ADVANCED PLASMAS - Lecture 3

LANDAU DAMPING

Last time we got to the plasma dielectric function

$$1 - \frac{\omega_{pe}^2}{n_e k^2} \int_{-\infty}^{\infty} dv_x \frac{1}{v_x - \omega/k} \frac{df_0}{dv_x}$$

To get the dispersion relation we need to do the integral.

This is problematic at $v_x = \omega/k$

Use the Landau procedure: Let ω be ~~imaginary~~ complex

$$\omega = \omega_R + i\delta$$

ω_R ← real part, $i\delta$ ← imaginary part

Then

$$\int_{-\infty}^{\infty} \frac{dv_x}{v_x - \omega/k} \frac{df_0}{dv_x} = \lim_{\delta \to 0} \left\{ \int_{-\infty}^{\infty} \frac{dv_x}{v_x - (\omega_R + i\delta)/k} \frac{df_0}{dv_x} \right\}$$

Change variables $\delta/k = \alpha$ $\quad v_x - \omega_R/k = x$

$$\int_{-\infty}^{\infty} \frac{dv_x}{v_x - \omega/k} \frac{df_0}{dv_x} = \lim_{\delta \to 0} \left\{ \int_{-\infty}^{\infty} dx \frac{1}{x - i\alpha} \frac{df_0}{dx} \right\}$$

Use the following standard integral

$$\lim_{\delta \to 0} \left\{ \int_{-\infty}^{\infty} dx \frac{f(x)}{x - i\alpha} \right\} = P\left[\int_{-\infty}^{\infty} \frac{f(x)}{x} \right] + i\pi f(0)$$

$P\left[\int_{-\infty}^{\infty} \frac{f(x)}{x}\, dx\right]$ is the principal part

~~s~~ i.e. the part where we ignore the pole

Using this result we get the dispersion relation

$$\omega = \omega_{pe}\left(1 + \frac{3k^2\lambda_D^2}{2}\right) + i\,\frac{\pi\,\omega_{pe}^2}{2k^2 n_{eo}}\left(\frac{df_0}{dv_x}\right)_{v_x = \omega/k}$$

Real part gives PLASMA WAVES

IMAGINARY PART GIVES LANDAU DAMPING

Show this...

Recall ~~E~~ - $E = E^A e^{i(kx-\omega t)}$

$\omega = \omega_R + i\omega_I$

So $E = E^A e^{i(kx - \omega_R t - i\omega_I t)}$

$= E^A e^{\omega_I t} e^{i(kx - \omega_R t)}$

exponential growth or damping

(1) CONSIDER TWO CASES

f_0 is Maxwellian $f_0 = \frac{n_e}{\pi^{3/2} V_T^3} e^{-V^2/V_T^2}$

Where V_T is the thermal velocity

First we need to find f_0.

$$f_0 = \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \, \frac{n_e}{\pi^{3/2} v_T^3} e^{-v^2/v_T^2}$$

$$= \frac{n_e}{\pi^{3/2} v_T^3} e^{-v_x^2/v_T^2} \int_{-\infty}^{\infty} dv_y \, e^{-v_y^2/v_T^2} \int_{-\infty}^{\infty} dv_z \, e^{-v_z^2/v_T^2}$$

Use the standard integral

$$\int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

So $$F_0 = \frac{n_e}{\pi^{3/2} v_T^3} e^{-v_x^2/v_T^2} \cdot \sqrt{\pi v_T^2} \cdot \sqrt{\pi v_T^2}$$

$$f_0 = \frac{n_e}{\sqrt{\pi} v_T} e^{-v_x^2/v_T^2}$$

Now find damping rate

$$\omega_I = \frac{\pi \omega_{pe}^2}{2k^2 n_{e0}} \left. \frac{dF_0}{dv_x} \right|_{v_x = \omega/k}$$

$$\frac{dF_0}{dv_x} = -\frac{2 v_x n_e e^{-v_x^2/v_T^2}}{\sqrt{\pi} v_T^3}$$

$$\cancel{\text{So } \omega_I = - \frac{\pi \omega_{pe}^2}{2k^2 n_{e0}} \cdot 2 v_x \omega /}$$

So $$\omega_I = \frac{\pi \omega_{pe}^2}{2k^2} \left(-\frac{2\omega}{\sqrt{\pi} v_T^3 k} e^{-\omega^2/k^2 v_T^2} \right)$$

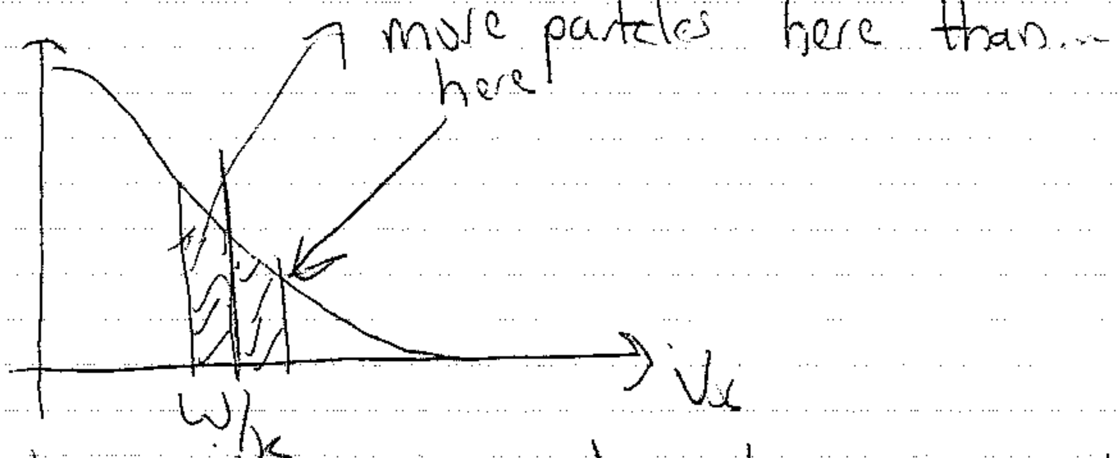
$$\omega_I = -\frac{\sqrt{\pi}\,\omega_{pe}^2\,\omega}{k^3 v_t^3} e^{-\omega^2/\omega_{pe}^2}$$

$\omega_I < 0$ Let $-\omega_I = \gamma$

So $E = E_A e^{-\gamma t} e^{i(k_x x - \omega_R t)}$

The wave damps

WHY?



Particles with v_x around ω/k can effectively exchange energy with the wave

Those with v_x just less than ω/k TAKE ENERGY FROM THE WAVE

Those with energy just greater than ω/k GIVE ENERGY TO THE WAVE

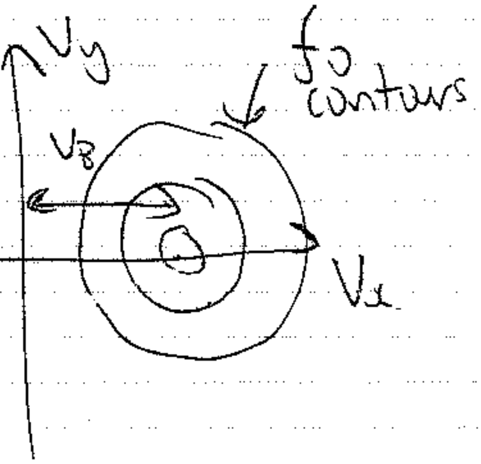
for $f_0 \propto e^{-v_x^2/v_t^2}$ more ~~particles~~ electrons with v_x less than $\omega/k \rightarrow$ net energy loss for the wave

$\Rightarrow$ DAMPING of the wave

(2) SECOND EXAMPLE

$$f_0 = \frac{n_B}{\pi^{3/2} V_{TB}^3} \exp\left[- \frac{(\underline{V} - V_B \hat{\underline{x}})^2}{V_{TB}^2} \right]$$

This is a beam centred on V_B moving in x-direction



NEED TO FIND F_0

$$\cancel{(\underline{V} - V_B \hat{\underline{x}})^2 = (V_y \hat{\underline{y}} +}$$

$$(\underline{V} - V_B \hat{\underline{x}})^2 = (\underline{V} - V_B \hat{\underline{x}}) \cdot (\underline{V} - V_B \hat{\underline{x}})$$

$$= (V_x \hat{\underline{x}} + V_y \hat{\underline{y}} + V_z \hat{\underline{z}} - V_B \hat{\underline{x}}) \cdot (V_x \hat{\underline{x}} + V_y \hat{\underline{y}} + V_z \hat{\underline{z}} - V_B \hat{\underline{x}})$$

$$= (V_x - V_B)^2 + V_y^2 + V_z^2$$

So we can find F_0

$$F_0 = \int_{-\infty}^{\infty} dV_y \int_{-\infty}^{\infty} dV_z \, f_0$$

$$= \frac{n_B}{\pi^{3/2} V_{TB}^3} e^{-(V_x - V_B)^2 / V_{TB}^2} \int_{-\infty}^{\infty} dV_y \, e^{-V_y^2 / V_{TB}^2} \int_{-\infty}^{\infty} dV_z \, e^{-V_z^2 / V_{TB}^2}$$

$$F_0 = \frac{n_B}{\sqrt{\pi} V_{TB}} e^{-(V_x - V_B)^2 / V_{TB}^2}$$

Now find $\dfrac{dF_0}{dV_x}$

$$\frac{df_0}{dv_x} = \frac{-2(v_x - V_B)n_B}{\sqrt{\pi}V_{TB}^3} e^{-(v_x - V_B)^2/V_{TB}^2}$$

~~So the damping (!?!) rate is~~

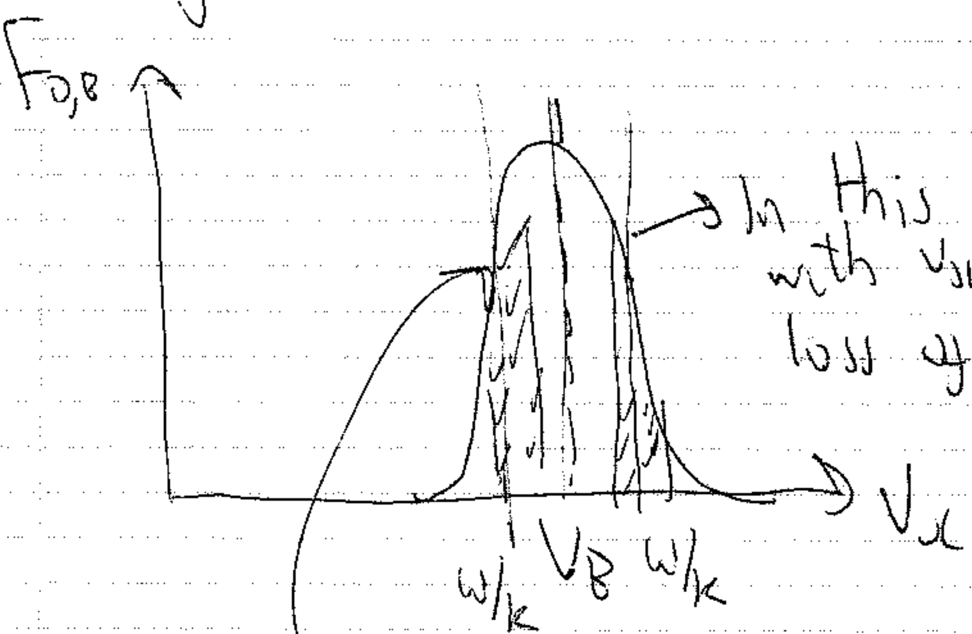
So $$\omega_I = \frac{\pi\omega_{pe}^2}{2k^2 n_{e0}}\left[\frac{-2(v_x - V_B)}{V_{TB}^3} n_B e^{-(v_x - V_B)^2/V_{TB}^2}\right]_{v_x=\omega/k}$$

$$= -\frac{\sqrt{\pi}\,\omega_{pe}^2 n_B}{k^3 V_{TB}^3 n_{e0}}\left(\frac{\omega}{k} - V_B\right) e^{-(\omega/k - V_B)^2/V_{TB}^2}$$

Two cases: (i) if $\frac{\omega}{k} > V_B$ ω_I is negative $\Rightarrow$ DAMPING

(ii) $\frac{\omega}{k} < V_B$ ω_I is positive $\Rightarrow$ GROWTH

Why



→ In this case more particles with $v_x < \omega/k$ → net energy loss of wave → DAMPING

In this case more particles have $v_x > \omega/k$ → net energy gain of wave → GROWTH & INSTABILITY