



## Why

We want to discuss real-valued random functions whose family of random variables have simple densities.<sup>1</sup>

## Definition

A *normal random function* is a real-valued random function whose family of real-valued random variables has the property that any subfamily is jointly normal.

For this reason, we call the family of random variables (or stochastic process) corresponding to the random function a *gaussian process* or *normal process*.

## Notation

Let  $(\Omega, \mathcal{A}, \mathbf{P})$  be a probability space and  $A$  a set. Let  $x : \Omega \rightarrow (A \rightarrow \mathbf{R})$  be a random function with family  $y : A \rightarrow (\Omega \rightarrow \mathbf{R})$ .

The random function  $x$  is a normal if, for all  $a^1, \dots, a^m \in A$ ,  $(y(a^1), \dots, y(a^m))$  is jointly normal.

## Mean and covariance function

**Proposition 1.** *Let  $x : \Omega \rightarrow (A \rightarrow \mathbf{R})$  be a normal random function with family  $X : A \rightarrow (\Omega \rightarrow \mathbf{R})$ . There exists unique functions  $m : A \rightarrow \mathbf{R}$  and  $k : A \times A \rightarrow \mathbf{R}$  so that the mean of the random variable  $X_a$  is  $m(a)$  for all  $A$  and the covariance of the random variables  $X_a$  and  $X_{a'}$  is  $k(a, a')$  for all  $a, a' \in A$ .<sup>2</sup>*

For this reason, we call  $m$  the *mean function* and  $k$  the *covariance function* of the random function.

Conversely, let  $m : A \rightarrow \mathbf{R}$  and  $k : A \times A \rightarrow \mathbf{R}$ . Then if  $k$  satisfies the

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<sup>1</sup>Future editions will expand.

<sup>2</sup>Future editions may include an account.

property that for all  $a^1, \dots, a^m$ , the  $m \times m$  matrix

$$\begin{pmatrix} k(a^1, a^1) & \cdots & k(a^1, a^m) \\ \vdots & \ddots & \vdots \\ k(a^m, a^1) & \cdots & k(a^m, a^m) \end{pmatrix}$$

is positive semidefinite, then we can construct a Gaussian process with mean function  $m$  and covariance function  $k$ .<sup>3</sup> For this reason, we call  $k$  with such a property *positive semidefinite* or a *covariance function*. Notice, of course, that  $k$  is symmetric. The matrix above is sometimes called the *Gram matrix* for  $k$  and  $a^1, \dots, a^m$ .

### Example

Let  $A = \{1, \dots, n\}$  and let  $K \in \mathbf{R}^{n \times n}$  be symmetric positive semidefinite. Define  $m : A \rightarrow \mathbf{R}$  to be  $m \equiv 0$  (the constant zero function) and  $k(i, j) = K_{ij}$ . Then the normal random function  $x : \Omega \rightarrow (A \rightarrow \mathbf{R})$  with mean  $m$  and covariance  $k$  is in one to one correspondence with the gaussian random vectors with mean zero.

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<sup>3</sup>Some authors belabor this point because of the natural inclination to want to specify an *inverse* covariance function, which need not satisfy the consistency property. The consistency property ensures that any marginal of a subfamily's density is the density of that further subfamily. Future editions may expand.

