

Why

Result

We bound below the measure that a non-negative measurable real-valued function exceeds some value by its integral.

Prop. 1. Let (X, \mathcal{A}, μ) be a measure space. Let $g: X \to [0, \infty]$ be measurable and square-integrable. Then for all t such that $\int t d\mu \in [0, \int g d\mu)$,

$$\mu(\lbrace x \in X \mid g(x) > t \rbrace) \ge \frac{(\int (g - t)d\mu)^2}{\int g^2 d\mu}.$$

Proof. Let t such that $\int t d\mu \in [0, \int g)$. We have selected t so that $\int (g - t) d\mu \ge 0$. Define $h = (g - t)^+$ and $A = \{x \in X \mid h(x) > 0\}$. Then

$$\int (g-t)d\mu \le \int hd\mu = \int h\chi_A d\mu \le \sqrt{\int h^2 d\mu \int \chi_A^2 d\mu}$$

Now $g^2 > h^2$, so $\int g^2 d\mu \ge \int h^2 d\mu$. Also $\chi_A^2 = \chi_A$ so $\int \chi_A^2 = \mu(A)$. h(x) > 0 if and only if $g(x) \ge t$ for all x. So $A = \{x \in X \mid g(x) \ge t\}$. Combining we have:

$$\int (g-t)d\mu \leq \sqrt{\left(\int g^2 d\mu\right)\mu(A)}.$$

Prop. 2. Let X be a random variable with $\mathbf{E}(X^2) \leq \infty$. Then for all $t \in [0, \mathbf{E}(X))$, we have

$$P(X > t) \ge \frac{\left(\mathbf{E}(X) - t\right)^2}{\mathbf{E}X^2}.$$

The above is also called the Paley-Zygmund Inequality.

