Why

We generalize our notion of norm on real vectors to abstract vector spaces.

Definition

A *norm* is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisifies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms on those vectors.

A *normed space* (or *norm space*) is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

Notation

Let $(X, \mathbf{F})$ be a vector space where $\mathbf{F}$ is the field of real numbers or the field of complex numbers. Let $f : X \to \mathbf{R}$. The functional f is a norm if

1. $f(v) \geq 0$ for all $x \in V$
2. $f(v) = 0$ if and only if $x = 0 \in X$.
3. $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in \mathbf{F}$, $x \in X$
4. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.

In this case, for $x \in X$, we denote $f(x)$ by $\|x\|$, read aloud "norm x". The notation follows the notation for the absolute value function is a norm on the vector space of real numbers. In some cases, we go further, and for a norm indexed by some parameter α or set A we write $\|x\|_\alpha$ or $\|x\|_A$.

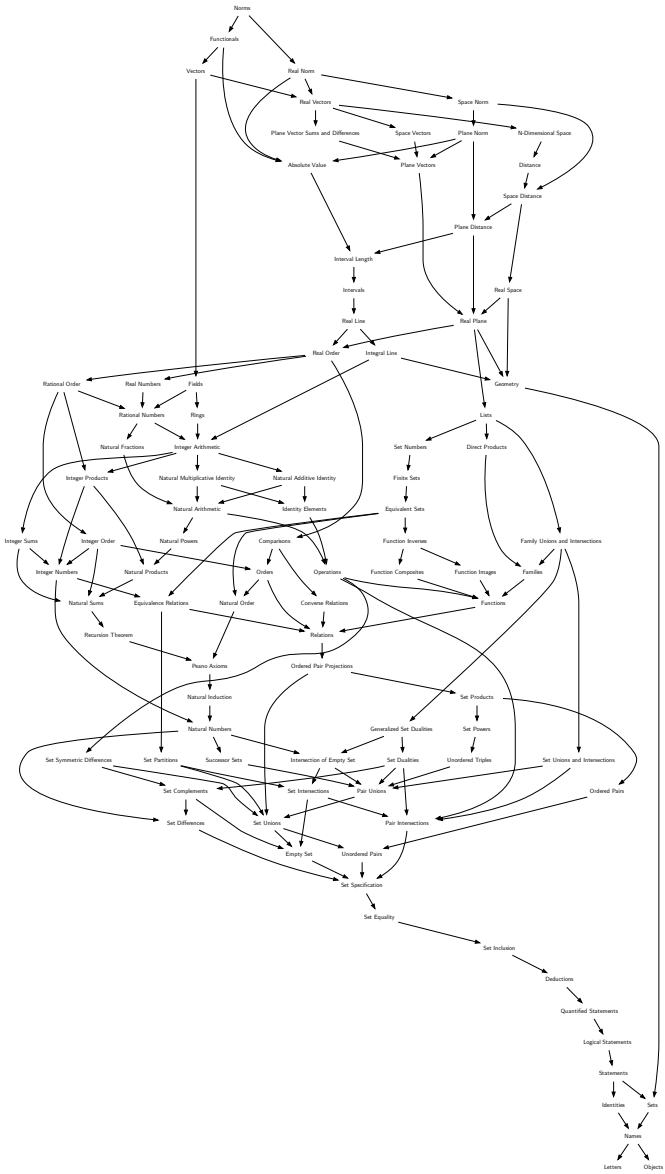
When the field is assumed or clear from context, it is succinct to say let $(V, \|\cdot\|)$ be a normed space.

Examples

The absolute value function is a norm on the vector space of real numbers. In addition, the (Euclidean norm) is a norm on the vector space $\mathbf{R}^n$.

Other terminology

The descriptive but slight more verbose *norm vector space* and *normed vector space* are also in usage.


Norms
Functionals
Vectors
Real Norm
Real Vectors
Space Norm
Plane Vector Sums and Differences
Space Vectors
Plane Norm
N-Dimensional Space
Absolute Value
Plane Vectors
Distance
Space Distance
Plane Distance
Interval Length
Intervals
Real Space
Real Line
Real Plane
Real Order
Integral Line
Geometry
Rational Order
Real Numbers
Fields
Rational Numbers
Rings
Lists
Natural Fractions
Integer Arithmetic
Set Numbers
Direct Products
Integer Products
Natural Multiplicative Identity
Natural Additive Identity
Finite Sets
Natural Arithmetic
Identity Elements
Equivalent Sets
Integer Sums
Integer Order
Natural Powers
Comparisons
Function Inverses
Family Unions and Intersections
Integer Numbers
Natural Products
Orders
Operations
Function Compositions
Function Images
Families
Natural Sums
Equivalence Relations
Natural Order
Converse Relations
Functions
Recursion Theorem
Relations
Peano Axioms
Ordered Pair Projections
Natural Induction
Set Products
Natural Numbers
Generalized Set Dualities
Set Powers
Set Symmetric Differences
Set Partitions
Successor Sets
Intersection of Empty Set
Set Dualities
Unordered Triples
Set Unions and Intersections
Set Complements
Set Intersections
Pair Unions
Ordered Pairs
Set Differences
Set Unions
Pair Intersections
Empty Set
Unordered Pairs
Set Specification
Set Equality
Set Inclusion
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Names
Letters
Objects