TEA013 Matemática Aplicada II
Curso de Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
F, 11 Dez 2017
Prof. Nelson Luís Dias

0

NOME: GABARITO Assinatura: ______________________

---

1 [25] Se

$$f(x) = \operatorname{sen}(x), \qquad 0 \le x \le \frac{\pi}{2},$$

obtenha a série de Fourier trigonométrica de $f_P(x)$, onde $f_P(x)$ é a extensão par de $f(x)$ entre $-\pi/2$ e $+\pi/2$.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
L &= \pi;\\
f_P(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi x}{\pi}\right);\\
A_n &= \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} f_P(x)\cos(2nx)\,\mathrm{d}x\\
&= \frac{4}{\pi}\int_{0}^{\pi/2} f_P(x)\cos(2nx)\,\mathrm{d}x\\
&= \frac{4}{\pi}\int_{0}^{\pi/2} \operatorname{sen}(x)\cos(2nx)\,\mathrm{d}x\\
&= -\frac{4}{\pi(4n^2-1)} \blacksquare
\end{aligned}$$

Continue a solução no verso ⟹

2 [25] Dada a EDO

$$\frac{d^2y}{dt^2} + \frac{1}{T_1}\frac{dy}{dt} + \frac{1}{T_0^2}y = \frac{1}{T_0^2}x,$$

onde todas as quantidades são reais, com T_0 e T_1 constantes, e sendo

$$\widehat{f}(\omega) = \frac{1}{2\pi}\int_{t=-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt$$

a transformada de Fourier de uma $f(t)$ genérica, obtenha $\widehat{y}(\omega)$ em função de $\widehat{x}(\omega)$.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
(i\omega)^2\widehat{y} + \frac{i\omega}{T_1}\widehat{y} + \frac{1}{T_0^2}\widehat{y} &= \frac{1}{T_0^2}\widehat{x};\\
\left[-\omega^2 + \frac{i\omega}{T_1} + \frac{1}{T_0^2}\right]\widehat{y} &= \frac{1}{T_0^2}\widehat{x};\\
\frac{\left[-\omega^2T_1T_0^2 + i\omega T_0^2 + T_1\right]}{T_1T_0^2}\widehat{y} &= \frac{T_1}{T_1T_0^2}\widehat{x};\\
\widehat{y} &= \frac{T_1}{\left[-\omega^2T_1T_0^2 + i\omega T_0^2 + T_1\right]}\widehat{x} \blacksquare
\end{aligned}$$

Continue a solução no verso ⟹

3 [25] (Bird, Stewart e Lightfoot, Transport Phenomena, Ex. 11.2-2) Dada a EDP e condições inicial e de contorno

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \phi}{\partial x}\right),\\
\phi(x,0) &= 0,\\
\phi(0,t) &= 1,\\
\phi(\infty,t) &= 0,
\end{aligned}
$$

obtenha uma EDO equivalente para $f(\xi)$ e suas condições de contorno $f(0)$ e $f(\infty)$, onde $\phi(x,t) = f(\xi)$, e

$$
\xi = \frac{x}{\sqrt[3]{9t}}
$$

é uma variável de similaridade.

SOLUÇÃO DA QUESTÃO:
Se $\phi(x,t) = f(\xi)$, então calculamos as derivadas parciais de ξ em relação a x e t para tê-las à mão:

$$
\begin{aligned}
\frac{\partial \xi}{\partial t} &= \frac{\partial}{\partial t}\left[x(9t)^{-1/3}\right]\\
&= x\left[-\frac{1}{3}(9t)^{-4/3}9\right]\\
&= x\left[-3(9t)^{-4/3}\right]\\
&= -3x(9t)^{-1/3}(9t)^{-1}\\
&= -3\xi(9t)^{-1}\\
&= \frac{-\xi}{3t}.
\end{aligned}
$$

Agora em x:

$$
\frac{\partial \xi}{\partial x} = (9t)^{-1/3}.
$$

Substituímos agora na EDP:

$$
\begin{aligned}
f(\xi) &= \phi(x,t) \;\Rightarrow\\
\frac{\partial f}{\partial t} &= \frac{\mathrm{d}f}{\mathrm{d}\xi}\frac{\partial \xi}{\partial t}\\
&= \frac{\mathrm{d}f}{\mathrm{d}\xi}\left(\frac{-\xi}{3t}\right)\\
&= -\frac{1}{3t}\xi\frac{\mathrm{d}f}{\mathrm{d}\xi}.
\end{aligned}
$$

Em x:

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{\mathrm{d}f}{\mathrm{d}\xi}\frac{\partial \xi}{\partial x}\\
&= \frac{\mathrm{d}f}{\mathrm{d}\xi}(9t)^{-1/3};\\
\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \phi}{\partial x}\right) &= \frac{1}{x}\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial x}\frac{1}{x^2}\\
&= \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}(9t)^{-1/3}\right) - \frac{1}{x^2}\frac{\mathrm{d}f}{\mathrm{d}\xi}(9t)^{-1/3};\\
&= \frac{(9t)^{-1/3}}{x}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}f}{\mathrm{d}\xi}\right]\frac{\partial \xi}{\partial x} - \frac{1}{x^2}\frac{\mathrm{d}f}{\mathrm{d}\xi}(9t)^{-1/3};\\
&= \frac{(9t)^{-2/3}}{x}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{(9t)^{-1/3}}{x^2}\frac{\mathrm{d}f}{\mathrm{d}\xi}.
\end{aligned}
$$

Continue a solução no verso $\Longrightarrow$

Juntando tudo:

$$
\begin{aligned}
-\frac{1}{3t}\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} &= \frac{(9t)^{-2/3}}{x}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{(9t)^{-1/3}}{x^2}\frac{\mathrm{d}f}{\mathrm{d}\xi},\\
-\frac{3}{(9t)}\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} &= \frac{(9t)^{-2/3}}{x}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{(9t)^{-1/3}}{x^2}\frac{\mathrm{d}f}{\mathrm{d}\xi},\\
-3\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} &= \frac{(9t)^{1/3}}{x}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{(9t)^{2/3}}{x^2}\frac{\mathrm{d}f}{\mathrm{d}\xi}\\
-3\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} &= \frac{1}{\xi}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{1}{\xi^2}\frac{\mathrm{d}f}{\mathrm{d}\xi}\\
-3\xi^3\frac{\mathrm{d}f}{\mathrm{d}\xi} &= \xi\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \frac{\mathrm{d}f}{\mathrm{d}\xi}\\
\xi\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + (3\xi^3 - 1)\frac{\mathrm{d}f}{\mathrm{d}\xi} &= 0,
\end{aligned}
$$

com condições de contorno

$$
\begin{aligned}
f(0) &= 1,\\
f(\infty) &= 0 \blacksquare
\end{aligned}
$$

Continue a solução no verso $\Longrightarrow$

4 [25] Obtenha a função de Green de

$$
\begin{aligned}
x\frac{\mathrm{d}y}{\mathrm{d}x} - \left[1 + x^2\right] y(x) &= f(x),\\
y(1) &= y_1.
\end{aligned}
$$

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
\xi\frac{\mathrm{d}y}{\mathrm{d}\xi} - \left[1+\xi^2\right] y(\xi) &= f(\xi)\\
\frac{\mathrm{d}y}{\mathrm{d}\xi} - \frac{1}{\xi}\left[1+\xi^2\right] y(\xi) &= \frac{1}{\xi} f(\xi)\\
G(x,\xi)\frac{\mathrm{d}y}{\mathrm{d}\xi} - \frac{G(x,\xi)}{\xi}\left[1+\xi^2\right] y(\xi) &= \frac{G(x,\xi)}{\xi} f(\xi)\\
\int_{\xi=1}^{\infty} G(x,\xi)\frac{\mathrm{d}y}{\mathrm{d}\xi}\,\mathrm{d}\xi - \int_{\xi=1}^{\infty}\frac{G(x,\xi)}{\xi}\left[1+\xi^2\right] y(\xi)\,\mathrm{d}\xi &= \int_1^{\infty}\frac{G(x,\xi)}{\xi} f(\xi)\,\mathrm{d}\xi\\
G(x,\xi)y(\xi)\bigg|_1^{\infty} - \int_1^{\infty} y(\xi)\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi - \int_{\xi=1}^{\infty}\frac{G(x,\xi)}{\xi}\left[1+\xi^2\right] y(\xi)\,\mathrm{d}\xi &= \int_1^{\infty}\frac{G(x,\xi)}{\xi} f(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

Faça $G(x,\infty) = 0$;

$$
\begin{aligned}
-G(x,1)y(1) - \int_1^{\infty} y(\xi)\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi - \int_{\xi=1}^{\infty}\frac{G(x,\xi)}{\xi}\left[1+\xi^2\right] y(\xi)\,\mathrm{d}\xi &= \int_1^{\infty}\frac{G(x,\xi)}{\xi} f(\xi)\,\mathrm{d}\xi\\
-G(x,1)y(1) - \int_1^{\infty}\left\{\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} + \frac{G(x,\xi)}{\xi}\left[1+\xi^2\right]\right\} y(\xi)\,\mathrm{d}\xi &= \int_1^{\infty}\frac{G(x,\xi)}{\xi} f(\xi)\,\mathrm{d}\xi.
\end{aligned}
$$

A função de Green é a solução de

$$
-\left[\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\xi} + \frac{G(x,\xi)}{\xi}\left[1+\xi^2\right]\right] = \delta(\xi - x).
$$

Tentemos $G(x,\xi) = u(x,\xi)v(x,\xi)$;

$$
\begin{aligned}
-u\frac{\mathrm{d}v}{\mathrm{d}\xi} - v\frac{\mathrm{d}u}{\mathrm{d}\xi} - \frac{uv}{\xi}\left[1+\xi^2\right] &= \delta(\xi - x);\\
u\left\{-\frac{\mathrm{d}v}{\mathrm{d}\xi} - \frac{v}{\xi}\left[1+\xi^2\right]\right\} - v\frac{\mathrm{d}u}{\mathrm{d}\xi} &= \delta(\xi - x)\\
\frac{\mathrm{d}v}{\mathrm{d}\xi} + \frac{v}{\xi}\left[1+\xi^2\right] &= 0\\
\frac{\mathrm{d}v}{v} &= -\left(\frac{\mathrm{d}\xi}{\xi} + \xi\right);\\
\int_{v(x,1)}^{v(x,\xi)}\frac{\mathrm{d}v}{v} &= -\int_1^{\xi}\frac{\mathrm{d}\eta}{\eta} - \int_1^{\xi}\eta\,\mathrm{d}\eta;\\
\ln\left(\frac{v(x,\xi)}{v(x,1)}\right) &= -\ln(\xi) + \frac{1}{2}[1-\xi^2]\\
v(x,\xi) &= \frac{v(x,1)}{\xi}\exp\left[\frac{1}{2}(1-\xi^2)\right].
\end{aligned}
$$

Prosseguimos:

$$
\begin{aligned}
-\frac{v(x,1)}{\xi}\exp\left[\frac{1}{2}(1-\xi^2)\right]\frac{\mathrm{d}u}{\mathrm{d}\xi} &= \delta(\xi - x)\\
\frac{\mathrm{d}u}{\mathrm{d}\xi} &= -\frac{\xi}{v(x,1)}\exp\left[\frac{1}{2}(\xi^2-1)\right]\delta(\xi - x)\\
u(x,\xi) - u(x,1) &= -\int_{\eta=1}^{\xi}\frac{\eta}{v(x,1)}\exp\left[\frac{1}{2}(\eta^2-1)\right]\delta(\eta - x)\,\mathrm{d}\eta\\
u(x,\xi) &= u(x,1) - \frac{x}{v(x,1)}\exp\left[\frac{1}{2}(x^2-1)\right]H(\xi - x).
\end{aligned}
$$

Continue a solução no verso ⟹

Agora,

$$
\begin{aligned}
G(x,\xi) &= u(x,\xi)v(x,\xi) \\
&= \left\{u(x,1) - \frac{x}{v(x,1)}\exp\left[\frac{1}{2}(x^2-1)\right]H(\xi-x)\right\}\left\{\frac{v(x,1)}{\xi}\exp\left[\frac{1}{2}(1-\xi^2)\right]\right\} \\
&= \frac{1}{\xi}\exp\left[\frac{1}{2}(1-\xi^2)\right]\left\{G(x,1) - x\exp\left[\frac{1}{2}(x^2-1)\right]H(\xi-x)\right\}.
\end{aligned}
$$

Para que $G(x,\infty) = 0$, devemos ter

$$
G(x,1) = x\exp\left[\frac{1}{2}(x^2-1)\right].
$$

Portanto,

$$
\begin{aligned}
G(x,\xi) &= \frac{1}{\xi}\exp\left[\frac{1}{2}(1-\xi^2)\right]x\exp\left[\frac{1}{2}(x^2-1)\right](1-H(\xi-x)) \\
&= \frac{x}{\xi}\exp\left[\frac{1}{2}(x^2-\xi^2)\right][1-H(\xi-x)] \;\blacksquare
\end{aligned}
$$

Continue a solução no verso ⟹