Submodular Optimization: From Discrete to Continuous and Back

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Slides + references: http://iid.yale.edu/icml/icml-20.md/

Submodularity: A General View

- Recall the two equivalent definitions of discrete submodularity:

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$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

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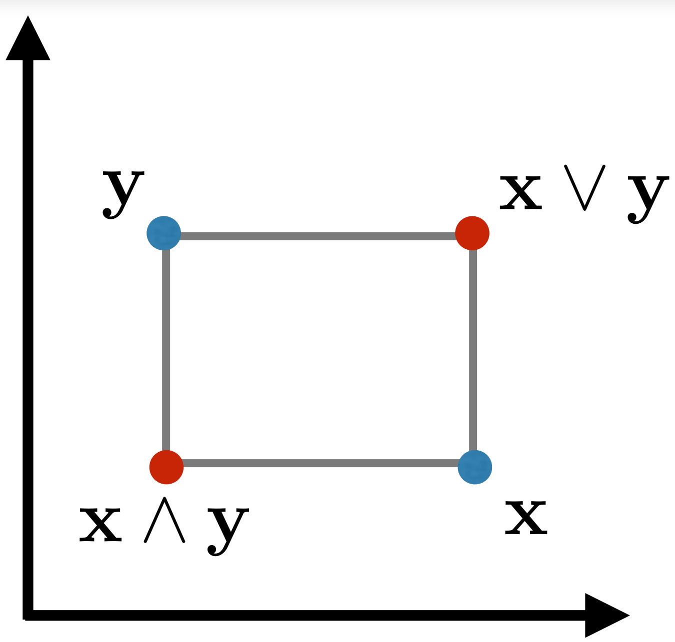

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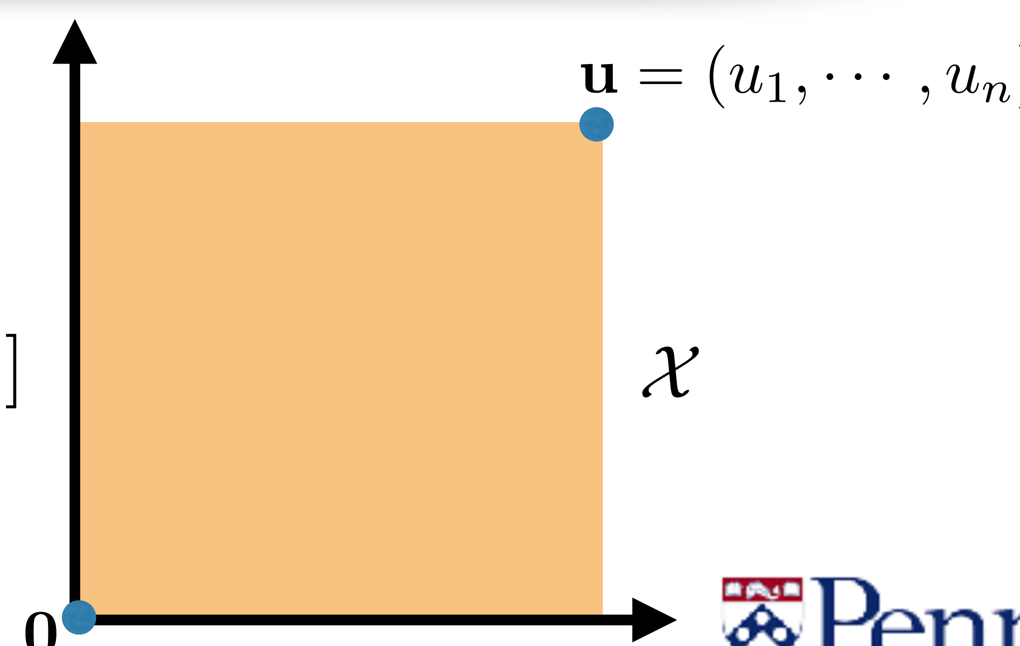

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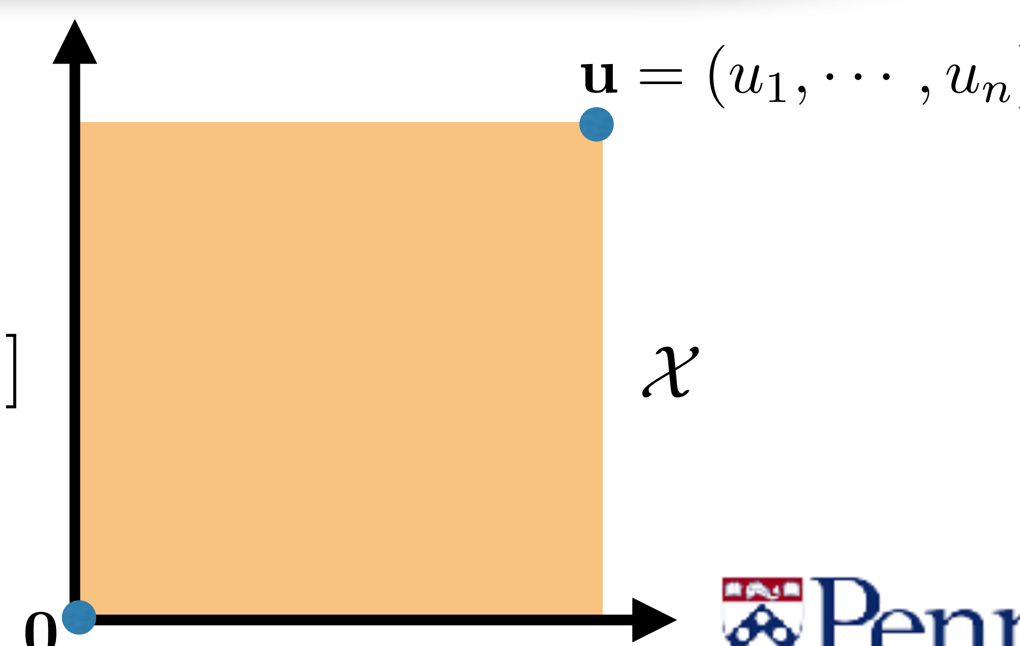

Continuous Submodular Functions

- A function $F : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is called **Continuous Submodular** if

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"Minimizing a submodular function on a lattice", Topkis, 1978

"Maximizing real-valued submodular functions: Primal and dual heuristics for location problems", Wolsey, 1982

"Submodular Functions and Optimization", Fujishige, 2005

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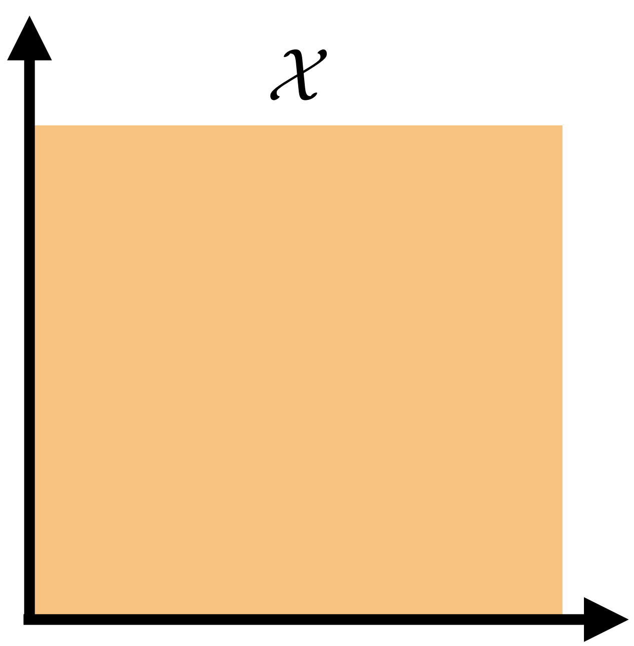

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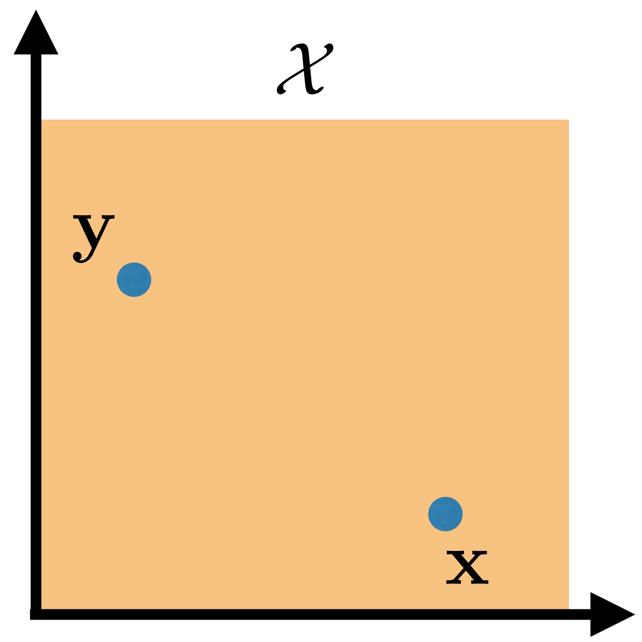

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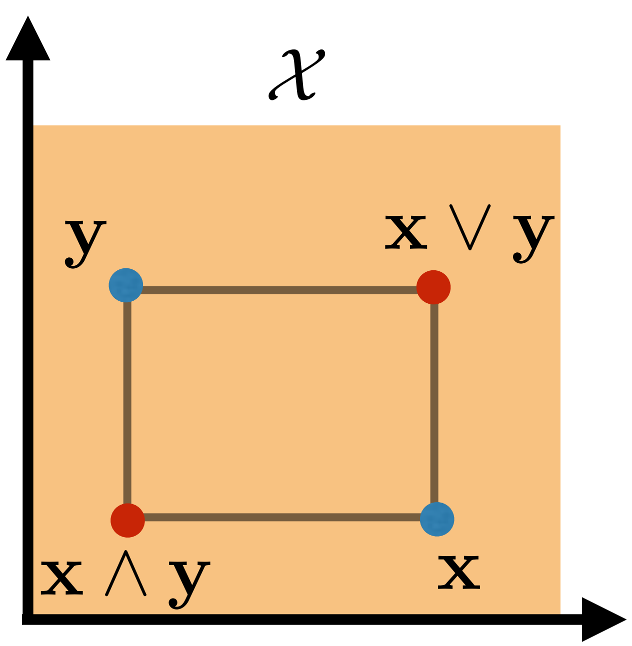

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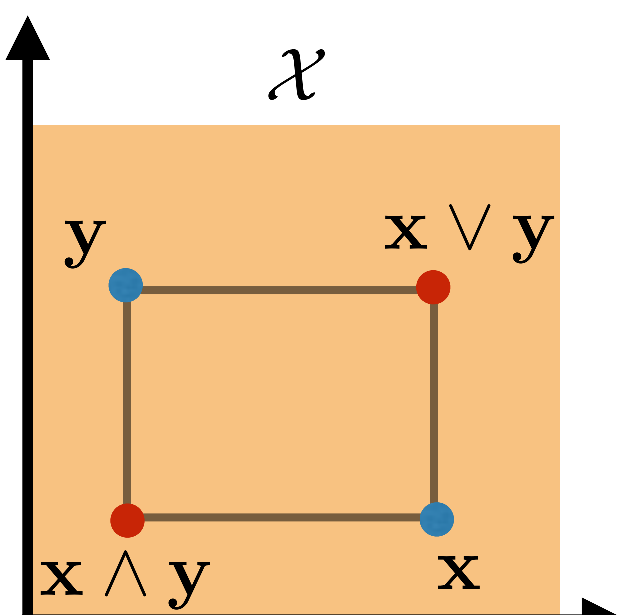

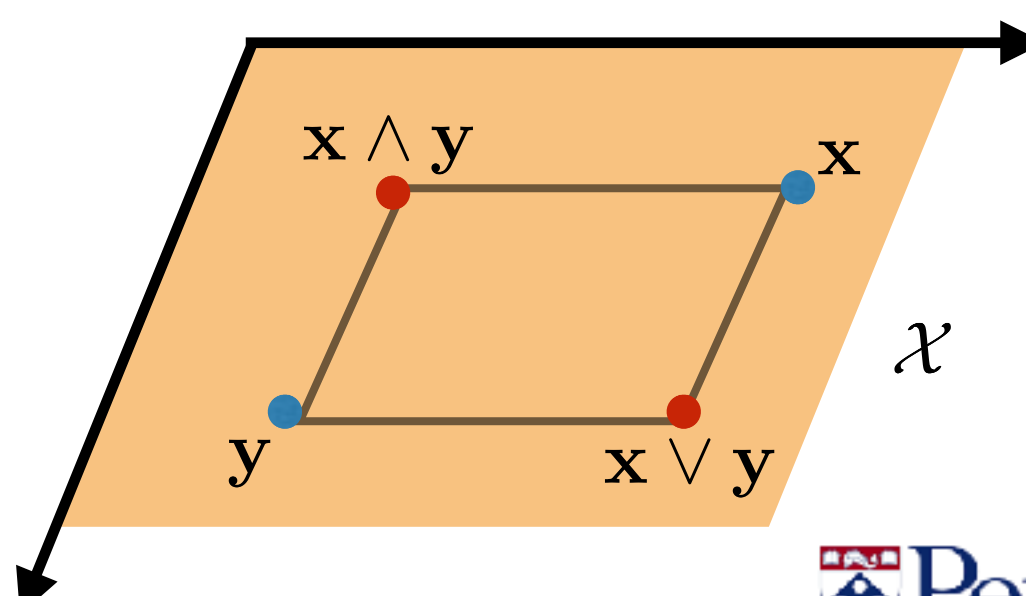

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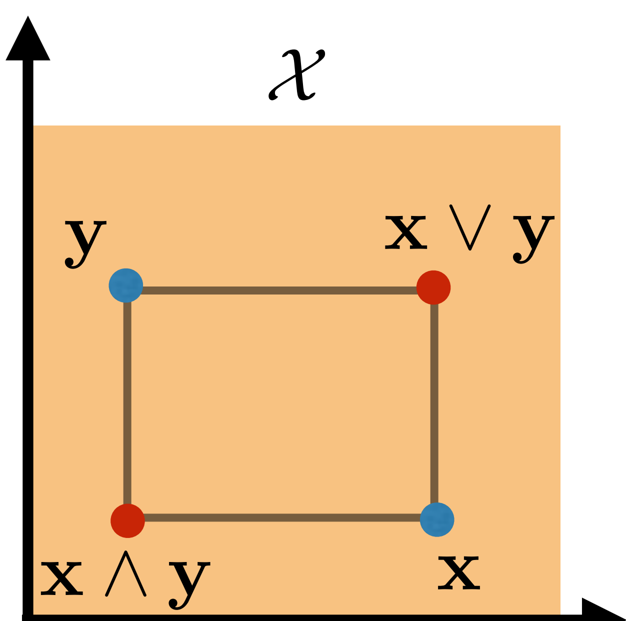

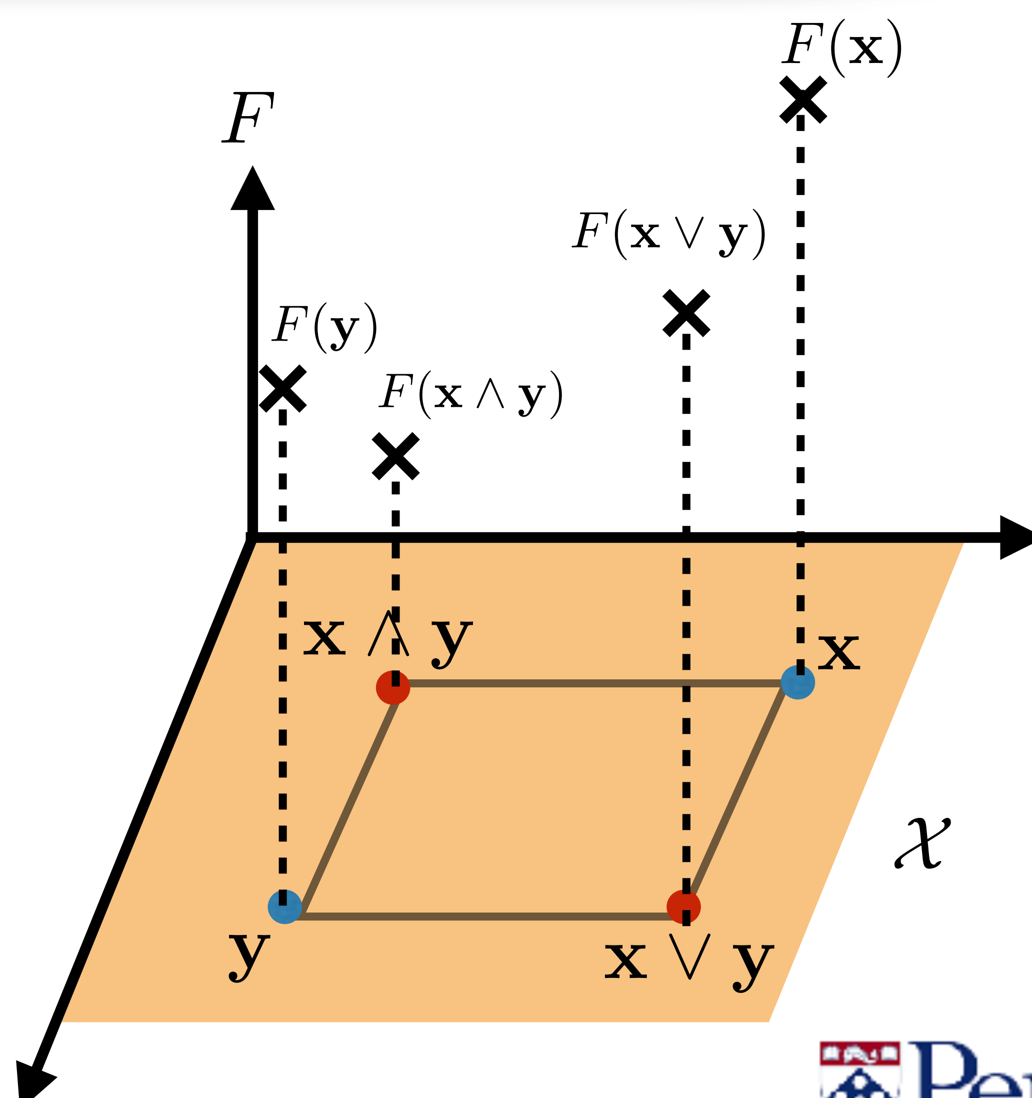

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$$\forall i \neq j, \forall \mathbf{x} \in \mathcal{X}, \quad \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j} \leq 0$$

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- BUT, does it mean Diminishing Returns (DR)?

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"Guaranteed non-convex optimization: Submodular maximization over continuous domains", Bian, Mirzasoleiman, Buhmann, Krause, 2017

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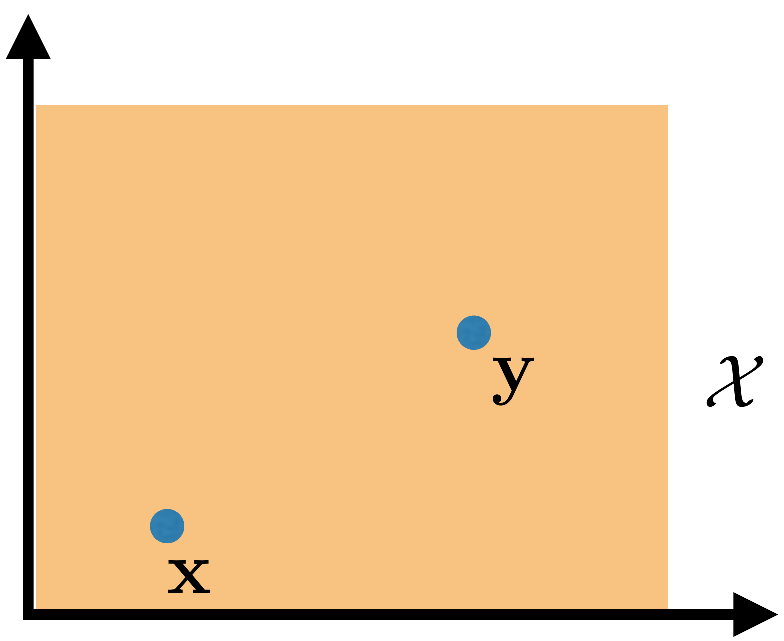

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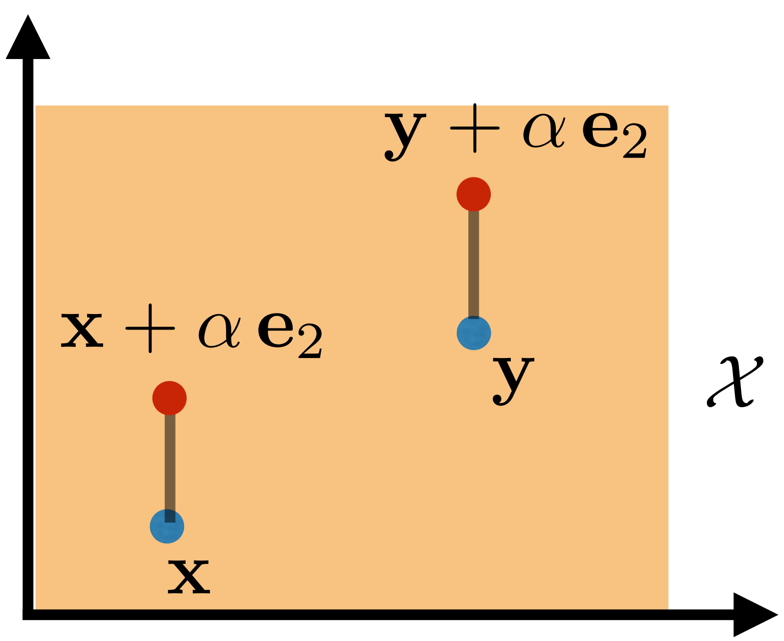

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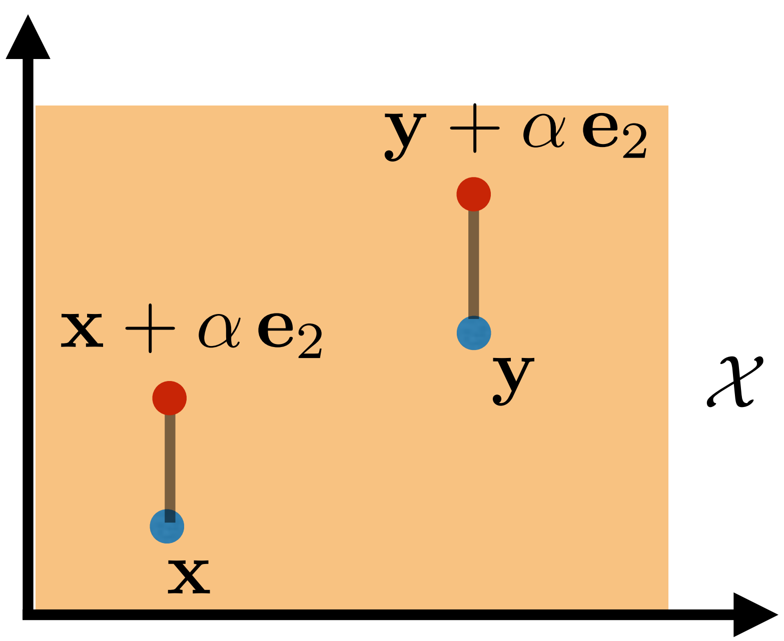

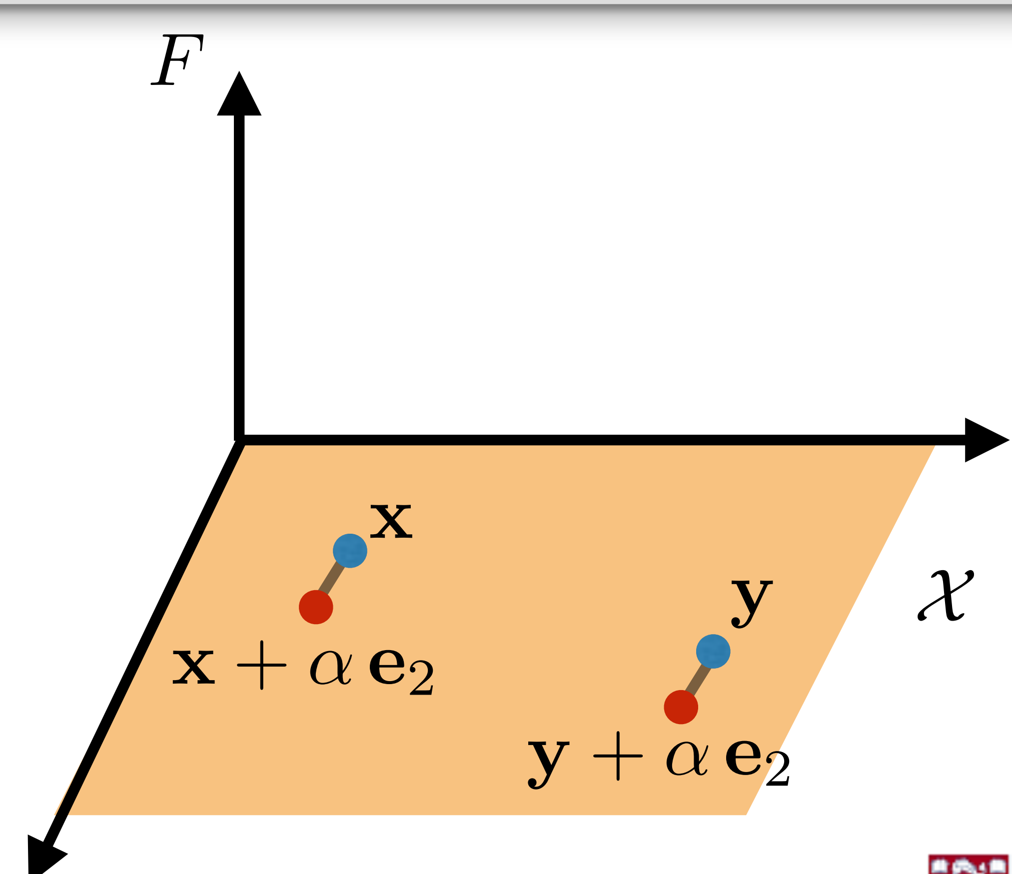

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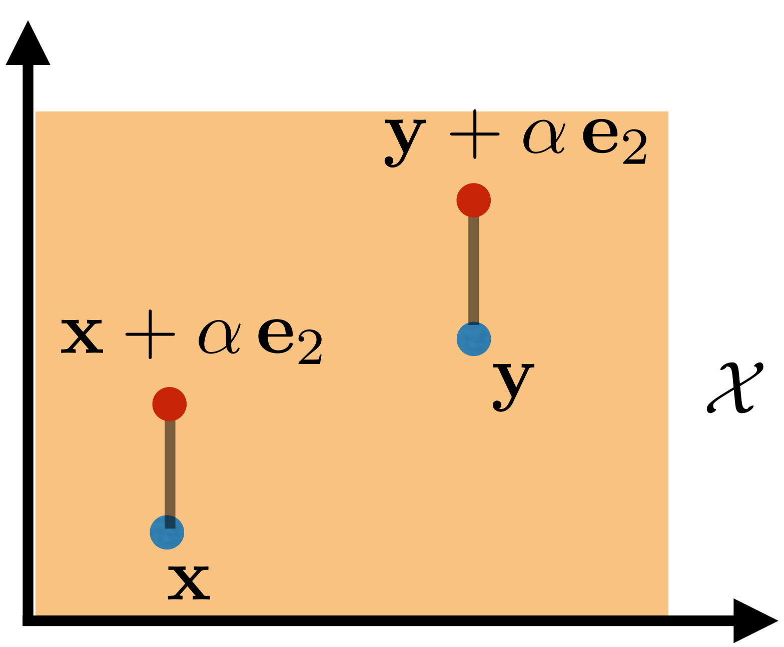

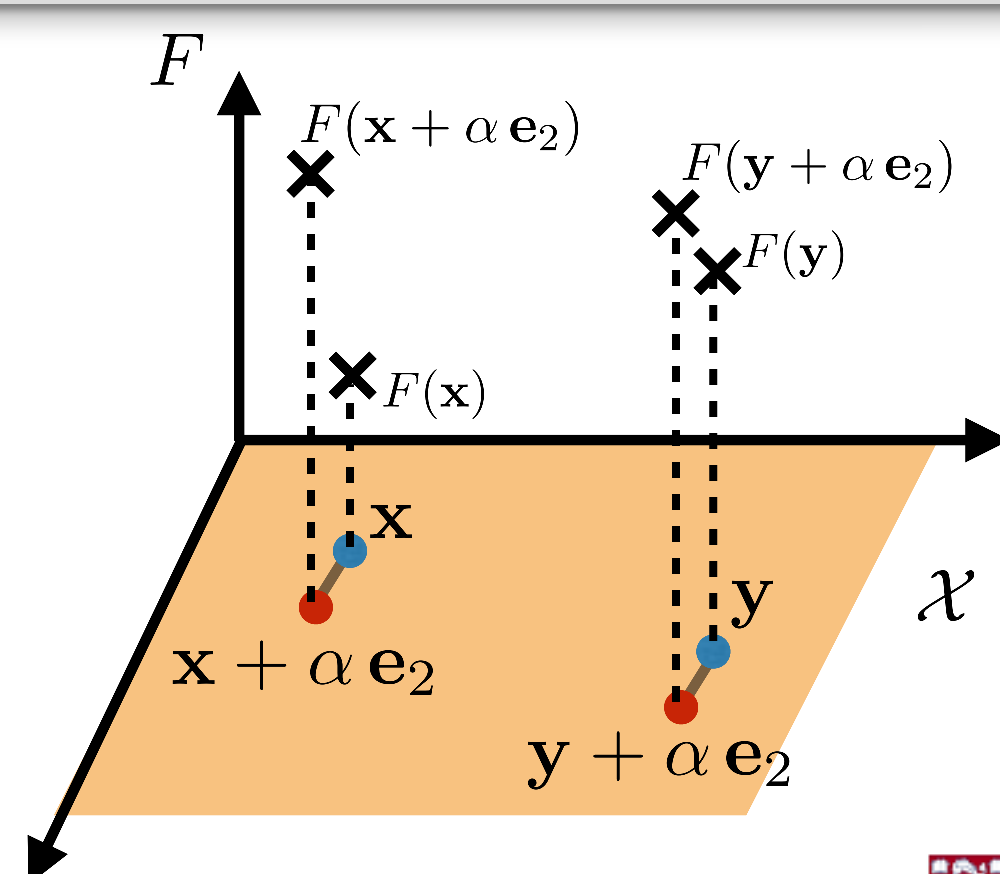

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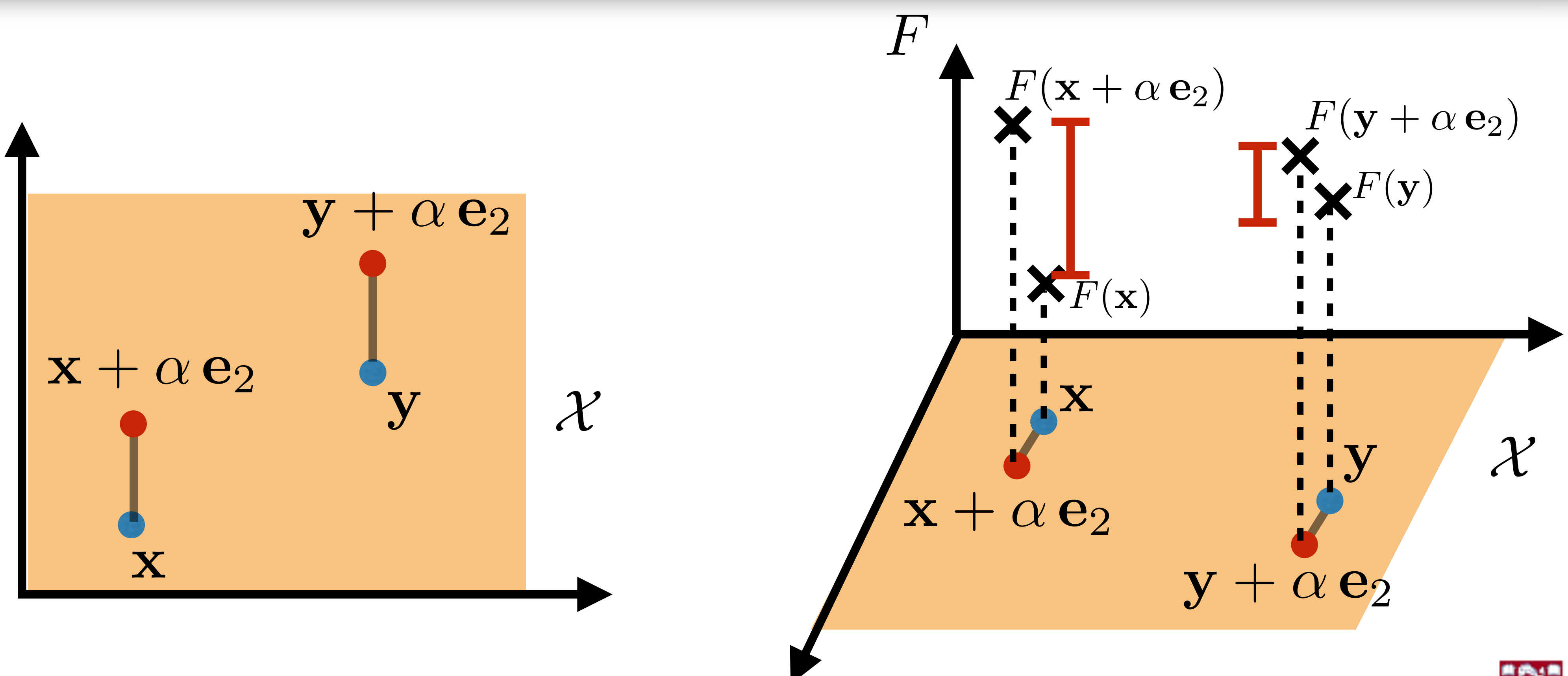

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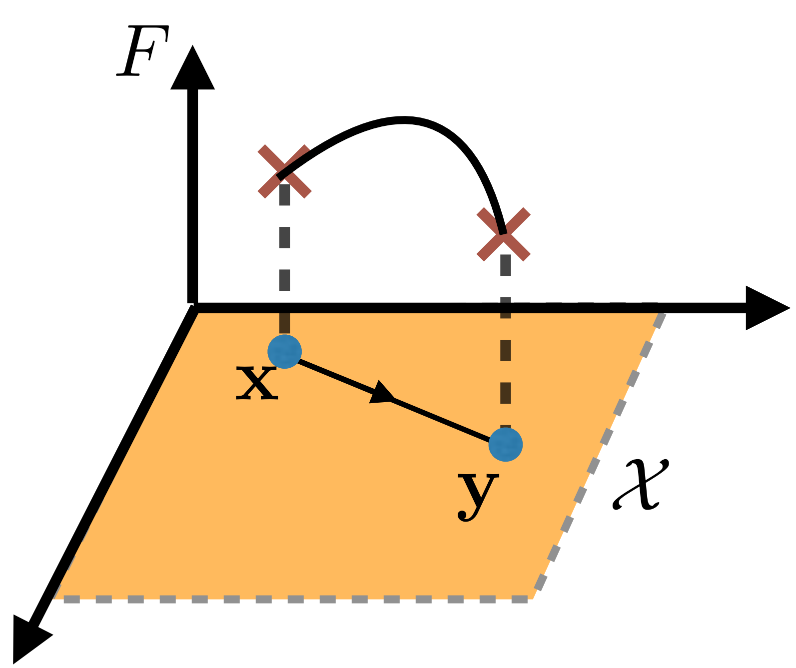

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$$\forall i, j,\ \forall \mathbf{x} \in \mathcal{X}, \quad \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j} \leq 0 \longrightarrow \nabla^2 F(\mathbf{x}) = \begin{pmatrix} \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \end{pmatrix}$$

Hessian

"Guaranteed non-convex optimization: Submodular maximization over continuous domains", Bian, Mirzasoleiman, Buhmann, Krause, 2017

Submodularity: A General View

Summary

cont. submodular

$$\begin{pmatrix} * & \leq 0 & \leq 0 \\ \leq 0 & * & \leq 0 \\ \leq 0 & \leq 0 & * \end{pmatrix}$$

DR-submodular

$$\begin{pmatrix} \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \end{pmatrix}$$

Submodularity: A General View

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Submodularity: A General View

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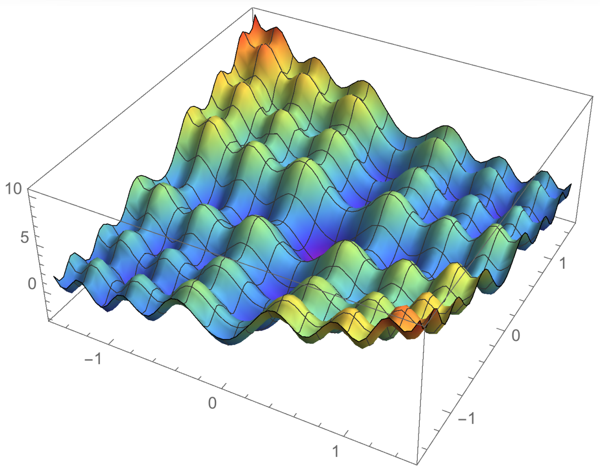
$$\begin{pmatrix} \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \end{pmatrix}$$

non-convex

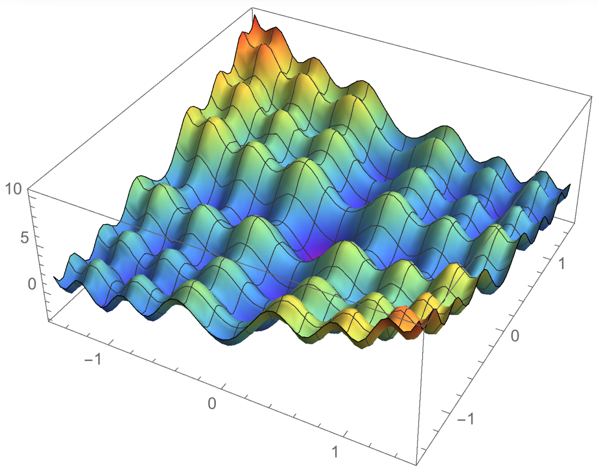


Submodularity: A General View

Summary

cont. submodular	DR-submodular	non-convex

$$\begin{pmatrix} * & \leq 0 & \leq 0 \\ \leq 0 & * & \leq 0 \\ \leq 0 & \leq 0 & * \end{pmatrix} \qquad \begin{pmatrix} \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \end{pmatrix}$$



- F is called monotone if for any $\mathbf{x} \leq \mathbf{y}$ we have $F(\mathbf{x}) \leq F(\mathbf{y})$

Submodularity: A General View

Summary

cont. submodular	DR-submodular	non-convex

$$\begin{pmatrix} * & \leq 0 & \leq 0 \\ \leq 0 & * & \leq 0 \\ \leq 0 & \leq 0 & * \end{pmatrix}$$

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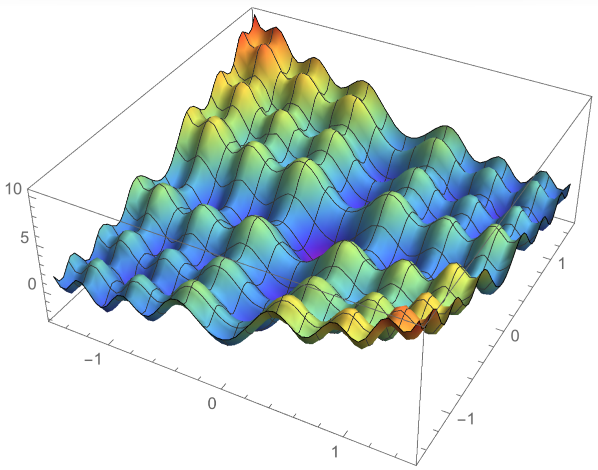
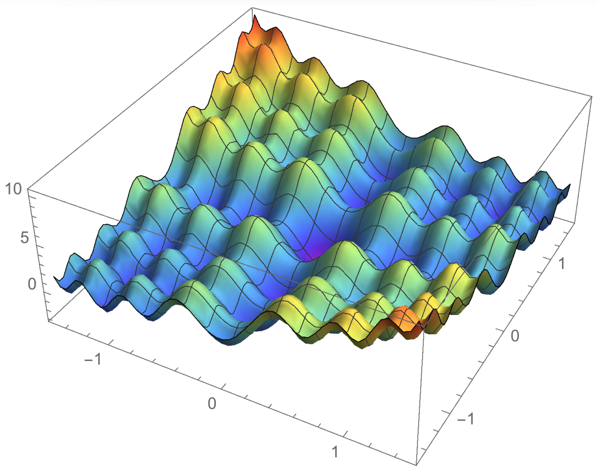


Examples and Applications

Submodularity: A General View

Summary

cont. submodular	DR-submodular	non-convex
$$\begin{pmatrix} * & \leq 0 & \leq 0 \\ \leq 0 & * & \leq 0 \\ \leq 0 & \leq 0 & * \end{pmatrix}$$	$$\begin{pmatrix} \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 \end{pmatrix}$$	

Examples and Applications

$$F(\mathbf{x}) = \mathbf{x}^T \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}$$

Yale Penn

Submodularity: A General View

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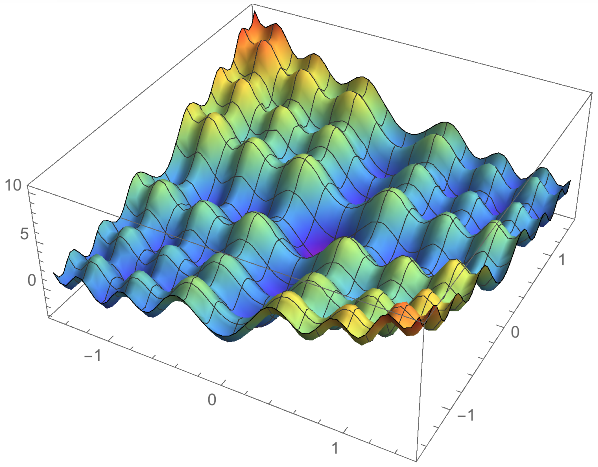

Examples and Applications

Calinescu et al., 2011

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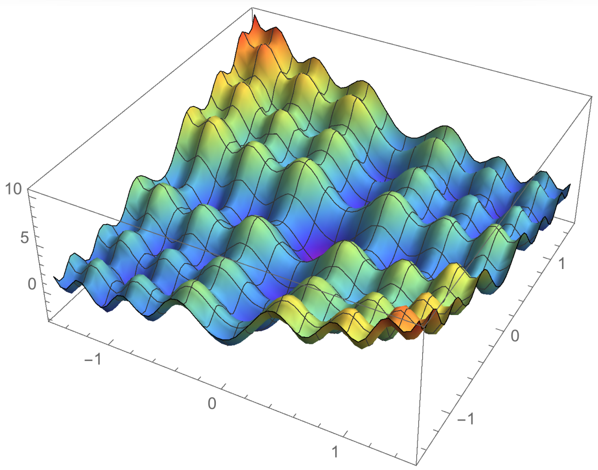
Multi-linear extension of submodular set functions

Yale

Penn

Submodularity: A General View

Summary

cont. submodular	DR-submodular	non-convex
$\begin{pmatrix} {\color{red}*} & \leq 0 & \leq 0 \\ \leq 0 & {\color{red}*} & \leq 0 \\ \leq 0 & \leq 0 & {\color{red}*} \end{pmatrix}$	$\begin{pmatrix} {\color{red}\leq 0} & \leq 0 & \leq 0 \\ \leq 0 & {\color{red}\leq 0} & \leq 0 \\ \leq 0 & \leq 0 & {\color{red}\leq 0} \end{pmatrix}$	

Examples and Applications

$F(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{x}$

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Multi-linear extension of submodular set functions

Bian et al, 2020

Influence maximization

Kulesza, Taskar, 2012

MAP Inference of DPPs

Hartline et al, 2014 Soma et al, 2017

Revenue Maximization

Soma et al, '14 Hotano et al, '15 Stalib, Jegelka, '17 Eghbali et al., '16

Resource Allocation

Bian et al, 2020

coverage, summarization, etc

Bian et al, 2018

Mean-Field Inference

Yale Penn

Submodular Maximization

- We consider the following optimization problem:

$$\text{maximize } F(\mathbf{x})$$
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Structure

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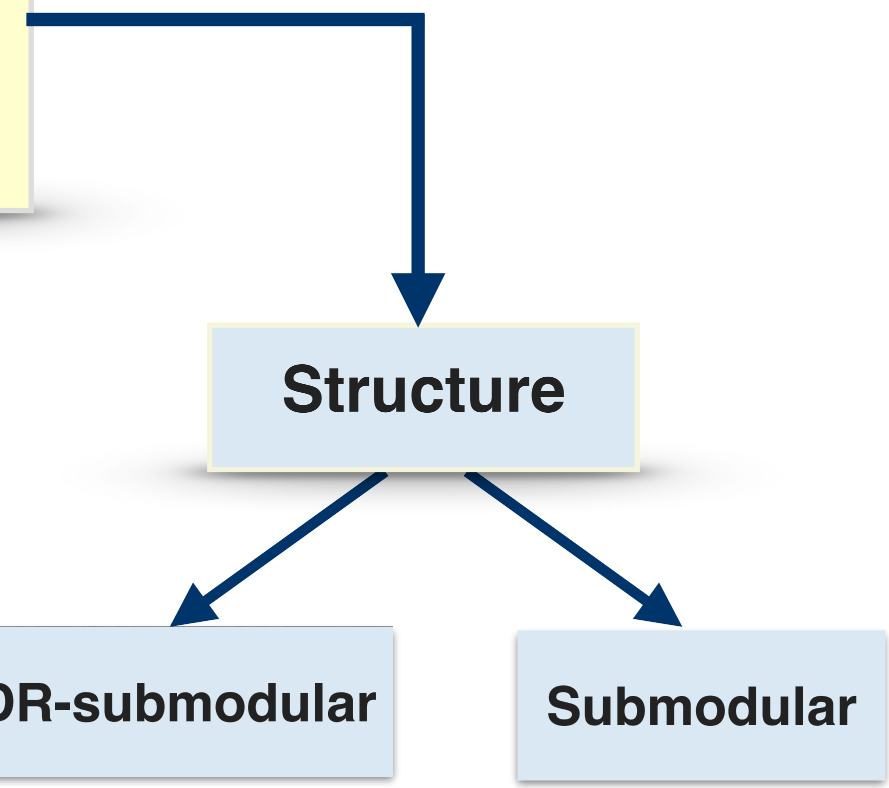

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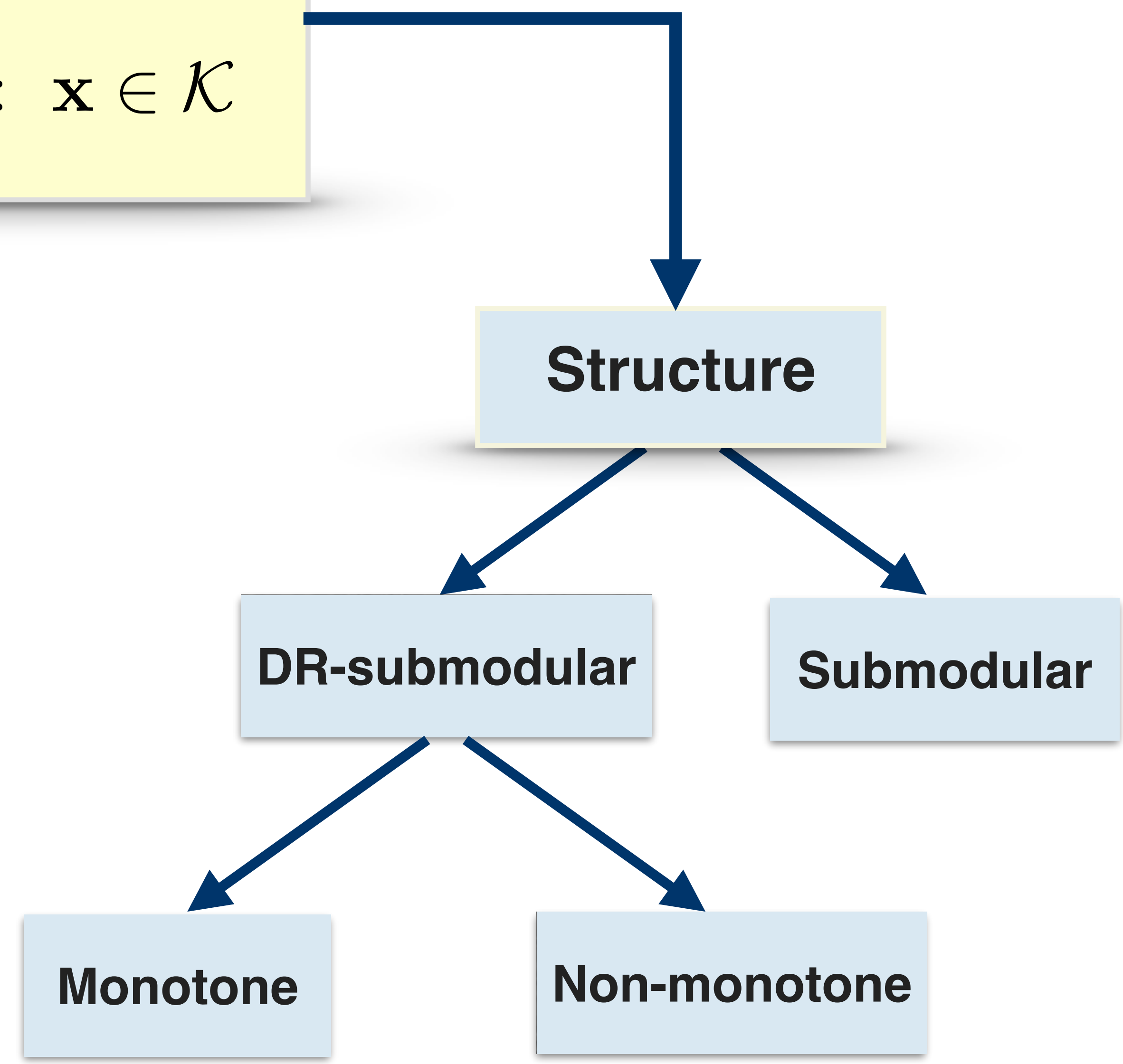

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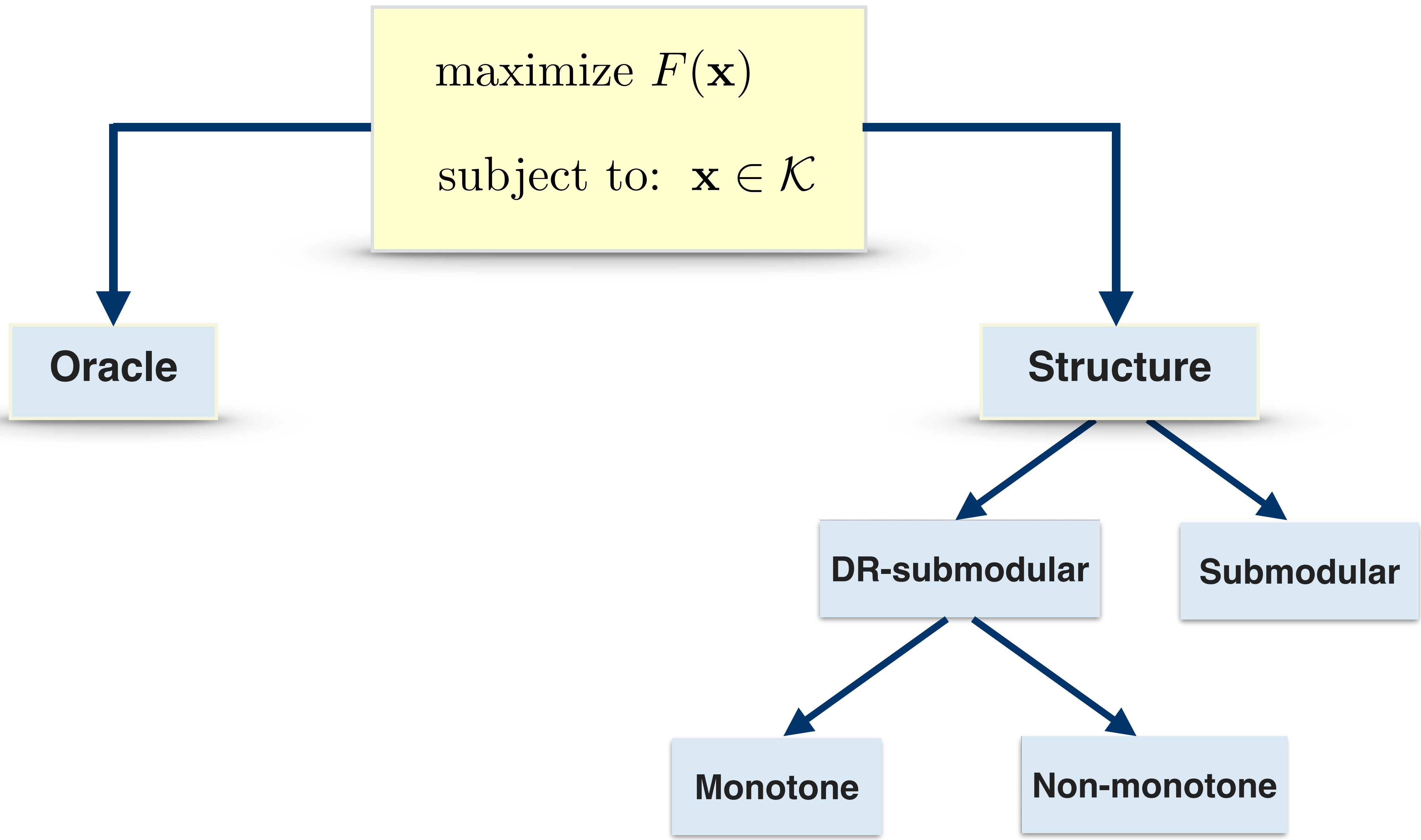

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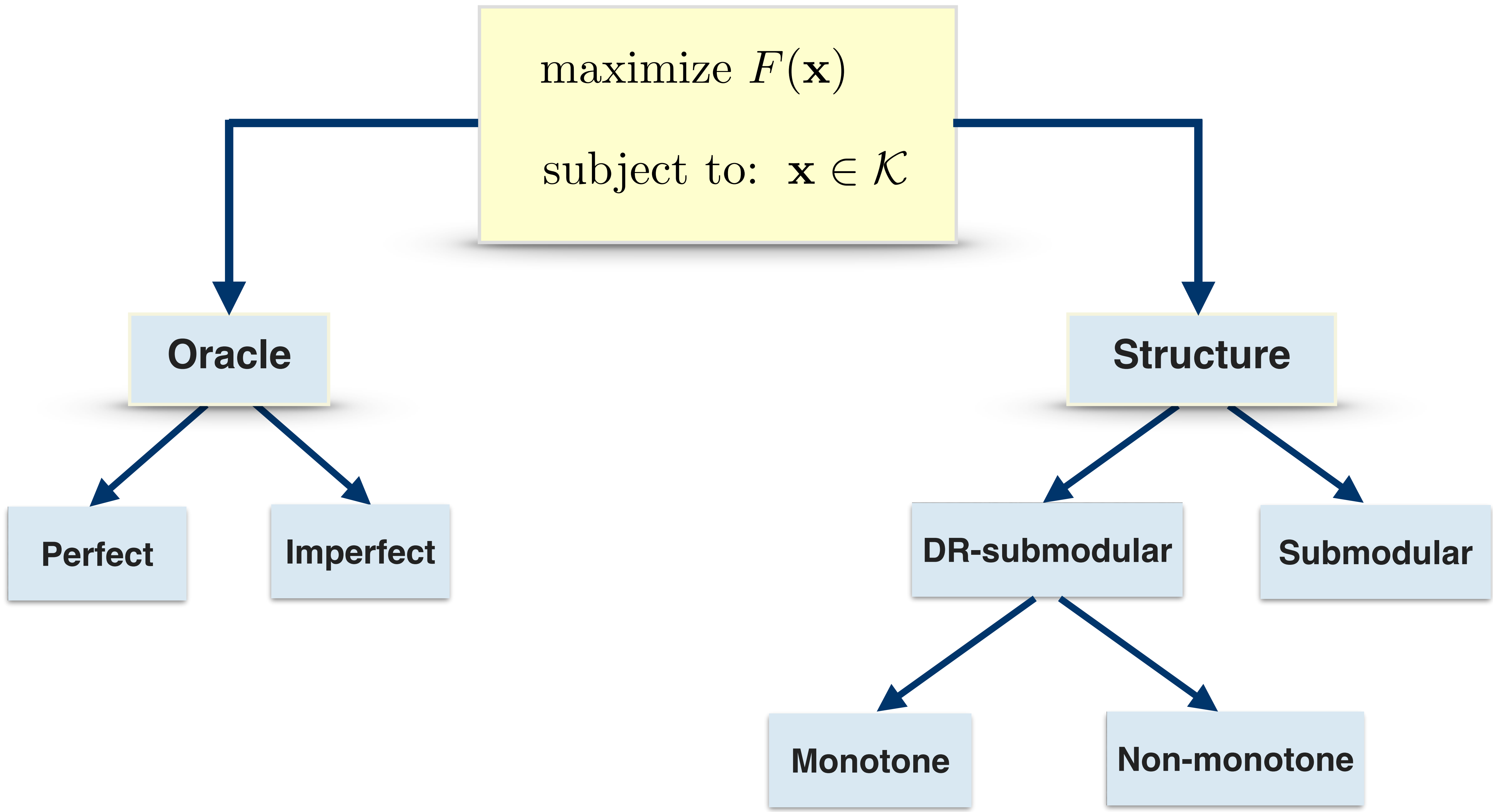

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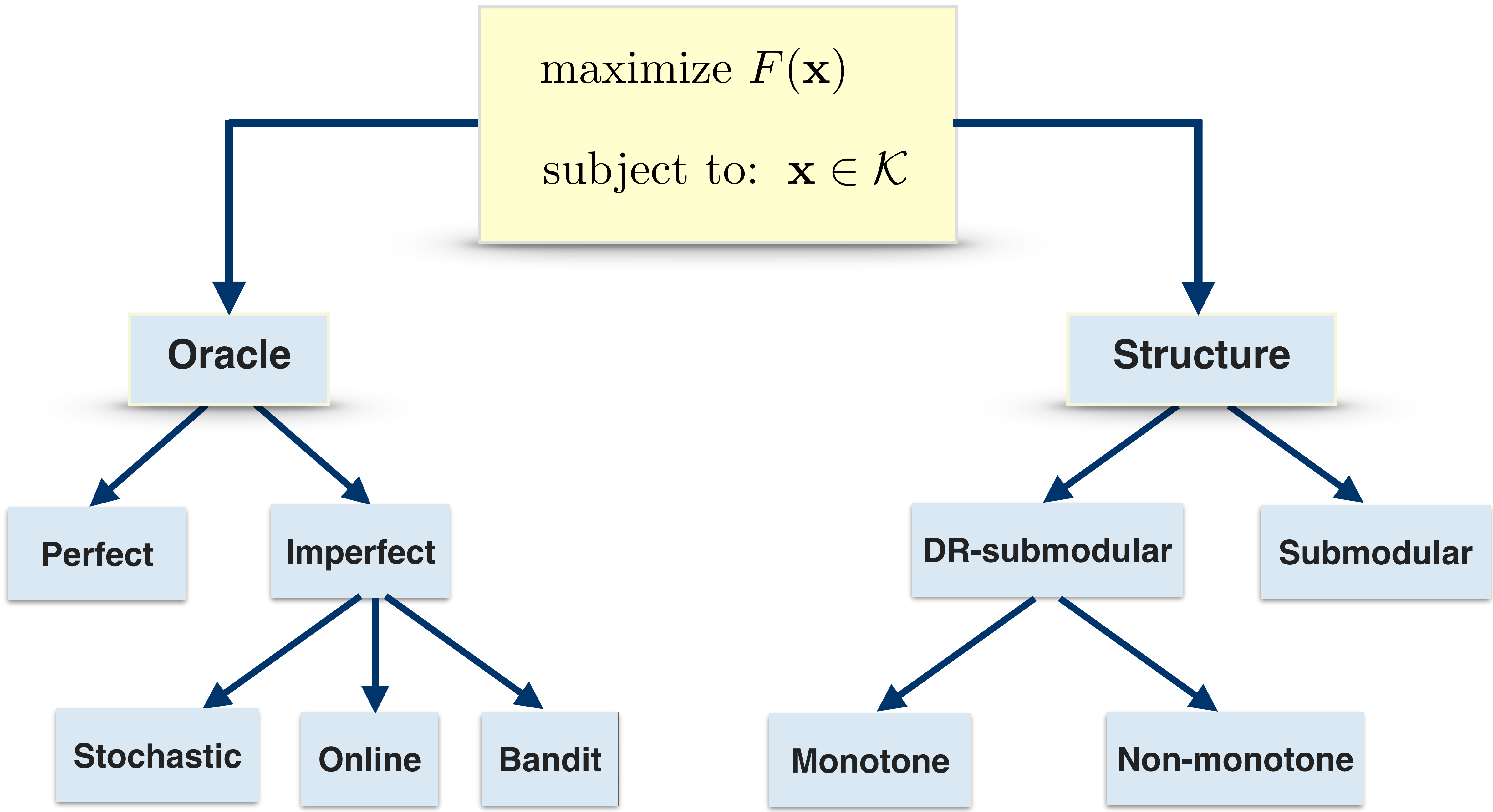

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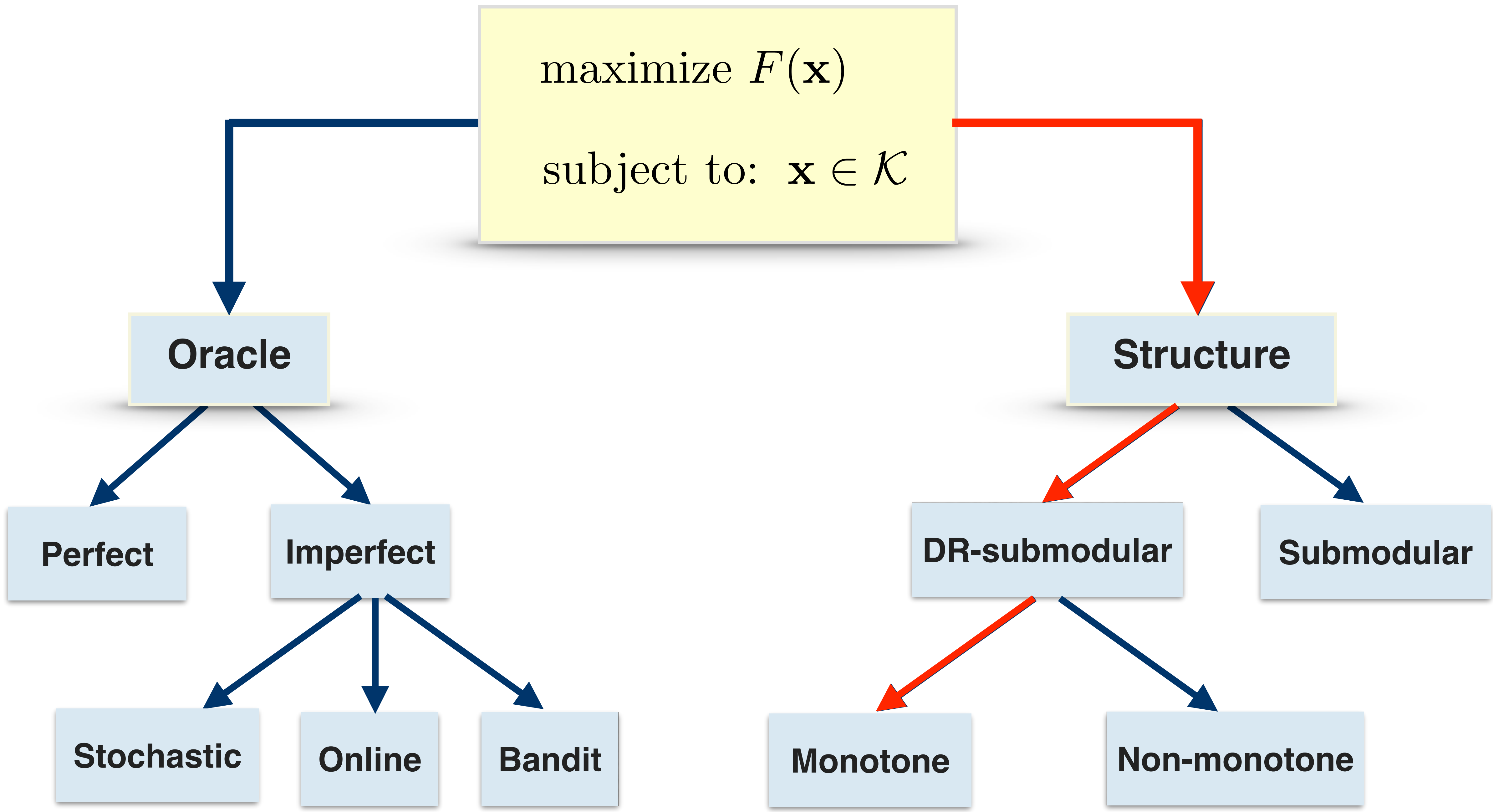

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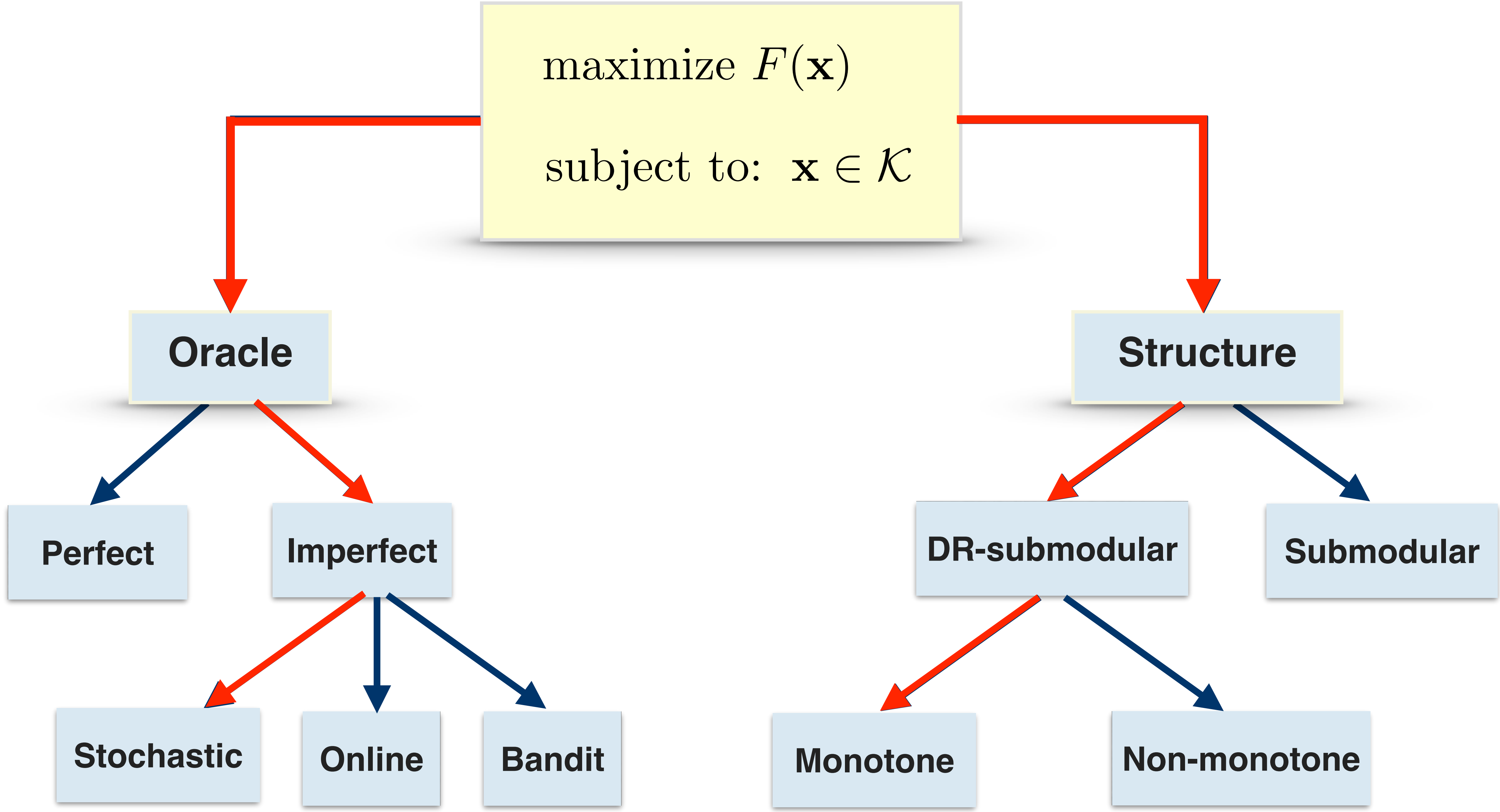

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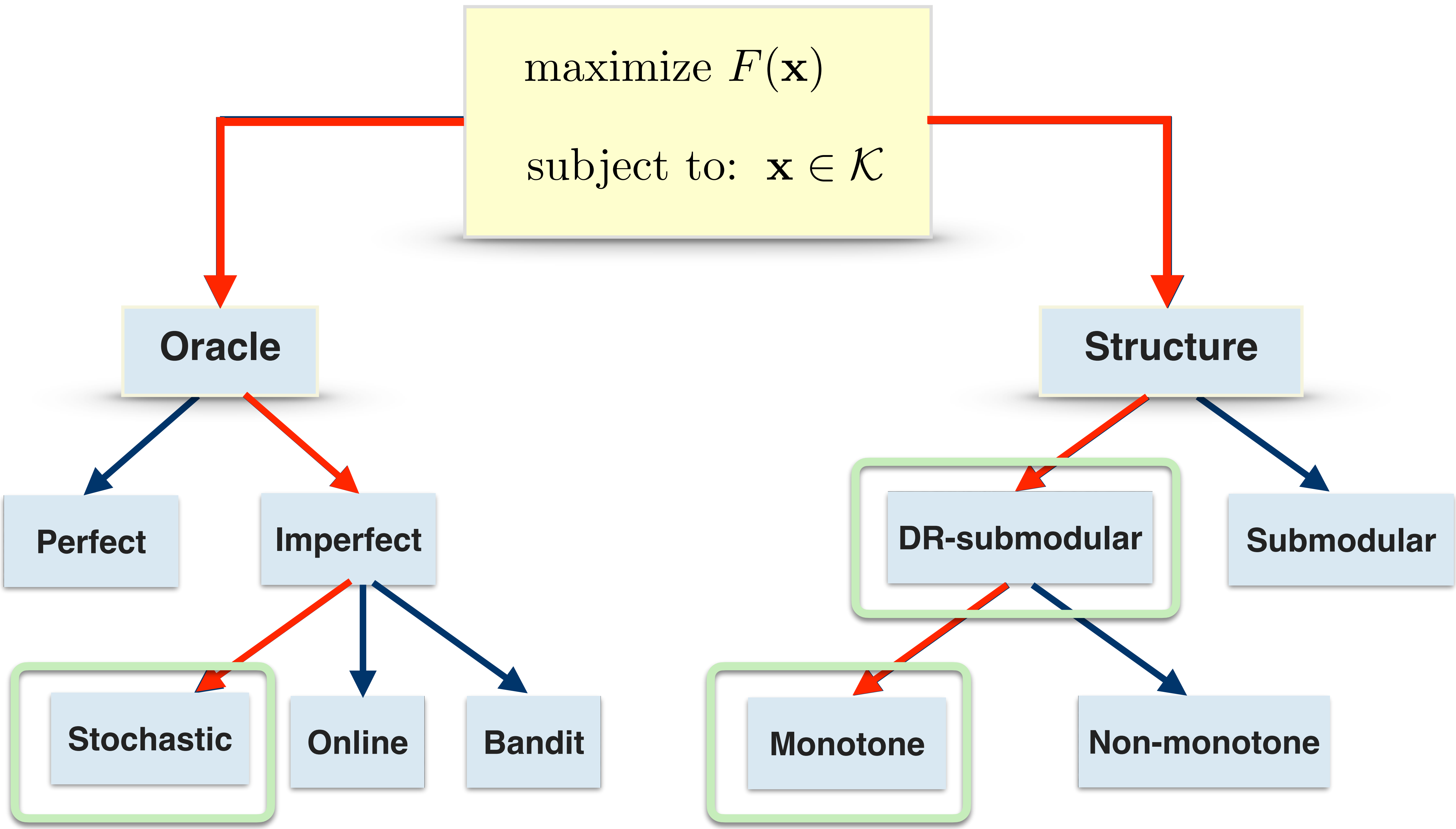

(Stochastic) Submodular Maximization

$$\max_{\mathbf{x} \in \mathcal{K}} F(\mathbf{x})$$

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Penn

(Stochastic) Submodular Maximization

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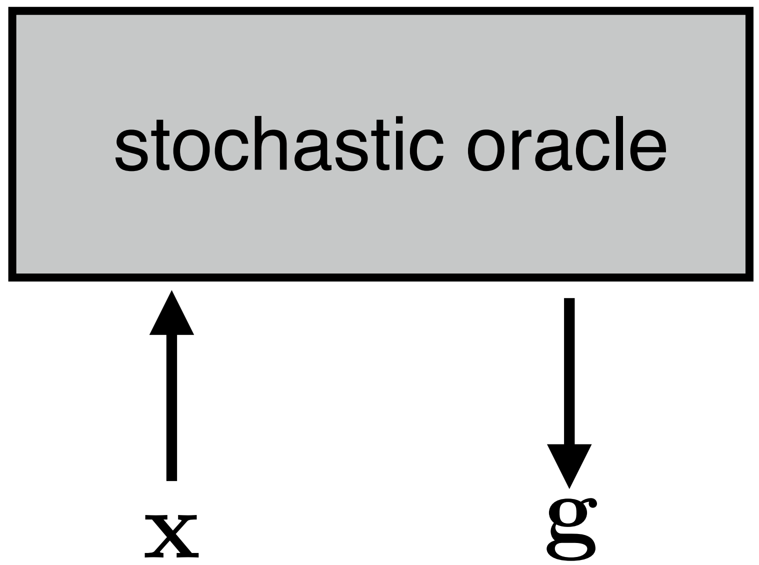

“Gradient Methods for Submodular Maximization”, Hassani, Soltanolkotabi, Karbasi, 2017

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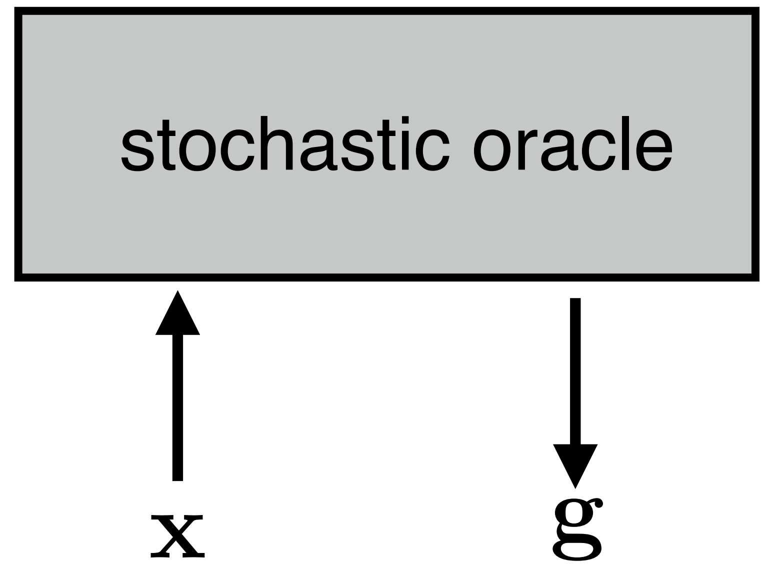

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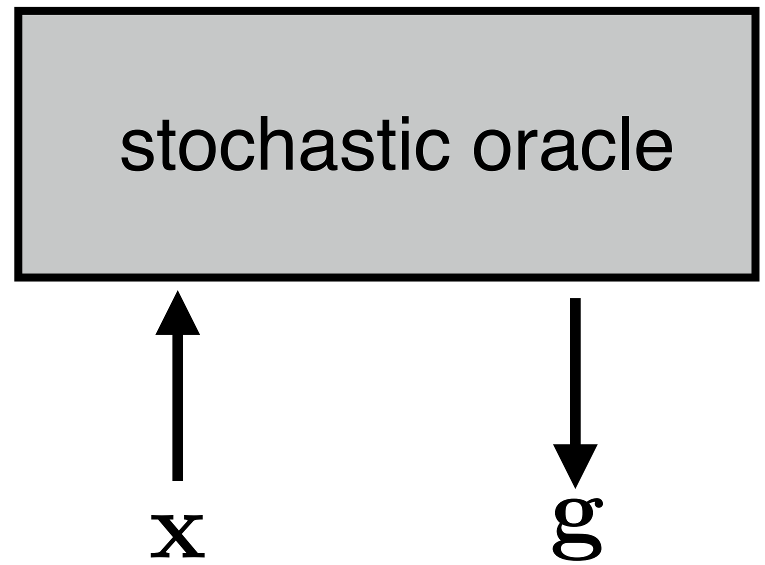

$\mathbf{x}$

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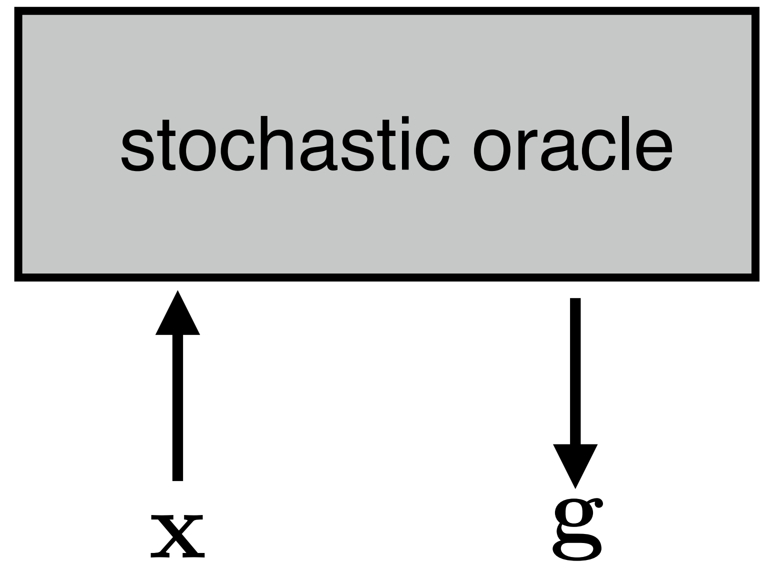

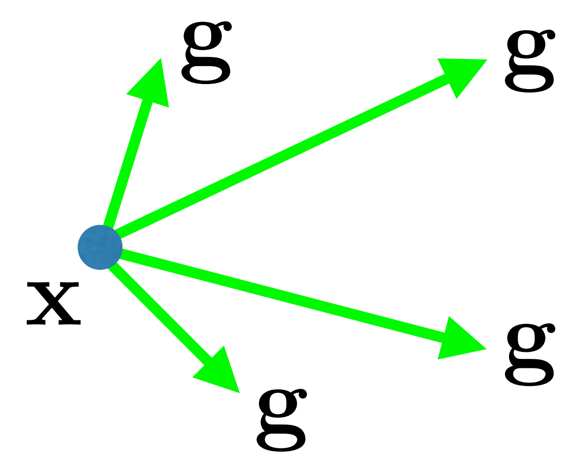

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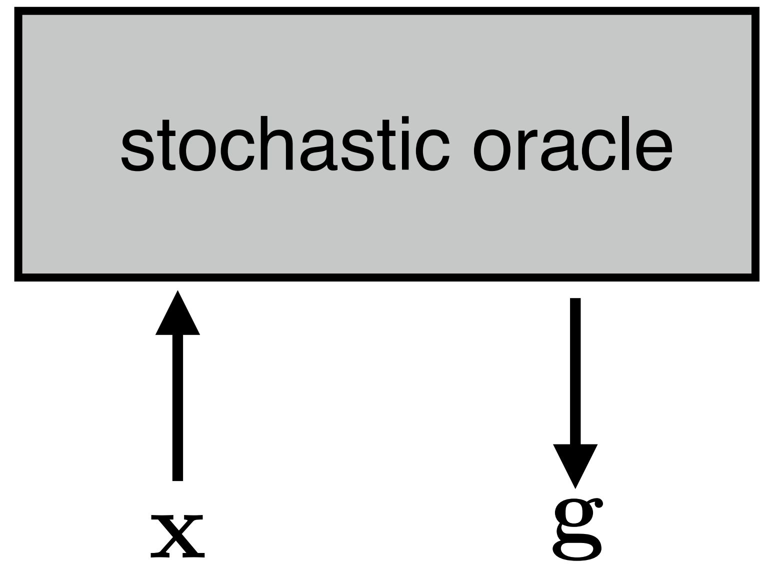

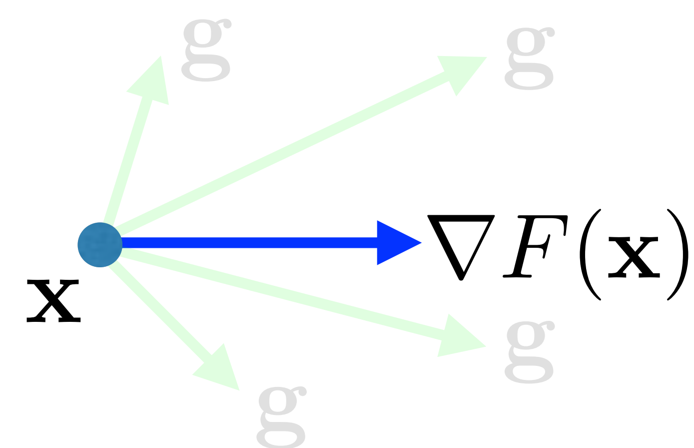

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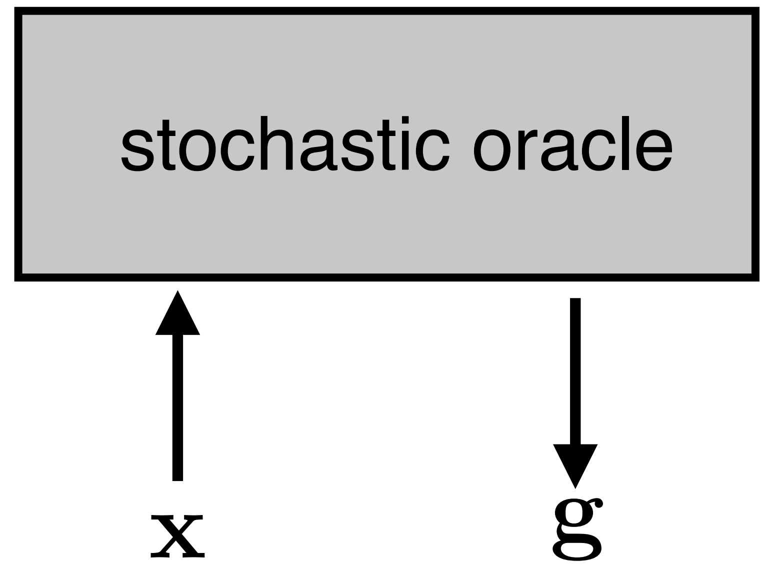

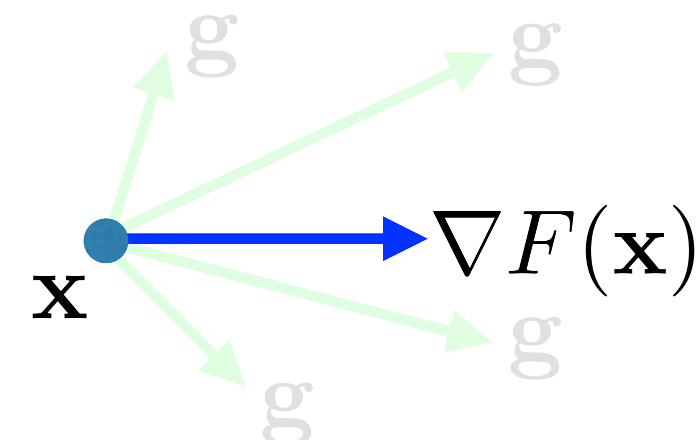

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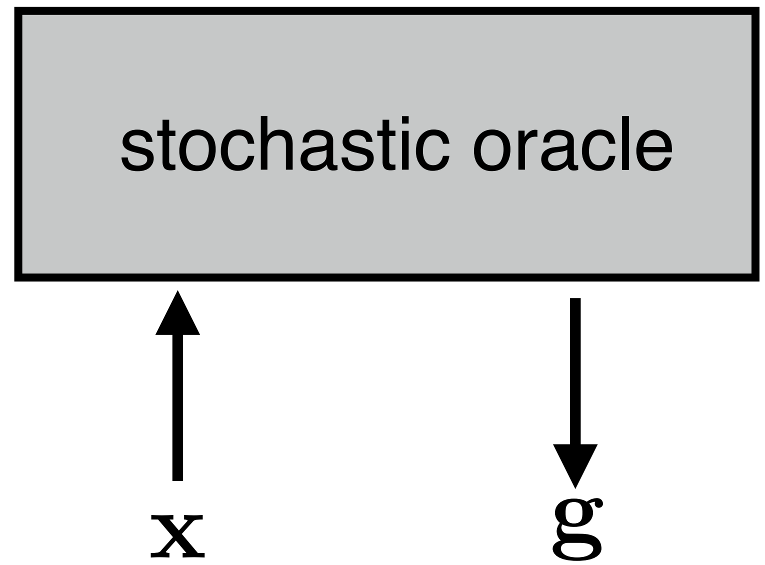

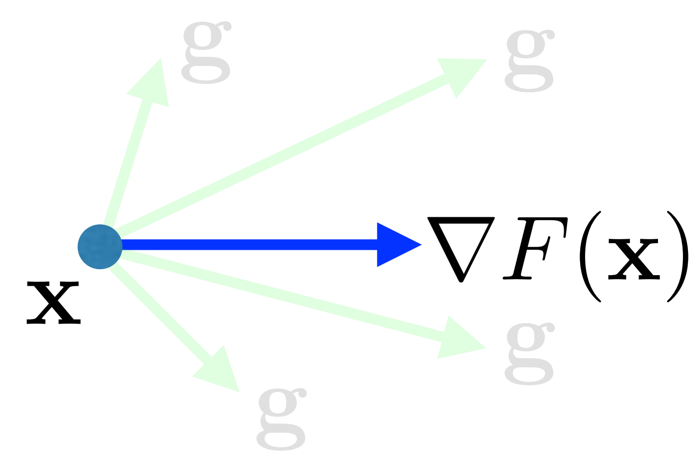

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Devise efficient algorithms to solve the problem above.

Yale Penn

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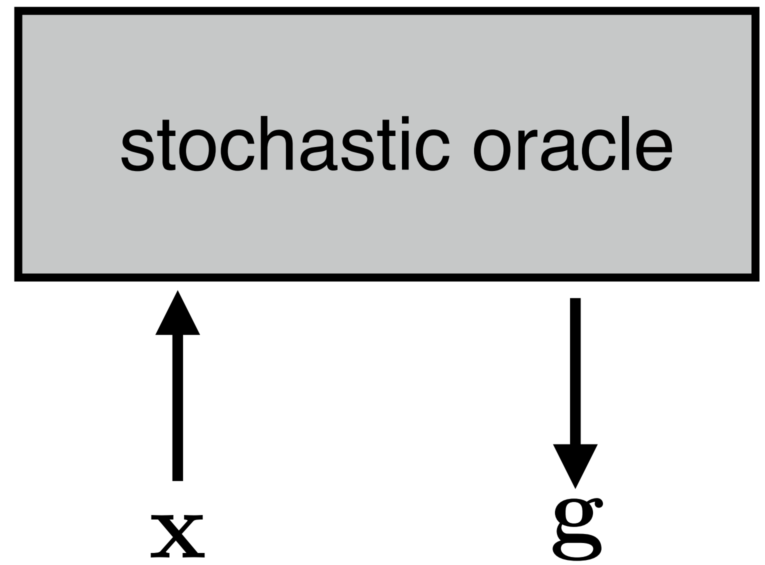

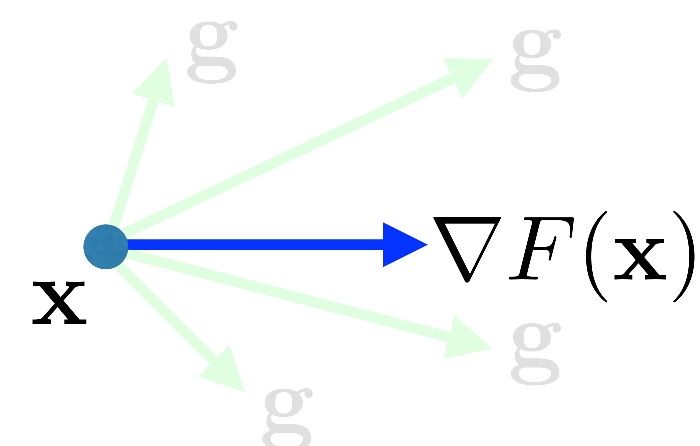

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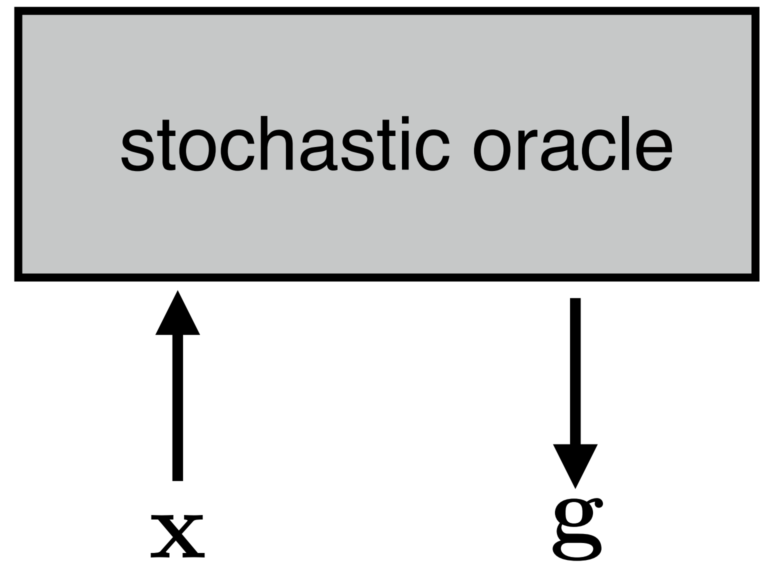

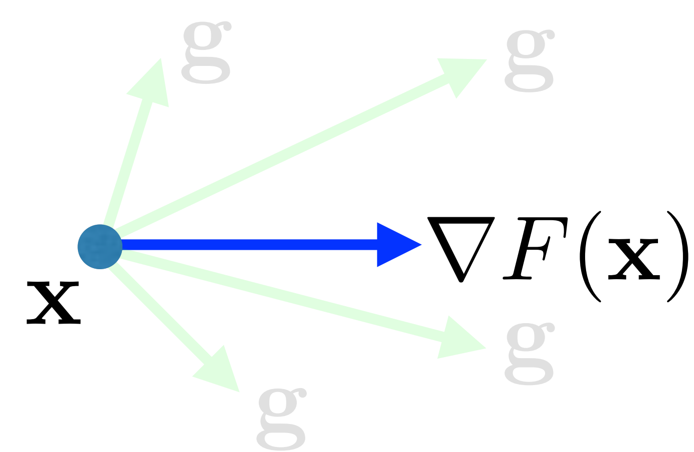

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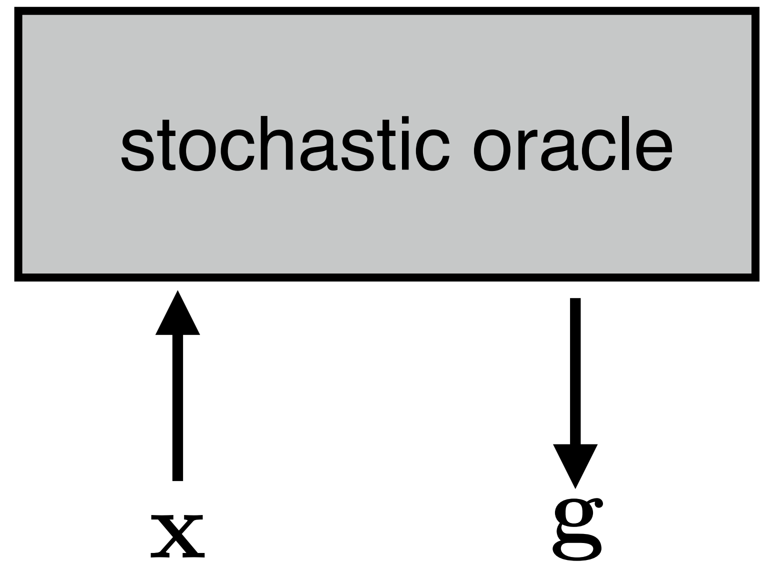

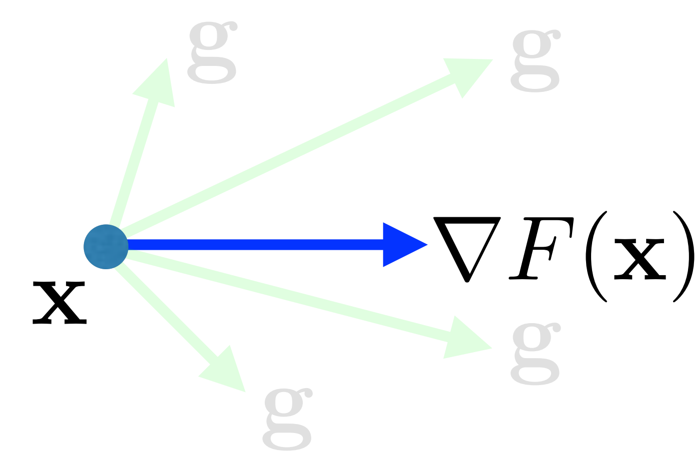

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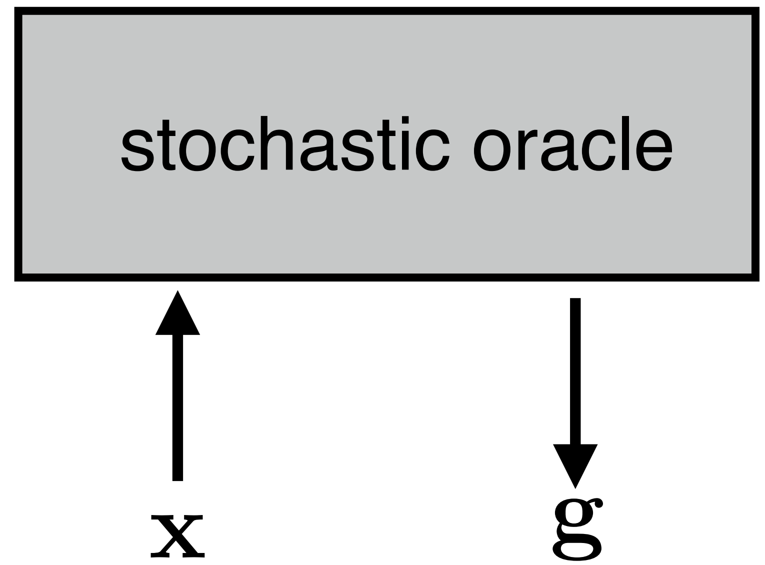
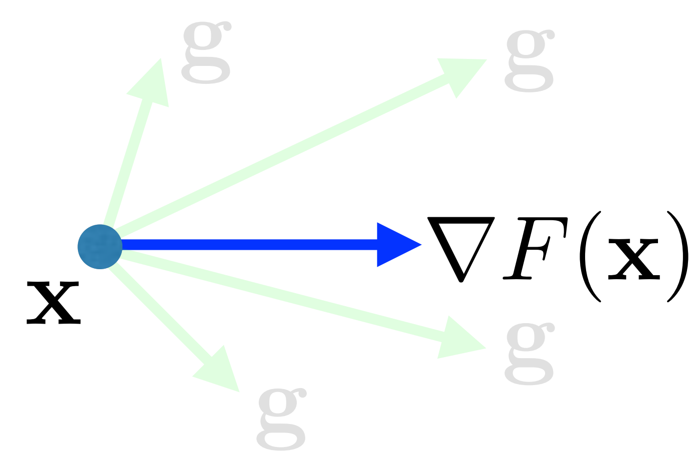
$$\mathrm{OPT} \triangleq \max_{\mathbf{x}\in\mathcal{K}} F(\mathbf{x})$$





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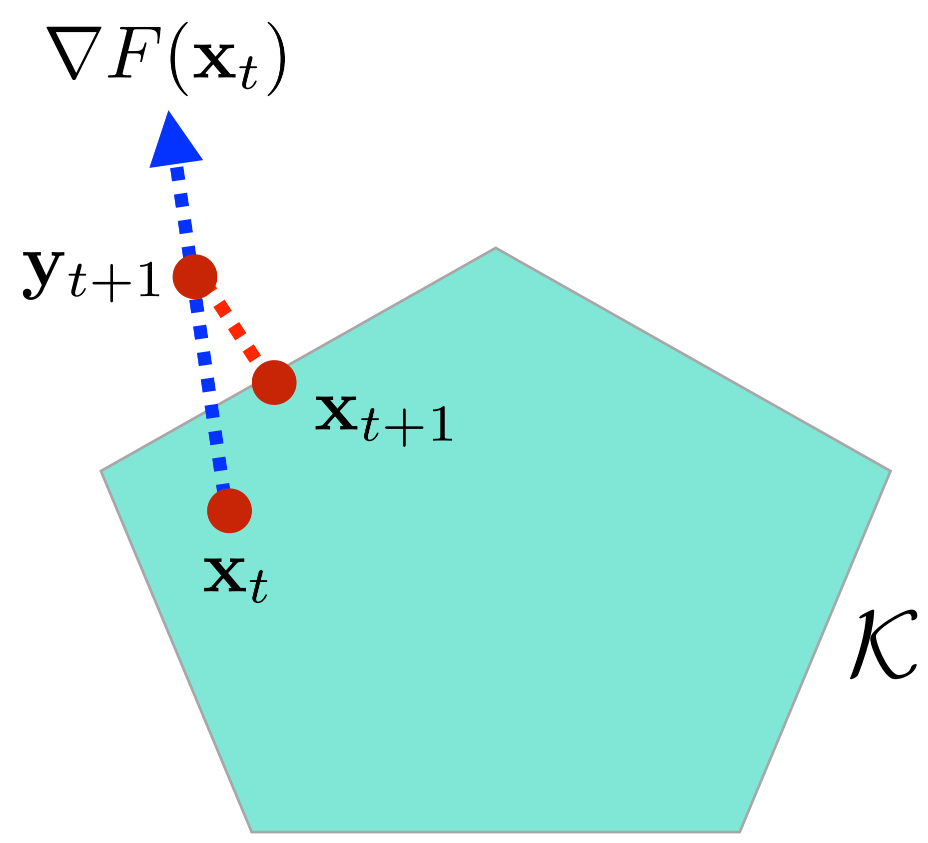
Projected Gradient Methods

- Repeat for T iterations:

$$\mathbf{y}_{t+1} = \mathbf{x}_t + \eta_t \nabla F(\mathbf{x}_t)$$

$$\mathbf{x}_{t+1} = \mathrm{Proj}_{\mathcal{K}}(\mathbf{y}_{t+1})$$



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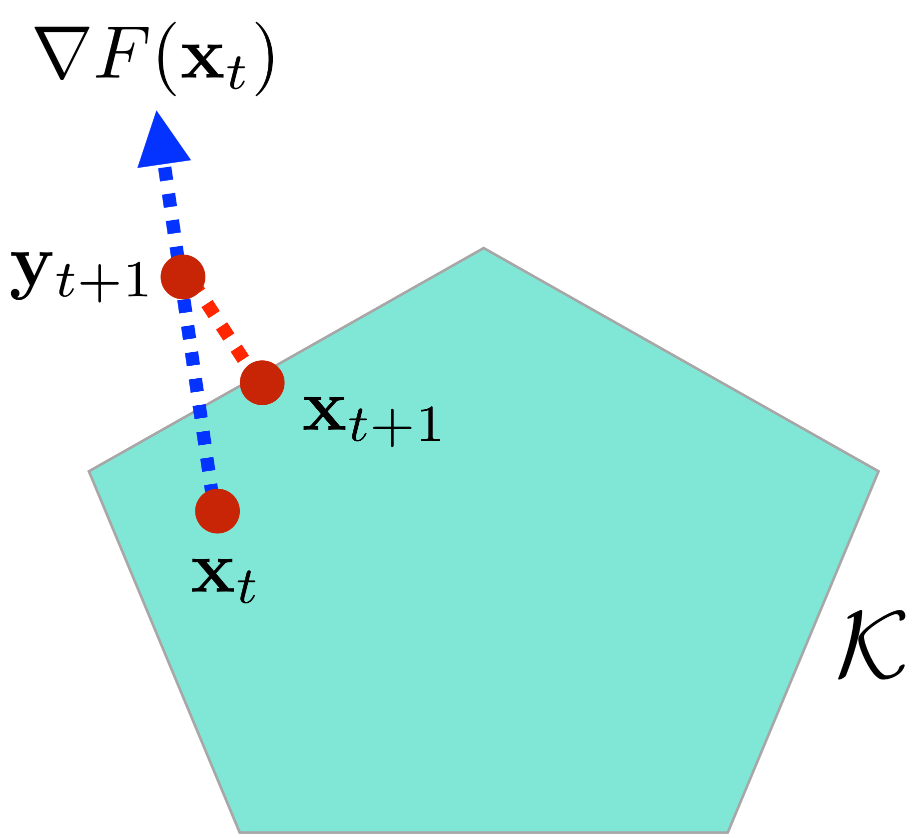

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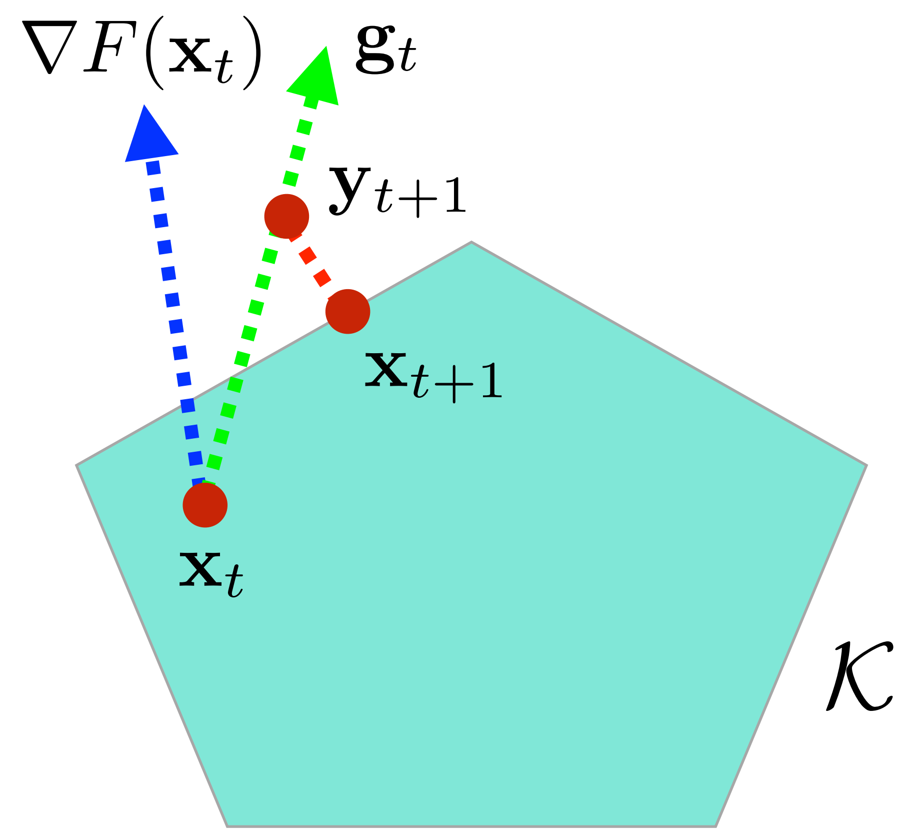

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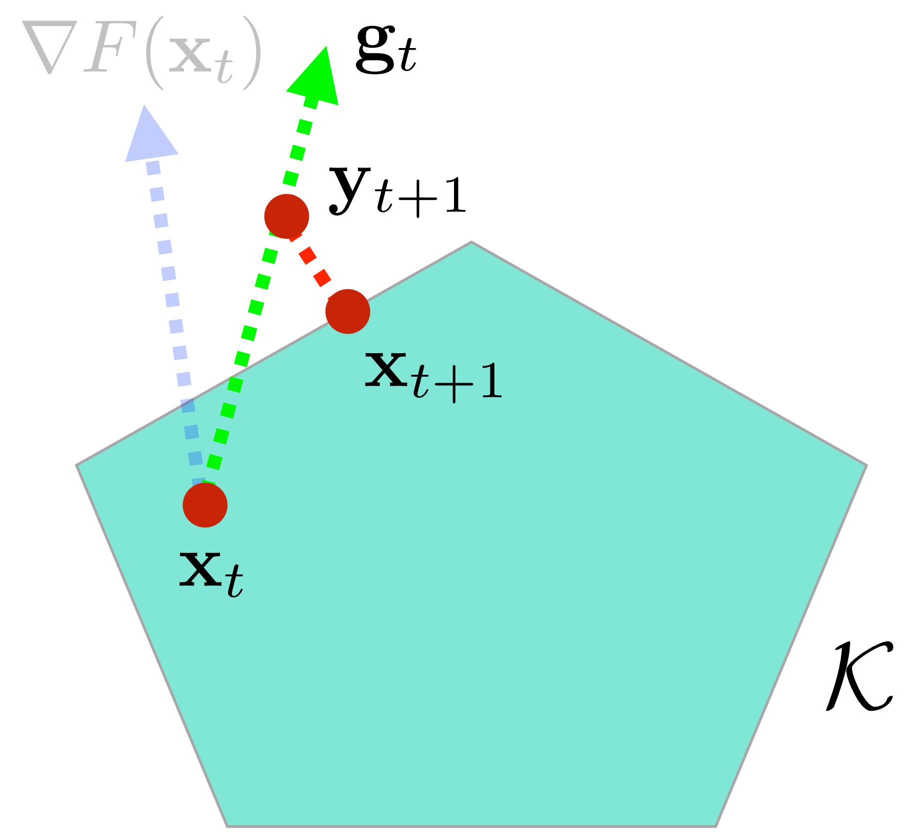

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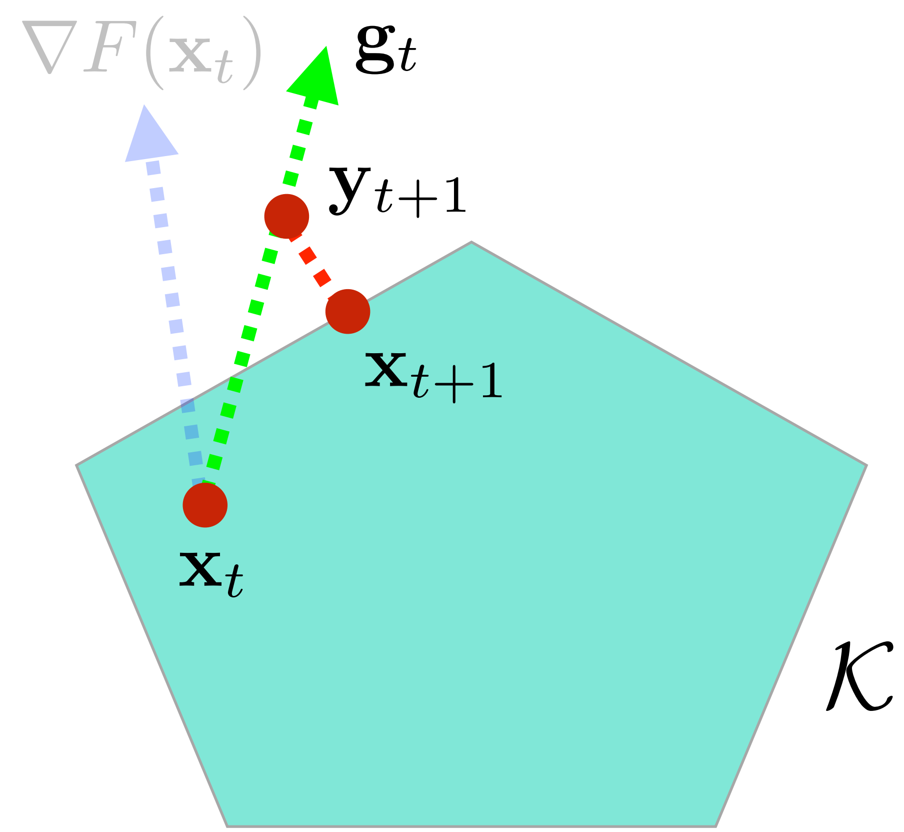

Do projected gradient methods lead to provably good solutions for continuous submodular maximization with general convex constraints?

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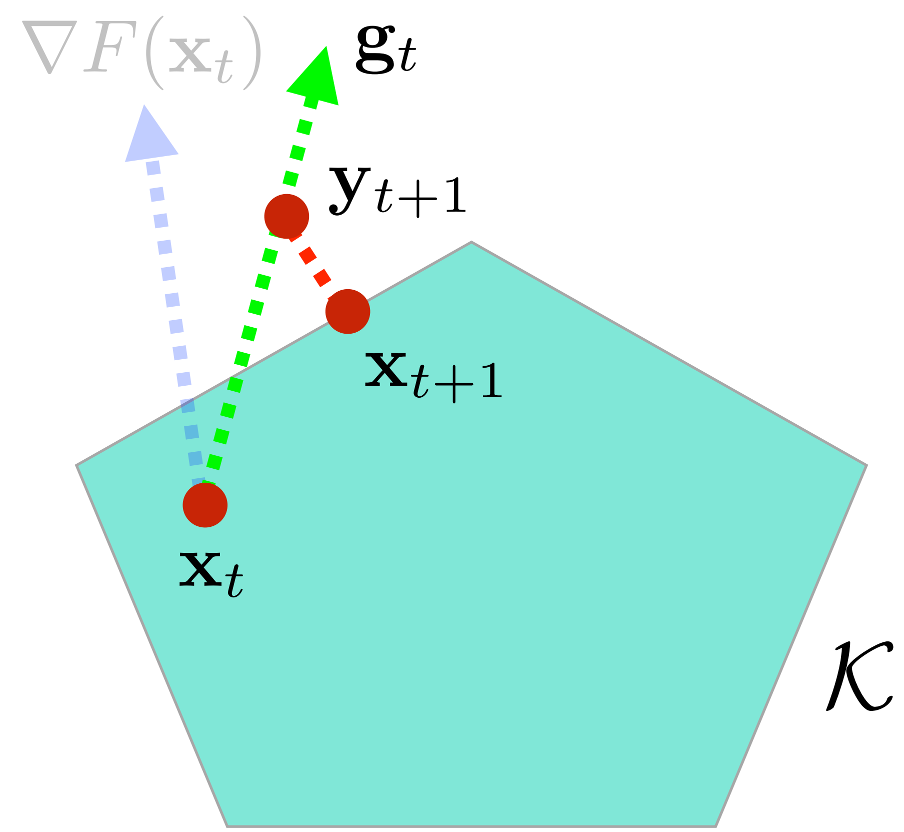

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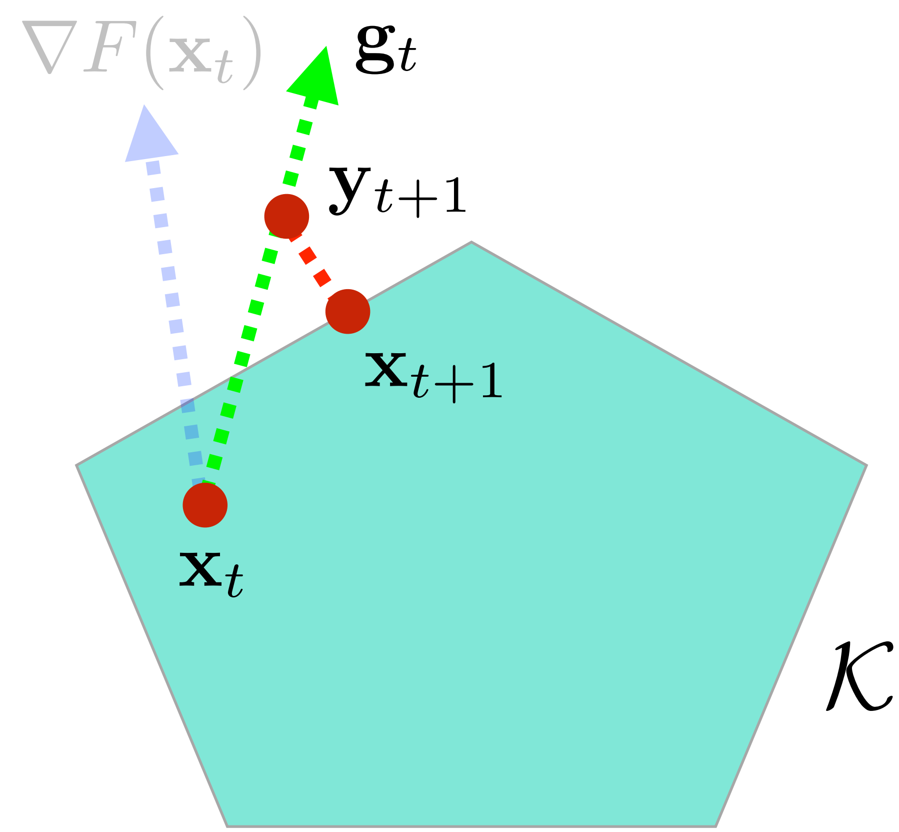

Do projected gradient methods lead to provably good solutions for continuous submodular maximization with general convex constraints?

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- Broader context of non-convex opt:
- Convergence to stationary points
- Structure matters

Stationary Points and Their Quality

- Let's first characterize the fixed point of gradient ascent

Stationary Points and Their Quality

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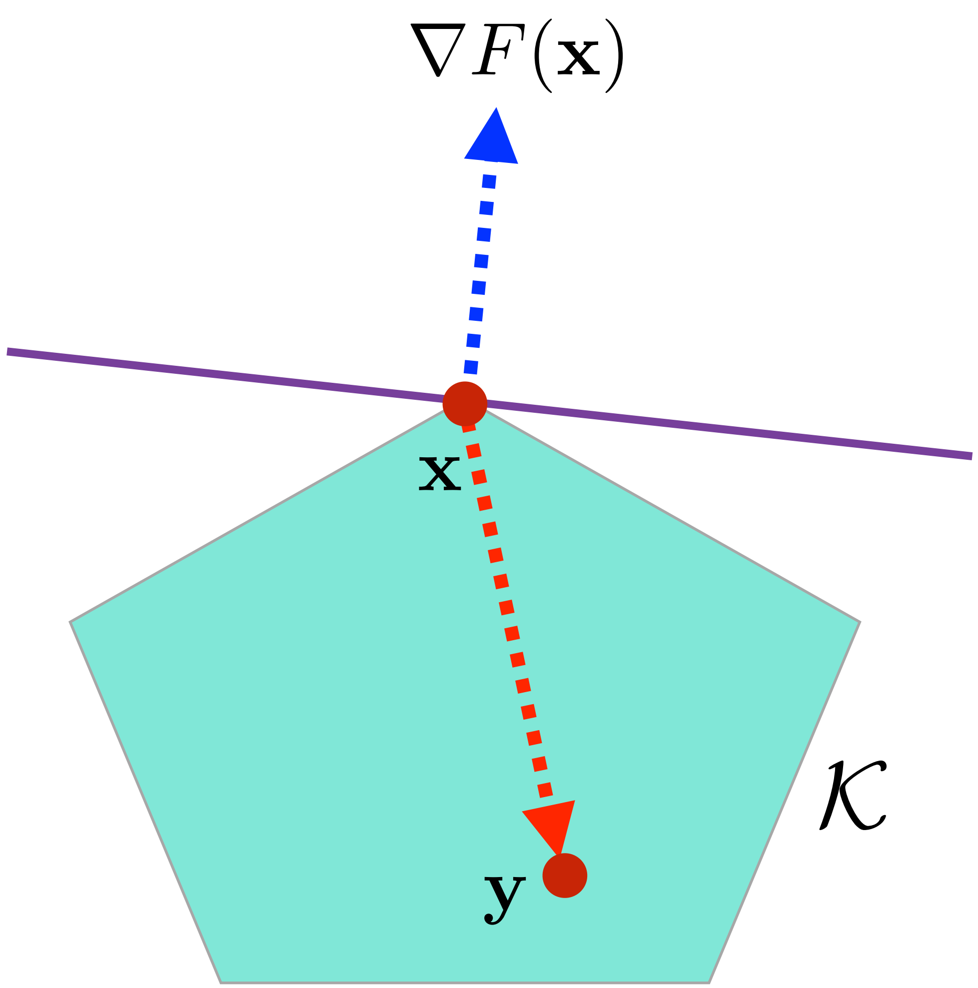
- A vector $\mathbf{x}$ is a **stationary point** of a function F over the set $\mathcal{K}$ if

$$\max_{\mathbf{y}\in\mathcal{K}} \langle \nabla F(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle \ \leq 0$$

Stationary Points and Their Quality

- Let's first characterize the fixed point of gradient ascent
- A vector $\mathbf{x}$ is a **stationary point** of a function F over the set $\mathcal{K}$ if

$$\max_{\mathbf{y}\in\mathcal{K}} \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \leq 0$$



Yale Penn

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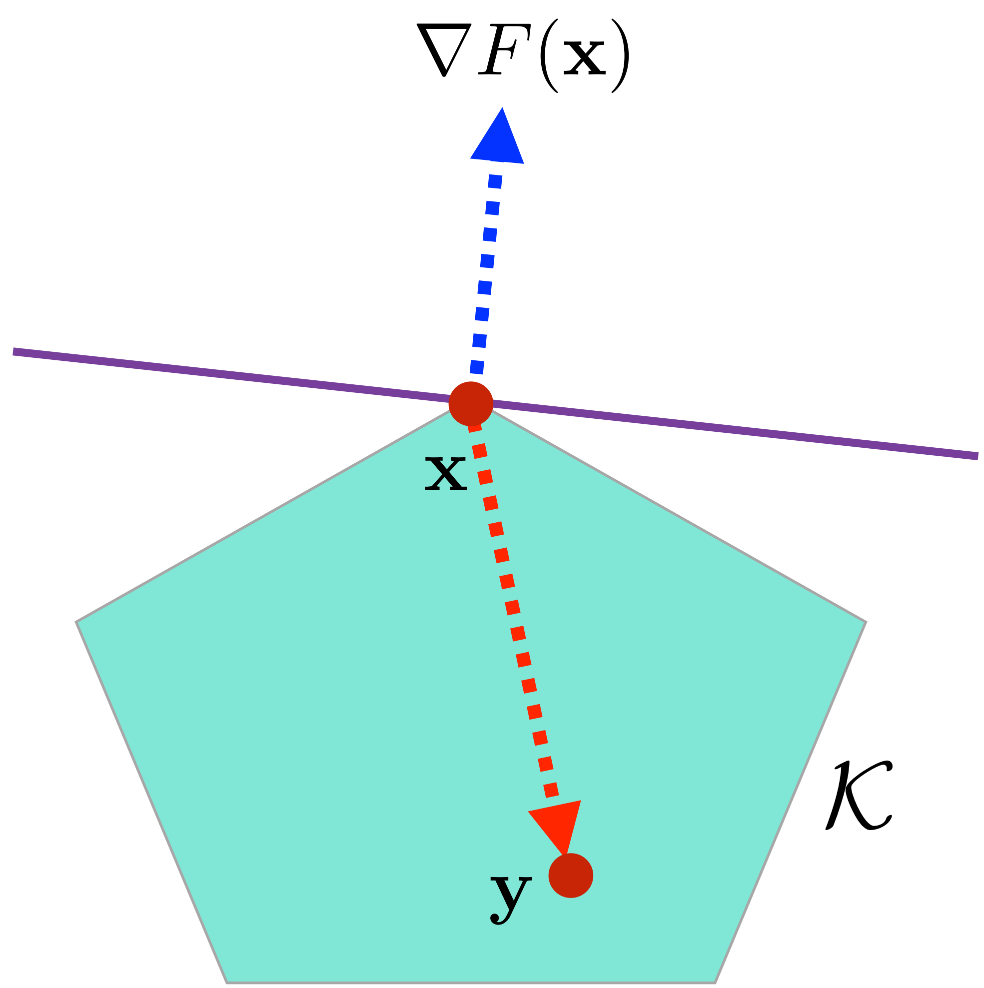

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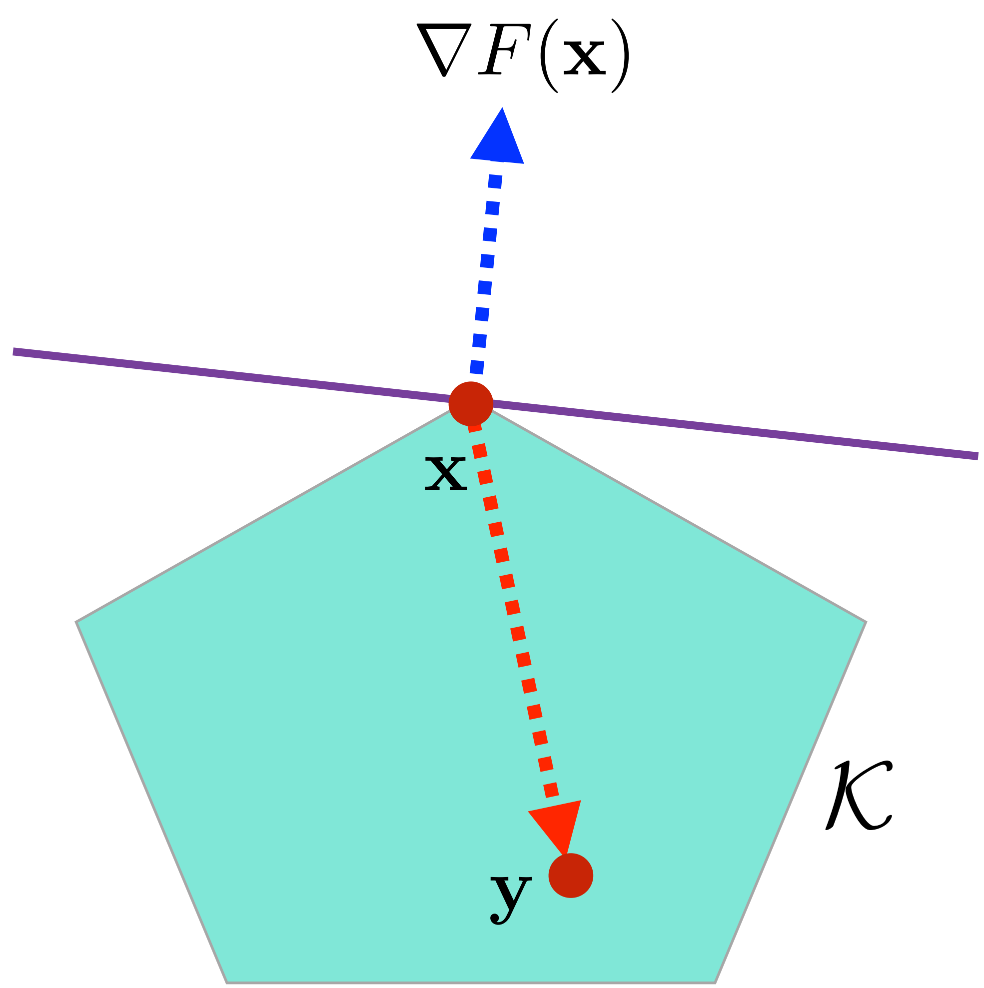

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$F(\mathbf{y})$

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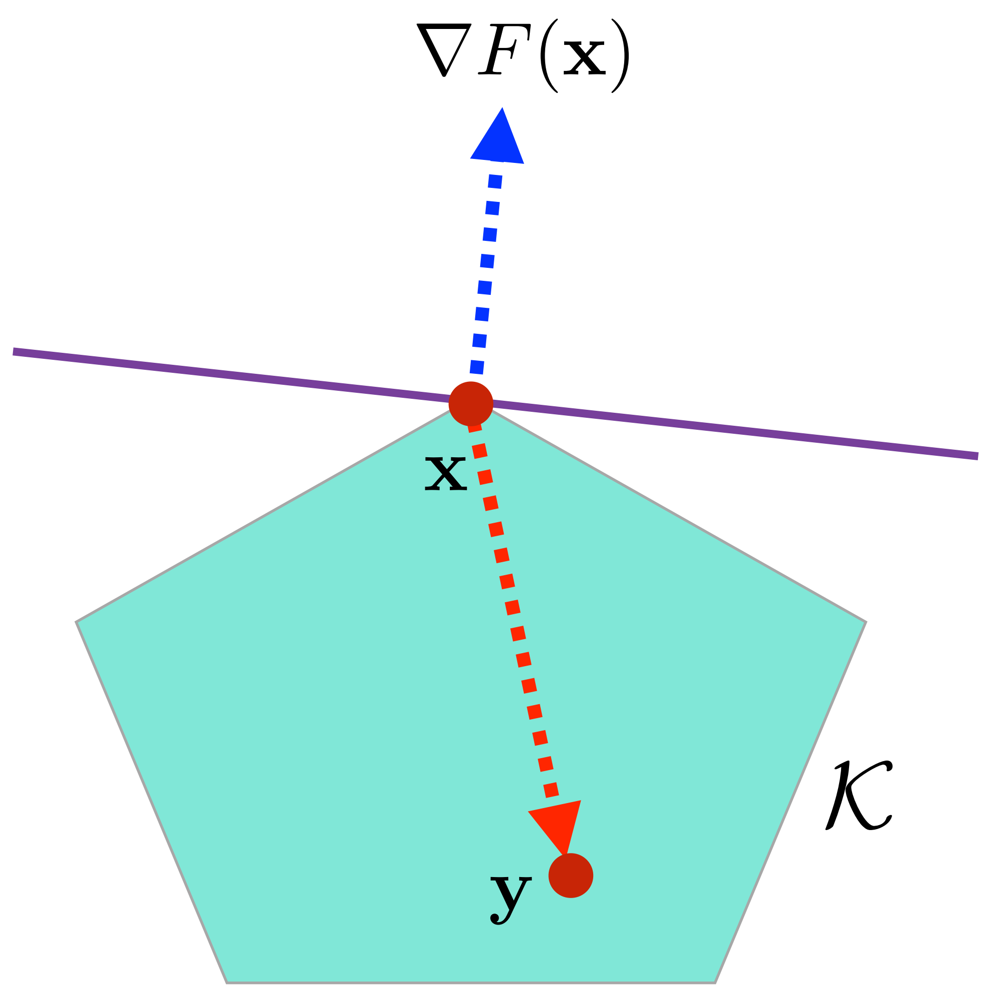

Intuition (first-order):

$$F(\mathbf{y}) = F(\mathbf{x} + (\mathbf{y} - \mathbf{x}))$$

$$\approx F(\mathbf{x}) + \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$$

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[Hassani, Soltanolkotabi, Karbasi]

If $\mathbf{x} \in \mathcal{K}$ is a stationary point of a monotone and DR-submodular function F, we have:

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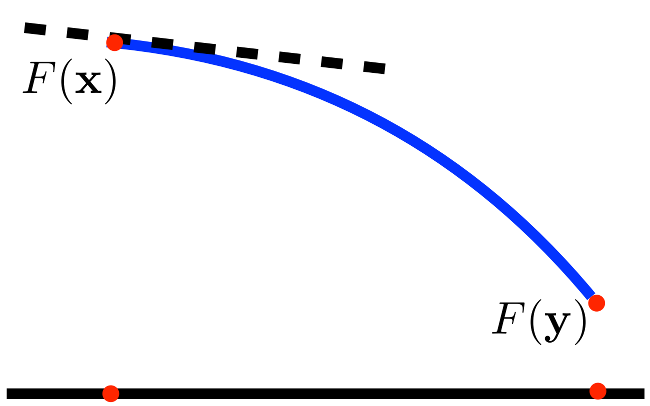

concave: $F(\mathbf{y}) - F(\mathbf{x}) \leq \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$

monotone DR-submodular: $F(\mathbf{y}) - 2F(\mathbf{x}) \leq \langle \nabla F(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$

Can We Reach Stationary Points?

[Hassani, Soltanolkotabi, Karbasi]

For the (stochastic) projected gradient ascent algorithm, we have:

$$\mathbb{E}[F(\mathbf{x}_T)] \geq \frac{1}{2}\mathrm{OPT} - (\frac{R^2 L}{2T} + \frac{R\sigma}{\sqrt{T}})$$

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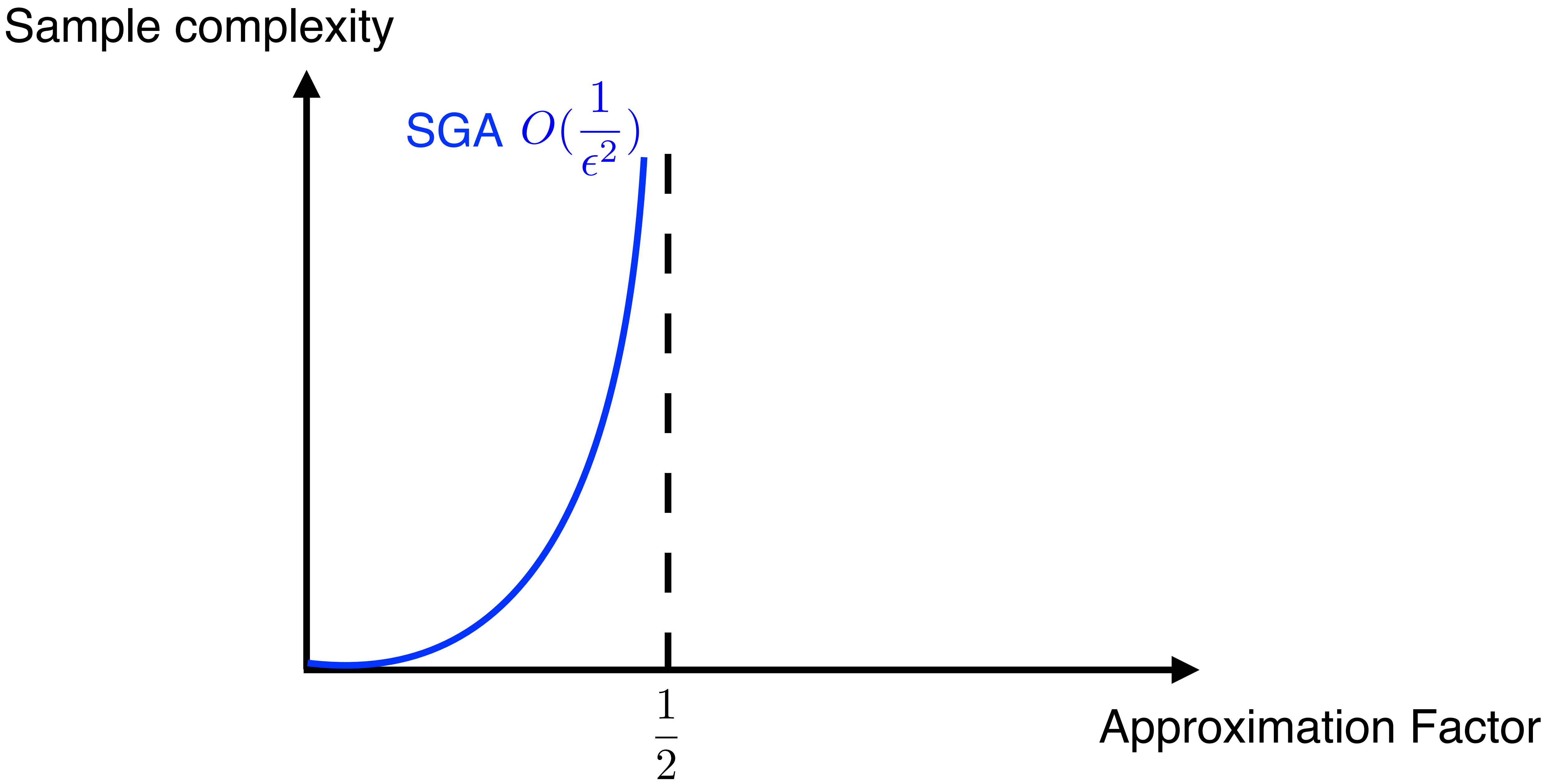
- Can stochastic gradient ascent do better than 1/2?
  - No. There exists a DR-Submodular function that attains OPT/2 at a stationary point which is also a *local maximum.*

Yale Penn

Maximizing DR-submodular Functions



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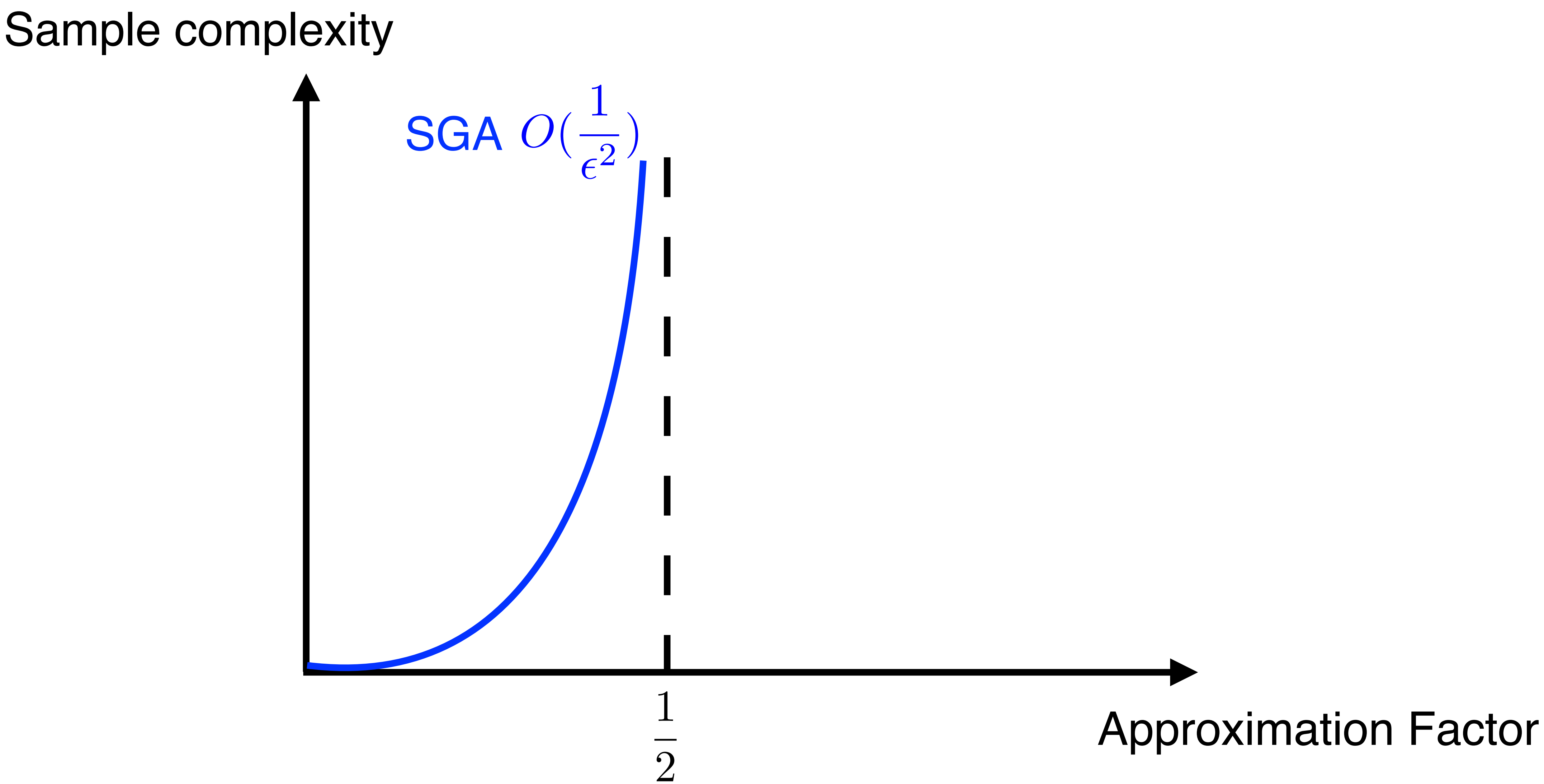

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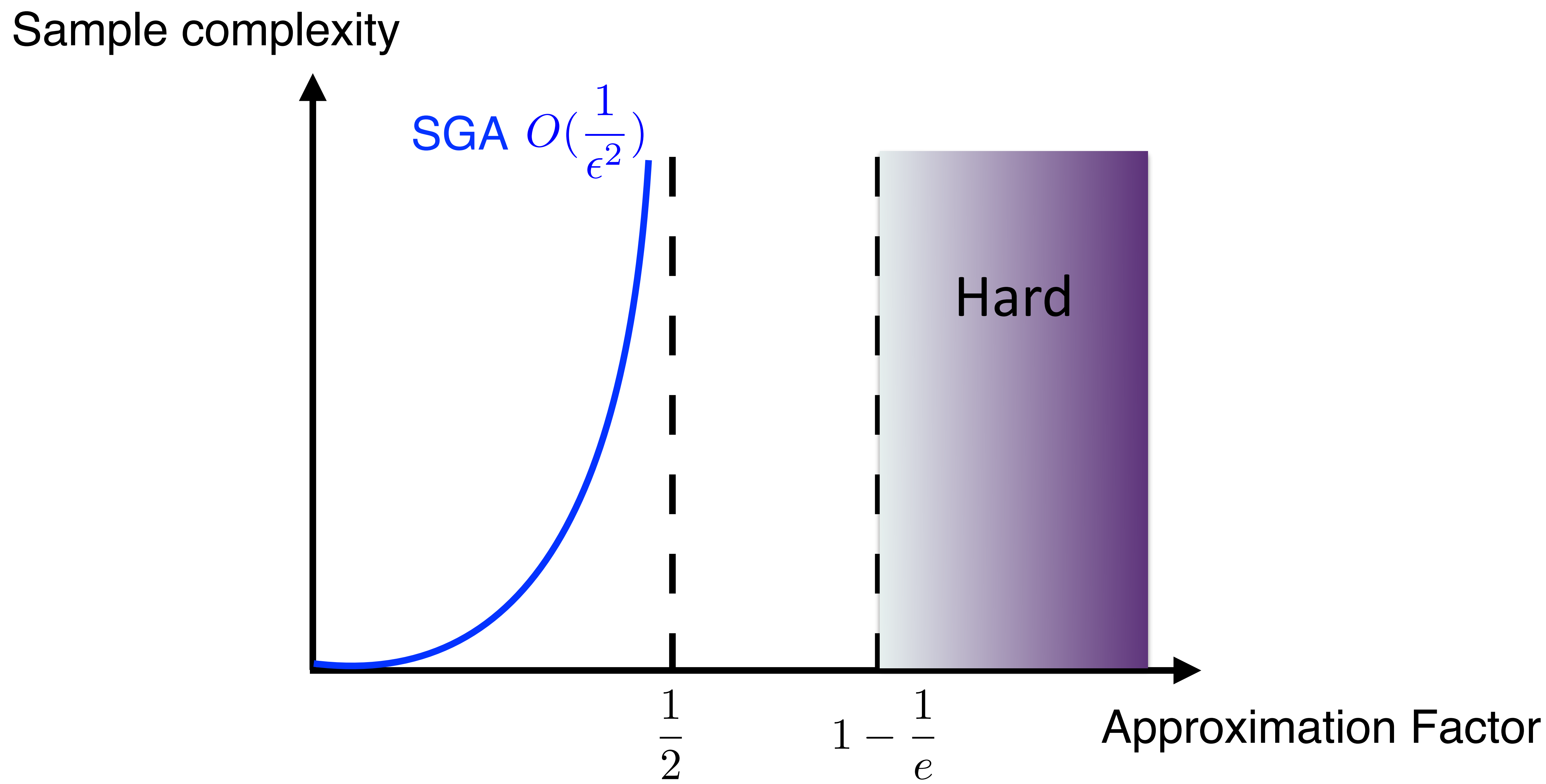

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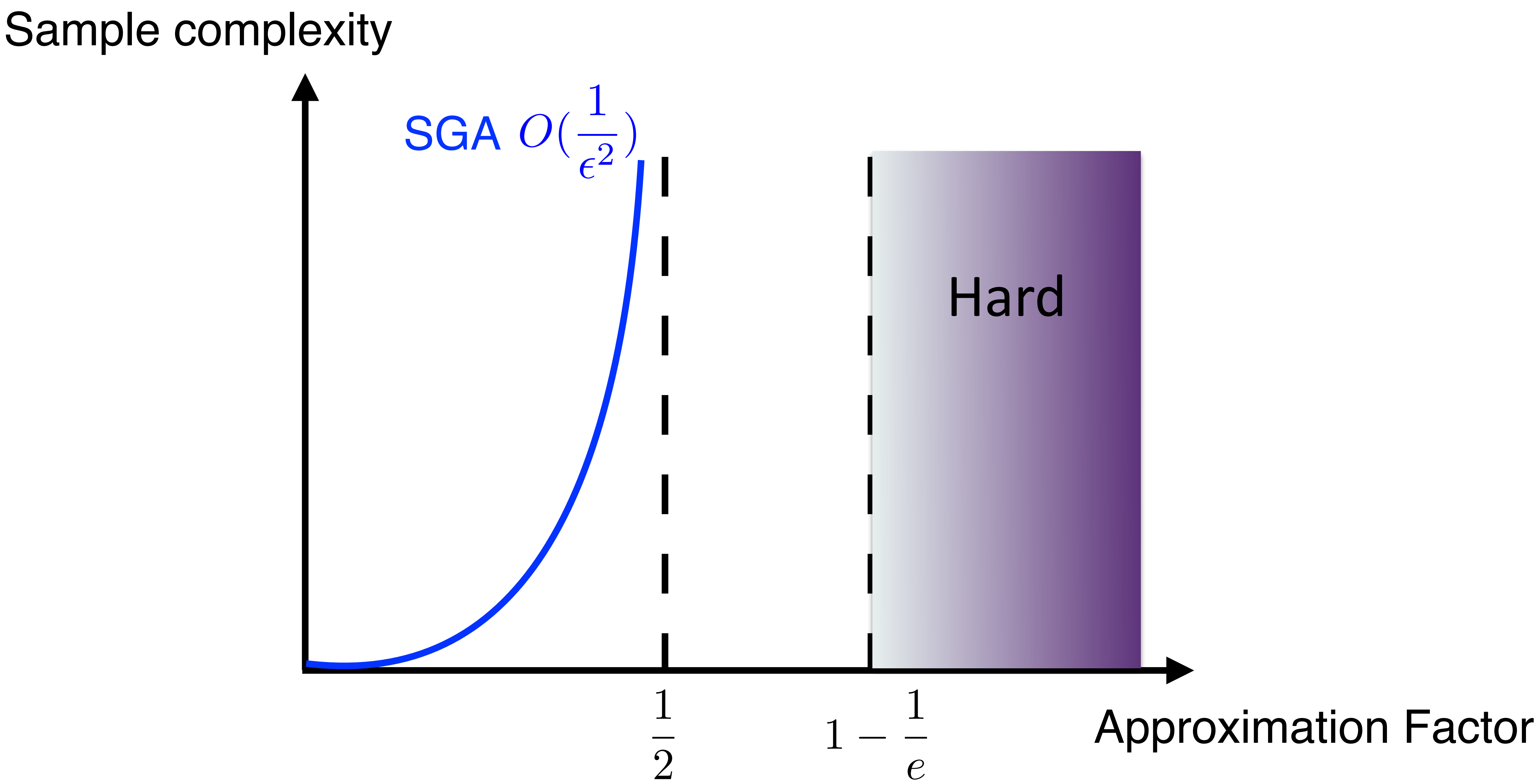

- Can we achieve better than 1/2? ▸ Yes

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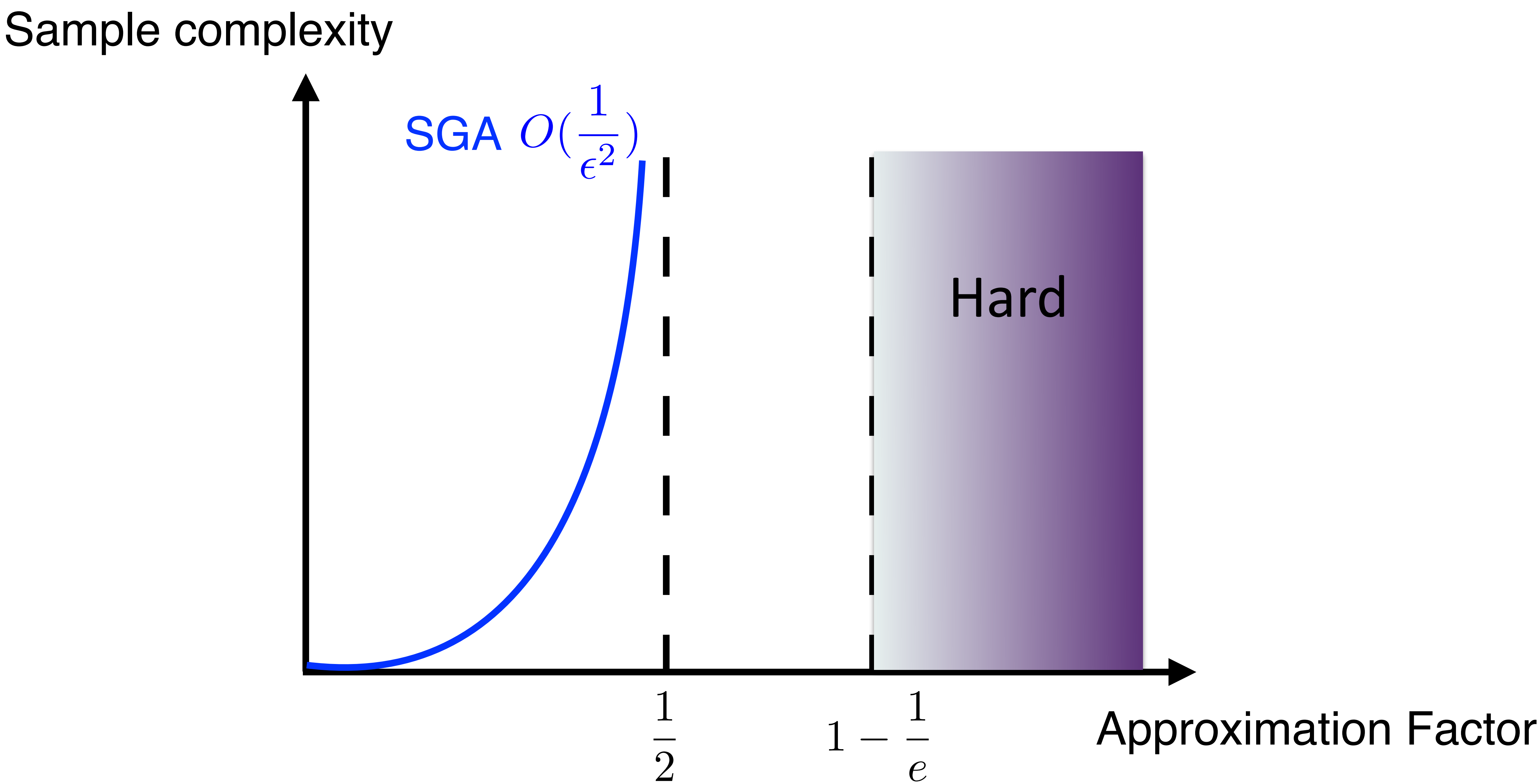

- Can we achieve better than 1/2? ▸ Yes
- How? ▸ Smart initialization + conditional gradient methods

Continuous Greedy

- Always initialize at $\mathbf{x}_0 = \mathbf{0}$
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$$\mathbf{v}_t = \arg\max_{\mathbf{v} \in \mathcal{K}} \langle \nabla F(\mathbf{x}_t), \mathbf{v} \rangle$$

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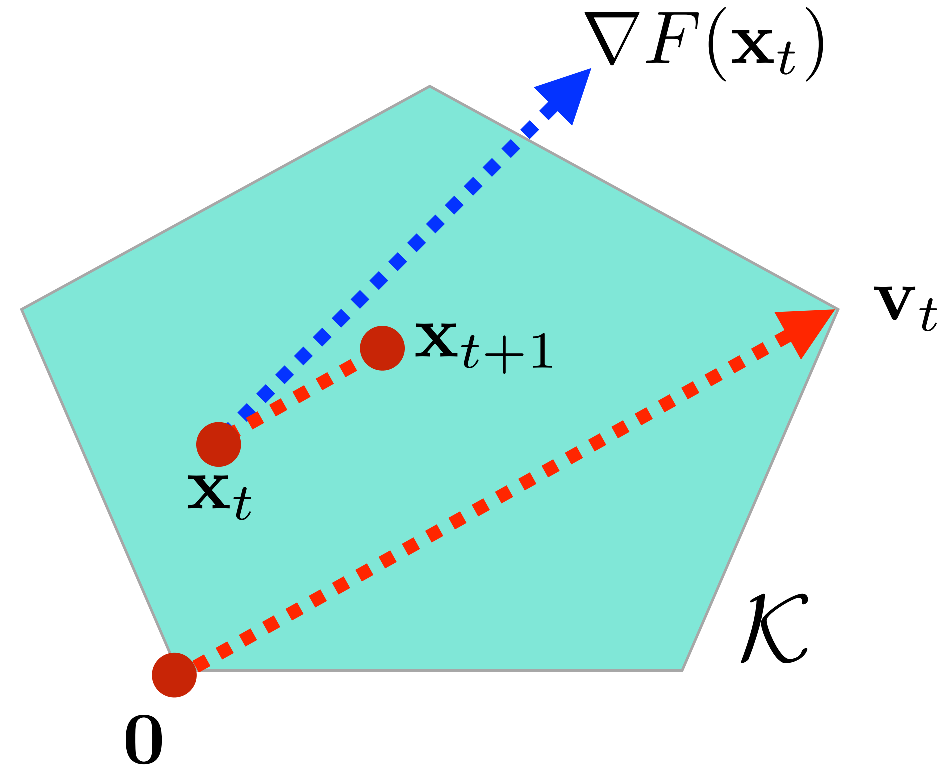

"Maximizing a monotone submodular function subject to a matroid constraint", Calinescu, Chekuri, Pal, Vondrák, 2011

"Guaranteed non-convex optimization: Submodular maximization over continuous domains", Bian, Mirzasoleiman, Buhmann, Krause, 2017

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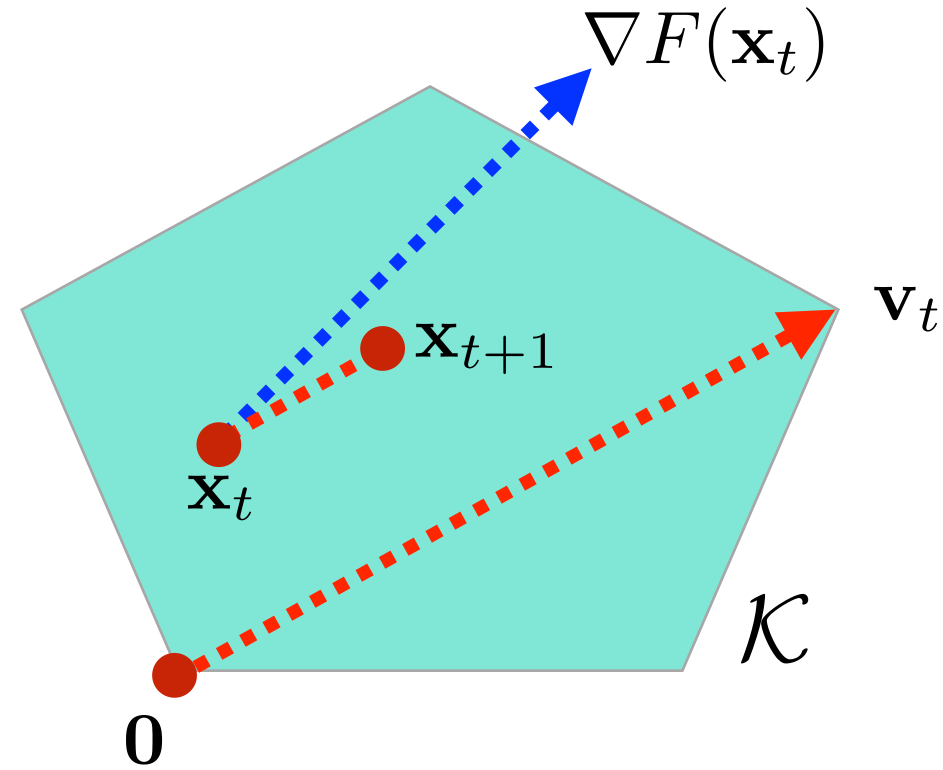

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For the continuous greedy algorithm, we have:

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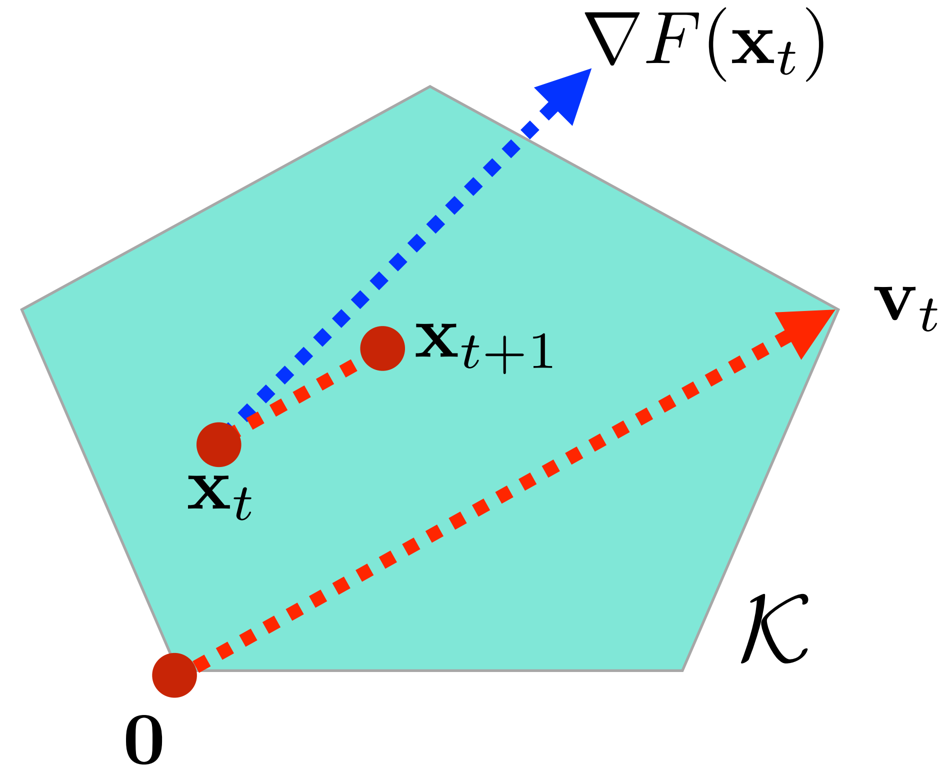

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- The continuous greedy is a variant of the Frank-Wolfe method

Yale Penn

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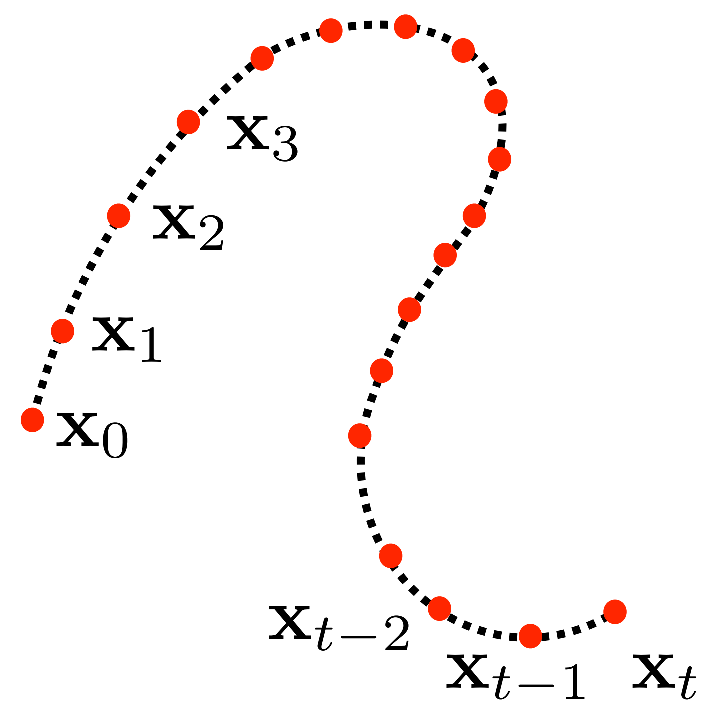

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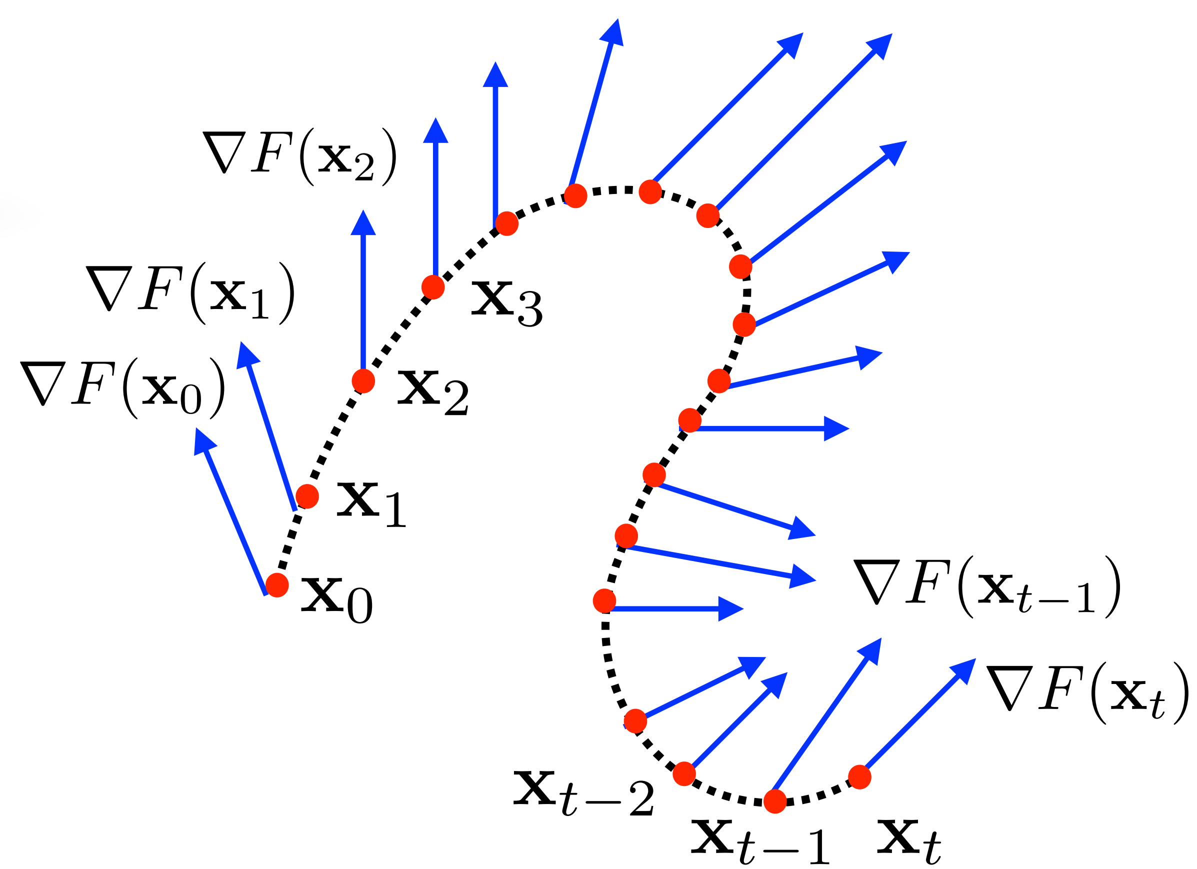

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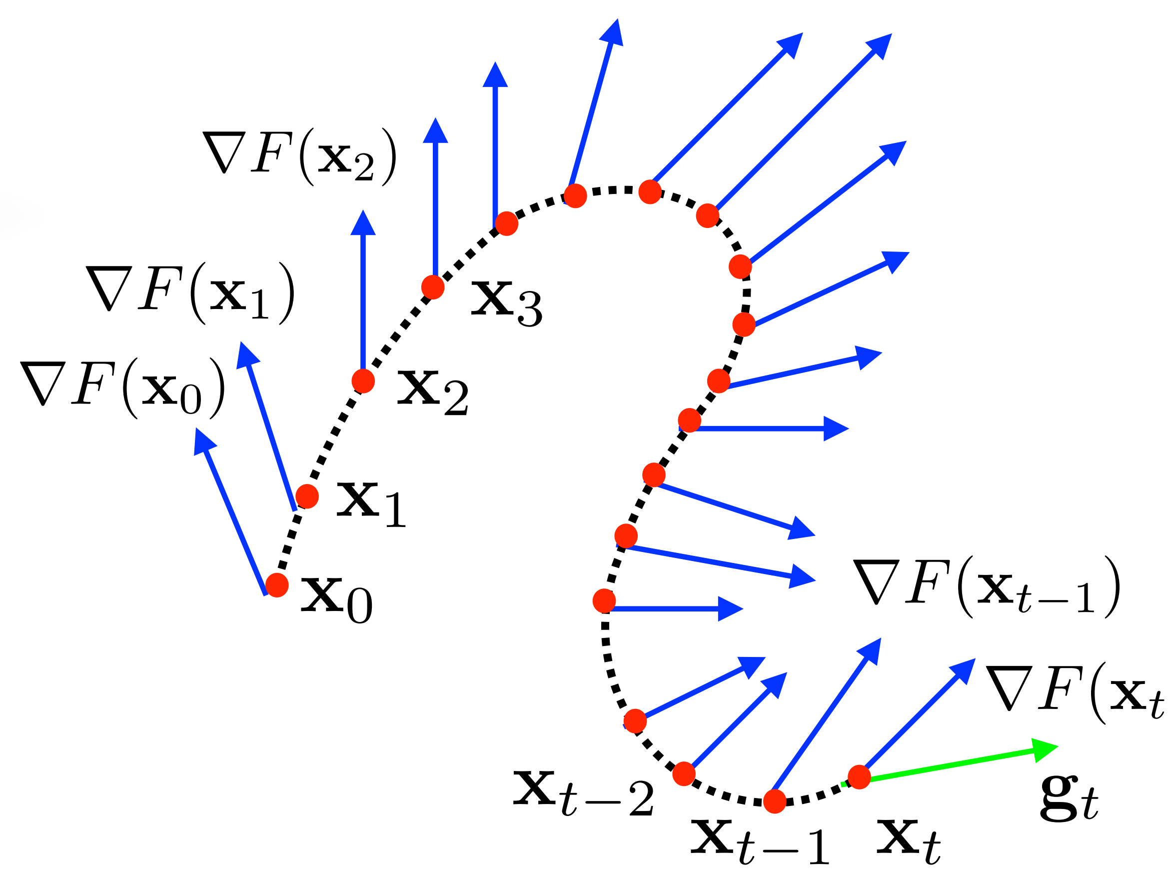

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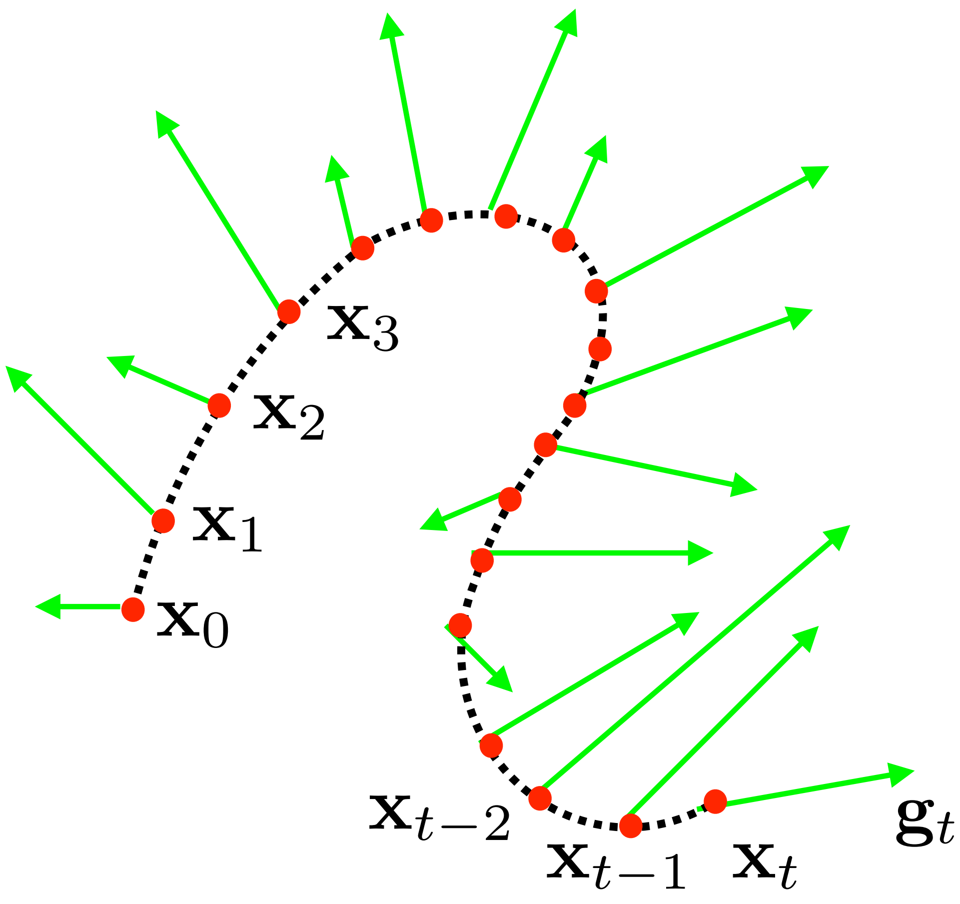

Yale Penn

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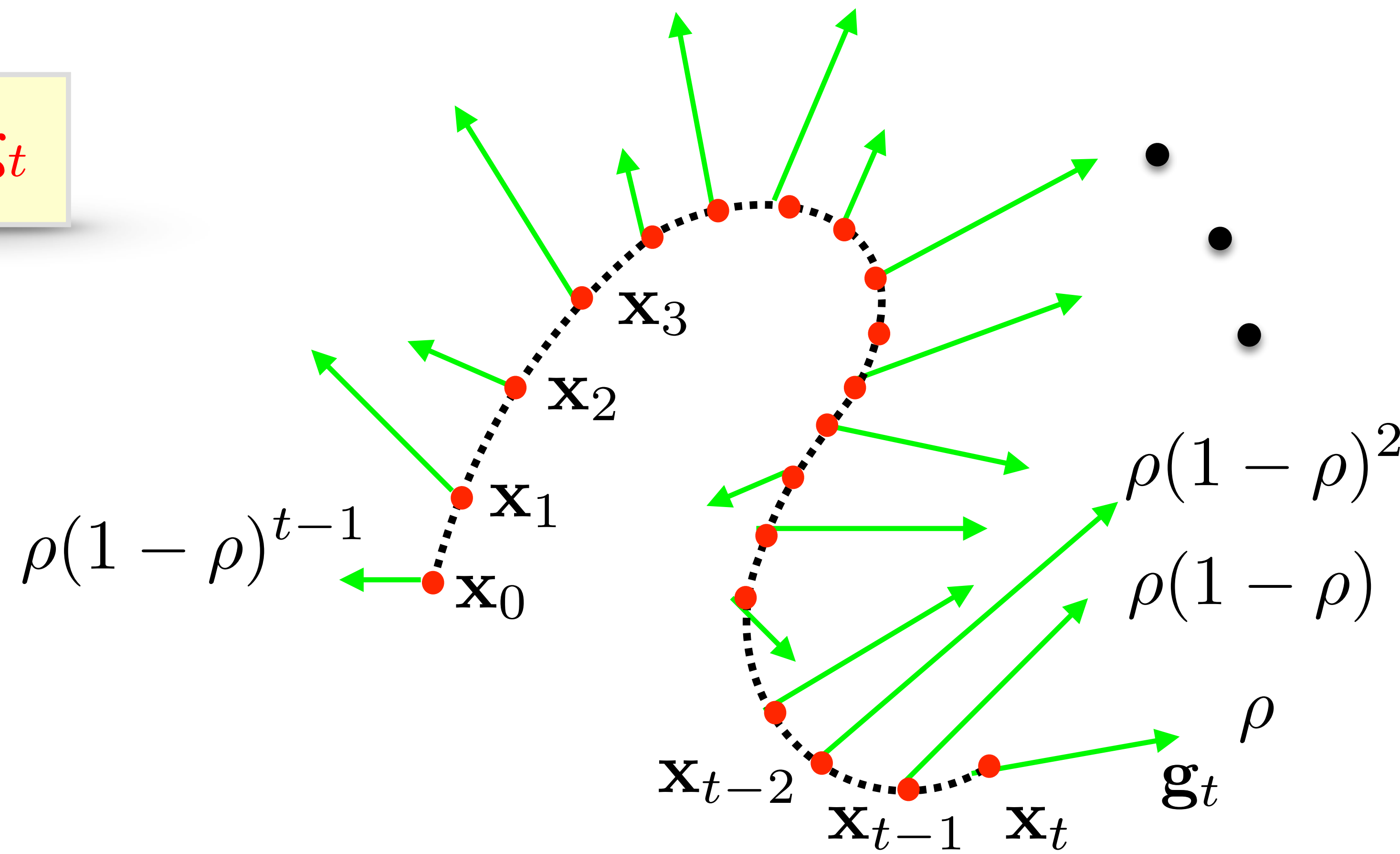

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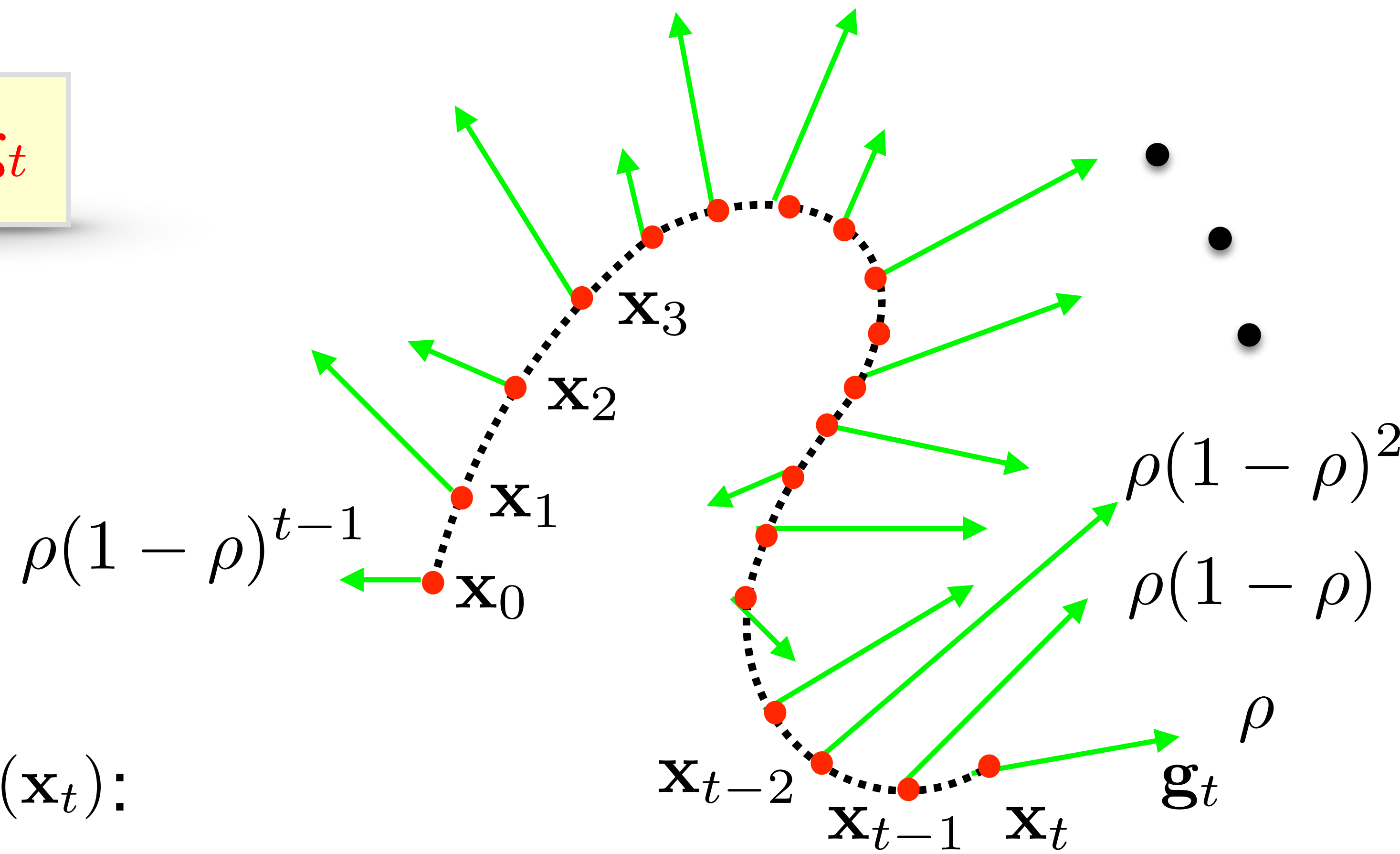

- $\mathbf{d}_t$ is a good approximation for $\nabla F(\mathbf{x}_t)$:
  - Computationally affordable $\Rightarrow$ single gradient evaluation
  - Reduces the noise of gradient approximation

Yale Penn

Stochastic Continuous Greedy

[Mokhtari, Hassani, Karbasi]

Assuming smoothness of F, by letting $\rho_t = \dfrac{4}{(t+8)^{2/3}}$, we have

$$\mathbb{E}[||\nabla F(\mathbf{x}_t) - \mathbf{d}_t||^2] \leq \frac{Q}{(t+9)^{2/3}}$$

where $Q := \max\left\{\, 5||\nabla F(\mathbf{x}_0) - \mathbf{d}_0||^2 \,,\, 16\sigma^2 + 2L^2R^2 \,\right\}$.

"Stochastic Conditional Gradient Methods: From Convex Minimization to Submodular Maximization", AISTATS '18, JMLR '20

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"Stochastic Conditional Gradient Methods: From Convex Minimization to Submodular Maximization", AISTATS '18, JMLR '20

$$\mathbf{v}_t = \text{argmax}_{\mathbf{v}\in\mathcal{K}}\{\langle \nabla F(\mathbf{x}_t), \mathbf{v}\rangle\} \quad \longrightarrow \quad \mathbf{v}_t = \text{argmax}_{\mathbf{v}\in\mathcal{K}}\{\langle \mathbf{d}_t, \mathbf{v}\rangle\}$$

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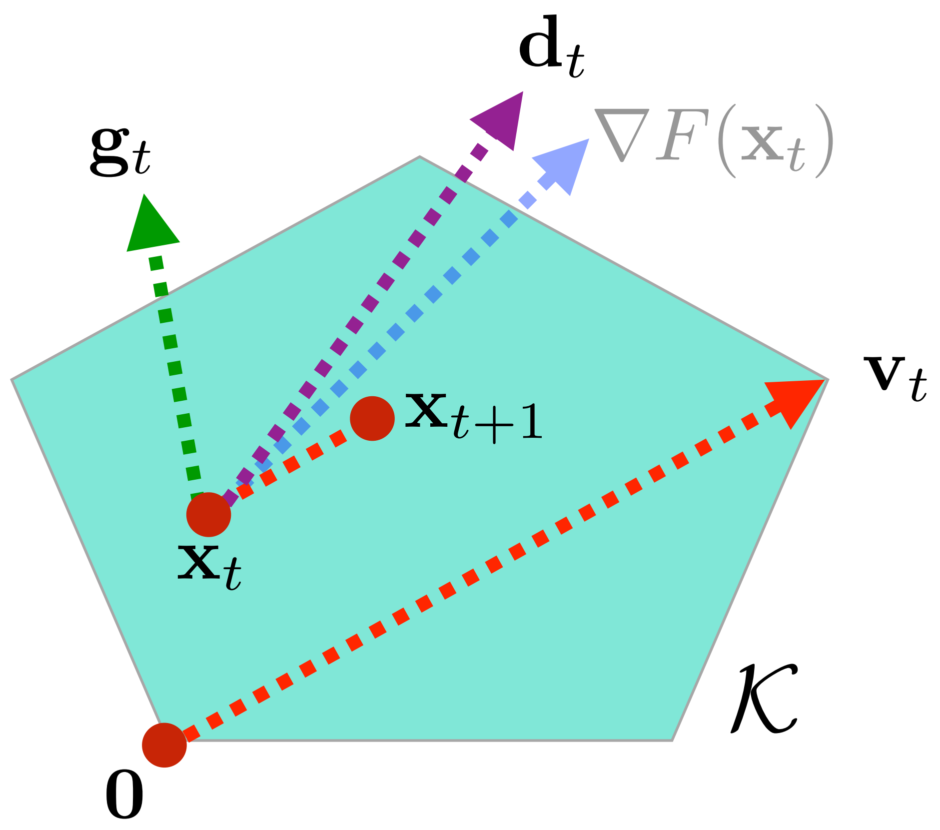

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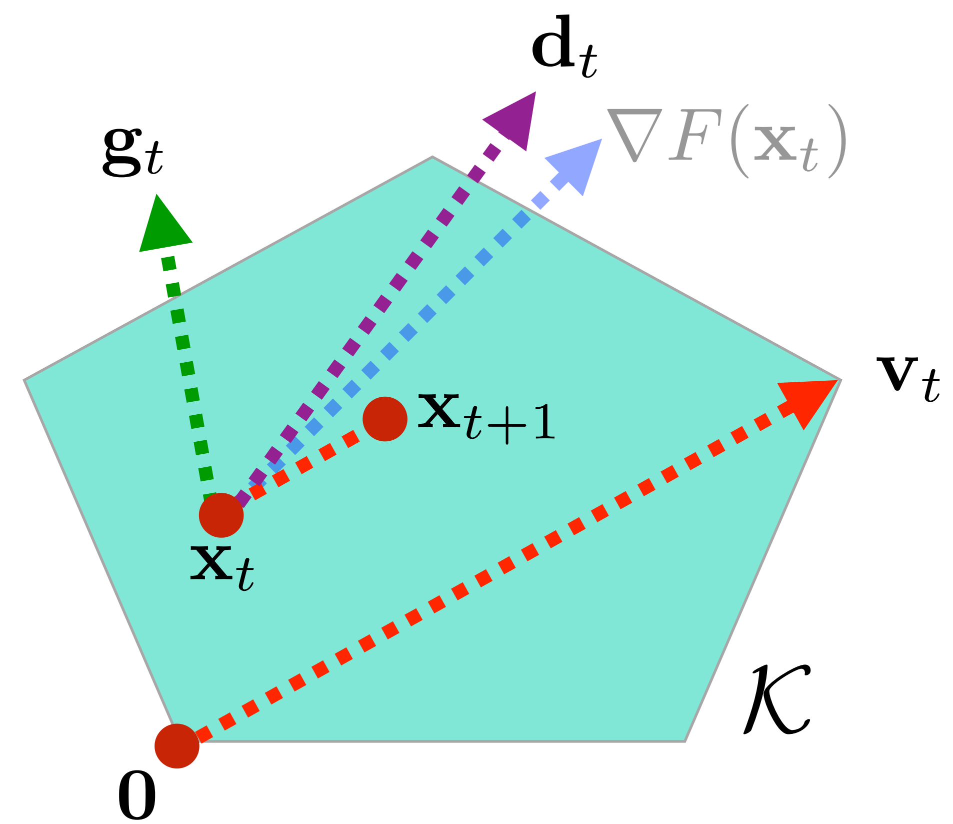

[Mokhtari, Hassani, Karbasi]

For the stochastic continuous greedy (SCG) algorithm, we have:

$$\mathbb{E}[F(\mathbf{x}_T)] \geq (1-\frac{1}{e})\mathrm{OPT} - \frac{15RQ^{\frac{1}{2}}}{T^{\frac{1}{3}}} - \frac{LR^2}{2T}$$

"Stochastic Conditional Gradient Methods: From Convex Minimization to Submodular Maximization", AISTATS '18, JMLR '20

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Penn

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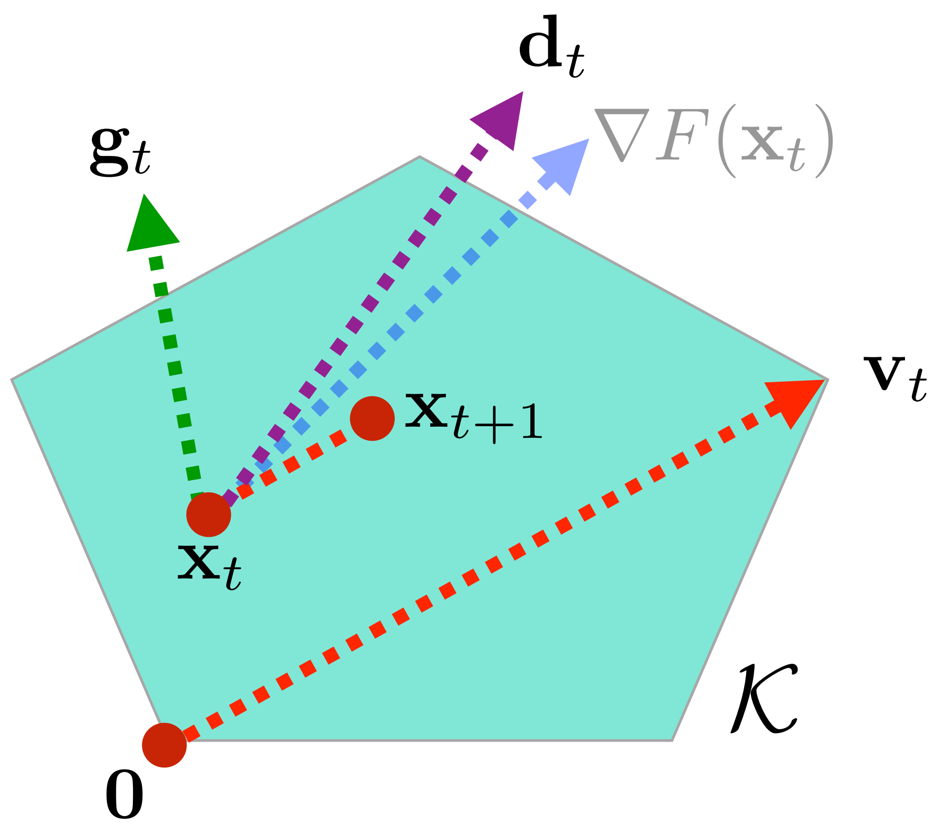

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"Stochastic Conditional Gradient Methods: From Convex Minimization to Submodular Maximization", AISTATS '18, JMLR '20

- To achieve $(1-\frac{1}{e})\mathrm{OPT} - \epsilon$, SCG requires $T = O(\frac{1}{\epsilon^3})$

Yale Penn

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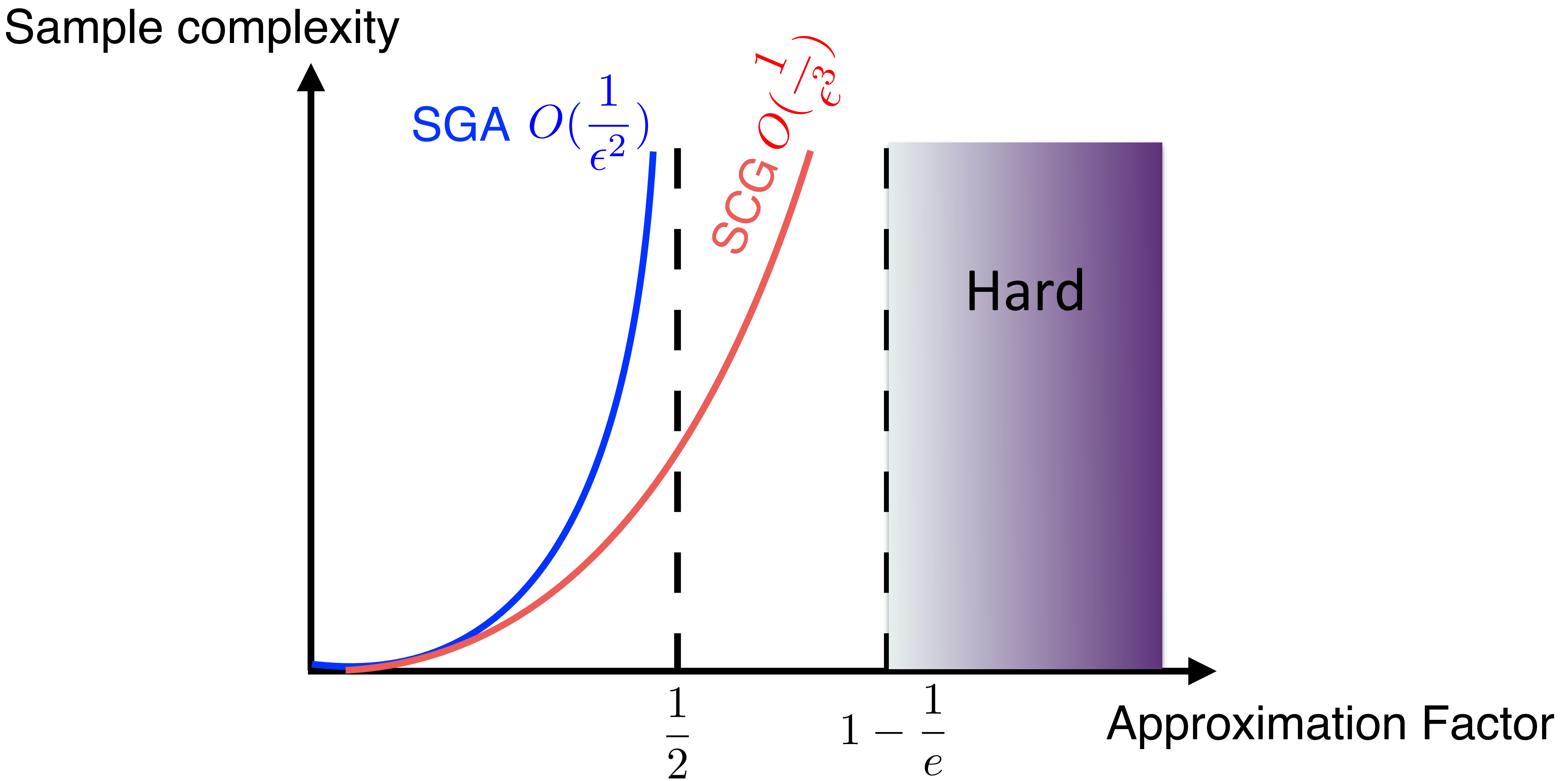

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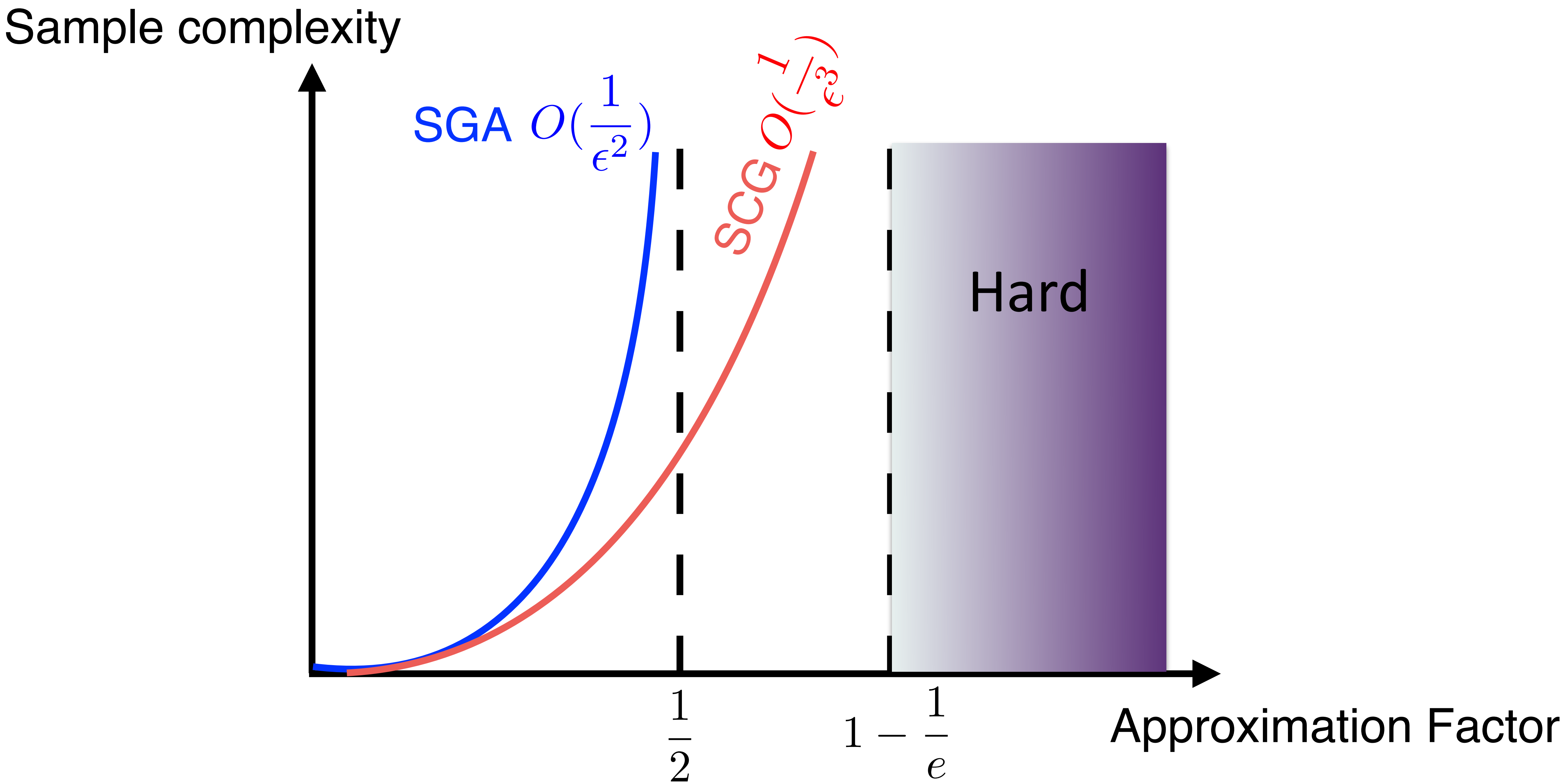
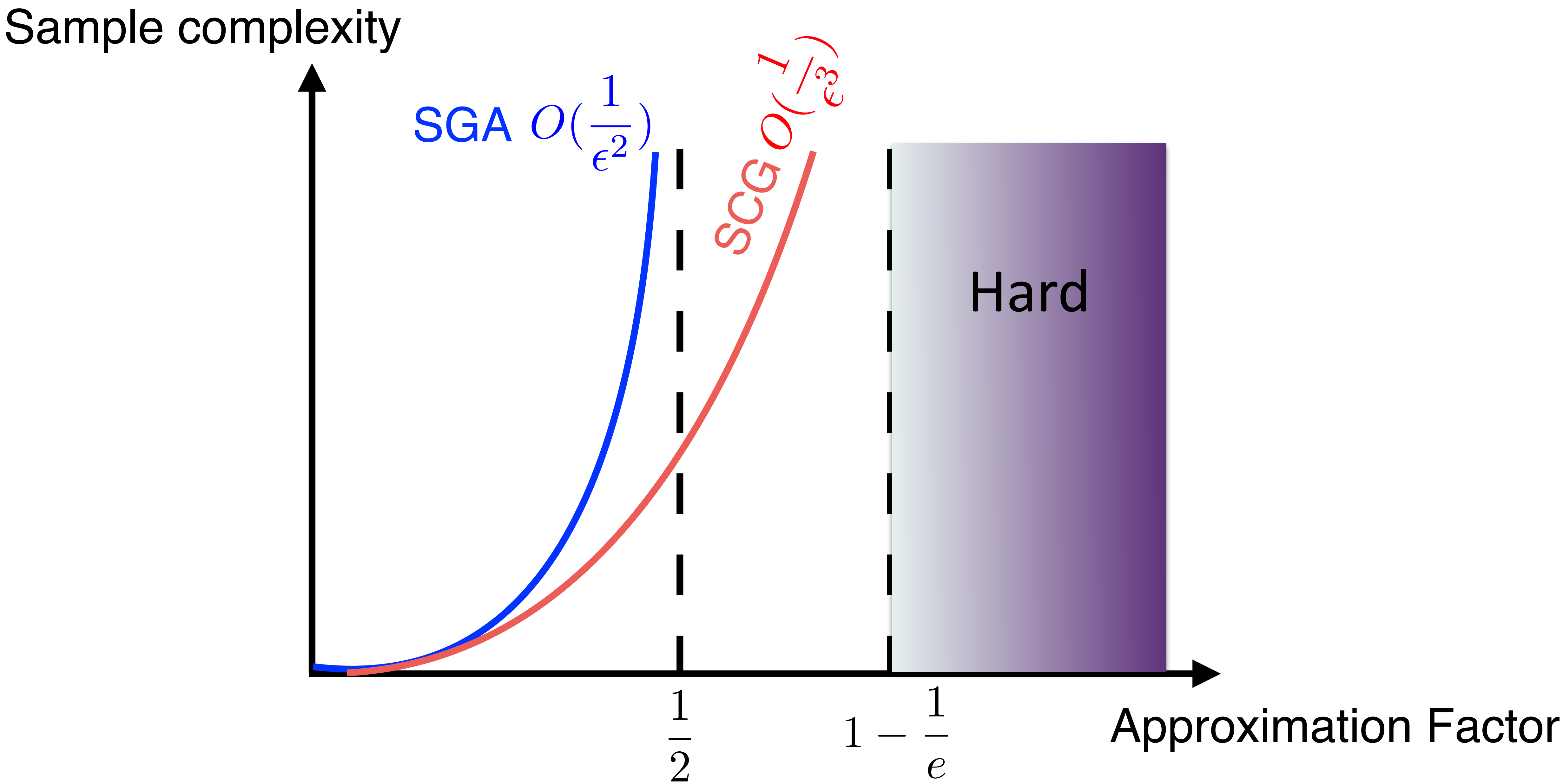


- Optimal sample complexity of stochastic first order methods:

convex: $O(\frac{1}{\epsilon^2})$ non-convex: $O(\frac{1}{\epsilon^3})$

Maximizing DR-submodular Functions



[Hassani, Karbasi, Mokhtari, Shen]

Stochastic continuous greedy++ (SCG++) achieves $(1 - \frac{1}{e})\mathrm{OPT} - \epsilon$ with optimal sample complexity $O(\frac{1}{\epsilon^2})$.

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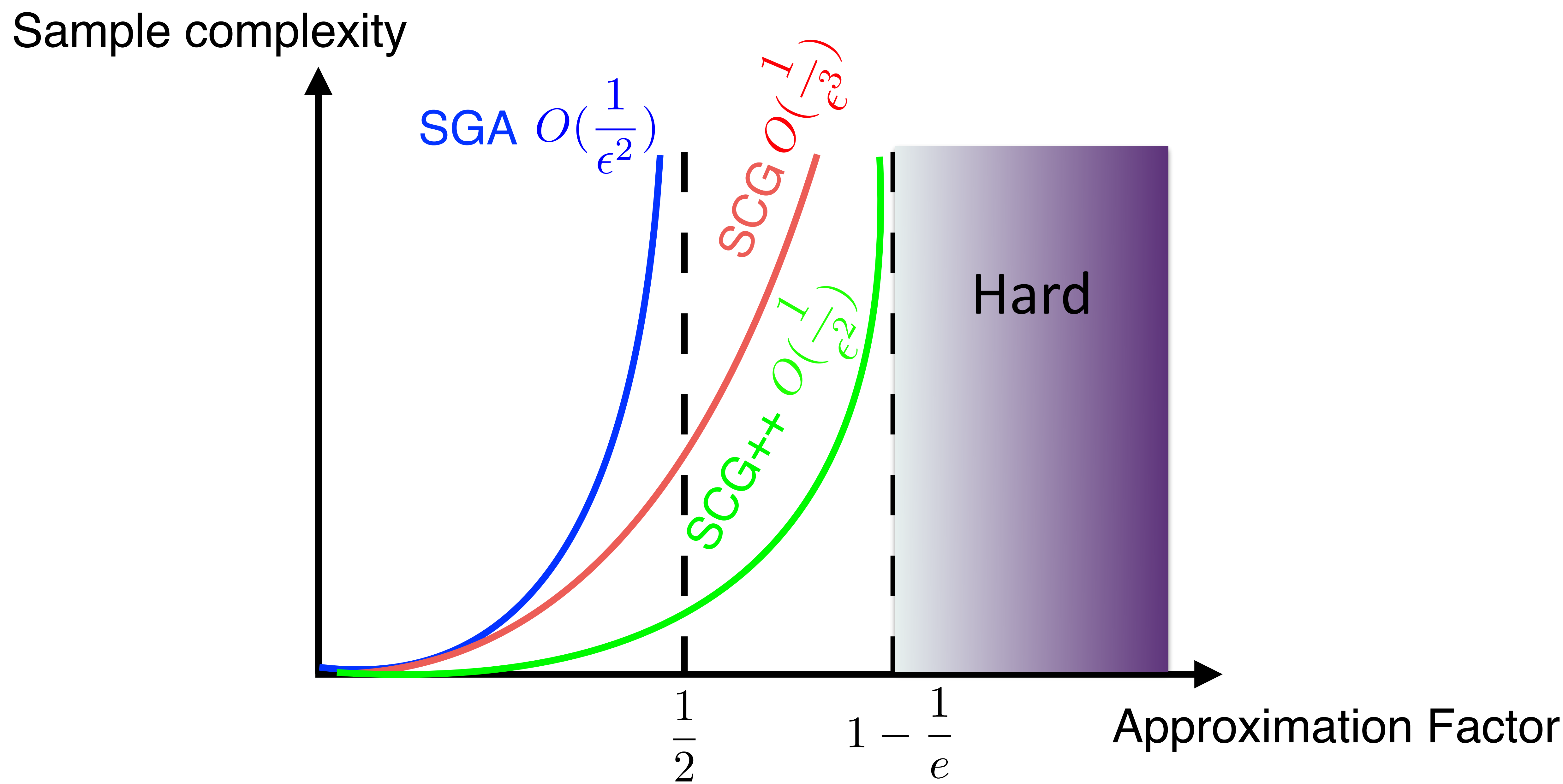

[Hassani, Karbasi, Mokhtari, Shen]

Stochastic continuous greedy++ (SCG++) achieves $(1-\frac{1}{e})\mathrm{OPT}-\epsilon$ with optimal sample complexity $O(\frac{1}{\epsilon^2})$.

Submodular Maximization

- We consider the following optimization problem: