



### Why

We can characterize the dependence of two events in terms of the rank of a particular matrix.

### Definition

Given a probability measure  $\mathbf{P} : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$  on the finite set  $\Omega$  and two events  $A, B \subset \Omega$ , the *joint probability matrix* of  $A$  and  $B$  is the matrix

$$M = \begin{bmatrix} \mathbf{P}(A \cap B) & \mathbf{P}(A \cap C_{\Omega}(B)) \\ \mathbf{P}(C_{\Omega}(A) \cap B) & \mathbf{P}(C_{\Omega}(A) \cap C_{\Omega}(B)) \end{bmatrix}.$$

### Characterization of independence

If  $A$  and  $B$  are independent, then so are  $A$  and  $C_{\Omega}(B)$ ,  $B$  and  $C_{\Omega}(A)$ , and  $C_{\Omega}(A)$  and  $C_{\Omega}(B)$ . In other words,

$$M = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_{\Omega}(A)) \end{bmatrix} \begin{bmatrix} \mathbf{P}(B) & \mathbf{P}(C_{\Omega}(B)) \end{bmatrix}.$$

In this case, we see that  $\text{rank}(M) = 1$ .

Conversely, suppose  $\text{rank}(M) = 1$ . Then, using the law of total probability, each row is a multiple of

$$M1 = \begin{bmatrix} \mathbf{P}(A) \\ \mathbf{P}(C_{\Omega}(A)) \end{bmatrix}.$$

In particular, we have  $\mathbf{P}(A \cap B) = \alpha \mathbf{P}(A)$  and  $\mathbf{P}(C_{\Omega}(A) \cap B) = \alpha \mathbf{P}(C_{\Omega}(A))$ . So

$$\mathbf{P}(A \cap B) + \mathbf{P}(C_{\Omega}(A) \cap B) = \alpha(\mathbf{P}(B) + \mathbf{P}(C_{\Omega}(A))),$$

from which we deduce  $\alpha = \mathbf{P}(B)$ . Likewise, the multiplier for the second column of  $M$  is  $\mathbf{P}(C_{\Omega}(B))$ . In other words,  $A$  and  $B$  are independent. We conclude that  $A$  and  $B$  are independent if and only if  $\text{rank}(M) = 1$ .



