Real Integral Monotone Convergence

1 Why

An integral is a limit. When can we exchange this limit with another? We give a first result in the search for sufficient conditions to do so.

2 Result

When context is clear, we refer to the following proposition as the *monotone convergence theorem*.

Proposition 1. *The integral of the almost everywhere limit of an almost-everywhere nondecreasing sequence of measurable, nonnegative, extended-real-valued functions is the limit of the sequence of integrals of the functions.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_n :\to [-\infty, \infty]$ a $\mathcal{A}$-measurable function for every natural number n and let $f : X \to [-\infty, \infty]$ a $\mathcal{A}$-measurable function. We want to show that if

$$f_n(x) \leq f_{n+1}(x) \quad \text{and} \quad f(x) = \lim_n f_n(x)$$

hold for all natural n and almost every x in X, then

$$\int f d\mu = \lim_n \int f_n d\mu.$$

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