



## Why

We have seen that the matrices are a vector space. Are they an inner product space?

## Definition

The *matrix scalar product* of  $A \in \mathbf{R}^{n \times k}$  and  $B \in \mathbf{R}^{n \times k}$  is the following product

$$\sum_{i=1}^n \sum_{j=1}^k a_{ij} b_{ij}.$$

Using the matrix trace, we can denote this as  $\text{tr } A^\top B$ . Some authors call this the *Euclidean matrix scalar product*, *matrix inner product* or *Frobenius inner product*.

**Proposition 1.** *The matrix scalar product is an inner product.*<sup>1</sup>

For example, symmetry of the product is a consequence of the fact that a square matrix and its transpose have identical traces. Commutativity of the trace yields  $\text{tr } A^\top B = \text{tr } B A^\top$ , where the LHS is the scalar product of  $B^\top$  and  $A^\top$ . In other words, transposition “preserves” the matrix scalar product.

With this inner product,  $\mathbf{R}^{n \times k}$  is a Euclidean vector space (see Inner Products) of dimension  $nk$ . For the case of  $k = 1$ , we recover a model<sup>2</sup> for the usual space  $\mathbf{R}^n$ .

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<sup>1</sup>Future editions will provide an account.

<sup>2</sup>Future editions will define this term.

## Notation

We commonly denote the matrix inner product by  $\langle A, B \rangle$ .

## Induced norm

The matrix inner product induces a norm in the usual way. This norm is sometimes called the *Frobenius norm* and is often denoted for a matrix  $A \in \mathbf{R}^{m \times n}$  by  $\|A\|_F$ .

