the previous paragraph, $\mathcal{P}(\mathbf{R})$ denotes the vector space of polynomials with coefficients in $\mathbf{R}$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ defined by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbf{R}, p \in \mathcal{P}(\mathbf{R})$$

is linear. (Prove this).

Sequence backward shift. Denote the space of infinite sequences in a field $\mathbf{F}$ by $\mathbf{F}^{\mathbf{N}}$ as usual. Define $T \in \mathcal{L}(\mathbf{F}^{\mathbf{N}}, \mathbf{F}^{\mathbf{N}})$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

T is called the *backward shift operator.*

From real space the the real plane. Define $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

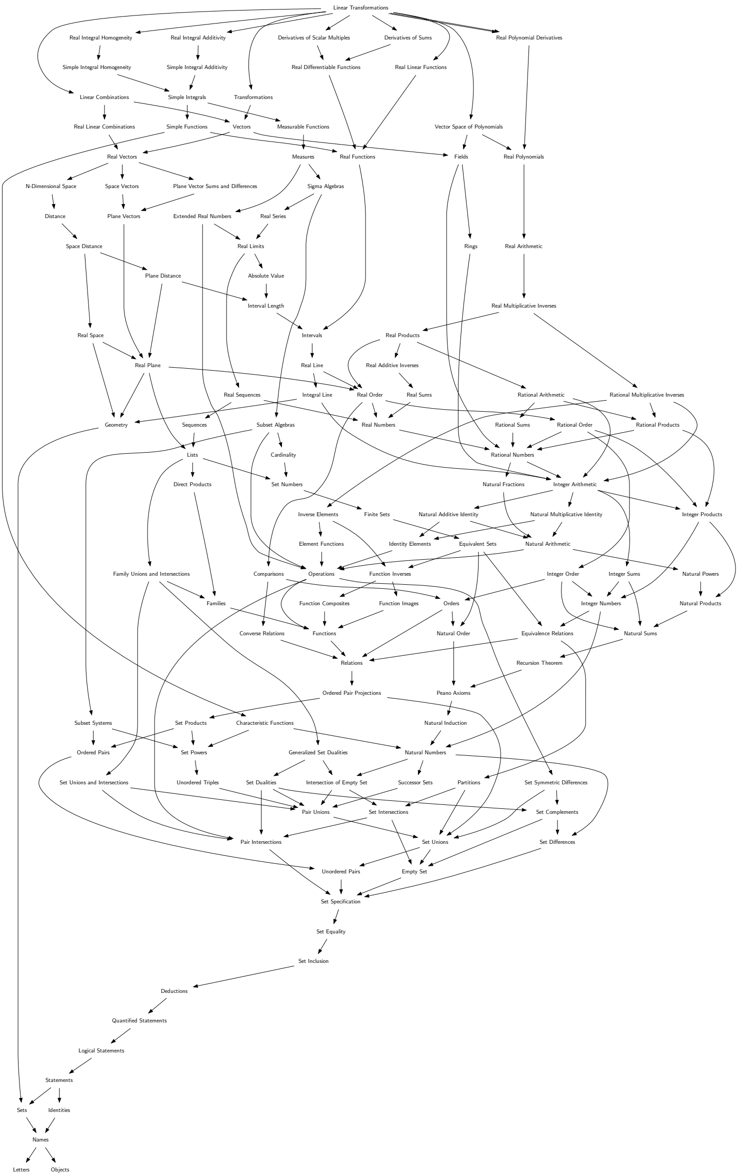
From $\mathbf{F}^n$ to $\mathbf{F}^m$. Generalizing the previous example, suppose m and n are natuarl numbers, and let $A_{i,j} \in \mathbf{F}$ for $i = 1, \dots, m$ and $j = 1, \dots, m$. Define $T \in \mathcal{L}(\mathbf{F}^3, \mathbf{F}^2)$ by

$$T(x_1, \cdots, x_n) = (A_{1,1}x + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

(It happens that every linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$ has this form.)

A counterexample: cos[1] Notice $\cos(x+y) = \cos(x) + \cos(y)$. True, cos is not homogenous. that $\cos 2x = 2\cos(x)$ and But this does not hold for all reals: $\cos \lambda x \neq \lambda \cos(x)$.

[1]Need to add a sheet for trigonometric functions.



Linear Transformations
Real Integral Homogeneity
Real Integral Additivity
Derivatives of Scalar Multiples
Derivatives of Sums
Real Polynomial Derivatives
Simple Integral Homogeneity
Simple Integral Additivity
Real Differentiable Functions
Real Linear Functions
Linear Combinations
Simple Integrals
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Space Vectors
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Extended Real Numbers
Real Series
Space Distance
Real Limits
Rings
Real Arithmetic
Plane Distance
Absolute Value
Interval Length
Real Multiplicative Inverses
Real Space
Intervals
Real Products
Real Plane
Real Line
Real Additive Inverses
Real Sequences
Integral Line
Real Order
Real Sums
Rational Arithmetic
Rational Multiplicative Inverses
Geometry
Sequences
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Real Numbers
Rational Sums
Rational Order
Rational Products
Lists
Cardinality
Rational Numbers
Direct Products
Set Numbers
Natural Fractions
Integer Arithmetic
Inverse Elements
Finite Sets
Natural Additive Identity
Natural Multiplicative Identity
Integer Products
Element Functions
Identity Elements
Equivalent Sets
Natural Arithmetic
Family Unions and Intersections
Comparisons
Operations
Function Inverses
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Integer Sums
Natural Powers
Families
Function Composites
Function Images
Orders
Integer Numbers
Natural Products
Converse Relations
Functions
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Equivalence Relations
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Relations
Recursion Theorem
Ordered Pair Projections
Peano Axioms
Subset Systems
Set Products
Characteristic Functions
Natural Induction
Ordered Pairs
Set Powers
Generalized Set Dualities
Natural Numbers
Set Unions and Intersections
Unordered Triples
Set Dualities
Intersection of Empty Set
Successor Sets
Partitions
Set Symmetric Differences
Pair Unions
Set Intersections
Set Complements
Pair Intersections
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Names
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Objects