



Why

The set of vector which map to 0 under a given linear transformation is a subspace.

Definition

The *nullspace* (or *kernel*) of a matrix $A \in \mathbf{R}^{m \times n}$ is the set

$$\{x \in \mathbf{R}^n \mid Ax = 0\}.$$

It is the set of vectors mapped to zero by A . Equivalently, it is the set of vectors orthogonal to the rows of A .

Notation

We denote the nullspace of $A \in \mathbf{R}^{m \times n}$ by $\text{null}(A) \subset \mathbf{R}^n$. Some authors denote the nullspace of A by $\mathcal{N}(A)$.

A subspace

Of course, the nullspace of a matrix is a subspace. There are a few routes to see this.

The first is direct. If $w, z \in \text{null}(A)$, then $Aw = 0$ and $Az = 0$. So then $A(w + z) = Aw + Az = 0$. So $\text{null}(A)$ is closed under vector addition. Also $A(\alpha w) = \alpha(Aw) = 0$ for all $\alpha \in \mathbf{R}$. [In particular $A0 = 0$, so $0 \in \text{null}(A)$; i.e., $\text{null}(A)$ contains the origin.] So $\text{null}(A)$ is closed under scalar multiplication.

The second is by thinking about orthogonal complements. Second, we can view the $\text{null}(A)$ as the set of vectors orthogonal to all the rows of A . In other words, $\text{null}(A) = \{\tilde{a}_1, \dots, \tilde{a}_m\}^\perp$. The orthogonal complement of any set is a subspace (see **Orthogonal Real Subspaces**).

The dimension of the nullspace is called the *nullity*.

Ambiguity in solutions

Suppose we have a solution to the system of linear equation with data (A, y) . In other words, we have a vector $x \in \mathbf{R}^n$ so that $y = Ax$. If we have a vector $z \in \text{null}(A)$, then $x + z$ is also a solution to the system (A, y) , since

$$A(x + z) = Ax + Az = Ax + 0 = y$$

Conversely, suppose there were another solution $\tilde{x} \in \mathbf{R}^n$ to the system (A, y) . Then $y = Ax = A\tilde{x}$, so

$$0 = y - y = Ax - A\tilde{x} = A(x - \tilde{x}).$$

Consequently, $(x - \tilde{x}) \in \text{null}(A)$, and so \tilde{x} is the solution x plus some vector in the null space of A . Consequently we are interested in whether A has vectors in its nullspace.

Zero nullspace

The origin 0 is always in the nullspace of A . However, this vector does not mean that we can find different solutions, since $x + 0 = x$ for all $x \in \mathbf{R}^n$. If, on the other hand, there is a nonzero vector $z \in \text{null}(A)$, then $x + z \neq x$, and $x + z$ is a solution for (A, y) . We think about A as a function from \mathbf{R}^n to \mathbf{R}^m . In the case that there is a nonzero element in the nullspace, A maps different vectors to the same vector. Here, x and $x + z$ both map to y . In this case, the function is *not invertible*, because it is not one-to-one. If, however, zero is the only element of the null space, the function is one-to-one. So call A *one-to-one* if $\text{null}(A) = 0$.

Equivalent statements

A matrix $A \in \mathbf{R}^{m \times n}$ is *one-to-one* if the linear function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ defined by $f(x) = Ax$ is one-to-one. In this case, if there exists $x \in \mathbf{R}^n$ so that $y = Ax$, then there is only one such x . Different elements in \mathbf{R}^n map to different elements in \mathbf{R}^m .

