



Measures

1 Why

We want to generalize the notion of length, area, volume beyond the Lebesgue measure on the product spaces of real numbers.

2 Definition

An extended-real-valued non-negative function on an algebra is *finitely additive* if the result of the function applied to the union of a disjoint finite family of distinguished sets is the sum of the results of the function applied to each of the sets individually.

An extended-real-valued non-negative function on a sigma algebra is *countably additive* if the result of the function applied to the union of a disjoint countable family of distinguished sets is the limit of the partial sums of the results of the function applied to each of the sets individually.

A *finitely additive measure* is an extended-real-valued non-negative finitely additive function which associates the empty set with the real number 0. A *countably additive measure* is an extended-real-valued non-negative countably additive function which associates the empty set with the real number 0. We call countably additive measures *measures*, for short.

Every countably additive measure is finitely additive. On the other hand, there exist finitely additive measures which are not countable

additive.

In the context of measure, we call a countably unital subset algebra a *measurable space*. We call the distinguished sets *measurable sets*. A *measure space* is triple. As a pair, the first two objects are a measurable space. The third object is a measure defined on the sigma algebra of the measurable space.

3 Notation

Let A a set. Let \mathcal{A} a sigma algebra on A . The pair (A, \mathcal{A}) is a measurable space.

Let $\mu : \mathcal{A} \rightarrow [0, \infty]$ a measure; thus: (a) $\mu(\emptyset) = 0$ and (b) for disjoint $\{A_n\} \subset \mathcal{A}$, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. The triple (A, \mathcal{A}, μ) is a measure space.

We use μ since it is a mnemonic for “measure”. We often also use ν to denote measures, since it is after μ in the Greek alphabet, and λ , since it is before μ in the Greek alphabet.

4 Examples

Example 1. Let (A, \mathcal{A}) a measurable space. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that $\mu(A)$ is $|A|$ if A is finite and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure. We call μ the counting measure.

Example 2. Let (A, \mathcal{A}) measurable. Fix $a \in A$. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that $\mu(A)$ is 1 if $a \in A$ and $\mu(A)$ is 0 otherwise. Then μ is a measure. We call μ the point mass concentrated at a .

Example 3. Let R denote the real numbers. The Lebesgue measure on

the measurable space $(R, \mathcal{B}(R))$ is a measure.

Example 4. Let N be the natural numbers. Let \mathcal{A} the finite co-finite algebra on N . Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be such that $\mu(A)$ is 1 if A is infinite or 0 otherwise. Then μ is a finitely additive measure. However it is impossible to extend μ to be a countably additive measure. Observe that if $A_n = \{n\}$ the $\mu(\cup_n A_n) = 1$ but $\sum_n \mu(A_n) = 0$.

Example 5. Let (A, \mathcal{A}) a measurable space. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be 0 if $A = \emptyset$ and $\mu(A)$ is $+\infty$ otherwise. Then μ is a measure.

Example 6. Let A be set with at least two elements ($|A| \geq 2$). Let $\mathcal{A} = A^*$. Let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that $\mu(A)$ is 0 if $A = \emptyset$ and $\mu(A) = 1$ otherwise. Then μ is not a measure, nor is μ finitely additive.

Proof. Let $B, C \in \mathcal{A}$, $B \cap C = \emptyset$ then using finite additivity we obtain a contradiction $1 = \mu(B \cup C) = \mu(B) + \mu(C) = 2$. \square

5 Why

We want to generalize the notions of length, area, and volume.

6 Definition

A *measurable space* is a sigma algebra. We call the distinguished subsets the *measurable sets*.

A *measure* on a measurable space is a function from the sigma algebra to the positive extended reals. A *measure space* is a measurable space and a measure.

6.1 Notation

6.2 Properties

Proposition 7. *Let (A, \mathcal{A}) be a measurable space and $m : \mathcal{A} \rightarrow [0, \infty]$ be a measure.*

If $B \subset C \subset A$, then $m(B) \leq m(C)$. We call this property the of measures monotonicity of measure.

Proposition 8. *For a measure space (A, \mathcal{A}, m) .*

If $B \subset C \subset A$, then $m(B) \leq m(C)$.

We call this property the monotonicity of measure.

Proposition 9. *For a measure space (A, \mathcal{A}, m) .*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

We this property the sub-additivity of measure.

Proposition 10. *For a measure space (A, \mathcal{A}, m) .*

If $\{A_n\} \subset \mathcal{A}$ a countable family, then $m(\cup A_n) \leq \sum_i m(A_i)$.

We this property the sub-additivity of measure.

Proposition 11. *For a measure space (A, \mathcal{A}, m) .*

$$m(\cup_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} m(A_n)$$

Proposition 12. *For a measure space (A, \mathcal{A}, m) .*

$$m(\cap_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} m(A_n)$$

6.3 Examples

Example 13. *counting measure*