



## Why

The set of vector which map to 0 under a given linear transformation is a subspace.

## Definition

The *nullspace* (or *kernel*) of a matrix  $A \in \mathbf{R}^{m \times n}$  is the set

$$\{x \in \mathbf{R}^n \mid Ax = 0\}.$$

It is the set of vectors mapped to zero by  $A$ . Equivalently, it is the set of vectors orthogonal to the rows of  $A$ .

## Notation

We denote the nullspace of  $A \in \mathbf{R}^{m \times n}$  by  $\text{null}(A) \subset \mathbf{R}^n$ . Some authors denote the nullspace of  $A$  by  $\mathcal{N}(A)$ .

## A subspace

Of course, the nullspace of a matrix is a subspace There are a few routes to see this.

The first is direct. If  $w, z \in \text{null}(A)$ , then  $Aw = 0$  and  $Az = 0$ . So then  $A(w + z) = Aw + Az = 0$ . So  $\text{null}(A)$  is closed under vector addition. Also  $A(\alpha w) = \alpha(Aw) = 0$  for all  $\alpha \in \mathbf{R}$ . [In particular  $A0 = 0$ , so  $0 \in \text{null}(A)$ ; i.e.,  $\text{null}(A)$  contains the origin.] So  $\text{null}(A)$  is closed under scalar multiplication.

The second is by thinking about orthogonal complements. Second, we can view the  $\text{null}(A)$  as the set of vectors orthogonal to all the rows of  $A$ . In other words,  $\text{null}(A) = \{\tilde{a}_1, \dots, \tilde{a}_m\}^\perp$ . The orthogonal complement of any set is a subspace (see **Orthogonal Real Subspaces**).

The dimension of the nullspace is called the *nullity*.

## Ambiguity in solutions

Suppose we have a solution to the system of linear equation with data  $(A, y)$ . In other words, we have a vector  $x \in \mathbf{R}^n$  so that  $y = Ax$ . If we have a vector  $z \in \text{null}(A)$ , then  $x + z$  is also a solution to the system  $(A, y)$ , since

$$A(x + z) = Ax + Az = Ax + 0 = y$$

Conversely, suppose there were another solution  $\tilde{x} \in \mathbf{R}^n$  to the system  $(A, y)$ . Then  $y = Ax = A\tilde{x}$ , so

$$0 = y - y = Ax - A\tilde{x} = A(x - \tilde{x}).$$

Consequently,  $(x - \tilde{x}) \in \text{null}(A)$ , and so  $\tilde{x}$  is the solution  $x$  plus some vector in the null space of  $A$ . Consequently we are interested in whether  $A$  has vectors in its nullspace.

## Zero nullspace

The origin  $0$  is always in the nullspace of  $A$ . However, this vector does not mean that we can find different solutions, since  $x + 0 = x$  for all  $x \in \mathbf{R}^n$ . If, on the other hand, there is a nonzero vector  $z \in \text{null}(A)$ , then  $x + z \neq x$ , and  $x + z$  is a solution for  $(A, y)$ . We think about  $A$  as a function from  $\mathbf{R}^n$  to  $\mathbf{R}^m$ . In the case that there is a nonzero element in the nullspace,  $A$  maps different vectors to the same vector. Here,  $x$  and  $x + z$  both map to  $y$ . In this case, the function is *not invertible*, because it is not one-to-one. If, however, zero is the only element of the null space, the function is one-to-one. So call  $A$  *one-to-one* if  $\text{null}(A) = 0$ .

## Equivalent statements

A matrix  $A \in \mathbf{R}^{m \times n}$  is *one-to-one* if the linear function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  defined by  $f(x) = Ax$  is one-to-one. In this case, if there exists  $x \in \mathbf{R}^n$  so that  $y = Ax$ , then there is only one such  $x$ . Different elements in  $\mathbf{R}^n$  map to different elements in  $\mathbf{R}^m$ .



