



# Hyperplanes

## 1 Why

## 2 Definition

A *hyperplane* in  $n$ -dimensional space is an  $(n - 1)$ -dimensional affine set.

Since the  $n - 1$ -dimensional subspaces are the orthogonal complements of the one-dimensional subspaces, they are the sets which can be specified by

$$\{x \in \mathbf{R}^n \mid x \perp b\}$$

for  $b \in \mathbf{R}^n$ . The hyperplanes are translates of these,

$$\begin{aligned} \{x \in \mathbf{R}^n \mid x \perp b\} + a &= \{x + a \mid \langle x, b \rangle = 0\} \\ &= \{y \mid \langle y - a, b \rangle = 0\} = \{y \mid \langle y, b \rangle = \beta\}, \end{aligned}$$

where  $\beta = \langle a, b \rangle$ .

## 3 Characterization

**Proposition 1.**  *$H \subset \mathbf{R}^n$  is a hyperplane if and only if there exists  $\beta \in \mathbf{R}$  and nonzero  $b \in \mathbf{R}^n$  so that*

$$H = \{x \in \mathbf{R}^n \mid \langle x, b \rangle = \beta\}.$$

**Remark 2.**  *$b$  and  $\beta$  are unique up to a common nonzero multiple. For example,  $b, \beta$  and  $2b, 2\beta$  give the same hyperplane.*

**Remark 3.** *The vector  $b$  is called a normal to the hyperplane.*