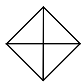


Why

Since one and only one outcome occurs, given a distribution on outcomes, we define the probability of a set of outcomes as the sum of their probabilities.

Definition

Suppose p is a distribution on a *finite* set of outcomes Ω. Given an event $E \subset \Omega$, the *event probability* of E *under* p as the sum of the probabilities of the outcomes in E.

Notation

It is common to define a function $P : \mathcal{P}(\Omega) \to \mathbf{R}$ by

$$P(A) = \sum_{a \in A} p(a) \quad \text{for all } A \subset \Omega$$

We call this function P *the event probability function* (or *the probability measure*) associated with p. Since it depends on the sample space Ω and the distribution p, we occasionally denote this dependence by $P_{\Omega,p}$ or P_p.

Example: a single six-sided die

We model rolling a single die with the set of outcomes $\Omega = \{1, \dots, 6\}$ as usual. We $p : \Omega \to \mathbf{R}$ by $p(\omega) = 1/6$ for $\omega = 1, \dots, 6$. In other words, p is the constant function at value $1/6$ on Ω. Now, we model the event that the number of pips showing is an even number by the set E defined by $E = \{2, 4, 6\}$. Given all this modeling, the probability of the event E is

$$\textstyle\sum_{\omega \in E} p(\omega) = p(2) + p(4) + p(6) = 1/2.$$

Properties of event probabilities

As a result of the conditions on p, $\mathbf{P}$ satisfies

1. $\mathbf{P}(A) \geq 0$ for all $A \subset \Omega$;

2. $\mathbf{P}(\Omega) = 1$ (and $\mathbf{P}(\varnothing) = 0$);

3. $\mathbf{P}(A) + \mathbf{P}(B)$ for all $A, B \subset \Omega$ and $A \cap B = \varnothing$. This statement follows from the more general identity

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

for $A, B \subset \Omega$, by using $\mathbf{P}(\varnothing) = 0$ of (2) above.

These three conditions are sometimes called the *axioms of probability for finite sets.* Do all such $\mathbf{P}$ satisfying (1)-(3) have a corresponding underlying probability distribution?

In other words, suppose $f : \mathcal{P}(\Omega) \to \mathbf{R}$ satisfies (1)-(3). Define $q : \Omega \to \mathbf{R}$ by $q(\omega) = f(\{\omega\})$. If f satisfies the axioms, then q is a probability distribution. For this reason we call any function satisfying (i)-(iii) an *event probability function* (or a *(finite) probability measure*).

Other basic consequences

Probability by cases

Let $\mathbf{P}$ be a probability event function. Suppose $A_1, \ldots, A_n$ partition Ω. Then for any $B \subset \Omega$,

$$\mathbf{P}(B) = \textstyle\sum_{i=1}^{n} \mathbf{P}(A_i \cap B).$$

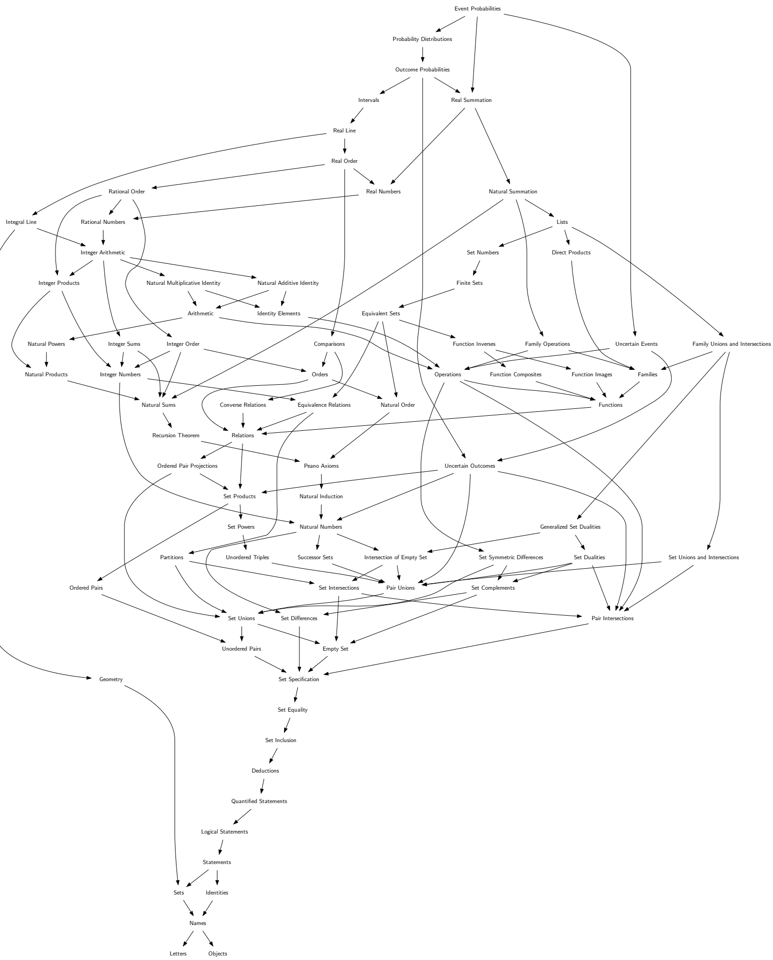
Some authors call this the *law of total probability.*

Monotonicity

If $A \subseteq B$, then $\mathbf{P}(A) \leq P(B)$. This is easy to see by splitting B into $A \cap B$ and $B - A$, and applying (1) and (3).

Subadditivity

For $A, B \subset \Omega$, $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$. This is easy to see from the more general identity in (3) above. This is sometimes referred to as a *union bound*, in reference to *bounding* the quantity $\mathbf{P}(A \cup B)$.



Event Probabilities
Probability Distributions
Outcome Probabilities
Intervals
Real Summation
Real Line
Real Order
Rational Order
Real Numbers
Natural Summation
Integral Line
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Natural Powers
Integer Sums
Integer Order
Comparisons
Function Inverses
Family Operations
Uncertain Events
Family Unions and Intersections
Natural Products
Integer Numbers
Orders
Operations
Function Composites
Function Images
Families
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Equivalence Relations
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Functions
Recursion Theorem
Relations
Ordered Pair Projections
Peano Axioms
Uncertain Outcomes
Set Products
Natural Induction
Set Powers
Natural Numbers
Generalized Set Dualities
Partitions
Unordered Triples
Successor Sets
Intersection of Empty Set
Set Symmetric Differences
Set Dualities
Set Unions and Intersections
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Geometry
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