CM1015 - Computational Mathematics

BSc Computer Science

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Number Bases

Decimal

Number bases relate to the amount of digits used to represent a number. The decimal system is base 10, meaning it uses 10 digits (0-9) to represent numbers. The value of a number is calculated by multiplying each digit by the base raised to the power of its position. For example, the number 123 in decimal is calculated as follows:

10^2	10^1	10^0
1	2	3
100	10	1

$$= 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$

$$= 123$$

Binary

The binary system is base 2, meaning it uses 2 digits (0-1) to represent numbers. Binary numbers are calculated in the same way as decimal numbers, but using powers of 2 instead of 10. For example, the binary number 101 in decimal is calculated as follows:

2^2	2^1	2^0
1	0	1
4	2	1

$$= 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$= 5$$

Hexadecimal

The hexadecimal system is base 16, meaning it uses 16 digits (0-9, A-F) to represent numbers. Hexadecimal numbers are calculated in the same way as decimal numbers, but using powers of 16 instead of 10. For example, the hexadecimal number 1A3 in decimal is calculated as follows:

16^2	16^1	16^0
1	10	3
256	16	1

$$= 1 \times 16^2 + 10 \times 16^1 + 3 \times 16^0$$

$$= 419$$

Generic Base Conversion to Decimal

To convert any base to decimal, the following method can be used:

1. Write the number in the base you are converting from.
2. Multiply each digit by the base raised to the power of its position.
3. Add the results together to get the decimal value.

Formula: $d_n \times b^n + d_{n-1} \times b^{n-1} + \ldots + d_1 \times b^1 + d_0 \times b^0$

Where: d_n is the digit at position n, b is the base, and n is the position of the digit.

Example, shown with above symbols:

$$\begin{array}{ccc} b^2 & b^1 & b^0 \\ \hline \multicolumn{1}{|c|}{d_2} & \multicolumn{1}{c|}{d_1} & \multicolumn{1}{c|}{d_0} \\ \hline \end{array} = d_2 \times b^2 + d_1 \times b^1 + d_0 \times b^0$$

Example with base 5 number 234, here $b = 5$, $d_2 = 2$, $d_1 = 3$, $d_0 = 4$:

$$\begin{array}{ccc} 5^2 & 5^1 & 5^0 \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline 25 & 5 & 1 \end{array} \begin{array}{l} = 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0 \\ = 69 \end{array}$$

Non-integer Conversion to Decimal

To convert a non-integer number to decimal, the following method can be used:

1. Write the number in the base you are converting from.
2. Multiply each digit by the base raised to the power of its position.
3. Add the results together to get the integer part of the decimal value.
4. Repeat the process for the fractional part of the number.

Formula: $d_n \times b^n + d_{n-1} \times b^{n-1} + \ldots + d_1 \times b^1 + d_0 \times b^0 + d_{-1} \times b^{-1} + d_{-2} \times b^{-2} + \ldots$

Where: d_n is the digit at position n, b is the base, and n is the position of the digit.

Example, shown with above symbols:

b^2	b^1	b^0		b^{-1}	b^{-2}
d_2	d_1	d_0	.	d_{-1}	d_{-2}

$$= d_2 \times b^2 + d_1 \times b^1 + d_0 \times b^0$$

$$+ \ d_{-1} \times b^{-1} \ + \ d_{-2} \times b^{-2}$$

Example with base 5 number 234.24, here $b = 5$, $d_2 = 2$, $d_1 = 3$, $d_0 = 4$, $d_{-1} = 2$, $d_{-2} = 5$:

5^2	5^1	5^0		5^{-1}	5^{-2}
2	3	4	.	2	4
25	5	1		0.2	0.04

$$= \ 2 \times 5^2 \ + \ 3 \times 5^1 \ + \ 4 \times 5^0$$

$$+ \quad 2 \times 5^{-1} \quad + \quad 4 \times 5^{-2} \quad = \quad 69.56$$

Binary Operations

Addition

Binary addition is similar to decimal addition, but with the base being 2.

Example: $101_2 + 11_2$

Step 1:

$$\begin{array}{cccc} & & 1 & \\ \hline & 1 & 0 & 1 \\ + & & 1 & 1 \\ \hline & & & 0 \end{array}$$

1 + 1 = 10, write 0 into the result row and carry 1 to the next left column

Step 2:

$$\begin{array}{cccc} & 1 & 1 & \\ \hline & 1 & 0 & 1 \\ + & & 1 & 1 \\ \hline & & 0 & 0 \end{array}$$

1 + 0 + 1 = 10, write 0 to the results row and carry 1 to the next left column

Step 3:

$$\begin{array}{cccc} & 1 & 1 & \\ \hline & 1 & 0 & 1 \\ + & & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}$$

1 + 1 = 10, write 1 0 in the result row.
Answer: 1000_2

Subtraction

Binary subtraction is similar to decimal subtraction, but with the base being 2.

Example: $101_2 - 11_2$

Step 1:

$$\begin{array}{cccc} & 1 & 0 & 1 \\ - & & 1 & 1 \\ \hline & & & 0 \end{array}$$

1 - 1 = 0, write 0 into the result row

Step 2:

$$\begin{array}{cccc} & & 2 & \\ \hline & \cancel{1} & 0 & 1 \\ - & & 1 & 1 \\ \hline & & 1 & 0 \end{array}$$

If the subtraction cannot be done, borrow 2 from the next left column, 2 - 1 = 1, write 1 in the result row
Answer: 10_2

Multiplication

Binary multiplication is similar to decimal multiplication, but with the base being 2. The general method is to multiply the top number by each digit of the bottom number, starting from the rightmost digit. From here we can perform addition to get the final result.

Example: $101_2 \times 11_2$

Step 1:

$$\begin{array}{ccccc} & & 1 & 0 & 1 \\ \times & & & 1 & 1 \\ \hline & & 1 & 0 & 1 \end{array}$$

Multiply the rightmost digit of the bottom number (1) by the top number (1 0 1), write the result in the first row (1 0 1)

Step 2:

$$\begin{array}{ccccc} & & 1 & 0 & 1 \\ \times & & & 1 & 1 \\ \hline & & 1 & 0 & 1 \\ & 1 & 0 & 1 & \end{array}$$

Multiply the next digit of the bottom number (1) by the top number (1 0 1), write the result in the second row (1 0 1), shift the second row one position to the left

Step 3:

$$\begin{array}{ccccc} & & 1 & 0 & 1 \\ \times & & & 1 & 1 \\ \hline & & 1 & 0 & 1 \\ + & 1 & 0 & 1 & \\ \hline & 1 & 1 & 1 & 1 \end{array}$$

Perform addition of the multiplication results rows to get the final answer
Answer: 1111_2

Self-assessment Questions from Foundation Maths Book

Worked Example 14.2

Convert 83_{10} to binary.

Solution:

The general method for converting a decimal number to binary is to repeatedly divide the number by 2 and record the remainder. The binary number is then read from the remainders in reverse order.

The steps for converting 83 to binary are as follows:

1. $83 \div 2 = 41$ remainder 1
2. $41 \div 2 = 20$ remainder 1
3. $20 \div 2 = 10$ remainder 0
4. $10 \div 2 = 5$ remainder 0
5. $5 \div 2 = 2$ remainder 1
6. $2 \div 2 = 1$ remainder 0
7. $1 \div 2 = 0$ remainder 1
8. The binary number is read from the remainders in reverse order: 1010011

2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	0	1	1
64	32	16	8	4	2	1

$$1 \times 64 + 1 \times 16 + 1 \times 2 + 1 \times 1$$

$$= 83$$

Exercise 14.2:

1. Convert the following decimal numbers to binary:

(a): 19 (b): 36 (c): 100 (d): 796 (e): 5000

Solutions:

(a): 19_{10}

1. $19 \div 2 = 9$ remainder 1
2. $9 \div 2 = 4$ remainder 1
3. $4 \div 2 = 2$ remainder 0
4. $2 \div 2 = 1$ remainder 0
5. $1 \div 2 = 0$ remainder 1
6. The binary number is read from the remainders in reverse order: 10011
7. **Answer:** $19_{10} = 10011_2$

(b): 36_{10}

1. $36 \div 2 = 18$ remainder 0
2. $18 \div 2 = 9$ remainder 0
3. $9 \div 2 = 4$ remainder 1
4. $4 \div 2 = 2$ remainder 0
5. $2 \div 2 = 1$ remainder 0
6. $1 \div 2 = 0$ remainder 1
7. The binary number is read from the remainders in reverse order: 100100
8. **Answer:** $36_{10} = 100100_2$

(c): 100_{10}

1. $100 \div 2 = 50$ remainder 0
2. $50 \div 2 = 25$ remainder 0
3. $25 \div 2 = 12$ remainder 1
4. $12 \div 2 = 6$ remainder 0
5. $6 \div 2 = 3$ remainder 0
6. $3 \div 2 = 1$ remainder 1
7. $1 \div 2 = 0$ remainder 1
8. The binary number is read from the remainders in reverse order: 1100100
9. **Answer:** $100_{10} = 1100100_2$

(d): 796_{10}

1. $796 \div 2 = 398$ remainder 0
2. $398 \div 2 = 199$ remainder 0
3. $199 \div 2 = 99$ remainder 1
4. $99 \div 2 = 49$ remainder 1
5. $49 \div 2 = 24$ remainder 1
6. $24 \div 2 = 12$ remainder 0
7. $12 \div 2 = 6$ remainder 0
8. $6 \div 2 = 3$ remainder 0
9. $3 \div 2 = 1$ remainder 1
10. $1 \div 2 = 0$ remainder 1
11. The binary number is read from the remainders in reverse order: 1100011100
12. **Answer:** $796_{10} = 1100011100_2$

(e): 5000_{10}

1. $5000 \div 2 = 2500$ remainder 0
2. $2500 \div 2 = 1250$ remainder 0
3. $1250 \div 2 = 625$ remainder 0
4. $625 \div 2 = 312$ remainder 1
5. $312 \div 2 = 156$ remainder 0
6. $156 \div 2 = 78$ remainder 0
7. $78 \div 2 = 39$ remainder 0
8. $39 \div 2 = 19$ remainder 1
9. $19 \div 2 = 9$ remainder 1
10. $9 \div 2 = 4$ remainder 1
11. $4 \div 2 = 2$ remainder 0
12. $2 \div 2 = 1$ remainder 0
13. $1 \div 2 = 0$ remainder 1
14. The binary number is read from the remainders in reverse order: 1001110001000
15. **Answer:** $5000_{10} = 1001110001000_2$

2. Convert the following binary numbers to decimal:

(a): 111 (b): 10101 (c): 111001 (d): 1110001 (e): 11111111

Solutions:

(a): 111_2

1. $1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$
2. $4 + 2 + 1 = 7$
3. **Answer:** $111_2 = 7_{10}$

(b): 10101_2

1. $1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
2. $16 + 0 + 4 + 0 + 1 = 21$
3. **Answer:** $10101_2 = 21_{10}$

(c): 111001_2

1. $1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
2. $32 + 16 + 8 + 0 + 0 + 1 = 57$
3. **Answer:** $111001_2 = 57_{10}$

(d): 1110001_2

1. $1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
2. $64 + 32 + 16 + 0 + 0 + 0 + 1 = 113$
3. **Answer:** $1110001_2 = 113_{10}$

(e): 11111111_2

1. $1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$
2. $128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 255$
3. **Answer:** $11111111_2 = 255_{10}$

3. What is the highest decimal number that can be written in binary form using a maximum of (a) two binary digits, (b) three binary digits, (c) four binary digits, (d) five binary digits? Can you spot a pattern? (e) Write a formula for the highest decimal number that can be written in binary form using N binary digits.

Solutions:

(a): Two binary digits

The highest decimal number that can be written in binary form using two binary digits is 11_2, which is equal to 3_{10}.

(b): Three binary digits

The highest decimal number that can be written in binary form using three binary digits is 111_2, which is equal to 7_{10}.

(c): Four binary digits

The highest decimal number that can be written in binary form using four binary digits is 1111_2, which is equal to 15_{10}.

(d): Five binary digits

The highest decimal number that can be written in binary form using five binary digits is 11111_2, which is equal to 31_{10}.

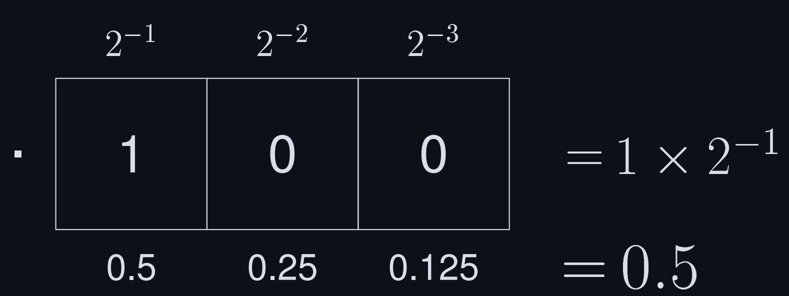
(e): Formula for highest decimal number using N binary digits

The pattern observed is that the highest decimal number that can be written in binary form using N binary digits is $2^N - 1$.

4. Write the decimal number 0.5 in binary.

Solution:

The decimal number 0.5 can be written in binary as 0.1_2.



Exercise 14.3:

1. Convert the following decimal numbers to octal numbers:

(a) 971 (b) 2841 (c) 5014 (d) 10000 (e) 17926

Solutions:

(a): 971_{10}

1. $971 \div 8 = 121$ remainder 3
2. $121 \div 8 = 15$ remainder 1
3. $15 \div 8 = 1$ remainder 7
4. $1 \div 8 = 0$ remainder 1
5. The octal number is read from the remainders in reverse order: 1733
6. **Answer:** $971_{10} = 1733_8$

(b): 2841_{10}

1. $2841 \div 8 = 355$ remainder 1
2. $355 \div 8 = 44$ remainder 3
3. $44 \div 8 = 5$ remainder 4
4. $5 \div 8 = 0$ remainder 5
5. The octal number is read from the remainders in reverse order: 5431
6. **Answer:** $2841_{10} = 5431_8$

(c): 5014_{10}

1. $5014 \div 8 = 626$ remainder 6
2. $626 \div 8 = 78$ remainder 2
3. $78 \div 8 = 9$ remainder 6
4. $9 \div 8 = 1$ remainder 1
5. $1 \div 8 = 0$ remainder 1
6. The octal number is read from the remainders in reverse order: 16126
7. **Answer:** $5014_{10} = 16126_8$

(d): 10000_{10}

1. $10000 \div 8 = 1250$ remainder 0
2. $1250 \div 8 = 156$ remainder 2
3. $156 \div 8 = 19$ remainder 4
4. $19 \div 8 = 2$ remainder 3
5. $2 \div 8 = 0$ remainder 2
6. The octal number is read from the remainders in reverse order: 23420
7. **Answer:** $10000_{10} = 23420_8$

(e): 17926_{10}

1. $17926 \div 8 = 2240$ remainder 6
2. $2240 \div 8 = 280$ remainder 0
3. $280 \div 8 = 35$ remainder 0
4. $35 \div 8 = 4$ remainder 3
5. $4 \div 8 = 0$ remainder 4
6. The octal number is read from the remainders in reverse order: 43006
7. **Answer:** $17926_{10} = 43006_8$

2. Convert the following octal numbers to decimal:

(a) 73 (b) 1237 (c) 7635 (d) 6677 (e) 67765

Solutions:

(a): 73_8

1. $7 \times 8^1 + 3 \times 8^0$
2. $56 + 3 = 59$
3. **Answer:** $73_8 = 59_{10}$

(b): 1237_8

1. $1 \times 8^3 + 2 \times 8^2 + 3 \times 8^1 + 7 \times 8^0$
2. $512 + 128 + 24 + 7 = 671$
3. **Answer:** $1237_8 = 671_{10}$

(c): 7635_8

1. $7 \times 8^3 + 6 \times 8^2 + 3 \times 8^1 + 5 \times 8^0$
2. $3584 + 384 + 24 + 5 = 3997$
3. **Answer:** $7635_8 = 3997_{10}$

(d): 6677_8

1. $6 \times 8^3 + 6 \times 8^2 + 7 \times 8^1 + 7 \times 8^0$
2. $3072 + 384 + 56 + 7 = 3519$
3. **Answer:** $6677_8 = 3519_{10}$

(e): 67765_8

1. $6 \times 8^4 + 7 \times 8^3 + 7 \times 8^2 + 6 \times 8^1 + 5 \times 8^0$
2. $24576 + 3584 + 448 + 48 + 5 = 28261$
3. **Answer:** $67765_8 = 28261_{10}$

3. What is the highest decimal number that can be written in octal form using a maximum of (a) one digit, (b) two digits, (c) three digits, (d) four digits (e) five digits, (f) N digits?

Solutions:

(a): One octal digit

The highest decimal number that can be written in octal form using one octal digit is 7_8, which is equal to 7_{10}.

(b): Two octal digits

The highest decimal number that can be written in octal form using two octal digits is 77_8, which is equal to 63_{10}.

(c): Three octal digits

The highest decimal number that can be written in octal form using three octal digits is 777_8, which is equal to 511_{10}.

(d): Four octal digits

The highest decimal number that can be written in octal form using four octal digits is 7777_8, which is equal to 4095_{10}.

(e): Five octal digits

The highest decimal number that can be written in octal form using five octal digits is 77777_8, which is equal to 32767_{10}.

(f): Formula for highest decimal number using N octal digits

The pattern observed is that the highest decimal number that can be written in octal form using N octal digits is $8^N - 1$.

Exercise 14.4:

1. Convert the following hexadecimal numbers to decimal:
(a) 91 (b) 6C (c) A1B (d) F9D4 (e) ABCD

Solutions:

(a): 91_{16}

1. $9 \times 16^1 + 1 \times 16^0$
2. $144 + 1 = 145$
3. **Answer:** $91_{16} = 145_{10}$

(b): $6C_{16}$

1. $6 \times 16^1 + 12 \times 16^0$
2. $96 + 12 = 108$
3. **Answer:** $6C_{16} = 108_{10}$

(c): $A1B_{16}$

1. $10 \times 16^2 + 1 \times 16^1 + 11 \times 16^0$
2. $2560 + 16 + 11 = 2587$
3. **Answer:** $A1B_{16} = 2587_{10}$

(d): $F9D4_{16}$

1. $15 \times 16^3 + 9 \times 16^2 + 13 \times 16^1 + 4 \times 16^0$
2. $61440 + 2304 + 208 + 4 = 63956$
3. **Answer:** $F9D4_{16} = 63956_{10}$

(e): $ABCD_{16}$

1. $10 \times 16^3 + 11 \times 16^2 + 12 \times 16^1 + 13 \times 16^0$
2. $40960 + 2816 + 192 + 13 = 43981$
3. **Answer:** $ABCD_{16} = 43981_{10}$

2. Convert the following decimal numbers to hexadecimal:
(a) 160 (b) 396 (c) 5010 (d) 25000 (e) 1000000

Solutions:

(a): 160_{10}

1. $160 \div 16 = 10$ remainder 0
2. $10 \div 16 = 0$ remainder 10
3. The hexadecimal number is read from the remainders in reverse order: $A0$
4. $160_{10} = A0_{16}$
5. **Answer:** $160_{10} = A0_{16}$

(b): 396_{10}

1. $396 \div 16 = 24$ remainder 12
2. $24 \div 16 = 1$ remainder 8
3. $1 \div 16 = 0$ remainder 1
4. The hexadecimal number is read from the remainders in reverse order: $18C$
5. $396_{10} = 18C_{16}$
6. **Answer:** $396_{10} = 18C_{16}$

(c): 5010_{10}

1. $5010 \div 16 = 313$ remainder 2
2. $313 \div 16 = 19$ remainder 9
3. $19 \div 16 = 1$ remainder 3
4. $1 \div 16 = 0$ remainder 1
5. The hexadecimal number is read from the remainders in reverse order: 1392
6. $5010_{10} = 1392_{16}$
7. **Answer:** $5010_{10} = 1392_{16}$

(d): 25000_{10}

1. $25000 \div 16 = 1562$ remainder 8
2. $1562 \div 16 = 97$ remainder 10
3. $97 \div 16 = 6$ remainder 1
4. $6 \div 16 = 0$ remainder 6
5. The hexadecimal number is read from the remainders in reverse order: $61A8$
6. $25000_{10} = 61A8_{16}$
7. **Answer:** $25000_{10} = 61A8_{16}$

(e): 1000000_{10}

1. $1000000 \div 16 = 62500$ remainder 0
2. $62500 \div 16 = 3906$ remainder 4
3. $3906 \div 16 = 244$ remainder 2
4. $244 \div 16 = 15$ remainder 4
5. $15 \div 16 = 0$ remainder 15
6. The hexadecimal number is read from the remainders in reverse order: $F4240$
7. $1000000_{10} = F4240_{16}$
8. **Answer:** $1000000_{10} = F4240_{16}$

3. Calculate the highest decimal number that can be represented by a hexadecimal with (a) one digit, (b) two digits, (c) three digits, (d) four digits, (e) N digits

Solutions:

(a): One hexadecimal digit

The highest decimal number that can be represented by a hexadecimal with one hexadecimal digit is F_{16}, which is equal to 15_{10}.

(b): Two hexadecimal digits

The highest decimal number that can be represented by a hexadecimal with two hexadecimal digits is FF_{16}, which is equal to 255_{10}.

(c): Three hexadecimal digits

The highest decimal number that can be represented by a hexadecimal with three hexadecimal digits is FFF_{16}, which is equal to 4095_{10}.

(d): Four hexadecimal digits

The highest decimal number that can be represented by a hexadecimal with four hexadecimal digits is $FFFF_{16}$, which is equal to 65535_{10}.

(e): N hexadecimal digits

The pattern observed is that the highest decimal number that can be represented by a hexadecimal with N hexadecimal digits is $16^N - 1$.

0.0.1 Challenge Exercise 14:

1. Convert the following decimal numbers to binary, octal and hexadecimal:
(a) 0.25 (b) 0.75

Solutions:

(a): 0.25_{10}

1. Binary: 0.01_2
2. Octal: 0.2_8
3. Hexadecimal: 0.4_{16}
4. **Answer:** $0.25_{10} = 0.01_2 = 0.2_8 = 0.4_{16}$

(b): 0.75_{10}

1. Binary: 0.11_2
2. Octal: 0.6_8
3. Hexadecimal: $0.C_{16}$
4. **Answer:** $0.75_{10} = 0.11_2 = 0.6_8 = 0.C_{16}$