



Why

When is a linear transformation *onto*? In other words, when is the range the whole space? This question is a bit more involved, but we will start with observing in this sheet that the range of a linear map happens to be a subspace.

Definition

For a *linear* transformation $T \in \mathcal{L}(V, W)$, we refer to $\text{range}(T)$ as the *range space* (or *image space*) of T .

Proposition 1. *Suppose $T \in \mathcal{L}(V, W)$. Then $\text{range } T$ is a subspace of W .*

Proof. We verify that $\text{range}(T)$ contains 0 and is closed under vector addition and scalar multiplication. Clearly $T(0) = 0$, so $0 \in \text{range } T$. Next, suppose $w_1, w_2 \in \text{range } T \subset W$. So there exists $v_1, v_2 \in V$ so that

$$Tv_1 = w_1 \text{ and } Tv_2 = w_2$$

We conclude $w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2)$. So $w_1 + w_2 \in \text{range } T$. Likewise, if $w \in \text{range } T$, then there exists $v \in V$ such that $w = Tv$. So then $\lambda w = \lambda Tv = T(\lambda v)$, and so $\lambda w \in \text{range } T$. \square

Examples

To come.

