



## Why

Can we represent the function associating a linear system with its row reduction by matrix multiplication?

## Main observation

The following proposition affirmatively answers the question.

**Proposition 1.** *Let  $(A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$  be a linear system with  $A_{kk} \neq 0$  and  $(C, d)$  the  $k$ th reduction of  $(A, b)$ . Then there exists a matrix  $L \in \mathbf{R}^{m \times m}$  so that  $C = LA$  and  $d = Lb$ .*

*Proof.* Define  $L \in \mathbf{R}^{m \times m}$  by  $L_{st} = 1$  if  $s = t$ ,  $-A_{sj}/A_{ik}$  if  $k < s \leq m$  and zero otherwise.  $\square$

For this reason, we call  $L$  in the above proposition a *row reducer matrix* or *row reducing matrix* or *row reducer*. The row reducer matrix for the  $k$ th reduction of  $(A, b)$  has the form

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & A_{ik}/A_{kk} & 1 & & \\ & & \vdots & & \ddots & \\ & & A_{mk}/A_{kk} & & & 1 \end{bmatrix}$$

So the following is immediate

**Proposition 2.** *Row reducing matrices are unit lower triangular.*

### Example

For example, the  $(1,1)$ -reduction of  $(A, b)$  in which

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 6 & 7 & 9 & 5 \\ 8 & 7 & 9 & 8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

is the linear system

$$A' = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and} \quad b' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The row reducer is  $L \in R^{4 \times 4}$  defined by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that  $A' = LA$  and  $b' = Lb$ , and clearly  $L$  is unit lower triangular.

