Why

What are the affine sets in terms of subspaces?

Affine sets which are subspaces

The subspaces of $\mathbf{R}^n$ are the affine sets which contain the origin.

Proposition 1. *$M \subset \mathbf{R}^n$ is a subspace if and only if M is affine and $0 \in M$.*

Proof. ($\Rightarrow$) Suppose M is a subspace. Then $0 \in M$. Also $\alpha x + \beta y \in M$ for all $\alpha, \beta \in \mathbf{R}$ and $x, y \in \mathbf{R}^n$ In particular, $(1-\lambda)x + \lambda y \in M$ for all $\lambda \in \mathbf{R}$, $x, y \in \mathbf{R}^n$. In other words, M contains the line through x and y.

($\Leftarrow$) Suppose M is affine and $0 \in M$. M is closed under scalar multiplication since

$$\alpha x = (1-\alpha)0 + \alpha x$$

is in the line through 0 and x. M is closed under vector addition since

$$(1/2)(x+y) = (1-1/2)x + (1/2)y$$

is in the line through x and y. Thus, $x + y = 2(1/2)(x+y) \in M$. $\square$

Affine sets as translated subspaces

Proposition 2. *Suppose $M \neq \varnothing$ is affine. Then there exists a unique subspace L and vector $a \in \mathbf{R}^n$ for which $M = L + a$. Moreover,*

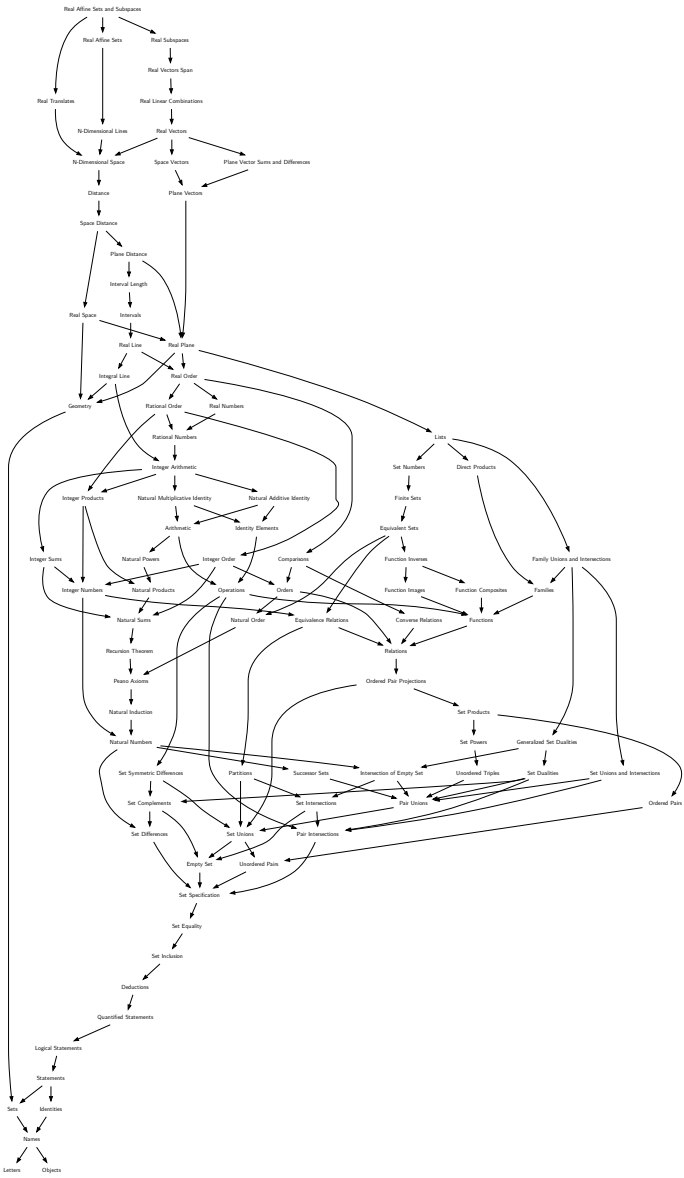
$$L = M - M = \{x - y \mid x, y \in M\}.$$

Proof. First, uniqueness. Suppose L_1 and L_2 are subspaces parallel to M. We will show that $L_2 \supset L_1$ (and similarly, $L_1 \supset L_2$).

Since L_1 and L_2 are both parallel to M, they are also parallel to each other. Consequently, there exists $a \in \mathbf{R}^n$ with $L_2 = L_1 + a$. Since $0 \in L_2$ (it is a subspace, after all), $-a \in L_1$. Since L_1 is a subspace, $a \in L_1$.

So $x + a \in L_1$ for every $a \in L_1$, and so $L_2 = L_1 + a \subset L_1$. A similar argument gives $L_1 \supset L_2$.

If $y \in M$, then $M + (-y) = M - y$ is a translate of M containing zero (since $y - y = 0$). In other words, the affine set $M - y$ is a subspace. This, then, is the unique subspace parallel to M. Since y was arbitrary, the subspace parallel to M is $L = \cup_{y \in M} M - y = \{x - y \mid x, y \in M\}$. $\square$


Real Affine Sets and Subspaces
Real Affine Sets
Real Subspaces
Real Vectors Span
Real Translates
Real Linear Combinations
N-Dimensional Lines
Real Vectors
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
Distance
Plane Vectors
Space Distance
Plane Distance
Interval Length
Real Space
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Real Plane
Integral Line
Real Order
Geometry
Rational Order
Real Numbers
Lists
Rational Numbers
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Arithmetic
Identity Elements
Equivalent Sets
Integer Sums
Natural Powers
Integer Order
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Operations
Orders
Function Images
Function Composites
Families
Natural Sums
Natural Order
Equivalence Relations
Converse Relations
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Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Partitions
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
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Statements
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