Why

We want to discuss real-valued random functions whose family of random variables have simple densities.[1]

Definition

A *normal random function* is a real-valued random function whose family of real-valued random variables has the property that any subfamily is jointly normal.

For this reason, we call the family of random variables (or stochastic process) corresponding to the random function a *gaussian process* or *normal process*.

Notation

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space and A a set. Let $x : \Omega \to (A \to \mathsf{R})$ be a random function with family $y : A \to (\Omega \to \mathsf{R})$.

The random function x is a normal if, for all $a^1, \ldots, a^m \in A$, $(y(a^1), \ldots, y(a^m))$ is jointly normal.

Mean and covariance function

Proposition 1. *Let $x : \Omega \to (A \to \mathsf{R})$ be a normal random function with family $X : A \to (\Omega \to \mathsf{R})$. There exists unique functions $m : A \to \mathsf{R}$ and $k : A \times A \to \mathsf{R}$ so that the mean of the random variable X_a is $m(a)$ for all A and the covariance of*

[1] Future editions will expand.

the random variables X_a and $X_{a'}$ is $k(a, a')$ for all $a, a' \in A$.[2]

For this reason, we call m the *mean function* and k the *covariance function* of the random function.

Conversely, let $m : A \to \mathbf{R}$ and $k : A \times A \to \mathbf{R}$. Then if k satisfies the property that for all $a^1, \ldots, a^m$, the $m \times m$ matrix

$$\begin{pmatrix} k(a^1, a^1) & \cdots & k(a^1, a^m) \\ \vdots & \ddots & \vdots \\ k(a^m, a^1) & \cdots & k(a^m, a^m) \end{pmatrix}$$

is positive semidefinite, then we can construct a Gaussian process with mean function m and covariance function k.[3] For this reason, we call k with such a property *positive semidefinite* or a *covariance function*. Notice, of course, that k is symmetric. The matrix above is sometimes called the *Gram matrix* for k and $a^1, \ldots, a^m$.

Example

Let $A = \{1, \ldots, n\}$ and let $K \in \mathbf{R}^{n \times n}$ be symmetric positive semidefinite. Define $m : A \to \mathbf{R}$ to be $m \equiv 0$ (the constant zero function) and $k(i, j) = K_{ij}$. Then the normal random function $x : \Omega \to (A \to \mathbf{R})$ with mean m and covariance k is in

[2] Future editions may include an account.

[3] Some authors belabor this point because of the natural inclination to want to specify an *inverse* covariance function, which need not satisfy the consistency property. The consistency property ensures that any marginal of a subfamilys density is the density of that further subfamily. Future editions may expand.

one to one correspondence with the gaussian random vectors with mean zero.

Normal Random Variables
Random Functions
Multivariate Normals
Positive Definite Matrices
Quadratic Forms
Symmetric Matrices
Statistical Distributions
Multivariate Real Densities
Matrix Determinants
Matrix Inverses
Matrix Traces
Probability Densities
Matrix-Matrix Products
Real Matrices
Matrix Transposes
Real Integrals
Random Vectors
Random Variables
Matrix-Vector Products
Identity Matrices
Nonnegative Integrals
Real Vectors
Linear Combinations
Matrices
Simple Integrals
Topological Sigma Algebra
Measurable Functions
Probability Measures
Measures
Exponential Function
N-Dimensional Spaces
Metrics
Real Square Roots
Distances
Extended Real Numbers
Outcome Variables
Real Squares
Real Limits
Space Distances
Real Additions
Absolute Value
Plane Distances
Simple Functions
Event Probabilities
Real Multiplication Inverses
Real Functions
Internal Sums
Probability Distributions
Real Products
Sums
Sigma Algebras
Real Addition Inverses
Real Subtractions
Real Lines
Real Plane
Real Numbers
Rational Multiplication Inverses
Rational Additions
Real Sums
Fields
Generalizations
Real Summations
Real Differences
Integral Line
Rational Products
Rational Sums
Real Numbers
Groups
Set Numbers
Geometry
Uncertain Numbers
Rational Numbers
Integer Addition Inverses
Finite Sets
Integer Additions
Integer Directions
Natural Summations
Direct Products
Natural Square Roots
Integer Products
Integer Sums
Reciprocals
Natural Multiplication Identity
Natural Addition Identity
Discrete Functions
Finite Cardinalities
Permutations
Subset Algebras
Square Numbers
Natural Equations
Function Inverses
Arithmetic
Identity Elements
Algebras
Orders
Finely Orders and Inversations
Cardinality
Set Cardinals
Natural Powers
Equations
Integer Squares
Natural Order
Function Images
Unions Cartesian
Natural Powers
Operations
Families
Natural Powers
Operations
Recreations Relations
Converse Relations
Relations
Natural Sums
Sums Theorem
Set Products
Subset Systems
Characteristic Functions
Ordered Pairs
Set Powers
Peano Axioms
Unordered Triples
Natural Induction
Natural Sums
Generalized Set Qualities
Partitions
Intersection Sets
Intersection of Empty Set
Set Qualities
Set Orders and Intersections
Set Symmetric Differences
Set Intersections
Set Unions
Set Complements
Set Differences
Set Unions
Set Intersections
Unordered Pairs
Empty Set
Set Specifications
Set Inclusion
Standardized Accounts
Accounts
Deductions
Quantified Statements
Logical Statements
Statements
Identities
Sets
Symbols
Letters
Digits