



## Why

We want to combine two groups.<sup>1</sup>

## Defining Result

**Proposition 1.** *For each natural number  $m$ , there exists a function  $s_m : \omega \rightarrow \omega$  which satisfies*

$$s_m(0) = m \quad \text{and} \quad s_m(n^+) = (s_m(n))^+$$

*for every natural number  $n$ .*

*Proof.* The proof uses the recursion theorem (see Recursion Theorem).<sup>2</sup> □

Let  $m$  and  $n$  be natural numbers. The value  $s_m(n)$  is the *sum* of  $m$  with  $n$ .

## Notation

We denote the sum  $s_m(n)$  by  $m + n$ .

## Properties

The properties of sums are direct applications of the principle of mathematical induction (see Natural Induction).<sup>3</sup>

**Proposition 2** (Associative). *Let  $k$ ,  $m$ , and  $n$  be natural numbers. Then*

$$(k + m) + n = k + (m + n).$$

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<sup>1</sup>Future editions will change this section.

<sup>2</sup>Future editions will give the entire account.

<sup>3</sup>Future editions will include the accounts.

**Proposition 3** (Commutative). *Let  $m$  and  $n$  be natural numbers.*  
*Then*

$$m + n = n + m.$$

### **Relation to Addition**

**Proposition 4** (Distributive). *Let  $k$ ,  $m$ , and  $n$  be natural numbers.*  
*Then*

$$k \cdot (m + n) = (k \cdot m) + (k \cdot n).$$

