Size of a direct product of sets

Proposition 1. *Suppose $A_1, \ldots, A_n$ is a list of finite sets. Then*

$$|\textstyle\prod_{i=1}^n| \, A_i = \prod_{i=1}^n |A_i|$$

Proof. As before, the proof will be by induction. Future editions will include. $\qquad\square$

We highlight one simple consequence: if $|A|$ is finite, then $|A^n| = |A|^n$.

Principle of counting

Proposition 2. *Suppose $X_1, \ldots, X_k$ is a list of nonempty sets and $f_2, \ldots, f_k$ are functions mapping $X_1, X_1 \times X_2, \ldots, X_1 \times \cdots X_{k-1}$ to $\mathcal{P}(X_2), \mathcal{P}(X_3), \cdots, \mathcal{P}(X_k)$ such that there exists natural numbers $n_1, n_2, \ldots, n_k$,*

$n_1 = |X_1|$

$n_2 = |f(x_1)| \quad$ *for all $x_1 \in X_1$*

$n_3 = |f(x_1, x_2)| = n_3 \quad$ *for all $(x_1, x_2) \in X_1 \times f(x_1)$*

$n_4 = |f(x_1, x_2, x_3)| = n_3 \quad$ *for all $(x_1, x_2, x_3) \in X_1 \times f_2(x_1) \times f_3(x_1, x_2)$*

$\quad \vdots$

$n_k = |f(x_1, x_2, \ldots, x_{k-1})|$

$\qquad$ *for all $(x_1, x_2, \ldots, x_{k-1}) \in X_1 \times f_2(x_1) \times \cdots \times f_k(x_1, \ldots, x_{k-1})$*

Then

$$\left| \{x : \{1, \ldots, k\} \to \cup_{i=1}^k \mid x_1 \in X_1, x_2 \in f_2(x_1), \ldots, x_k \in f_k(x_1, \ldots, x_{k-1})\} \right|$$

is $\prod_{i=1}^k n_i$

Proof. Future editions will include the proof. $\qquad\square$

The English is much clearer. Suppose we have to do k *tasks* (here, the term task is left undefined). The first task can be done in n_1 ways,

and after it has been completed, no matter how, the second task can be done in n_2 ways. Further, after the first two tasks have been completed, in whatever manner, there are n_3 ways of doing the third task. And so on. Then the number of ways to complete all tasks is $n_1 n_2 \cdots n_k$. This conclusion, often taken as a principle, or axiom, is called the *fundamental principle of counting*.

Example

Consider the usual52 playing cards cards. How many ways are there to stack them as a deck? First the bottom card, we have 52 choices. For the next to bottom card, we have 51 choices—we can pick any card except the bottom one. For the third to bottom card, we have 50 choices—and so on. By applying the above proposition, we can deduce that there are $52 \cdot 51 \cdot 50 \cdots 1$ ways of arranging the deck.

Denote the set of cards by C. For the above proposition, $k = 52$ and $X_1, \ldots, X_k$ are so that $X_i = C$ for all $i = 1, \ldots, 52$. The function $f_2(c_1) = C - \{c_1\}$. The function $f_3(c_1, c_2) = C - \{c_1, c_2\}$. And so on, so that the function $f_k(c_1, \ldots, c_{k-1}) = C - \{c_1, \ldots, c_{k-1}\}$. Notice that given distinct arguments to these functions, the size of the sets $f_1(c_1)$ is always 51. Likewise, the size of $f_2(c_1, c_2)$ is 50, $f_3(c_1, c_2, c_3)$ is 49 and so on, so that the size of $f_k(c_1, \ldots, c_k)$ is 1, regardless of the choice of c_i, $i = 1, \ldots, 51$.

Size of Direct Product

Finite Set Examples

Number of Set Products

Lists

Number of Disjoint Unions

Natural Products

Direct Products

Set Numbers

Natural Sums

Finite Sets

Equivalent Sets

Recursion Theorem

Function Inverses

Natural Order

Family Unions and Intersections

Function Composites

Function Images

Families

Peano Axioms

Functions

Equivalence Relations

Relations

Natural Induction

Ordered Pair Projections

Set Products

Set Powers

Natural Numbers

Generalized Set Dualities

Partitions

Unordered Triples

Successor Sets

Intersection of Empty Set

Set Unions and Intersections

Set Dualities

Ordered Pairs

Set Intersections

Pair Unions

Set Complements

Set Unions

Pair Intersections

Set Differences

Unordered Pairs

Empty Set

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects

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