Simple Integral Additivity

1 Why

If we stack two rectangles, with equal base lengths but different heights, on top of each other, the additivity principle says that the area of the so-formed rectangle is the sum of the areas of the stacked rectangles. Our definition of integral for simple functions has this property, as it ought to.

2 Result

Proposition 1. *The simple non-negative integral operator is additive.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{SF}_+(X)$ denote the non-negative real-valued simple functions on X. Define $s : \mathcal{SF}_+(X) \to [0, \infty]$ by $s(f) = \int f d\mu$ for $f \in \mathcal{SF}_+(X)$.

In this notation, we want to show that $s(f+g) = s(f)+s(g)$ for all $f, g \in \mathcal{SF}_+(X)$. Toward this end, let $f, g \in \mathcal{SF}_+(X)$ with the simple partitions:

$$\{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n \subset \mathcal{A} \quad \text{and} \quad \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset [0, \infty].$$

We consider the refinement of the two partitions. TODO: this is why you don't do the unique maximal partition business. $\{A_i \cap B_j\}_{i,j=1}^{i=m,j=n}$.

First, let $\alpha \in (0, \infty)$. Then $\alpha f \in \mathcal{SF}_+(X)$, with the simple partition $\{A_n\} \subset \mathcal{A}$ and $\{\alpha a_n\} \subset [0, \infty]$.

$$s(\alpha f) = \sum_{i=1}^{n} \alpha a_n \mu(A_i) \quad = \alpha \sum_{i=1}^{n} a_n \mu(A_i) = \alpha s(f).$$

If $\alpha = 0$, then αf is uniformly zero; it is the non-negative simple with partition $\{X\}$ and $\{0\}$. Regardless of the measure of X, this non-negative simple function is zero Recall that we define $0 \cdot \infty = \infty \cdot 0 = 0$. $\square$

Real F

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