Real Length Impossible

1 Why

Given a subset of the real line, what is its length?

2 Background

Let $a, b \in R$ with $a \leq b$. The *length* of the closed interval of the real numbers $[a, b]$ is $b - a$. The length is non-negative.

A family $\{A_\alpha\}_{\alpha \in I}$ is *disjoint* if for $\alpha, \beta \in I$, $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \varnothing$. A set A can be *partioned* into a family if there exists a disjoint family whose union is A. A set $A \subset R$ is *simple* if it can be partitioned into a countable family whose members are closed intervals. The above discussion suggests that we should define the length of a simple set as the sum of the lengths of sets which parition it.

The above discussion suggests that if we wish to define a function length : $2^R \to R \cup \{-\infty, \infty\}$, we should ask that (1) $\text{length}(A) \geq 0$, (2) $\text{length}([a, b]) = b - a$, (3) for disjoint closed intervals $\{A_n\}_{n \in N}$, $\text{length}(A_i) = \sum_i \text{length}(A_i)$, and (4) for all $A \subset R$ and $a \in R$, $\text{length}(A + x) = \text{length}(A)$.

3 Converse

Define the equivalence relation $\sim$ on R by by $x \sim y$ if $x \sim y \in Q$

3.1 Notation

Let A be a set and $\mathcal{A} \subset A^*$. We denote the subset algebra of A and $\mathcal{A}$ by $(A, \mathcal{A})$, read aloud as "A, script A."

4 Properties

Proposition 1. *For any set A,* 2^A *is a sigma algebra.*

Proposition 2. *The intersection of a family of sigma algebras is a sigma algebra.*

5 Generation

Proposition 3. *Let* A *a set and* $\mathcal{B}$ *a set of subsets. There is a unique smallest sigma algebra* $(A, \mathcal{A})$ *with* $\mathcal{B} \subset \mathcal{A}$.

We call the unique smallest sigma algebra containing B the *generated sigma algebra* of B.