Why

We want to compute integrals with respect to arbitrary measures. If we find a special measure with which we can compute integrals, would there be some way to translate integrals against another measure into integrals against this special measure?

It would suffice to produce a special function whose integral against the base measure would over any measurable set gave the measure of the set under the alternate measure. In this case, the monotone convergence theorem would say that the integral of any function under the alternate measure was the integral of the function against the special function under the base measure.

No small miracle is that the condition for this exchange is extremely simple. In this sheet, we will define the condition. In a later sheet we will construct the special function.

Definition

Consider two measures. Suppose there exists a measurable set for which the first was positive but the second was zero. Then we have no hope of exchanging these measures, because the integral of any function over this set against the second measure is zero.

So let us rule out this case. Suppose that whenever the first was positive on a set, the second was positive on that set. The contrapositive is that whenever the second is zero on a set, the first is zero on the same. This condition is sufficient for exchangeability.

A first measure is *exchangeable* with respect to a second measure if the first measure of any measurable set is zero whenever the second measure of that set is zero.[1] As we said earlier, this condition means that the first measure is never nonzero on a set for which the second measure is zero.

[1] The language "exchangeable" is not at all standard. We have coined it for the reasons outlined in the why of this sheet.

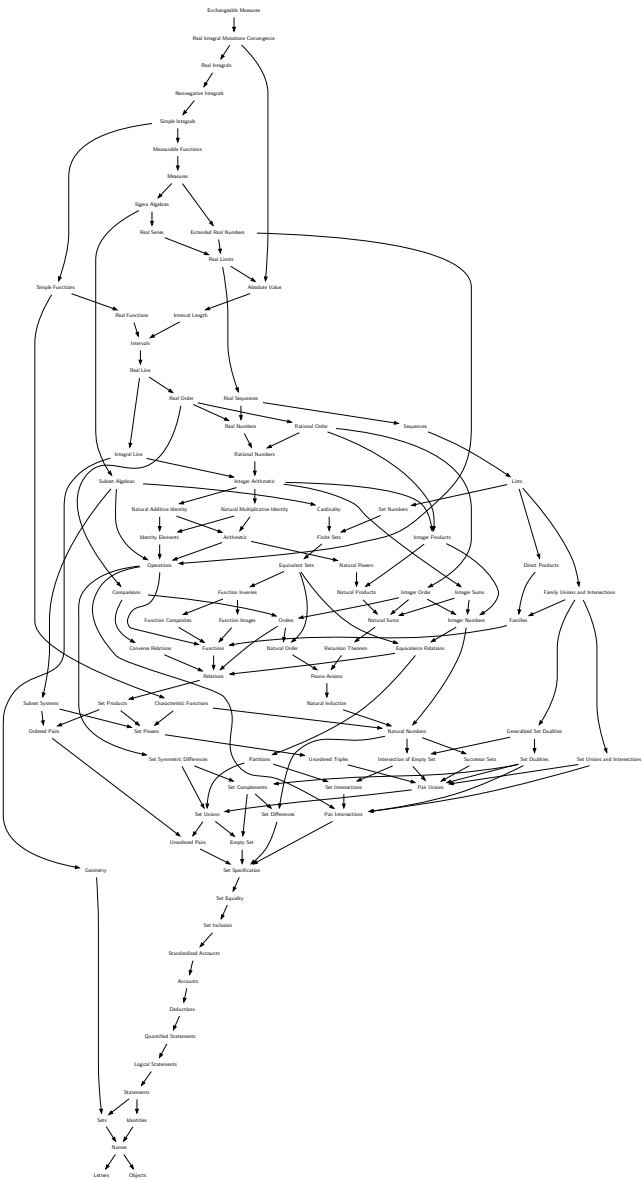
It is common to say that the first measure is *absolutely continuous* with respect to the second measure.[2]

Notation

Let $(X, \mathcal{A})$ be a measurable space. Let μ and ν be two measures on $\mathcal{A}$. Then ν is exchangeable with respect to μ if for all measurable $A \in \mathcal{A}$, $\mu(A) = 0 \longrightarrow \nu(A) = 0$. Equivalently, $\nu(A) > 0 \longrightarrow \mu(A) > 0$ for all $A \in \mathcal{A}$. Notice that in this second statement, instead of the relation of equality $(=)$ we are using the relation of inequality $(<)$.

There is often used notation for a measure μ being exchangeable (absolutely continuous) with respect to a second measure ν. It is $\mu \ll \nu$. The notation is a reminder of the fact that $\nu(A) = 0 \longrightarrow \mu(A) = 0$ for all $A \in \mathcal{A}$.

[2]This needs more justification, perhaps, as this naming seems to be an artifact of a notion of absolutely continuous functions.


Exchangeable Measures
Real Integral Monotone Convergence
Real Integrals
Nonnegative Integrals
Simple Integrals
Measurable Functions
Measures
Sigma Algebras
Real Series
Extended Real Numbers
Real Limits
Simple Functions
Absolute Value
Real Functions
Interval Length
Intervals
Real Line
Real Order
Real Sequences
Real Numbers
Rational Order
Sequences
Rational Numbers
Integral Line
Integer Arithmetic
Lists
Subset Algebras
Natural Addition Identity
Natural Multiplicative Identity
Cardinality
Set Numbers
Identity Elements
Arithmetic
Finite Sets
Integer Products
Operations
Equivalent Sets
Natural Powers
Direct Products
Comparisons
Function Inverses
Natural Products
Integer Order
Integer Sums
Family Unions and Intersections
Function Composites
Function Images
Order
Natural Sums
Integer Numbers
Families
Converse Relations
Functions
Natural Order
Recursion Theorem
Equivalence Relations
Relations
Peano Axioms
Subset Systems
Set Products
Characteristic Functions
Natural Induction
Ordered Pairs
Set Powers
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Geometry
Set Specification
Set Equality
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Latches
Objects