Why

Can we characterize positive (semi-)definite matrices in terms of their eigenvalues?

Main Result

Using eigenvalue decompositions, we can answer in the affirmative.

Proposition 1. *Suppose $A \in \mathsf{S}^d$ has smallest eigenvalue $\lambda_{\min}(A)$. Then*

$$A \in \mathsf{S}^d_+ \quad \& \longleftrightarrow \quad \lambda_{\min}(A) \geq 0$$
$$\& \longleftrightarrow \quad \operatorname{tr} AB \geq 0 \text{ for all } B \in \mathsf{S}^d_+.$$

and

$$A \in \mathsf{S}^d_{++} \quad \& \longleftrightarrow \quad \lambda_{\min}(A) > 0$$
$$\& \longleftrightarrow \quad \operatorname{tr} AB > 0 \text{ for all nonzero } B \in \mathsf{S}^d_{++}.$$

Eigenvalues and Definiteness

Positive Definite Matrices

Symmetric Real Matrices — Eigenvalue Decomposition — Quadratic Forms — Identity Matrices

Orthogonal Matrices — Matrix Trace

Orthonormal Set of Vectors — Matrix Transpose — Real Matrix-Matrix Products

Inner Product Norms — Real Matrix-Vector Products — Eigenvalues and Eigenvectors — Matrices

Orthonormal Set of Real Vectors — Inner Products — Norms — Vector as Matrices — Matrix Determinants

Real Inner Product — Functionals — Real Norm — Vectors — Real Linear Combinations — Real Matrices

Multivariate Polynomials — Space Norm — Multivariate Functions — Real Vectors — Linear Equations

Plane Inner Product — Plane Norm — N-Dimensional Space — Plane Vector Sums and Differences — Linear Functions — Space Vectors

Real Square Roots — Right Triangle Sides Relation — Absolute Value — Distance — Plane Vectors

Real Squares — Real Polynomials — Space Distance — Analyst

Real Arithmetic — Plane Distance — Monic Polynomials

Natural Square Roots — Real Multiplicative Inverse — Interval Length

Intervals — Real Products — Real Space

Square Numbers — Integral Line — Real Line — Real Additive Inverse — Real Plane

Squares — Geometry — Rational Multiplicative Inverse — Rational Arithmetic — Real Sums — Real Order — Real Summation — Permutations — Polynomials

Rational Products — Rational Order — Rational Sums — Real Numbers — Fields — Groups — Lists

Integer Arithmetic — Rational Numbers — Rings — Set Numbers — Direct Products

Integer Products — Natural Additive Identity — Natural Multiplicative Identity — Finite Sets

Natural Summation — Equivalent Sets — Inverse Elements

Integer Sums — Integer Order — Natural Equation — Arithmetic — Identity Elements — Comparisons — Element Functions — Family Operations — Function Inverses — Family Unions and Intersections

Integer Numbers — Natural Powers — Orders — Equations — Operations — Function Composites — Function Images — Families

Natural Products — Natural Order — Converse Relations — Equivalence Relations — Functions

Natural Sums — Relations

Recursion Theorem

Peano Axioms — Set Products

Natural Induction — Set Powers — Generalized Set Dualities

Natural Numbers — Set Symmetric Differences — Successor Sets — Unordered Triples — Set Dualities — Intersection of Empty Set — Partitions — Set Union and Intersections

Set Complements — Pair Unions — Ordered Pairs — Set Intersections

Set Differences — Set Union — Unordered Pairs — Pair Intersections

Empty Set

Set Specification

Set Equality

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects