Why

We speak of objects which we can add and scale. Of course, our model is the real vectors.

Definition

A *vector space* (or *linear space*, *linear class*) is an ordered triple consisting of (1) a commutative group, (2) a field, and (3) a function from the product of the field ground set and group ground set to the group ground set.

We call the ground set of the commutative group the *vectors* and we call the group operation *vector addition*. We call the field the *scalars*. The field has addition and multiplication of scalars which we call *scalar addition* and *scalar multiplication*, respectively. The *scaling map* is the map on pairs of scalars and vectors. We call these two functions the *vector operations*.

The first property is that the result of scaling a vector by a scalar, and then scaling the result by a second scalar is the same as first multiplying the scalars and then scaling the vector. The second property is that the result of scaling a vector by the scalar sum of two scalars is the same as scaling the vector by each scalar separately and then adding the vectors. The third property is that the result of scaling the vector sum of two vectors is the same as scaling each individually and then adding the two vectors.

Notation

Let $(V, +)$ be a commutative group and $(F, \tilde{+}, \cdot)$ be a field. Let $s : F \times V \to V$. $((V, +), (F, \tilde{+}, \cdot), s)$ is a vector space if

1. $s(b, s(a, v)) = s(b \cdot a, v)$ for all $a, b \in F$ and $v \in V$.

2. $s(a \tilde{+} b, v) = s(a, v) + s(b, v)$ for all $a, b \in F$ and $v \in V$.

3. $s(a, u + v) = s(a, u) + s(a, b)$ for all $a \in F$ and $u, v \in V$.

We can simplify (contract) the notation above with four conventions. First, we denote both vector and scalar addition by $+$. So $a + b$ denotes $a \,\tilde{+}\, b$ for all $a, b \in F$. Second, we contract $\cdot$. So ab denotes $a \cdot b$ for all $a, b \in F$. Third, we contract the scaling map. So av denotes $s(a, v)$ for $a \in F$ and $v \in V$. Fourth, we conventionally take the scaling map to have precedence so that $au + av$ means $(au) + (av)$ for all $a \in F$ and $u, v \in V$.

We denote: (1) $b(av) = (ba)v$; (2) $(a+b)v = av+bv$; and (3) $a(u+v) = au + av$ for all $a, b \in F$ and $u, v \in V$.

Finally, we say: let (V, F) be a vector space, assuming the above notational conventions. Sometimes we will say let V be a vector space over the field F. Often the field will be assumed, in which case we will refer to the vector space V and define functions on V.

Vectors

Real Vectors

Space Vectors Plane Vector Sums and Differences N-Dimensional Space

Plane Vectors

Distance

Space Distance

Plane Distance

Interval Length

Intervals Real Space

Real Line Real Plane

Real Order Integral Line

Rational Order Fields Real Numbers Geometry

Rings Rational Numbers

Integer Arithmetic Set Numbers Lists Direct Products

Integer Products Natural Multiplicative Identity Natural Additive Identity Finite Sets

Arithmetic Identity Elements Equivalent Sets

Natural Powers Integer Sums Integer Order Comparisons Function Inverses Family Unions and Intersections

Natural Products Integer Numbers Orders Operations Function Composite Function Images Families

Natural Sums Equivalence Relations Natural Order Converse Relations Functions

Recursion Theorem Relations

Peano Axioms Ordered Pair Projections

Natural Induction Set Products

Natural Numbers Set Powers Generalized Set Dualities

Set Symmetric Differences Partitions Intersection of Empty Set Successor Sets Unordered Triples Set Dualities Set Unions and Intersections

Set Complements Set Intersections Pair Unions Ordered Pairs

Set Differences Set Unions Pair Intersections

Empty Set Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities Sets

Names

Letters Objects

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