Why

1

Result

This result is called sometimes called the *probability inverse transform.*

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $X : \Omega \to \mathbf{R}$ be a random variable with cumulative distribution function $F_X : \mathbf{R} \to [0, 1]$. Suppose $F_X{}^{-1} : [0, 1] \to \mathbf{R}$ exists, then $Y = F_X{}^{-1} \circ X$ is a random variable with cumulative distribution function $F_Y : [0, 1] \to [0, 1]$ satisfying $F_Y(y) = y$.*

Remark 1. *The conclusion is equivalent to the following: Y has a density and that density is the the standard unform density (see Uniform Densities).*

Proof. Express $F_Y(\gamma) = \mathbf{P}[Y \leq \gamma] = \mathbf{P}(Y^{-1}([0, \gamma]))$ Notice

$$
\begin{aligned}
Y^{-1}([0, \gamma]]) &= \{\omega \in \Omega \mid Y(\omega) \leq \gamma\} \\
&= \{\omega \in \Omega \mid F_X(X(\omega)) \leq \gamma\} \\
&= \{\omega \in \Omega \mid X(\omega) \leq F_X^{-1}(\gamma)\}. \quad = X^{-1}(\cdots).^2
\end{aligned}
$$

$\square$

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[1]Future editions will include.

Remark 2. *Using different notation the above can be expressed succinctly as*

$$F_Y(\gamma) = \mathbf{P}[Y \leq \gamma] = \mathbf{P}[F_X \circ X \leq \gamma]$$
$$= \mathbf{P}[X \leq F_X^{-1}(\gamma)] = F_X(F_X^{-1}(\gamma)) = \gamma.$$

Future editions will discuss *inverse transform sampling.*

Cumulative Distribution Function Inverse Transform

Cumulative Distribution Functions — Uniform Densities

Probability Densities

Real Integrals

Nonnegative Integrals

Simple Integrals

Random Variables

Topological Sigma Algebras — Measurable Functions — Probability Measure

Topological Spaces

Measure — Event Probabilities

Metrics

Sigma Algebras

Real Series — Extended Real Numbers

Distance

Real Limits

Space Distance

Absolute Value — Simple Functions

Plane Distance

Interval Length

Real Functions — Probability Distributions

Intervals

Real Space

Real Plane

Real Line

Geometry

Real Sequences — Real Order — Integral Line — Real Summation

Rational Order — Real Numbers

Sequences

Rational Numbers

Subset Algebras — Lists — Integer Arithmetic

Cardinality

Set Numbers

Natural Additive Identity — Natural Multiplicative Identity

Finite Sets

Identity Elements — Arithmetic

Integer Products

Equivalent Sets — Direct Products

Natural Power — Natural Products — Natural Summation

Function Inverses

Family Unions and Intersections — Comparisons — Integer Order — Family Operations — Integer Sums

Function Composites — Function Images — Families — Orders — Integer Numbers — Operations — Natural Sums

Functions

Equivalence Relations — Converse Relations — Natural Order — Recursion Theorem

Relations

Peano Axioms

Subset Systems — Set Products — Characteristic Functions — Natural Induction — Uncertain Outcomes

Ordered Pairs — Set Powers — Generalized Set Dualities — Natural Numbers

Partitions — Set Unions and Intersections — Unordered Triples — Intersection of Empty Set — Successor Sets — Set Dualities — Set Symmetric Differences

Set Intersection — Pair Unions — Set Complements

Set Union — Pair Intersections — Set Differences

Unordered Pairs — Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets — Identities

Names

Letters — Objects