



Definition

The *covariance* between two random variables which are each integrable and whose product is integrable is the expectation of their product less the product of their expectation.

Notation

Let f and g be two integrable random variables with fg integrable. Denote the covariance of f with g by $\mathbf{cov}(f, g)$. We defined it:

$$\mathbf{cov}(f, g) = \mathbf{E}(fg) - \mathbf{E}(f)\mathbf{E}(g).$$

Properties

Proposition 1. *Covariance is symmetric and bilinear.*

Proposition 2. *The covariance of a random variable with itself is its variance.*

Proof. Let f be a square-integrable real-valued random variable, then

$$\mathbf{cov}(f, f) = \mathbf{E}(ff) - \mathbf{E}(f)\mathbf{E}(f) = \mathbf{E}(f^2) - (\mathbf{E}(f))^2 = \mathbf{var}(f).$$

□

Proposition 3. *The variance of a sum of integrable real-valued random variables whose pairwise products are integrable is the double sum of the pairwise covariances.*

Proof. Let f_1, \dots, f_n be integrable random variables with $f_i f_j$ integrable for all $i, j = 1, \dots, n$. Using the bilinearity,

$$\begin{aligned} \mathbf{var}\left(\sum_{i=1}^n f_i\right) &= \mathbf{cov}\left(\sum_{i=1}^n f_i, \sum_{i=1}^n f_i\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \mathbf{cov}(f_i, f_j) \end{aligned}$$

□

