Why

We want to talk about a set with a prescribed quantitative degree of closeness (or distance) between its elements.

Definition

The correspondences which serve as a degree of closeness, or measure of distance, must satisfy our previously developed (see Distance) notion of distance.

A function on ordered pairs which does not depend on the order of the elements so considered is *symmetric*. A function into the real numbers which takes only nonnegative values is *nonnegative*. A repeated pair is an ordered pair of the same element twice. A function which satisfies a triangle inequality for any three elements is *triangularly transitive*.

A *metric* (or *distance function*) is a function on ordered pairs of elements of a set which is symmetric, non-negative, zero only on repeated pairs, and triangularly transitive. A *metric space* is an ordered pair whose first coordinate is a nonempty set and whose second coordinate is a metric.

In a metric space, we say that one pair of objects is *closer* together if the metric of the first pair is smaller than the metric value of the second pair.

Notice that a set can be made into different metric spaces by using different metrics.

Notation

Let A be a set. We commonly denote a metric by the letter d, as a mnemonic for "distance." Let $d : A \times A \to \mathbf{R}$. Then d is a metric if:

1. it is non-negative, which we tend to denote by

$$d(a, b) \geq 0 \quad \forall a, b \in A.$$

2. it is 0 only on repeated pairs, which we tend to denote by

$$d(a, b) = 0 \longleftrightarrow a = b, \quad \forall a, b \in A.$$

3. it is symmetric, which we tend to denote by:

$$d(a, b) = d(b, a), \quad \forall a, b \in A.$$

4. it is triangularly transitive, which we tend to denote by

$$d(a, b) \leq d(a, c) + d(c, b), \quad \forall a, b, c \in A.$$

As usual, we denote the metric space of A with d by (A, d). Another common choice of letter for a metric is ρ.

Examples

$\mathbf{R}$ with the absolute value distance is a metric space. As is $\mathbf{R}^2$ and $\mathbf{R}^3$ with the Euclidean distance. $\mathbf{R}^n$ with Euclidean metric is an example of a metric space for which the objects (n-dimensional tuples of real numbers) are impossible to visualize.

Metrics

Distance

Absolute Value

Space Distance

Plane Distance

Interval Length

Real Space

Intervals

Real Plane

Real Line

Real Order

Integral Line

Geometry

Rational Order

Real Numbers

Lists

Rational Numbers

Direct Products

Natural Fractions

Integer Arithmetic

Set Numbers

Integer Products

Natural Multiplicative Identity

Natural Additive Identity

Finite Sets

Natural Arithmetic

Identity Elements

Equivalent Sets

Integer Order

Integer Sums

Natural Powers

Comparisons

Function Inverses

Family Unions and Intersections

Integer Numbers

Natural Products

Orders

Operations

Function Composites

Function Images

Families

Natural Sums

Equivalence Relations

Natural Order

Converse Relations

Functions

Recursion Theorem

Relations

Peano Axioms

Ordered Pair Projections

Natural Induction

Set Products

Natural Numbers

Generalized Set Dualities

Set Powers

Set Symmetric Differences

Partitions

Successor Sets

Intersection of Empty Set

Set Dualities

Unordered Triples

Set Unions and Intersections

Set Complements

Set Intersections

Pair Unions

Ordered Pairs

Set Differences

Set Unions

Pair Intersections

Empty Set

Unordered Pairs

Set Specification

Set Equality

Set Inclusion

Deductions

Quantified Statements

Logical Statements

Statements

Identities

Sets

Names

Letters

Objects