

# CM1015 - Computational Mathematics

BSc Computer Science

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## Number Bases

### Decimal

Number bases relate to the amount of digits used to represent a number. The decimal system is base 10, meaning it uses 10 digits (0-9) to represent numbers. The value of a number is calculated by multiplying each digit by the base raised to the power of its position. For example, the number 123 in decimal is calculated as follows:

$10^2$	$10^1$	$10^0$
1	2	3
100	10	1

$$= 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$
$$= 123$$

### Binary

The binary system is base 2, meaning it uses 2 digits (0-1) to represent numbers. Binary numbers are calculated in the same way as decimal numbers, but using powers of 2 instead of 10. For example, the binary number 101 in decimal is calculated as follows:

$2^2$	$2^1$	$2^0$
1	0	1
4	2	1

$$= 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 5$$

### Hexadecimal

The hexadecimal system is base 16, meaning it uses 16 digits (0-9, A-F) to represent numbers. Hexadecimal numbers are calculated in the same way as decimal numbers, but using powers of 16 instead of 10. For example, the hexadecimal number 1A3 in decimal is calculated as follows:

$16^2$	$16^1$	$16^0$
1	10	3
256	16	1

$$= 1 \times 16^2 + 10 \times 16^1 + 3 \times 16^0$$
$$= 419$$

## Generic Base Conversion to Decimal

To convert any base to decimal, the following method can be used:

1. Write the number in the base you are converting from.
2. Multiply each digit by the base raised to the power of its position.
3. Add the results together to get the decimal value.

Formula:  $d_n \times b^n + d_{n-1} \times b^{n-1} + \dots + d_1 \times b^1 + d_0 \times b^0$

Where:  $d_n$  is the digit at position  $n$ ,  $b$  is the base, and  $n$  is the position of the digit.

Example, shown with above symbols:

$b^2$	$b^1$	$b^0$
$d_2$	$d_1$	$d_0$

$$= d_2 \times b^2 + d_1 \times b^1 + d_0 \times b^0$$

Example with base 5 number 234, here  $b = 5$ ,  $d_2 = 2$ ,  $d_1 = 3$ ,  $d_0 = 4$ :

$5^2$	$5^1$	$5^0$
2	3	4

$$= 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0$$
$$= 69$$

$25$

$5$

$1$

## Non-integer Conversion to Decimal

To convert a non-integer number to decimal, the following method can be used:

1. Write the number in the base you are converting from.
2. Multiply each digit by the base raised to the power of its position.
3. Add the results together to get the integer part of the decimal value.
4. Repeat the process for the fractional part of the number.

Formula:  $d_n \times b^n + d_{n-1} \times b^{n-1} + \dots + d_1 \times b^1 + d_0 \times b^0 + d_{-1} \times b^{-1} + d_{-2} \times b^{-2} + \dots$

Where:  $d_n$  is the digit at position  $n$ ,  $b$  is the base, and  $n$  is the position of the digit.

Example, shown with above symbols:

$b^2$	$b^1$	$b^0$	.	$b^{-1}$	$b^{-2}$
$d_2$	$d_1$	$d_0$		$d_{-1}$	$d_{-2}$

$$= d_2 \times b^2 + d_1 \times b^1 + d_0 \times b^0$$

$$+ d_{-1} \times b^{-1} + d_{-2} \times b^{-2}$$

Example with base 5 number 234.24, here  $b = 5$ ,  $d_2 = 2$ ,  $d_1 = 3$ ,  $d_0 = 4$ ,  $d_{-1} = 2$ ,  $d_{-2} = 4$ :

$5^2$	$5^1$	$5^0$	.	$5^{-1}$	$5^{-2}$
<b>2</b>	<b>3</b>	<b>4</b>		<b>2</b>	<b>4</b>
<b>25</b>	<b>5</b>	<b>1</b>		<b>0.2</b>	<b>0.04</b>

$$= 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0$$

$$+ 2 \times 5^{-1} + 4 \times 5^{-2} = 69.56$$

## Binary Operations

### Addition

Binary addition is similar to decimal addition, but with the base being 2.

Example:  $101_2 + 11_2$

<p><b>Step 1:</b></p> $\begin{array}{r} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ 1 \phantom{0} \phantom{1} \phantom{0} \\ + \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ \hline \phantom{1} \phantom{0} \phantom{1} \phantom{0} \end{array}$ <p><math>1 + 1 = 10</math>, write 0 into the result row and carry 1 to the next left column</p>	<p><b>Step 2:</b></p> $\begin{array}{r} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ 1 \phantom{0} \phantom{1} \phantom{0} \\ + \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ \hline \phantom{1} \phantom{0} \phantom{1} \phantom{0} \end{array}$ <p><math>1 + 0 + 1 = 10</math>, write 0 to the results row and carry 1 to the next left column</p>	<p><b>Step 3:</b></p> $\begin{array}{r} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ 1 \phantom{0} \phantom{1} \phantom{0} \\ + \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ \hline \phantom{1} \phantom{0} \phantom{1} \phantom{0} \end{array}$ <p><math>1 + 1 = 10</math>, write 1 0 in the result row. Answer: <math>1000_2</math></p>
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### Subtraction

Binary subtraction is similar to decimal subtraction, but with the base being 2.

Example:  $101_2 - 11_2$

<p><b>Step 1:</b></p> $\begin{array}{r} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ 1 \phantom{0} \phantom{1} \phantom{0} \\ - \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ \hline \phantom{1} \phantom{0} \phantom{1} \phantom{0} \end{array}$ <p><math>1 - 1 = 0</math>, write 0 into the result row</p>	<p><b>Step 2:</b></p> $\begin{array}{r} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ 1 \phantom{0} \phantom{1} \phantom{0} \\ - \phantom{1} \phantom{0} \phantom{1} \phantom{0} \\ \hline \phantom{1} \phantom{0} \phantom{1} \phantom{0} \end{array}$ <p>If the subtraction cannot be done, borrow 2 from the next left column, <math>2 - 1 = 1</math>, write 1 in the result row Answer: <math>10_2</math></p>
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## Self-assessment Questions from Foundation Maths Book

### Worked Example 14.2

Convert  $83_{10}$  to binary.

#### Solution:

The general method for converting a decimal number to binary is to repeatedly divide the number by 2 and record the remainder. The binary number is then read from the remainders in reverse order.

The steps for converting 83 to binary are as follows:

1.  $83 \div 2 = 41$  remainder 1
2.  $41 \div 2 = 20$  remainder 1
3.  $20 \div 2 = 10$  remainder 0
4.  $10 \div 2 = 5$  remainder 0
5.  $5 \div 2 = 2$  remainder 1
6.  $2 \div 2 = 1$  remainder 0
7.  $1 \div 2 = 0$  remainder 1
8. The binary number is read from the remainders in reverse order: 1010011

$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
1	0	1	0	0	1	1

  

64	32	16	8	4	2	1
$1 \times 64$	+	$1 \times 16$		+	$1 \times 2$	$+ 1 \times 1$

  
$$= 83$$

### Exercise 14.2:

1. Convert the following decimal numbers to binary:

(a): 19 (b): 36 (c): 100 (d): 796 (e): 5000

#### Solutions:

(a):  $19_{10}$

1.  $19 \div 2 = 9$  remainder 1
2.  $9 \div 2 = 4$  remainder 1
3.  $4 \div 2 = 2$  remainder 0
4.  $2 \div 2 = 1$  remainder 0
5.  $1 \div 2 = 0$  remainder 1
6. The binary number is read from the remainders in reverse order: 10011
7. **Answer:**  $19_{10} = 10011_2$

(b):  $36_{10}$

1.  $36 \div 2 = 18$  remainder 0
2.  $18 \div 2 = 9$  remainder 0
3.  $9 \div 2 = 4$  remainder 1
4.  $4 \div 2 = 2$  remainder 0
5.  $2 \div 2 = 1$  remainder 0
6.  $1 \div 2 = 0$  remainder 1
7. The binary number is read from the remainders in reverse order: 100100
8. **Answer:**  $36_{10} = 100100_2$

(c):  $100_{10}$

1.  $100 \div 2 = 50$  remainder 0
2.  $50 \div 2 = 25$  remainder 0
3.  $25 \div 2 = 12$  remainder 1
4.  $12 \div 2 = 6$  remainder 0
5.  $6 \div 2 = 3$  remainder 0
6.  $3 \div 2 = 1$  remainder 1
7.  $1 \div 2 = 0$  remainder 1
8. The binary number is read from the remainders in reverse order: 1100100
9. **Answer:**  $100_{10} = 1100100_2$

(d):  $796_{10}$

1.  $796 \div 2 = 398$  remainder 0
2.  $398 \div 2 = 199$  remainder 0
3.  $199 \div 2 = 99$  remainder 1
4.  $99 \div 2 = 49$  remainder 1
5.  $49 \div 2 = 24$  remainder 1
6.  $24 \div 2 = 12$  remainder 0
7.  $12 \div 2 = 6$  remainder 0
8.  $6 \div 2 = 3$  remainder 0
9.  $3 \div 2 = 1$  remainder 1
10.  $1 \div 2 = 0$  remainder 1
11. The binary number is read from the remainders in reverse order: 1100011100
12. **Answer:**  $796_{10} = 1100011100_2$

(e):  $5000_{10}$

1.  $5000 \div 2 = 2500$  remainder 0
2.  $2500 \div 2 = 1250$  remainder 0
3.  $1250 \div 2 = 625$  remainder 0
4.  $625 \div 2 = 312$  remainder 1
5.  $312 \div 2 = 156$  remainder 0
6.  $156 \div 2 = 78$  remainder 0
7.  $78 \div 2 = 39$  remainder 0
8.  $39 \div 2 = 19$  remainder 1
9.  $19 \div 2 = 9$  remainder 1
10.  $9 \div 2 = 4$  remainder 1
11.  $4 \div 2 = 2$  remainder 0
12.  $2 \div 2 = 1$  remainder 0
13.  $1 \div 2 = 0$  remainder 1
14. The binary number is read from the remainders in reverse order: 1001110001000
15. **Answer:**  $5000_{10} = 1001110001000_2$



2. Convert the following binary numbers to decimal:

(a): 111 (b): 10101 (c): 111001 (d): 1110001 (e): 11111111

**Solutions:**

(a):  $111_2$

1.  $1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

2.  $4 + 2 + 1 = 7$

3. **Answer:**  $111_2 = 7_{10}$

(b):  $10101_2$

1.  $1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

2.  $16 + 0 + 4 + 0 + 1 = 21$

3. **Answer:**  $10101_2 = 21_{10}$

(c):  $111001_2$

1.  $1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

2.  $32 + 16 + 8 + 0 + 0 + 1 = 57$

3. **Answer:**  $111001_2 = 57_{10}$

(d):  $1110001_2$

1.  $1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

2.  $64 + 32 + 16 + 0 + 0 + 0 + 1 = 113$

3. **Answer:**  $1110001_2 = 113_{10}$

(e):  $11111111_2$

1.  $1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

2.  $128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 255$

3. **Answer:**  $11111111_2 = 255_{10}$

3. What is the highest decimal number that can be written in binary form using a maximum of (a) two binary digits, (b) three binary digits, (c) four binary digits, (d) five binary digits? Can you spot a pattern? (e) Write a formula for the highest decimal number that can be written in binary form using  $N$  binary digits.

**Solutions:**

(a): Two binary digits

The highest decimal number that can be written in binary form using two binary digits is  $11_2$ , which is equal to  $3_{10}$ .

(b): Three binary digits

The highest decimal number that can be written in binary form using three binary digits is  $111_2$ , which is equal to  $7_{10}$ .

(c): Four binary digits

The highest decimal number that can be written in binary form using four binary digits is  $1111_2$ , which is equal to  $15_{10}$ .

(d): Five binary digits

The highest decimal number that can be written in binary form using five binary digits is  $11111_2$ , which is equal to  $31_{10}$ .

(e): Formula for highest decimal number using  $N$  binary digits

The pattern observed is that the highest decimal number that can be written in binary form using  $N$  binary digits is  $2^N - 1$ .

4. Write the decimal number 0.5 in binary.

**Solution:**

The decimal number 0.5 can be written in binary as  $0.1_2$ .

$2^{-1}$	$2^{-2}$	$2^{-3}$	
1	0	0	$= 1 \times 2^{-1}$ $= 0.5$
0.5	0.25	0.125	

**Exercise 14.3:**

1. Convert the following decimal numbers to octal numbers:

(a) 971 (b) 2841 (c) 5014 (d) 10000 (e) 17926

**Solutions:**

(a):  $971_{10}$

1.  $971 \div 8 = 121$  remainder 3
2.  $121 \div 8 = 15$  remainder 1
3.  $15 \div 8 = 1$  remainder 7
4.  $1 \div 8 = 0$  remainder 1
5. The octal number is read from the remainders in reverse order: 1733
6. **Answer:**  $971_{10} = 1733_8$

(b):  $2841_{10}$

1.  $2841 \div 8 = 355$  remainder 1
2.  $355 \div 8 = 44$  remainder 3
3.  $44 \div 8 = 5$  remainder 4
4.  $5 \div 8 = 0$  remainder 5
5. The octal number is read from the remainders in reverse order: 5431
6. **Answer:**  $2841_{10} = 5431_8$

(c):  $5014_{10}$

1.  $5014 \div 8 = 626$  remainder 6
2.  $626 \div 8 = 78$  remainder 2
3.  $78 \div 8 = 9$  remainder 6
4.  $9 \div 8 = 1$  remainder 1
5.  $1 \div 8 = 0$  remainder 1
6. The octal number is read from the remainders in reverse order: 16126
7. **Answer:**  $5014_{10} = 16126_8$

(d):  $10000_{10}$

1.  $10000 \div 8 = 1250$  remainder 0
2.  $1250 \div 8 = 156$  remainder 2
3.  $156 \div 8 = 19$  remainder 4
4.  $19 \div 8 = 2$  remainder 3
5.  $2 \div 8 = 0$  remainder 2
6. The octal number is read from the remainders in reverse order: 23420
7. **Answer:**  $10000_{10} = 23420_8$

(e):  $17926_{10}$

1.  $17926 \div 8 = 2240$  remainder 6
2.  $2240 \div 8 = 280$  remainder 0
3.  $280 \div 8 = 35$  remainder 0
4.  $35 \div 8 = 4$  remainder 3
5.  $4 \div 8 = 0$  remainder 4
6. The octal number is read from the remainders in reverse order: 43006
7. **Answer:**  $17926_{10} = 43006_8$

2. Convert the following octal numbers to decimal:

(a) 73 (b) 1237 (c) 7635 (d) 6677 (e) 67765

**Solutions:**

(a):  $73_8$

1.  $7 \times 8^1 + 3 \times 8^0$
2.  $56 + 3 = 59$
3. **Answer:**  $73_8 = 59_{10}$

(b):  $1237_8$

1.  $1 \times 8^3 + 2 \times 8^2 + 3 \times 8^1 + 7 \times 8^0$
2.  $512 + 128 + 24 + 7 = 671$
3. **Answer:**  $1237_8 = 671_{10}$

(c):  $7635_8$

1.  $7 \times 8^3 + 6 \times 8^2 + 3 \times 8^1 + 5 \times 8^0$

2.  $3584 + 384 + 24 + 5 = 3997$

3. **Answer:**  $7635_8 = 3997_{10}$

(d):  $6677_8$

1.  $6 \times 8^3 + 6 \times 8^2 + 7 \times 8^1 + 7 \times 8^0$

2.  $3072 + 384 + 56 + 7 = 3519$

3. **Answer:**  $6677_8 = 3519_{10}$

(e):  $67765_8$

1.  $6 \times 8^4 + 7 \times 8^3 + 7 \times 8^2 + 6 \times 8^1 + 5 \times 8^0$

2.  $24576 + 3584 + 448 + 48 + 5 = 28261$

3. **Answer:**  $67765_8 = 28261_{10}$

3. What is the highest decimal number that can be written in octal form using a maximum of  
(a) one digit, (b) two digits, (c) three digits, (d) four digits (e) five digits, (f)  $N$  digits?

**Solutions:**

(a): One octal digit

The highest decimal number that can be written in octal form using one octal digit is  $7_8$ , which is equal to  $7_{10}$ .

(b): Two octal digits

The highest decimal number that can be written in octal form using two octal digits is  $77_8$ , which is equal to  $63_{10}$ .

(c): Three octal digits

The highest decimal number that can be written in octal form using three octal digits is  $777_8$ , which is equal to  $511_{10}$ .

(d): Four octal digits

The highest decimal number that can be written in octal form using four octal digits is  $7777_8$ , which is equal to  $4095_{10}$ .

(e): Five octal digits

The highest decimal number that can be written in octal form using five octal digits is  $77777_8$ , which is equal to  $32767_{10}$ .

(f): Formula for highest decimal number using  $N$  octal digits

The pattern observed is that the highest decimal number that can be written in octal form using  $N$  octal digits is  $8^N - 1$ .

**Exercise 14.4:**

1. Convert the following hexadecimal numbers to decimal:  
(a) 91 (b) 6C (c) A1B (d) F9D4 (e) ABCD

**Solutions:**

(a):  $91_{16}$

1.  $9 \times 16^1 + 1 \times 16^0$
2.  $144 + 1 = 145$
3. **Answer:**  $91_{16} = 145_{10}$

(b):  $6C_{16}$

1.  $6 \times 16^1 + 12 \times 16^0$
2.  $96 + 12 = 108$
3. **Answer:**  $6C_{16} = 108_{10}$

(c):  $A1B_{16}$

1.  $10 \times 16^2 + 1 \times 16^1 + 11 \times 16^0$
2.  $2560 + 16 + 11 = 2587$
3. **Answer:**  $A1B_{16} = 2587_{10}$

(d):  $F9D4_{16}$

1.  $15 \times 16^3 + 9 \times 16^2 + 13 \times 16^1 + 4 \times 16^0$
2.  $61440 + 2304 + 208 + 4 = 63956$
3. **Answer:**  $F9D4_{16} = 63956_{10}$

(e):  $ABCD_{16}$

1.  $10 \times 16^3 + 11 \times 16^2 + 12 \times 16^1 + 13 \times 16^0$
2.  $40960 + 2816 + 192 + 13 = 43981$
3. **Answer:**  $ABCD_{16} = 43981_{10}$

2. Convert the following decimal numbers to hexadecimal:  
(a) 160 (b) 396 (c) 5010 (d) 25000 (e) 1000000

**Solutions:**

(a):  $160_{10}$

1.  $160 \div 16 = 10$  remainder 0
2.  $10 \div 16 = 0$  remainder 10
3. The hexadecimal number is read from the remainders in reverse order:  $A0$
4.  $160_{10} = A0_{16}$
5. **Answer:**  $160_{10} = A0_{16}$

(b):  $396_{10}$

1.  $396 \div 16 = 24$  remainder 12
2.  $24 \div 16 = 1$  remainder 8
3.  $1 \div 16 = 0$  remainder 1
4. The hexadecimal number is read from the remainders in reverse order:  $18C$
5.  $396_{10} = 18C_{16}$
6. **Answer:**  $396_{10} = 18C_{16}$

(c):  $5010_{10}$

1.  $5010 \div 16 = 313$  remainder 2
2.  $313 \div 16 = 19$  remainder 9
3.  $19 \div 16 = 1$  remainder 3
4.  $1 \div 16 = 0$  remainder 1
5. The hexadecimal number is read from the remainders in reverse order:  $1392$
6.  $5010_{10} = 1392_{16}$
7. **Answer:**  $5010_{10} = 1392_{16}$

(d):  $25000_{10}$

1.  $25000 \div 16 = 1562$  remainder 8
2.  $1562 \div 16 = 97$  remainder 10
3.  $97 \div 16 = 6$  remainder 1
4.  $6 \div 16 = 0$  remainder 6
5. The hexadecimal number is read from the remainders in reverse order:  $61A8$
6.  $25000_{10} = 61A8_{16}$
7. **Answer:**  $25000_{10} = 61A8_{16}$

(e):  $1000000_{10}$

1.  $1000000 \div 16 = 62500$  remainder 0
2.  $62500 \div 16 = 3906$  remainder 4
3.  $3906 \div 16 = 244$  remainder 2
4.  $244 \div 16 = 15$  remainder 4
5.  $15 \div 16 = 0$  remainder 15
6. The hexadecimal number is read from the remainders in reverse order:  $F4240$
7.  $1000000_{10} = F4240_{16}$
8. **Answer:**  $1000000_{10} = F4240_{16}$

3. Calculate the highest decimal number that can be represented by a hexadecimal with  
(a) one digit, (b) two digits, (c) three digits, (d) four digits, (e)  $N$  digits

**Solutions:**

(a): One hexadecimal digit

The highest decimal number that can be represented by a hexadecimal with one hexadecimal digit is  $F_{16}$ , which is equal to  $15_{10}$ .

(b): Two hexadecimal digits

The highest decimal number that can be represented by a hexadecimal with two hexadecimal digits is  $FF_{16}$ , which is equal to  $255_{10}$ .

(c): Three hexadecimal digits

The highest decimal number that can be represented by a hexadecimal with three hexadecimal digits is  $FFF_{16}$ , which is equal to  $4095_{10}$ .



(d): Four hexadecimal digits

The highest decimal number that can be represented by a hexadecimal with four hexadecimal digits is  $FFF_{16}$ , which is equal to  $65535_{10}$ .

(e):  $N$  hexadecimal digits

The pattern observed is that the highest decimal number that can be represented by a hexadecimal with  $N$  hexadecimal digits is  $16^N - 1$ .

#### 0.0.1 Challenge Exercise 14:

1. Convert the following decimal numbers to binary, octal and hexadecimal:  
(a) 0.25 (b) 0.75

#### Solutions:

(a):  $0.25_{10}$

1. Binary:  $0.01_2$
2. Octal:  $0.2_8$
3. Hexadecimal:  $0.4_{16}$
4. **Answer:**  $0.25_{10} = 0.01_2 = 0.2_8 = 0.4_{16}$

(b):  $0.75_{10}$

1. Binary:  $0.11_2$
2. Octal:  $0.6_8$
3. Hexadecimal:  $0.C_{16}$
4. **Answer:**  $0.75_{10} = 0.11_2 = 0.6_8 = 0.C_{16}$