

## Row Reducer Matrices

## Why

Can we represent the function associating a linear system with its row reduction by matrix multiplication?

## Main observation

The following proposition affirmatively answers the question.

**Proposition 1.** Let  $(A \in \mathbb{R}^{m \times m}, b \in \mathbb{R}^m)$  be a linear system with  $A_{kk} \neq 0$  and (C, d) the kth reduction of (A, b). Then there exists a matrix  $L \in \mathbb{R}^{m \times m}$  so that C = LA and d = Lb.

*Proof.* Define 
$$L \in \mathbb{R}^{m \times m}$$
 by  $L_{st} = 1$  if  $s = t$ ,  $-A_{sj}/A_{ij}$  if  $k < s \le m$  and zero otherwise.

For this reason, we call L in Proposition 1 a row reducer matrix or row reducing matrix or row reducer. The row reducer matrix for the kth reduction of (A, b) has the form

$$L_{k} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & A_{ik}/A_{kk} & 1 & & \\ & & \vdots & & \ddots & \\ & & & A_{mk}/A_{kk} & & 1 \end{bmatrix}$$

So the following is immediate

**Prop. 2.** Row reducing matrices are unit lower triangular.

## Example

For example, the (1,1)-reduction of (A,b) in which

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 6 & 7 & 9 & 5 \\ 8 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

is the linear system

$$A' = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \text{ and } b' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The row reducer is  $L \in \mathbb{R}^{4 \times 4}$  defined by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that A' = LA and b' = Lb, and clearly L is unit lower triangular.

