Supremum Norm Complete

1 Why

We want a complete norm on the vector space of continuous functions.

2 Result

Proposition 1. *The supremum norm is complete.*

Proof. Let R denote the real numbers. Let $\{f_n\}_n$ be an egoprox sequence in $C[a, b]$. Then $\forall \epsilon > 0, \exists N$ so that

$$m, n > N \implies |f_n - f_m|_{\sup} < \epsilon.$$

Since $|f_n - f_m|_{\sup} < \epsilon \implies |f_n(x) - f_m(x)| < \epsilon$ for all $x \in [a, b]$, the sequence of real numbers $\{f_n(x)\}_n$ is egoprox for each $x \in [a, b]$. Since the metric space $(R, |\cdot|)$ is complete, there is a limit l_x such that $f_n(x) \longrightarrow l_x$ as $n \longrightarrow \infty$, for each $x \in [a, b]$. Define $f : [a, b] \to R$ by $f(x) = l_x$ for each $x \in [a, b]$.

First, we argue that f is continous. Let $x_0 \in [a, b]$ and let $\epsilon > 0$. For each n, f_n is a continuous function on a closed interval, and therefore is uniformly continuous: $\forall \epsilon > 0, \exists \delta > 0$

so that $\forall x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

For $\epsilon/3 > 0$, there exists an n_1 so that

$$n > n_1 \implies |f_n(x_0) - f(x_0)| < \epsilon/3.$$

For $\epsilon/3 > 0$, there exists an n_2 so that

$$n > n_2 \implies |f_n(x_0) - f(x_0)| < \epsilon/3.$$

Let $n_0 = \max\{n_1, n_2\}$. The function f_{n_0} is continuous, so for $\epsilon/3$, there is a $\delta > 0$ so that for all $x \in [a, b]$,

$$|x_0 - x| < \delta \implies |f_{n_0}(x_0) - f_{n_0}(x_0)| < \epsilon/3.$$

By the triangle inequality,

$$|f(x_0) - f(x)| \leq |f(x_0) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x)|$$

Since $n_0 \geq n_1$, $|f(x_0) - f_{n_0}(x_0)| < \epsilon/3$. Using this fact, and the triangle inequality

$$\begin{aligned} |f(x_0) - f(x)| &< \epsilon/3 + |f_{n_0}(x_0) - f(x)| \\ &\leq \epsilon/3 + |f_{n_0}(x_0) - f_{n_0}(x)| + |f_{n_0}(x_0) - f(x)| \end{aligned}$$

Since $|x_0 - x| < \delta$, $|f_{n_0}(x_0) - f_{n_0}(x)| < \epsilon/3$. Since $n_0 > n_2$, $|f_{n_0}(x_0) - f(x)| < \epsilon/3$. We conclude

$$|f(x_0) - f(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

for all $|x_0 - x| < \delta$. Since ϵ was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is continuos everywhere. $\square$