



Why

For each value of a random variable's codomain, the set of outcomes corresponding to that value is the inverse image of the random variable. We can speak of the probability that a random variable takes a value then, by assigning it the probability of the set of outcomes corresponding to that value.

Definition

Let $p : \Omega \rightarrow \mathbf{R}$ be a probability distribution with corresponding probability measure $\mathbf{P} : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$. Suppose $x : \Omega \rightarrow V$ is an outcome variable. The *probability* $x = a$, for $a \in \Omega$, is

$$\mathbf{P}(\{\omega \in \Omega \mid x(\omega) = a\}).$$

From the definition of \mathbf{P} , we express the above as

$$\sum_{\omega \in \Omega \mid x(\omega) = a} p(\omega).$$

We refer to the probability of the *event* that $x = a$.

Notation

We denote the probability that $x = a$ by $\mathbf{P}[x = a]$. Our square brackets deviate from the slightly slippery but universally standard notation $\mathbf{P}(x = a)$. We prefer the square brackets, since $x = a$ is not itself an argument to \mathbf{P} , but shorthand for $(\{\omega \in \Omega \mid x(\omega) = a\})$.

There are many similar notations. For example, $\mathbf{P}[x \in C]$ means $\mathbf{P}(\{x \in \Omega \mid x(\omega) \in C\})$. In particular, if $x : \Omega \rightarrow \mathbf{R}$, $\mathbf{P}[x \geq a]$ means $\mathbf{P}(\{\omega \in \Omega \mid x(\omega) \geq a\})$. Since the *event* that $x = a$ is the inverse image of $\{a\}$ under x , we also use the notations $\mathbf{P}(x^{-1}(a))$ and $\mathbf{P}(x^{-1}(C))$.

Example: sum of two dice

Define $\Omega = \{1, \dots, 6\}^2$ and define $p : \Omega \rightarrow \mathbf{R}$ with $p(\omega) = 1/36$ for each $\omega \in \Omega$. Define $x : \Omega \rightarrow \mathbf{N}$ by $x(\omega_1, \omega_2) = \omega_1 + \omega_2$. Then

$$\mathbf{P}[x = 4] = p((2, 2)) + p(1, 3) + p(3, 1) = 1/12.$$

