Norms

1 Why

We want to measure the size of an element in a vector space.

2 Definition

A **norm** is a real-valued functional that is (a) non-negative, (b) definite, (c) absolutely homogeneous, (d) and satisifies a triangle inequality. The triangle inequality property requires that the norm applied to the sum of any two vectors is less than the sum of the norms.

A **norm space** is an ordered pair: a vector space whose field is the real or complex numbers and a norm on the space. We require the vector space to be over the field of real or complex numbers because of absolute homogeneity: the absolute value of a scalar must be defined.

2.1 Notation

Let (X, F) be a vector space where F is the field of real numbers or the field of complex numbers. Let R denote the set of real numbers. Let $f : X \to R$. The functional f is a norm if

1. $f(v) \geq 0$ for all $x \in V$
2. $f(v) = 0$ if and only if $x = 0 \in X$.
3. $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in F$, $x \in X$
4. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.

In this case, for $x \in X$, we denote $f(x)$ by $|x|$, read aloud "norm x". The notation follows the notation of absolute value as a norm. When we wish to distinguish the norm from the absolute value function, we may write $\|x\|$. In some cases, we go further, and for a norm indexed by some parameter α or set A we write $\|x\|_\alpha$ or $\|x\|_A$.