Como denotar uma dimensão "à mão livre":

$$\llbracket T \rrbracket = \mathbb{T},$$
$$\llbracket L \rrbracket = \mathbb{L},$$
$$\llbracket M \rrbracket = \mathbb{M}$$

$$1 = \mathsf{M}^{1+c}\mathsf{L}^{1+a+b-3c}\mathsf{T}^{-2-b},$$
$$0 = 1 + c,$$
$$0 = 1 + a + b - 3c,$$
$$0 = -2 - b$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Um parâmetro adimensional elevado a qualquer potência continua a ser (é óbvio) adimensional:

$$\left\llbracket \frac{F}{\rho U^2 D^2} \right\rrbracket = 1,$$
$$\left\llbracket \left[\frac{F}{\rho U^2 D^2}\right]^p \right\rrbracket = 1,$$
$$\left\llbracket \frac{\nu}{UD} \right\rrbracket = 1,$$
$$\left\llbracket \left[\frac{\nu}{UD}\right]^p \right\rrbracket = 1,$$
$$\left\llbracket \frac{UD}{\nu} \right\rrbracket = 1,$$

Refaço agora o Exemplo 1.1 escolhendo como variáveis comuns a todos os parâmetros (grupos) adimensionais ρ, ν, e D.

$$\Pi_1 = F^1 \rho^a \nu^b D^c,$$
$$\llbracket \Pi_1 \rrbracket = \mathsf{M}\mathsf{L}\mathsf{T}^{-2} \left[\mathsf{M}\mathsf{L}^{-3}\right]^a \left[\mathsf{L}^2\mathsf{T}^{-1}\right]^b [\mathsf{L}]^c$$
$$1 = \mathsf{M}^{1+a}\mathsf{L}^{1-3a+2b+c}\mathsf{T}^{-2-b}$$

Isso produz o sistema de equações lineares

$$a = -1,$$
$$-3a + 2b + c = -1,$$
$$-b = 2$$

1

Com solução
$$a = -1, \qquad b = -2, c = 0.$$

$$\Pi_1 = \frac{F}{\rho \nu^2},$$

$$[\![\Pi_1]\!] = \frac{\mathsf{MLT^{-2}}}{\mathsf{ML^{-3}(L^2T^{-1})^2}} = \frac{\mathsf{MLT^{-2}}}{\mathsf{ML^{-3}L^4T^{-2}}} = 1 \;\blacksquare$$

Procuramos agora o Π_2:

$$\Pi_2 = U^1 \rho^a \nu^b D^c,$$

$$[\![\Pi_1]\!] = \mathsf{LT^{-1}} \left[\mathsf{ML^{-3}}\right]^a \left[\mathsf{L^2T^{-1}}\right]^b [\mathsf{L}]^c$$

$$1 = \mathsf{M}^a \mathsf{L}^{1-3a+2b+c} \mathsf{T}^{-1-b}$$

Isso produz o sistema de equações lineares

$$a = 0,$$
$$-3a + 2b + c = -1,$$
$$-b = 1.$$

A solução é
$$a = 0, \qquad b = -1, \qquad c = 1.$$

Isso produz o segundo grupo adimensional

$$\Pi_2 = \frac{UD}{\nu},$$

que é o número de Reynolds! Antes eu tinha, agora eu tenho:

$$\Pi_1 = \frac{F}{\rho U^2 D^2}, \qquad\qquad \Pi_1' = \frac{F}{\rho \nu^2},$$

$$\Pi_2 = \frac{\nu}{UD}, \qquad\qquad \Pi_2' = \frac{UD}{\nu}$$

Faço

$$\Pi_1 \times \Pi_2^{-2} = \frac{F}{\rho U^2 D^2} \times \left(\frac{UD}{\nu}\right)^2 = \frac{F}{\rho \nu^2} = \Pi_1',$$

$$\Pi_2^{-1} = \Pi_2'.$$

Extensões naturais de "dimensões fundamentais" para problemas específicos:

1. Quando há espécies químicas diferentes *sem* reações químicas entre elas: M é a massa total, M_v é a massa de vapor d'água, M_c é a massa de CO_2, etc..

2. Quando há transferência de *calor sem* que haja conversão significativa de energia mecânica em energia interna. Neste caso, além de energia mecânica, $\mathsf{ML^2T^2}$, "calor" deve se expresso em termos de uma dimensão extra correspondente, em geral temperatura Θ.

3. Quando existem diferenças grandes do tamanho de grandezas características em direções diferentes: em lugar de um único L, muitas vezes é proveitoso utilizar uma dimensão física para cada dimensão do espaço, ou seja: X, Y, Z.

Eu gostaria de ter um método numérico que funcionasse para "qualquer" equação diferencial ordinária (EDO) do tipo

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \operatorname{sen}(x),$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} + \operatorname{sen}(x).$$

O que significam os k's

$$y_{n+1} \approx y_n + h\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_n$$
$$k_1 = h\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_n = hf(x_n, y_n)$$
$$y_{n+1} = y_n + k_1$$

Um sistema de equações diferenciais ordinárias já colocado na forma "clássica" do método de Runge-Kutta é algo do tipo: um conjunto de incógnitas $y_1, y_2, \ldots, y_n$ que evoluem com uma dinâmica acoplada:

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = f_1(x, y_1, \ldots, y_n),$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = f_2(x, y_1, \ldots, y_n),$$
$$\vdots$$
$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = f_n(x, y_1, \ldots, y_n).$$

Vetores, convenções, etc.

$$\boldsymbol{i} = (1, 0, 0),$$
$$\boldsymbol{j} = (0, 1, 0),$$
$$\boldsymbol{k} = (0, 0, 1).$$

$$\boldsymbol{u} = (1, 2, 3),$$
$$\boldsymbol{u} = 1\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}.$$

$$c = (d + e)$$
$$c^2 = (d + e)^2$$
$$c^2 = d^2 + e^2 + 2de \ \blacksquare$$

Notação indicial ou de Einstein

$$y_i = A_{ij} x_j$$

quer dizer

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \qquad \forall i = 1, \dots n.$$

$$\begin{aligned} \delta_{ii} &= \sum_{i=1}^{3} \delta_{(i)(i)} \\ &= \delta_{11} + \delta_{22} + \delta_{33} \\ &= 1 + 1 + 1 \\ &= 3. \end{aligned}$$

$$\begin{aligned} [\boldsymbol{\delta}][\boldsymbol{x}] &= [\delta_{ij}][x_j] \\ &= \delta_{ij} x_j = x_i \\ &= [x_i] = [\boldsymbol{x}]. \end{aligned}$$

No caso unidimensional,

$$[\boldsymbol{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\boldsymbol{x}]^\top = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

O significado de $[\boldsymbol{y}]^\top [\boldsymbol{\delta}]$ é

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

Em notação indicial:

$$y_i \delta_{ij} = y_j$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

O que são dois vetores ortogonais no $\mathbb{R}^3$?

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos \left(\frac{\pi}{2} \right) = 0.$$

No $\mathbb{R}^n$, nós generalizamos sem poder "ver" os vetores:

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

Uma base ortogonal é uma base $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ tal que, se $i \neq j$,

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0.$$

Se o índice for repetido, obviamente,

$$\boldsymbol{v}_{(i)} \cdot \boldsymbol{v}_{(i)} = \left|\boldsymbol{v}_{(i)}\right|^2.$$

Uma base *ortogonal* é tudo isso aí em cima somado a

$$\boldsymbol{v}_{(i)} \cdot \boldsymbol{v}_{(i)} = 1.$$

Dada uma base ortonormal E,

$$\boldsymbol{u} = u_{Ei}\boldsymbol{e}_i,$$
$$\boldsymbol{v} = v_{Ej}\boldsymbol{e}_j.$$

onde

$$\boldsymbol{u} = \begin{bmatrix} u_{E1} \\ u_{E2} \\ \vdots \\ u_{En} \end{bmatrix}_E$$

Lembre-se! $u = (u_1, u_2, \ldots, u_n)$, e em geral $u_1 \neq u_{E1}$, $u_2 \neq u_{E2}$, etc.

É possível usar as coordendas u_{Ei} e v_{Ej} para calcular o produto escalar $\boldsymbol{u} \cdot \boldsymbol{j}$? Sim:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}\boldsymbol{e}_i \cdot v_{Ei}\boldsymbol{e}_i$$

Paro, e recomeço:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}\boldsymbol{e}_i \cdot v_{Ej}\boldsymbol{e}_j$$
$$\alpha\boldsymbol{x} \cdot \beta\boldsymbol{y} = \alpha\beta(\boldsymbol{x} \cdot \boldsymbol{y});$$
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}v_{Ej}(\boldsymbol{e}_i \cdot \boldsymbol{e}_j)$$
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}v_{Ej}\delta_{ij}$$
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}[v_{Ej}\delta_{ij}]$$
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}v_{Ei} = \sum_{i=1}^{n} u_{Ei}v_{Ei};$$
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i = \sum_{i=1}^{n} u_i v_i.$$

Às vezes escrever $\boldsymbol{u} = u_{Ei}\boldsymbol{e}_i$ é demais. Quando não houver motivo de confusão, nós simplicaremos a notação e escreveremos

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_E$$

ou simplesmente $\boldsymbol{u} = u_i\boldsymbol{e}_i$, etc.

Reforçando: elemento de vetor × coordenada de vetor.

$$\boldsymbol{v} = (3, 4)$$

Se eu escolher a base $((1,1), (1,-1))$ para representar o $\mathbb{R}^2$, então

$$\boldsymbol{v} = \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}_{((1,1),(1,-1))}$$

$$\begin{aligned}
(3,4) &= 7/2(1,1) - 1/2(1,-1) \\
&= (7/2, 7/2) + (-1/2, 1/2) \\
&= (6/2, 8/2) \\
&= (3,4) \ \blacksquare
\end{aligned}$$

Bases!

$A = (\boldsymbol{i}, \boldsymbol{j})$, $B = (\boldsymbol{j}, \boldsymbol{i})$.

O símbolo de permutação de Levi-Civita é calculado sempre com n entradas. Exemplos:

$$\begin{aligned}
\epsilon_{1234} &= +1, \\
\epsilon_{1243} &= -1, \\
\epsilon_{1223} &= 0.
\end{aligned}$$

Permutações cíclicas (apenas com 3 elementos) de 1, 2, 3:

$$\begin{aligned}
&1, 2, 3, \\
&3, 1, 2, \\
&2, 3, 1.
\end{aligned}$$

Permutações cíclicas (apenas com 3 elementos) de 3, 2, 1:

$$\begin{aligned}
&3, 2, 1, \\
&1, 3, 2, \\
&2, 1, 3.
\end{aligned}$$

Para definir o produto vetorial eu parto dos "absolutos"

$$\begin{aligned}
\boldsymbol{u} &= (u_1, u_2, u_3), \\
\boldsymbol{v} &= (v_1, v_2, v_3), \\
\boldsymbol{u} \times \boldsymbol{v} &\equiv (\epsilon_{ij1} u_i v_j, \epsilon_{ij2} u_i v_j, \epsilon_{ij3} u_i v_j).
\end{aligned}$$

A forma "antiga" de calcular o produto vetorial é

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= (u_2 v_3 - u_3 v_2)\boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3)\boldsymbol{e}_2 + (u_1 v_2 - u_2 v_1)\boldsymbol{e}_3$$

A forma "nova" de calcular o produto vetorial é

$$\boldsymbol{u} \times \boldsymbol{v} = \epsilon_{ijk} u_i v_j \boldsymbol{e}_k$$
$$= (\epsilon_{231} u_2 v_3 + \epsilon_{321} u_3 v_2)\boldsymbol{e_1} + (\epsilon_{132} u_1 v_3 + \epsilon_{312} u_3 v_1)\boldsymbol{e_2} + (\epsilon_{123} u_1 v_2 + \epsilon_{213} u_2 v_1)\boldsymbol{e_3}$$
$$= (u_2 v_3 - u_3 v_2)\boldsymbol{e_1} + (u_3 v_1 - u_1 v_3)\boldsymbol{e_2} + (u_1 v_2 - u_2 v_1)\boldsymbol{e_3}$$

Vamos provar agora que $\boldsymbol{u} \times \boldsymbol{v}$ é perpendicular a $\boldsymbol{u}$.

$$[\boldsymbol{u} \times \boldsymbol{v}] \cdot \boldsymbol{u} = \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \cdot u_l \boldsymbol{e}_l$$
$$\alpha \boldsymbol{a} \cdot \beta \boldsymbol{b} = (\alpha\beta)(\boldsymbol{a} \cdot \boldsymbol{b})$$
$$= \epsilon_{ijk} u_i v_j u_l (\boldsymbol{e}_k \cdot \boldsymbol{e}_l)$$
$$= \epsilon_{ijk} u_i v_j u_l \delta_{kl}$$
$$= \epsilon_{ijk} u_i v_j u_k$$
$$= \frac{1}{2} \left[\epsilon_{ijk} u_i v_j u_k + \epsilon_{ijk} u_i v_j u_k \right]$$
$$= \frac{1}{2} \left[\epsilon_{ijk} u_i v_j u_k + \epsilon_{kji} u_k v_j u_i \right]$$
$$= \frac{1}{2} \left[\epsilon_{ijk} + \epsilon_{kji} \right] u_i v_j u_k$$
$$= \frac{1}{2} \left[\epsilon_{ijk} + \epsilon_{jik} \right] u_i v_j u_k$$
$$= 0.$$

O produto vetorial é anti-simétrico:

$$\boldsymbol{v} \times \boldsymbol{u} = \epsilon_{ijk} v_i u_j \boldsymbol{e}_k$$
$$= \epsilon_{ijk} u_j v_i \boldsymbol{e}_k$$
$$= -\epsilon_{jik} u_j v_i \boldsymbol{e}_k$$
$$= -\boldsymbol{u} \times \boldsymbol{v} \ \blacksquare$$

Em seguida:

$$|\boldsymbol{u} \times \boldsymbol{v}|^2 = [\boldsymbol{u} \times \boldsymbol{v}] \cdot [\boldsymbol{u} \times \boldsymbol{v}]$$

$$= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \cdot \epsilon_{lmn} u_l v_m \boldsymbol{e}_n$$

$$\alpha \boldsymbol{a} \cdot \beta \boldsymbol{b} = \alpha\beta(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m (\boldsymbol{e}_k \cdot \boldsymbol{e}_n)$$

$$= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m \delta_{kn}$$

$$= \epsilon_{ijk} u_i v_j (\epsilon_{lmn} \delta_{kn}) u_l v_m$$

$$= \epsilon_{ijk} u_i v_j \epsilon_{lmk} u_l v_m$$

$$= [\epsilon_{ijk} \epsilon_{lmk}] u_i v_j u_l v_m$$

$$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}] u_i v_j u_l v_m$$

$$= \delta_{il}\delta_{jm} u_i v_j u_l v_m - \delta_{im}\delta_{jl} u_i v_j u_l v_m$$

$$= u_i v_j u_i v_j - u_i v_j u_j v_i$$

$$= (u_i u_i)(v_j v_j) - (u_i v_i)(u_j v_j)$$

$$= (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})(\boldsymbol{u} \cdot \boldsymbol{v})$$

$$= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 - (|\boldsymbol{u}||\boldsymbol{v}| \cos(\theta))(|\boldsymbol{u}||\boldsymbol{v}| \cos(\theta))$$

$$= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \left(1 - \cos^2(\theta)\right)$$

$$= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \operatorname{sen}^2(\theta); \; \Rightarrow$$

$$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \operatorname{sen}(\theta) \; \blacksquare$$

1. Um "1-volume" é um comprimento.

2. Um "2-volume" é uma área.

3. Um "3-volume" é um volume usual.

4. Um "4-volume" é um "volume quadridimensional", ou um hipervolume.

Determinantes e hipervolumes.

Considere n vetores no $\mathbb{R}^n$:

$$\boldsymbol{u}_1 = (u_{11}, u_{21}, \ldots, u_{n1});$$
$$\boldsymbol{u}_2 = (u_{12}, u_{22}, \ldots, u_{n2});$$
$$\vdots$$
$$\boldsymbol{u}_n = (u_{1n}, u_{2n}, \ldots, u_{nn}).$$

O determinante associado a esses n vetores é definido como

$$V^n \equiv \det(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_n)$$
$$\equiv \epsilon_{i_1, i_2, \ldots, i_n} u_{1i_1} u_{2i_2} \ldots u_{ni_n}.$$

O determinante está associado à matriz

$$[\boldsymbol{u}] = [u_{ij}] = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{bmatrix}$$

Suponha que uma função f é linear:

$$f(\alpha x) = \alpha f(x),$$
$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Desejamos provar que, então,

$$f(x) = cx.$$

Prova:

$$\frac{\mathrm{d}f(\alpha x)}{\mathrm{d}x} = \alpha f'(\alpha x),$$

mas f é linear:

$$f(\alpha x) = \alpha f(x)$$
$$\alpha f'(\alpha x) = \alpha f'(x).$$

Portanto, no caso de f ser linear, teremos que

$$f'(\alpha x) = f'(x), \qquad \forall \alpha \in \mathbb{R},$$

ou seja: a derivada é constante!

$$f'(x) = c,$$
$$f(x) = cx + b,$$
$$f(0) = 0 \Rightarrow b = 0 \blacksquare$$

Transformações lineares. Pergunta: será possível definir uma operação entre $\boldsymbol{A}$ e $\boldsymbol{x}$ que nos permita substituir

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{x})$$

por

$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}?$$

$$
\begin{aligned}
A_i \delta_{im} &= \sum_{i=1}^{3} A_i \delta_{im} \\
&= A_1 \delta_{1m} + A_2 \delta_{2m} + A_3 \delta_{3m} \\
&= A_m \\
A_{ijkl} B_{ijkp} \delta_{ln} &= A_{ijkn} B_{ijkp}
\end{aligned}
$$

Transformações lineares

$$E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n);$$

$$\boldsymbol{x} = \sum_{j=1}^{3} x_j \boldsymbol{e}_j = x_j \boldsymbol{e}_j$$

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}(x_j \boldsymbol{e}_j)$$

$$= x_j \boldsymbol{A}(\boldsymbol{e}_j)$$

mas

$$\boldsymbol{A}(\boldsymbol{e}_j) \in \mathbb{V};$$

$$\boldsymbol{A}(\boldsymbol{e}_j) = A_{ij} \boldsymbol{e}_i$$

portanto

$$\boldsymbol{A}(\boldsymbol{x}) = x_j A_{ij} \boldsymbol{e}_i$$

$$= A_{ij} x_j \boldsymbol{e}_i$$

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{y};$$

$$\boldsymbol{y} = y_i \boldsymbol{e}_i = A_{ij} x_j \boldsymbol{e}_i :$$

$$y_i = A_{ij} x_j,$$

$$[\boldsymbol{y}]_E = [\boldsymbol{A}]_{E,E} [\boldsymbol{x}]_E.$$

Agora, eu mudo um pouco o jogo: a base do domínio continua a ser $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$; e a base do contra-domínio é $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n)$. Agora,

$$\boldsymbol{A}(\boldsymbol{x}) = x_j \boldsymbol{A}(\boldsymbol{e}_j)$$

$$\boldsymbol{A}(\boldsymbol{e}_j) = A_{ij} \boldsymbol{f}_i$$

$$\boldsymbol{A}(\boldsymbol{x}) = A_{ij} x_j \boldsymbol{f}_i$$

$$\boldsymbol{y} = y_i \boldsymbol{f}_i$$

$$y_i = A_{ij} x_j$$

$$[\boldsymbol{y}]_F = [\boldsymbol{A}]_{F,E} [\boldsymbol{x}]_E$$

Vamos agora nos lembrar de que um funcional linear em sua forma mais geral é dado por

$$f(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = c_j x_j = \boldsymbol{c} \cdot \boldsymbol{x}.$$

Mas

$$\boldsymbol{A}(\boldsymbol{x}) = A_{ij}x_j\boldsymbol{f}_i,$$
$$\boldsymbol{A}_i = \boldsymbol{c};$$
$$\boldsymbol{A}_i = A_{ij}\boldsymbol{e}_j$$
$$\boldsymbol{A}(\boldsymbol{x}) = [\boldsymbol{c}\cdot\boldsymbol{x}]_i\boldsymbol{f}_i$$
$$= [A_{ij}\boldsymbol{e}_j\cdot\boldsymbol{x}]\boldsymbol{f}_i$$
$$= [A_{ij}\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k]\boldsymbol{f}_i$$
$$= [A_{ij}x_k\boldsymbol{e}_j\cdot\boldsymbol{e}_k]\boldsymbol{f}_i$$
$$= [A_{ij}x_k\delta_{jk}]\boldsymbol{f}_i$$
$$= [A_{ij}x_j]\boldsymbol{f}_i$$
$$= [A_{ij}\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k]\boldsymbol{f}_i$$
$$= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k$$
$$= [A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j]\cdot[x_k\boldsymbol{e}_k]$$
$$\boldsymbol{y} = \boldsymbol{A}\cdot\boldsymbol{x}$$
$$\boldsymbol{A} = A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j$$

Uma transformação de $\mathbb{R}^3$ em $\mathbb{R}^2$ é representada matricialmente por

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}_{2\times1} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\end{bmatrix}_{2\times3}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}_{3\times1}$$

Bases!

$$E = (\boldsymbol{e}_1,\boldsymbol{e}_2) = ((1,0),(0,1));$$
$$\boldsymbol{v} = (3,4)$$
$$\boldsymbol{v} = 3\boldsymbol{e}_1 + 4\boldsymbol{e}_2$$
$$[\boldsymbol{v}]_E = \begin{bmatrix}3\\4\end{bmatrix}$$
$$F = (\boldsymbol{f}_1,\boldsymbol{f}_2) = ((1,1),(1,-1))$$
$$\boldsymbol{v} = (7/2)\boldsymbol{f}_1 - (1/2)\boldsymbol{f}_2$$
$$[\boldsymbol{v}]_F = \begin{bmatrix}7/2\\-1/2\end{bmatrix}$$

Às vezes é preciso "disciplinar" a notação indicial:

$$\sum_{i=1}^{3}C_{ij}^2 = \sum_{i=1}^{3}C_{ij}C_{ij} \equiv C_{i(j)}C_{i(j)} = 1$$

Notação preferida para transformações lineares:

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{x}),$$
$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}$$
$$\boldsymbol{y} = y_i \boldsymbol{f}_i,$$
$$\boldsymbol{x} = x_j \boldsymbol{e}_j,$$
$$\boldsymbol{A} = A_{ij} \boldsymbol{f}_i \boldsymbol{e}_j$$
$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}$$
$$= A_{ij} \boldsymbol{f}_i \boldsymbol{e}_j \cdot x_k \boldsymbol{e}_k$$
$$= A_{ij} x_k \boldsymbol{f}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_k)$$
$$= A_{ij} x_k \boldsymbol{f}_i \delta_{jk}$$
$$y_i \boldsymbol{f}_i = A_{ij} x_j \boldsymbol{f}_i$$

Rotações e suas matrizes/transformações

$$\boldsymbol{u} = u'_j \boldsymbol{e}'_j = u_i \boldsymbol{e}_i$$
$$u_i = (\boldsymbol{u} \cdot \boldsymbol{e}_i)$$

mas

$$\boldsymbol{e}_i = C_{ij} \boldsymbol{e}'_j$$
$$u_i = (\boldsymbol{u} \cdot C_{ij} \boldsymbol{e}'_j)$$
$$= C_{ij}(\boldsymbol{u} \cdot \boldsymbol{e}'_j)$$
$$u_i = C_{ij} u'_j$$
$$[\boldsymbol{u}]_E = [\boldsymbol{C}][\boldsymbol{u}]_{E'}$$
$$[\boldsymbol{u}]'_E = [\boldsymbol{C}]^\top [\boldsymbol{u}]_E$$

Podemos mostrar que $[\boldsymbol{C}]$ e $[\boldsymbol{C}]^\top$ são inversas operando diretamente:

$$\boldsymbol{u} = u_k \boldsymbol{e}_k = u_i \boldsymbol{e}_i = u_i C_{ij} \boldsymbol{e}'_j = u_i C_{ij} C_{kj} \boldsymbol{e}_k$$
$$u_k = u_i C_{ij} C_{kj} = C_{ij} C^\top_{jk} u_i$$
$$u_k = \delta_{ki} u_i,$$
$$C_{ij} C^\top_{jk} = \delta_{ki}$$
$$[\boldsymbol{C}][\boldsymbol{C}]^\top = [\boldsymbol{\delta}].$$

O determinante de uma rotação é igual a 1:

$$1 = [\boldsymbol{e}_1 \times \boldsymbol{e}_2] \cdot \boldsymbol{e}_3$$
$$= [C_{1l} \boldsymbol{e}'_l \times C_{2m} \boldsymbol{e}'_m] \cdot C_{3n} \boldsymbol{e}'_n$$
$$= C_{1l} C_{2m} C_{3n} \underbrace{[\boldsymbol{e}'_l \times \boldsymbol{e}'_m] \cdot \boldsymbol{e}'_n}_{\epsilon_{lmn}}$$
$$1 = \epsilon_{lmn} C_{1l} C_{2m} C_{3n} = \det[\boldsymbol{C}].$$

13

Qual é a regra para a mudança das matrizes de uma transformação $\boldsymbol{A}$ quando eu mudo da base E para a base E' (ou vice-versa)?

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j,$$
$$\boldsymbol{A} = A'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j,$$
$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{kl}\boldsymbol{e}'_k\boldsymbol{e}'_l$$
$$A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{kl}C_{ik}\boldsymbol{e}_iC_{jl}\boldsymbol{e}_j$$
$$A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{kl}C_{ik}C_{jl}\boldsymbol{e}_i\boldsymbol{e}_j$$
$$A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = C_{ik}A'_{kl}C^{\top}_{lj}\boldsymbol{e}_i\boldsymbol{e}_j$$
$$[\boldsymbol{A}]_E = [\boldsymbol{C}][\boldsymbol{A}]_{E'}[\boldsymbol{C}]^{\top},$$
$$[\boldsymbol{A}]_{E'} = [\boldsymbol{C}]^{\top}[\boldsymbol{A}]_E[\boldsymbol{C}].$$

Exemplo:

Parto da base canônica $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ e desejo construir uma outra base ortonormal dextrógira $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Os dois primeiros vetores são

$$\boldsymbol{f}_1 = \frac{1}{\sqrt{3}}(1, 1, 1),$$
$$\boldsymbol{f}_2 = \frac{1}{\sqrt{6}}(2, -1, -1).$$

Obviamente

$$\boldsymbol{f}_3 = \boldsymbol{f}_1 \times \boldsymbol{f}_2.$$

$$\boldsymbol{f}_3 = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$
$$= \frac{1}{\sqrt{18}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$
$$= \frac{1}{3\sqrt{3}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$
$$= \frac{1}{3\sqrt{2}}[0\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 3\boldsymbol{e}_3]$$
$$= \frac{1}{\sqrt{2}}(0, 1, -1).$$

A matriz de rotaçao de E para F é

$$\boldsymbol{f}_j = C_{ij}\boldsymbol{e}_i,$$
$$C_{ij} = \boldsymbol{f}_j \cdot \boldsymbol{e}_i.$$

Portanto,

$$[\boldsymbol{C}] = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$$

Se eu quero rodar em torno do eixo 2, devo rodar de 1 para 3 ou de 3 para 1?

$$\epsilon_{132} = \epsilon_{321} = -1,$$
$$\epsilon_{312} = \epsilon_{123} = +1.$$

Resposta: de 3 para 1!

Um sistema de equações lineares é algo da forma

$$A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n = y_1,$$
$$A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n = y_2,$$
$$\vdots$$
$$A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{nn}x_n = y_n.$$

A transformação

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

projeta qq vetor do $\mathbb{R}^3$ no plano que passa pela origem e cujo normal é $\boldsymbol{k}$. Note que a equação desse plano é

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0.$$

$$\boldsymbol{b} = \boldsymbol{P} \cdot \boldsymbol{a},$$
$$b_i = \left[\delta_{ij} - \frac{k_i k_j}{k^2}\right] a_j$$
$$b_i = \delta_{ij} a_j - \frac{k_i k_j}{k^2} a_j$$
$$= a_i - \frac{k_i k_j}{k^2} a_j;$$
$$\boldsymbol{b} \cdot \boldsymbol{k} = b_i k_i$$
$$= \left[a_i - \frac{k_i k_j}{k^2} a_j\right] k_i$$
$$= a_i k_i - \left(\frac{k_i k_i}{k^2}\right) k_j a_j$$
$$= a_i k_i - a_j k_j = 0 \blacksquare$$

Teorema dos Pis:

$$\left(M^A\right)^x = M^{Ax}.$$
$$1 = \prod_{j=1}^{n} \left(\mathsf{M}^{A_{ij}}\right)_j^x \times \ldots$$

Vamos voltar!

$$\Pi = \prod_{j=1}^{n} v_j^{x_j},$$

$$\llbracket \Pi \rrbracket = \left\llbracket \prod_{j=1}^{n} v_j^{x_j} \right\rrbracket$$

$$\llbracket \Pi \rrbracket = \prod_{j=1}^{n} \llbracket v_j^{x_j} \rrbracket$$

$$\llbracket \Pi \rrbracket = \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}} \mathsf{L}^{A_{2j}} \mathsf{T}^{A_{3j}} \right)^{x_j}$$

$$\llbracket \Pi \rrbracket = \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}} \right)^{x_j} \left(\mathsf{L}^{A_{2j}} \right)^{x_j} \left(\mathsf{T}^{A_{3j}} \right)^{x_j}$$

$$\llbracket \Pi \rrbracket = \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}} \right)^{x_j} \prod_{j=1}^{n} \left(\mathsf{L}^{A_{2j}} \right)^{x_j} \prod_{j=1}^{n} \left(\mathsf{T}^{A_{3j}} \right)^{x_j}$$

Se $n = 4$, teremos por exemplo

$$\prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}} \right)^{x_j} = \left(\mathsf{M}^{A_{11}} \right)^{x_1} \times \left(\mathsf{M}^{A_{12}} \right)^{x_2} \times \left(\mathsf{M}^{A_{13}} \right)^{x_3} \times \left(\mathsf{M}^{A_{14}} \right)^{x_4} = 1$$

$$\mathsf{M}^{A_{11}x_1} \times \mathsf{M}^{A_{12}x_2} \times \mathsf{M}^{A_{13}x_3} \times \mathsf{M}^{A_{14}x_4} = 1$$

$$\mathsf{M}^{A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4} = 1$$

$$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 = 0$$

Autovalores e autovetores

Se $\boldsymbol{v}$ é autovetor de $\boldsymbol{A}$, então $\alpha\boldsymbol{v}$, $\alpha \neq 0$, também é:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda\boldsymbol{v};$$
$$\boldsymbol{A} \cdot [\alpha\boldsymbol{v}] = \alpha\boldsymbol{A} \cdot \boldsymbol{v}$$
$$= \alpha\lambda\boldsymbol{v}$$
$$= \lambda[\alpha\boldsymbol{v}].$$

No exemplo em tela, $\lambda = 1$ possui multiplicidade 2 e também está associado a um subespaço de dimensão 2. A equação

$$x_1 + 2x_2 + x_3 = 0$$

é a equação de um plano (pela origem), e quaisquer dois vetores desse plano são autovetores. De fato, sejam $\boldsymbol{v}_1, \boldsymbol{v_2}$ dois autovetores LI associados a λ; então

$$\boldsymbol{A} \cdot [\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2] = \alpha_1\boldsymbol{A} \cdot \boldsymbol{v}_1 + \alpha_2\boldsymbol{A} \cdot \boldsymbol{v}_2$$
$$= \alpha_1\lambda\boldsymbol{v}_1 + \alpha_2\lambda\boldsymbol{v}_2$$
$$= \lambda[\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2].$$

No caso univariado,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x,$$
$$x = \mathrm{e}^{4t},$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4\mathrm{e}^{4t} = 4x.$$

Transposta de uma transformação linear

$$\boldsymbol{x} \cdot [\boldsymbol{A}^\top \cdot \boldsymbol{y}] \equiv \boldsymbol{y} \cdot [\boldsymbol{A} \cdot \boldsymbol{x}]$$
$$x_k\boldsymbol{e}_k \cdot [A_{ij}^\top\boldsymbol{e}_i\boldsymbol{e}_j \cdot y_l\boldsymbol{e}_l] = y_k\boldsymbol{e}_k \cdot [A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j \cdot x_l\boldsymbol{e}_l]$$
$$x_k\boldsymbol{e}_k \cdot [A_{ij}^\top\boldsymbol{e}_i(\boldsymbol{e}_j \cdot \boldsymbol{e}_l)y_l] = y_k\boldsymbol{e}_k \cdot [A_{ij}\boldsymbol{e}_i(\boldsymbol{e}_j \cdot \boldsymbol{e}_l)x_l]$$
$$x_k\boldsymbol{e}_k \cdot [A_{ij}^\top\boldsymbol{e}_i\delta_{jl}y_l] = y_k\boldsymbol{e}_k \cdot [A_{ij}\boldsymbol{e}_i\delta_{jl}x_l]$$
$$x_k\boldsymbol{e}_k \cdot [A_{ij}^\top y_j\boldsymbol{e}_i] = y_k\boldsymbol{e}_k \cdot [A_{ij}x_j\boldsymbol{e}_i]$$
$$x_k A_{ij}^\top y_j(\boldsymbol{e}_k \cdot \boldsymbol{e}_i) = y_k A_{ij}x_j(\boldsymbol{e}_k \cdot \boldsymbol{e}_i)$$
$$x_k A_{ij}^\top y_j\delta_{ki} = y_k A_{ij}x_j\delta_{ki}$$
$$x_i A_{ij}^\top y_j = y_i A_{ij}x_j$$
$$x_i A_{ij}^\top y_j = y_j A_{ji}x_i$$
$$x_i A_{ij}^\top y_j = x_i A_{ji}y_j$$
$$A_{ij}^\top = A_{ji} \blacksquare$$

Até agora nós vimos (e demos significado a)

$$\boldsymbol{A} \cdot \boldsymbol{x} = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j \cdot x_k\boldsymbol{e}_k$$
$$= A_{ij}x_k\boldsymbol{e}_i(\boldsymbol{e}_j \cdot \boldsymbol{e}_k)$$
$$= A_{ij}x_k\boldsymbol{e}_i\delta_{jk}$$
$$= A_{ij}x_j\boldsymbol{e}_i$$
$$= [\boldsymbol{A}][\boldsymbol{x}]$$

Eu também posso definir a operação

$$
\begin{aligned}
\boldsymbol{x} \cdot \boldsymbol{A} &= x_k \boldsymbol{e}_k \cdot A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \\
&= x_k A_{ij} (\boldsymbol{e}_k \cdot \boldsymbol{e}_i) \boldsymbol{e}_j \\
&= x_k A_{ij} \delta_{ki} \boldsymbol{e}_j \\
&= x_i A_{ij} \boldsymbol{e}_j \\
&= [x_1 A_{11} + x_2 A_{21} + x_3 A_{31}] \boldsymbol{e}_1 \\
&+ [x_1 A_{12} + x_2 A_{22} + x_3 A_{32}] \boldsymbol{e}_2 \\
&+ [x_1 A_{13} + x_2 A_{23} + x_3 A_{33}] \boldsymbol{e}_3 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \\
&= [\boldsymbol{x}]^\top [\boldsymbol{A}]
\end{aligned}
$$

Portanto eu posso definir a transformação simétrica via

$$
\boldsymbol{x} \cdot \boldsymbol{A}^\top \equiv \boldsymbol{A} \cdot \boldsymbol{x},
$$

$$
\vdots
$$

$$
A_{ij}^\top = A_{ji}.
$$

Na equação

$$
A_{ij} x_j = \lambda x_i
$$

Os A_{ij}s são reais. O conjugado da equação é

$$
\begin{aligned}
(A_{ij} x_j)^* &= (\lambda x_i)^* \\
A_{ij}^* x_j^* &= \lambda^* x_i^* \\
A_{ij} x_j^* &= \lambda^* x_i^*
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{e}_1 &= (1, 0, 0), \\
\boldsymbol{f}_1 &= \frac{1}{\sqrt{3}}(1, 1, 1), \\
\boldsymbol{e}_1 \cdot \boldsymbol{f}_1 &= \frac{1}{\sqrt{3}} =
\end{aligned}
$$

$$
\boldsymbol{e}_j \qquad j
$$

Quando existe uma base de autovetores para a transformação $\boldsymbol{A}$, a matriz de $\boldsymbol{A}$ nessa base é uma matriz diagonal composta pelos autovalores.

Olhe para o caso 3×3

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_F$$

Cada coluna representa a imagem de um dos vetores da base e **se** eu estiver na base dos autovetores:

$$\boldsymbol{A} \cdot \boldsymbol{f}_1 = A_{11}\boldsymbol{f}_1 + A_{21}\boldsymbol{f}_2 + A_{31}\boldsymbol{f}_3 = \lambda_1 \boldsymbol{f}_1$$
$$\boldsymbol{A} \cdot \boldsymbol{f}_2 = A_{12}\boldsymbol{f}_1 + A_{22}\boldsymbol{f}_2 + A_{32}\boldsymbol{f}_3 = \lambda_2 \boldsymbol{f}_2$$
$$\boldsymbol{A} \cdot \boldsymbol{f}_3 = A_{13}\boldsymbol{f}_1 + A_{23}\boldsymbol{f}_2 + A_{33}\boldsymbol{f}_3 = \lambda_3 \boldsymbol{f}_3$$

Portanto, concluímos que

$$\boldsymbol{A}_F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_F$$

Exercício 5.31

$$\begin{aligned} \boldsymbol{S} : \boldsymbol{A} &= S_{ij}A_{lm}(\boldsymbol{e}_j \cdot \boldsymbol{e}_l)(\boldsymbol{e}_i \cdot \boldsymbol{e}_m) \\ &= S_{ij}A_{lm}\delta_{jl}\delta_{im} \\ &= S_{ij}A_{ji} \\ &= \frac{1}{2}S_{ij}A_{ji} + \frac{1}{2}S_{ij}A_{ji} \\ &= \frac{1}{2}S_{ij}A_{ji} + \frac{1}{2}S_{ji}A_{ij} \\ &= \frac{1}{2}S_{ij}(A_{ji} + A_{ij}) = 0. \end{aligned}$$

Uma EDO de ordem 2 pode ser escrita como um sistema de duas EDOs acopladas de ordem 1.

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + u = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -u$$

$$
\begin{aligned}
u(t) &= \frac{A - \mathrm{i}B}{2}\left[\cos t + \mathrm{i}\operatorname{sen} t\right] + \frac{A + \mathrm{i}B}{2}\left[\cos t - \mathrm{i}\operatorname{sen} t\right] \\
&= \frac{A}{2}\left[\cos t + \mathrm{i}\operatorname{sen} t\right] - \frac{\mathrm{i}B}{2}\left[\cos t + \mathrm{i}\operatorname{sen} t\right] + \frac{A}{2}\left[\cos t - \mathrm{i}\operatorname{sen} t\right] + \frac{\mathrm{i}B}{2}\left[\cos t - \mathrm{i}\operatorname{sen} t\right] \\
&= \frac{A}{2}\cos t + \frac{A}{2}\cos t + \frac{-\mathrm{i}^2 B}{2}\operatorname{sen} t + \frac{-\mathrm{i}^2 B}{2}\operatorname{sen} t + \mathrm{i}\left[\frac{A}{2}\operatorname{sen} t - \frac{B}{2}\cos t - \frac{A}{2}\operatorname{sen} t + \frac{B}{2}\cos t\right] \\
&= A\cos t + B\operatorname{sen} t \ \blacksquare
\end{aligned}
$$

Ordem de diferenciação (ou de derivação):

$$\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Em geral,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Série de Taylor

$$f(x) = \frac{1}{0!}f(x_0) + \frac{1}{1!}\frac{\mathrm{d}f(x_0)}{\mathrm{d}x}(x - x_0) + \frac{1}{2!}\frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2}(x - x_0)^2 + \ldots + \frac{1}{n!}\frac{\mathrm{d}^n f(x_0)}{\mathrm{d}x^n}(x - x_0)^n + \ldots$$

No caso de $n = 2$, e de uma função (também) de 2 variáveis

$$
\begin{aligned}
\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_i \partial x_j}(x_i - x_{0i})(x_j - x_{0j}) &= \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_i \partial x_j}(x_i - x_{0i})(x_j - x_{0j}) \\
&= \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_1}(x_1 - x_{01})(x_1 - x_{01}) + \\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_2}(x_1 - x_{01})(x_2 - x_{02}) + \\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2 \partial x_1}(x_2 - x_{02})(x_1 - x_{01}) + \\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2 \partial x_2}(x_2 - x_{02})(x_2 - x_{02}).
\end{aligned}
$$

Verifique o número de índices repetidos em uma expressão

$$\boldsymbol{P} \cdot \boldsymbol{a} = P_{il}a_l\boldsymbol{e}_i$$

$$= \left[\delta_{il} - \frac{k_ik_l}{k^2}\right]a_l\boldsymbol{e}_i$$

$$[\boldsymbol{P} \cdot \boldsymbol{a}] \times \boldsymbol{k} = \epsilon_{ijk}\left[\delta_{il} - \frac{k_ik_l}{k^2}\right]a_lk_j\boldsymbol{e}_k$$

Teorema da função implícita

$$\frac{\mathrm{d}u(x_0)}{\mathrm{d}x} = -\frac{\frac{\partial f(x_0, u_0)}{\partial x}}{\frac{\partial f(x_0, u_0)}{\partial u}}$$

$$u(x) = u_0 + \frac{\mathrm{d}u(x_0)}{\mathrm{d}x}(x - x_0) + \dots$$

$$
\begin{aligned}
x &= x(u, v) & u &= u(x, y) \\
y &= y(u, v) & v &= v(x, y)
\end{aligned}
$$

A regra de Leibnitz:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x, t)\,\mathrm{d}x = \int_a^b \frac{\partial f(x, t)}{\partial t}\,\mathrm{d}x, \qquad \text{porém}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x \neq \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t}\,\mathrm{d}x.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t}\,\mathrm{d}x + f(b, t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a, t)\frac{\mathrm{d}a}{\mathrm{d}t};$$

$$\frac{\mathrm{D}N}{\mathrm{D}t} = \frac{\partial}{\partial t}\int_{VC} \eta\rho\,\mathrm{d}V + \oint_{SC} (\boldsymbol{n} \cdot \boldsymbol{v})\eta\rho\,\mathrm{d}S$$

A idéia é mudar a variável de integração de x para ξ:

$$x = X(\xi, t)$$

$$\mathrm{d}x = \frac{\partial X}{\partial \xi}\mathrm{d}\xi + \frac{\partial X}{\partial t}\mathrm{d}t,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta f(X(\xi, t), t)\frac{\partial X}{\partial \xi}\mathrm{d}\xi$$

Revisite a regra da cadeia!

$$f(x, u);$$

$$F(x) = f(x, u(x));$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$$

Quero o módulo de $\boldsymbol{v} = (1, \mathrm{i}, 0)$:

$$|\boldsymbol{v}| = \sqrt{\sum_{i=1}^{3} v_i^* v_i}$$
$$= \sqrt{(1 \times 1 + (-\mathrm{i} \times \mathrm{i}) + 0 \times 0)}$$
$$= \sqrt{1 - \mathrm{i}^2 + 0}$$
$$= \sqrt{1 + 1 + 0}$$
$$= \sqrt{2}$$

Regra de Leibnitz:
Considere

$$\frac{\partial}{\partial \xi} \frac{\partial X}{\partial t} = ?$$
$$\frac{\partial X}{\partial t} = h(\xi, t)$$
$$\frac{\partial}{\partial \xi} \frac{\partial X}{\partial t} = \frac{\partial h}{\partial \xi} = \frac{\partial h}{\partial x} \frac{\partial X}{\partial \xi}$$

Surgimento de uma "velocidade"

$$x = X(\xi, t),$$
$$u = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial X}{\partial t}$$

A regra de Leibnitz é

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)}^{b(t)} f(x, t) \, \mathrm{d}x = f(b, t) \frac{\mathrm{d}b}{\mathrm{d}t} - f(a, t) \frac{\mathrm{d}b}{\mathrm{d}t} + \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} \, \mathrm{d}x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(x, t) \, \mathrm{d}t = f(x, b) \frac{\mathrm{d}b}{\mathrm{d}x} - f(x, a) \frac{\mathrm{d}a}{\mathrm{d}x} + \int_{a(x)}^{b(x)} \frac{\partial f(x, t)}{\partial x} \, \mathrm{d}t$$
$$b(x) = x,$$
$$a(x) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \left[xt + \mathrm{e}^{-t^2} \right] \mathrm{d}t = x^2 + \mathrm{e}^{-x^2} + \int_0^x \frac{\partial}{\partial x} \left[xt + \mathrm{e}^{-t^2} \right] \mathrm{d}t$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \left[xt + \mathrm{e}^{-t^2} \right] \mathrm{d}t = x^2 + \mathrm{e}^{-x^2} + \int_0^x t \, \mathrm{d}t$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \left[xt + \mathrm{e}^{-t^2} \right] \mathrm{d}t = \frac{3x^2}{2} + \mathrm{e}^{-x^2}$$

Quero calcular um comprimento de arco sobre a curva (entre $t = 0$ e $t = \tau$)

$$\boldsymbol{r}(t) = \cos(t)[\boldsymbol{e}_x + \boldsymbol{e}_y] - \sqrt{2}\operatorname{sen}(t)\boldsymbol{e}_z$$

$$\ell_{\mathscr{L}} = \int_{\boldsymbol{r} \in \mathscr{L}} \left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]^{1/2} \mathrm{d}t$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -\operatorname{sen}(t)[\boldsymbol{e}_x + \boldsymbol{e}_y] - \sqrt{2}\cos(t)\boldsymbol{e}_z$$

$$= (-\operatorname{sen}(t), -\operatorname{sen}(t), -\sqrt{2}\cos(t))$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \operatorname{sen}^2(t) + \operatorname{sen}^2(t) + 2\cos^2(t) = 2$$

$$\ell_{\mathscr{L}} = \int_0^\tau \sqrt{2}\,\mathrm{d}t = \sqrt{2}\tau.$$

Exemplo 7.11: Calcule a área da superfície externa do parabolóide de revolução $z = x^2 + y^2$, $x^2 + y^2 \leq 1$

Procuro a teoria:

$$z = g(x, y) = x^2 + y^2,$$

$$A_{\mathscr{S}} = \int_{R_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y$$

$$\frac{\partial g}{\partial x} = 2x,$$

$$\frac{\partial g}{\partial y} = 2y,$$

$$A_{\mathscr{S}} = \int_{R_{xy}} \sqrt{1 + (2x)^2 + (2y)^2}\,\mathrm{d}x\mathrm{d}y$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1 + 4(x^2 + y^2)}\,r\mathrm{d}r\mathrm{d}\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r\mathrm{d}\theta$$

$$= \left[\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r\right]\left[\int_{\theta=0}^{2\pi} \mathrm{d}\theta\right]$$

$$= 2\pi \int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r$$

$$= 2\pi \frac{5\sqrt{5} - 1}{12} = \frac{5\sqrt{5} - 1}{6}.$$

Virtudes acadêmicas **e** profissionais:

1. Trabalho.

2. Persistência.

3. Iniciativa.

Divergência, gradiente, rotacional

$$\operatorname{div}\boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = ?$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(u, v, w)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

$\boldsymbol{\nabla}$ é o **operador** "nabla".

Posso repetir, de forma mais resumida, em notação indicial. Suponha que $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ seja uma base **ortonormal** do $\mathbb{R}^3$.

$$\boldsymbol{\nabla} \equiv \boldsymbol{e}_i\frac{\partial}{\partial x_i},$$

e não

$$\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x_i}\boldsymbol{e}_i,$$

$$\boldsymbol{u} = u_j\boldsymbol{e}_j$$

$$\operatorname{div}\boldsymbol{u} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{u}$$

$$= \boldsymbol{e}_i\frac{\partial}{\partial x_i}\cdot u_j\boldsymbol{e}_j$$

$$= (\boldsymbol{e}_i\cdot\boldsymbol{e}_j)\frac{\partial u_j}{\partial x_i}$$

$$= \delta_{ij}\frac{\partial u_j}{\partial x_i}$$

$$= \frac{\partial u_i}{\partial x_i}.$$

Lei de Fourier é

$$\boldsymbol{q} = -\rho c_p \alpha \boldsymbol{\nabla} T,$$

$$\dot{Q} = -\oint_{\text{casca do ovo}} (\boldsymbol{n}\cdot\boldsymbol{q})\,\mathrm{d}A$$

Gradiente, e sua "definição" (em coordenadas cartesianas) com o uso de $\boldsymbol{\nabla}$:

$$\alpha\boldsymbol{u},$$
$$\boldsymbol{u}\alpha,$$
$$\mathbf{grad}\, f = \nabla f,$$
$$\mathbf{grad}\, f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f$$
$$\mathbf{grad}\, f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$\mathbf{grad}\, f = \boldsymbol{e}_i \frac{\partial}{\partial x_i} f$$
$$= \boldsymbol{e}_i \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \boldsymbol{e}_i$$

O rotacional:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \epsilon_{ijk}\nabla_i u_j \boldsymbol{e}_k$$
$$= \epsilon_{ijk}\frac{\partial}{\partial x_i}u_j \boldsymbol{e}_k$$
$$= \epsilon_{ijk}\frac{\partial u_j}{\partial x_i}\boldsymbol{e}_k$$

Em resumo, as operações relacionados a

$$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

são

1. A divergência: $\boldsymbol{u} = (u, v, w)$ e

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z};$$

2. O gradiente: $f(x, y, z)$ e

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

3. O rotacional:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}.$$

O laplaciano!

$$\nabla^2 f \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f$$
$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= \frac{\partial^2 f}{\partial x_i \partial x_i}.$$

Por extensão, também podemos tratar do laplaciano de um vetor:

$$\nabla^2 \boldsymbol{u} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\boldsymbol{i}$$
$$+ \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\boldsymbol{j}$$
$$+ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\boldsymbol{k}$$
$$= \frac{\partial^2 u_i}{\partial x_k \partial x_k}\boldsymbol{e}_i.$$

Aplicações em mecânica dos fluidos:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0,$$

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} \right] = \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u},$$

Importante: a forma das operações muda quando nós mudamos de sistema de coordenadas. Por exemplo, em coordenadas cilíndricas, **não** é verdade que

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

$$r = (x^2 + y^2)^{1/2},$$
$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \times 2x$$
$$= \frac{x}{(x^2 + y^2)^{1/2}} = \frac{x}{r} = \cos(\theta).$$
$$\theta = \operatorname{arctg}\left(\frac{y}{x}\right),$$
$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x} \operatorname{arctg}\left(\frac{y}{x}\right),$$
$$\frac{\operatorname{darctg} u}{\operatorname{d} u} = \frac{1}{1 + u^2},$$
$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \times -\frac{y}{x^2}$$

Teorema da divergência, aplicações em Mecânica dos Fluidos.

$$\frac{\partial}{\partial t} \int_{\mathscr{C}} \rho \, \mathrm{d}V + \oint_{\mathscr{S}} \rho(\boldsymbol{n} \cdot \boldsymbol{v}) \, \mathrm{d}A = 0$$
$$\frac{\partial}{\partial t} \int_{\mathscr{C}} \rho \, \mathrm{d}V + \oint_{\mathscr{S}} (\boldsymbol{n} \cdot [\rho \boldsymbol{v}]) \, \mathrm{d}A = 0$$
$$\int_{\mathscr{C}} \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_{\mathscr{C}} \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] \, \mathrm{d}V = 0$$
$$\int_{\mathscr{C}} \left\{ \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] \right\} \, \mathrm{d}V = 0$$
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] = 0$$

Exercício 7.9

$$x^2 + y^2 = 1$$
$$z = 1 - y^2,$$
$$z = u,$$
$$x = \cos v,$$
$$y = \operatorname{sen} v,$$
$$\boldsymbol{r} = (\cos v, \operatorname{sen} v, u),$$
$$\frac{\partial \boldsymbol{r}}{\partial u} = (0, 0, 1),$$
$$\frac{\partial \boldsymbol{r}}{\partial v} = (-\operatorname{sen} v, \cos v, 0)$$

A área da superfície é

$$A = \iint_{R_{uv}} \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| \mathrm{d}u\mathrm{d}v$$

No nosso caso a área é simplesmente

$$A = \iint_{R_{uv}} \mathrm{d}u\mathrm{d}v$$
$$= \int_{v=0}^{2\pi} \int_{u=0}^{1-v^2} \mathrm{d}u\mathrm{d}v$$

Campos irrotacionais

$$\boldsymbol{F} = (F_1, F_2, F_3)$$

A integral de log:

$$\int \frac{\mathrm{d}u}{u} = \ln|u|;$$

$$\int_a^b \frac{\mathrm{d}u}{u} = \ln|b| - \ln|a|$$

$$= \ln \frac{|b|}{|a|}$$

O que acontece se os sinais de a e b forem iguais: teremos somente duas possibilidades:

$$|a| = a \qquad\qquad\qquad \mathrm{e}\,|b| = b,$$
$$|a| = -a \qquad\qquad\qquad \mathrm{e}\,|b| = -b.$$

Em ambos os casos,

$$\ln \frac{|b|}{|a|} = \ln \frac{b}{a}.$$

Ao tentarmos resolver

$$\mathscr{D}y = f(x)$$

queremos encontrar a **função** $y(x)$.

$$\mathscr{D} = \frac{\mathrm{d}}{\mathrm{d}x},$$

$$\mathscr{D}y = \frac{\mathrm{d}y}{\mathrm{d}x},$$

$$\mathscr{D} = x + y\frac{\mathrm{d}}{\mathrm{d}x},$$

$$\mathscr{D}y = xy + y\frac{\mathrm{d}y}{\mathrm{d}x},$$

$$\mathscr{D} = x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2,$$

$$\mathscr{D}y = x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2y,$$

$$\mathscr{D} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x} + 1,$$

$$\mathscr{D}y = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y.$$

$$\mathscr{D}y = f(x)$$

Uma EDO com coeficientes não constantes é

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

EDOs lineares, coeficientes constantes, ordem 2.

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

$$y = \mathrm{e}^{\lambda x},$$

$$y' = \lambda\mathrm{e}^{\lambda x},$$

$$y'' = \lambda^2\mathrm{e}^{\lambda x},$$

$$a\lambda^2\mathrm{e}^{\lambda x} + b\lambda\mathrm{e}^{\lambda x} + c\mathrm{e}^{\lambda x} = 0,$$

$$\mathrm{e}^{\lambda x}\left[a\lambda^2 + b\lambda + c\right] = 0$$

Se

$$ay'' + by' + cy = 0,$$

a equação característica é

$$a\lambda^2 + b\lambda + c = 0.$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

$$= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \alpha \pm \beta$$

$$y = C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x}$$

$$= C_1\mathrm{e}^{(\alpha+\beta)x} + C_2\mathrm{e}^{(\alpha-\beta)x}$$

$$= C_1\mathrm{e}^{\alpha x}\mathrm{e}^{\beta x} + C_2\mathrm{e}^{\alpha x}\mathrm{e}^{-\beta x}$$

$$= \mathrm{e}^{\alpha x}\left[C_1\mathrm{e}^{\beta x} + C_2\mathrm{e}^{-\beta x}\right]$$

$$C_1 = \frac{1}{2}\left(D_1 + D_2\right)$$

$$C_2 = \frac{1}{2}\left(D_1 - D_2\right)$$

$$y = \mathrm{e}^{\alpha x}\left[\frac{1}{2}\left(D_1 + D_2\right)\mathrm{e}^{\beta x} + \frac{1}{2}\left(D_1 - D_2\right)\mathrm{e}^{-\beta x}\right]$$

$$y = \mathrm{e}^{\alpha x}\left[D_1\frac{\mathrm{e}^{\beta x} + \mathrm{e}^{-\beta x}}{2} + D_2\frac{\mathrm{e}^{\beta x} - \mathrm{e}^{-\beta x}}{2}\right]$$

$$\cosh(x) \equiv \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2},$$

$$\mathrm{senh}(x) \equiv \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2},$$

$$y = \mathrm{e}^{\alpha x}\left[D_1\cosh(\beta x) + D_2\,\mathrm{senh}(\beta x)\right] \blacksquare$$

Problema "real", raízes "complexas": temos

$$ay'' + by' + cy = 0$$

e desejamos $y(x) \in \mathbb{R}$. Porém,

$$b^2 - 4ac < 0.$$

A solução geral será a mesma de antes:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
$$= C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$
$$= C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$$
$$= e^{\alpha x} \left[C_1 e^{i\beta x} + C_2 e^{-i\beta x} \right];$$
$$C_1 = \frac{1}{2}(A - iB),$$
$$C_2 = \frac{1}{2}(A + iB),$$
$$y = e^{\alpha x} \left[\frac{1}{2}(A - iB)e^{i\beta x} + \frac{1}{2}(A + iB)e^{-i\beta x} \right]$$
$$y = e^{\alpha x} \left[A \left(\frac{e^{i\beta x} + e^{-i\beta x}}{2} \right) - iB \left(\frac{e^{i\beta x} - e^{-i\beta x}}{2} \right) \right]$$
$$e^{i\beta x} = \cos(\beta x) + i\,\mathrm{sen}(\beta x),$$
$$e^{-i\beta x} = \cos(\beta x) - i\,\mathrm{sen}(\beta x),$$
$$\cosh(i\beta x) = \frac{e^{i\beta x} + e^{-i\beta x}}{2} = \frac{1}{2} \left[(\cos(\beta x) + i\,\mathrm{sen}(\beta x)) + (\cos(\beta x) - i\,\mathrm{sen}(\beta x)) \right]$$
$$= \cos(\beta x);$$
$$\mathrm{senh}(i\beta x) = \frac{e^{i\beta x} - e^{-i\beta x}}{2} = \frac{1}{2} \left[(\cos(\beta x) + i\,\mathrm{sen}(\beta x)) - (\cos(\beta x) - i\,\mathrm{sen}(\beta x)) \right]$$
$$= i\,\mathrm{sen}(\beta x);$$
$$y = e^{\alpha x} \left[A\cos(\beta x) - iB \times i\,\mathrm{sen}(\beta x) \right]$$
$$= e^{\alpha x} \left[A\cos(\beta x) + B\,\mathrm{sen}(\beta x) \right] \quad \blacksquare$$

O método de variação de constantes, equações diferenciais ordinárias de ordem 2.

$$ay'' + by' + cy = f(x),$$
$$ay_h'' + by_h' + cy_h = 0,$$

ache y_1 e y_2 de tal maneira que

$$ay_1'' + by_1' + cy_1 = 0,$$
$$ay_2'' + by_2' + cy_2 = 0,$$

Equações diferenciais ordinárias de ordem 2, não-homogêneas: solução pelo método de "variação de constantes".

$$y'' - 3y' + 2y = x,$$
$$y_h'' - 3y_h' + 2y_h = 0,$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = \frac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2}$$
$$\lambda = \frac{3 \pm 1}{2}$$
$$= 2, \qquad 1;$$
$$y_h = Ae^x + Be^{2x};$$

Procure a solução na forma

$$y(x) = A(x)e^x + B(x)e^{2x};$$
$$y'(x) = Ae^x + 2Be^{2x} + \left[A'e^x + B'e^{2x}\right]$$

Controle as derivadas de A e B, e impeça derivadas de ordem 2

$$y'(x) = Ae^x + 2Be^{2x} + \underbrace{\left[A'e^x + B'e^{2x}\right]}_{=0}$$
$$A'e^x + B'e^{2x} = 0;$$
$$y'(x) = Ae^x + 2Be^{2x},$$
$$y''(x) = Ae^x + 4Be^{2x} + A'e^x + 2B'e^{2x}$$

Substituindo na equação original, obtemos

$$Ae^x + 4Be^{2x} + A'e^x + 2B'e^{2x} - 3\left[Ae^x + 2Be^{2x}\right] + 2\left[Ae^x + Be^{2x}\right] = x,$$
$$e^x\left[A - 3A + 2A\right] + e^{2x}\left[4B - 6B + 2B\right] + A'e^x + 2B'e^{2x} = x$$
$$A'e^x + 2B'e^{2x} = x$$

Reúno agora duas EDOs em A e B:

$$A'e^x + B'e^{2x} = 0;$$
$$A'e^x + 2B'e^{2x} = x$$
$$2A'e^x + 2B'e^{2x} = 0;$$
$$A'e^x + 2B'e^{2x} = x$$
$$A'e^x = -x$$
$$\frac{\mathrm{d}A}{\mathrm{d}x} = -xe^{-x}$$

Integrando,

$$A(x) = (x+1)\mathrm{e}^{-x} + C_1$$
$$A'\mathrm{e}^x + B'\mathrm{e}^{2x} = 0;$$
$$A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} = x$$
$$B'\mathrm{e}^{2x} = x,$$
$$\frac{\mathrm{d}B}{\mathrm{d}x} = x\mathrm{e}^{-2x},$$
$$B(x) = -\frac{1}{4}(2x+1)\mathrm{e}^{-2x} + C_2,$$
$$y(x) = A(x)\mathrm{e}^x + B(x)\mathrm{e}^{2x},$$
$$= \left[(x+1)\mathrm{e}^{-x} + C_1\right]\mathrm{e}^x + \left[-\frac{1}{4}(2x+1)\mathrm{e}^{-2x} + C_2\right]\mathrm{e}^{2x}$$
$$= C_1\mathrm{e}^x + C_2\mathrm{e}^{2x} + x + 1 - \frac{1}{4}(2x+1)$$
$$y(x) = \underbrace{C_1\mathrm{e}^x + C_2\mathrm{e}^{2x}}_{y_h(x)} + \underbrace{\frac{2x+3}{4}}_{y_p(x)} \; \blacksquare$$

Uma equação de Euler com raízes **complexas**:

$$x^2 y'' + y = 0,$$
$$y = x^m,$$
$$y' = mx^{m-1},$$
$$y'' = (m-1)mx^{m-2}$$
$$x^2 \underbrace{(m-1)mx^{m-2}}_{y''} + \underbrace{x^m}_{y} = 0$$
$$[(m-1)m + 1]\, x^m = 0,$$
$$m^2 - m + 1 = 0,$$
$$m = \frac{1 \pm \mathrm{i}\sqrt{3}}{2}$$
$$y = C_1 x^{\frac{1+\mathrm{i}\sqrt{3}}{2}} + C_2 x^{\frac{1-\mathrm{i}\sqrt{3}}{2}}$$
$$x^a = \exp(\ln x^a) = \exp(a \ln x);$$
$$x^{\frac{1 \pm \mathrm{i}\sqrt{3}}{2}} = x^{\frac{1}{2}} x^{\frac{\pm \mathrm{i}\sqrt{3}}{2}}$$
$$x^{\frac{\pm \mathrm{i}\sqrt{3}}{2}} = \exp\left(\mathrm{i}\frac{\pm\sqrt{3}}{2} \ln x \right)$$
$$\exp(\mathrm{i}\theta) = \cos(\theta) + \mathrm{i}\,\mathrm{sen}(\theta)$$
$$x^{\frac{\pm \mathrm{i}\sqrt{3}}{2}} = \cos\left(\frac{\sqrt{3}}{2} \ln x \right) \pm \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2} \ln x \right)$$
$$y = C_1 x^{\frac{1+\mathrm{i}\sqrt{3}}{2}} + C_2 x^{\frac{1-\mathrm{i}\sqrt{3}}{2}}$$
$$y = C_1 x^{1/2} x^{+\frac{\mathrm{i}\sqrt{3}}{2}} + C_2 x^{1/2} x^{-\frac{\mathrm{i}\sqrt{3}}{2}}$$
$$y = C_1 x^{1/2} \left[\cos\left(\frac{\sqrt{3}}{2} \ln x \right) + \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2} \ln x \right) \right]$$
$$+ C_2 x^{1/2} \left[\cos\left(\frac{\sqrt{3}}{2} \ln x \right) - \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2} \ln x \right) \right]$$

Agora, escolha C_1 e C_2 conjugados complexos:
$$C_1 = \frac{1}{2}\left(D_1 - \mathrm{i}D_2 \right),$$
$$C_2 = \frac{1}{2}\left(D_1 + \mathrm{i}D_2 \right),$$

$$\frac{y}{x^{1/2}} = \frac{1}{2}\left(D_1 - \mathrm{i}D_2 \right)\left[C + \mathrm{i}S \right] + \frac{1}{2}\left(D_1 + \mathrm{i}D_2 \right)\left[C - \mathrm{i}S \right]$$
$$= \frac{1}{2}\left\{ D_1 C - \mathrm{i}^2 D_2 S + \mathrm{i}(-D_2 C + D_1 S) + D_1 C - \mathrm{i}^2 D_2 S + \mathrm{i}(D_2 C - D_1 S) \right\}$$
$$= D_1 C + D_2 S;$$
$$y = x^{1/2} \left[D_1 \cos\left(\frac{\sqrt{3}}{2} \ln x \right) + D_2 \,\mathrm{sen}\left(\frac{\sqrt{3}}{2} \ln x \right) \right] \ \blacksquare$$

Desigualdades importantes com números complexos
Resultado auxiliar:

$$z_1^* z_2 + z_1 z_2^* = (z_1^* z_2) + (z_1^* z_2)^* = 2\operatorname{Re}(z_1^* z_2) \le 2|z_1^* z_2| = 2|z_1^*||z_2| = 2|z_1||z_2|$$

$$\begin{aligned}
|x + \mathrm{i}y|^2 &= x^2 + y^2 \\
&= (x - \mathrm{i}y)(x + \mathrm{i}y) \\
&= x^2 - (\mathrm{i}y)^2 \\
&= x^2 - \mathrm{i}^2 y^2 \\
&= x^2 + y^2 \ \blacksquare
\end{aligned}$$

A desigualdade do triângulo, agora, é

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)^*(z_1 + z_2) \\
&= z_1^* z_1 + z_2^* z_2 + z_1^* z_2 + z_2^* z_1 \\
&= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1^* z_2) \\
&\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
&= (|z_1| + |z_2|)^2; \ \Rightarrow
\end{aligned}$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Variáveis complexas têm mil (e uma) utilidades

$$I = \int_0^\infty \mathrm{e}^{-x} \cos(ax)\,\mathrm{d}x;$$

$$\mathrm{e}^{\mathrm{i}ax} = \cos(ax) + \mathrm{i}\operatorname{sen}(ax);$$

$$I = \int_0^\infty \mathrm{e}^{-x} \operatorname{Re}(\mathrm{e}^{\mathrm{i}ax})\,\mathrm{d}x;$$

$$I = \operatorname{Re}\int_0^\infty \mathrm{e}^{-x}\mathrm{e}^{\mathrm{i}ax}\,\mathrm{d}x;$$

$$I = \operatorname{Re}\int_0^\infty \mathrm{e}^{(\mathrm{i}a-1)x}\,\mathrm{d}x;$$

$$I = \operatorname{Re}\frac{1}{\mathrm{i}a - 1}\int_0^\infty \mathrm{e}^{(\mathrm{i}a-1)x}\,\mathrm{d}(\mathrm{i}a - 1)x;$$

$$I = \operatorname{Re}\frac{1}{\mathrm{i}a - 1}\mathrm{e}^{(\mathrm{i}a-1)x}\Big|_{x=0}^{x=\infty}$$

$$I = \operatorname{Re}\frac{1}{\mathrm{i}a - 1}\left[\mathrm{e}^{(\mathrm{i}a-1)\infty} - 1\right]$$

Agora avalio o limite cuidadosamente:

$$\lim_{x\to\infty}\mathrm{e}^{(\mathrm{i}a-1)x} = \lim_{x\to\infty}\mathrm{e}^{-x}(\cos(ax) + \mathrm{i}\operatorname{sen}(ax))$$

$$\lim_{x\to\infty}|\mathrm{e}^{(\mathrm{i}a-1)x}| = \lim_{x\to\infty}|\mathrm{e}^{-x}||(\cos(ax) + \mathrm{i}\operatorname{sen}(ax))|;$$

$$|\cos(ax) + \mathrm{i}\operatorname{sen}(ax)| = 1$$

$$|x + \mathrm{i}y| = \sqrt{x^2 + y^2}$$

$$\lim_{x\to\infty}|\mathrm{e}^{(\mathrm{i}a-1)x}| = \lim_{x\to\infty}|\mathrm{e}^{-x}| = 0;$$

$$\lim_{x\to\infty}\mathrm{e}^{(\mathrm{i}a-1)x} = 0 \ \blacksquare$$

Portanto,

$$
\begin{aligned}
I &= \operatorname{Re} \frac{1}{\mathrm{i}a - 1}\,[-1] \\
&= \operatorname{Re} \frac{1}{1 - \mathrm{i}a} \\
&= \operatorname{Re} \frac{1 + \mathrm{i}a}{(1 - \mathrm{i}a)(1 + \mathrm{i}a)} \\
&= \operatorname{Re} \frac{1 + \mathrm{i}a}{1 + a^2} = \frac{1}{1 + a^2} \ \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
\cos(\mathrm{i}x) &= \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2} \\
&= \frac{\cos \mathrm{i}x + \mathrm{i}\operatorname{sen}\mathrm{i}x + \cos \mathrm{i}x - \mathrm{i}\operatorname{sen}\mathrm{i}x}{2} \\
&= \cos \mathrm{i}x. \\
\operatorname{sen}(\mathrm{i}x) &= \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} \\
&= \frac{\cos \mathrm{i}x + \mathrm{i}\operatorname{sen}\mathrm{i}x - (\cos \mathrm{i}x - \mathrm{i}\operatorname{sen}\mathrm{i}x)}{2\mathrm{i}} \\
\cos^2(\mathrm{i}x) + \operatorname{sen}^2(\mathrm{i}x) &= \frac{\mathrm{e}^{2\mathrm{i}x} + 2 + \mathrm{e}^{-2\mathrm{i}x}}{4} + \frac{\mathrm{e}^{2\mathrm{i}x} - 2 + \mathrm{e}^{-2\mathrm{i}x}}{-4} \\
&= \frac{\mathrm{e}^{2\mathrm{i}x} + 2 + \mathrm{e}^{-2\mathrm{i}x}}{4} + \frac{-\mathrm{e}^{2\mathrm{i}x} + 2 - \mathrm{e}^{-2\mathrm{i}x}}{4} \\
&= \frac{4}{4} = 1 \ \blacksquare
\end{aligned}
$$

Variáveis complexas:

$$
\frac{1}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}^2} = \frac{\mathrm{i}}{-1} = -\mathrm{i}.
$$

Uma integral no plano complexo:

$$
\begin{aligned}
\zeta - z &= R\mathrm{e}^{\mathrm{i}\theta}, \\
\mathrm{d}\zeta &= \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta
\end{aligned}
$$

Encontrando coeficientes de Laurent:

$$
\begin{aligned}
f(z) &= \frac{1}{(z - \mathrm{i})^3 (z + \mathrm{i})^2} \\
f(z) &= \sum_{n=-\infty}^{+\infty} c_n (z - \mathrm{i})^n = \sum_{n=-3}^{+\infty} c_n (z - \mathrm{i})^n
\end{aligned}
$$

"Bem perto" de $z = \mathrm{i}$:

$$
\begin{aligned}
f(z) &\sim \frac{1}{(z - \mathrm{i})^3 (2\mathrm{i})^2} = -\frac{1}{4}\frac{1}{(z - \mathrm{i})^3}; \\
f(z) &= \frac{1}{(z - \mathrm{i})^3 (z + \mathrm{i})^2} = c_{-3}(z - \mathrm{i})^{-3} + c_{-2}(z - \mathrm{i})^{-2} + c_{-1}(z - \mathrm{i})^{-1} + \ldots; \\
(z - \mathrm{i})^3 f(z) &= \frac{1}{(z + \mathrm{i})^2} = c_{-3} + c_{-2}(z - \mathrm{i}) + c_{-1}(z - \mathrm{i})^2 + \ldots;
\end{aligned}
$$

Exemplo 9.9

$$f(z) = \frac{z}{(z^2 + 1)^2}$$

Ache os polos:

$$z^2 + 1 = 0,$$
$$z^2 = -1,$$
$$z = \pm i.$$

Tente colocar $\pm i$ explicitamente na fórmula da função:

$$z^2 + 1 = z^2 - (-1)$$
$$= z^2 - (i^2)$$
$$= (z + i)(z - i)$$
$$= z^2 - (i)^2$$
$$= z^2 - (-1)$$

Reescrevo agora a função:

$$f(z) = \frac{z}{[(z - i)(z + i)]^2}$$
$$= \frac{z}{(z - i)^2(z + i)^2}$$

Exemplo 9.10:

$$f(z) = \frac{1}{z + i}$$

É importante nos lembrarmos da série geométrica:

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots; \qquad |x| < 1$$
$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \dots; \qquad |x| < 1$$

Dentro do círculo unitário, $|z/i| < 1$:

$$\frac{1}{z + i} = \frac{1}{i\left(\frac{z}{i} + 1\right)}$$
$$= \frac{1}{i\left(1 + \frac{z}{i}\right)}$$
$$= \frac{1}{i\left(1 - (iz)\right)}$$
$$= \frac{1}{i}\left[1 + (iz) + (iz)^2 + (iz)^3 + \dots\right]$$
$$= -i\left[1 + iz - z^2 - iz^3 + \dots\right]$$
$$= -i + z + iz^2 - z^3 + \dots$$

Procuramos agora a série de Laurent de $f(z)$ **fora** do círculo unitário; nessa região, $|\mathrm{i}/z| < 1$.

$$\frac{1}{z+\mathrm{i}} = \frac{1}{z\left(1+\frac{\mathrm{i}}{z}\right)}$$

$$= \frac{1}{z}\left[1 - \frac{\mathrm{i}}{z} + \left(\frac{\mathrm{i}}{z}\right)^2 - \left(\frac{\mathrm{i}}{z}\right)^3 + \ldots\right]$$

$$= \frac{1}{z} - \frac{\mathrm{i}}{z^2} - \frac{1}{z^3} + \frac{\mathrm{i}}{z^4} + \ldots$$

Próximo exemplo!

$$f(z) = \frac{1}{(z-2)(z-3)};$$

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1;$$

$$|z| > 2 \Rightarrow \frac{|z|}{|z|} > \frac{2}{|z|} \Rightarrow \left|\frac{2}{z}\right| < 1.$$

$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$\frac{1}{(z-2)(z-3)} = \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}$$

$$= \frac{(A+B)z - 3A - 2B}{(z-2)(z-3)}$$

$$A + B = 0,$$

$$-3A - 2B = 1,$$

$$\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2};$$

$$\frac{1}{z-3} = \frac{1}{3\left(\frac{z}{3}-1\right)} = \frac{-1/3}{1-\frac{z}{3}}$$

$$= -\frac{1}{3}\left[1 + (z/3) + (z/3)^2 + (z/3)^3 + \ldots\right]$$

$$\frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z}\left[1 + (2/z) + (2/z)^2 + (2/z)^3\right]$$

Integração de Contorno e o Teorema dos Resíduos:

$$\int_{\mathscr{L}} f(z)\,\mathrm{d}z$$

$$\mathrm{e}^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\operatorname{sen}(\theta);$$
$$|\mathrm{e}^{\mathrm{i}\theta}| = \sqrt{\cos^2(\theta) + \operatorname{sen}^2(\theta)} = 1.$$

$$\frac{a}{|1+b|} \le ?$$
$$|b| \le |1+b|; \Rightarrow$$
$$\frac{a}{|1+b|} \le \frac{a}{|b|}?$$

$$\int_{\theta=0}^{\pi} \frac{R}{|R^2 \mathrm{e}^{2\mathrm{i}\theta}|}\,\mathrm{d}\theta = \int_{\theta=0}^{\pi} \frac{R}{|R^2||\mathrm{e}^{2\mathrm{i}\theta}|}\,\mathrm{d}\theta$$
$$= \int_{\theta=0}^{\pi} \frac{R}{|R^2|}\,\mathrm{d}\theta$$
$$= \int_{\theta=0}^{\pi} \frac{1}{R}\,\mathrm{d}\theta$$
$$= \frac{1}{R}\int_{\theta=0}^{\pi}\,\mathrm{d}\theta = \frac{\pi}{R}$$

Como achar as raízes de $z^4 = -1$?

$$z = r\mathrm{e}^{\mathrm{i}\theta};$$
$$z^4 = r^4 \mathrm{e}^{4\mathrm{i}\theta_k} = -1 = 1\mathrm{e}^{\mathrm{i}(\pi + 2(k-1)\pi)}$$
$$r = 1,$$
$$\theta_k = \frac{\pi}{4} + \frac{(k-1)\pi}{2}$$

Equações de ordem 2 que eu sei resolver:

$$ay'' + by' + cy = f(x),$$
$$ax^2 y'' + bxy' + cy = f(x)$$

Eu gostaria de saber resolver equações diferenciais lineares de ordem 2 mais gerais, de coeficientes não constantes:

$$y'' + p(x)y' + q(x)y = f(x),$$

Considere a equação diferencial

$$y' + y = 0;$$
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$
$$y'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$\sum_{n=0}^{\infty} \left[a_n + (n+1)a_{n+1}\right] x^n = 0$$

Método de Frobenius:

$$y'' + p(x)y' + q(x)y = 0;$$
$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0;$$

Um exemplo de singularidade fora da origem:

$$2(\xi - 1)\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \frac{\mathrm{d}y}{\mathrm{d}\xi} + y = 0$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \frac{1}{2(\xi - 1)}\frac{\mathrm{d}y}{\mathrm{d}\xi} + \frac{1}{2(\xi - 1)}y = 0$$

$$x = \xi - 1;$$
$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\xi};$$
$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\mathrm{d}y}{\mathrm{d}x};$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]\frac{\mathrm{d}x}{\mathrm{d}\xi};$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

$$2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

O método de Frobenius
Caso (i) $r_1 \neq r_2$, e $r_1 - r_2 \notin \mathbb{Z}$.

$$2xy'' + y' + y = 0,$$
$$y'' + \frac{1}{2x}y' + \frac{1}{2x}y = 0,$$
$$xp(x) = \frac{1}{2},$$
$$x^2 q(x) = \frac{x}{2}.$$

Portanto, $x = 0$ é um ponto singular regular, e o método de Frobenius aplica-se, ou seja: podemos encontrar uma solução em série em torno de $x = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{r+n},$$
$$y' = \sum_{n=0}^{\infty} (r + n)a_n x^{r+n-1},$$
$$y'' = \sum_{n=0}^{\infty} (r + n - 1)(r + n)a_n x^{r+n-2}.$$

$$2x \left[\sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2} \right] + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

$$\left[\sum_{n=0}^{\infty} 2(r+n-1)(r+n)a_n x^{r+n-1} \right] + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\sum_{n=0}^{\infty} \left[2(r+n-1)(r+n) + (r+n) \right] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$r + n - 1 = r + m,$$
$$n - 1 = m,$$
$$n = m + 1.$$

$$\sum_{m=-1}^{\infty} \left[2(r+m)(r+m+1) + (r+m+1) \right] a_{m+1} x^{r+m} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\sum_{n=-1}^{\infty} \left[2(r+n)(r+n+1) + (r+n+1) \right] a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$[2(r-1)r+r]a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[2(r+n)(r+n+1) + (r+n+1) \right] a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$[2(r-1)r+r]a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[(r+n+1)(2(r+n)+1) \right] a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$[2(r-1)r+r]a_0 x^{r-1} + \sum_{n=0}^{\infty} \left\{ (r+n+1)(2(r+n)+1)a_{n+1} + a_n \right\} x^{r+n} = 0$$

A equação indicial é

$$2(r-1)r + r = 0,$$
$$2(r^2 - r) + r = 0,$$
$$2r^2 - 2r + r = 0,$$
$$2r^2 - r = 0,$$
$$r(2r-1) = 0,$$
$$r_2 = 0,$$
$$r_1 = \frac{1}{2}.$$

A relação de recorrência será

$$(r + n + 1)(2(r + n) + 1)a_{n+1} + a_n = 0,$$
$$(r + n + 1)(2(r + n) + 1)a_{n+1} = -a_n,$$
$$a_{n+1} = -\frac{a_n}{(r + n + 1)(2(r + n) + 1)},$$
$$r_1 = 1/2,$$
$$a_{n+1} = -\frac{a_n}{(n + 3/2)(2(1/2 + n) + 1)},$$
$$a_{n+1} = -\frac{a_n}{(n + 3/2)(2(1 + n))},$$
$$a_{n+1} = -\frac{a_n}{(2n + 3)(1 + n)},$$
$$b_{n+1} = -\frac{b_n}{(r + n + 1)(2(r + n) + 1)},$$
$$r_2 = 0,$$
$$b_{n+1} = -\frac{b_n}{(n + 1)(2n + 1)},$$

Agora eu calculo alguns dos termos de cada série:

$$a_0,$$
$$a_1 = -\frac{a_0}{3},$$
$$a_2 = -\frac{a_1}{10} = +\frac{a_0}{30},$$
$$a_3 = -\frac{a_2}{21} = -\frac{a_0}{630},$$
$$y_1(x) = x^{1/2}\left[1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \dots\right]$$

$$b_0,$$
$$b_1 = -b_0,$$
$$b_2 = -\frac{b_1}{6} = \frac{b_0}{6},$$
$$b_3 = -\frac{b_2}{15} = -\frac{b_0}{90},$$
$$b_4 = -\frac{b_3}{28} = \frac{b_0}{2520},$$
$$y_2(x) = \left[1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \dots\right]$$

A solução geral é da forma

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Caso (ii): $r_1 = r_2$

$$x^2 y'' - (x + x^2)y' + y = 0,$$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n},$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1},$$

$$y'' = \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}$$

$$x^2 \left[\sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2} \right] - (x + x^2) \left[\sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \right] + \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) - (r+n) + 1 \right] a_n x^{r+n} - \sum_{n=0}^{\infty} (r+n) a_n x^{r+n+1} = 0,$$

$$r + m = r + n + 1,$$
$$m = n + 1,$$
$$n = m - 1$$

$$\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) - (r+n) + 1 \right] a_n x^{r+n} - \sum_{m=1}^{\infty} (r+m-1) a_{m-1} x^{r+m} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) - (r+n) + 1 \right] a_n x^{r+n} - \sum_{n=1}^{\infty} (r+n-1) a_{n-1} x^{r+n} = 0,$$

$$\left[(r-1)r - r + 1 \right] a_0 x^r + \sum_{n=1}^{\infty} \left\{ \left[(r+n-1)(r+n) - (r+n) + 1 \right] a_n - (r+n-1) a_{n-1} \right\} x^{r+n} = 0,$$

A equação indicial será

$$(r-1)r - r + 1 = 0,$$
$$r^2 - r - r + 1 = 0,$$
$$r^2 - 2r + 1 = 0,$$
$$(r-1)^2 = 0.$$

Portanto, $r = 1$ é uma raiz dupla.

Preciso inicialmente encontrar uma solução. A relação de recorrência é

$$[(r+n-1)(r+n) - (r+n) + 1]\, a_n - (r+n-1)a_{n-1} = 0$$
$$[(r+n-1)(r+n) - (r+n) + 1]\, a_n = (r+n-1)a_{n-1}$$
$$a_n = \frac{(r+n-1)a_{n-1}}{(r+n-1)(r+n) - (r+n) + 1}$$
$$a_n = \frac{na_{n-1}}{n(1+n) - (1+n) + 1}$$
$$a_n = \frac{na_{n-1}}{n + n^2 - 1 - n + 1}$$
$$a_n = \frac{na_{n-1}}{n^2}$$
$$a_n = \frac{a_{n-1}}{n}.$$

$$a_0 = 1 = \frac{1}{0!},$$
$$a_1 = \frac{1}{1} = 1 = \frac{1}{1!},$$
$$a_2 = \frac{1}{2} = \frac{1}{2!}$$
$$a_3 = \frac{1/2}{3} = \frac{1}{6} = \frac{1}{3!},$$
$$a_4 = \frac{1/6}{4} = \frac{1}{24} = \frac{1}{4!},$$
$$a_n = \frac{1}{n!}$$

A 1ª solução será

$$y_1(x) = x\left[\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$$
$$= x\mathrm{e}^x \; \blacksquare$$

A segunda solução, LI da primeira, deve ser procurada na forma

$$y_2 = y_1\ln(x) + \sum_{n=1}^{\infty} c_n x^{n+1},$$
$$y_2' = y_1'\ln(x) + y_1\frac{1}{x} + \sum_{n=1}^{\infty}(n+1)c_n x^n,$$
$$y_2'' = y_1''\ln(x) + y_1'\frac{1}{x} + y_1'\frac{1}{x} - y_1\frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1}$$
$$= y_1''\ln(x) + \frac{2y_1'}{x} - y_1\frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1}$$

A equação diferencial é

$$x^2 y'' - (x + x^2)y' + y = 0.$$

Substituindo,

$$x^2 \left[y_1'' \ln(x) + \frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] -$$

$$(x + x^2) \left[y_1' \ln(x) + y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n \right] + y_1 \ln(x) + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$\ln(x) \underbrace{\left[x^2 y_1'' - (x + x^2)y_1' + y_1 \right]}_{=0} + x^2 \left[\frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] -$$

$$(x + x^2) \left[+y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n \right] + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$x^2 \left[\frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] -$$

$$(x + x^2) \left[+y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n \right] + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2xy_1' - y_1 - y_1 - xy_1 + x^2 \left[\sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] -$$

$$(x + x^2) \left[\sum_{n=1}^{\infty} (n+1)c_n x^n \right] + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2xy_1' - 2y_1 - xy_1 +$$

$$\sum_{n=1}^{\infty} n(n+1)c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2xy_1' - 2y_1 - xy_1 + \sum_{n=1}^{\infty} \left[n(n+1) - (n+1) + 1 \right] c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} = 0$$

$$2xy_1' - 2y_1 - xy_1 + \sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} = 0,$$

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} + 2xy_1' - (2+x)y_1 = 0$$

$$m + 1 = n + 2,$$
$$m = n + 1,$$
$$n = m - 1$$

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$$\sum_{n=1}^{\infty}(n+1)c_n x^{n+2} = \sum_{m=2}^{\infty} m c_{m-1} x^{m+1} = \sum_{n=2}^{\infty} n c_{n-1} x^{n+1}.$$

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=2}^{\infty} n c_{n-1} x^{n+1} + 2xy_1' - (2+x)y_1 = 0,$$

Seja criativo e faça $c_0 = 0$:

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} n c_{n-1} x^{n+1} + 2xy_1' - (2+x)y_1 = 0,$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - n c_{n-1} \right] x^{n+1} + 2xy_1' - (2+x)y_1 = 0,$$

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!},$$

$$y_1' = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!},$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - n c_{n-1} \right] x^{n+1} + 2x \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} - (2+x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = 0,$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - n c_{n-1} \right] x^{n+1} = (2+x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - 2x \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Concentremo-nos nos somatórios!

$$(2+x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - 2x \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} = \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty} \frac{2n x^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty} \frac{2n x^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=1}^{\infty} \frac{2n x^{n+1}}{n!}$$

$$n! = n \times \underbrace{n-1 \times n-2 \times \dots 1}_{=(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!}$$

$$n+2 = m+1,$$

$$n+1 = m,$$

$$n = m-1,$$

$$= \sum_{m=1}^{\infty} \frac{x^{m+1}}{(m-1)!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!}$$

$$= - \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!}$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - nc_{n-1} \right] x^{n+1} = -\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!}$$

$$n^2 c_n - nc_{n-1} = -\frac{1}{(n-1)!}$$

$$c_n - \frac{n}{n^2} c_{n-1} = -\frac{1}{n^2(n-1)!}$$

$$c_n = \frac{1}{n^2} \left[nc_{n-1} - \frac{1}{(n-1)!} \right]$$

$$= \frac{1}{n^2} \left[\frac{n(n-1)!c_{n-1} - 1}{(n-1)!} \right]$$

$$= \frac{1}{n^2} \left[\frac{n!c_{n-1} - 1}{(n-1)!} \right]$$

$$= \frac{1}{n} \left[\frac{n!c_{n-1} - 1}{n(n-1)!} \right]$$

$$= \frac{1}{n} \left[\frac{n!c_{n-1} - 1}{n!} \right]$$

$$= \frac{1}{n} \left[c_{n-1} - \frac{1}{n!} \right]$$

Caso (iii)-1: $r_1 - r_2 \in \mathbb{Z}$, e a menor raiz leva a duas soluções.

$$x^2 y'' + xy' + (x^2 - 1/4)y = 0.$$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n},$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1},$$

$$y'' = \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}.$$

$$x^2 \sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2} + x \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \left(x^2 - \frac{1}{4}\right) \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

$$\sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) + (r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n-1+1)(r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n-1+1)(r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0,$$

$$r + n + 2 = r + m,$$
$$n + 2 = m,$$
$$n = m - 2$$

$$\sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{1}{4} \right] a_n x^{r+n} + \sum_{m=2}^{\infty} a_{m-2} x^{r+m} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0,$$

$$(r^2 - 1/4)a_0 x^r + [(r+1)^2 - 1/4]a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ \left[(r+n)^2 - \frac{1}{4} \right] a_n + a_{n-2} \right\} x^{r+n} = 0$$

Faça

$$r^2 = 1/4,$$
$$r_1 = +1/2,$$
$$r_2 = -1/2.$$

Se $r = -1/2$,

$$(r+1)^2 - 1/4 = (-1/2+1)^2 - 1/4$$
$$(1/2)^2 - 1/4 = 0.$$

A fórmula de recorrência será

$$\left[(r+n)^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0,$$
$$[(-1/2+n)^2 - 1/4]a_n + a_{n-2} = 0,$$
$$[1/4 - n + n^2 - 1/4]a_n + a_{n-2} = 0,$$
$$[-n + n^2]a_n + a_{n-2} = 0,$$
$$a_n = -\frac{a_{n-2}}{n(n-1)}.$$

A solução geral é

$$y_1 = a_0 \left\{ x^{-1/2} \left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots \right] \right\} = x^{-1/2}\cos(x),$$

$$y_2 = a_1 \left\{ x^{-1/2} \left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \ldots \right] \right\} = x^{-1/2}\operatorname{sen}(x).$$

Na literatura, nós encontramos a notação

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi}} x^{-1/2}\cos(x),$$

$$J_{+1/2}(x) \equiv \sqrt{\frac{2}{\pi}} x^{-1/2}\operatorname{sen}(x).$$

Finalmente, vamos para o caso iii-b: Duas raízes que diferem por um número inteiro, mas a menor raiz não leva a nenhuma solução.

$$x^2 y'' + xy' + (x^2 - 1)y = 0.$$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n},$$

$$y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1},$$

$$y'' = \sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2}.$$

$$x^2 \sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2} + x \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + x^2 \sum_{n=0}^{\infty} a_n x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

$$\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) + (r+n) - 1 \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

$$r + m = r + n + 2,$$
$$m = n + 2,$$
$$n = m - 2.$$

$$\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) + (r+n) - 1 \right] a_n x^{r+n} + \sum_{m=2}^{\infty} a_{m-2} x^{r+m} = 0$$

$$\sum_{n=0}^{\infty} \left[(r+n)(r+n) - 1 \right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0,$$

$$(r^2 - 1)a_0 x^r + [(r+1)^2 - 1]a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ \left[(r+n)(r+n) - 1 \right] a_n + a_{n-2} \right\} x^{r+n}$$

Fazendo $a_0 \neq 0$:

$$r^2 = 1,$$
$$r_1 = 1,$$
$$r_2 = -1$$

Discuto o que acontece quando $r_2 = -1$:

$$[(-1+1)^2 - 1]a_1 x^{r+1} = -a_1 x^0 = 0; \Rightarrow a_1 = 0.$$
$$[(-1+n)(-1+n) - 1]\, a_n + a_{n-2} = 0,$$
$$\left[n^2 - 2n + 1 - 1\right] a_n + a_{n-2} = 0,$$
$$a_n = -\frac{a_{n-2}}{n(n-2)},$$
$$a_2 = -\frac{a_0}{2 \times 0}.$$

Portanto, a menor raiz não leva a nenhuma solução. Faço "tudo de novo" com $r = 1$:

$$(r^2 - 1)a_0 x^r + [(r+1)^2 - 1]a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{[(r+n)(r+n) - 1]\, a_n + a_{n-2}\right\} x^{r+n}$$
$$(r+1)^2 - 1]a_1 = 0,$$
$$[4-1]a_1 = 0,$$
$$a_1 = 0.$$

A relação de recorrência fica

$$[(r+n)(r+n) - 1]\, a_n + a_{n-2} = 0,$$
$$[(n+1)^2 - 1]a_n + a_{n-2} = 0,$$
$$[n^2 + 2n + 1 - 1]a_n + a_{n-2} = 0$$
$$[n^2 + 2n]a_n + a_{n-2} = 0,$$
$$a_n = -\frac{a_{n-2}}{n(n+2)}.$$

$$x^2 \left[y_1'' \ln(x) + \frac{2y_1'}{x} - \frac{y_1}{x^2} + v''\right] + x \left[y_1' \ln(x) + \frac{y_1}{x} + v'\right] + (x^2 - 1)\left[y_1 \ln(x) + v\right] = 0$$

Os termos em $\ln(x)$ são
$$\left[x^2 y_1'' + x y_1' + (x^2 - 1)y_1\right] \ln(x)$$