Why

We want to describe how fast a function grows or declines.[1]

Definition

Let $f : \mathbf{R} \to \mathbf{R}$. The *lower growth class* of f (*toward infinity*) is the set of all functions $g : \mathbf{R} \to \mathbf{R}$ for which there exists $C, M > 0$ so that $|g(x)| \leq C|f(x)|$ for all $x > M$. The intuition is that if $h : \mathbf{R} \to \mathbf{R}$ is in the lower growth class of f, h does not grow faster than f. In this case we say that h *grows at order* f.

The *lower limit class of* f *at* x_0 is the set of all functions $g : \mathbf{R} \to \mathbf{R}$ for which there exists $C, \varepsilon > 0$ so that $|g(x)| \leq C|f(x)|$ for all $|x - x_0| < \varepsilon$. The intuition is that for x sufficiently close to x_0, the magnitude of f is bounded by a constant times the magnitude of g. Often x_0 is 0.

The *upper growth class* of f (*toward infinity*) is the set of all functions $g : \mathbf{R} \to \mathbf{R}$ for which there exists $C, M > 0$ so that $|g(x)| \geq C|f(x)|$ for all $x > M$. The intuition is that if h is in the upper growth class of f, h grows at least as fast as f. We similarly define the *upper growth class at a limit* x_0.

The *(exact) growth class* of f is the set of all functions $g : \mathbf{R} \to \mathbf{R}$ for which there exists C_1, C_2, M so that $C_1|f(x)| \leq |g(x)| \leq C_2|f(x)|$ for all $x > M$. The intuition is that if h is in the growth class of f, then h and f grow at the same rate. Again, we similarly define the *growth class at limit* x_0.

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[1]Future editions will expand this vague introduction.

Notation

We denote the upper, lower and exact growth classes of a function $f : \mathbf{R} \to \mathbf{R}$ by of f by $O(f)$, $\Omega(f)$ and $\Theta(f)$, respectively. We read the notation $O(f)$ as "order at most f," we read $\Omega(f)$ as "order at least f," and $\Theta(f)$ as "order exactly f."

The letter O is a mnemonic for order, and Ω and Θ build on this mnemonic. The term order appears to arise from the use of growth classes when discussing Taylor approximations. In this case of small x (i.e., $|x| < 1$), $|x^p| < |x^q|$ if $q < p$ and so higher order terms are "smaller" and "negligible." This notation is sometimes called *Big O notation*, *Landau's symbol*, *Landau notation* or *Landau's notation*.

Let $\phi, \psi : \mathbf{R} \to \mathbf{R}$. Many authors use $\phi = O(\psi)$ or $\phi(t) = O(\psi(t))$ to assert that ϕ is in the upper growth class of ψ at some understood limit (e.g., 0 or ∞). In other words, the equation asserts that there exists some positive constant $C > 0$ sot that, for all t sufficiently close to the understood limit, $|\phi(t)| \leq C|\psi(t)|$.[2] For example, the statement $\sin^2(t) = P(t^2)$ as $t \to 0$ (or for $t \to 0$) means that there exists constants $C, \varepsilon > 0$ so that, $|t| < \varepsilon \longrightarrow |\sin^2(t)| \leq Ct^2$.

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[2]Often also defined $|\phi(t)| < C\psi(t)$, with no absolute value on ψ.

Function Growth Classes

Absolute Value

Interval Length   Real Functions

Intervals

Real Line

Real Order

Real Numbers   Rational Order

Integral Line   Rational Numbers

Integer Arithmetic

Natural Additive Identity   Natural Multiplicative Identity   Integer Products

Identity Elements   Arithmetic

Natural Powers

Natural Products   Integer Sums   Integer Order   Comparisons

Operations   Natural Sums   Integer Numbers   Orders

Functions   Recursion Theorem   Equivalence Relations   Natural Order   Converse Relations

Peano Axioms   Relations

Natural Induction   Set Products

Natural Numbers   Set Powers   Ordered Pairs

Partitions   Intersection of Empty Set   Successor Sets   Unordered Triples

Set Symmetric Differences   Set Intersections   Pair Unions

Pair Intersections   Set Complements   Set Unions   Unordered Pairs

Empty Set   Set Differences

Geometry   Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Sets   Identities

Names

Letters   Objects

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