



## SUPREMUM NORM

### Why

We want a norm on the vector space of continuous functions.

### Definition

Consider a function from a closed real interval to the real numbers. The *absolute supremum* of the function is the absolute value of its results on the interval. Since the function is continuous and defined on a closed interval, the supremum is finite.

**Prop. 1.** *The functional mapping  $f \in C[a, b]$  to its absolute supremum is a norm.*

*Proof.* Let  $R$  denote the set of real numbers. Define  $\phi : C[a, b] \rightarrow R$  by:

$$\phi(f) = \sup\{|f(x)| \mid x \in [a, b]\}.$$

1.  $|f(x)| \geq 0$  for all  $x \in [a, b]$ , so  $\phi(f) \geq 0$ .
2. If  $\phi(f) = 0$  then  $|f(x)| \leq 0$  for all  $x$  and so  $f(x) = 0$  for all  $x \in [a, b]$ . If  $f = 0$ , then  $|f(x)| = 0$  for all  $x \in [a, b]$
3. For all  $\alpha$  real,  $|\alpha f(x)| = |\alpha| |f(x)|$ . so  $\phi(\alpha f) = |\alpha| \phi(f)$
4. For all  $f, g \in C[a, b]$ , and  $x \in [a, b]$ ,  $|f(x) + g(x)| \leq |f(x)| + |g(x)|$  by the triangle inequality for absolute value. Thus,

$$\begin{aligned}\phi(f + g) &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a, b]\} \\ &\leq \sup\{|f(x)| \mid x \in [a, b]\} + \sup\{|g(x)| \mid x \in [a, b]\} \\ &= \phi(f) + \phi(g)\end{aligned}$$

□

We call the functional  $\phi$  defined above the *supremum norm*.

**Notation**

Let  $f \in C[a, b]$ . We denote the supremum norm of  $f$  by  $\|f\|_{\text{sup}}$ .

