



Why

Many functions of interest are additive and homogenous.

Definition

A transformation is *linear* (a *linear transformation*, *linear map*) if the result of a linear combination of the two vectors is the linear combination of the results of the vectors (using the same coefficients). The transformation is linear *with respect to* the field of the two vector spaces.

We use the term transformation (see **Transformations**) to emphasize and remind that the function is defined on a *vector* space. Of course, \mathbf{R} is a vector space and so a function $f : \mathbf{R} \rightarrow \mathbf{R}$ may be linear. The linear maps from \mathbf{R} to \mathbf{R} are the *linear functions* (see **Real Linear Functions**).

Notation

Let (V, \mathbf{F}) and (W, \mathbf{F}) be two vector spaces over the same field. Suppose $T : V \rightarrow W$. T is linear means

$$T(\alpha u + \beta v) = \beta T(u) + \alpha T(v) \quad \text{for all } \alpha, \beta \in F \text{ and } u, v \in V.$$

As usual, the condition that T is linear condition is equivalent to the two conditions:

1. $T(u + v) = f(u) + T(v)$ for all $u, v \in V$, and
2. $f(\lambda u) = \lambda f(u)$ for all $\lambda \in \mathbf{F}$ and $u \in V$.

If T satisfies (1), we call T *additive* (has the property of *additivity*). If T satisfies (2) we call T *homogeneous* (has the property of *homogeneity*).

For linear maps, it is common to denote $T(v)$ by Tv ; notice that we have dropped the usual parentheses.

We denote the set of all linear maps by $\mathcal{L}(V, W)$. It is understood when using this notation that V and W are vector spaces with respect to the same field \mathbf{F} .

Examples

Throughout, we consider vector spaces V and W over some fixed field \mathbf{F} .

Constant zero map. The map $T \in \mathcal{L}(V, W)$ defined by

$$T(v) = 0 \in W \quad \text{for all } v \in V$$

is called the *zero map* (or *zero transformation*). It is common to overload the symbol 0 so that $0 \in \mathcal{L}(V, W)$ denotes the zero map. In other words, the map 0 is defined by

$$0v = 0$$

Some care is required to interpret this equation. The 0 on the left hand side refers to a function, from V to W . The 0 on the right hand side is the additive identity in W . Usually context disambiguates the overloaded notation.

The identity map. The map $T \in \mathcal{L}(V, V)$ defined by

$$Tv = v \quad \text{for all } v \in V$$

is called the *identity map* (or *identity transformation*). It is common to denote this map by I .

Differentiation of polynomials Suppose P is the set of all polynomials with coefficients in \mathbf{R} . (Some authors denote this set by $\mathcal{P}(\mathbf{R})$. Recall that every $p \in \mathcal{P}(\mathbf{R})$ is differentiable and $p' \in \mathcal{P}(\mathbf{R})$. The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ defined by

$$Tp = p'$$

is linear. To see this, recall $(f + g)' = f' + g'$ and $(\lambda f)' = \lambda f'$ whenever f, g are differentiable and $\lambda \in \mathbf{R}$ (see Derivative of Sums and Derivatives of Scalar Multiples).

Integration of polynomials As in the previous paragraph, $\mathcal{P}(\mathbf{R})$ denotes the vector space of polynomials with coefficients in \mathbf{R} . The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathbf{R})$ defined by

$$Tp = \int_{[0,1]} p$$

is linear To see this, recall that $\int(f+g) = \int f + \int g$ and $\int \lambda f = \lambda \int f$ whenever f, g are differentiable and $\lambda \in \mathbf{R}$ (see Real Integral Additivity and Real Integral Homogeneity).

Multiplication by a quadratic. As in the previous paragraph, $\mathcal{P}(\mathbf{R})$ denotes the vector space of polynomials with coefficients in \mathbf{R} . The map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ defined by

$$(Tp)(x) = x^2 p(x) \quad \text{for all } x \in \mathbf{R}, p \in \mathcal{P}(\mathbf{R})$$

is linear. (Prove this).

Sequence backward shift. Denote the space of infinite sequences in a field \mathbf{F} by $\mathbf{F}^{\mathbf{N}}$ as usual. Define $T \in \mathcal{L}(\mathbf{F}^{\mathbf{N}}, \mathbf{F}^{\mathbf{N}})$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

T is called the *backward shift operator*.

From real space to the real plane. Define $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$$

From \mathbf{F}^n to \mathbf{F}^m . Generalizing the previous example, suppose m and n are natural numbers, and let $A_{i,j} \in \mathbf{F}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. Define $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ by

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

(It happens that every linear map from \mathbf{F}^n to \mathbf{F}^m has this form.)

A counterexample: ¹Notice $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$. True, \cos is not homogenous. that $\cos 2x = 2\cos(x)\cos(x) - \sin(x)\sin(x)$ and But this does not hold for all reals: $\cos \lambda x \neq \lambda \cos(x)$.

¹Need to add a sheet for trigonometric functions.

