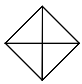

Why

We integrate over a product space by integrating one coordinate at a time.

Result

Proposition 1. *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{A}, \nu)$ are σ-finite measurable spaces. Let $f : X \times Y \to \bar{\mathbf{R}}$ be $\mathcal{A} \times \mathcal{B}$-measurable and $\mu \times \nu$-integrable. Then*

1. *For μ-almost every x in X the section f_x is ν-integrable and for ν-almost every y in Y the section f^y is μ-integrable,*
2. *the functions I_f and J_f defined by*
$$I_f(x) = \begin{cases} \int_Y f_x d\nu & \text{if } f_x \text{ is } \nu\text{-integrable,} \\ 0 & \text{otherwise} \end{cases}$$
and
$$J_f(y) = \begin{cases} \int_X f^y d\mu & \text{if } f^y \text{ is } \mu\text{-integrable,} \\ 0 & \text{otherwise} \end{cases}$$
belong to $\mathcal{L}(X, \mathcal{A}, \mu, \mathbf{R})$ and $\mathcal{L}(Y, \mathcal{A}, \nu, \mathbf{R})$ respectively, and
3. *the relation*
$$\int_{X \times Y} f d(\mu \times \nu) = \int_X I_f d\mu = \int_Y J_f d\nu$$
holds.

The above is called *Fubini's Theorem.* Next: *Tonelli's theorem.*