Why

We want to discuss real-valued random functions whose family of random variables have simple densities.[1]

Definition

A *normal random function* is a real-valued random function whose family of real-valued random variables has the property that any subfamily is jointly normal.

For this reason, we call the family of random variables (or stochastic process) corresponding to the random function a *gaussian process* or *normal process*.

Notation

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and A a set. Let $x : \Omega \to (A \to \mathsf{R})$ be a random function with family $y : A \to (\Omega \to \mathsf{R})$.

The random function x is a normal if, for all $a^1, \ldots, a^m \in A$, $(y(a^1), \ldots, y(a^m))$ is jointly normal.

Mean and covariance function

Proposition 1. *Let $x : \Omega \to (A \to \mathsf{R})$ be a normal random function with family $X : A \to (\Omega \to \mathsf{R})$. There exists unique functions $m : A \to \mathsf{R}$ and $k : A \times A \to \mathsf{R}$ so that the mean of the random variable X_a is $m(a)$ for all A and the covariance of*

[1] Future editions will expand.

the random variables X_a and $X_{a'}$ is $k(a, a')$ for all $a, a' \in A$.[2]

For this reason, we call m the *mean function* and k the *covariance function* of the random function.

Conversely, let $m : A \to \mathbf{R}$ and $k : A \times A \to \mathbf{R}$. Then if k satisfies the property that for all $a^1, \ldots, a^m$, the $m \times m$ matrix

$$\begin{pmatrix} k(a^1, a^1) & \cdots & k(a^1, a^m) \\ \vdots & \ddots & \vdots \\ k(a^m, a^1) & \cdots & k(a^m, a^m) \end{pmatrix}$$

is positive semidefinite, then we can construct a Gaussian process with mean function m and covariance function k.[3] For this reason, we call k with such a property *positive semidefinite* or a *covariance function*. Notice, of course, that k is symmetric. The matrix above is sometimes called the *Gram matrix* for k and $a^1, \ldots, a^m$.

Example

Let $A = \{1, \ldots, n\}$ and let $K \in \mathbf{R}^{n \times n}$ be symmetric positive semidefinite. Define $m : A \to \mathbf{R}$ to be $m \equiv 0$ (the constant zero function) and $k(i, j) = K_{ij}$. Then the normal random function $x : \Omega \to (A \to \mathbf{R})$ with mean m and covariance k is in

[2] Future editions may include an account.

[3] Some authors belabor this point because of the natural inclination to want to specify an *inverse* covariance function, which need not satisfy the consistency property. The consistency property ensures that any marginal of a subfamilys density is the density of that further subfamily. Future editions may expand.

one to one correspondence with the gaussian random vectors with mean zero.

Normal Random Functions
Multivariate Normals
Random Functions
Positive Definite Matrices
Symmetric Matrices
Quadratic Forms
Matrix Sums
Matrix Trace
Identity Matrices
Matrix Multiplication
Matrix Determinants
Matrix Transpose
Random Vectors
Multivariate Real Derivatives
Matrix Inverse Product
Real Matrices
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Normal Distribution
Topological Sigma Algebra
Linear Combinations
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Probability Densities
Exponential Function
Topological Spaces
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Absolute Value
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Real Square Roots
Probability Measure
Cumulative Functions
Metric Spaces
Real Numbers
Real Arithmetic
Event Probability
Measure
Plane Distance
Real Multiplication Increments
Estimate Real Numbers
Probability Distributions
Interval Length
Real Predicates
Sigma Algebra
Real Lines
Intervals
Real Addition Increments
Real Sequences
Real Line
Real Parts
Real Space
Integral Line
Intervals
Real Inequalities
Permutations
Real Sums
Pairs
Rational Multiplication Increments
Rational Arithmetic
Set Orders
Set Numbers
Real Numbers
Cosine
Rational Products
Relational Sums
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Integer Addition Sums
Rational Sums
Direct Products
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Integer Differences
Integer Addition
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Set Unions and Intersections
Set Unions
Intersection of Empty Set
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Set Unions
Set Intersections
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Set Specification
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