



Why

What does it mean for two random variables to be independent? What are the events associated with a random variable?¹

Defining Result

Proposition 1. *The set of inverse images the distinguished sets of a measurable space under a function from a set to that space is a sigma algebra.*

If the first set and the function are measurable, the sigma algebra is a sub sigma algebra of the domain sigma algebra.

Proof. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be measurable spaces. Let $f : X \rightarrow Y$ be a measurable function. Define

$$\mathcal{C} := \{f^{-1}(B) \mid B \in \mathcal{B}\}.$$

First, since $f^{-1}(Y) = X$, $X \in \mathcal{C}$.

Second, let $C \in \mathcal{C}$. Then there is B such that $C = f^{-1}(B)$. Then $X - C = X - f^{-1}(B) = f^{-1}(Y - B)$. Since \mathcal{B} is a sigma algebra, $B \in \mathcal{B}, Y - B \in \mathcal{B}$ and so $X - C \in \mathcal{C}$.

Finally, let $(C_n)_n \subset \mathcal{C}$. Then for every n there exists a $B_n \in \mathcal{B}$ so that $C_n = f^{-1}(B_n)$. Then:

$$\cup_n C_n = \cup_n f^{-1}(B_n) = f^{-1}(\cup_n B_n).$$

Since \mathcal{B} is a sigma algebra, $\cup_n B_n \in \mathcal{B}$ and so $\cup_n C_n \in \mathcal{C}$.

Since f is measurable, $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$, and so $\mathcal{C} \subset \mathcal{A}$. \square

The sigma algebra *generated by a random variable* is the sigma algebra consisting of the inverse images of every measurable set of the codomain.

¹Future editions will expand.

The sigma algebra generated by a family of random variables is the sigma algebra generated by the union of the sigma algebras generated individually by each of the random variables.

Notation

Let (X, \mathcal{A}, μ) be a probability space and (Y, \mathcal{B}) be a measurable space. Let $f : X \rightarrow Y$ be a random variable. Denote by $\sigma(f)$ the sigma algebra generated by f .

Results

Proposition 2. *The sigma algebra generated by a family of random variables is the smallest sigma algebra for with respect to which each random variable is measurably.*

