$$\diamondplus$$

Tree Density Approximators

1   Why

We can approximate a density with a tree density similar to how we can approximate a distribution with a tree distribution.

2   Definition

We use the differential relative entropy as a criterion of approximation. An *optimal tree approximator* of density for a tree is a density which factors according to a tree and minimizes its differential relative entropy with the given density.

3   Notation

Let $g : \mathsf{R}^n \to \mathsf{R}$ be a density and T be a tree on $\{1, \ldots, n\}$. An optimal tree approximator of g for T is a density f that factors according to T and minimizes $d(g, f)$. In other words, given g and T we want to find f to

$$\text{minimize} \quad d(g, f)$$
$$\text{subject to} \quad f \text{ factors according to } T.$$

4   Result

Proposition 1. *Let* $g : \mathsf{R}^n \to \mathsf{R}$ *be a density and* T *be a tree on* $\{1, \ldots, n\}$. *The density* $f_T^* : \mathsf{R}^d \to \mathsf{R}$ *defined by*

$$f_T^* = g_1 \prod_{i \neq 1} g_{i|\mathbf{pa}_i}$$

minimizes the differential relative entropy with g among all densities on $\mathbf{R}^n$ which factor according to T ($\mathbf{pa}_i$ *is the parent of i in T rooted at vertex 1, $i = 2, \ldots, n$).*

Proof. Let $f : \mathbf{R}^d \to \mathbf{R}$ be a density factoring according to T. First, express

$$f = f_1 \prod_{i=1} f_{i|\mathbf{pa}_i}.$$

Second, recall that $d(g, f) = h(g, f) - h(g)$. Since $h(g)$ does not depend on f, f is a minimizer of $d(g, f)$ if and only if f is a minimizer of $h(g, f)$.

Third, express

$$
\begin{aligned}
h(g, f) &= -\int_{\mathbf{R}^d} g \log f \\
&= -\int_{\mathbf{R}^d} g(x) \left(\log f_i(x_i) + \sum_{i \neq 1} \log f_{i|\mathbf{pa}_i}(x_i, x_{\mathbf{pa}_i}) \right) dx \\
&= h(g_1, f_1) + \sum_{i \neq 1} \left(\int_{\mathbf{R}} g_{\mathbf{pa}_i}(\xi) h\big(g_{i|\mathbf{pa}_i}(\cdot, \xi), f_{i|\mathbf{pa}_i}(\cdot, \xi)\big) d\xi \right)
\end{aligned}
$$

which separates across f_1 and $f_{i|\mathbf{pa}_i}(\cdot, \xi)$ for $i = 1, \ldots, n$ and $\xi \in \mathbf{R}$. In particular, since $g_{pai} \geq 0$, we can minimize the integrand pointwise.

Fourth, recall $h(\phi, \psi) \geq 0$ for densities ϕ, ψ of any dimension, and zero if $\phi = \psi$. So $f_1 = g_1$ and $f_{i|\mathbf{pa}_i} = g_{i|\mathbf{pa}_i}$ are solutions.

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