Why

We want to select a normal density which summarizes well a dataset.

Formulation

Let $D = (x^1, \dots, x^n)$ be a dataset in $\mathbf{R}$. We want to select a density from among normal densities, which require specifying a mean and covariance.

Following the principle of maximum likelihood, we want to solve

$$\begin{array}{rl} \textbf{find} & \mu, \sigma \in \mathbf{R} \\ \textbf{to maximize} & \displaystyle\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x^k - \mu}{\sigma}\right)^2\right) \\ \textbf{subject to} & \sigma > 0 \end{array}$$

We call a solution to the above problem a *maximum likelihood normal density* with respect to the dataset.

Solution

Prop. 1. *Let $(x^1, \dots, x^n)$ be a dataset in $\mathbf{R}$. Let f be a normal density with mean*

$$\frac{1}{n}\sum_{k=1}^{n} x^k$$

and covariance

$$\frac{1}{n}\sum_{k=1}^{n}\left(x^k - \frac{1}{n}\sum_{k=1}^{n} x^k\right)^2.$$

Then f is a maximum likelihood normal density.

Proof. Every normal density has two parameters: the mean and the covariance. If the likelihood of one normal is less than or equal to the likelihood of another, then so is are their log likelihoods. Let f be a normal density with parameter μ and σ^2. We express the log likelihood of f by

$$\sum_{k=1}^{n} \left(\frac{1}{2\sigma^2}(x^k - \mu)^2 - \frac{1}{2} \log 2\pi\sigma^2 \right)$$

The partial derivative of the log likelihood with respect to the mean $(\partial_\mu \ell) : \mathsf{R}^2 \to \mathsf{R}$ is

$$(\partial_\mu \ell)(\mu, \sigma^2) = -\sum_{k=1}^{n} \frac{1}{\sigma^2}(x - \mu)$$

and with respect to the covariance $(\partial_{\sigma^2} \ell) : \mathsf{R}^2 \to \mathsf{R}$ is

$$(\partial_{\sigma^2} \ell)(\mu, \sigma^2) = \left(\frac{-1}{2(\sigma^2)^2} \sum_{k=1}^{n} (x^k - \mu)^2 \right) - \frac{1}{2\sigma^2}$$

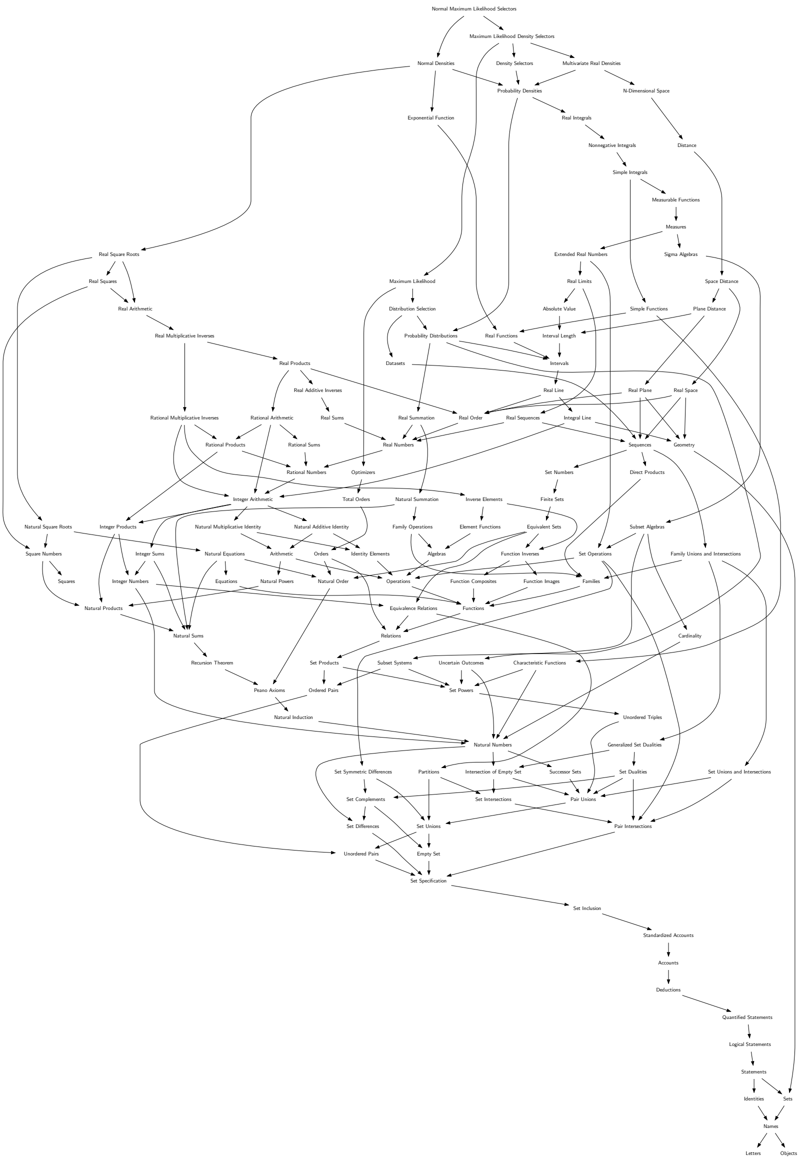
We are interested in finding $\mu_0 \in \mathsf{R}$ and $\sigma_0^2 > 0$, at which $\partial_\mu \ell(\mu_0, \sigma_0^2) = 0$ and $\partial_{\sigma^2} \ell(\mu_0, \sigma_0^2) = 0$. So we have two equations. First, notice that $\partial_\mu \ell$ is zero if an only if its first argument (the mean) is $\frac{1}{n} \sum_{k=1}^{n} x^k$. Second, notice that for all μ, σ^2, $\partial_{\sigma^2} \ell$ is zero if and only if

$$\sigma^2 = \sum_{k=1}^{n} (x^k - \mu)^2.$$

So the pair

$$\left(\frac{1}{n}\sum_{k=1}^{k} x^k, \frac{1}{n}\sum_{k=1}^{n}(x_k - \frac{1}{n}\sum_{k=1}^{n} x^k)^2\right)$$

is a stationary point of ℓ. $\square$



Normal Maximum Likelihood Selectors
Maximum Likelihood Density Selectors
Normal Densities
Density Selectors
Multivariate Real Densities
Probability Densities
N-Dimensional Space
Exponential Function
Real Integrals
Nonnegative Integrals
Distance
Simple Integrals
Measurable Functions
Measures
Real Square Roots
Extended Real Numbers
Sigma Algebras
Real Squares
Maximum Likelihood
Real Limits
Space Distance
Real Arithmetic
Distribution Selection
Absolute Value
Simple Functions
Plane Distance
Real Multiplicative Inverses
Probability Distributions
Real Functions
Interval Length
Real Products
Datasets
Intervals
Real Additive Inverses
Real Line
Real Plane
Real Space
Rational Multiplicative Inverses
Rational Arithmetic
Real Sums
Real Summation
Real Order
Real Sequences
Integral Line
Rational Products
Rational Sums
Real Numbers
Sequences
Geometry
Rational Numbers
Optimizers
Set Numbers
Direct Products
Integer Arithmetic
Total Orders
Natural Summation
Inverse Elements
Finite Sets
Natural Square Roots
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Family Operations
Element Functions
Equivalent Sets
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Square Numbers
Integer Sums
Natural Equations
Arithmetic
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Identity Elements
Algebras
Function Inverses
Set Operations
Family Unions and Intersections
Squares
Integer Numbers
Equations
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Natural Order
Operations
Function Composites
Function Images
Families
Natural Products
Equivalence Relations
Functions
Natural Sums
Relations
Cardinality
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Set Products
Subset Systems
Uncertain Outcomes
Characteristic Functions
Peano Axioms
Ordered Pairs
Set Powers
Natural Induction
Unordered Triples
Natural Numbers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Intersection of Empty Set
Successor Sets
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects