Why

We explore matrix-vector multiplication.

Definition

Given a matrix $A \in \mathbf{R}^{m \times n}$ and a vector $x \in \mathbf{R}^n$, the *product* of A *with* x is the vector $y \in \mathbf{R}^m$ defined by

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \quad i = 1, \ldots, m.$$

Notation

We denote the product of A with x by Ax. With which we concisely write the system of linear equations (A, b) as $b = Ax$.

This notation suggests both algebraic and geometric interpretations of solving systems of linear equations. The algebraic interpretation is that we are interested in the invertibility of the function $x \mapsto Ax$. In other words, we are interested in the existence of an inverse element of A. The geometric interpretation is that A transforms the vector x.

Conversely, we can view x as transforming (acting on) A. Let $a^j \in \mathbf{R}^m$ denote the jth column of A, then

$$Ax = \sum_{j=1}^{n} x_j a^j.$$

In other words, y is linear combination of the columns of A.

Properties

We call the function $f : \mathbf{R}^n \to \mathbf{R}^m$ defined by $f(x) = Ax$ the *matrix multiplication function* (or *matrix-vector product function*) associated with A. f is satisfies the following two important properties:

1. $A(x + y) = Ax + Ay$

2. $A(\alpha x) = \alpha A x$.

We call such a function f *linear*. In other words, the matrix multiplication function is linear. Conversely, if $g : \mathbf{R}^n \to \mathbf{R}^m$ is linear, there exists a matrix inducing g.

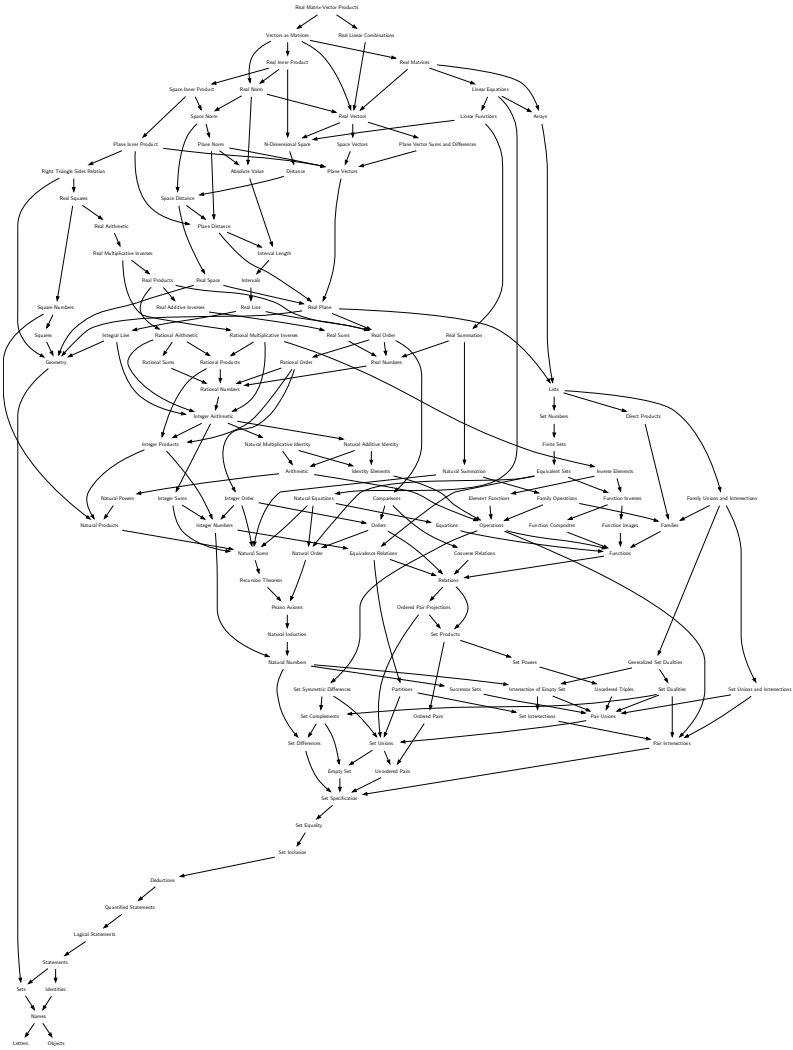
Proposition 1. *Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be linear. Then there exists a unique $A \in \mathbf{R}^{m \times n}$ satisfying $f(x) = Ax$ for all $x \in \mathbf{R}^n$.*

Proof. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. □

Moreover, this matrix representation of f is unique.

Proposition 2. *If $A, B \in \mathbf{R}^{m \times n}$ are two matrices so that $f(x) = Ax = Bx$, then $A = B$.*

Proof. We have $Ax - Bx = 0$ so $(A - B)x = 0$ for every x. In particular $y^\top (A - B)x = 0$ for every $x \in \mathbf{R}^n, y \in \mathbf{R}^m$. In particular, $e_i^\top (A - b)e_j = 0$. Conclusion: $A_{ij} - B_{ij} = 0$, and conclude that $A_{ij} = B_{ij}$. Thus, $A = B$. Evaluate f at the standard unit vectors e_i. The ith component of e_i is 1 and all other components are 0. □


Real Matrix-Vector Products
Vectors as Matrices
Real Linear Combinations
Real Inner Product
Real Matrices
Space Inner Product
Real Norm
Linear Equations
Space Norm
Real Vectors
Arrays
Linear Functions
Plane Inner Product
Plane Norm
N-Dimensional Space
Space Vectors
Plane Vector Sums and Differences
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Plane Distance
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Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Set Powers
Generalized Set Dualities
Set Symmetric Differences
Successor Sets
Intersection of Empty Set
Unordered Triples
Set Dualities
Set Unions and Intersections
Set Complements
Functions
Ordered Pairs
Set Intersections
Pair Unions
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deduction
Quantified Statements
Logical Statements
Statements
Sets
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Names
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Objects