



## Why

There is a natural predictor corresponding to a normal linear model.

## Definition

Let  $(x : \Omega \rightarrow \mathbf{R}^d, A \in \mathbf{R}^{n \times d}, e : \Omega \rightarrow \mathbf{R}^n)$  be a normal linear model over the probability space  $(\Omega, \mathcal{A}, \mathbf{P})$ .

## Predictive density

We are modeling  $h_\omega : \mathbf{R}^d \rightarrow \mathbf{R}$  by  $h_w(a) = x(\omega)^\top a$ . The *predictive density* for a dataset  $c^1, \dots, c^m \in \mathbf{R}^d$  is the conditional density of the random vector  $(h_{(\cdot)}(c^1), \dots, h_{(\cdot)}(c^m))$  given  $y$ .

**Proposition 1.** *The predictive density for  $c^1, \dots, c^m \in \mathbf{R}^d$  (with data matrix  $C \in \mathbf{R}^{m \times d}$ ) is normal with mean*

$$g(a) = (C\Sigma_x A^\top) \left( A\Sigma_x A^\top + \Sigma_e \right)^{-1} \gamma.$$

*and covariance*

$$C\Sigma_x C^\top - C\Sigma_x A^\top \left( A\Sigma_x A^\top + \Sigma_e \right)^{-1} A\Sigma_x C^\top.$$

*Proof.* Define (as usual)  $y : \Omega \rightarrow \mathbf{R}^n$  and  $z : \Omega \rightarrow \mathbf{R}^m$  by

$$y = Ax + e$$

$$z = Cx.$$

Recognize  $(x, y, z)$  as jointly normal, and use Normal Conditionals). □

## Predictor

The *normal linear model predictor* or *normal linear model regressor* for the normal linear model  $(x, A, e)$  is the predictor which assigns to a new point  $a \in \mathbf{R}^d$  the mean of the predictive density at  $a$ . That is, the predictor  $g : \mathbf{R}^d \rightarrow \mathbf{R}$  defined by

$$g(a) = a^\top \Sigma_x A^\top \left( A \Sigma_x A^\top + \Sigma_e \right)^{-1} \gamma.$$

In the above we have substituted  $a^\top$  for  $C$ . In the case of normal random vectors this corresponds with the MAP estimate and the MMSE estimate.<sup>1</sup> Notice that  $g$  is *linear* in its argument,  $a$ .

The use of a normal linear model predictor is often called *Bayesian linear regression*. The word Bayesian is used in reference to treating the parameters of the function,  $x$ , as a random variable.

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<sup>1</sup>Future editions will have discussed this and include a reference to the sheet.

