PAC LEARNABLE HYPOTHESES

Why

We want to talk about an algorithm which can learn from a hypothesis class, regardless of the underlying distribution.

Definition

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable input and output spaces.

A hypothesis class $\mathcal{H}$ of measurable functions from X to Y is *probably approximately correct learnable* (or *PAC learnable*) if (a) there exists an inductor $\mathcal{A} : (X \times Y)^n \to \mathcal{H}$, so that (b) for every underlying measure μ and labeling function $f : X \to Y$ (c) for every $\varepsilon, \delta \in (0, 1)$ (d) there exists $m_0 \in \mathsf{N}$ so that (e) for all $m \geq m_0$

$$\mu^m \left[\mu \left[\mathcal{A}((x_i, f(x_i))_{i=1}^n)(\xi) \neq f(\xi) \right] \leq \varepsilon \right] \geq 1 - \delta \qquad (1)$$

where $x \in X^m$ and $\xi \in X$. In this case we say that the inductor (or learning algorithm) $\mathcal{A}$ *PAC learns* $\mathcal{H}$.

We interpret this as follows: "no matter the underlying distribution and correct labeling function, if someone specifies an accuracy and confidence we can tell them the number of samples they need so that the inductor outputs a hypothesis which is probably approximately correct."

Some authors require that the hypothesis class be realizable with respect to the underlying distribution and correct labeling function. This is natural, because if the hypothesis class includes the correct labeling function f, then it is realizable. In this case they refer to the above definition as the *agnostic PAC model*. We emphasize here that there this definition includes the notion of realizability. In other words. We emphasize again that this definition

contains to "approximation parameters." The accuracy parameter ε corresponds to the "approximately correct" piece and the confidence parameters δ corresponds to the "probably" piece.

Sample complexity

Note that the existence of an m_0 above for each ε and δ is equivalent to requiring that there exists $m_0 : (0,1)^2 \to \mathsf{N}$ so that for all $m \geq m_0(\varepsilon, \delta)$ the condition in Equation 1 holds. There may exist multiple such m_0, so we define $\tilde{m} : (0,1)^2 \to \mathsf{N}$ so that $\tilde{m}(\varepsilon, \delta)$ is the smallest integer so that Equation (1) holds. We call $\tilde{m}$ the *sample complexity*. Clearly it is a function of ε and δ. It also depends on the hypothesis class $\mathcal{H}$ and the learning algorithm $\mathcal{A}$.

PAC Learnable Hypotheses

Hypothesis Classes   Probably Approximately Correct Inductors

Supervised Probabilistic Data Models

Independent and Identically Distributed Random Variables

Predictors   Independent Random Variables

Independent Sigma Algebras   Random Variables

Dependent Events   Measurable Functions   Probability Measure   Topological Sigma Algebra   Approximation

Measures   Conditional Event Probabilities   Topological Space   Similarity Functions

Sigma Algebras   Event Probabilities   Metrics

Real Series   Extended Real Numbers   Distance

Real Limits   Space Distance

Absolute Value   Plane Distance

Probability Distribution   Interval Length   Real Space

Intervals

Real Sequences   Real Line   Real Plane

Sequences   Real Summation   Real Optimizers   Real Order   Integral Line

Lists   Subset Algebra   Real Numbers   Rational Order   Geometry

Direct Products   Set Numbers   Cardinality   Rational Numbers

Finite Sets   Optimizers   Integer Arithmetic

Equivalent Sets   Natural Summation   Natural Multiplicative Identity   Natural Additive Identity   Integer Products

Family Unions and Intersections   Function Inverses   Family Operations   Comparisons   Identity Elements   Arithmetic   Integer Order   Integer Sums

Families   Function Composites   Function Images   Operations   Orders   Natural Powers   Integer Numbers

Functions   Converse Relations   Equivalence Relations   Natural Order   Natural Products

Relations   Natural Sums

Subset Systems   Set Products   Recursion Theorem

Ordered Pairs   Set Powers   Peano Axioms

Uncertain Outcomes   Natural Induction

Generalized Set Dualities   Unordered Triples   Natural Numbers

Set Unions and Intersections   Intersection of Empty Set   Set Dualities   Partitions   Successor Sets   Set Symmetric Differences

Set Intersections   Pair Unions   Set Complements

Pair Intersections   Set Unions   Set Differences

Unordered Pairs   Empty Set

Set Specification

Set Inclusion

Standardized Accounts

Accounts

Deductions

Quantified Statements

Logical Statements

Statements

Identities   Sets

Names

Letters   Objects

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