Submodular Optimization: From Discrete to Continuous and Back

Hamed Hassani

Amin Karbasi

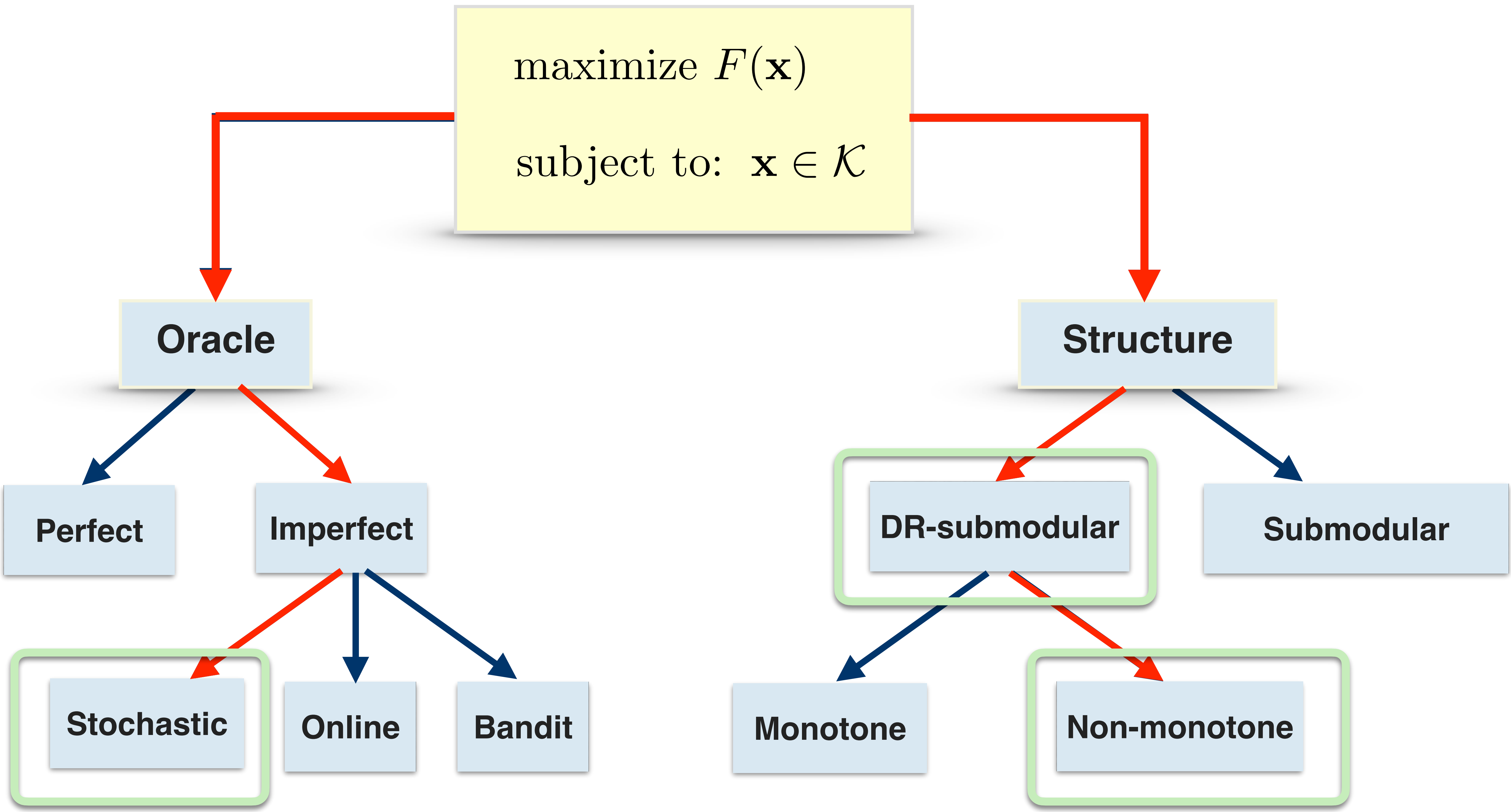
Penn

Yale

Slides + references: http://iid.yale.edu/icml/icml-20.md/

Submodular Maximization

- We consider the following optimization problem:



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Non-monotone and DR-submodular

- measured continuous greedy:

> - Initialize at $\mathbf{x}_0 = \mathbf{0}$
> - Repeat for T iterations:
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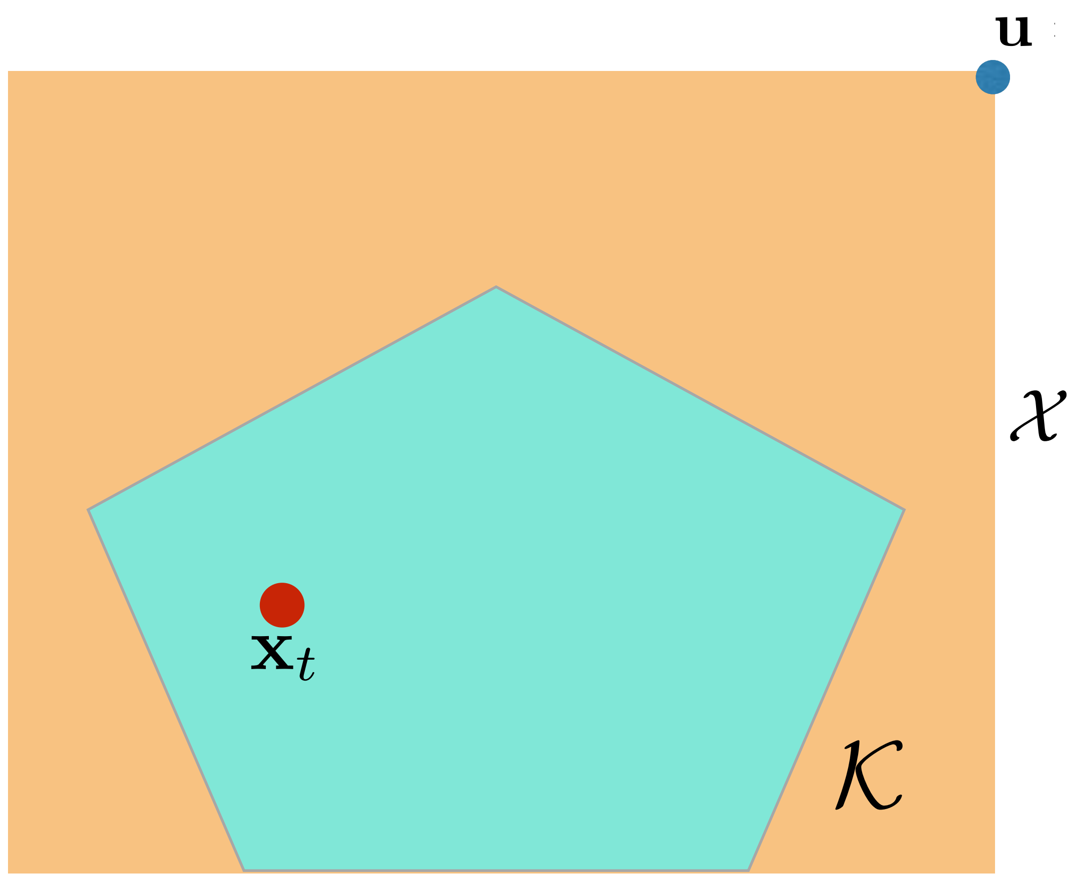

"A Unified Continuous Greedy Algorithm for Submodular Maximization", Feldmann, Noar, Schwartz, 2011

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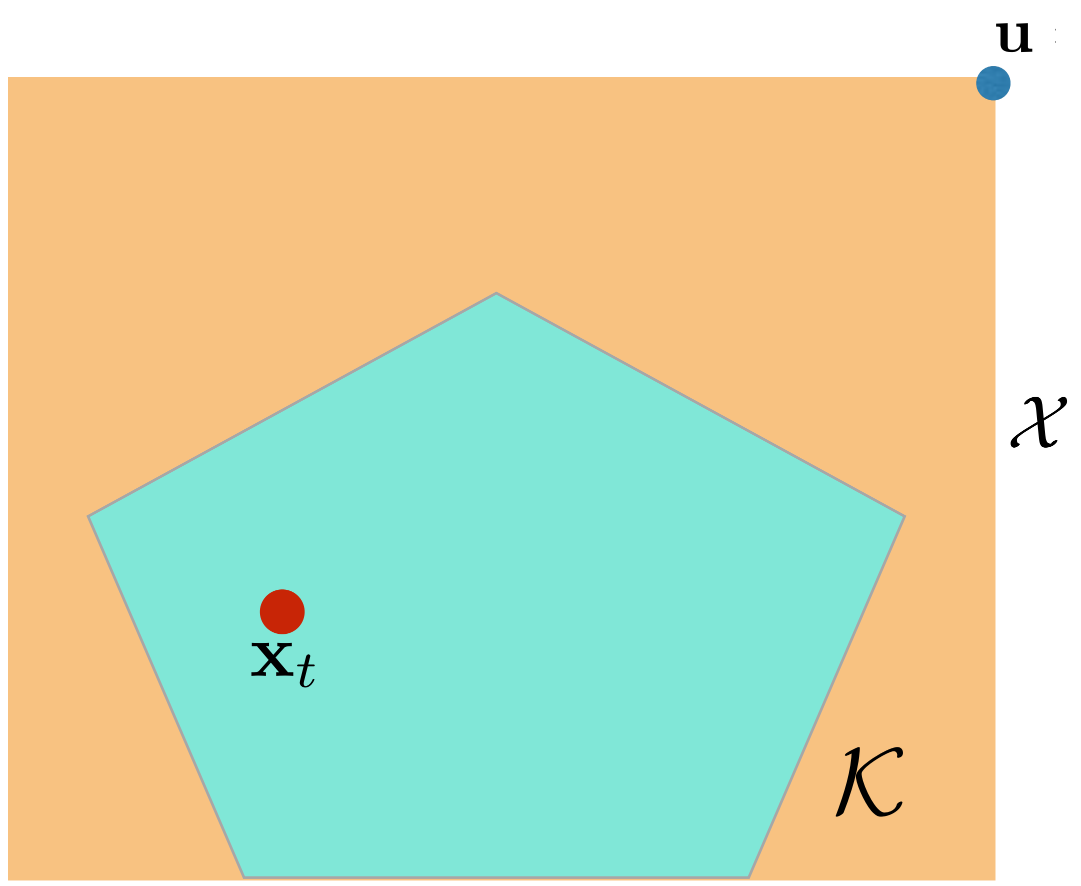

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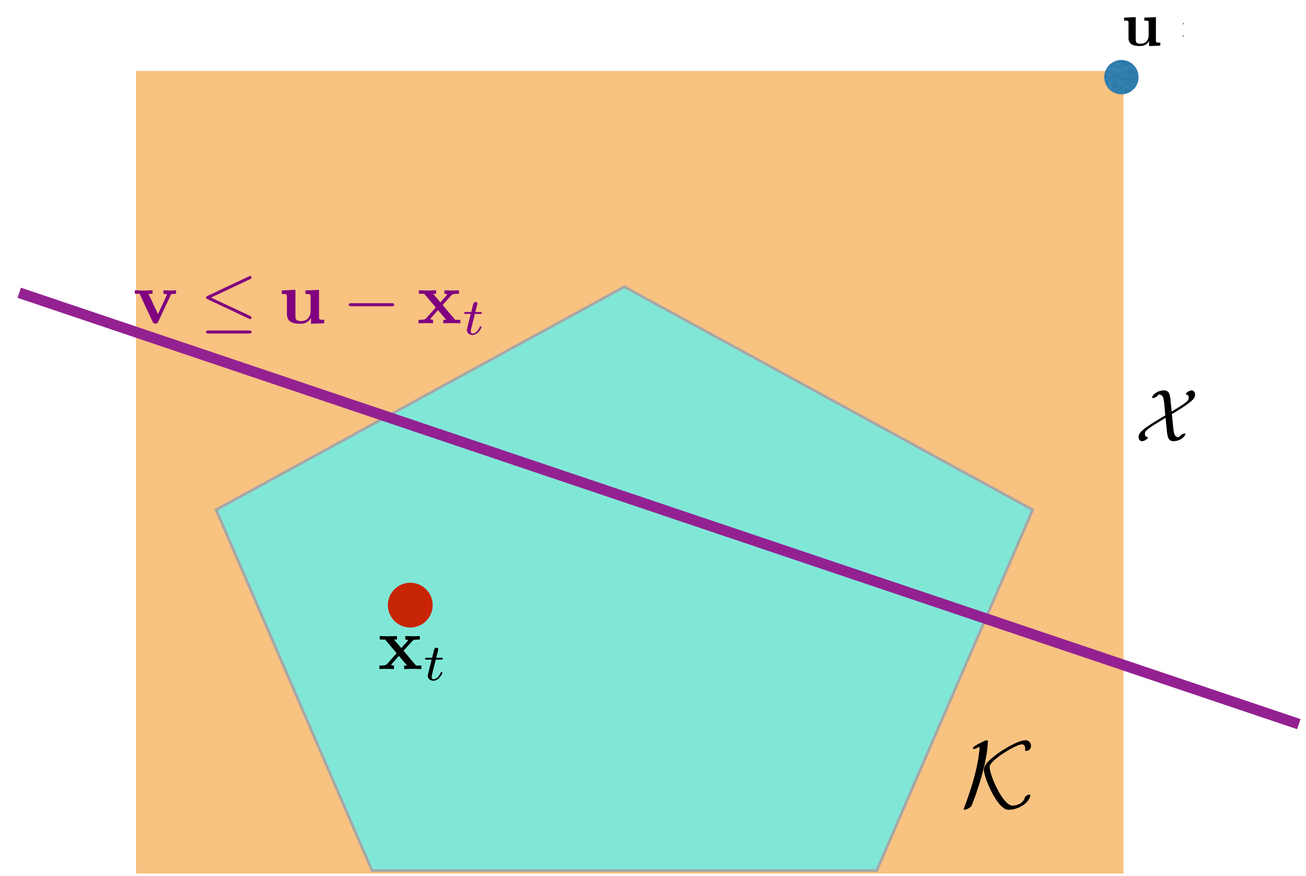

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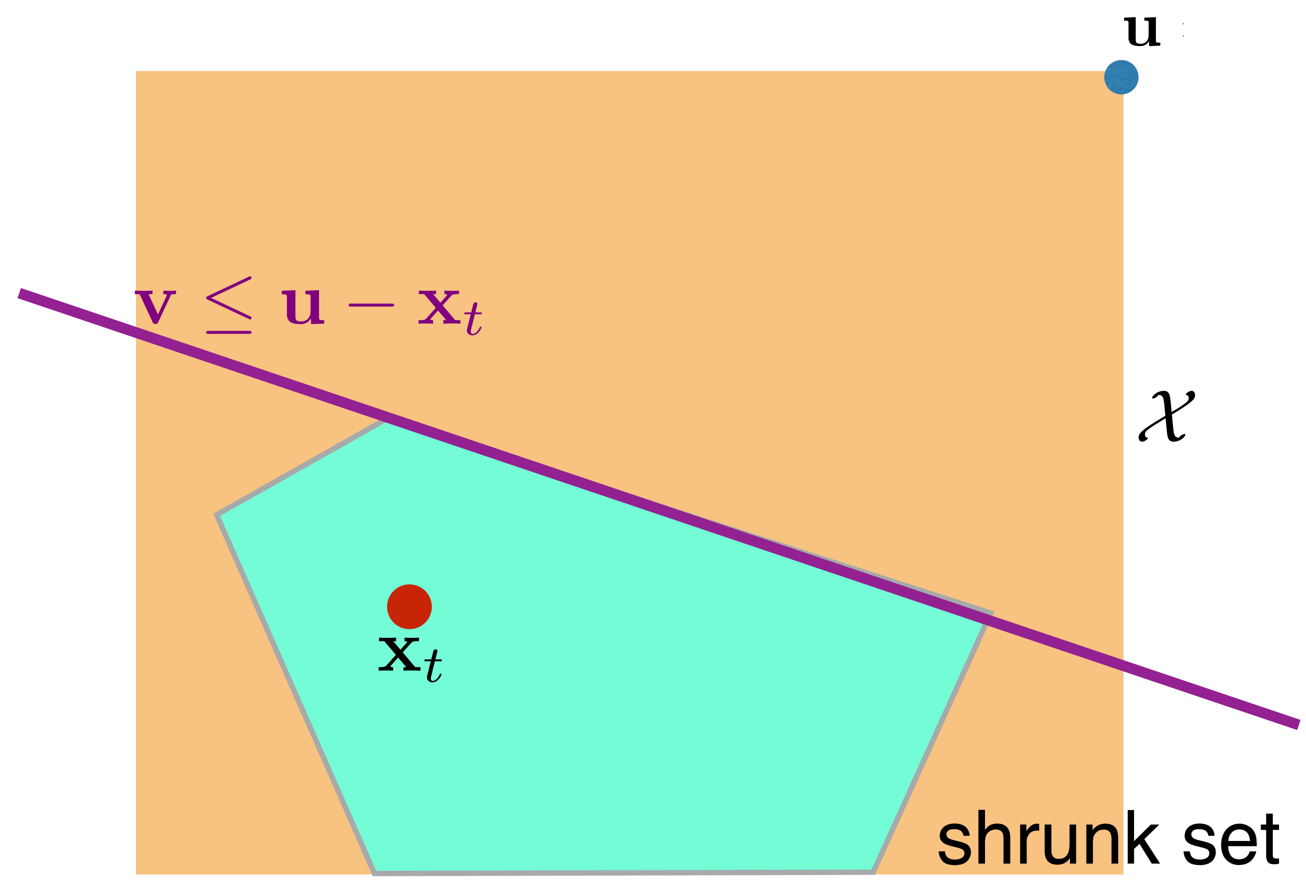

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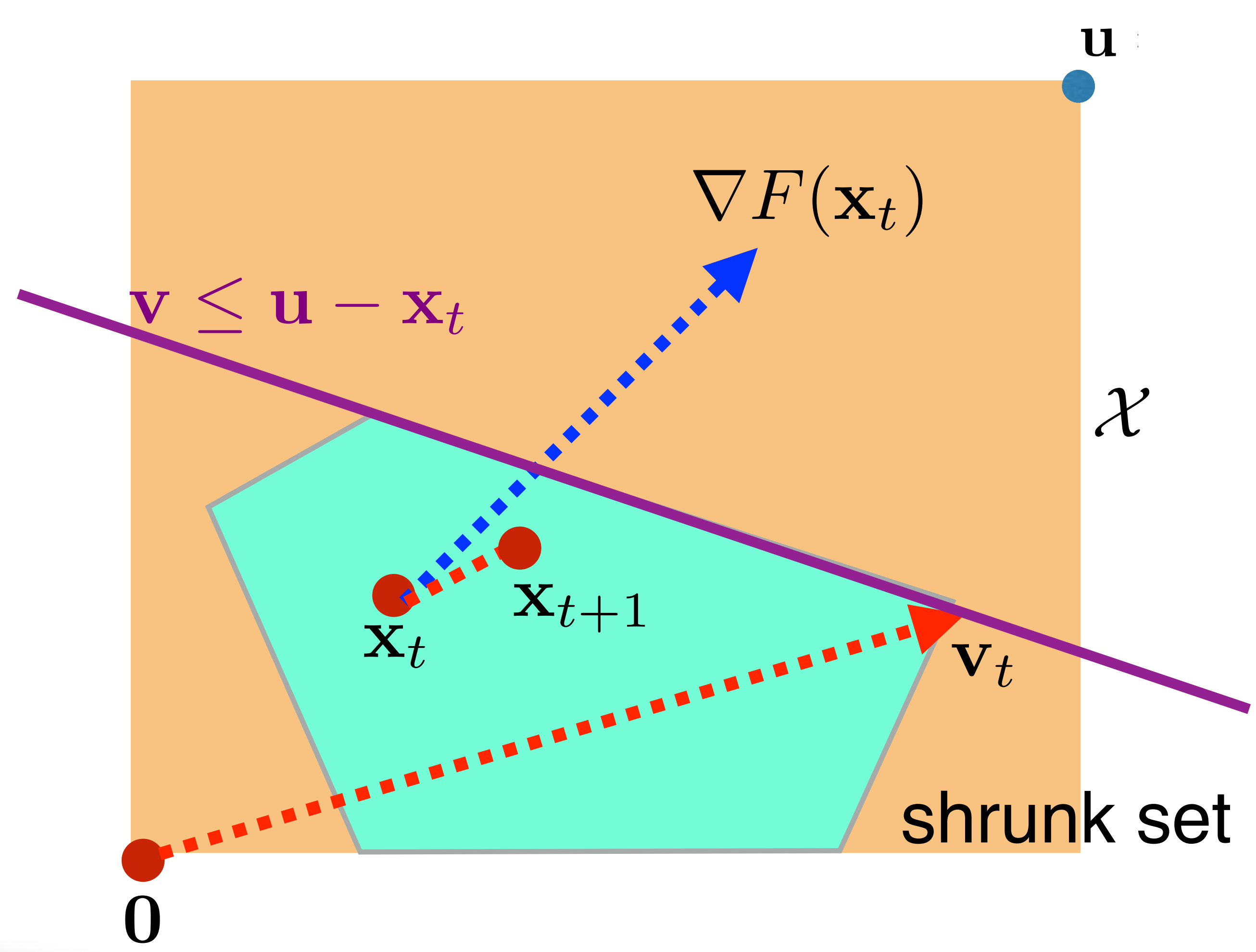

[Bian, Levy, Krause, Buhmann]

The measured continuous greedy algorithm achieves $\frac{1}{e}\mathrm{OPT}$ for constrained maximization of non-monotone DR-submodular functions

"Continuous DR-submodular Maximization: Structure and Algorithms", NIPS'17

Non-monotone and DR-submodular

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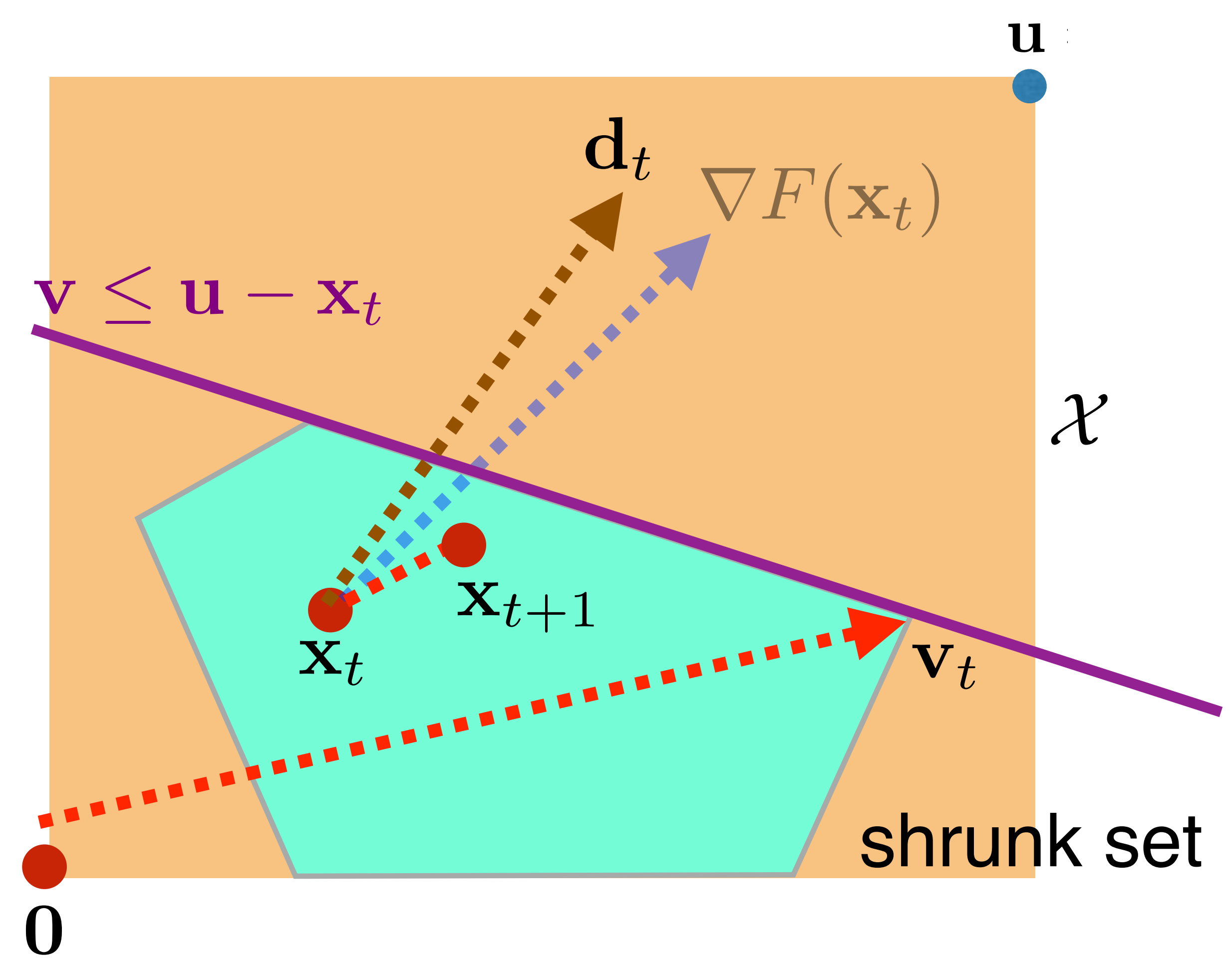

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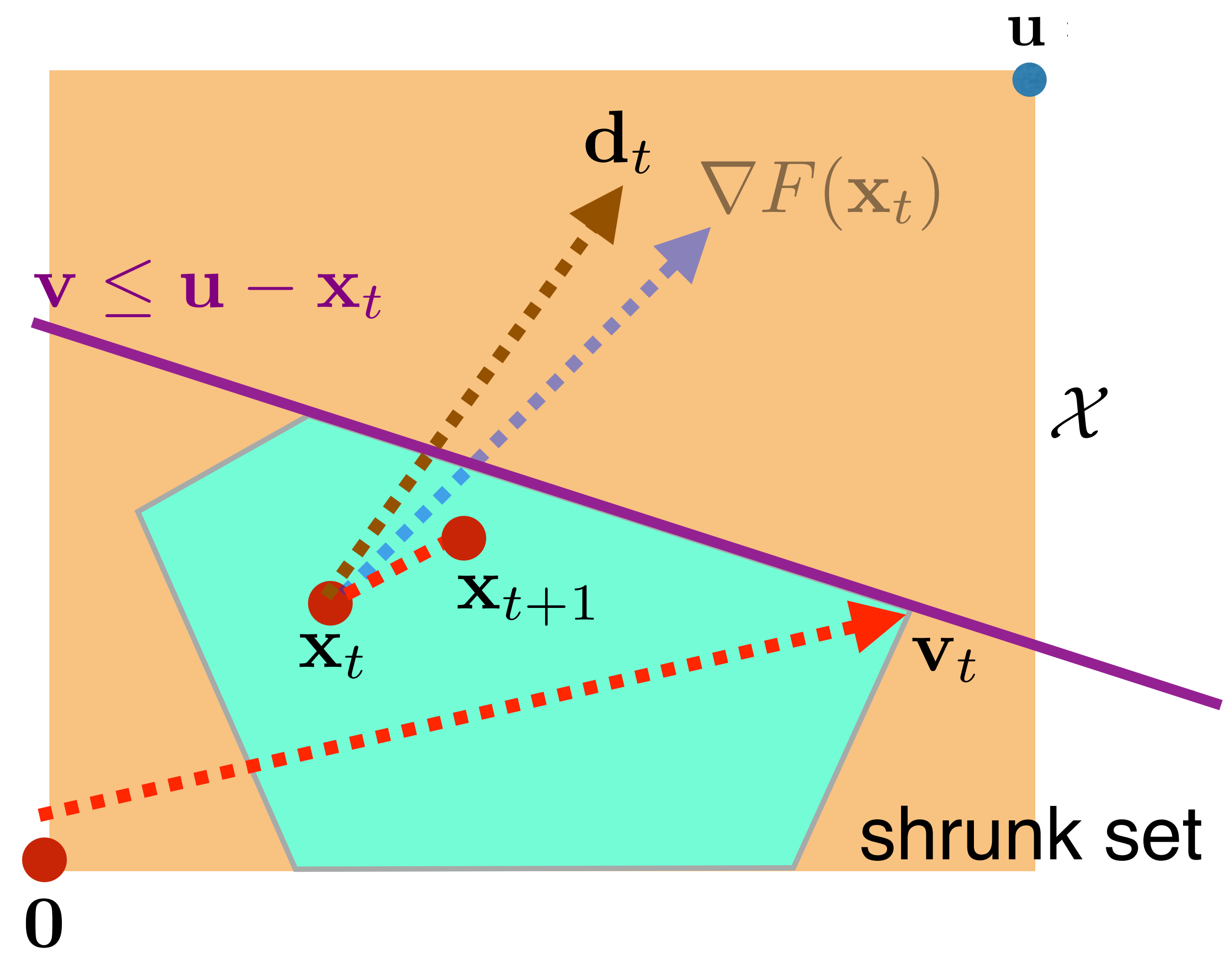

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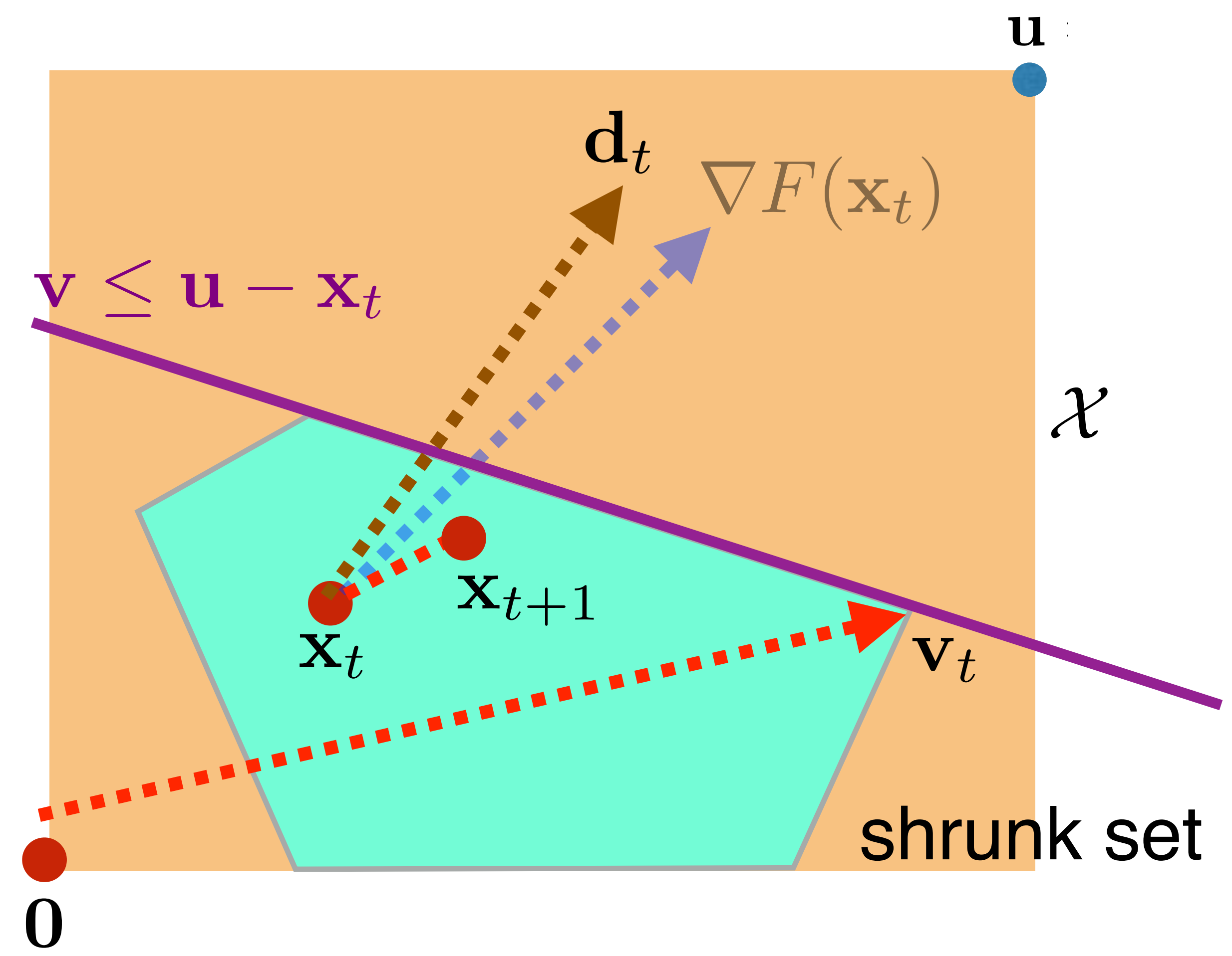

[Hassani, Karbasi, Mokhtari, Shen]

The stochastic measured continuous greedy algorithm achieves $\frac{1}{e}\mathrm{OPT} - \epsilon$ with sample complexity $O(\frac{1}{\epsilon^2})$

"Stochastic Conditional Gradient Methods: From Convex Minimization to Submodular Maximization", JMLR '20

"Stochastic Conditional Gradient++: (Non-)Convex Minimization and Continuous Submodular Maximization", preprint '20

- Relevant work:

"Non-monotone DR-submodular Maximization: Approximation and Regret Guarantees", Dürr, Nguyen Kim Thang, Abhinav Srivastav, Léo Tible, 2019

Non-monotone and DR-submodular

- Constrained case:

Non-monotone and DR-submodular

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- Constrained case:

- Unconstrained case:

[Bian, Buhmann, Krause]

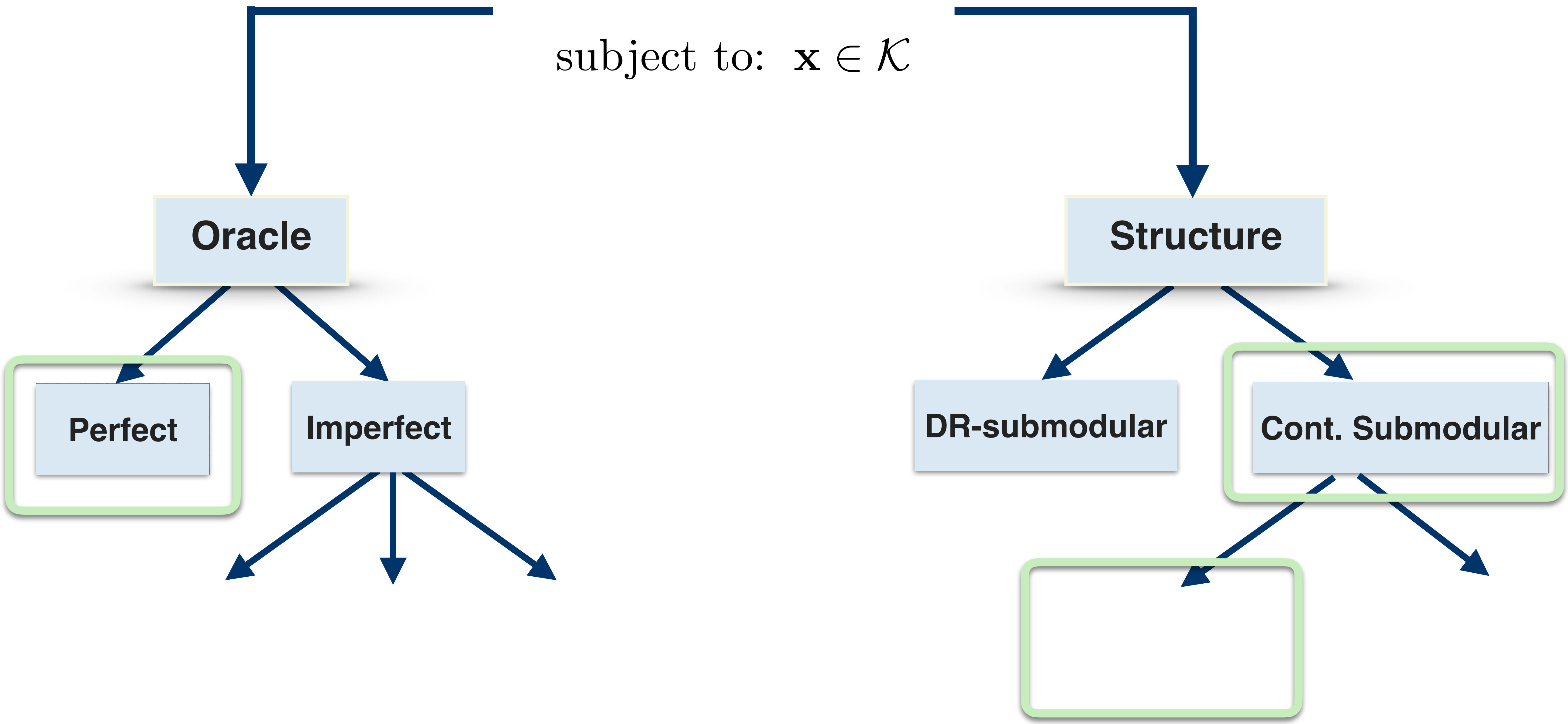
For the the case where $\mathcal{K}$ is a box, an extension of the double-greedy method (covered in part 1) achieves a tight guarantee $\frac{1}{2}\mathrm{OPT}$.

"Optimal Continuous DR-Submodular Maximization and Applications to Provable Mean Field Inference", JMLR '20

Yale Penn

Submodular Maximization

$$\text{maximize } F(\mathbf{x})$$
$$\text{subject to: } \mathbf{x} \in \mathcal{K}$$



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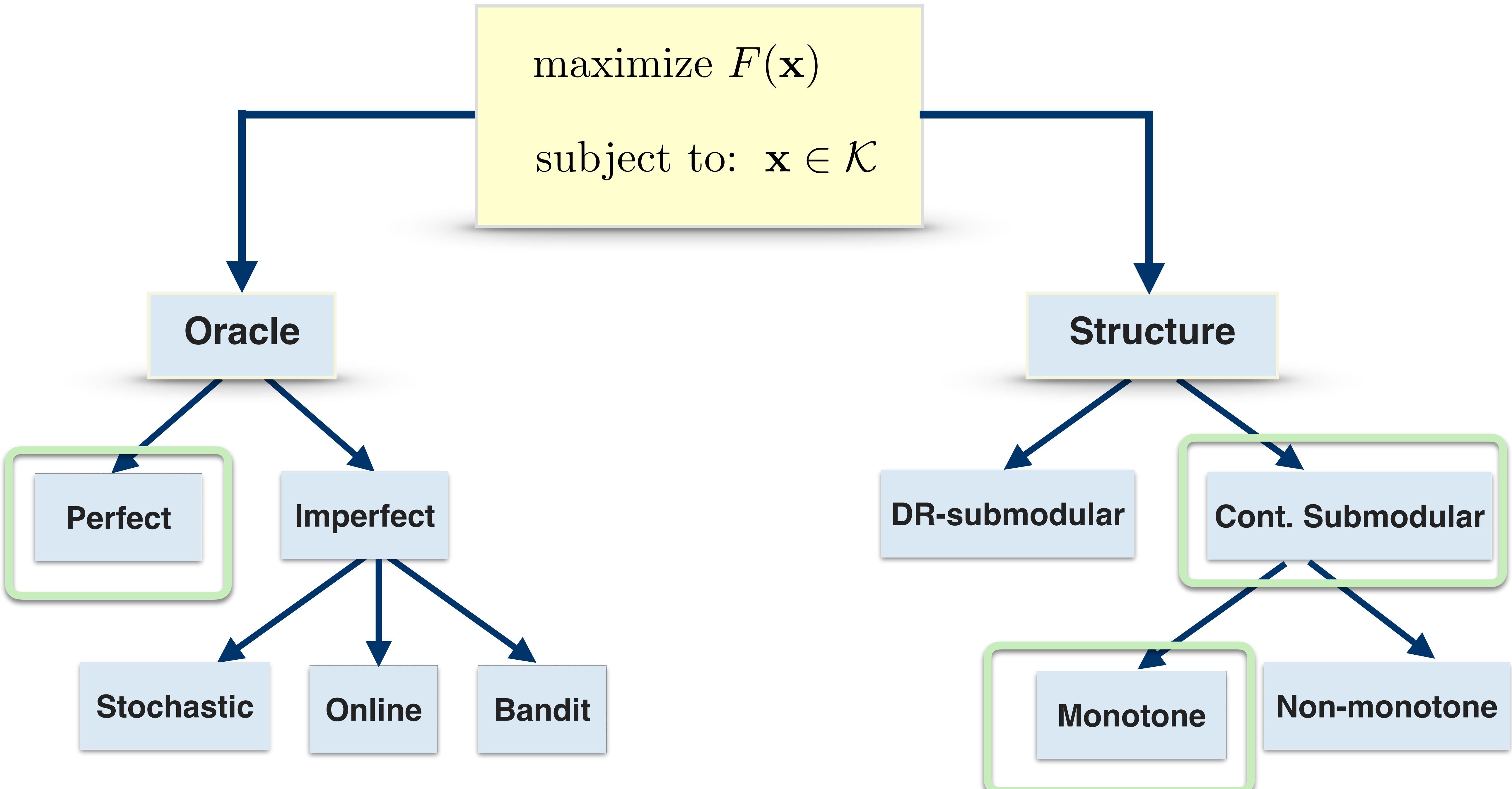

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- Very recent progress: Coordinate-ascent methods

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- Basic idea:

while the budget B is not exhausted

for each coordinate i:

find: $\alpha_i = \arg\max_{\alpha \geq 0} \dfrac{F(\mathbf{x} + \alpha \mathbf{e}_i) - F(\mathbf{x})}{\alpha}$

choose coordinate i^* with the largest α_{i^*}

update $\mathbf{x} = \mathbf{x} + \alpha_{i^*} e_{i^*}$

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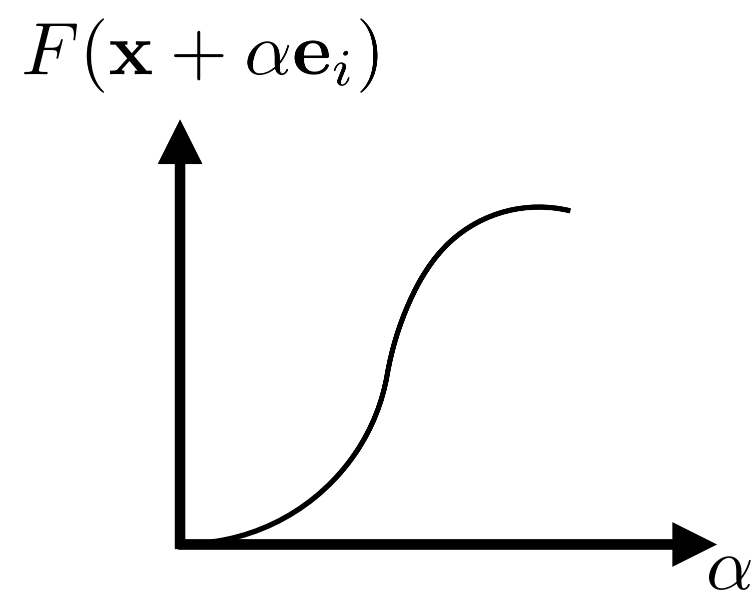

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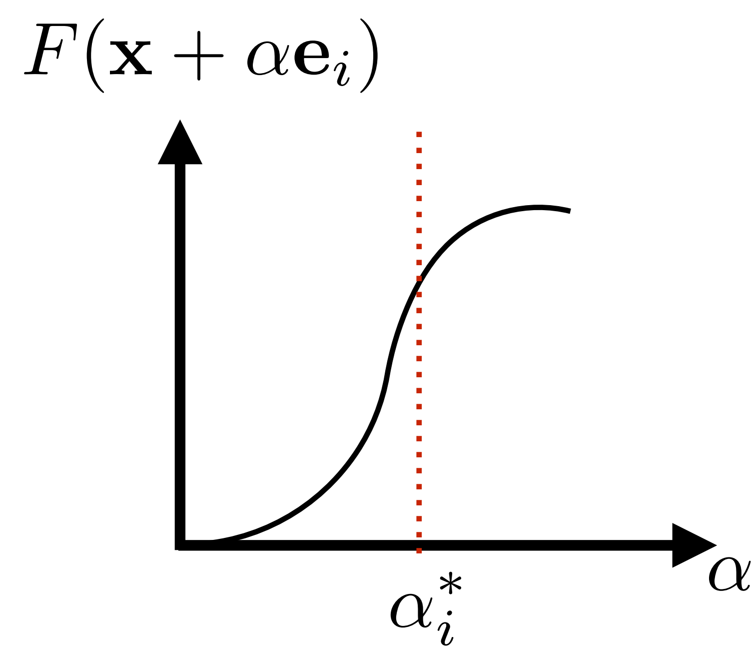

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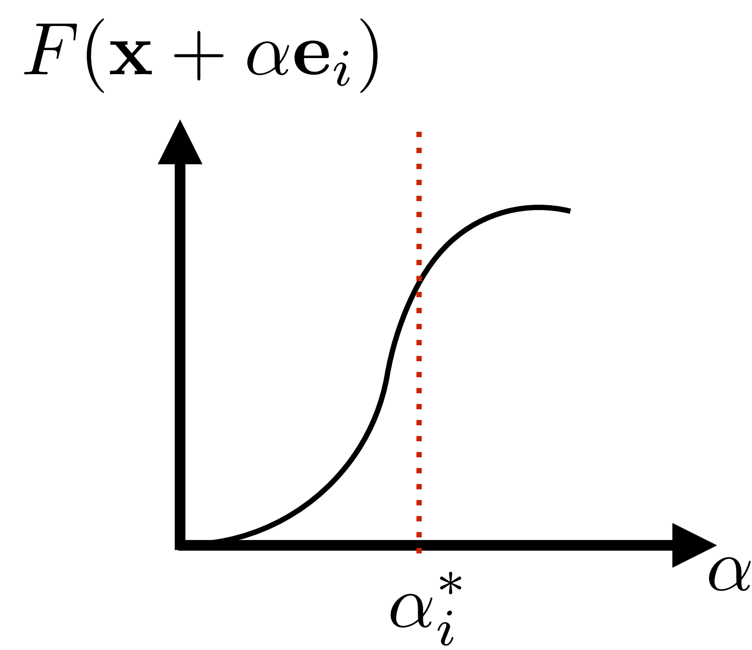

[Feldmann, Karbasi]

Coordinate descent methods obtain a $(1 - \frac{1}{e} - \epsilon)$ OPT solution to the above problem with computational complexity $O(\frac{n^3}{\epsilon^{2.5}})$

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- Many open questions:
  General convex constraints, stochastic, better algorithms, etc

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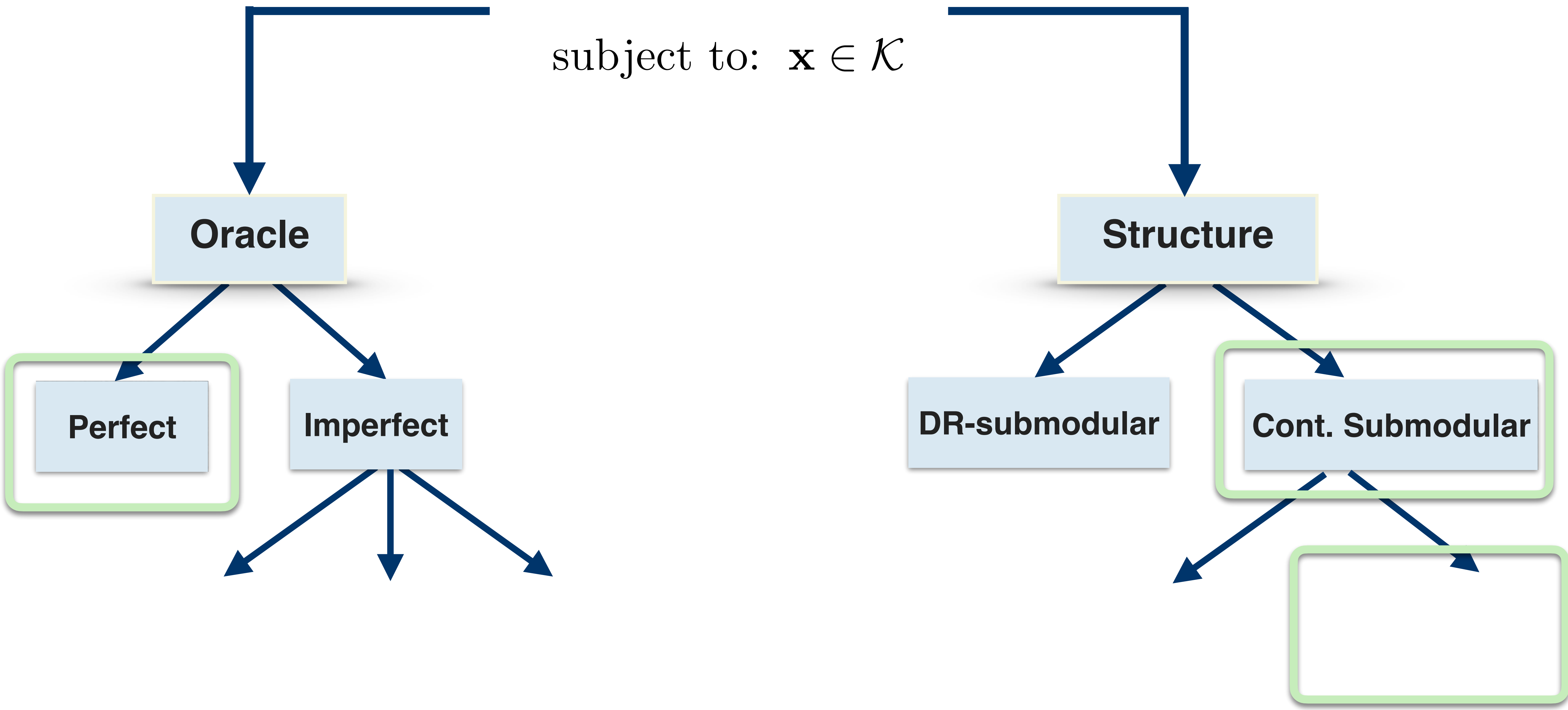

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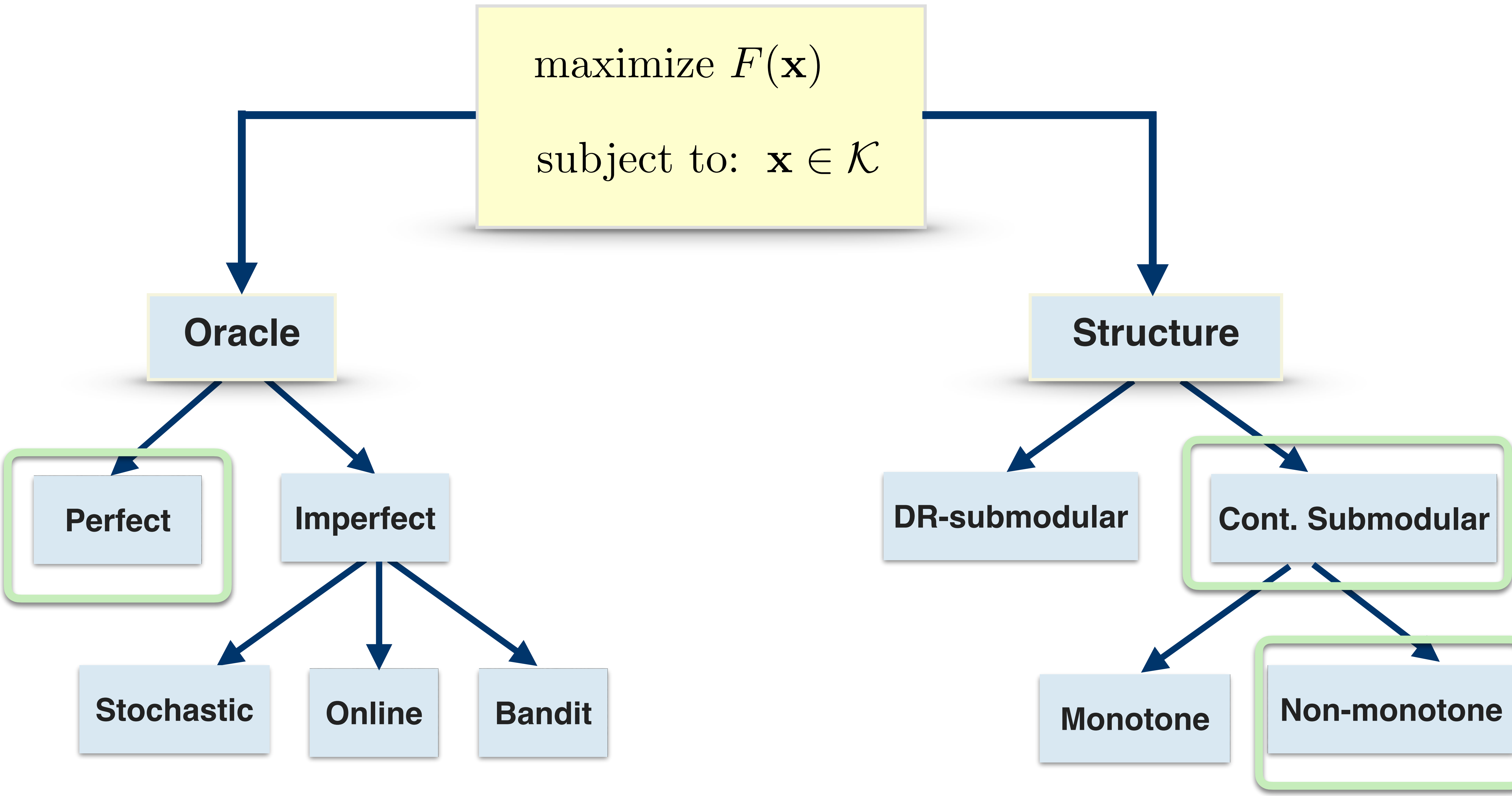

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- For $i = 1$ to n

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"Optimal Algorithms for Continuous Non-monotone Submodular and DR-Submodular Maximization", Niazadeh, Roughgarden, Wang '18

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- Output: $(z_1, z_2, \cdots, z_n)$

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[Niazadeh, Roughgarden, Wang]

For the the case where $\mathcal{K}$ is a box, the above algorithm achieves a tight guarantee $\frac{1}{2}\text{OPT} - \epsilon$ with complexity $O(\frac{n^2}{\epsilon})$.

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non-monotone and constrained, stochastic, etc

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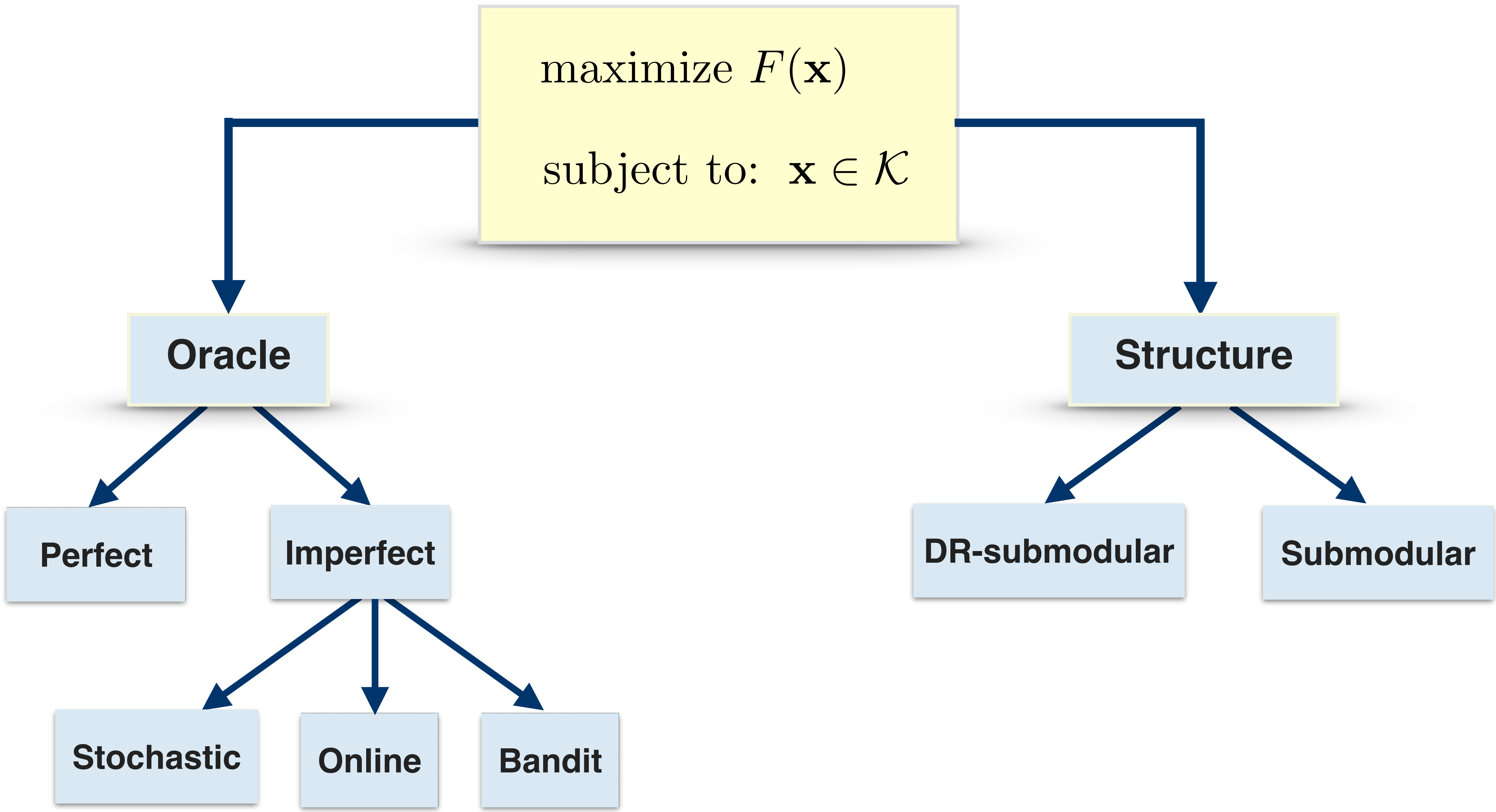

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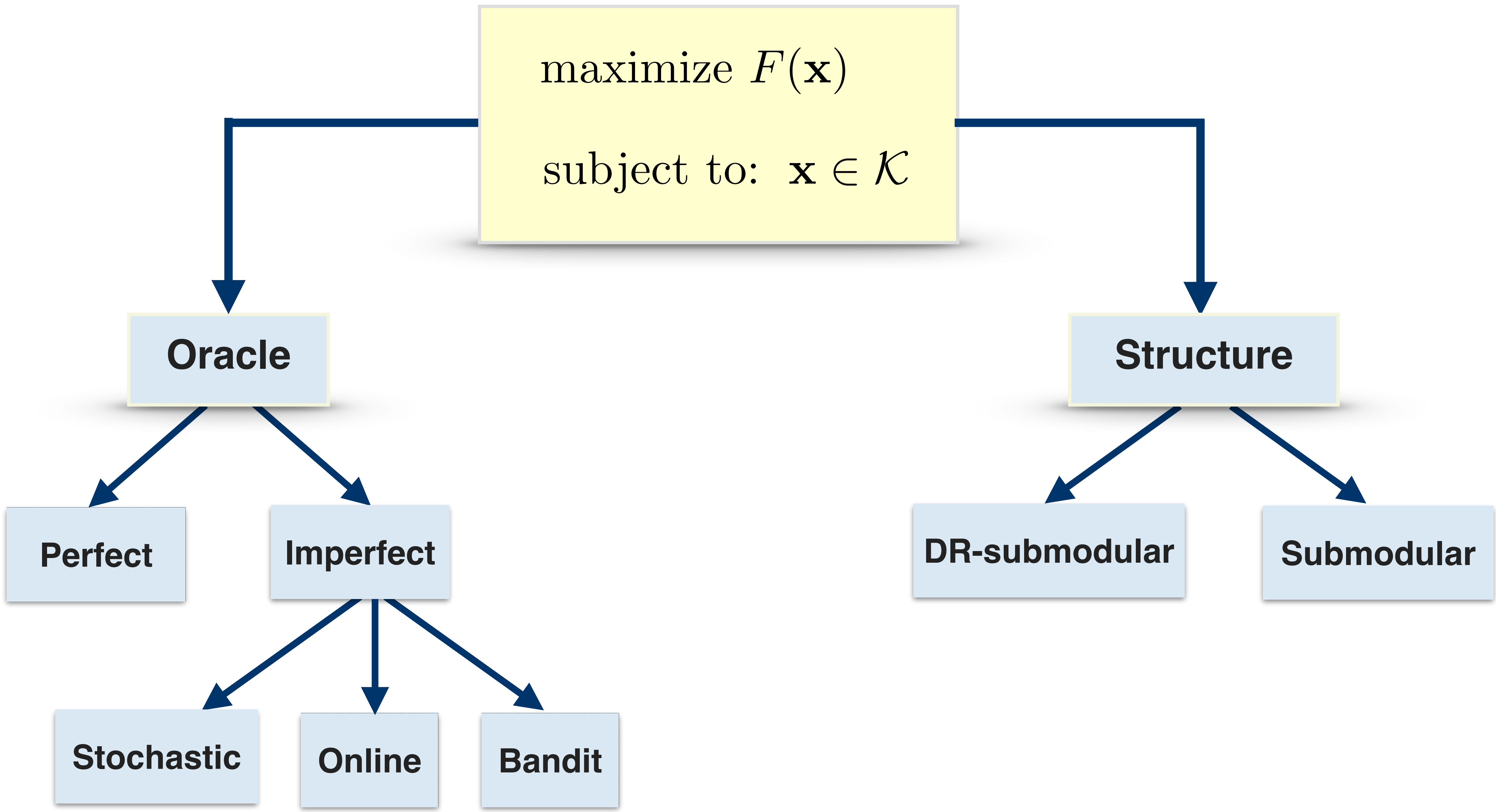
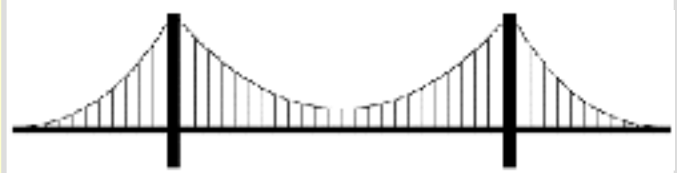


Bridging Discrete and Continuous Settings

$$\max_{S \in \mathcal{I}} f(S)$$



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"Maximizing a monotone submodular function subject to a matroid constraint", Calinescu, Chekuri, Pal, Vondrák, 2011

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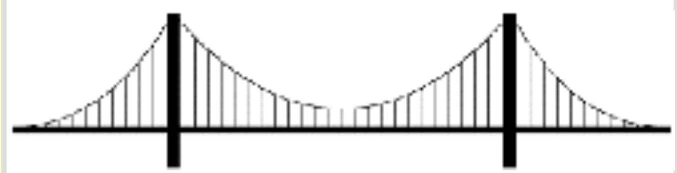

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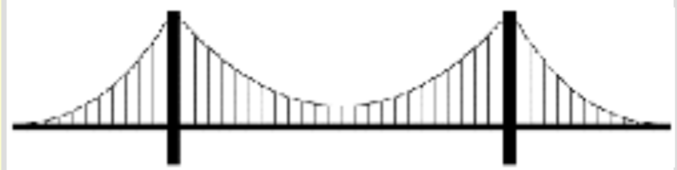

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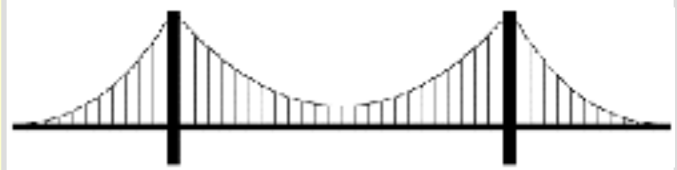

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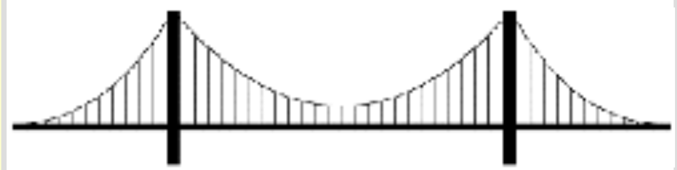

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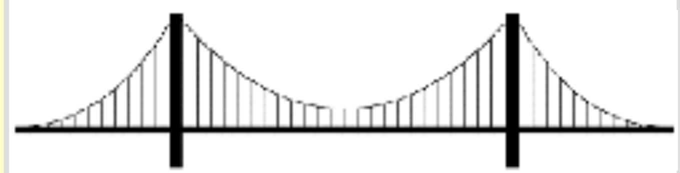

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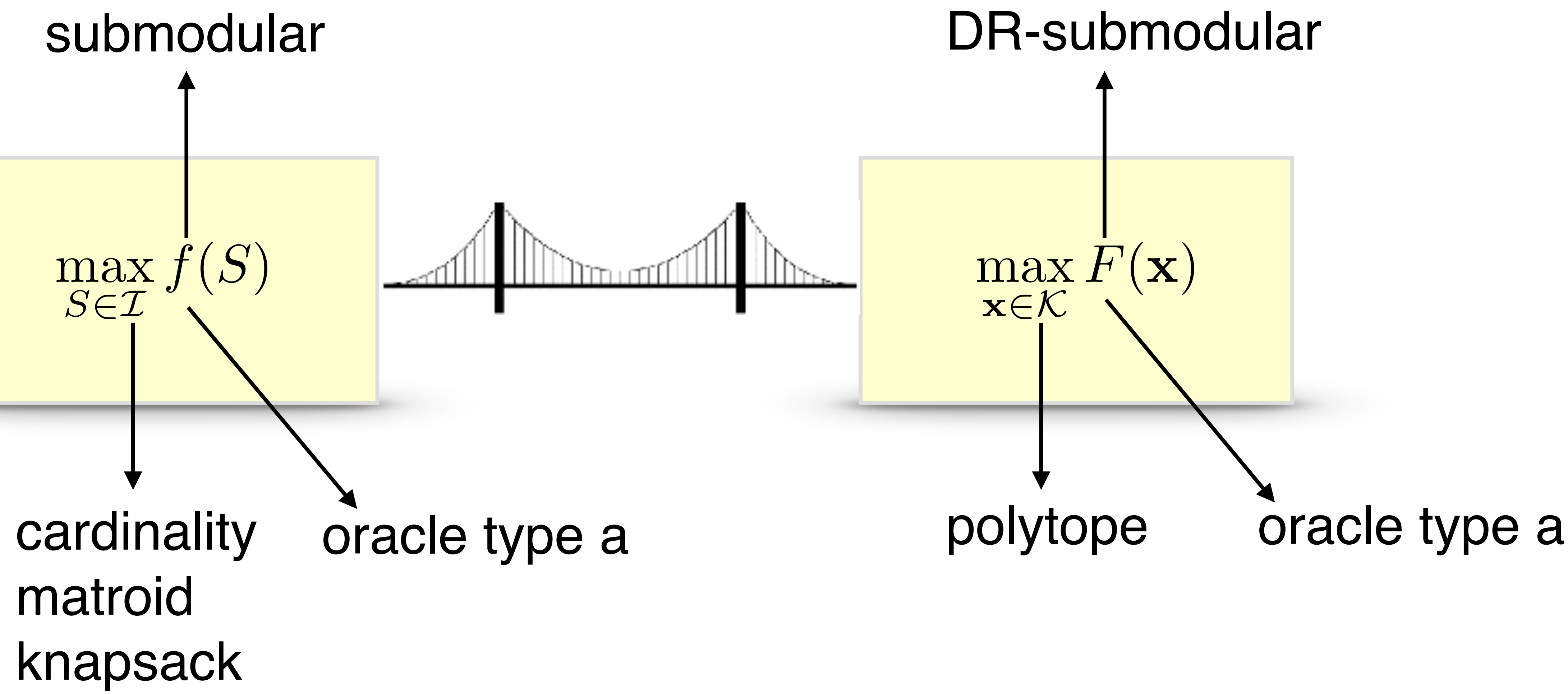

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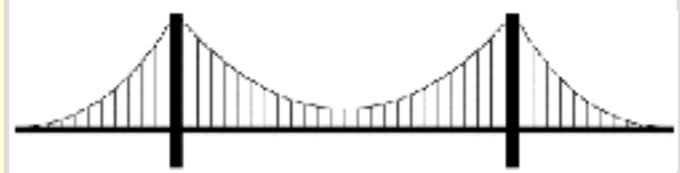
Bridging Discrete and Continuous Settings

submodular

$$\max_{S \in \mathcal{I}} f(S)$$

cardinality
matroid
knapsack

oracle type a



DR-submodular

$$\max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x})$$

polytope

oracle type a

Building blocks:

- Multi-linear extension
- Equivalent formulations
- Rounding

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Multi-linear Extension

$f : 2^V \to \mathbb{R}$ $\Rightarrow$ $F : \mathcal{X} \to \mathbb{R}$

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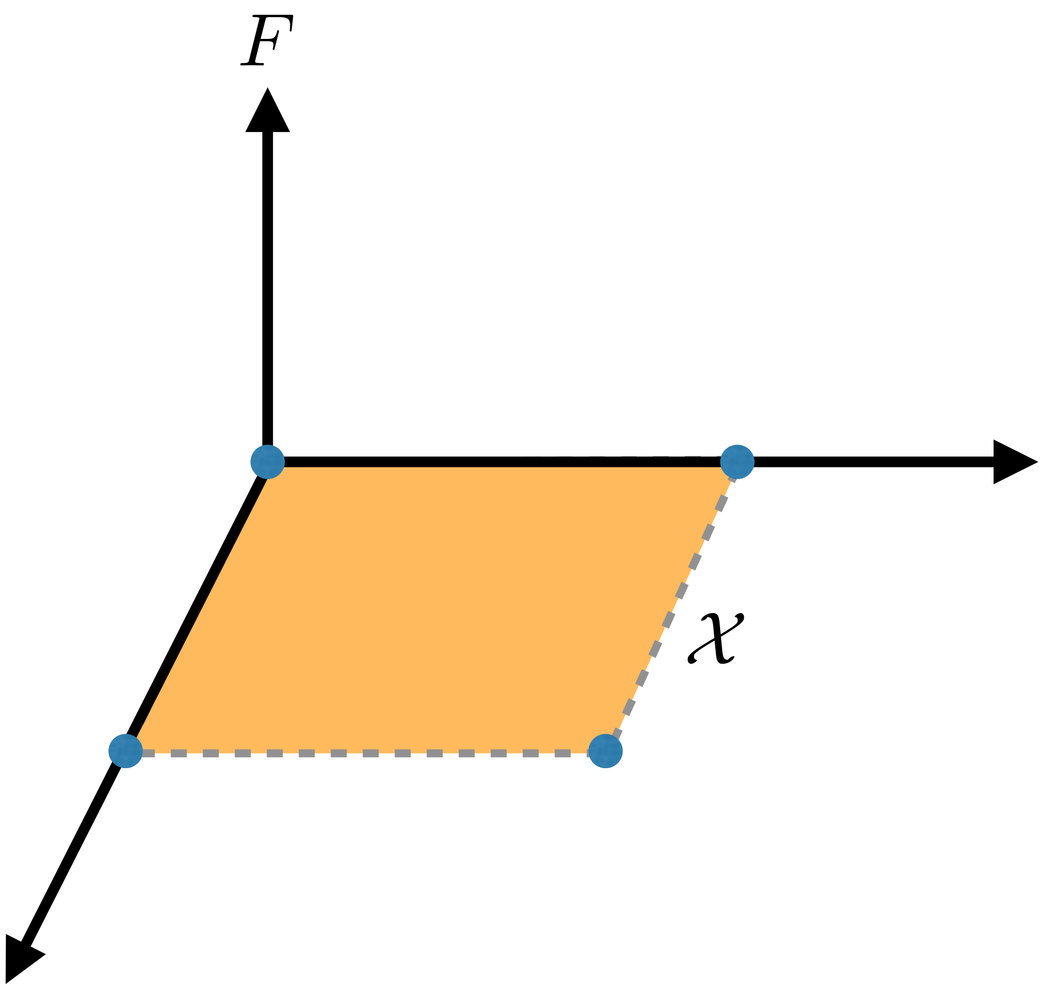

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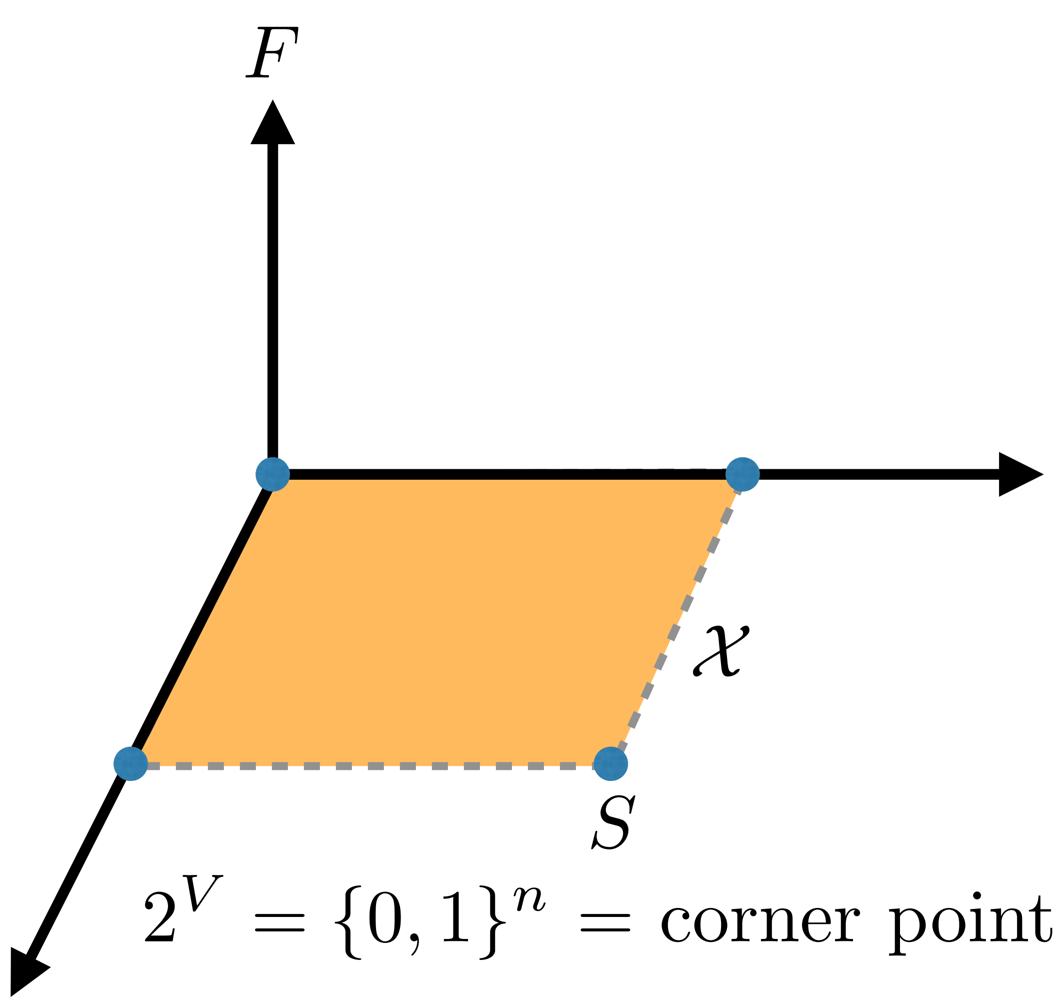

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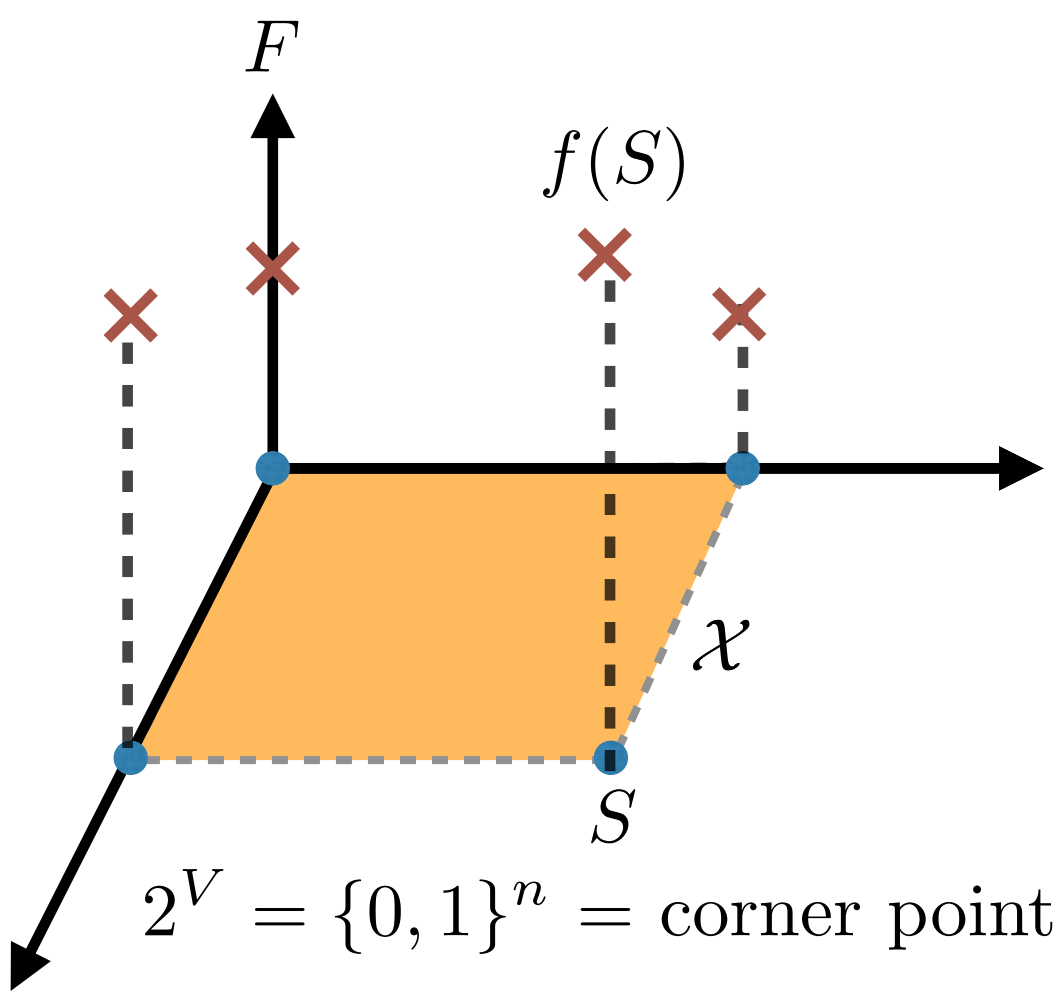

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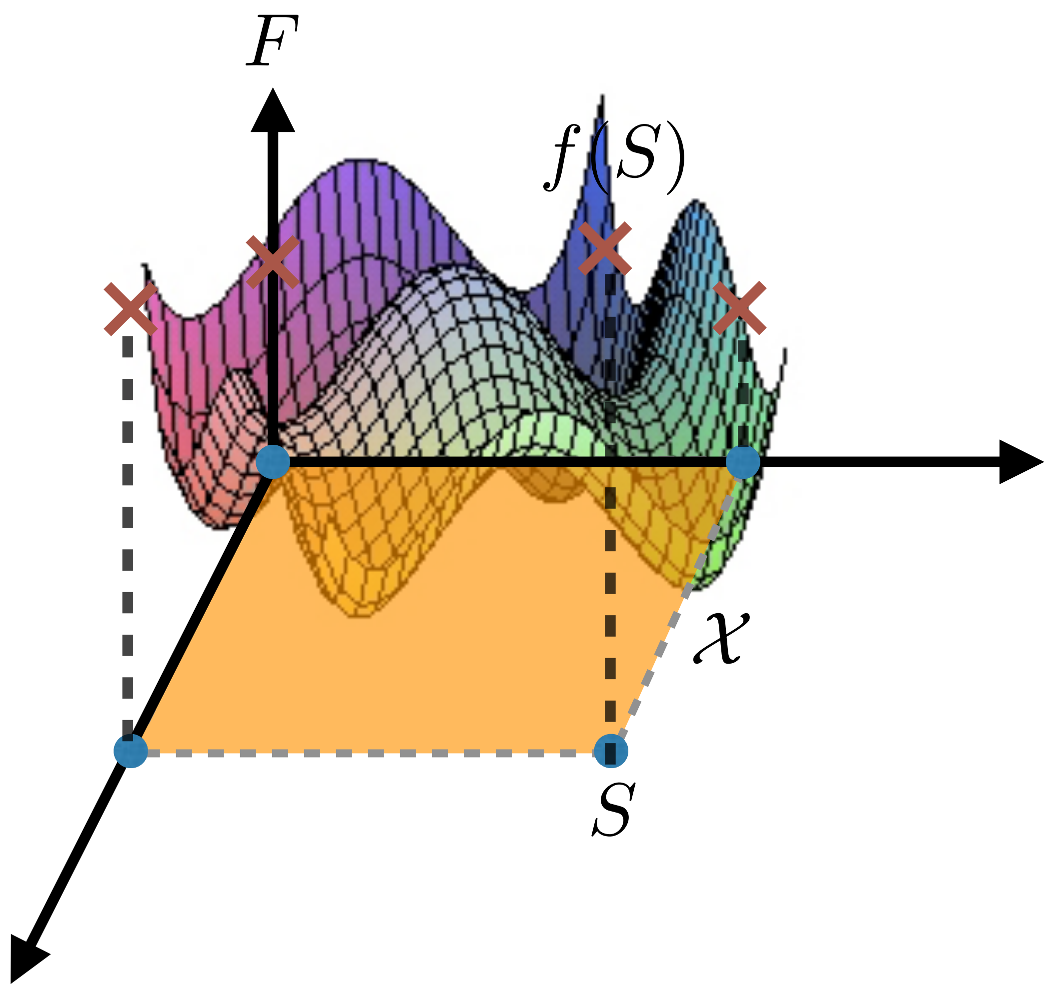

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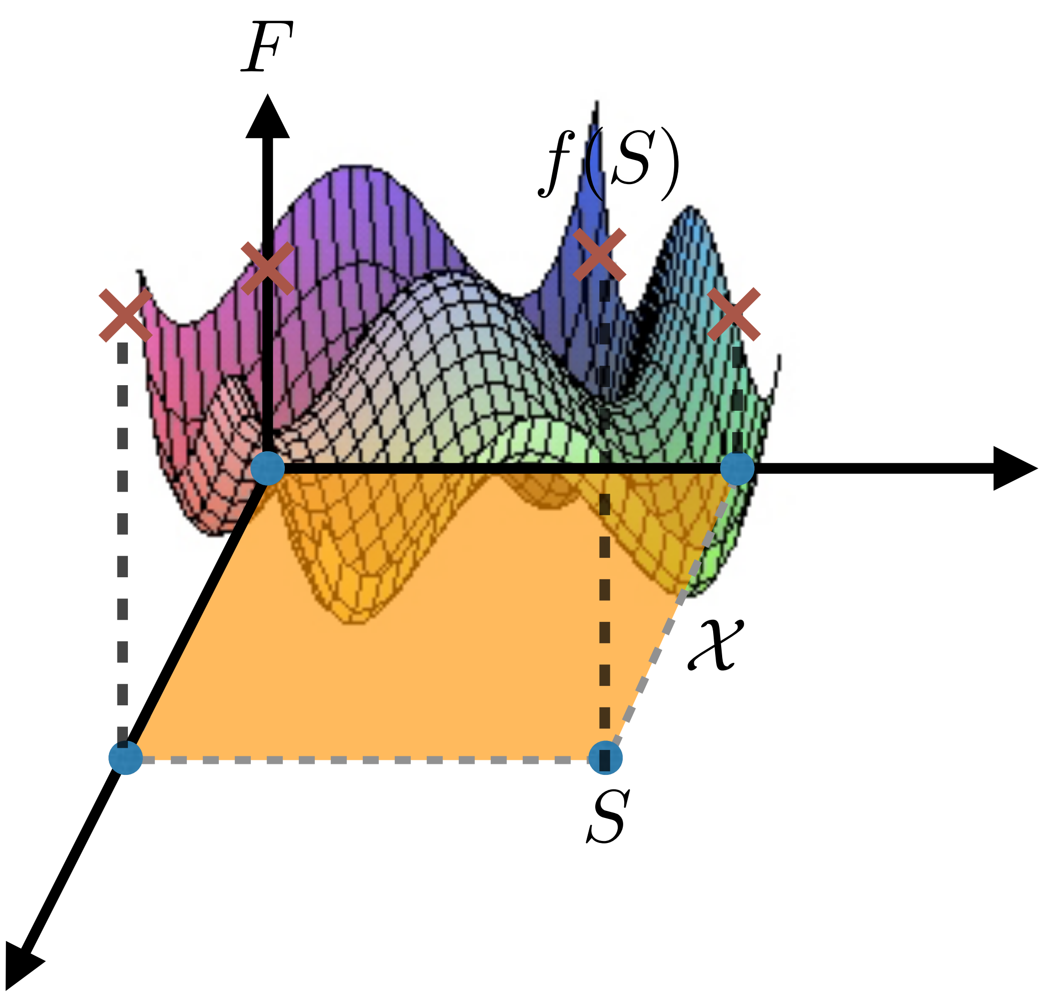

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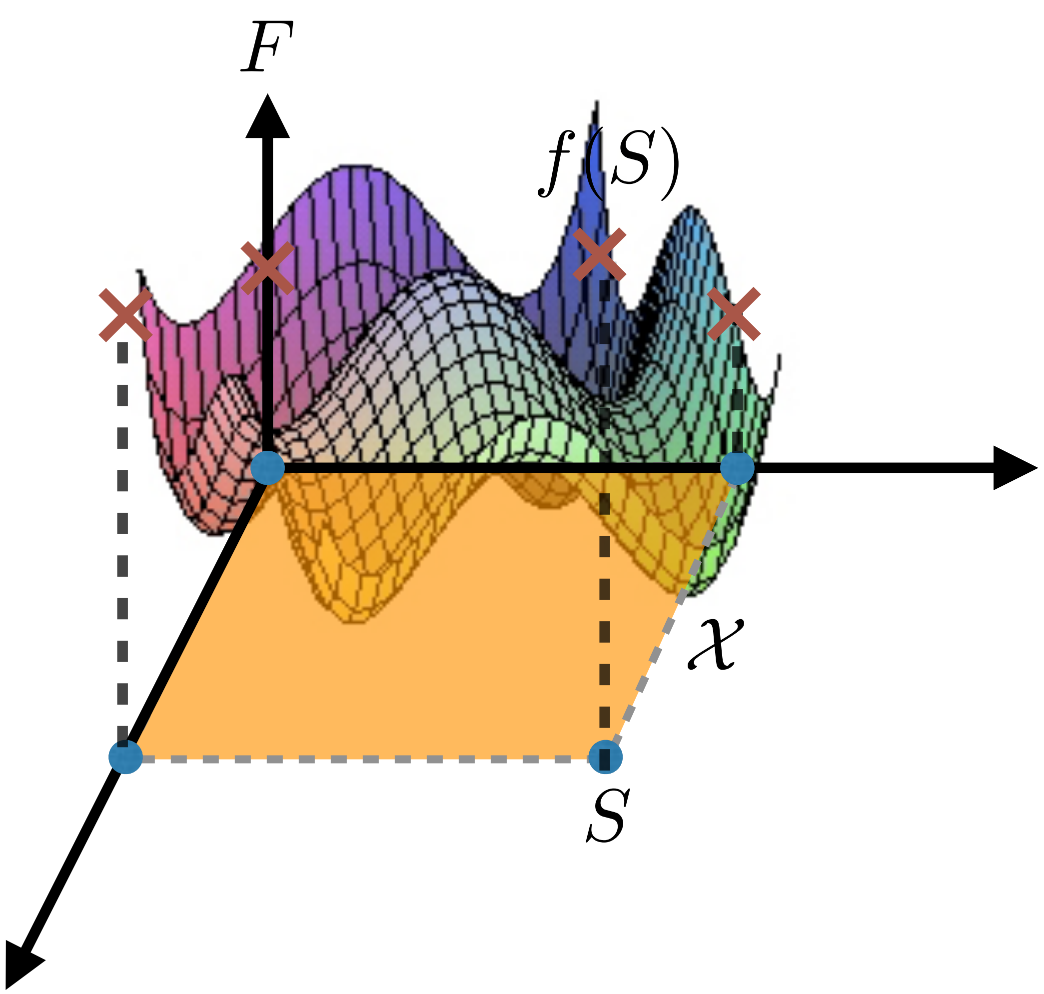

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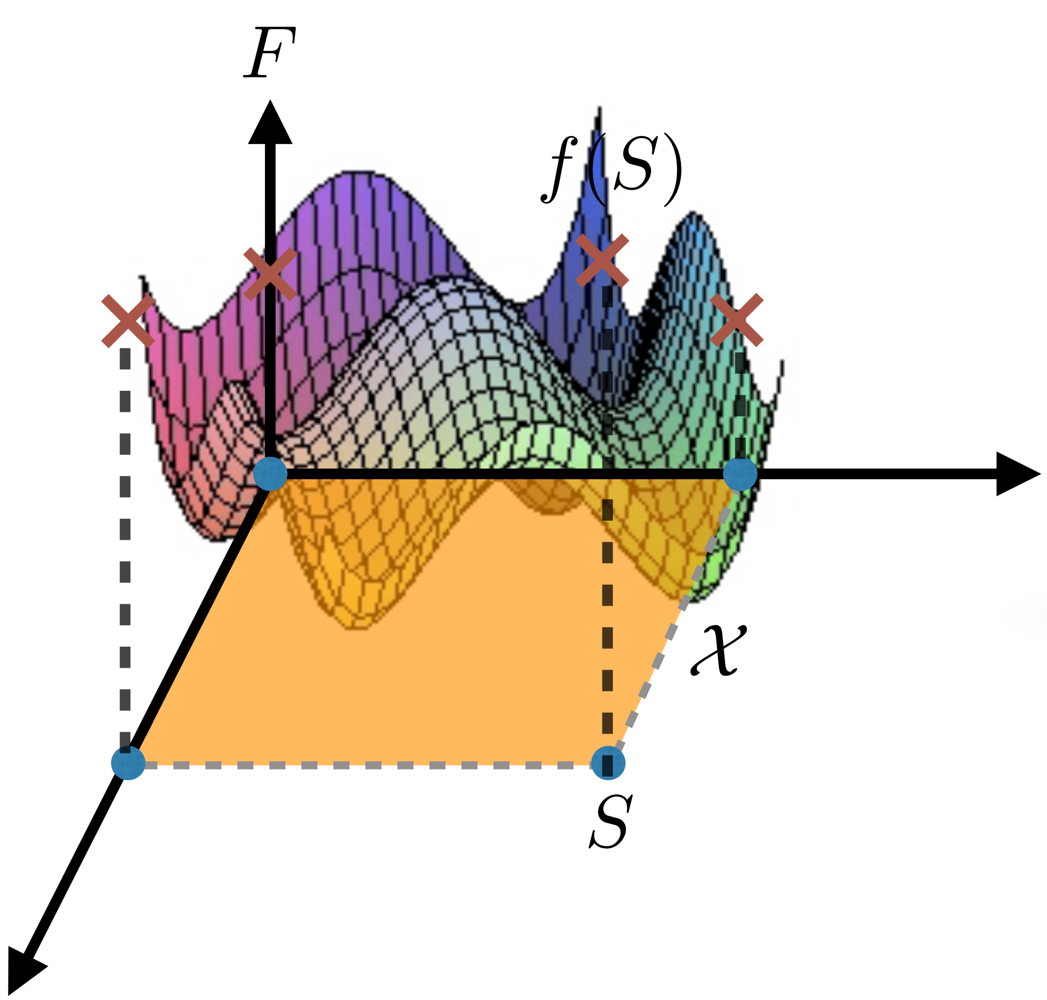

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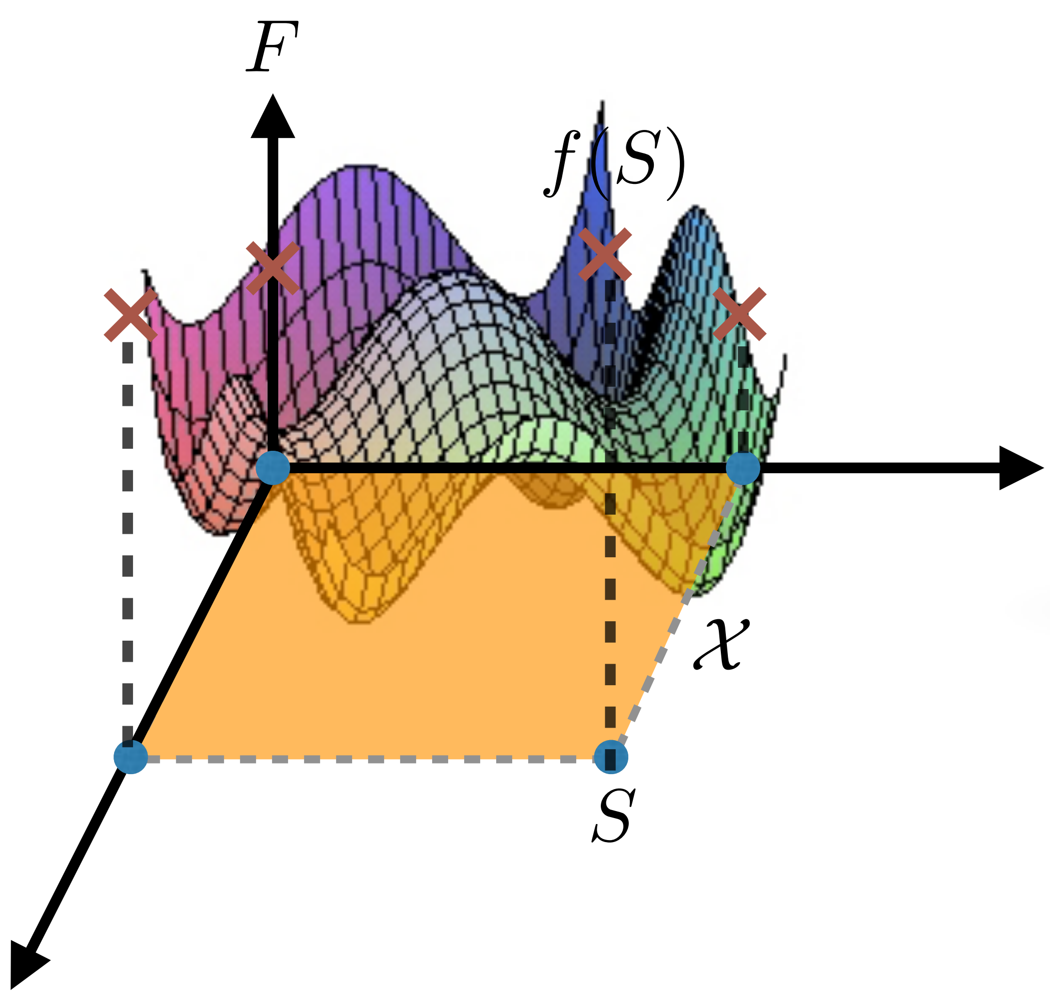

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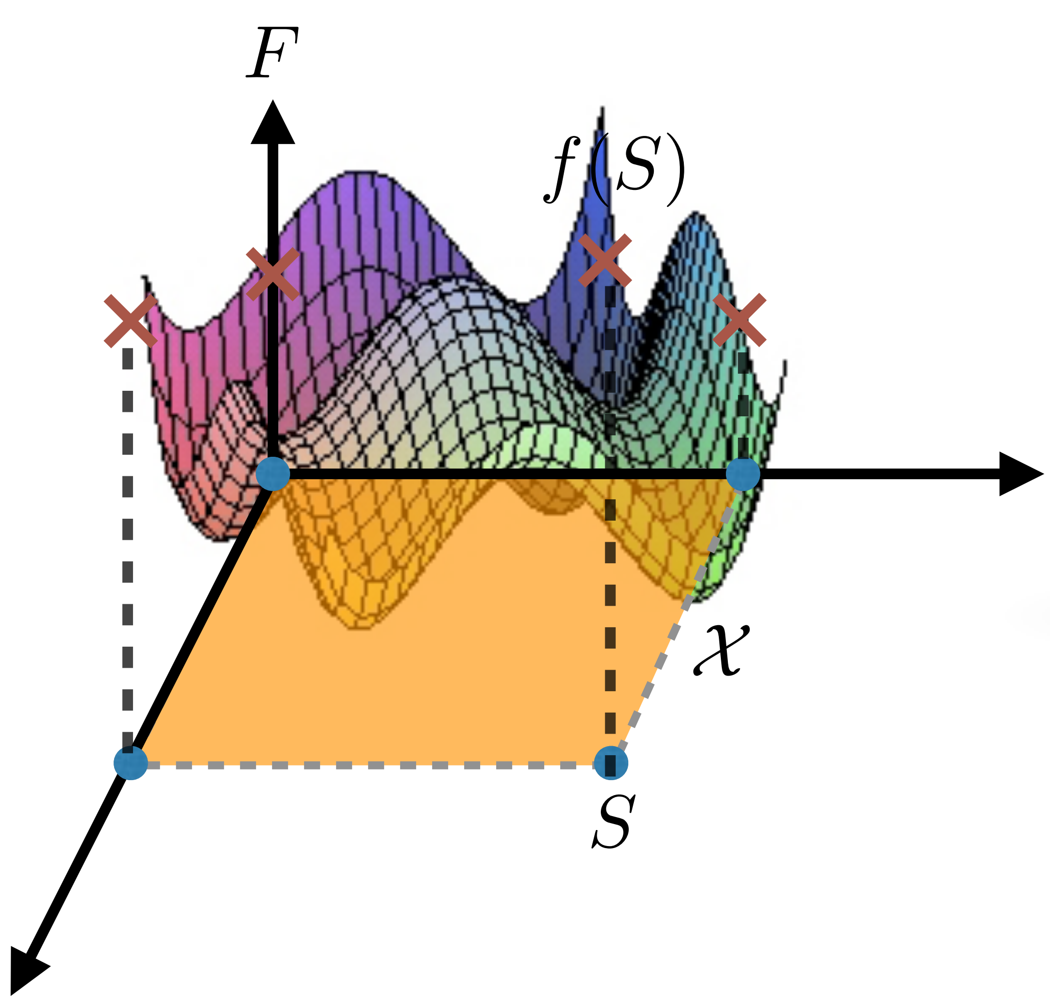

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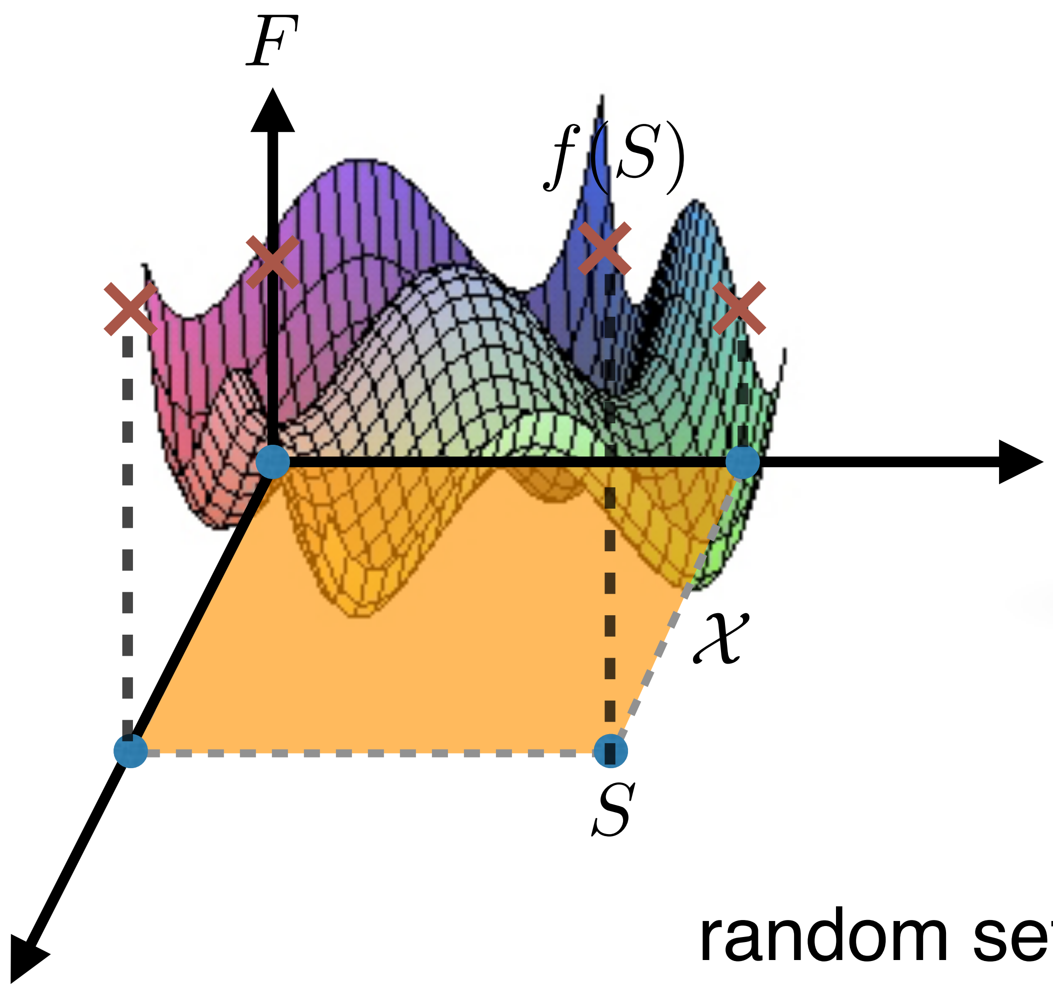

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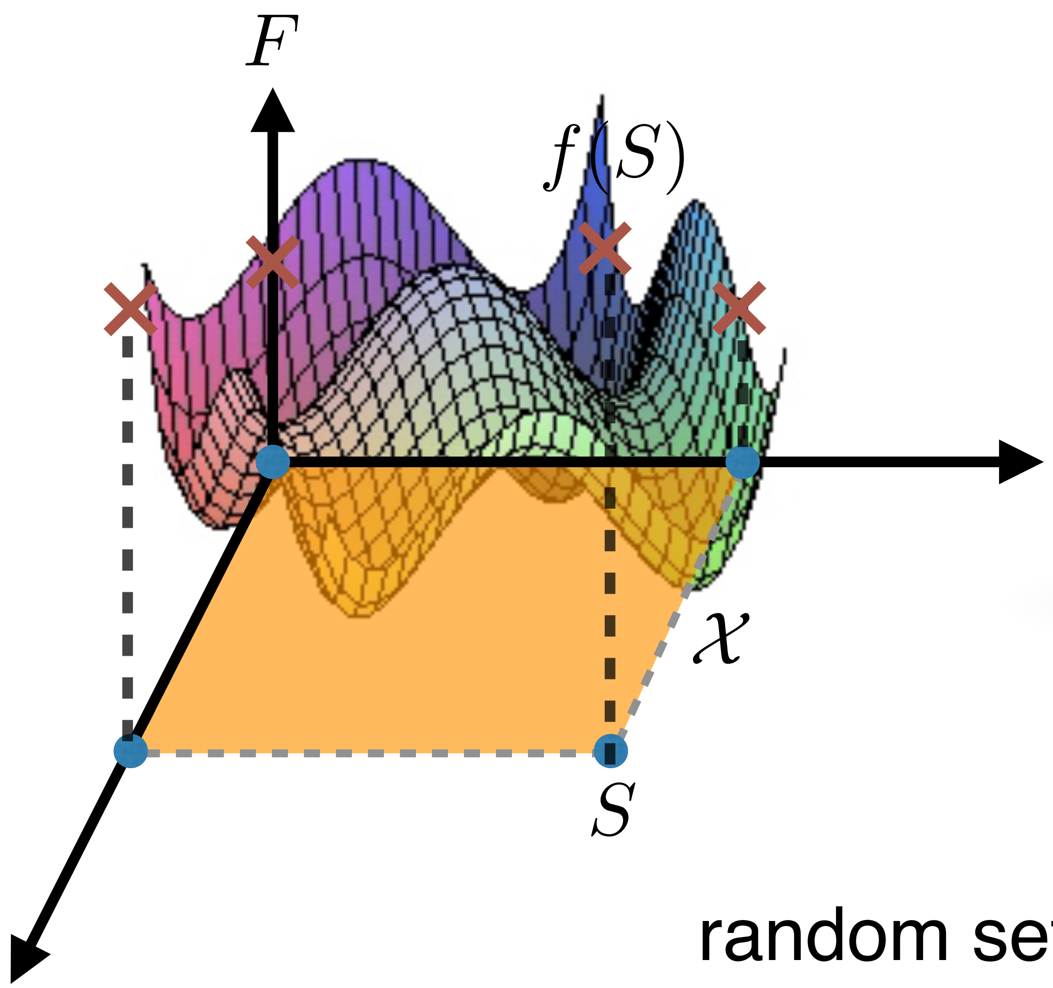

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Yale $\qquad$ Penn

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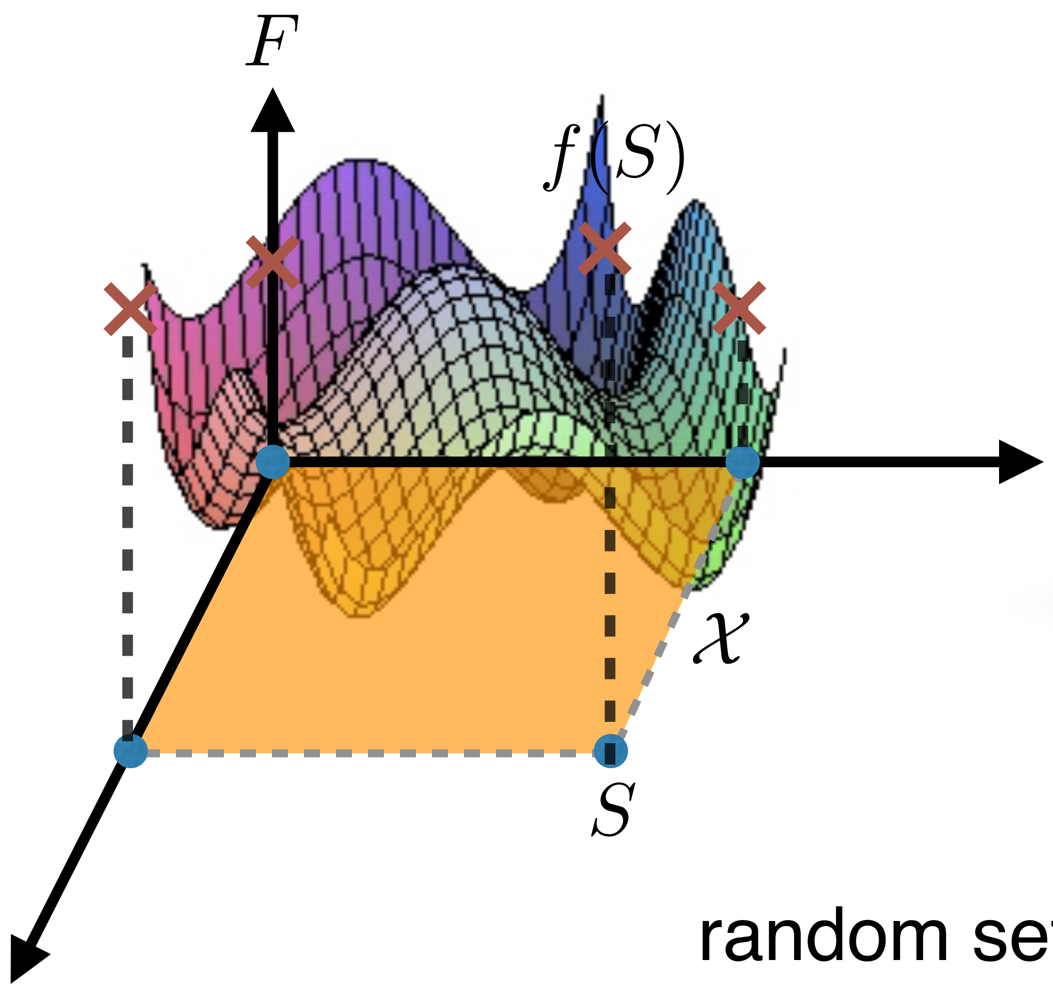

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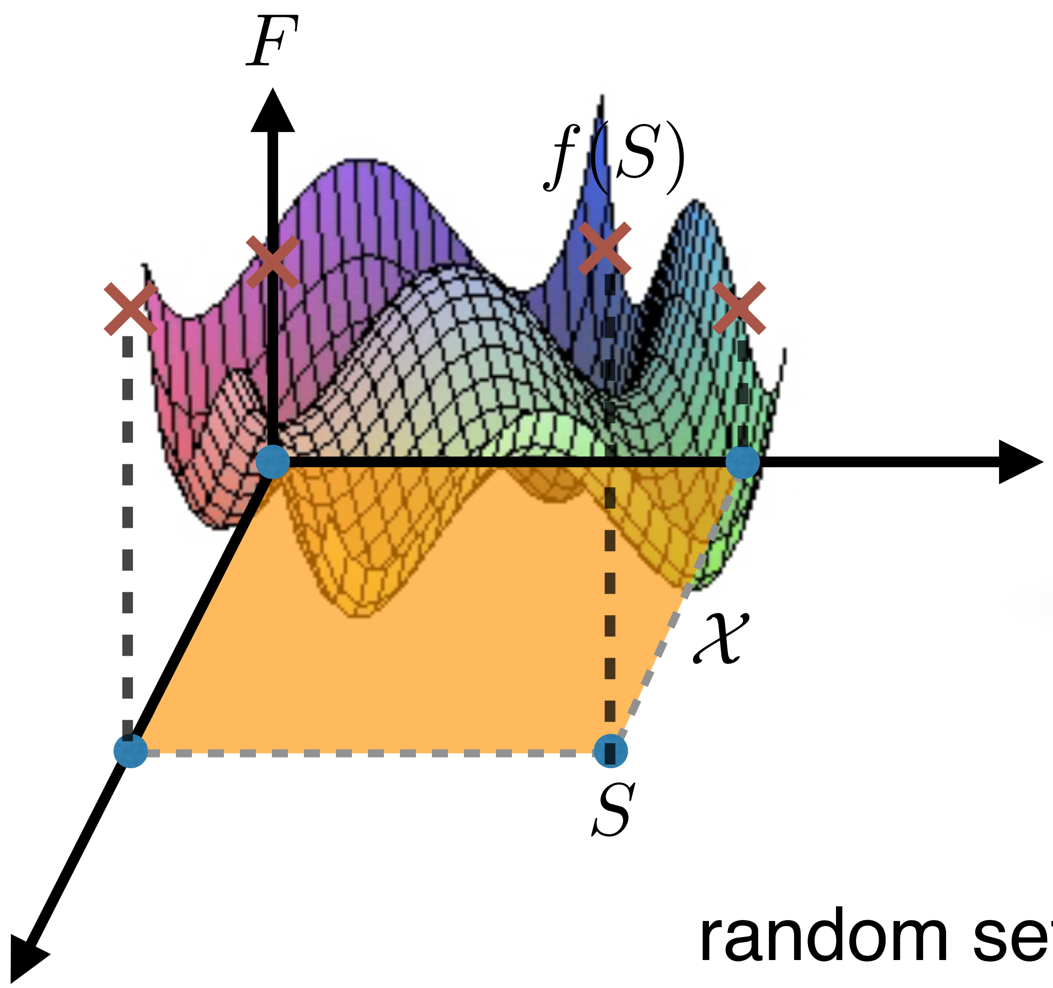

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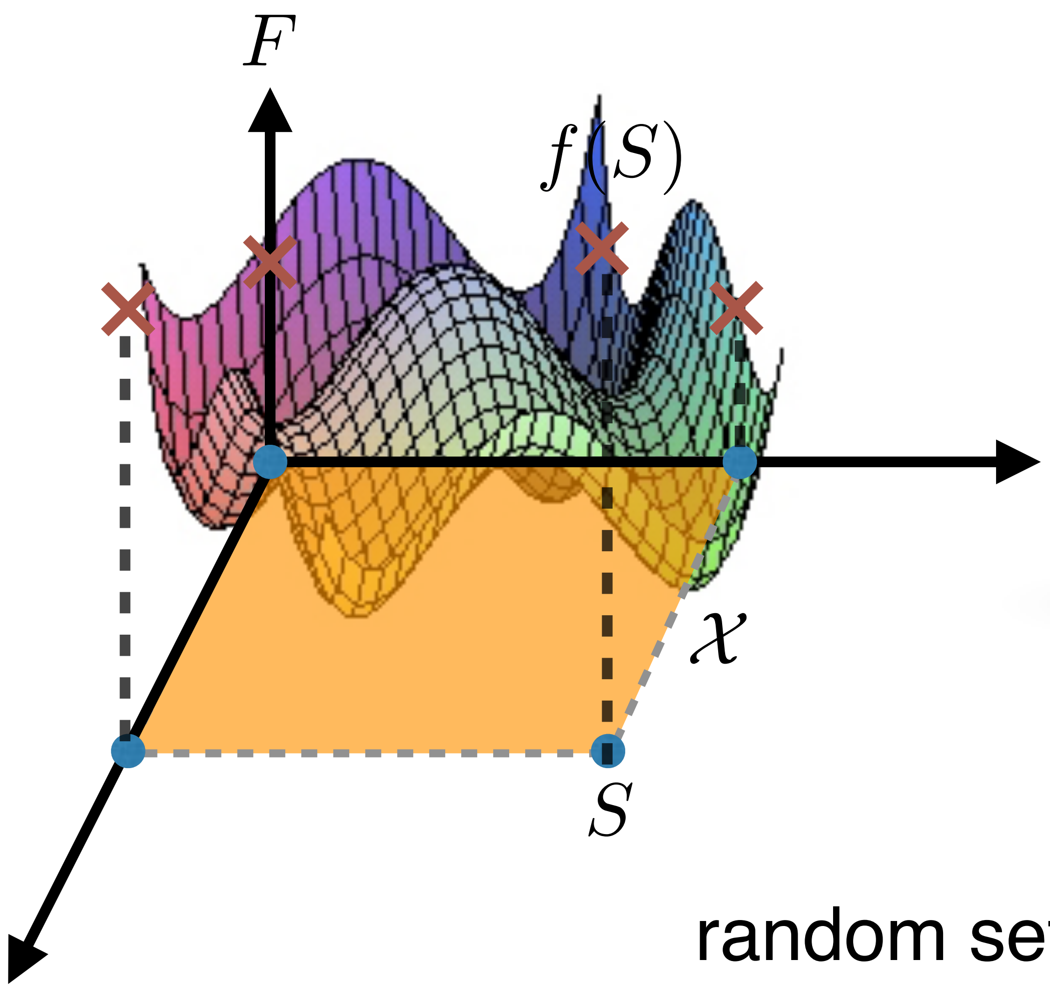

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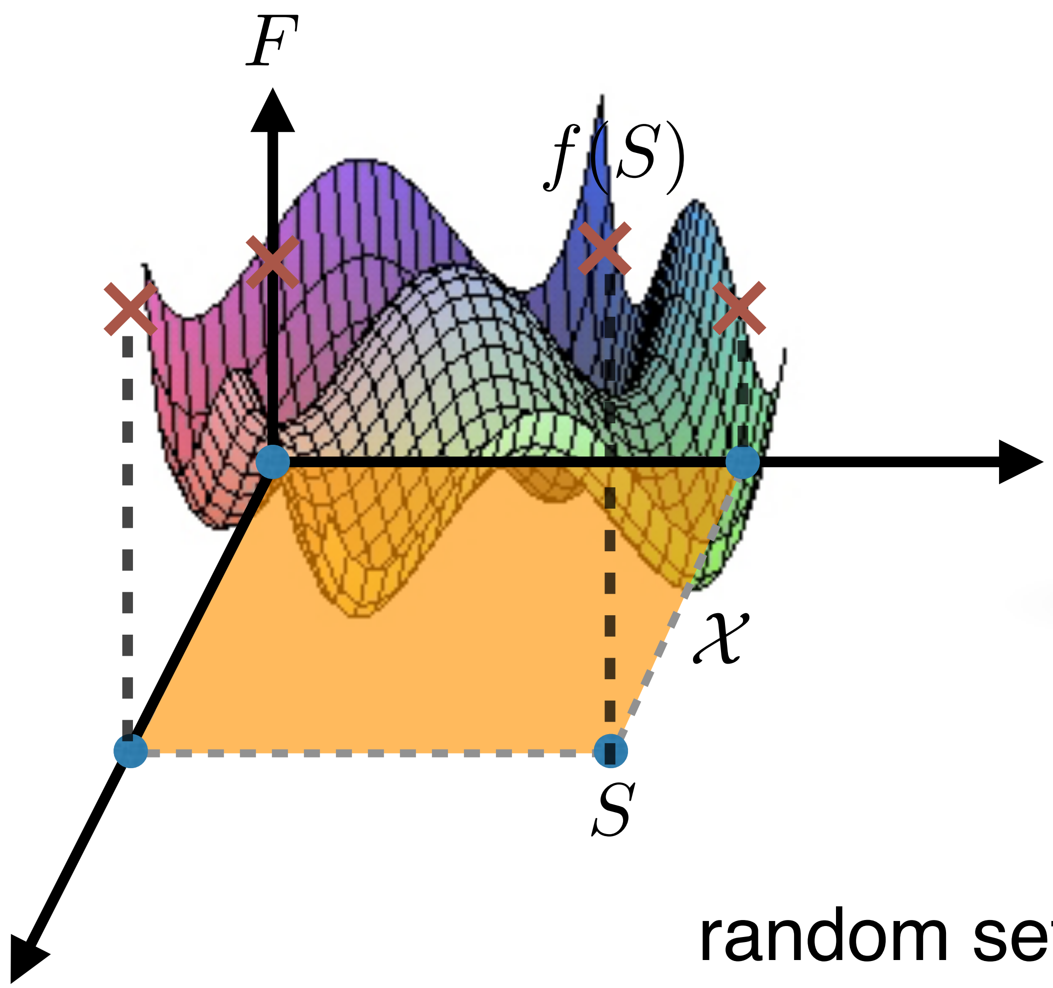

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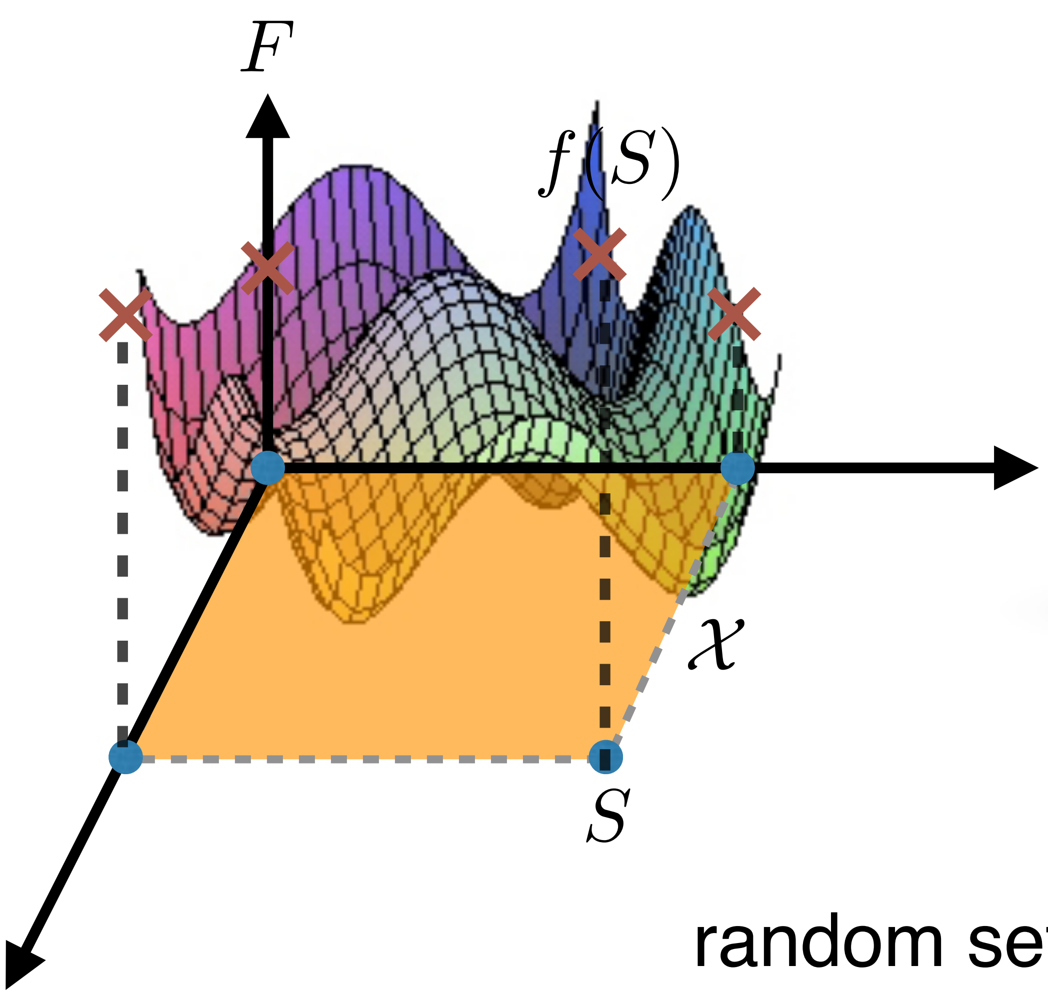

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$$\nabla^2 F(\mathbf{x}) = \begin{pmatrix} 0 & \leq 0 & \leq 0 \\ \leq 0 & 0 & \leq 0 \\ \leq 0 & \leq 0 & 0 \end{pmatrix}$$

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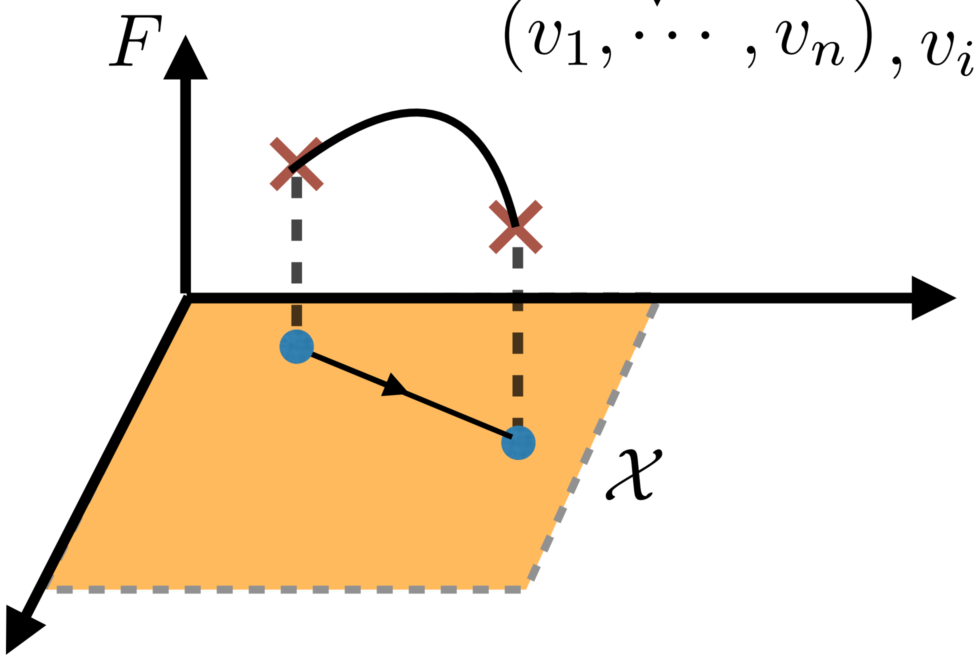

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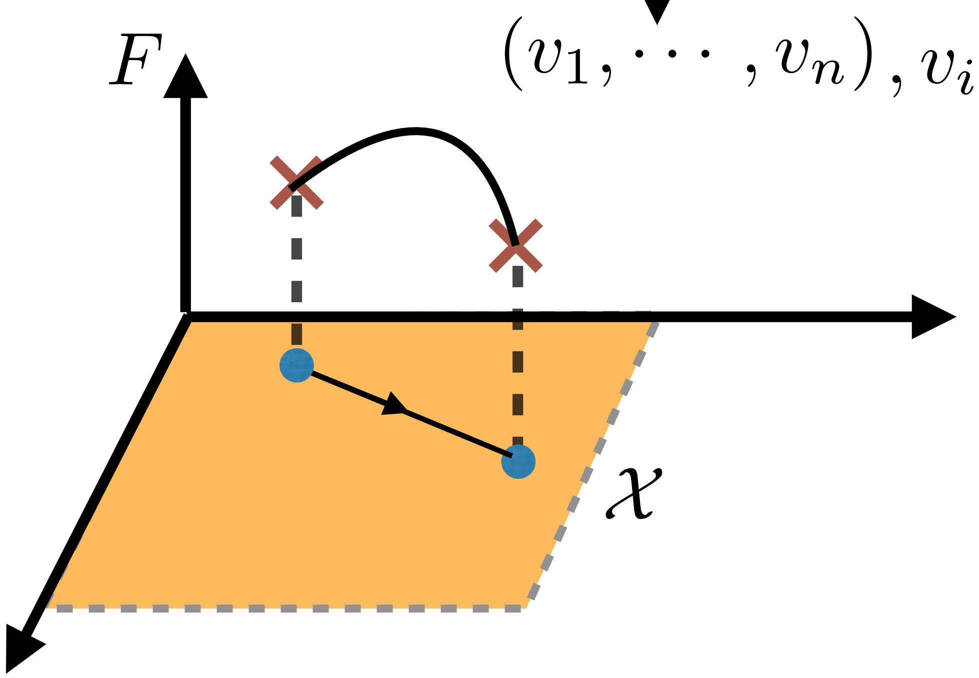

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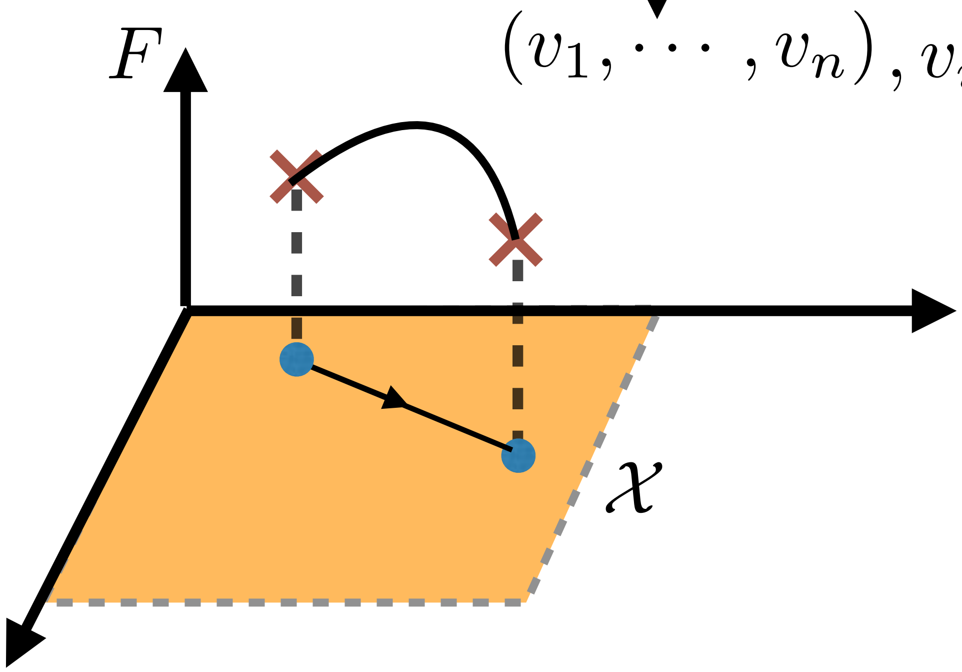

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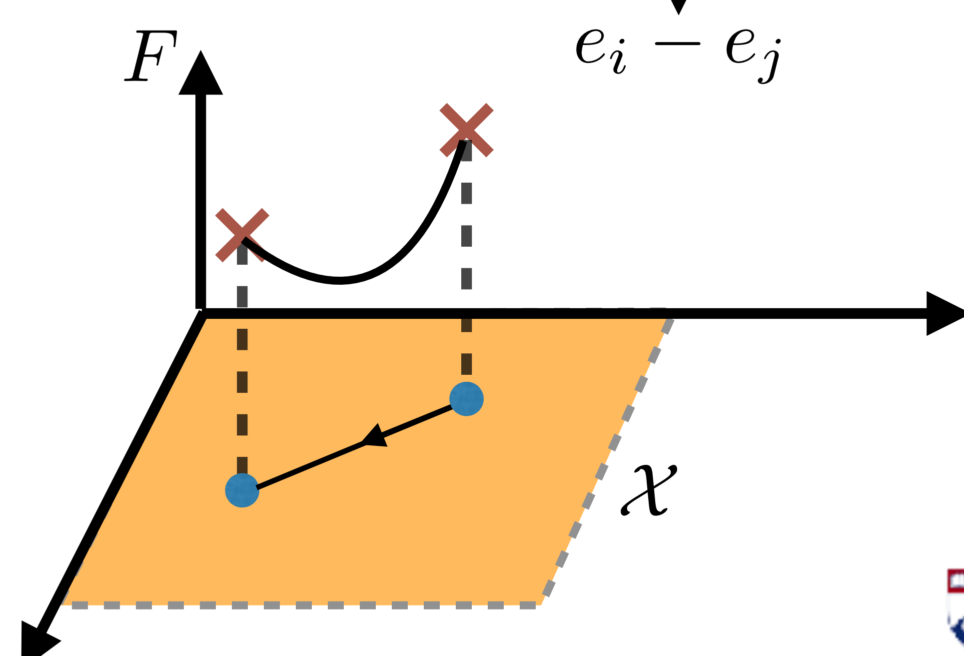

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(matroid)

$$\begin{tikzcd}
\boxed{\max_{S \in \mathcal{I}} f(S)} \arrow[r] & \boxed{\max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x})} \arrow[d] \\
 & \boxed{\mathbf{x}_{\text{sol}} \in \mathcal{K}}
\end{tikzcd}$$

Equivalent Formulations

Let f be a submodular set function with multi-linear extension F, then:

$$\max_{S \in \mathcal{I}} f(S) \quad = \quad \max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x})$$

$$\mathcal{I} = \{S \subseteq V : |S| \leq k\} \quad \longrightarrow \quad \mathcal{K} = \{(x_1, \cdots, x_n) \in [0,1]^n : \sum_{i=1}^{n} x_i \leq k\}$$

(cardinality)

$$\mathcal{I} = \{S \subseteq V : \sum_{i \in S} c_i \leq B\} \quad \longrightarrow \quad \mathcal{K} = \{(x_1, \cdots, x_n) \in [0,1]^n : \sum_{i=1}^{n} c_i x_i \leq B\}$$

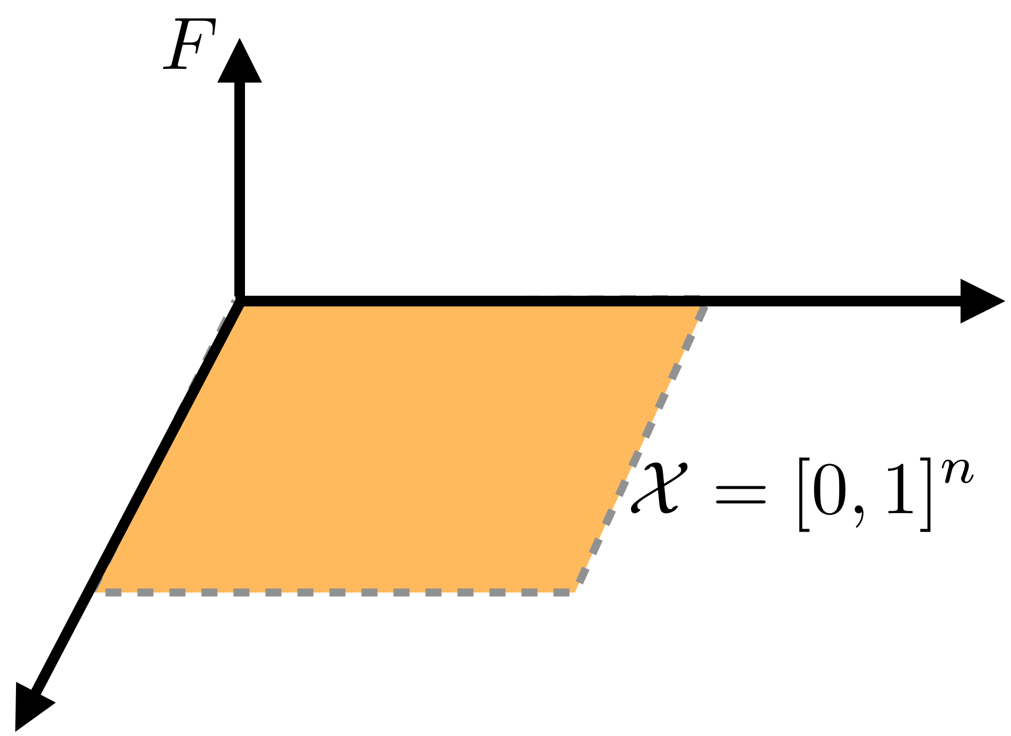
(knapsack)

$\mathcal{I}$ is a matroid $\longrightarrow$ $\mathcal{K}$ is the base polytope of the matroid

(matroid)

$$\begin{tikzcd}
\max_{S \in \mathcal{I}} f(S) \arrow[r] & \max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x}) \arrow[d] \\
S_{\text{sol}} \in \mathcal{I} \arrow[u] & \mathbf{x}_{\text{sol}} \in \mathcal{K} \arrow[l, "\text{rounding}"']
\end{tikzcd}$$

Rounding



$$
\begin{tikzcd}
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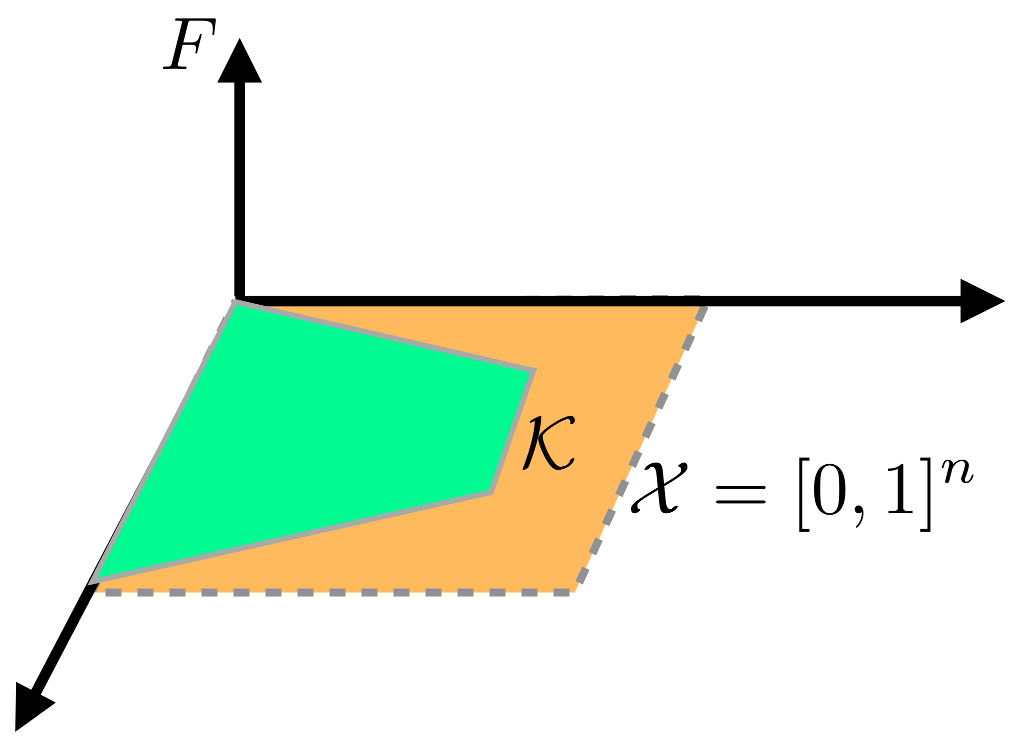
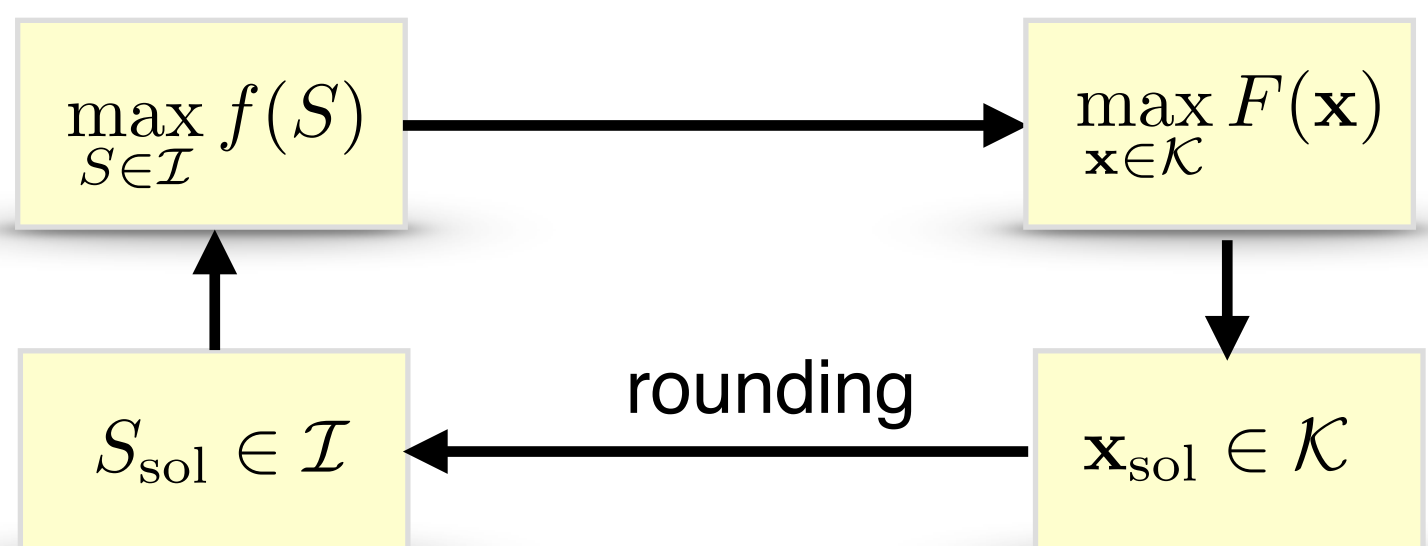
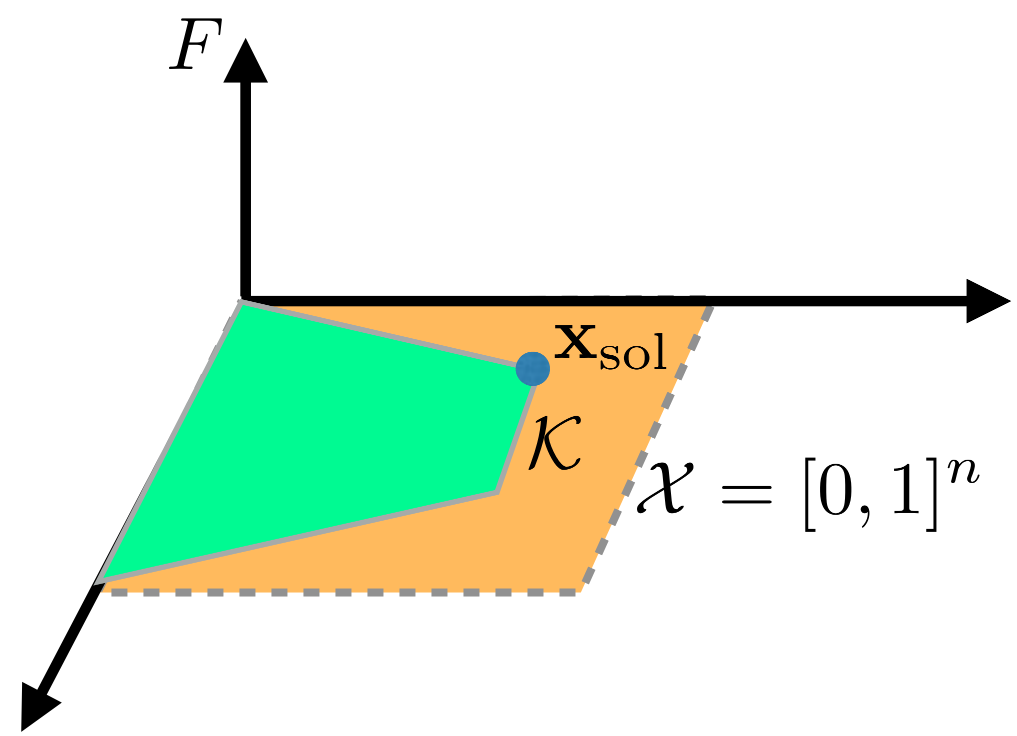
S_{\text{sol}} \in \mathcal{I} \arrow[u] & \mathbf{x}_{\text{sol}} \in \mathcal{K} \arrow[l, "\text{rounding}"']
\end{tikzcd}
$$

Rounding


F
$\mathcal{K}$
$\mathcal{X} = [0,1]^n$

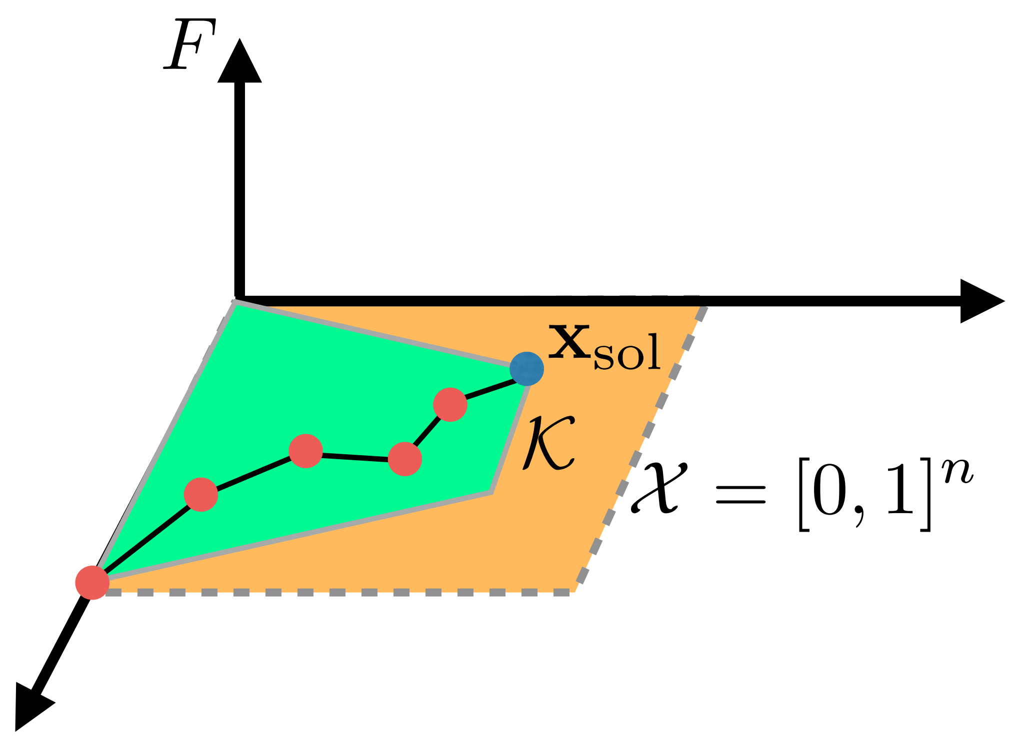

$\max_{S \in \mathcal{I}} f(S)$
$\max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x})$
$\mathbf{x}_{\text{sol}} \in \mathcal{K}$
rounding
$S_{\text{sol}} \in \mathcal{I}$

Rounding

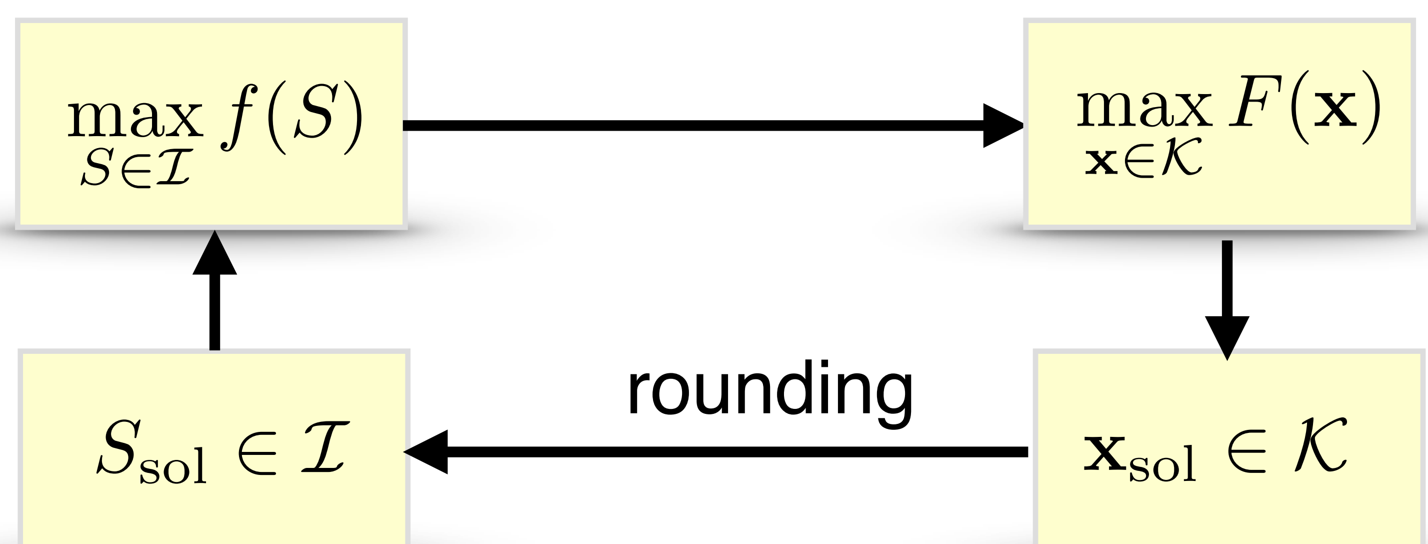


$$
\begin{tikzcd}
\max\limits_{S\in\mathcal{I}} f(S) \arrow[r] & \max\limits_{\mathbf{x}\in\mathcal{K}} F(\mathbf{x}) \arrow[d] \\
S_{\rm sol} \in \mathcal{I} \arrow[u] & \mathbf{x}_{\rm sol} \in \mathcal{K} \arrow[l, "\text{rounding}"']
\end{tikzcd}
$$

Rounding

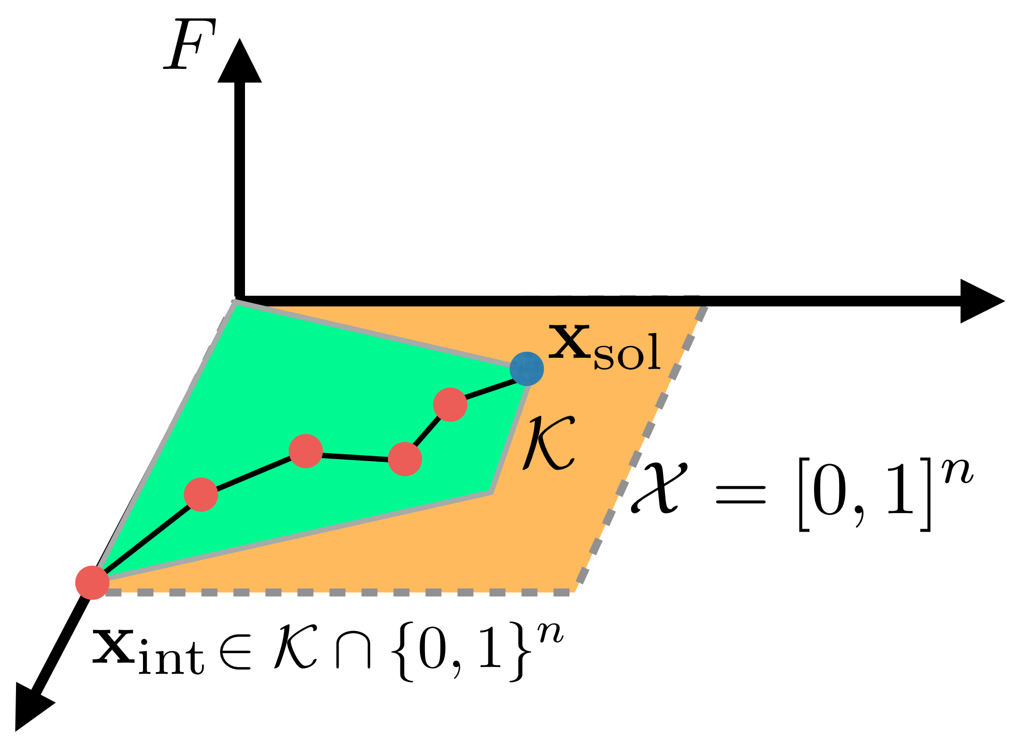


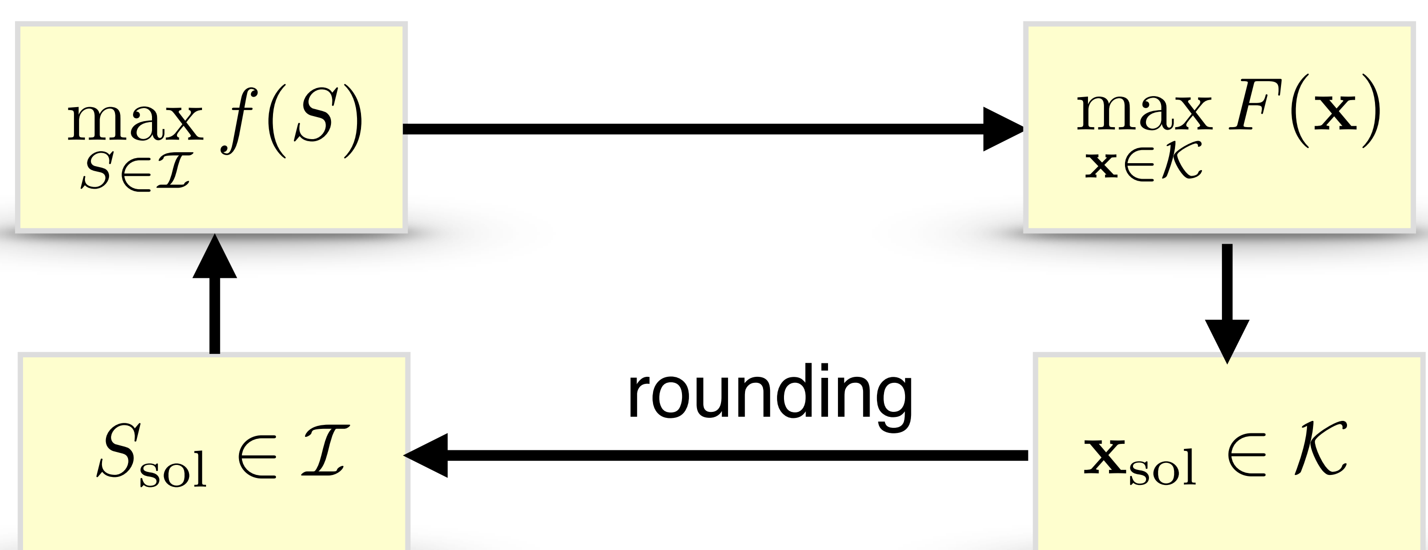
F
$\mathbf{x}_{\text{sol}}$
$\mathcal{K}$
$\mathcal{X} = [0,1]^n$



$\max_{S \in \mathcal{I}} f(S)$
$\max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x})$
$\mathbf{x}_{\text{sol}} \in \mathcal{K}$
rounding
$S_{\text{sol}} \in \mathcal{I}$

Rounding


F
$\mathbf{x}_{\text{sol}}$
$\mathcal{K}$
$\mathcal{X} = [0,1]^n$
$\mathbf{x}_{\text{int}} \in \mathcal{K} \cap \{0,1\}^n$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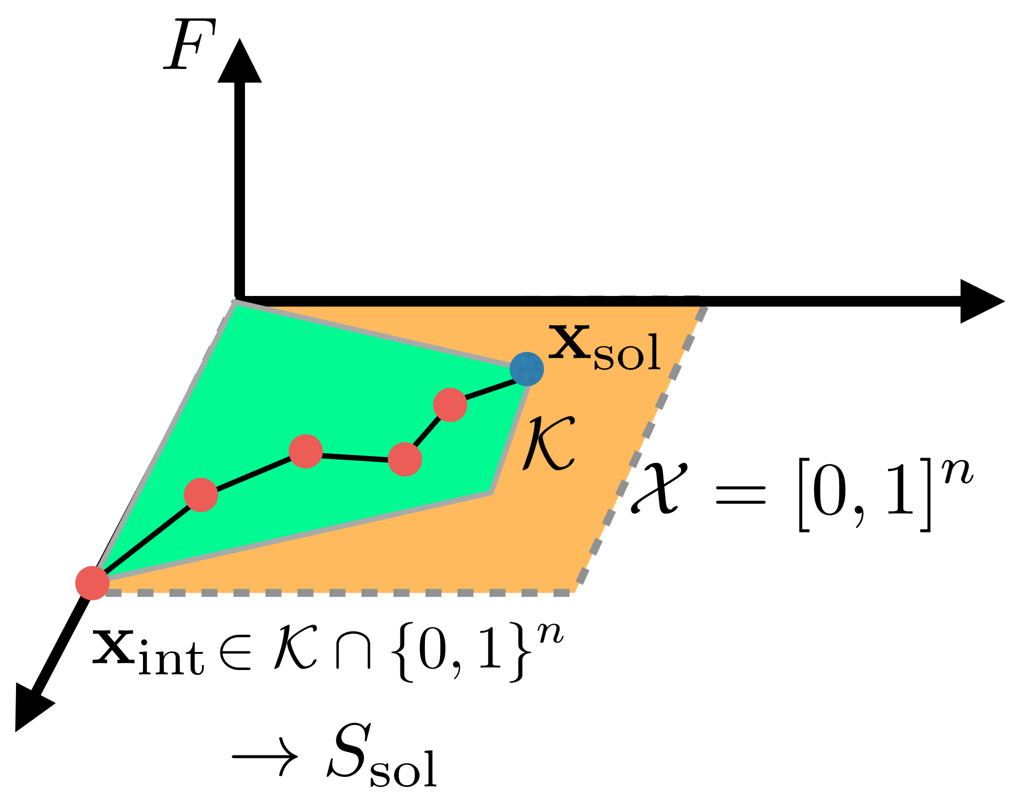
Rounding



$$
\begin{tikzcd}
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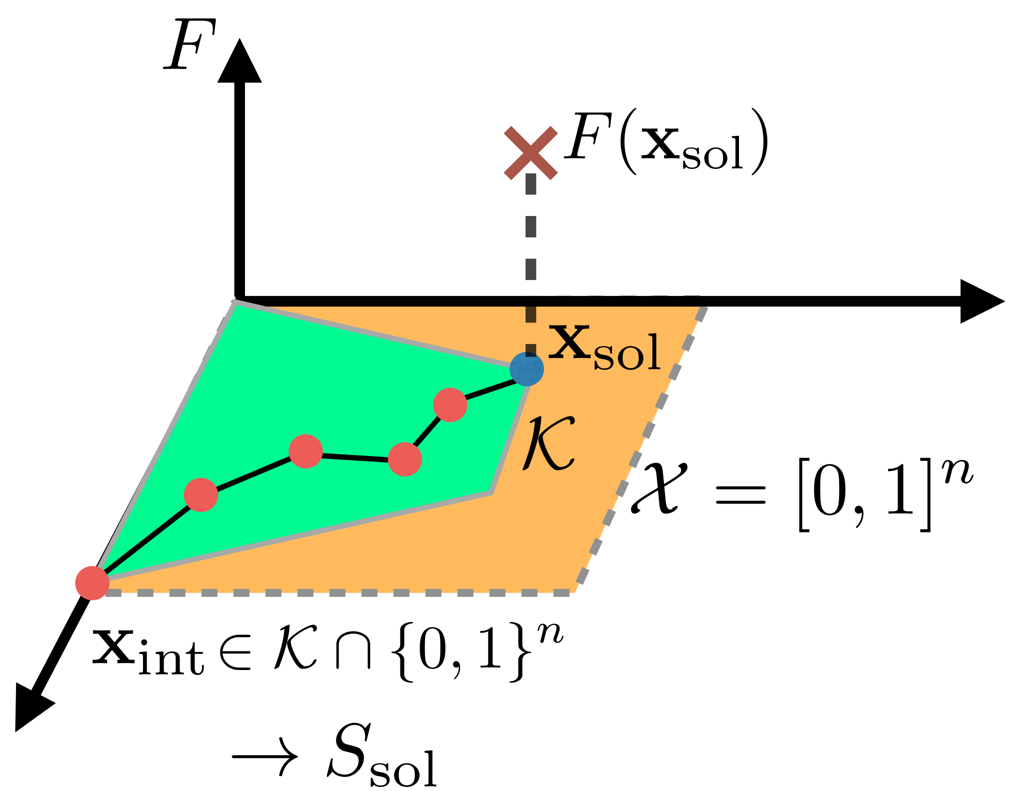
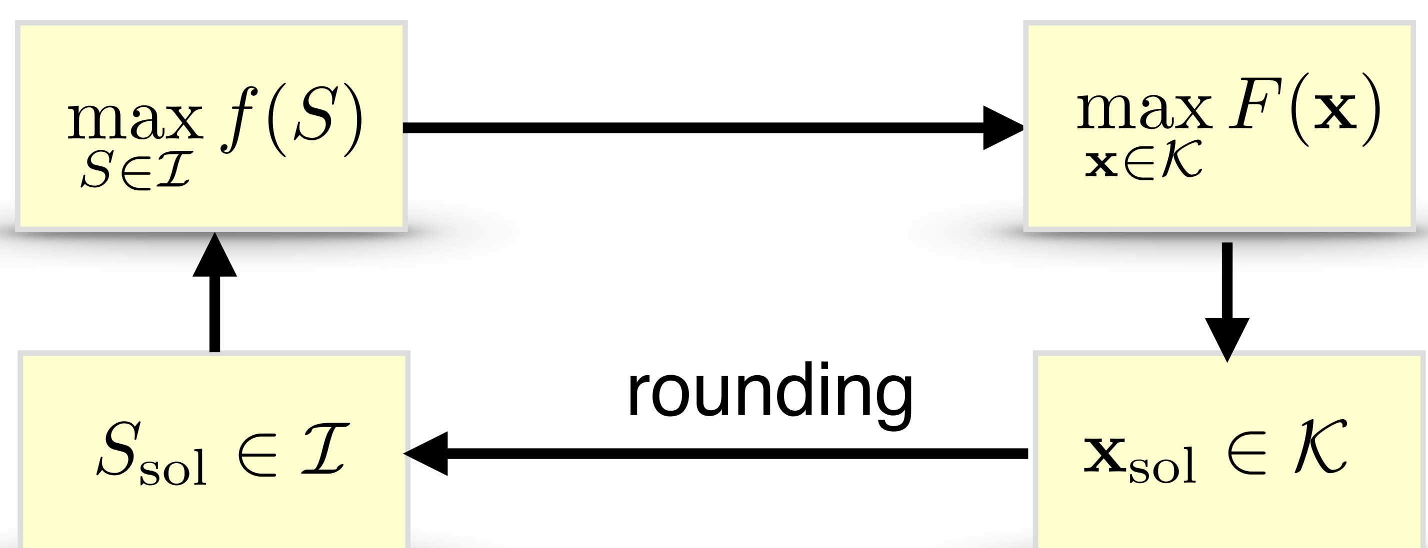
$$

Rounding


F
$F(\mathbf{x}_{\mathrm{sol}})$
$\mathbf{x}_{\mathrm{sol}}$
$\mathcal{K}$
$\mathcal{X} = [0,1]^n$
$\mathbf{x}_{\mathrm{int}} \in \mathcal{K} \cap \{0,1\}^n$
$\to S_{\mathrm{sol}}$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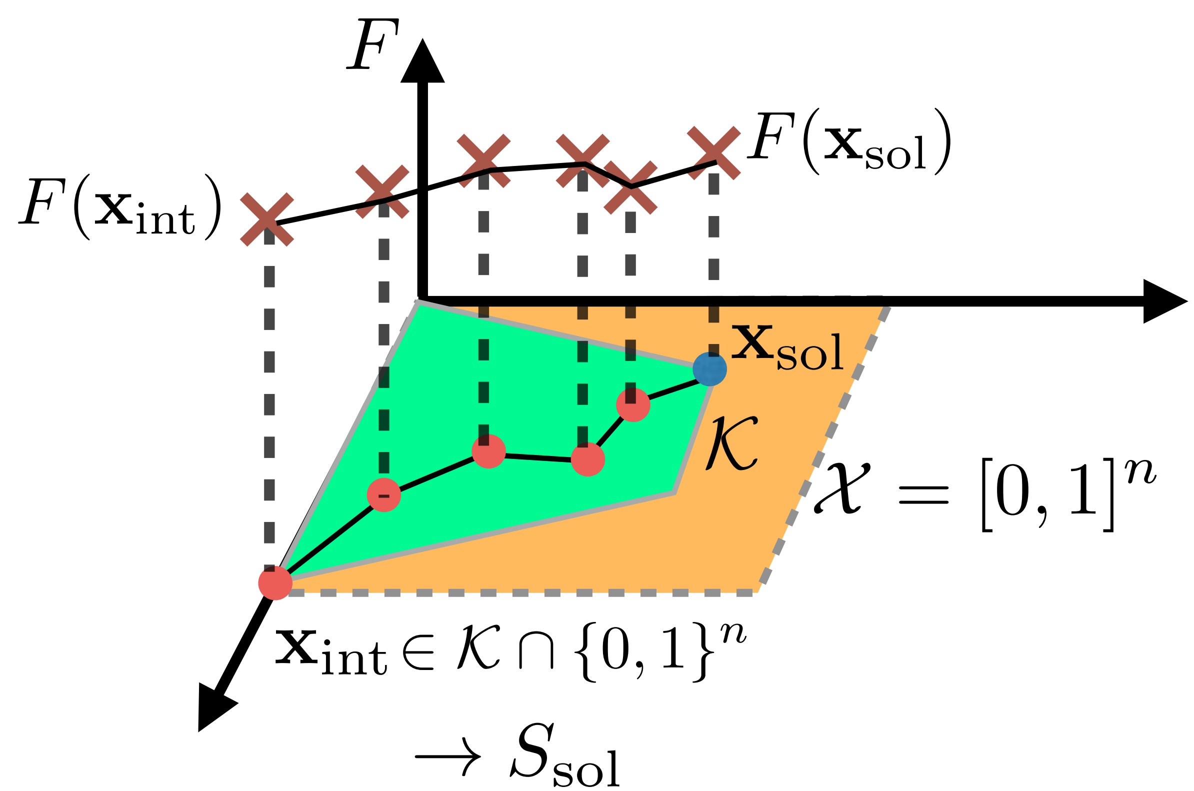
Rounding


F
$F(\mathbf{x}_{\text{int}})$
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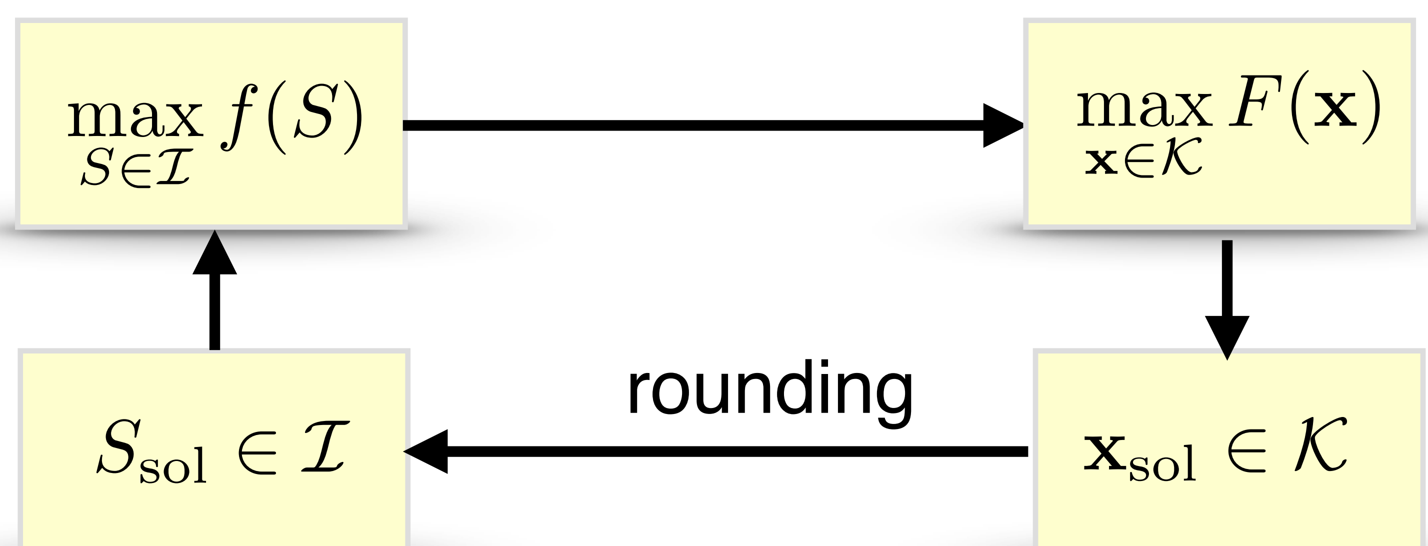
$\max_{S \in \mathcal{I}} f(S)$
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rounding
$S_{\text{sol}} \in \mathcal{I}$
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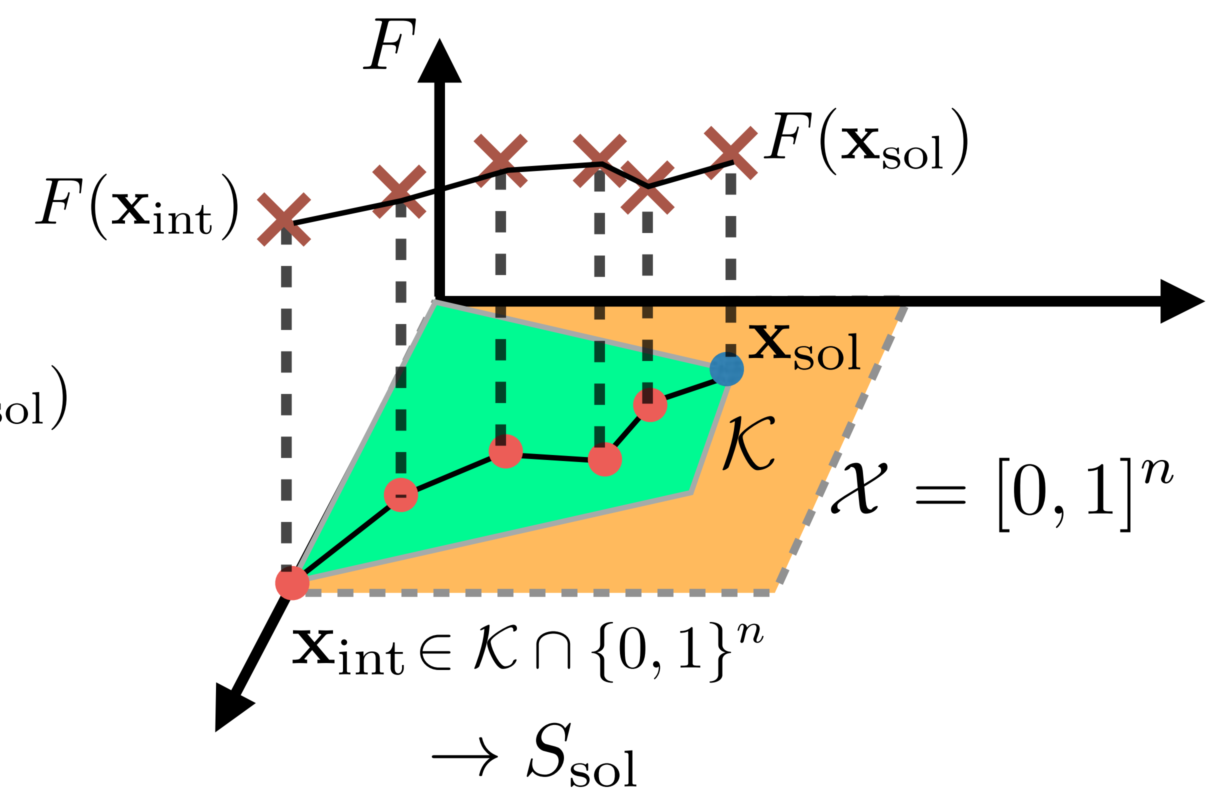
Rounding

$$f(S_{\text{sol}}) = F(\mathbf{x}_{\text{int}}) \geq F(\mathbf{x}_{\text{sol}})$$



$$\begin{tikzcd}
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\end{tikzcd}$$

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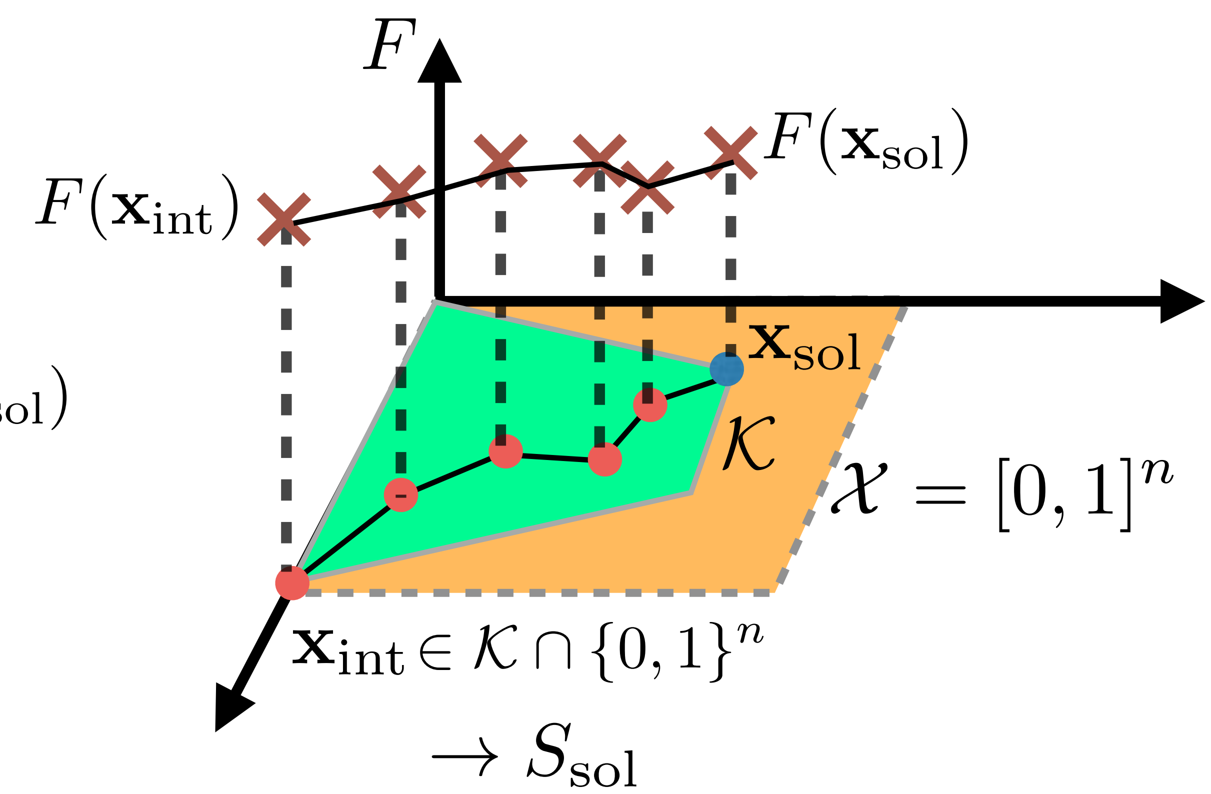
Rounding

$$f(S_{\text{sol}}) = F(\mathbf{x}_{\text{int}}) \geq F(\mathbf{x}_{\text{sol}})$$



Several rounding techniques:

- Pipage rounding **Calinescu et al, 2007, 2011**
- Swap rounding **Chekuri et al, 2010**
- Contention resolution schemes **Chekuri et al, 2014**

Pipage Rounding

- Used for Matroid constraints
- We describe the special case of cardinality constraints (for simplicity)

Pipage Rounding

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Pipage Rounding

$\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$ → $\mathbf{x}_{\text{int}} = (x_1, \cdots, x_n) \in \{0,1\}^n$

$$\sum_{i=1}^{n} x_i = k$$

$$\sum_{i=1}^{n} x_i = k$$

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Pipage Rounding

$\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$ → $\mathbf{x}_{\text{int}} = (x_1, \cdots, x_n) \in \{0,1\}^n$

$$\sum_{i=1}^{n} x_i = k \qquad \sum_{i=1}^{n} x_i = k$$

while $\mathbf{x}$ is not integer-valued:

pick two non-integer coordinates x_i, x_j

if $x_i + x_j \leq 1$:

Let
$$\mathbf{x}_1 = \mathbf{x} + x_j(\mathbf{e}_i - \mathbf{e}_j)$$
$$\mathbf{x}_2 = \mathbf{x} - x_i(\mathbf{e}_i - \mathbf{e}_j)$$

if $F(\mathbf{x}_1) \geq F(\mathbf{x}_2)$: $\mathbf{x} \leftarrow \mathbf{x}_1$

else: $\mathbf{x} \leftarrow \mathbf{x}_2$

Pipage Rounding

$$\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n \quad \longrightarrow \quad \mathbf{x}_{\text{int}} = (x_1, \cdots, x_n) \in \{0,1\}^n$$

$$\sum_{i=1}^{n} x_i = k \qquad\qquad \sum_{i=1}^{n} x_i = k$$

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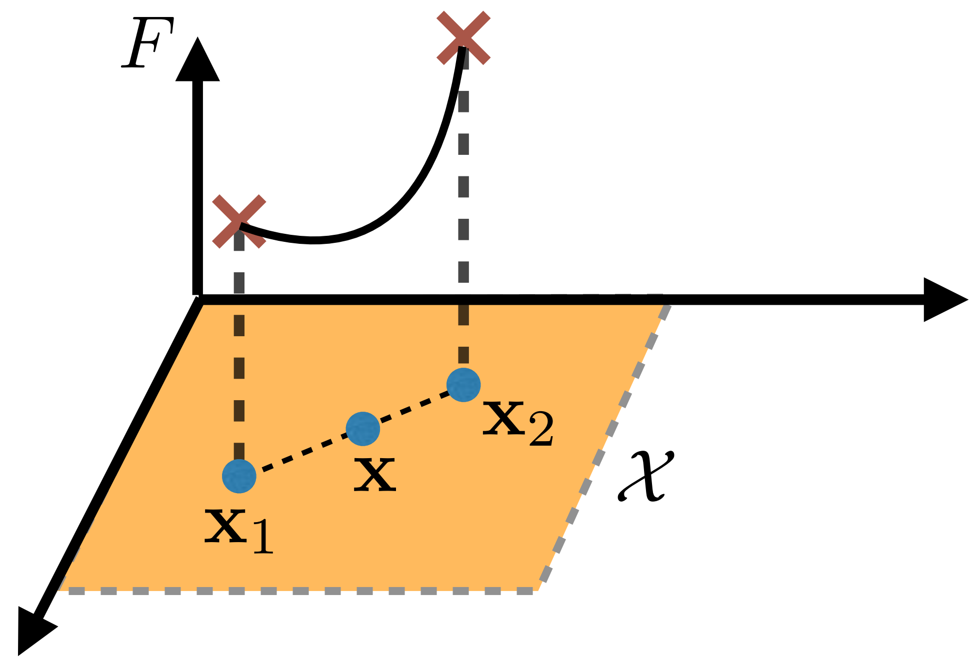


$$\begin{array}{c} \\ \mathbf{x}_1 = \end{array}\begin{array}{c} \\ (\cdots, \end{array}\begin{array}{c} i \\ \downarrow \\ x_i + x_j, \end{array}\begin{array}{c} \\ \\ \cdots, \end{array}\begin{array}{c} j \\ \downarrow \\ 0, \end{array}\begin{array}{c} \\ \\ \cdots) \end{array}$$

$$\begin{array}{c} \\ \mathbf{x}_2 = \end{array}\begin{array}{c} \\ (\cdots, \end{array}\begin{array}{c} i \\ \downarrow \\ 0, \end{array}\begin{array}{c} \\ \\ \cdots, \end{array}\begin{array}{c} j \\ \downarrow \\ x_i + x_j, \end{array}\begin{array}{c} \\ \\ \cdots) \end{array}$$

Pipage Rounding

$$\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n \qquad \rightarrow \qquad \mathbf{x}_{\text{int}} = (x_1, \cdots, x_n) \in \{0,1\}^n$$

$$\sum_{i=1}^{n} x_i = k \qquad\qquad \sum_{i=1}^{n} x_i = k$$

while $\mathbf{x}$ is not integer-valued:

pick two non-integer coordinates x_i, x_j

if $x_i + x_j \leq 1$:

Let
$$\mathbf{x}_1 = \mathbf{x} + x_j(\mathbf{e}_i - \mathbf{e}_j)$$
$$\mathbf{x}_2 = \mathbf{x} - x_i(\mathbf{e}_i - \mathbf{e}_j)$$

if $F(\mathbf{x}_1) \geq F(\mathbf{x}_2)$: $\mathbf{x} \leftarrow \mathbf{x}_1$

else: $\mathbf{x} \leftarrow \mathbf{x}_2$

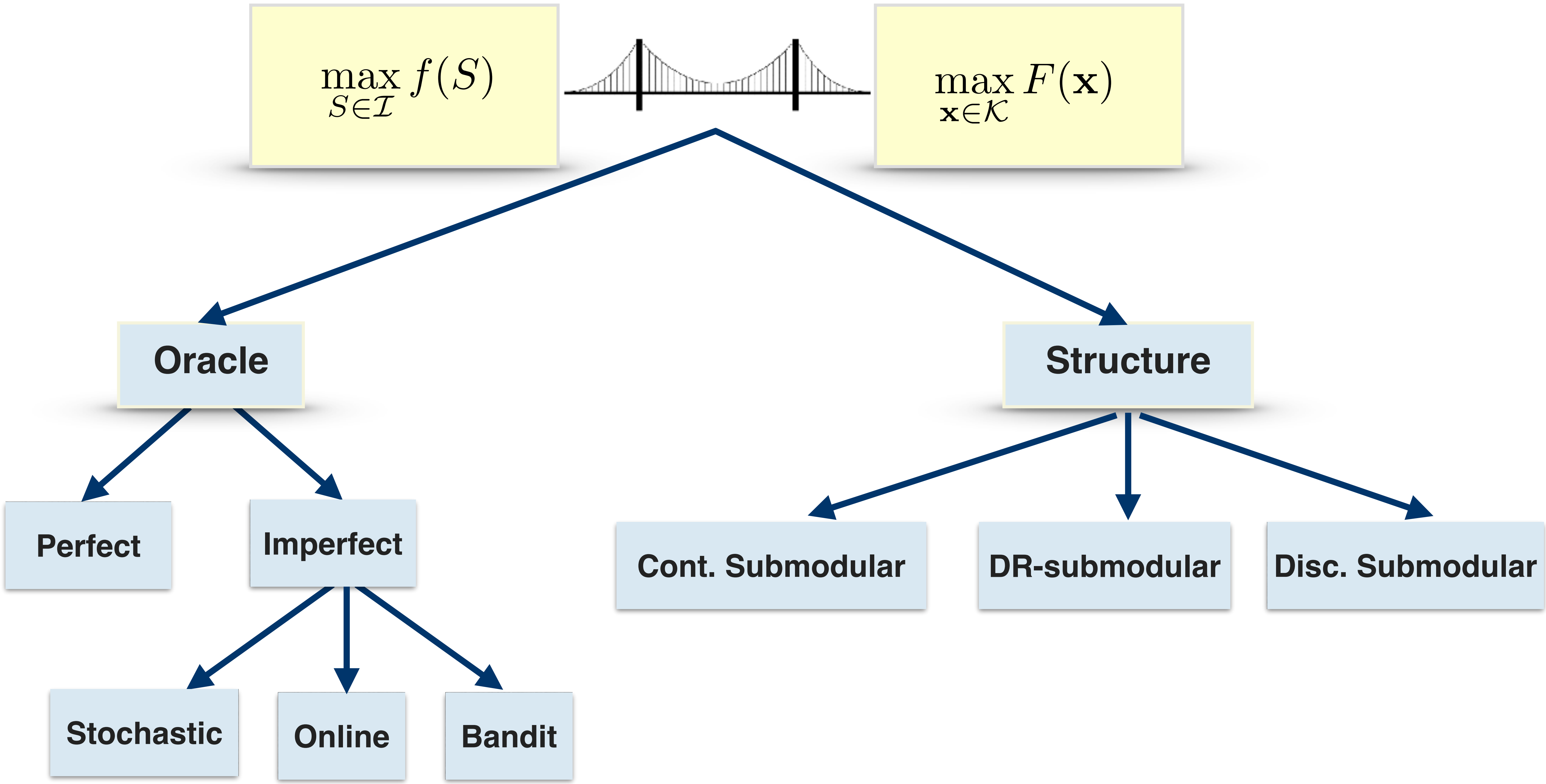
if $x_i + x_j > 1$:

Let
$$\mathbf{x}_1 = \mathbf{x} + (x_j - 1)(\mathbf{e}_i - \mathbf{e}_j)$$
$$\mathbf{x}_2 = \mathbf{x} - (x_i - 1)(\mathbf{e}_i - \mathbf{e}_j)$$

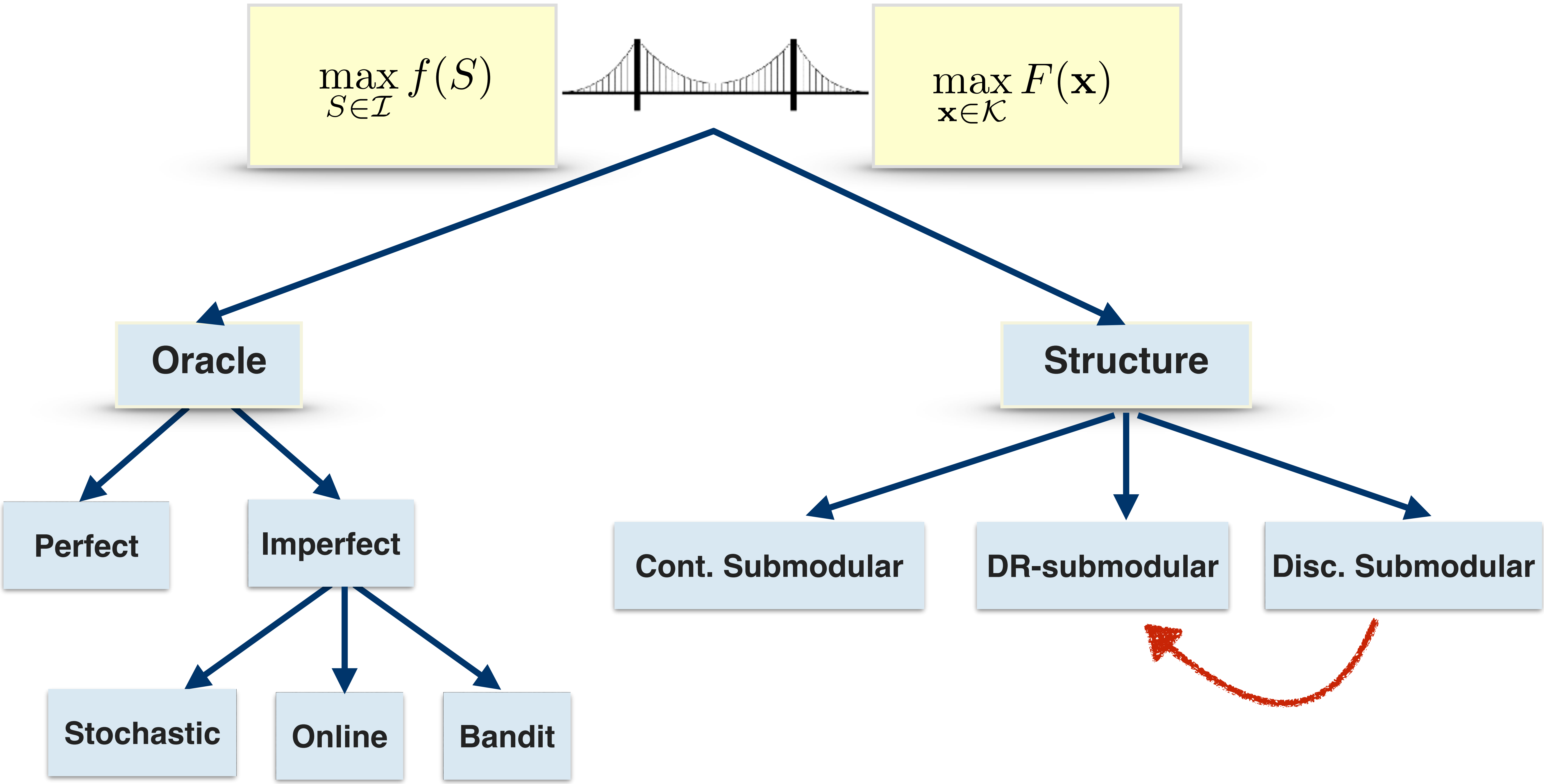
if $F(\mathbf{x}_1) \geq F(\mathbf{x}_2)$: $\mathbf{x} \leftarrow \mathbf{x}_1$

else: $\mathbf{x} \leftarrow \mathbf{x}_2$

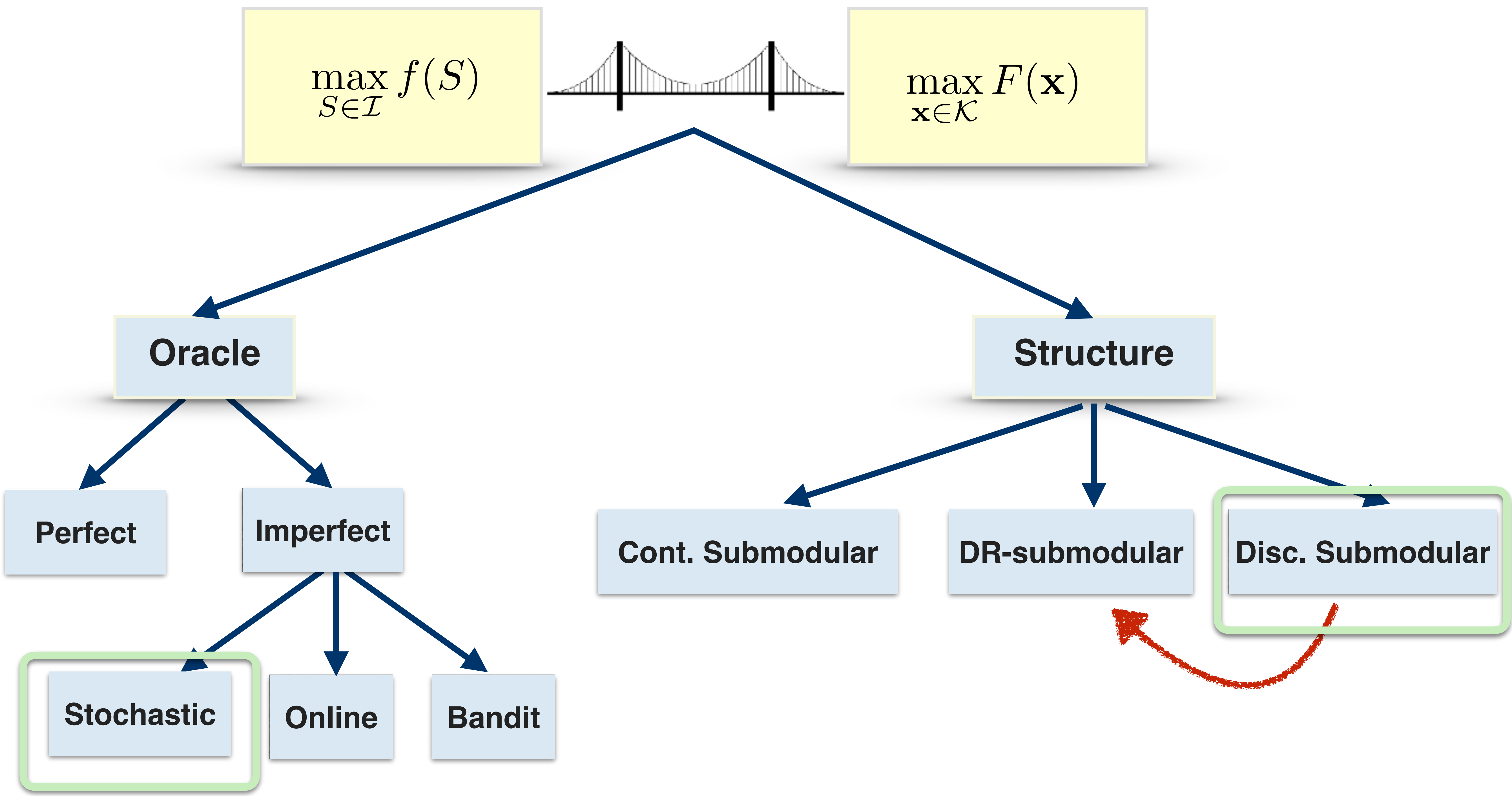
Submodular Maximization: Discrete and Continuous



Submodular Maximization: Discrete and Continuous



Submodular Maximization: Discrete and Continuous



Stochastic Submodular Maximization: Discrete

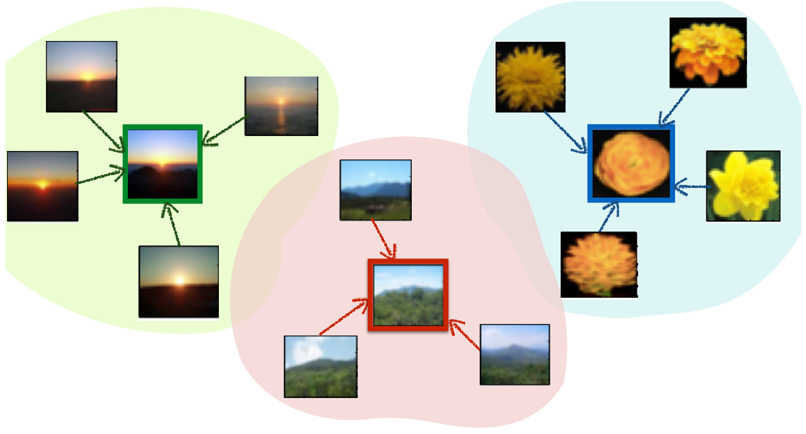
$$\max_{S \in \mathcal{I}} f(S) \doteq \max_{S \in \mathcal{I}} \mathop{\mathbb{E}}_{\theta \sim D} [f_\theta(S)]$$

Stochastic Submodular Maximization: Discrete

$$\max_{S\in\mathcal{I}} f(S) \doteq \max_{S\in\mathcal{I}} \mathop{\mathbb{E}}_{\theta\sim D}\left[f_\theta(S)\right]$$

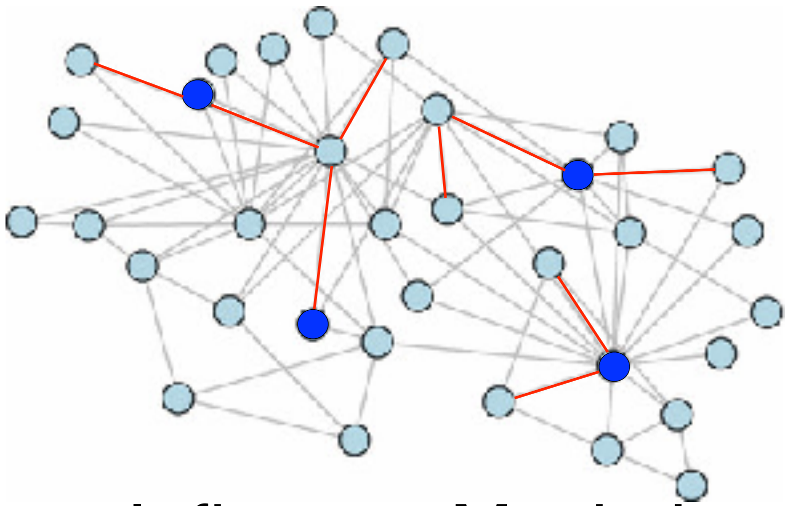
Stochastic Submodular Maximization: Discrete

$$\max_{S \in \mathcal{I}} f(S) \doteq \max_{S \in \mathcal{I}} \mathop{\mathbb{E}}_{\theta \sim D} [f_\theta(S)]$$



Data Summarization

$$f(S) = \frac{1}{|V|} \sum_{i \in V} f_i(S)$$



Influence Maximization

$$f(S) = \mathbb{E}_{H \sim D}[f_H(S)]$$

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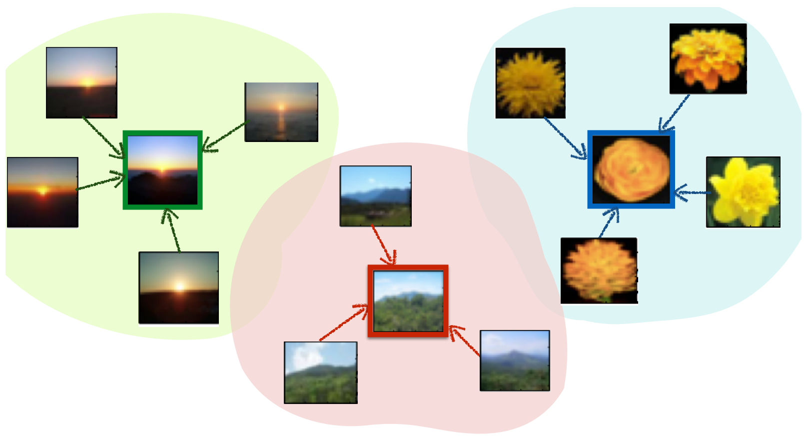
Stochastic Submodular Maximization: Discrete

$$\max_{S \in \mathcal{I}} f(S) \doteq \max_{S \in \mathcal{I}} \mathop{\mathbb{E}}_{\theta \sim D} [f_\theta(S)]$$

- We see f through its samples, i.e., $f_{\theta_1}, f_{\theta_2}, \cdots$



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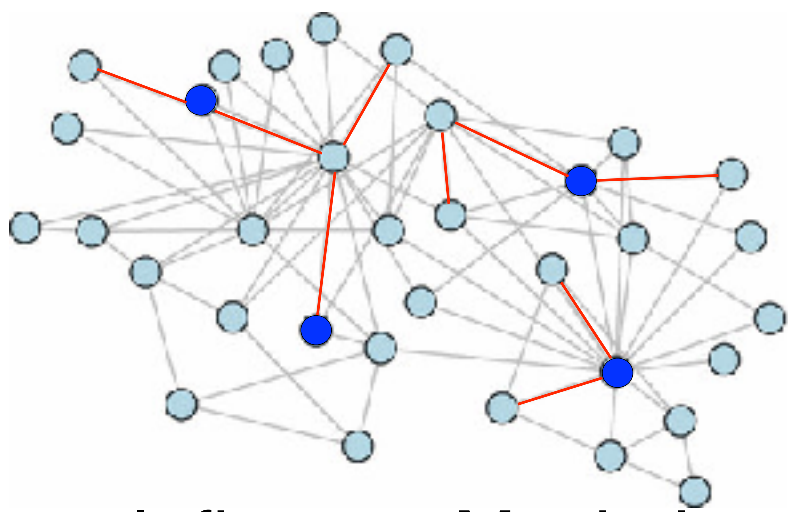


Influence Maximization

$$f(S) = \mathbb{E}_{H \sim D}[f_H(S)]$$

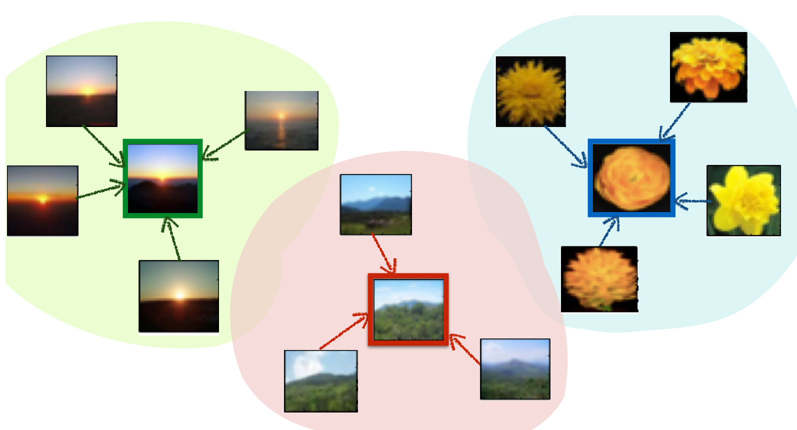
Stochastic Submodular Maximization: Discrete

$$\max_{S \in \mathcal{I}} f(S) \doteq \max_{S \in \mathcal{I}} \mathbb{E}_{\theta \sim D} [f_\theta(S)]$$

- We see f through its samples, i.e., $f_{\theta_1}, f_{\theta_2}, \cdots$

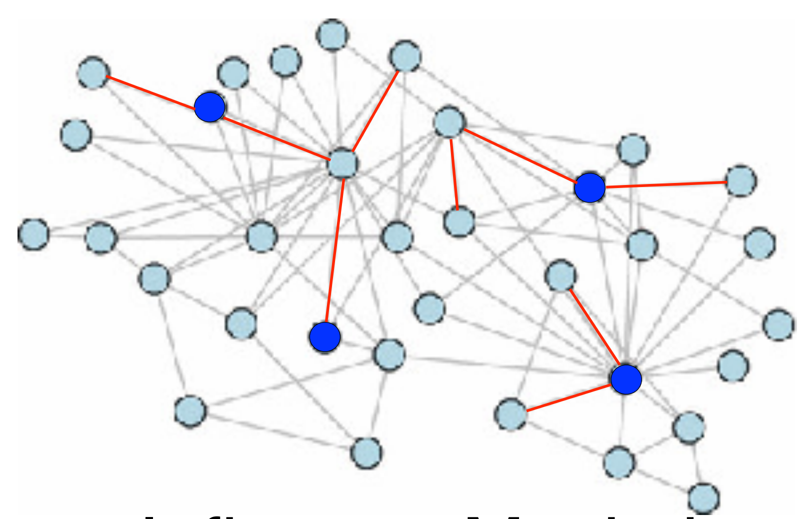
stochastic oracle

$S \uparrow \qquad \downarrow f_\theta(S)$



Data Summarization

$$f(S) = \frac{1}{|V|} \sum_{i \in V} f_i(S)$$



Influence Maximization

$$f(S) = \mathbb{E}_{H \sim D}[f_H(S)]$$

Yale Penn

Stochastic Submodular Maximization: Discrete

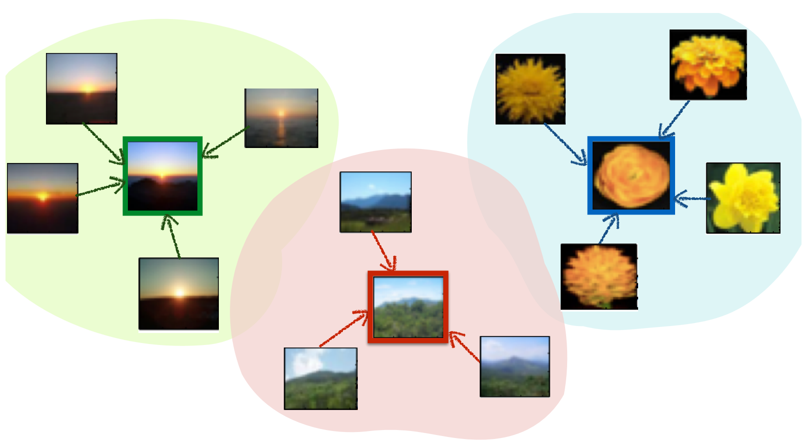
$$\max_{S \in \mathcal{I}} f(S) \doteq \max_{S \in \mathcal{I}} \mathop{\mathbb{E}}_{\theta \sim D} [f_\theta(S)]$$

- We see f through its samples, i.e., $f_{\theta_1}, f_{\theta_2}, \cdots$

stochastic oracle

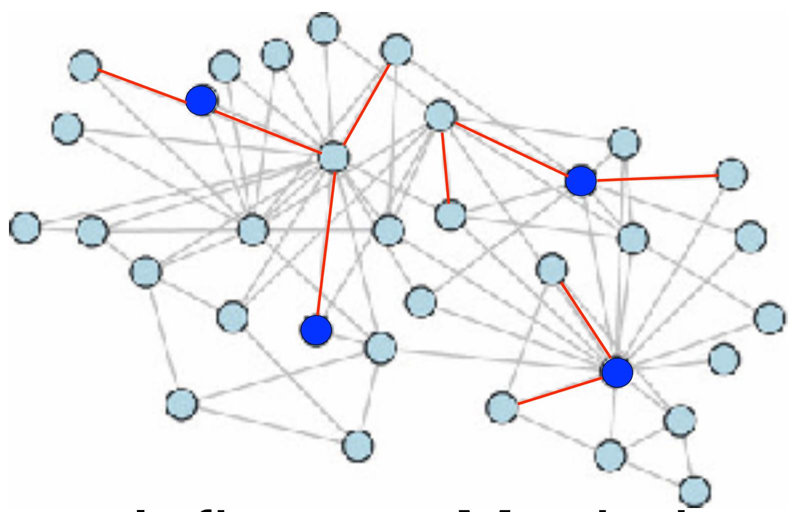
$S \uparrow \quad \downarrow f_\theta(S)$

$$\mathbb{E}_{\theta \sim D}[f_\theta(S)] = f(S)$$



Data Summarization

$$f(S) = \frac{1}{|V|} \sum_{i \in V} f_i(S)$$



Influence Maximization

$$f(S) = \mathbb{E}_{H \sim D}[f_H(S)]$$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta\sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta\sim D}[F_\theta(\mathbf{x})]$$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta\sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta\sim D}[F_\theta(\mathbf{x})]$$

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$$\nabla F(\mathbf{x}) = \mathbb{E}_{\theta \sim D}[\nabla F_\theta(\mathbf{x})]$$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta \sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta \sim D}[F_\theta(\mathbf{x})]$$

$$\nabla F(\mathbf{x}) = \mathbb{E}_{\theta \sim D}[\nabla F_\theta(\mathbf{x})]$$

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \mathbb{E}_{\theta \sim D, S \sim \mathbf{x}}\left[f_\theta(S \cup \{i\}) - f_\theta(S \setminus \{i\})\right]$$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta \sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta \sim D}[F_\theta(\mathbf{x})]$$

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Stochastic Submodular Maximization: Discrete and Continuous

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stochastic oracle

$S \uparrow$ $\quad\downarrow f_\theta(S)$

$[f_\theta(S)] = f(S)$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta \sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta \sim D}[F_\theta(\mathbf{x})]$$

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stochastic oracle $\longleftrightarrow$ stochastic first order oracle

$[f_\theta(S)] = f(S)$ $\quad S \uparrow \quad \downarrow f_\theta(S)$

$\mathbf{x} \uparrow \quad \downarrow \mathbf{g} \quad \mathbb{E}[\mathbf{g}] = \nabla F(\mathbf{x})$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta\sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta\sim D}[F_\theta(\mathbf{x})]$$

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stochastic oracle $\longleftrightarrow$ stochastic first order oracle

$[f_\theta(S)] = f(S)$ $\quad S \uparrow \quad \downarrow f_\theta(S)$

$$\max_{S\in\mathcal{I}} f(S)$$

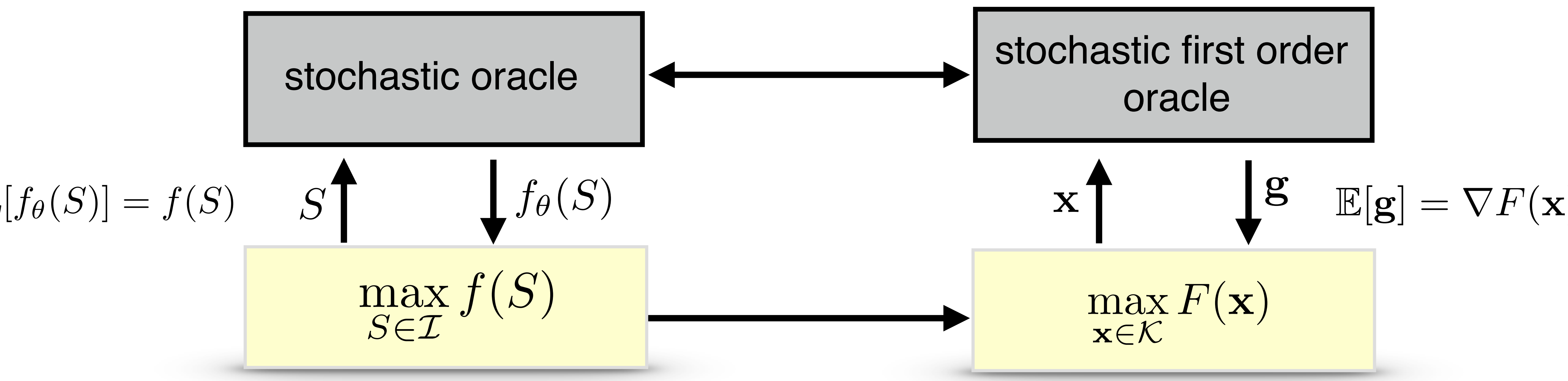
$\mathbf{x} \uparrow \quad \downarrow \mathbf{g} \quad \mathbb{E}[\mathbf{g}] = \nabla F(\mathbf{x})$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta\sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta\sim D}[F_\theta(\mathbf{x})]$$

$$\nabla F(\mathbf{x}) = \mathbb{E}_{\theta\sim D}[\nabla F_\theta(\mathbf{x})]$$

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \mathbb{E}_{\theta\sim D, S\sim\mathbf{x}}\big[\overbrace{f_\theta(S\cup\{i\}) - f_\theta(S\setminus\{i\})}^{\text{unbiased estimate } \mathbf{g}_i}\big]$$



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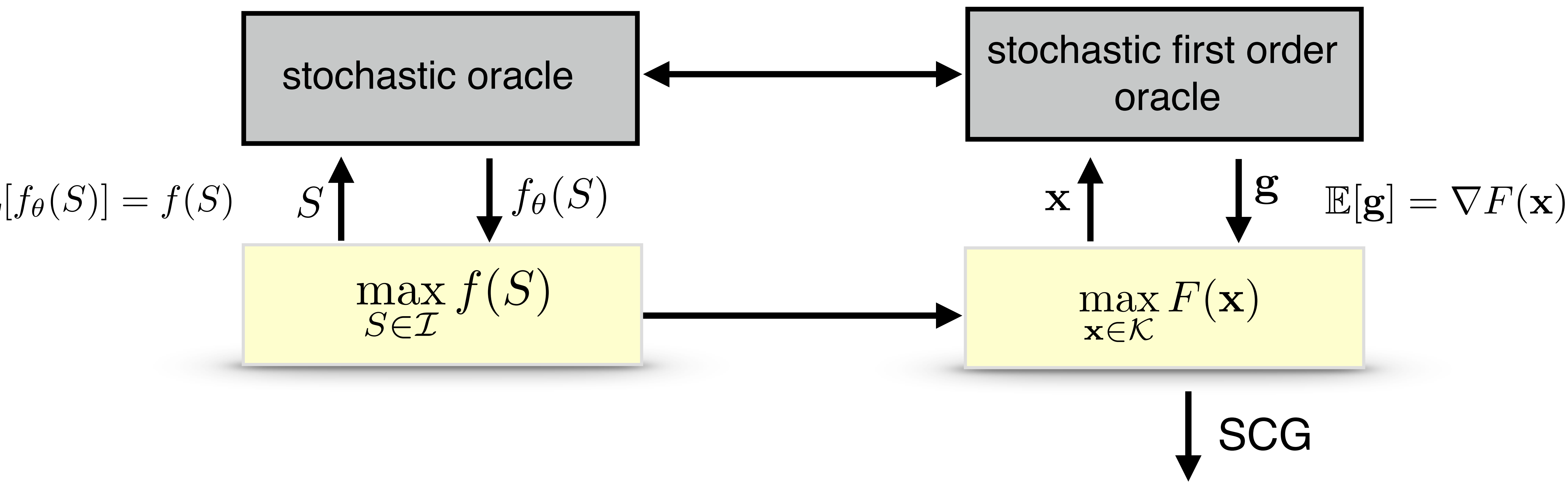

$$F(\mathbf{x}_{\mathrm{sol}}) \geq (1 - \frac{1}{e})\mathrm{OPT} - \epsilon$$

Stochastic Submodular Maximization: Discrete and Continuous

$$f(S) = \mathbb{E}_{\theta\sim D}[f_\theta(S)] \xrightarrow{\text{multi-linear ext.}} F(\mathbf{x}) = \mathbb{E}_{\theta\sim D}[F_\theta(\mathbf{x})]$$

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stochastic oracle $\longleftrightarrow$ stochastic first order oracle

$[f_\theta(S)] = f(S)$ $S \uparrow$ $\downarrow f_\theta(S)$ $\mathbf{x} \uparrow$ $\downarrow \mathbf{g}$ $\mathbb{E}[\mathbf{g}] = \nabla F(\mathbf{x})$

$$\max_{S\in\mathcal{I}} f(S) \longrightarrow \max_{\mathbf{x}\in\mathcal{K}} F(\mathbf{x})$$

SCG

$$f(S_{\text{sol}}) \geq (1-\frac{1}{e})\text{OPT} - \epsilon \xleftarrow{\text{rounding}} F(\mathbf{x}_{\text{sol}}) \geq (1-\frac{1}{e})\text{OPT} - \epsilon$$

Yale   Penn

Discrete vs. Continuous

- Discrete submodular optimization with matroid constraints:

$$O(\frac{n^8}{\epsilon^4}) \longrightarrow O(\frac{n^2}{\epsilon^4}) \longrightarrow O(\frac{n^2}{\epsilon^2})$$

Calinescu et al, 2011 **Badanidiyuru, Vondrak, 2013** **SCG++, 2019**

"Maximizing a monotone submodular function subject to a matroid constraint", Calinescu, Chekuri, Pal, Vondrák, 2011

"Fast algorithms for maximizing submodular functions", Badanidiyuru, Vondrak, 2013

"Stochastic Continuous Greedy++: When Upper and Lower Bounds Match", Hassani, Karbasi, Mokhtari, Shen, 2019

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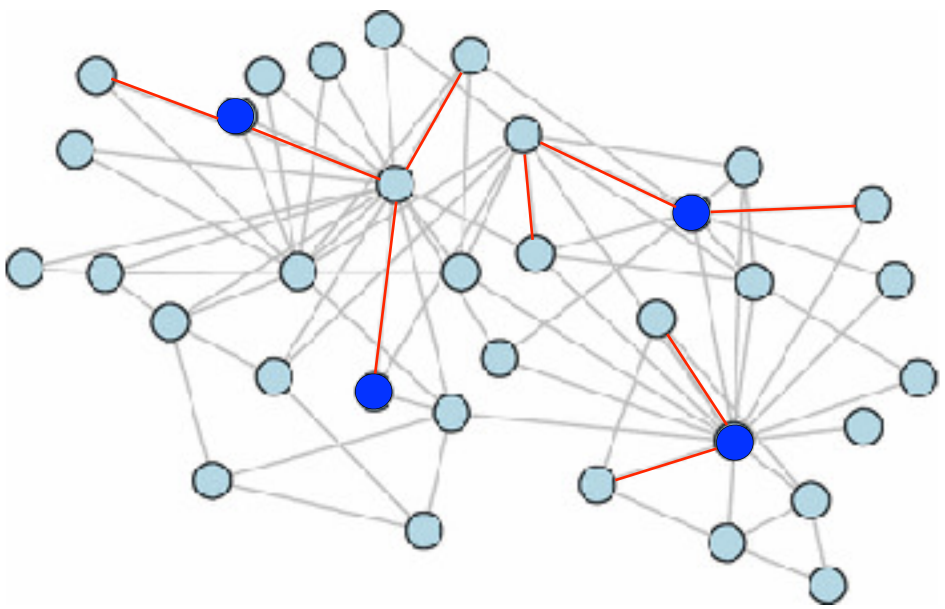
- Epinions social network (79k nodes, 580k edges)



Influence Maximization

$$f(S) = \mathbb{E}_{H\sim D}[f_H(S)]$$

Discrete vs. Continuous

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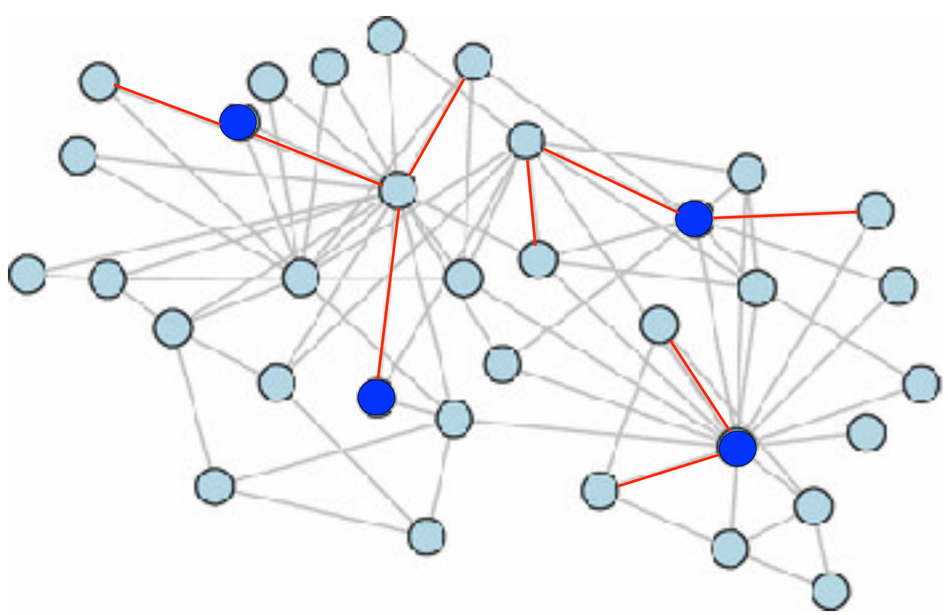

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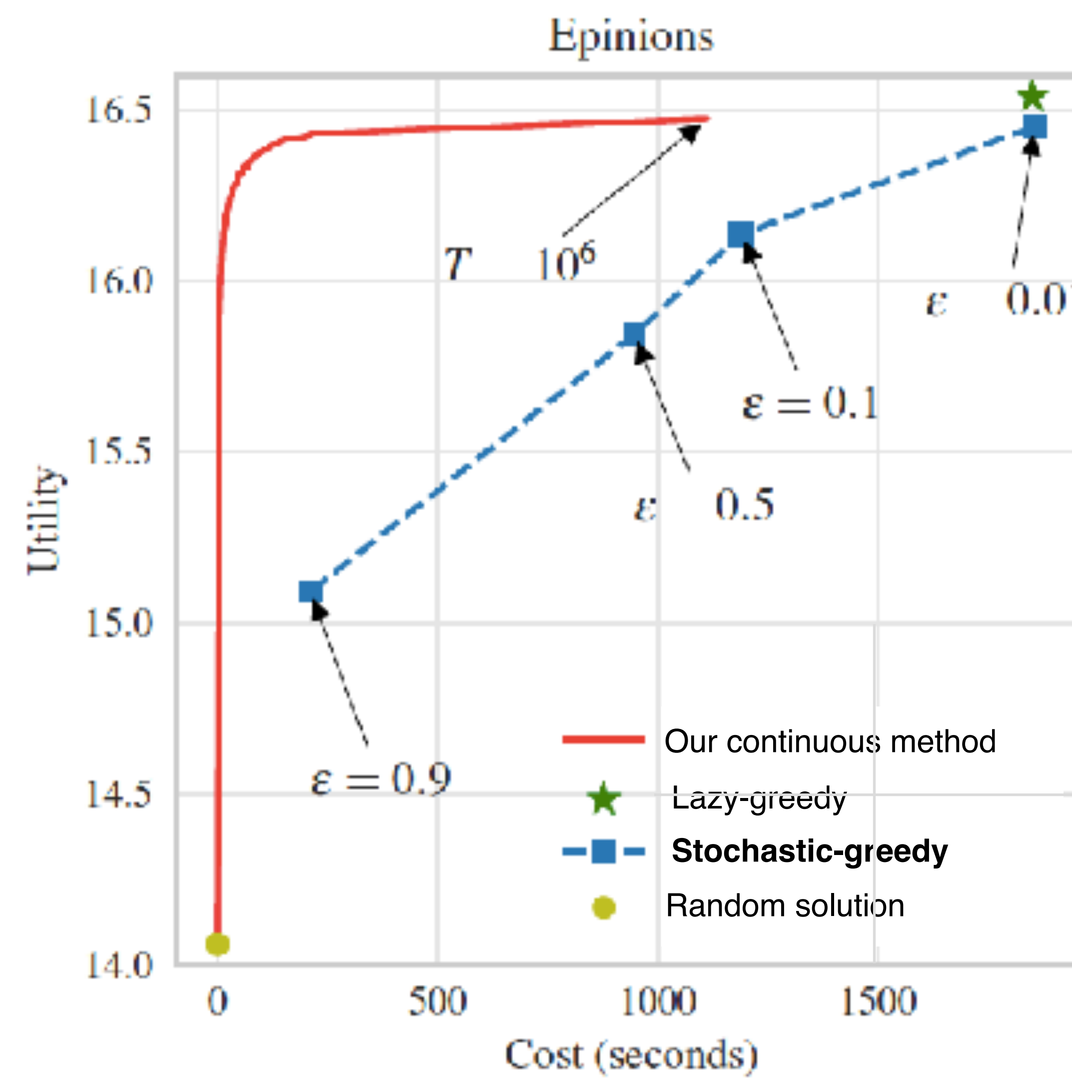

"Stochastic Submodular Maximization: The Case of Coverage Functions", Karimi et al.

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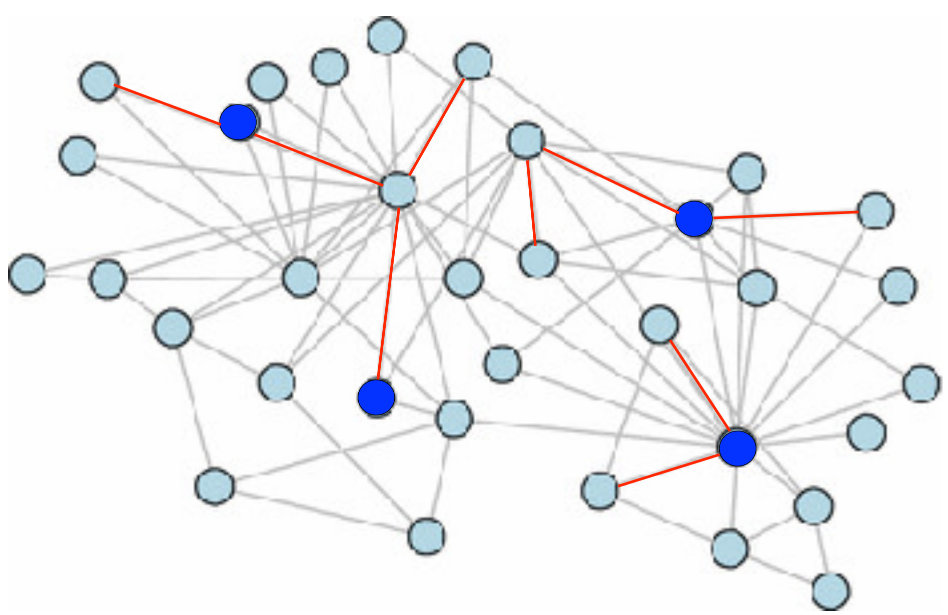

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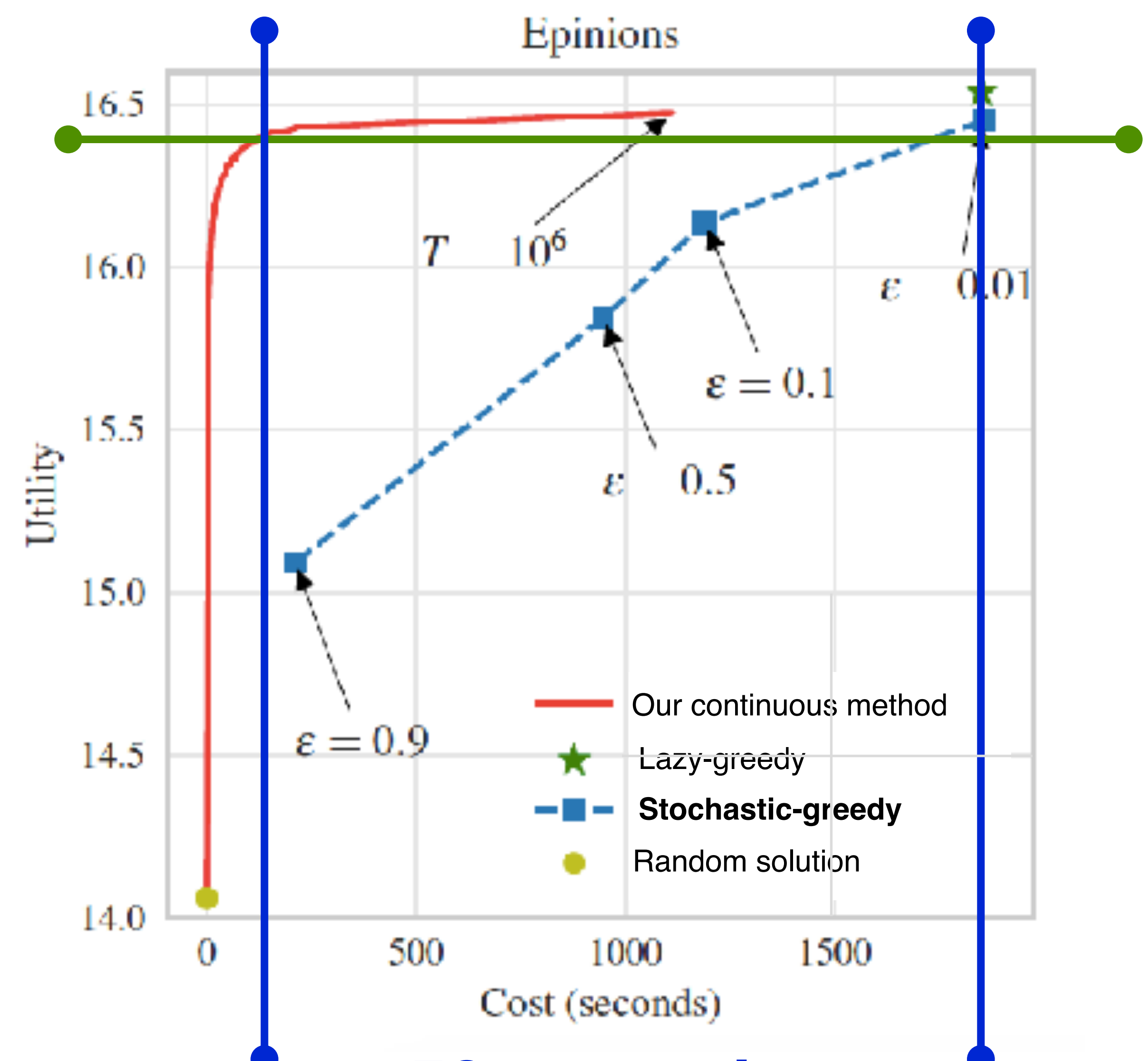
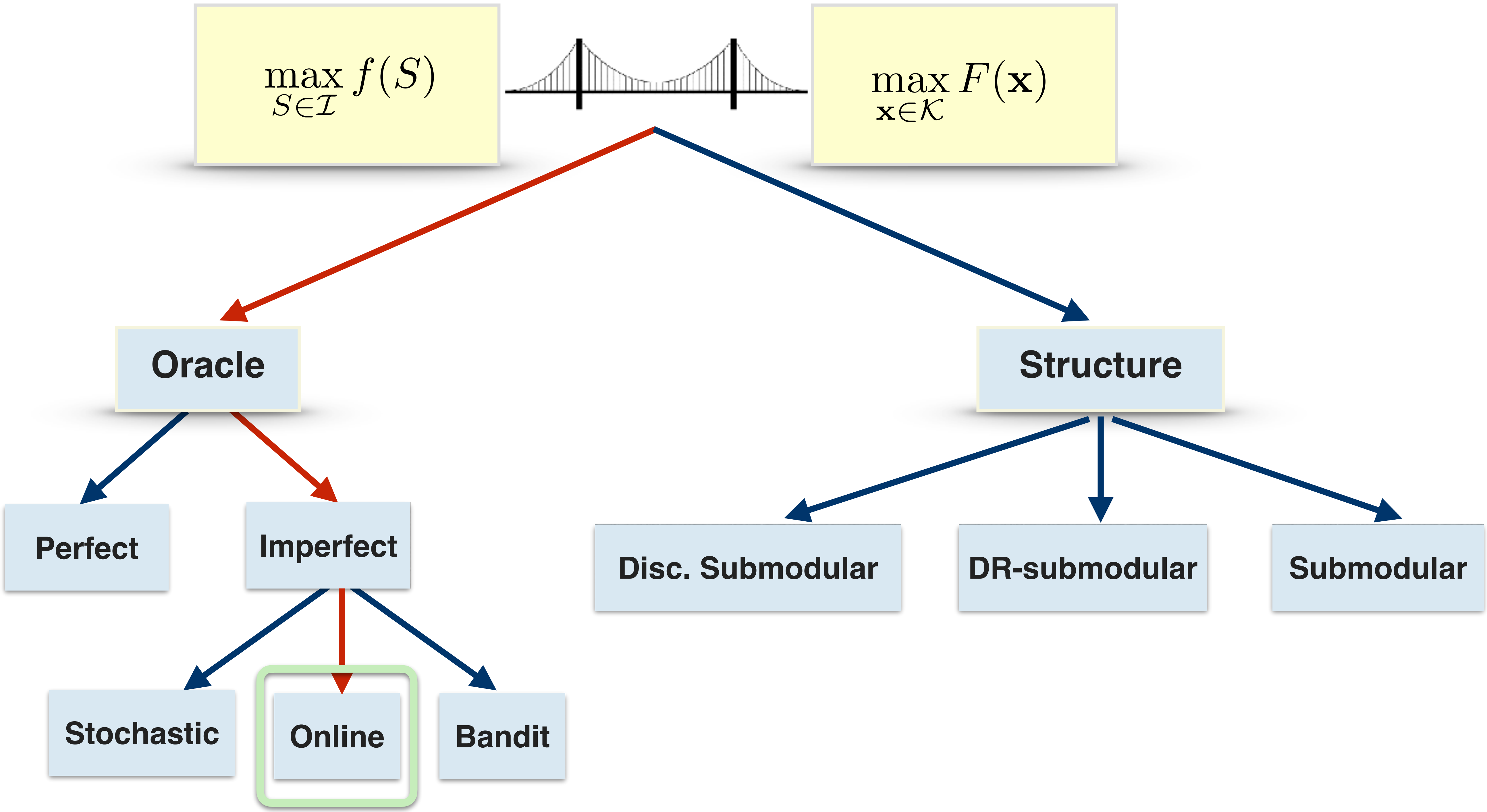


Yale

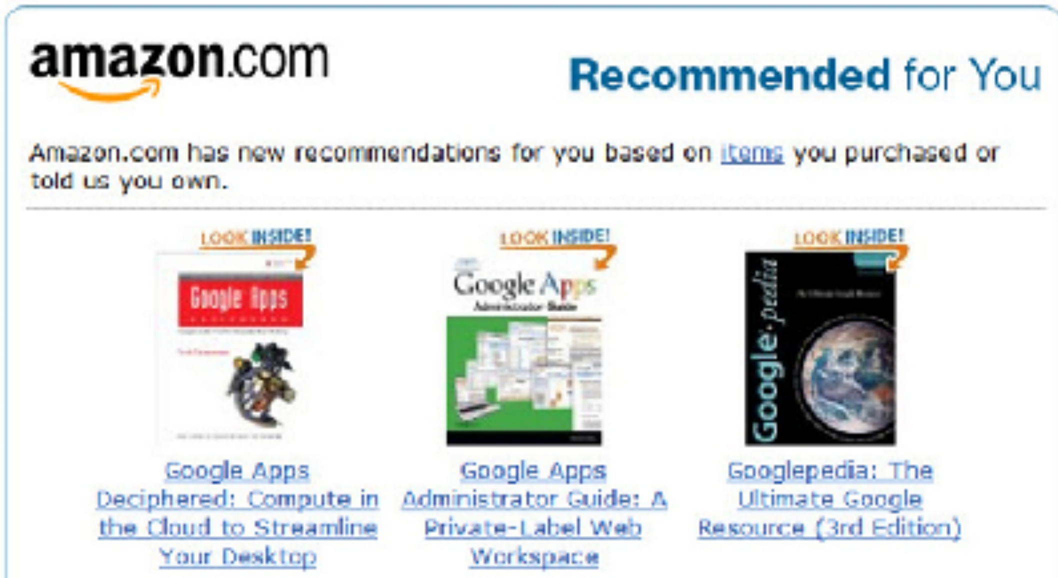
"Stochastic Submodular Maximization: The Case of Coverage Functions", Karimi et al.

Penn

Submodular Maximization: Discrete and Continuous

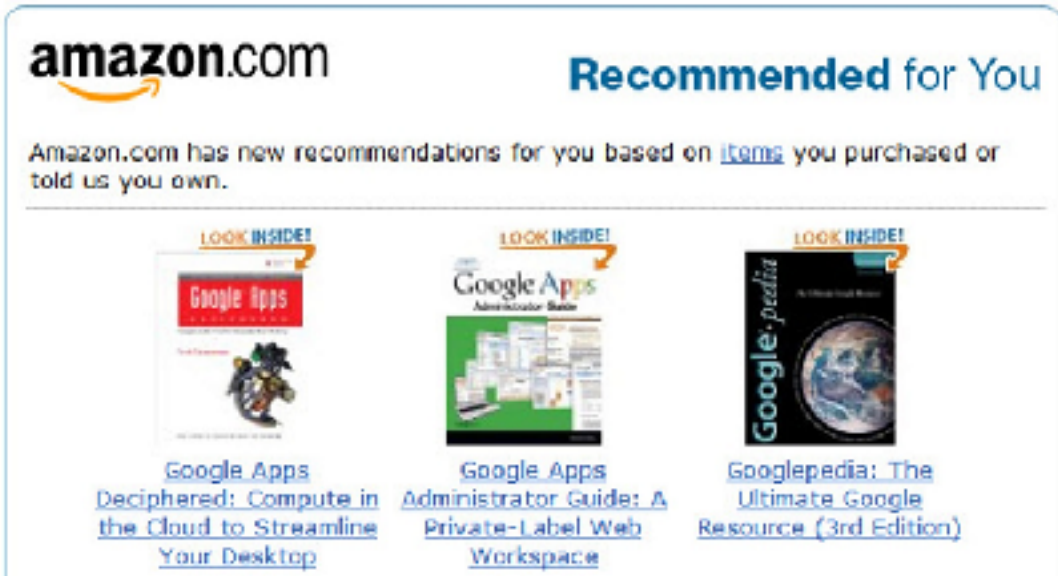


Online Submodular Maximization



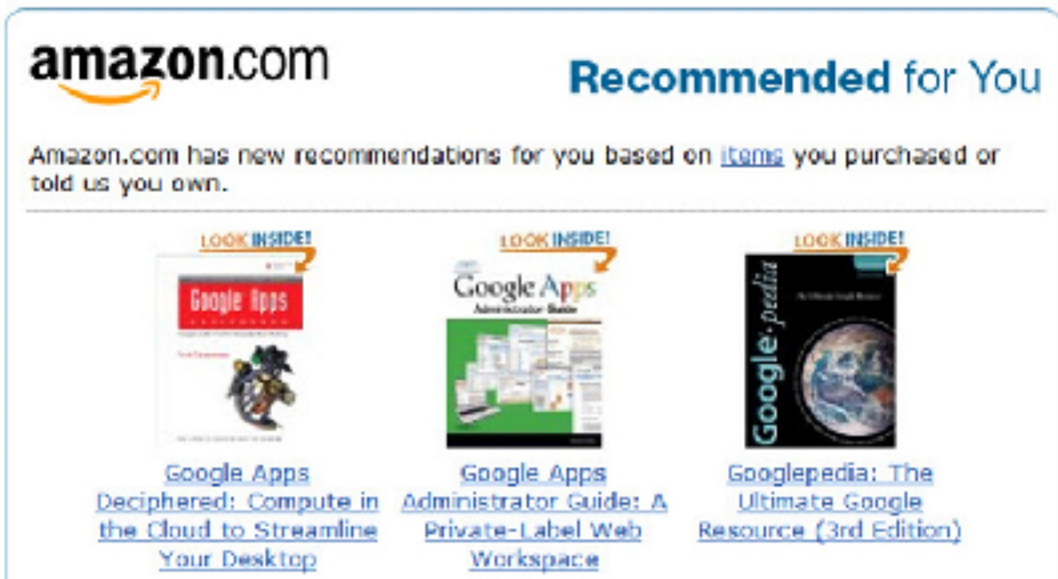
Online recommendation, advertising, …

Online Submodular Maximization



Online recommendation, advertising, …

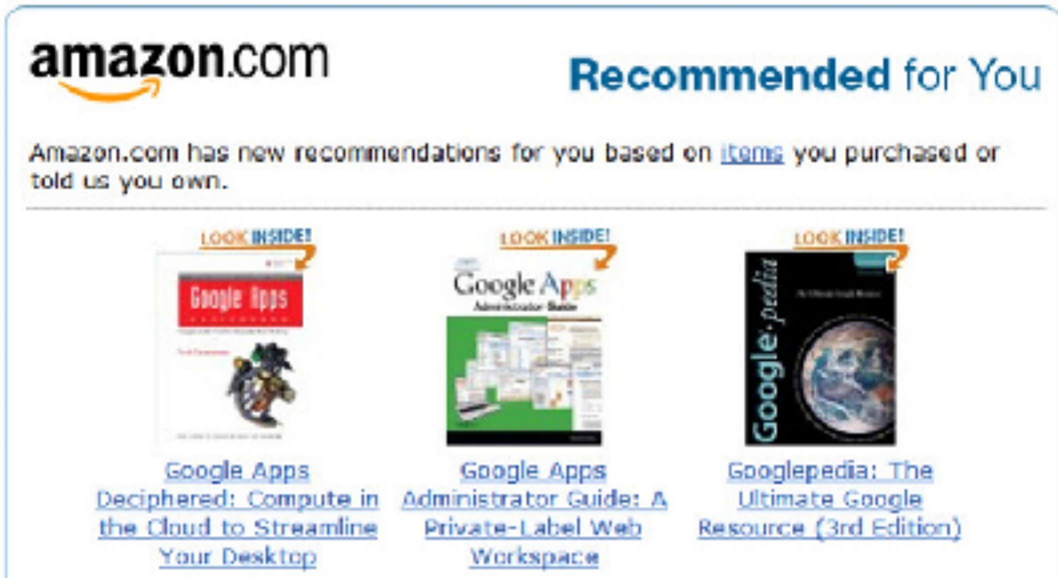
Online Submodular Maximization



Online recommendation, advertising, …

additional dimension:

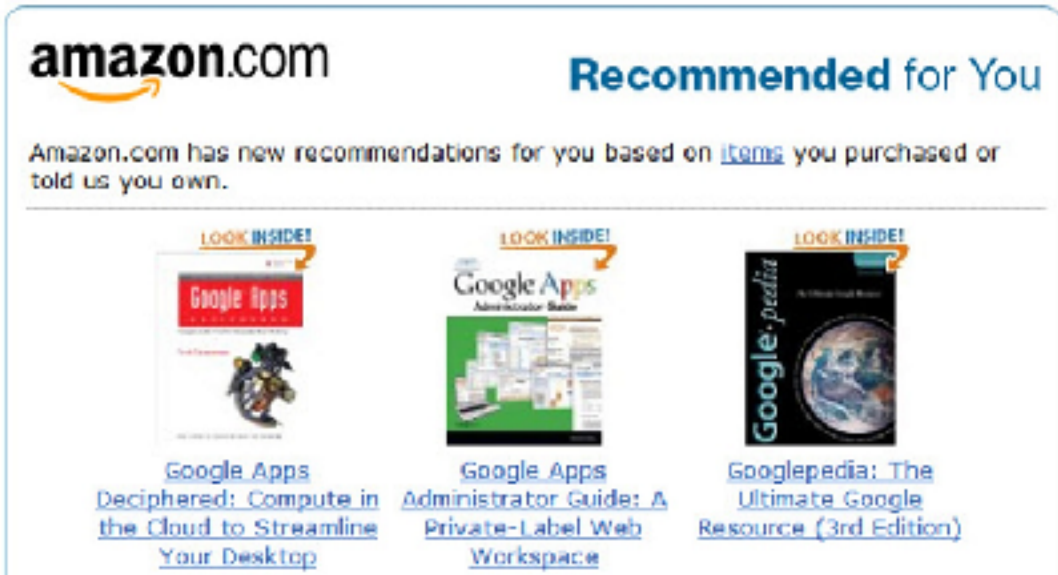
Online Submodular Maximization



Online recommendation, advertising, …

additional dimension: ⟶ time

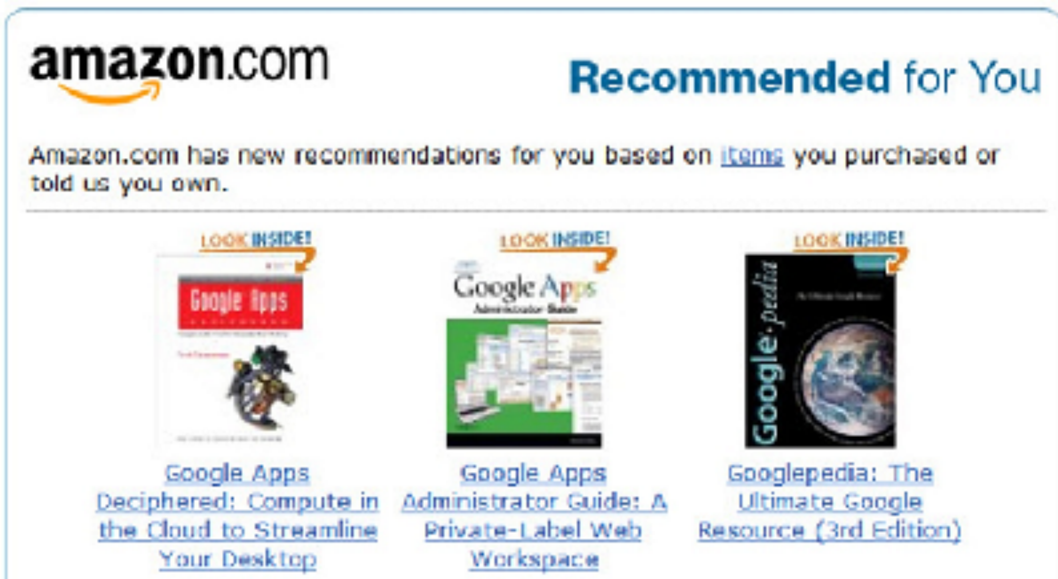
Online Submodular Maximization



Online recommendation, advertising, …

additional dimension: $\xrightarrow{f_1, f_2, f_3, \cdots, f_T}$ time

Online Submodular Maximization



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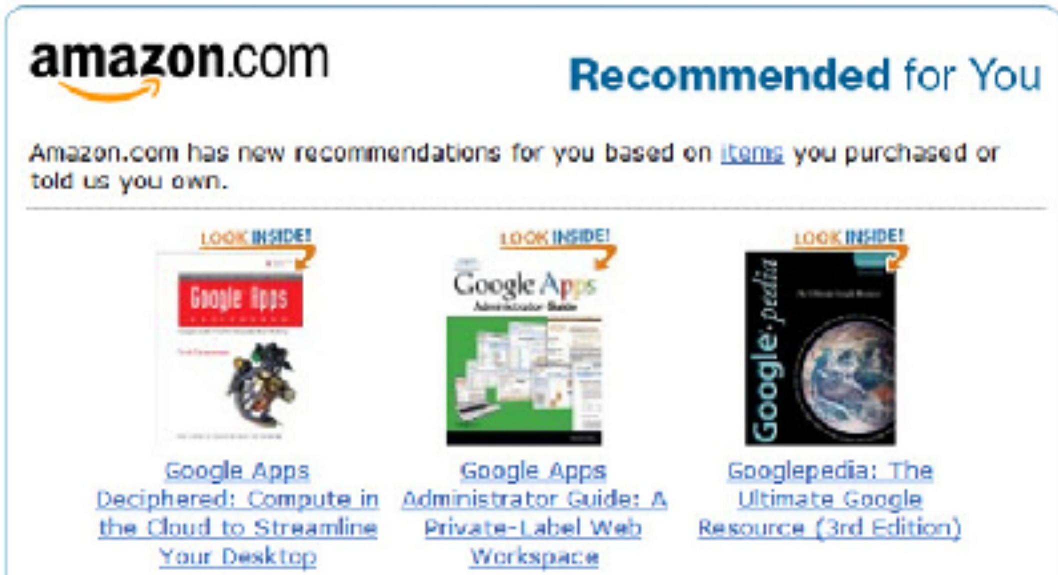
- at the beginning of each round t, a set $S_t \in \mathcal{I}$ is selected

Online Submodular Maximization



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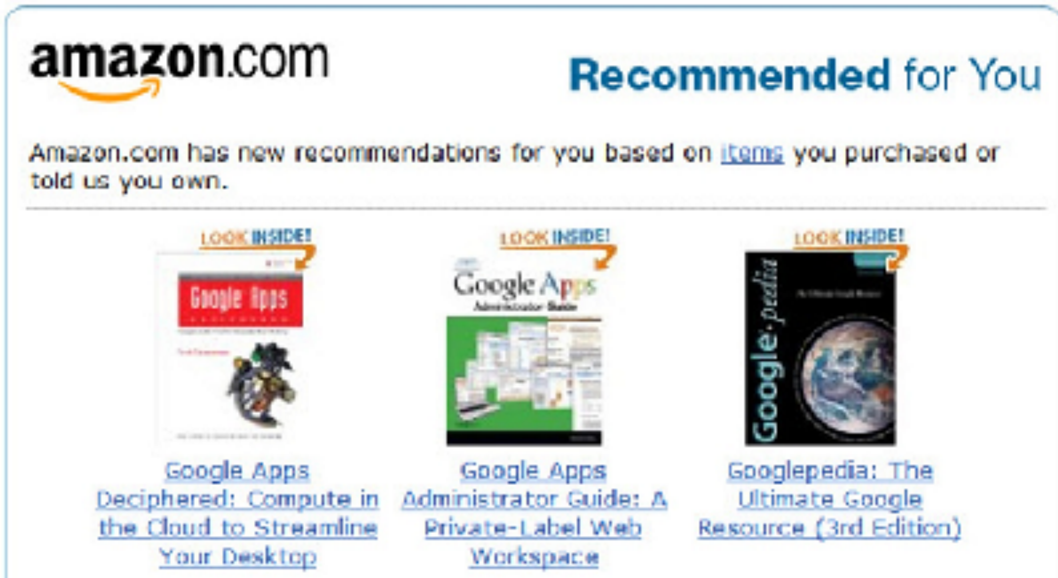

Online recommendation, advertising, …

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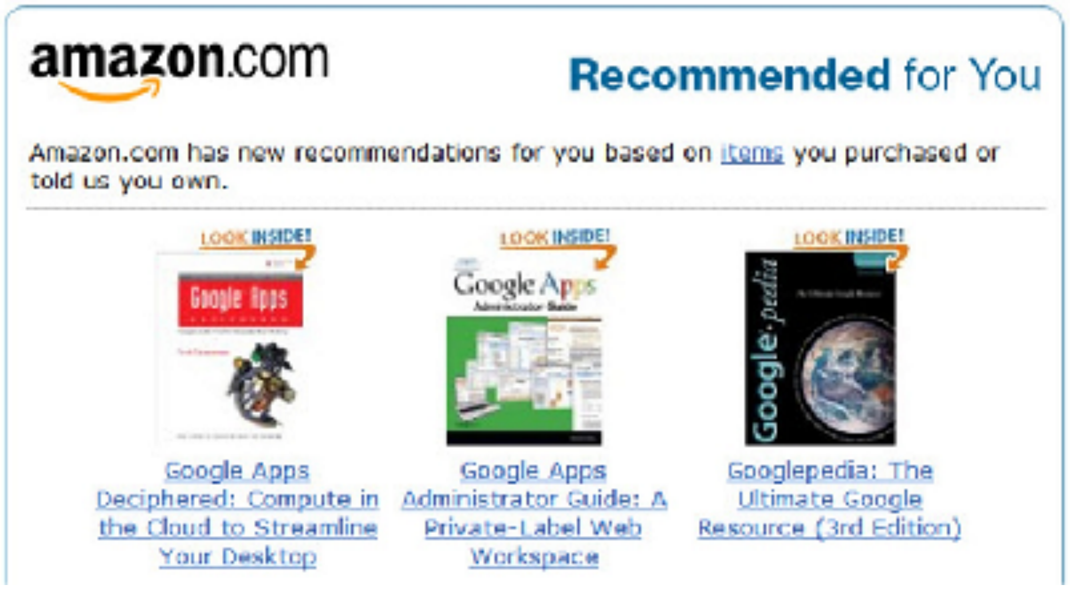
- $f_t : 2^V \to \mathbb{R}$; the reward of round t is $f_t(S_t)$

Online Submodular Maximization



Online recommendation, advertising, …

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time-varying oracle

action S_t ↑ reward ↓ $f_t(S_t)$ feedback ↓ $f_t(\cdot)$

Yale Penn

Online Submodular Maximization: Discrete and Continuous

Discrete:

- performance metric:

$$\text{regret} := \max_{S \in \mathcal{I}} \sum_{t=1}^{T} f_t(S) - \sum_{t=1}^{T} f_t(S_t)$$

online oracle

action: $S_t \in \mathcal{I}$

reward: $f_t(S_t)$

feedback: $f_t(\cdot)$

"An online algorithm for maximizing submodular functions", Streeter, Golovin, 2008

Online Submodular Maximization: Discrete and Continuous

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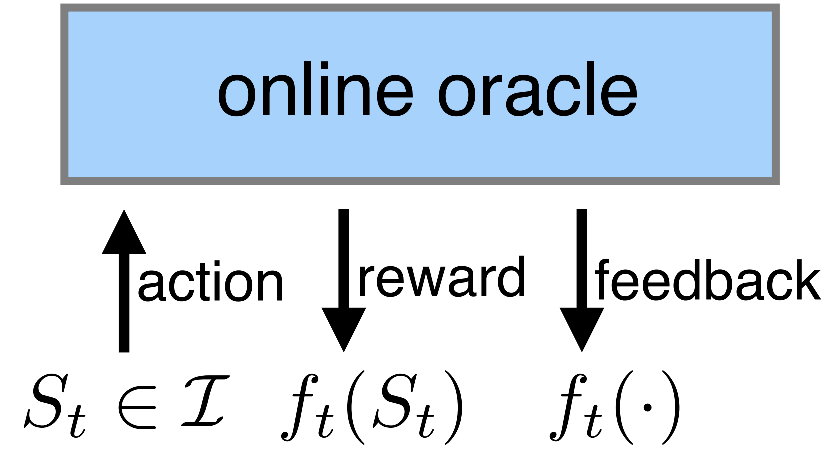

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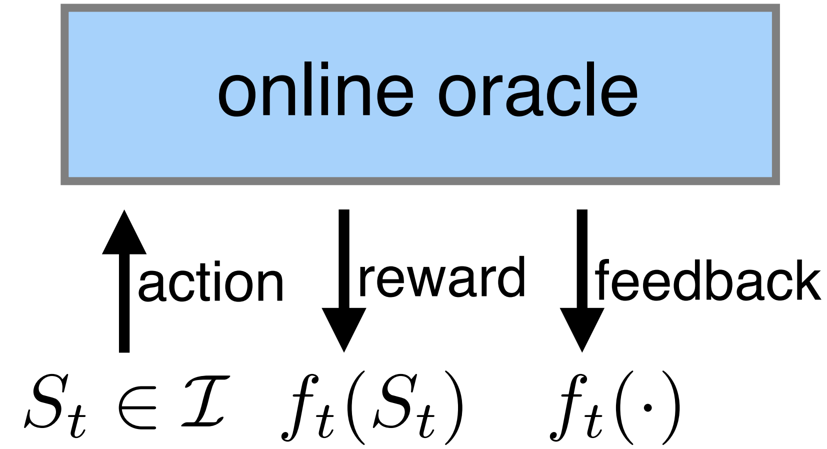

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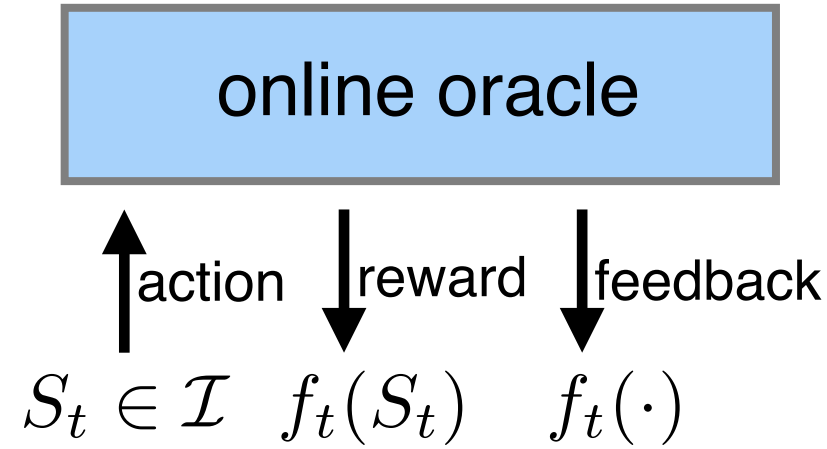

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Online Submodular Maximization: Discrete and Continuous

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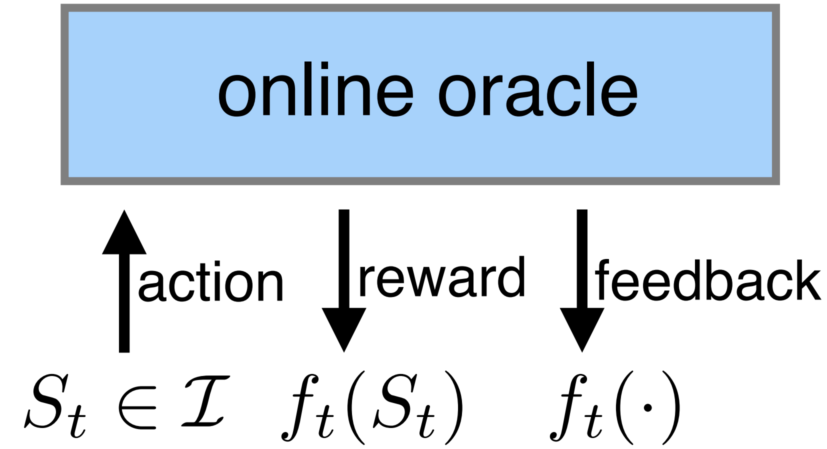
$$\color{red}{\alpha} - \text{regret} := \color{red}{\alpha} \underbrace{\max_{S \in \mathcal{I}} \sum_{t=1}^{T} f_t(S)}_{\text{best action in hindsight}} - \underbrace{\sum_{t=1}^{T} f_t(S_t)}_{\text{algorithm}}$$



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Yale Penn

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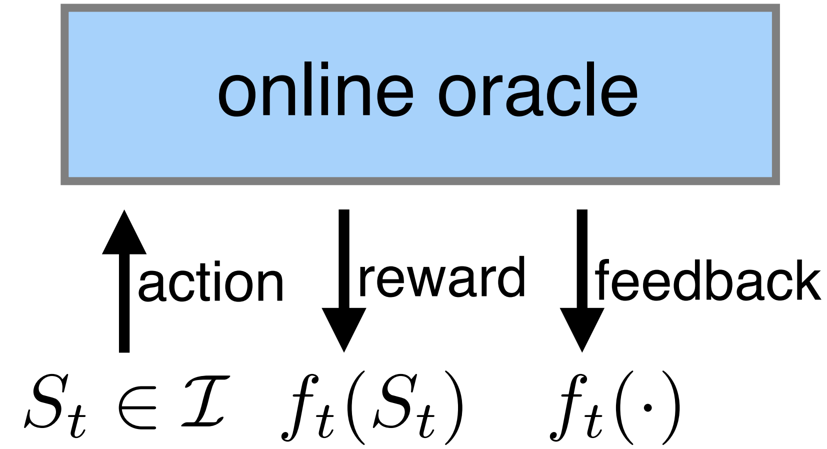

Continuous:

"Online Continuous Submodular Maximization", Chen, Hassani, Karbasi, 2018

Online Submodular Maximization: Discrete and Continuous

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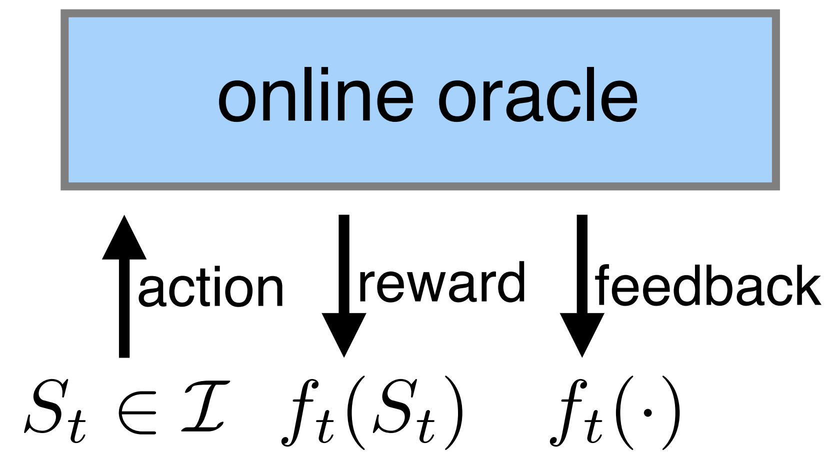

Continuous:

- $F_1, F_2, \cdots, F_T \,;\, F_i : \mathcal{X} \to \mathbb{R}$

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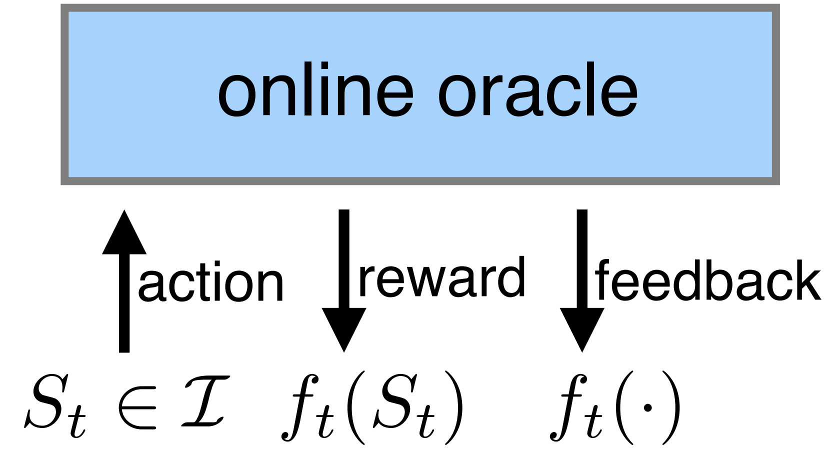

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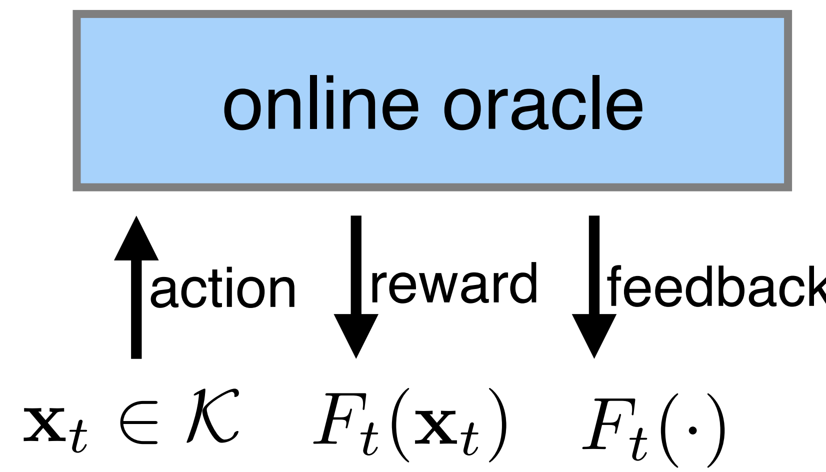

Online Submodular Maximization: Discrete and Continuous

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$$\color{red}{\alpha} - \text{regret} := \color{red}{\alpha} \max_{S \in \mathcal{I}} \underbrace{\sum_{t=1}^{T} f_t(S)}_{\text{best action in hindsight}} - \underbrace{\sum_{t=1}^{T} f_t(S_t)}_{\text{algorithm}}$$

online oracle

action $\uparrow$ $S_t \in \mathcal{I}$ reward $\downarrow$ $f_t(S_t)$ feedback $\downarrow$ $f_t(\cdot)$

Continuous:

- $F_1, F_2, \cdots, F_T$; $F_i : \mathcal{X} \to \mathbb{R}$

- performance metric:

$$\alpha - \text{regret} := \alpha \max_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^{T} F_t(\mathbf{x}) - \sum_{t=1}^{T} F_t(\mathbf{x}_t)$$

online oracle

action $\uparrow$ $\mathbf{x}_t \in \mathcal{K}$ reward $\downarrow$ $F_t(\mathbf{x}_t)$ feedback $\downarrow$ $F_t(\cdot)$

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online oracle

action: $\mathbf{x}_t \in \mathcal{K}$ reward: $F_t(\mathbf{x}_t)$ feedback: $F_t(\cdot)$

Devise algorithms for the online setting with α-regret being sub-linear in T

best value of α? lowest regret?

Yale Penn

Online Submodular Maximization: Discrete and Continuous

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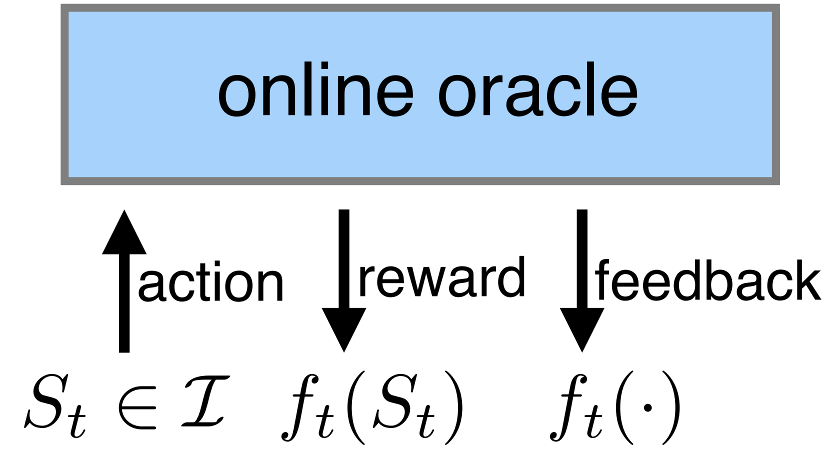

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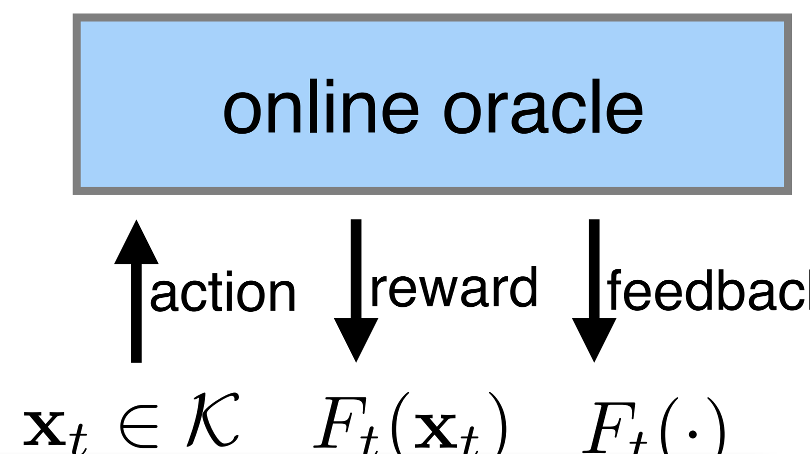

Devise algorithms for the online setting with α-regret being sub-linear in T

best value of α?　　lowest regret?

Online Gradient Ascent

- At each time t :

  play $\mathbf{x}_t$ and receive reward $F_t(\mathbf{x}_t)$

$$\mathbf{x}_{t+1} = \mathrm{Proj}_{\mathcal{K}}\{\mathbf{x}_t + \eta_t \nabla F_t(\mathbf{x}_t)\}$$

Online Gradient Ascent

- At each time t :

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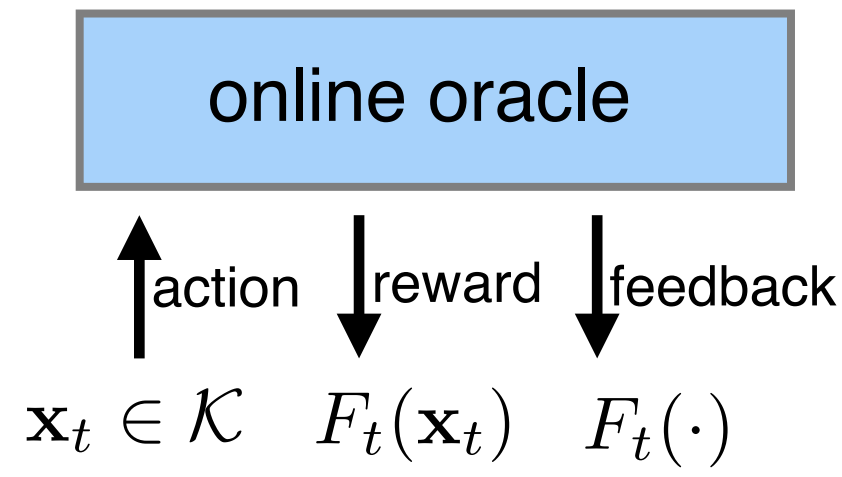

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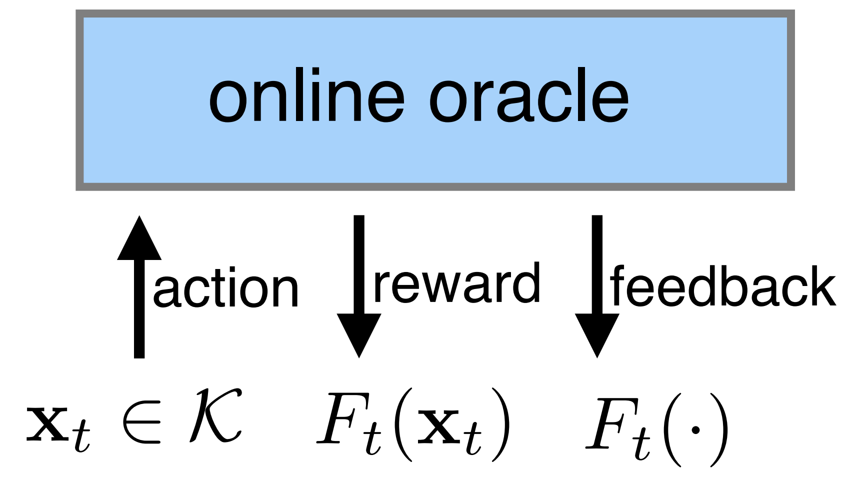

[Chen, Hassani, Karbasi]

If F_t's are monotone and DR-submodular, then for the online gradient ascent algorithm we have

$$\frac{1}{2} - \mathrm{regret} = O(\sqrt{T})$$

"Online Continuous Submodular Maximization", AISTATS '18

Online Gradient Ascent

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  play $\mathbf{x}_t$ and receive reward $F_t(\mathbf{x}_t)$

$$\mathbf{x}_{t+1} = \mathrm{Proj}_{\mathcal{K}}\{\mathbf{x}_t + \eta_t \nabla F_t(\mathbf{x}_t)\}$$

online oracle

action ↑ $\mathbf{x}_t \in \mathcal{K}$

reward ↓ $F_t(\mathbf{x}_t)$

feedback ↓ $F_t(\cdot)$

[Chen, Hassani, Karbasi]

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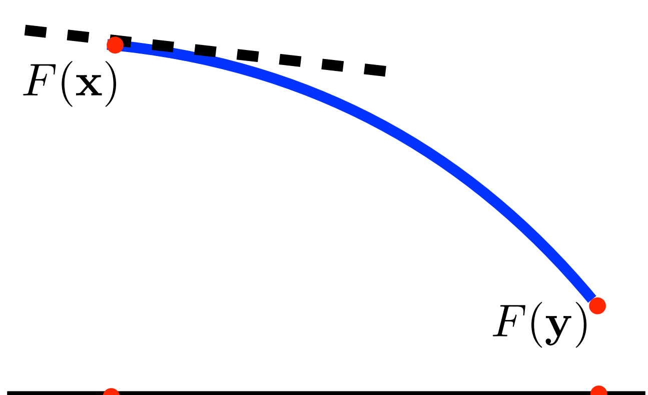
"Online Continuous Submodular Maximization", AISTATS '18



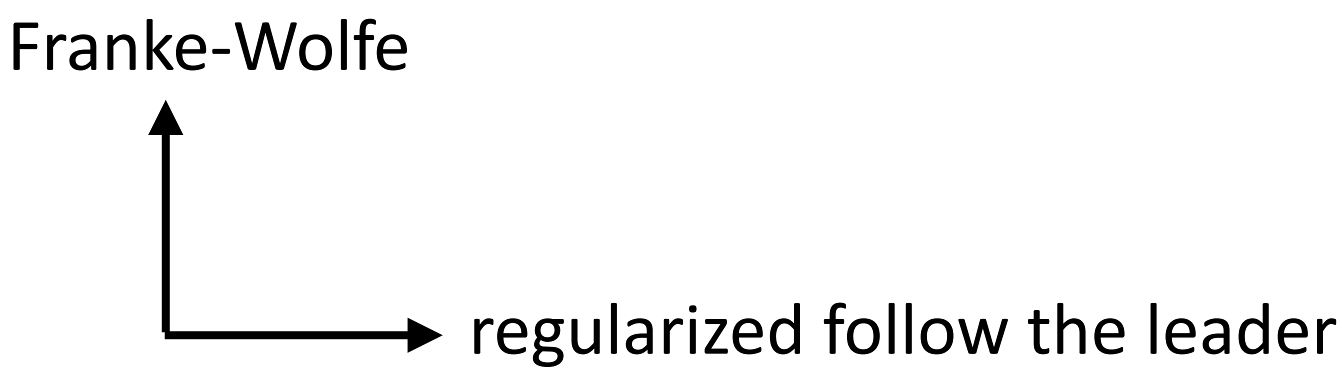
concave: $F(\mathbf{y}) - F(\mathbf{x}) \leq \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$

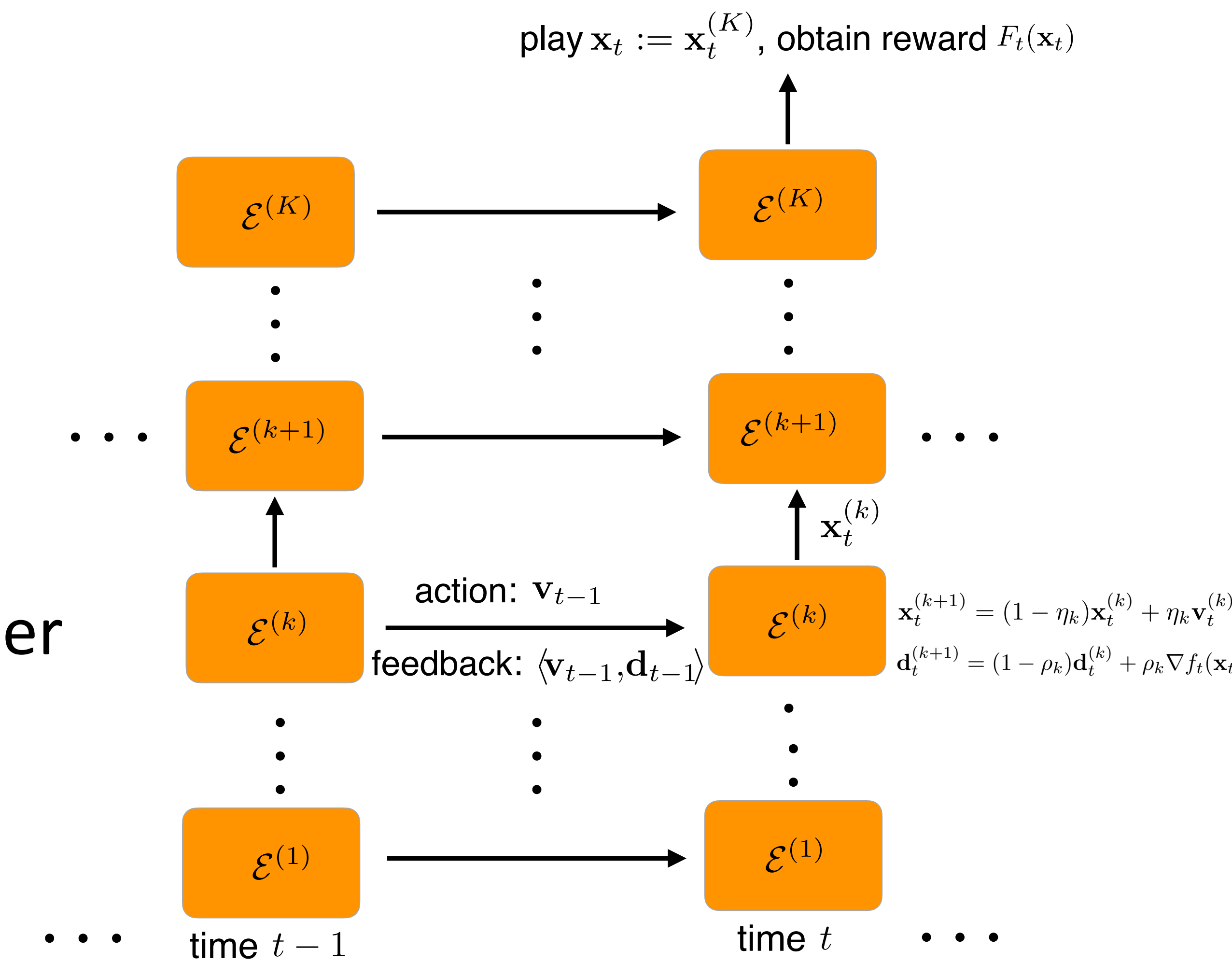
monotone DR-submodular: $F(\mathbf{y}) - 2F(\mathbf{x}) \leq \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$

Yale Penn

Meta Franke-Wolfe

- The Meta-FW Framework:





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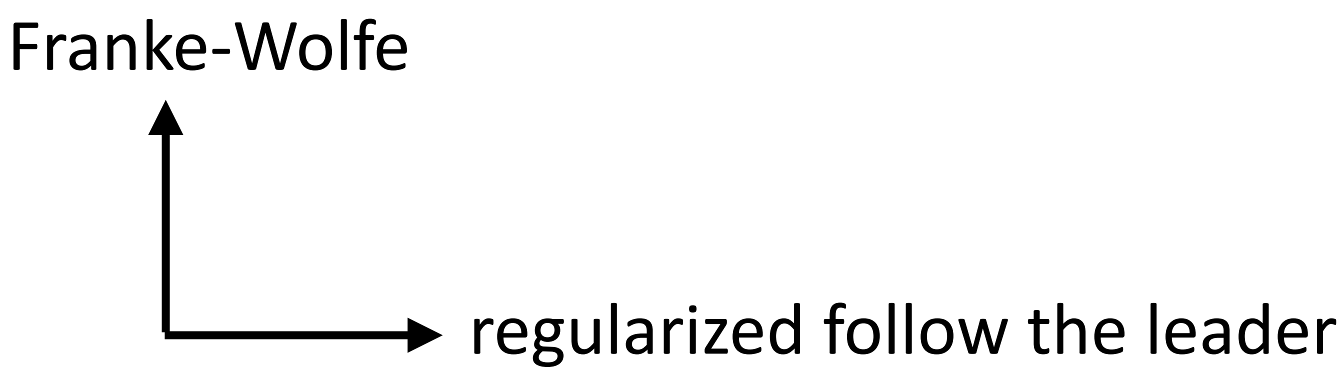
"Online Submodular Maximization under a Matroid Constraint with Application to Learning Assignments", Golovin, Krause, Streeter, 2014

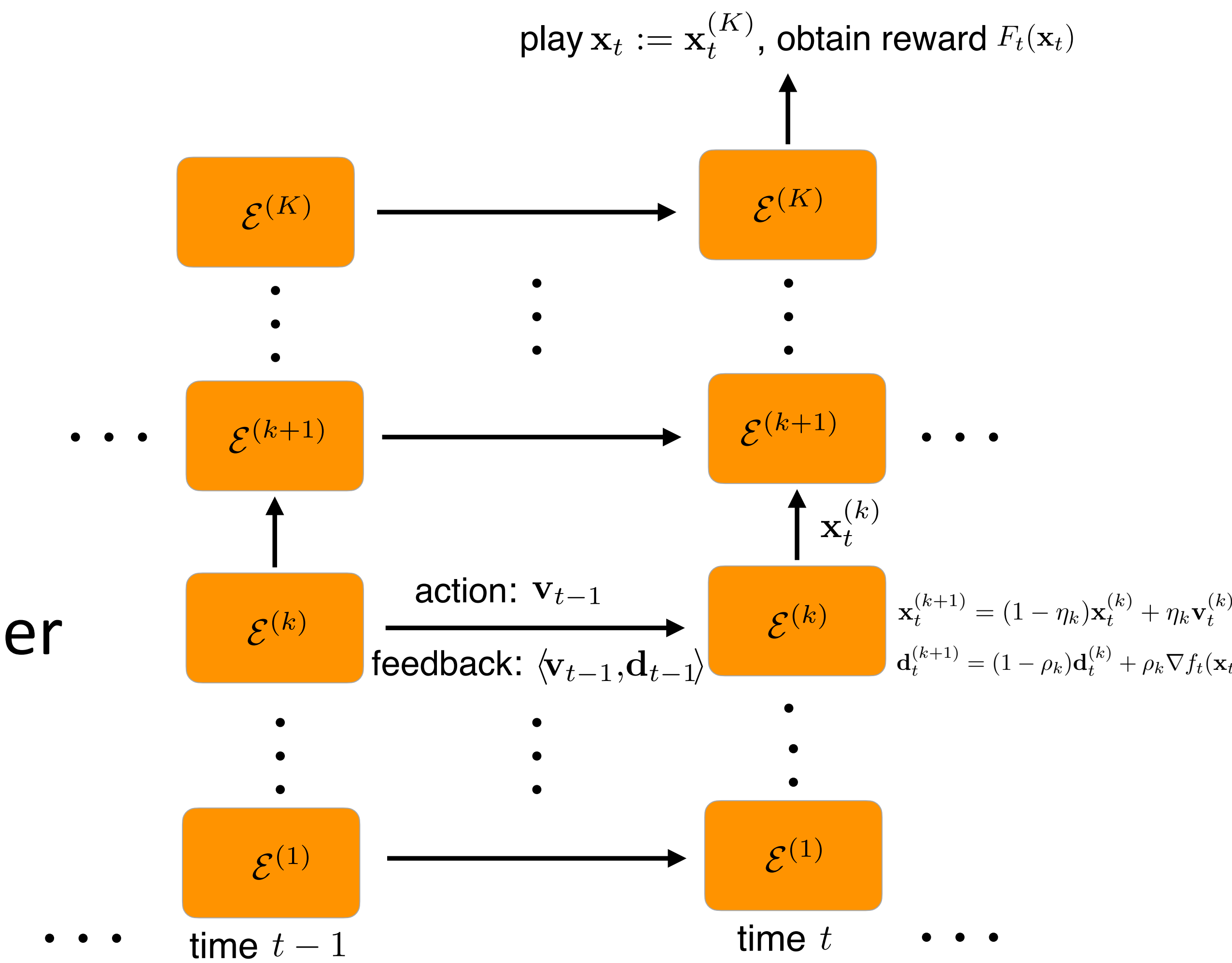
"Online Continuous Submodular Maximization", Chen, Hassani, Karbasi, 2018

"Projection-Free Online Optimization with Stochastic Gradient: From Convexity to Submodularity", Chen, Harshaw, Hassani, Karbasi, 2018

Meta Franke-Wolfe

- The Meta-FW Framework:





"An online algorithm for maximizing submodular functions", Streeter, Golovin, 2008

"Online Submodular Maximization under a Matroid Constraint with Application to Learning Assignments", Golovin, Krause, Streeter, 2014

"Online Continuous Submodular Maximization", Chen, Hassani, Karbasi, 2018

"Projection-Free Online Optimization with Stochastic Gradient: From Convexity to Submodularity", Chen, Harshaw, Hassani, Karbasi, 2018

Meta Franke-Wolfe

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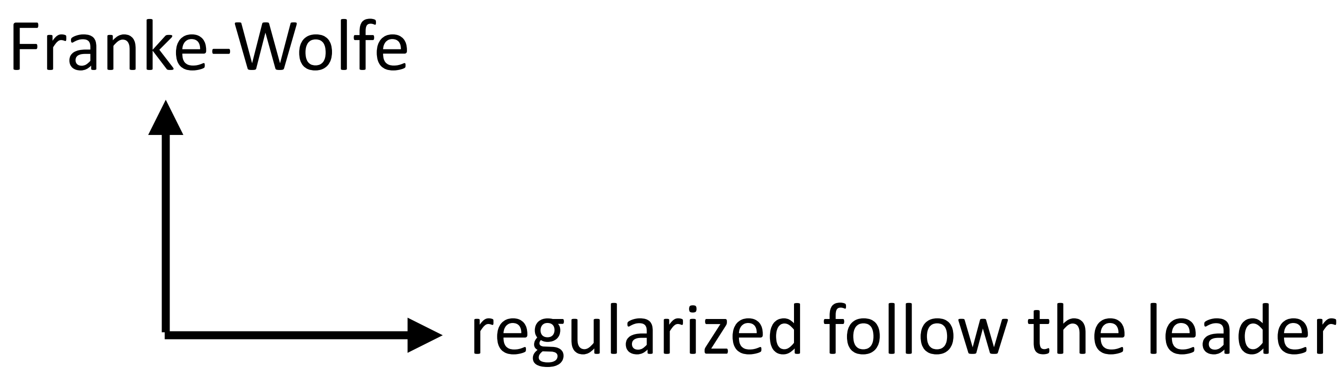

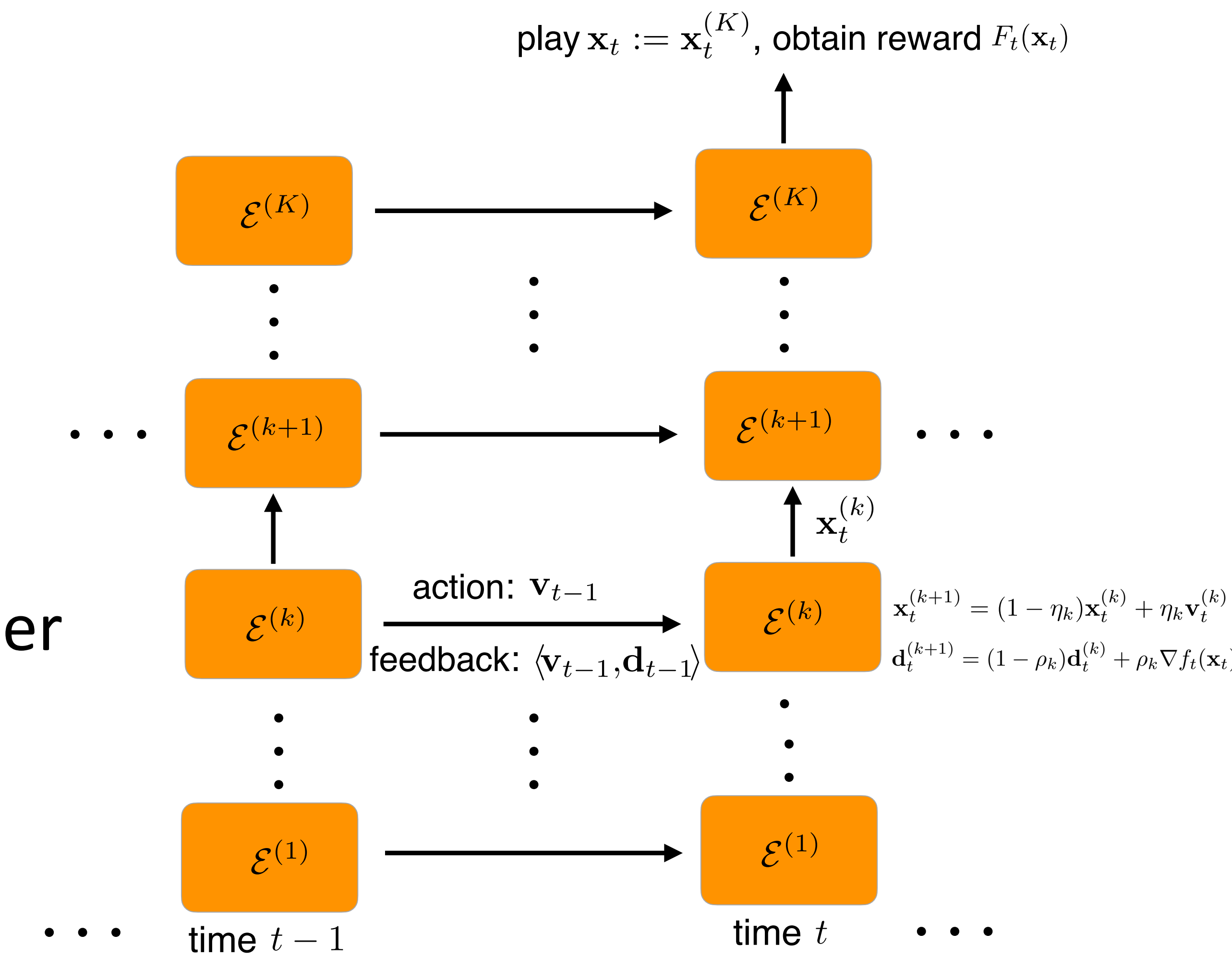

[Chen, Hassani, Karbasi]

If F_t's are monotone and DR-submodular, then for the Meta-Frank-Wolfe algorithm we have

$$(1-\frac{1}{e}) - \text{regret} = O(\sqrt{T})$$

"Online Continuous Submodular Maximization", AISTATS '18

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The Online Setting

- Many open questions:
  continuous submodular, non-monotone, other constraints
  simpler algorithms, better regret bounds, etc

- Related work/ Recent Progress:

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  "Designing smoothing functions for improved worst-case competitive ratio in online optimization", Eghbali, Fazel, 2016

  "Online Non-Monotone DR-submodular Maximization", Thang, Srivastav, 2019

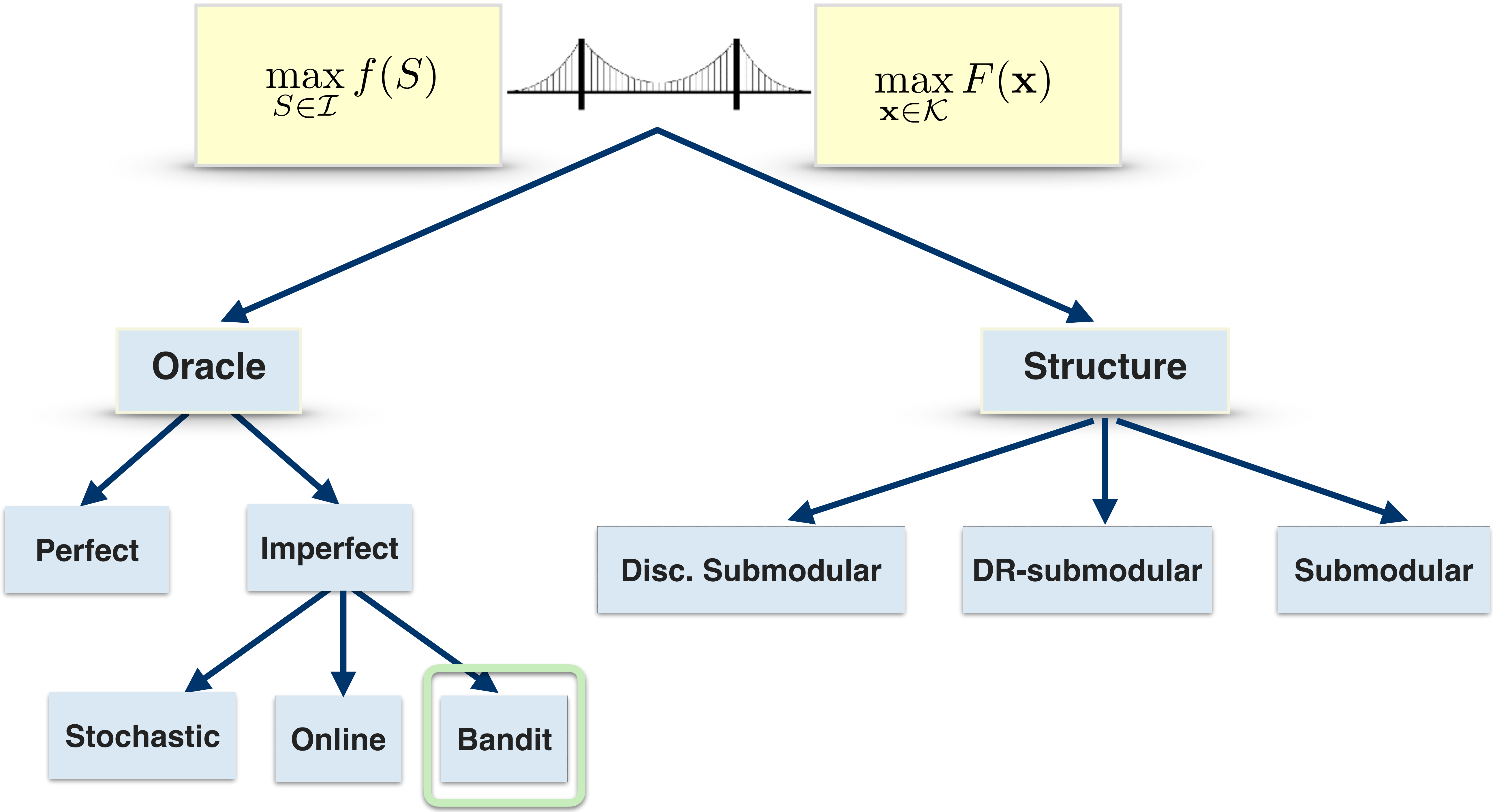
  "No-regret Algoriths for Online k-submodular optimization", Soma, 2019

  "Consistent Online Optimization: Convex and Submodular", Karimi, Krause, Lattanzi, Vassilvtiskii, 2019

  "Online DR-Submodular Maximization with Stochastic Cumulative Constraints", Raut, Sadeghi, Fazel, 2020

  "Online continuous DR-submodular maximization with long-term budget constraints", Sadeghi, Fazel, 2020

Submodular Maximization: Discrete and Continuous



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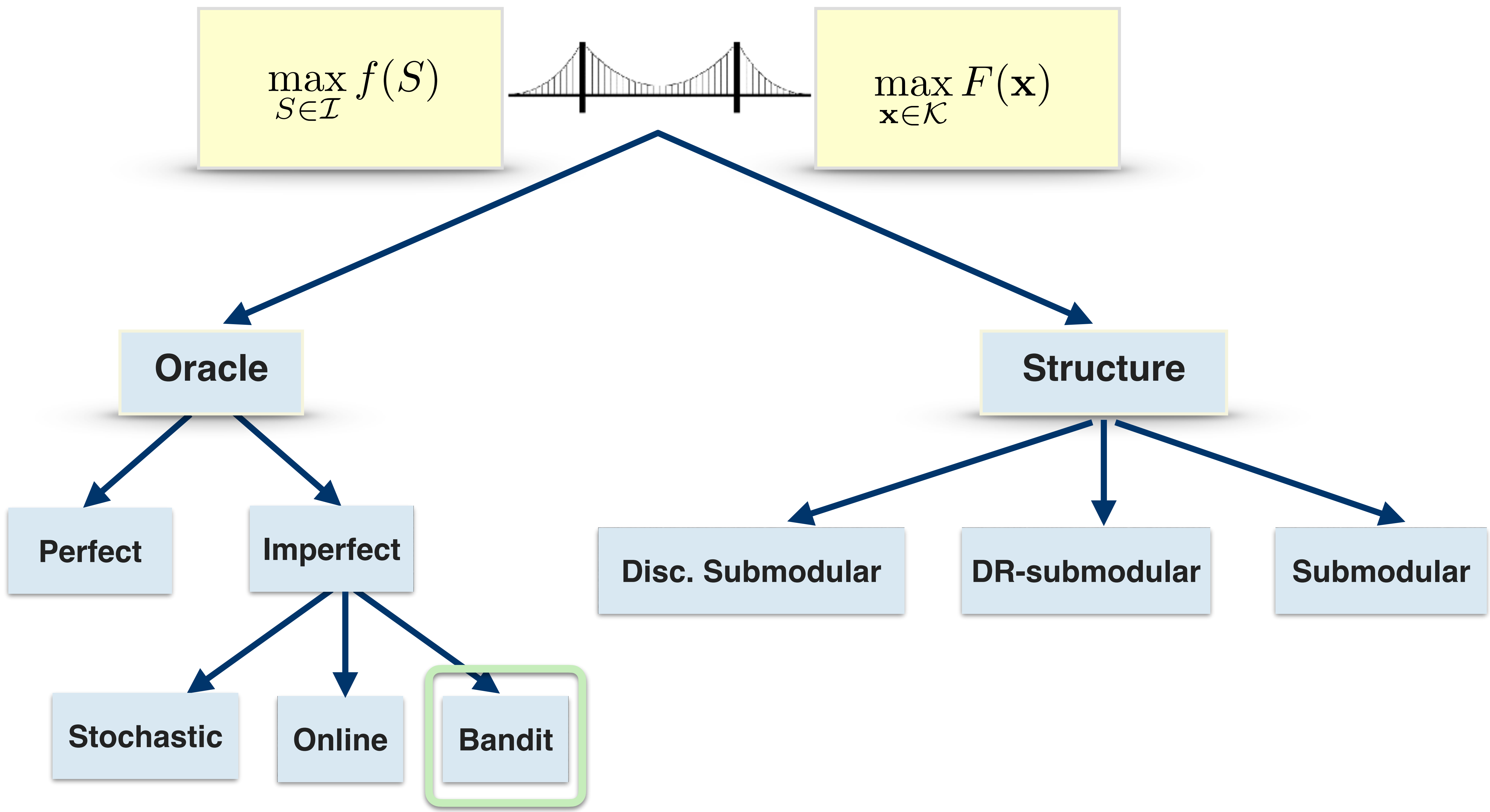

Bandit Submodular Maximization

Bandit oracle

action $\uparrow$ reward & feedback $\downarrow$

$S_t \in \mathcal{I}$ $\quad$ $f_t(S_t)$

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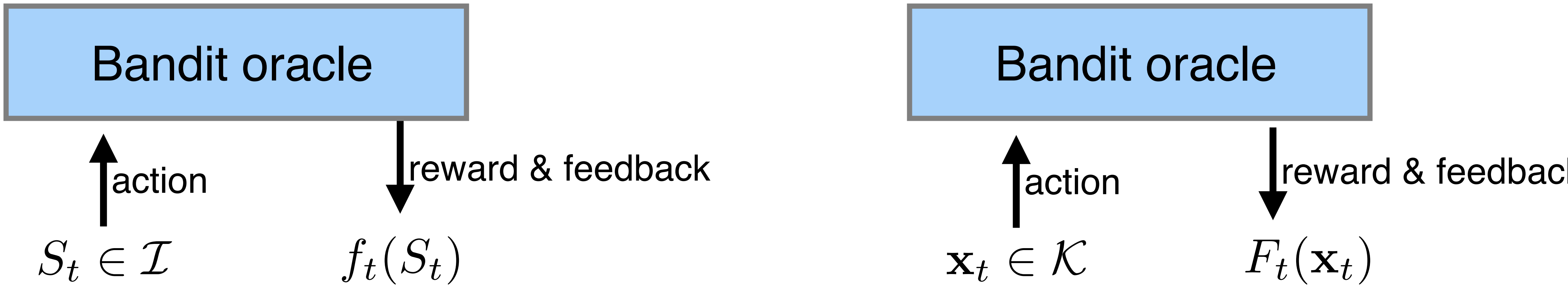

[Zhang, Chen, Hassani, Karbasi]

If F_t's are monotone and DR-submodular, then for the Bandit Frank-Wolfe algorithm we have

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Bandit Submodular Maximization

Bandit oracle		Bandit oracle	
↑ action	↓ reward & feedback	↑ action	↓ reward & feedback
$S_t \in \mathcal{I}$	$f_t(S_t)$	$\mathbf{x}_t \in \mathcal{K}$	$F_t(\mathbf{x}_t)$

[Zhang, Chen, Hassani, Karbasi]

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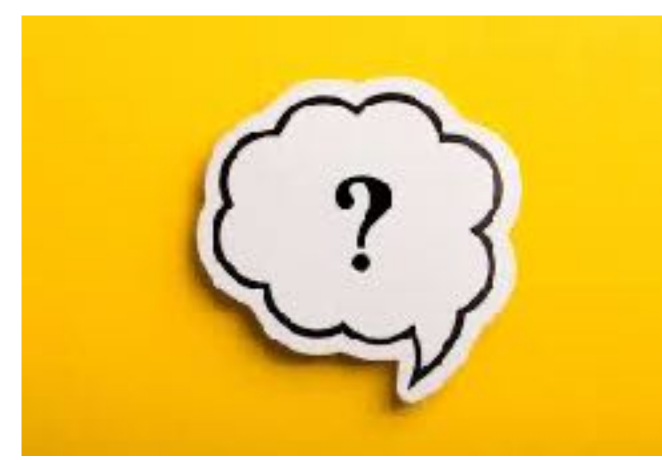
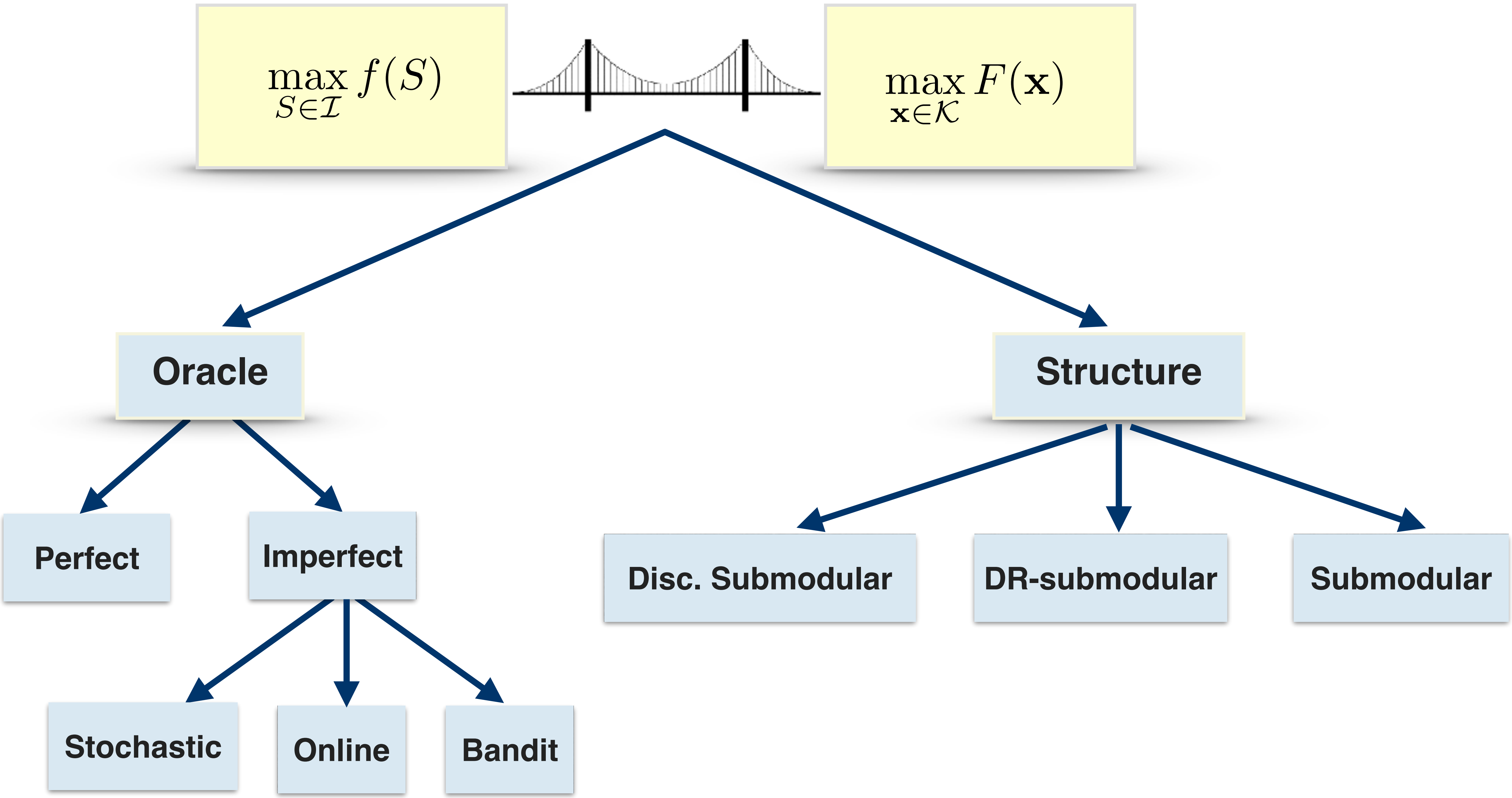
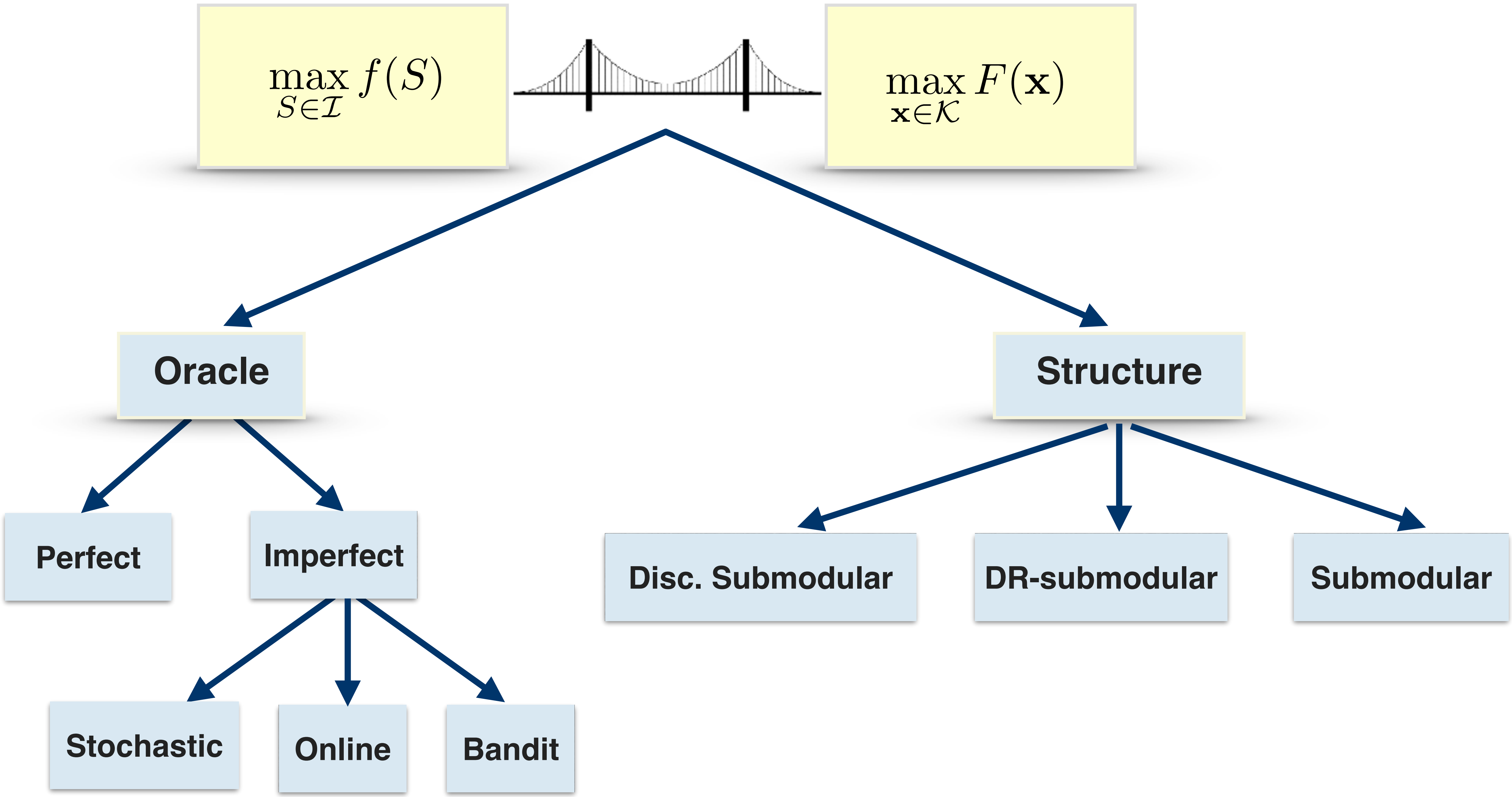


Final Remarks



Final Remarks



Final Remarks

- Minimization of continuous submodular functions is another story, see:

"Submodular Functions: from Discrete to Continuous Domains", Bach, 2016

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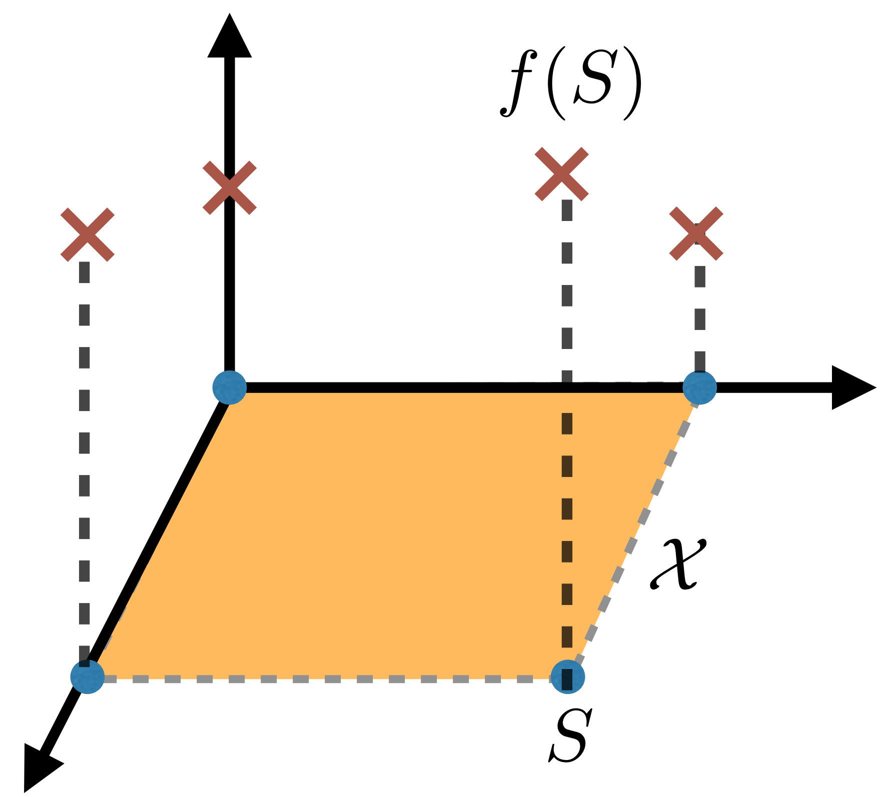
- Other continuous extensions of submodular set functions:

  F^+ : concave extension

  F^-: Lovasz (convex) extension

  F : Multi-linear extension



"Submodularity in Combinatorial Optimization", Vondrak, 2006

Final Remarks

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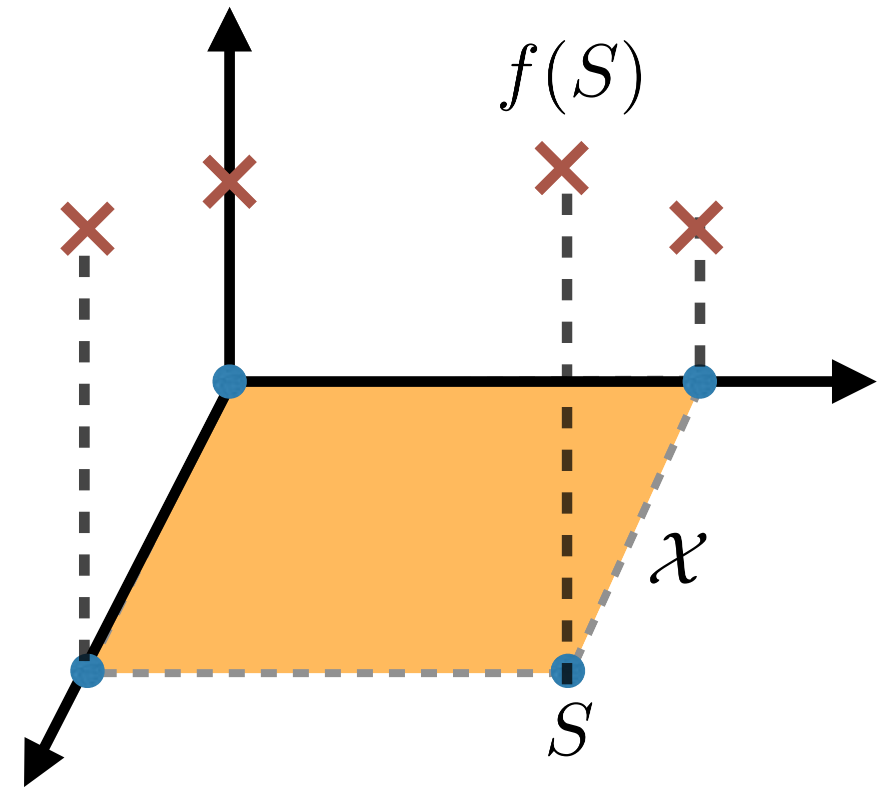

$$F^-(\mathbf{x}) \leq F(\mathbf{x}) \leq F^+(\mathbf{x}) \leq (1-\frac{1}{e})^{-1}F(\mathbf{x})$$

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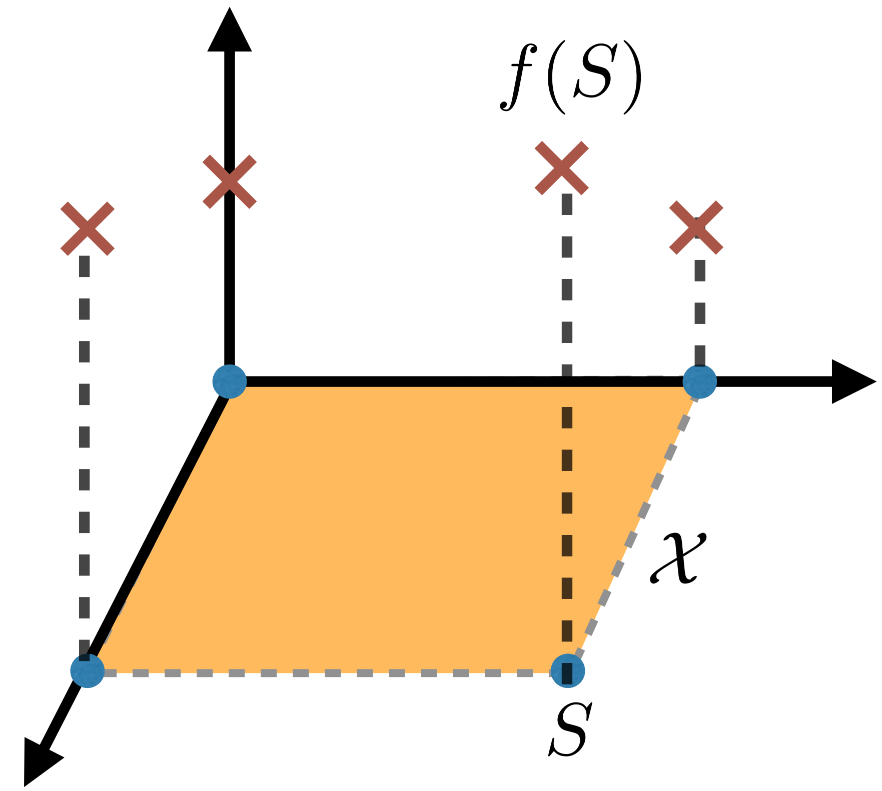

- For coverage and deep submodular functions, tight concave extensions are possible which are also efficiently computable

"Stochastic Submodular Maximization: The Case of coverage Functions", Karimi, Lucic, Hassani, Krause, 2019

"Deep Submodular Functions", Bilmes, Bai, 2017

Other Resources on Submodularity

- To watch:

  (tutorial) S. Jegelka and A. Krause: Submodularity: Theory and Applications, Simons Institute, 2017 (available on YouTube)

  (course) J. Bilmes: Submodular Functions, Optimization, & Applications to Machine Learning, University of Washington, 2014 (available on YouTube)

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Thank you!

Submodular Optimization: From Discrete to Continuous and Back

Hamed Hassani

Amin Karbasi

Penn

Yale

Slides + references: http://iid.yale.edu/icml/icml-20.md/