Why

Given a probability measure on the events of a set of outcomes, we can discuss probabilities of outcome variables.

Definition

Let $p : \Omega \to \mathsf{R}$ be a probability distribution with corresponding probability measure $\mathsf{P} : \mathcal{P}(\Omega) \to \mathsf{R}$. Suppose $x : \Omega \to V$ is an outcome variable. The *probability* $x = a$, for $a \in \Omega$, is

$$\mathsf{P}(\{\omega \in \Omega \mid x(\omega) = a\}).$$

From the definition of P, we express the above as

$$\textstyle\sum_{\omega \in \Omega | x(\omega) = a} p(\omega).$$

We refer to the probability of the *event* that $x = a$.

Notation

We denote the probability that $x = a$ by $\mathsf{P}[x = a]$. Our square brackets deviate from the slightly slippery but universally standard notation $\mathsf{P}(x = a)$. We prefer the square brackets, since $x = a$ is not itself an argument to P, but shorthand for $(\{\omega \in \Omega \mid x(\omega) = a\})$.

Accordingly, there are many similar notations. For example $V = \mathsf{R}$, $\mathsf{P}[x \geq a]$ is $\mathsf{P}(\{\omega \in \Omega \mid x(\omega) \geq a\})$. Or, $\mathsf{P}[x \in C]$ means $\mathsf{P}(\{x \in \Omega \mid x(\omega) \in C\})$. Since the *event* that $x = a$ is the inverse image of $\{a\}$ under x, we also use the notation $x^{-1}(a)$. Or generally, $x^{-1}(C)$.

Example: sum of two dice

Define $\Omega = \{1, \dots, 6\}^2$ and define $p : \Omega \to \mathsf{R}$ with $p(\omega) = 1/36$ for each $\omega \in \Omega$. Define $x : \Omega \to \mathsf{N}$ by $x(\omega_1, \omega_2) = \omega_1 + \omega_2$. Then $\mathsf{P}[x = 4] = p((2,2)) + p(1,3) + p(3,1) = 1/12$.

Induced probabiliity

For $x : \Omega \to V$, the events $x^{-1}(a)$ for $a \in V$ partition Ω. Define $q : V \to \mathsf{R}$ by

$$q(a) = \mathsf{P}[x = a].$$

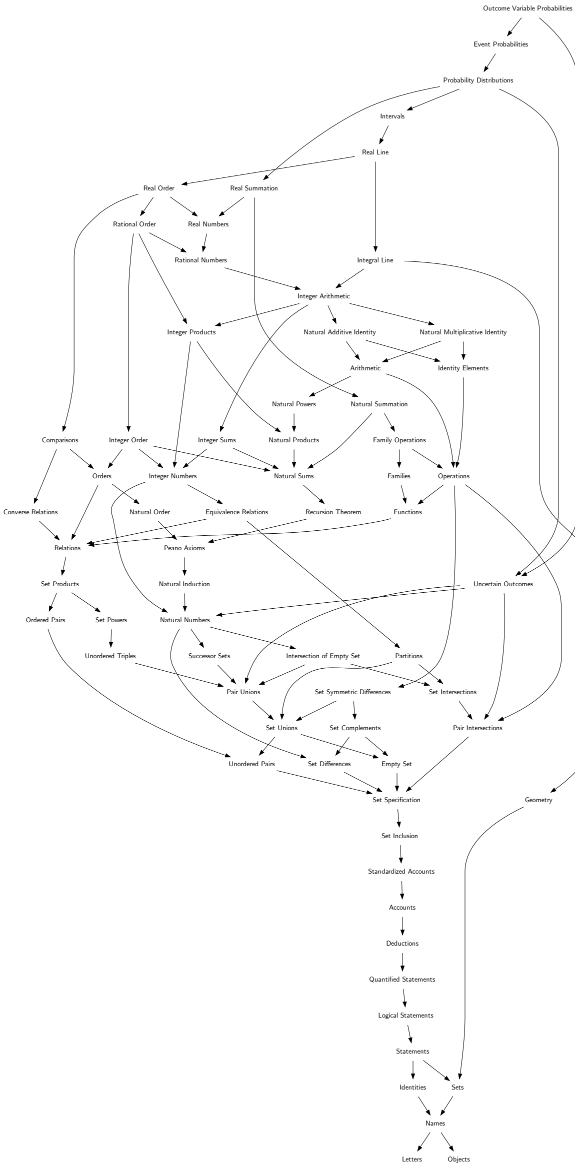
Since $\cup_{a \in V} x^{-1}(a) = \Omega$, $\sum_{a \in A} q(a) = 1$.

We call q the *induced distribution* (or *induced probability mass function*) of the random variable x. It is common to denote it by p_x. Thus we can think of V as a set of outcomes, which we call the outcomes *induced* by x.

If $x : \Omega \to V$ is a random variable and $f : V \to U$, then if we define $y : \Omega \to V$ so that $y \equiv f(x)$, y is a random variable with induced distribution $p_y : \Omega \to \mathsf{R}$ satisfying

$$p_y(b) = \textstyle\sum_{a \in V | y(a) = b} p_x(a).$$

As a matter of practical computation, we can evaluate probabilities having to do with the outcome variable x using p_x instead of p. For example with x as in the example above, $\mathsf{P}(x = 4 \text{ or } x = 5) = p_x(4) + p_x(5)$, rather than $\sum_{\omega \in \Omega | x(\omega) = 4 \text{ or } x(\omega) = 5} p(\omega)$.


Outcome Variable Probabilities
Event Probabilities
Probability Distributions
Intervals
Real Line
Real Order
Real Summation
Rational Order
Real Numbers
Rational Numbers
Integral Line
Integer Arithmetic
Integer Products
Natural Additive Identity
Natural Multiplicative Identity
Arithmetic
Identity Elements
Natural Powers
Natural Summation
Comparisons
Integer Order
Integer Sums
Natural Products
Family Operations
Orders
Integer Numbers
Natural Sums
Families
Operations
Converse Relations
Natural Order
Equivalence Relations
Recursion Theorem
Functions
Relations
Peano Axioms
Set Products
Natural Induction
Uncertain Outcomes
Ordered Pairs
Set Powers
Natural Numbers
Unordered Triples
Successor Sets
Intersection of Empty Set
Partitions
Pair Unions
Set Symmetric Differences
Set Intersections
Set Unions
Set Complements
Pair Intersections
Unordered Pairs
Set Differences
Empty Set
Set Specification
Geometry
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects