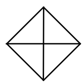


Why

We want to solve linear equations. Our approach is to "eliminate" variables from equations in our system. Once we reach an equation in one variable, we will back-substitute to solve.

Two-variable example

Suppose we want to find $x_1, x_2 \in \mathbf{R}$ to satisfy

$$3x_1 + 2x_2 = 10, \text{ and}$$
$$6x_1 + 5x_2 = 20.$$

We can list the coefficients in a two-dimensional array $A = (3, 2; 6, 5)$ and $b = (10, 20)$. We can eliminate x_1 from the second equation by subtracting twice the first equation from the second. In doing so we obtain the system of equations

$$3x_1 + 2x_2 = 10 \text{ and}$$
$$x_2 = 0.$$

The key insight is that this system has the *same solution set.* We call the process of moving between these two systems a *row reduction.*

Four-variable example

What if instead we have four unknowns? Suppose

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

We might first eliminate x_1 (the variable associated with the first column of coefficients) from the remaining three equations to obtain the linear system $S_1 = (A^1, b^1)$ in which

$$A^1 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \text{ and } b^1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The trick is that, since $A'_{22} \neq 0$, we can take the same route to eliminate x_2, to obtain the system $S_2 = (A^2, b^2)$ in which

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \text{ and } b^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Likewise for x_3, we obtain $S_3 = (A^3, b^3)$ in which

$$A^3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } b^3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Here, as in the two-variable case, the key insight is that all these systems have the same solution set and the last one, (A^3, b^3), is easy to solve. We solve it by *back substitution.* First, since $2x_4 = 3$, we find $x_4 = {}^3\!/\!_2$. Second, since $2x_3 + 2x_4 = -1$, we find $x_3 = -2$. Similarly we find $x_2 = {}^1\!/\!_2$ and $x_3 = {}^5\!/\!_4$.

Definition

Let $S = (A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system. The lower *row reduction* of S for index i with $A_{ii} \neq 0$ (or the i-row reduction) is the linear system $\tilde{A}_{st} = A_{st} - (A_{sj}/A_{ij})A_{it}$ if $i < s \leq m$ and A_{st} otherwise. We say that the system (A, b) is *ordinarily reducible.*

Let $a^k, \tilde{a}^k \in \mathbf{R}^n$ denote the kth row of A and $\tilde{A}$, respectively. Then if $k \neq i$, $\tilde{a}^k = a^k - \alpha_k a^i$ where $\alpha_k = A_{kj}/A_{ij}$. In other words, a row k

of the matrix $\tilde{A}$ is obtained by subtracting a multiple of the ith row of matrix A from row k of matrix A. We are "reducing" the rows of A.

Proposition 1. *Let $(A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^n)$ be a linear system which row reduces to (C, d). Then $x \in \mathbf{R}^n$ is a solution of (A, b) if and only if it is a solution of (C, d).*[1]

First we reduce by subtracting twice row 1 from row 2, four times row 1 from row 3, and three times row 1 from row 4.

$$S_1 = (\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}).$$

We then subtract three times row 2 from row 3 and four times row 2 from row 4 to obtain

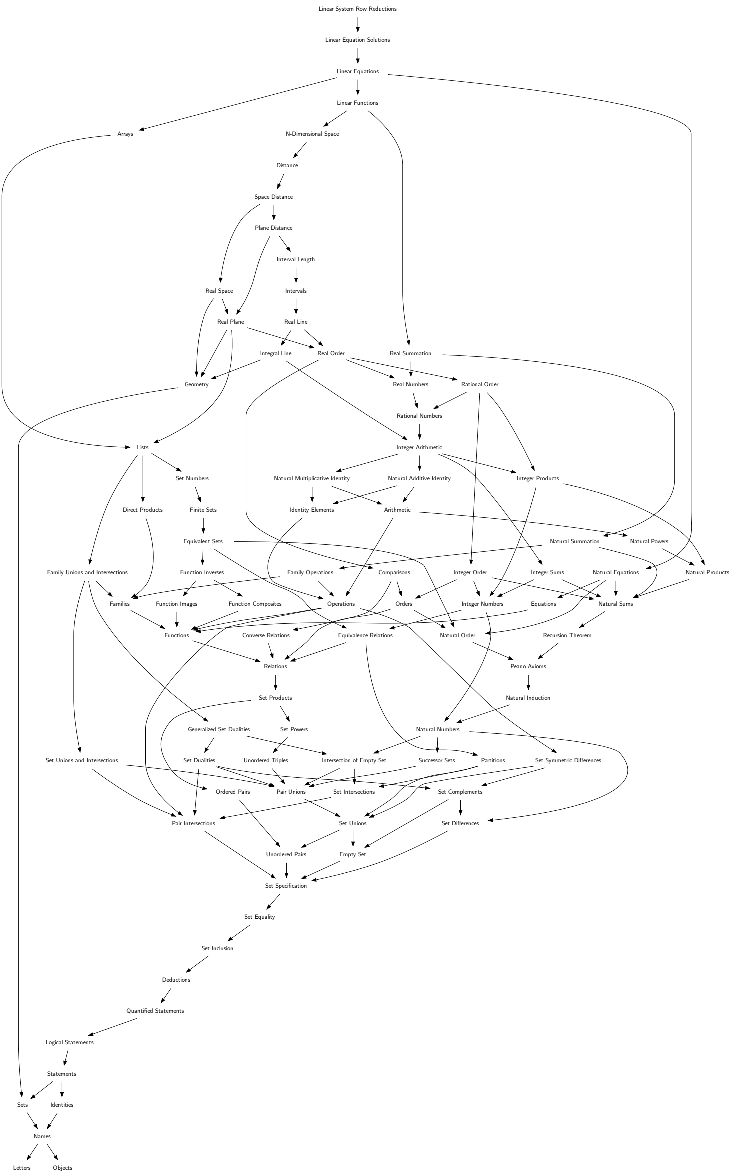
$$S_2 = (\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}).$$

Finally, we subtract two times row 3 from row 4 to obtain S_4, which we can write as

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1, \\ x_2 + x_3 + x_4 &= 0, \\ 2x_3 + 2x_4 &= -1, \quad \text{and} \\ 2x_4 &= 3. \end{aligned}$$

We can now back-substitute to find $x_4 = 3/2$, $x_3 = -2$, $x_2 = 1/2$ and $x_1 = 5/4$. Proposition **??** says that this is the only solution of S, as well.

[1] Future editions will include an account.


Linear System Row Reductions
Linear Equation Solutions
Linear Equations
Linear Functions
Arrays
N-Dimensional Space
Distance
Space Distance
Plane Distance
Interval Length
Real Space
Intervals
Real Plane
Real Line
Integral Line
Real Order
Real Summation
Geometry
Real Numbers
Rational Order
Rational Numbers
Lists
Integer Arithmetic
Set Numbers
Natural Multiplicative Identity
Natural Additive Identity
Integer Products
Direct Products
Finite Sets
Identity Elements
Arithmetic
Equivalent Sets
Natural Summation
Natural Powers
Family Unions and Intersections
Function Inverses
Family Operations
Comparisons
Integer Order
Integer Sums
Natural Equations
Natural Products
Families
Function Images
Function Composites
Operations
Orders
Integer Numbers
Equations
Natural Sums
Functions
Converse Relations
Equivalence Relations
Natural Order
Recursion Theorem
Relations
Peano Axioms
Set Products
Natural Induction
Generalized Set Dualities
Set Powers
Natural Numbers
Set Unions and Intersections
Set Dualities
Unordered Triples
Intersection of Empty Set
Successor Sets
Partitions
Set Symmetric Differences
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Empty Set
Set Specification
Set Equality
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Deductions
Quantified Statements
Logical Statements
Statements
Sets
Identities
Names
Letters
Objects