



## Why

We want to solve linear equations. Our approach is to “eliminate” variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

## Two-variable example

Suppose we want to find  $x_1, x_2 \in \mathbf{R}$  to satisfy

$$3x_1 + 2x_2 = 10, \text{ and}$$

$$6x_1 + 5x_2 = 20.$$

We can list the coefficients in a two-dimensional array  $A = (3, 2; 6, 5)$  and  $b = (10, 20)$ . We can eliminate  $x_1$  from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$3x_1 + 2x_2 = 10 \text{ and}$$

$$x_2 = 0.$$

The key insight is that this system has the *same solution set*. We call the process of moving between these two systems a *row reduction*.

## Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate  $x_1$  (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear

system  $S_1 = (A^1, b^1)$  in which

$$A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \quad \text{and} \quad b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The trick is that, since  $A'_{22} \neq 0$ , we can take the same route to eliminate  $x_2$ , to obtain the system  $S_2 = (A^2, b^2)$  in which

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \text{and} \quad b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Likewise for  $x_3$ , we obtain  $S_3 = (A^3, b^3)$  in which

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one,  $(A^3, b^3)$ , is easy to solve. We solve it by *back substitution*. First, since  $2x_4 = 3$ , we find  $x_4 = 3/2$ . Second, since  $2x_3 + 2x_4 = -1$ , we find  $x_3 = -2$ . Similarly we find  $x_2 = 1/2$  and  $x_1 = 5/4$ .

### Definition

Let  $S = (A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$  be a linear system. The lower *row reduction* of  $S$  for index  $i$  with  $A_{ii} \neq 0$  (or the  $i$ -row reduction) is the linear system  $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$  if  $i < s \leq m$  and  $A_{st}$  otherwise. We say that the system  $(A, b)$  is *ordinarily reducible*.

Let  $a^k, \tilde{a}^k \in \mathbf{R}^n$  denote the  $k$ th row of  $A$  and  $\tilde{A}$ , respectively. Then if  $k \neq i$ ,  $\tilde{a}^k = a^k - \alpha_k a^i$  where  $\alpha_k = A_{kj}/A_{ij}$ . In other words, a row  $k$

of the matrix  $\tilde{A}$  is obtained by subtracting a multiple of the  $i$ th row of matrix  $A$  from row  $k$  of matrix  $A$ . We are “reducing” the rows of  $A$ .

**Proposition 1.** *Let  $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$  be a linear system which row reduces to  $(C, d)$ . Then  $x \in \mathbf{R}^n$  is a solution of  $(A, b)$  if and only if it is a solution of  $(C, d)$ .<sup>1</sup>*

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = \left( \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

We then subtract three times row 2 from row 3 and four times row 2 from row 4 to obtain

$$S_2 = \left( \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

Finally, we subtract two times row 3 from row 4 to obtain  $S_4$ , which we can write as

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ x_2 + x_3 + x_4 &= 0, \\ 2x_3 + 2x_4 &= -1, \quad \text{and} \\ 2x_4 &= 3. \end{aligned}$$

We can now back-substitute to find  $x_4 = 3/2$ ,  $x_3 = -2$ ,  $x_2 = 1/2$  and  $x_1 = 5/4$ . Proposition ?? says that this is the only solution of  $S$ , as well.

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<sup>1</sup>Future editions will include an account.

