



Why

We generalize our notion of *size* to n -dimensional space.

Definition

The *norm* (or *Euclidean norm*) of $x \in \mathbf{R}^n$ is

$$\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Notation

We denote the norm of x by $\|x\|$. In other words, $\|\cdot\| : \mathbf{R}^n \rightarrow \mathbf{R}$ is a function from vectors to real numbers. The notation follows the notation of absolute value, the *magnitude* of a real number, and the double verticals remind us that x is a vector. A warning: some authors write $|x|$ for the norm of x when it is understood that $x \in \mathbf{R}^n$.

We understand the norm of x by comparison with the distance function $d : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$. On one hand, the norm of x is $d(x, 0)$. So $\|x\|$ measures the length of the vector x from the origin 0. On the other hand, $d(x, y) = \|x - y\|$. So $\|x - y\|$ measures the distance between x and y .

Properties

The norm has several important properties:

1. $\|\alpha x\| = |\alpha| \|x\|$, called (*absolute*) *homogeneity*,
2. $\|x + y\| \leq \|x\| + \|y\|$, called the *triangle inequality*,
3. $\|x\| \geq 0$, called *non-negativity*, and

4. $\|x\| = 0 \longleftrightarrow x = 0$, called *definiteness*.

