



## Why

If we model some measured value as a random variable with induced distribution  $p : V \rightarrow \mathbf{R}$ , then one interpretation of  $p(v)$  for  $v \in V$  is the *proportion* of times in a large number of trials that we *expect* to measure the value  $v$ .<sup>1</sup>

## Definition

Given a distribution  $p : \Omega \rightarrow \mathbf{R}$  and a *real-valued* outcome variable  $x : \Omega \rightarrow \mathbf{R}$ , the *expectation* (or *mean*) of  $x$  under  $p$  is  $\sum_{\omega \in \Omega} p(\omega)x(\omega)$ .

## Notation

We denote the expectation of  $x$  under  $p$  by  $\mathbf{E}(x)$ . When there is no chance of ambiguity, we write  $\mathbf{E}(x)$ .

## Properties

Let  $x, y : \Omega \rightarrow \mathbf{R}$  be two outcome variables and  $p : \Omega \rightarrow \mathbf{R}$  a distribution. Let  $\alpha, \beta \in \mathbf{R}$ . Define  $z = \alpha x + \beta y$  by  $z(\omega) = \alpha x(\omega) + \beta y(\omega)$ . Then  $\mathbf{E}(z) = \alpha \mathbf{E}(x) + \beta \mathbf{E}(y)$ .

## Example: expectation

Suppose  $\Omega = \{1, 2, 3, 4, 5\}$  with  $p(1) = 0.1$ ,  $p(2) = 0.15$ ,  $p(3) = 0.1$ ,  $p(4) = 0.25$  and  $p(5) = 0.4$ . Define  $x : \Omega \rightarrow \mathbf{R}$  by

$$x(a) = \begin{cases} -1 & \text{if } a = 1 \text{ or } a = 2, \\ 1 & \text{if } a = 3 \text{ or } a = 4, \\ 2 & \text{if } a = 5. \end{cases}$$

The expectation of  $x$  under  $p$  is

$$\mathbf{E}x = -1 - 0.15 + 0.1 + 0.25 + 2(0.4) = 0.9.$$

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<sup>1</sup>Future editions may modify this explanation, and take a genetic approach via summary statistics.

## Two routes for computation

Denote by  $p_x : V \rightarrow \mathbf{R}$  the induced distribution of  $x : \Omega \rightarrow V$  (where  $V \subset \mathbf{R}$ ). Then  $\mathbf{E}(x) = \sum_{v \in V} p_x(v)v$  since

$$\begin{aligned}\sum_{\omega \in \Omega} p(\omega)x(\omega) &= \sum_{v \in V} \sum_{\omega \in x^{-1}(v)} x(\omega)p(\omega) \\ &= \sum_{v \in V} v \sum_{\omega \in x^{-1}(v)} p(\omega) \\ &= \sum_{v \in V} x(v)p_x(v).\end{aligned}$$

## Interpretations

We interpret the mean as the center of mass of the induced distribution.



