Why

We characterize chordal graphs using vertex separators, and vice versa.[1]

Main Result

Proposition 1 (Chordal Graphs and Vertex Separators)**.** *An undirected graph is chordal if and only if all minimal vertex separators are complete.*

Proof. Let $G = (V, E)$ be an undirected graph.

First, suppose that all minimal vertex separators of G are complete. Let c be a cycle of length greater than 3. Let v, w be nonconsecutive vertices in c. If v and w are adjacent in G, then $\{v, w\} \in E$ is a chord. If v and w are nonadjacent, then vw-separator exists.

The key insight is that there exists two non-consecutive vertices in the cycle that are also included in any vw-separator T. Split the cycle into the path from v to w, call it p_1 and the path from w to v, call it p_2. T must include an interior point of p_1, call it u_1, otherwise v and w are connected. Similarly, T must include an interior point of p_2, call it u_2. u_1 and u_2 are not consecutive in c, since they are distinct from x and y.

Let S be a minimal vw-separator. Let $s, t \in S$ be two non-consecutive vertices in the cycle different from v and w

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[1]Future editions will expand and may include graphics.

By assumption S is complete, so s and t are adjacent in G.

Second, let $G = (V, E)$ be a chordal graph. Let S be a minimal vw-separator. Let C_v and C_w be the connected components containing v and w of the subgraph induced by $V - S$.

If $|S| = 1$, then S is complete. Otherwise, let $x, y \in S$ by distinct. We want to show $\{x, y\} \in E$. The key insight is that x is adjacent to vertices in C_v and C_w. If there were no such vertex, $S - \{x\}$ would be a vw-separator and S would not be minimal. Similarly with y. Also, $|C_v|, |C_w| \geq 1$.

With these observations, there exists a path from x to y through C_v. Let $p_v = (x, v_1, \ldots, v_k, y)$ be a path of shortest length with at least one interior vertex (so $k \geq 1$) from x to y using interior vertices in $v_1, \ldots, v_k \in C_v$. Let $p_w = (y, w_1, \ldots, w_l, x)$ be a path of shortest length with at least one interior vertex (so $l \geq 1$) from y to x using interior vertices $w_1, \ldots, w_k \in C_w$. Use p_v and p_w to define the cycle $c = (x, v_1, \ldots, v_k, y, w_1, \ldots, w_l, x)$ which has length at least four. G is chordal, so c has a chord.

We argue that the chord of c is $\{x, y\}$. Since C_w and C_V are different connected components (whose vertices are not included in S), there are no edges $\{v_i, w_j\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Since p_v and p_w are paths of shortest length, they have no chords. In particular, there is no edge $\{v_i, v_j\}$ for $|i - j| > 1$ or $\{v_i, x\}$ for $i = 1, \ldots, l$. Similarly, there is no edge $\{w_i, w_j\}$ for $|i - j| > 1$ or $\{w_i, y\}$ for $i = 1, \ldots, l$. The only remaining pair is $\{x, y\}$, and so it must be the chord. $\square$

Chordal Graphs and Vertex Separators

Vertex Separators

Chordal Graphs

Integer Divisors

Undirected Subgraphs

Trees

Undirected Paths

Integral Line

Integer Arithmetic

Undirected Graphs

Sequences

Natural Multiplicative Identity

Natural Additive Identity

Set Numbers

Direct Products

Arithmetic

Identity Elements

Finite Sets

Integer Products

Natural Powers

Operations

Equivalent Sets

Integer Sums

Natural Products

Function Inverses

Family Unions and Intersections

Integer Numbers

Natural Sums

Function Composites

Function Images

Families

Recursion Theorem

Natural Order

Equivalence Relations

Functions

Peano Axioms

Relations

Natural Induction

Cartesian Products

Natural Numbers

Set Powers

Generalized Set Dualities

Partitions

Successor Sets

Intersection of Empty Set

Unordered Triples

Set Dualities

Set Unions and Intersections

Set Intersections

Pair Unions

Ordered Pairs

Set Complements

Set Differences

Set Unions

Pair Intersections

Unordered Pairs

Empty Set

Set Specification

Geometry

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects