Rotação de coordenadas:

$$\boldsymbol{u} = u'_j \boldsymbol{e}'_j = u_i \boldsymbol{e}_i$$

Mas

$$u_i = (\boldsymbol{u} \cdot \boldsymbol{e}_i)$$

é a projeção de $\boldsymbol{u}$ sobre $\boldsymbol{e}_i$, que é dada pelo produto escalar

$$\boldsymbol{u} \cdot \boldsymbol{e}_i = |\boldsymbol{u}| \underbrace{|\boldsymbol{e}_i|}_{=1} \cos\theta = |\boldsymbol{u}| \cos\theta$$

Voltando agora ao fio da meada,

$$\begin{aligned} u_i &= (\boldsymbol{u} \cdot \boldsymbol{e}_i) = \boldsymbol{u} \cdot C_{ij} \boldsymbol{e}'_j = C_{ij}(\boldsymbol{u} \cdot \boldsymbol{e}'_j) = C_{ij} u'_j, \\ u_i &= C_{ij} u'_j \\ [\boldsymbol{u}]_E &= [\boldsymbol{C}][\boldsymbol{u}]_{E'} \end{aligned}$$

Também posso fazer isso “ao contrário”:

$$\begin{aligned} u'_j &= \boldsymbol{u} \cdot \boldsymbol{e}'_j = \boldsymbol{u} \cdot C_{ij} \boldsymbol{e}_i = C_{ij}(\boldsymbol{u} \cdot \boldsymbol{e}_i) = C_{ij} u_i; \\ u'_j &= C_{ij} u_i = C^\top_{ji} u_i \\ [\boldsymbol{u}]_{E'} &= [\boldsymbol{C}]^\top [\boldsymbol{u}]_E \end{aligned}$$

Isso me dá imediatamente dois resultados:

$$\begin{aligned} [\boldsymbol{u}]_E &= [\boldsymbol{C}][\boldsymbol{u}]_{E'} \\ &= [\boldsymbol{C}][\boldsymbol{C}]^\top [\boldsymbol{u}]_E \;\Rightarrow \\ [\boldsymbol{C}][\boldsymbol{C}]^\top &= [\boldsymbol{\delta}] \end{aligned}$$

e

$$\begin{aligned} [\boldsymbol{u}]_{E'} &= [\boldsymbol{C}]^\top [\boldsymbol{u}]_E \\ &= [\boldsymbol{C}]^\top [\boldsymbol{C}][\boldsymbol{u}]_{E'} \;\Rightarrow \\ [\boldsymbol{C}]^\top [\boldsymbol{C}] &= [\boldsymbol{\delta}] \end{aligned}$$

Se $[\boldsymbol{C}]$ representa uma rotação de coordenadas, sempre teremos

$$\det [\boldsymbol{C}] = +1.$$

No $\mathbb{R}^3$,

$$\begin{aligned} 1 &= [\boldsymbol{e}_1 \times \boldsymbol{e}_2] \cdot \boldsymbol{e}_3 \\ &= [C_{1l} \boldsymbol{e}'_l \times C_{2m} \boldsymbol{e}'_m] \cdot C_{3n} \boldsymbol{e}'_n \\ &= C_{1l} C_{2m} C_{3n} \underbrace{[\boldsymbol{e}'_l \times \boldsymbol{e}'_m] \cdot \boldsymbol{e}'_n}_{\epsilon_{lmn}} \\ &= \epsilon_{lmn} C_{1l} C_{2m} C_{3n} = \det [\boldsymbol{C}] \;\blacksquare \end{aligned}$$

O que acontece com a matriz de uma transformação linear quando se rotaciona a base?

$$
\begin{aligned}
\boldsymbol{A} = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= A'_{kl}\boldsymbol{e}'_k\boldsymbol{e}'_l \\
&= A'_{kl}C_{ik}\boldsymbol{e}_iC_{jl}\boldsymbol{e}_j \\
&= A'_{kl}C_{ik}C_{jl}\boldsymbol{e}_i\boldsymbol{e}_j \\
&= C_{ik}A'_{kl}C^{\top}_{lj}\boldsymbol{e}_i\boldsymbol{e}_j \\
[\boldsymbol{A}]_E &= [\boldsymbol{C}][\boldsymbol{A}]_{E'}[\boldsymbol{C}]^{\top} \ \blacksquare
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}
$$