Como denotar uma dimensão "à mão livre":

$$
\begin{aligned}
[\![T]\!] &= \mathbb{T}, \\
[\![L]\!] &= \mathbb{L}, \\
[\![M]\!] &= \mathbb{M}
\end{aligned}
$$

$$
\begin{aligned}
1 &= \mathsf{M}^{1+c}\mathsf{L}^{1+a+b-3c}\mathsf{T}^{-2-b}, \\
0 &= 1 + c, \\
0 &= 1 + a + b - 3c, \\
0 &= -2 - b
\end{aligned}
$$

$$
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}
$$

Um parâmetro adimensional elevado a qualquer potência continua a ser (é óbvio) adimensional:

$$
\begin{aligned}
\left[\!\!\left[\frac{F}{\rho U^2 D^2} \right]\!\!\right] &= 1, \\
\left[\!\!\left[\left[\frac{F}{\rho U^2 D^2}\right]^p \right]\!\!\right] &= 1, \\
\left[\!\!\left[\frac{\nu}{UD} \right]\!\!\right] &= 1, \\
\left[\!\!\left[\left[\frac{\nu}{UD}\right]^p \right]\!\!\right] &= 1, \\
\left[\!\!\left[\frac{UD}{\nu} \right]\!\!\right] &= 1,
\end{aligned}
$$

Refaço agora o Exemplo 1.1 escolhendo como variáveis comuns a todos os parâmetros (grupos) adimensionais ρ, ν, e D.

$$
\begin{aligned}
\Pi_1 &= F^1 \rho^a \nu^b D^c, \\
[\![\Pi_1]\!] &= \mathsf{MLT}^{-2}\left[\mathsf{ML}^{-3}\right]^a \left[\mathsf{L}^2\mathsf{T}^{-1}\right]^b [\mathsf{L}]^c \\
1 &= \mathsf{M}^{1+a}\mathsf{L}^{1-3a+2b+c}\mathsf{T}^{-2-b}
\end{aligned}
$$

Isso produz o sistema de equações lineares

$$
\begin{aligned}
a &= -1, \\
-3a + 2b + c &= -1, \\
-b &= 2
\end{aligned}
$$

Com solução

$$a = -1, \qquad b = -2, c = 0.$$

$$\begin{aligned}
\Pi_1 &= \frac{F}{\rho\nu^2},\\
[\![\Pi_1]\!] &= \frac{\mathsf{MLT}^{-2}}{\mathsf{ML}^{-3}(\mathsf{L}^2\mathsf{T}^{-1})^2} = \frac{\mathsf{MLT}^{-2}}{\mathsf{ML}^{-3}\mathsf{L}^4\mathsf{T}^{-2}} = 1 \blacksquare
\end{aligned}$$

Procuramos agora o Π_2:

$$\begin{aligned}
\Pi_2 &= U^1\rho^a\nu^b D^c,\\
[\![\Pi_1]\!] &= \mathsf{LT}^{-1}\left[\mathsf{ML}^{-3}\right]^a\left[\mathsf{L}^2\mathsf{T}^{-1}\right]^b[\mathsf{L}]^c\\
1 &= \mathsf{M}^a\mathsf{L}^{1-3a+2b+c}\mathsf{T}^{-1-b}
\end{aligned}$$

Isso produz o sistema de equações lineares

$$\begin{aligned}
a &= 0,\\
-3a + 2b + c &= -1,\\
-b &= 1.
\end{aligned}$$

A solução é

$$a = 0, \qquad b = -1, \qquad c = 1.$$

Isso produz o segundo grupo adimensional

$$\Pi_2 = \frac{UD}{\nu},$$

que é o número de Reynolds! Antes eu tinha, agora eu tenho:

$$\begin{aligned}
\Pi_1 &= \frac{F}{\rho U^2 D^2}, & \Pi_1' &= \frac{F}{\rho\nu^2},\\
\Pi_2 &= \frac{\nu}{UD}, & \Pi_2' &= \frac{UD}{\nu}
\end{aligned}$$

Faço

$$\begin{aligned}
\Pi_1 \times \Pi_2^{-2} &= \frac{F}{\rho U^2 D^2} \times \left(\frac{UD}{\nu}\right)^2 = \frac{F}{\rho\nu^2} = \Pi_1',\\
\Pi_2^{-1} &= \Pi_2'.
\end{aligned}$$

Extensões naturais de “dimensões fundamentais” para problemas específicos:

1. Quando há espécies químicas diferentes *sem* reações químicas entre elas: M é a massa total, M_v é a massa de vapor d’água, M_c é a massa de CO_2, etc..

2. Quando há transferência de *calor sem* que haja conversão significativa de energia mecânica em energia interna. Neste caso, além de energia mecânica, $\mathsf{ML}^2\mathsf{T}^2$, “calor” deve se expresso em termos de uma dimensão extra correspondente, em geral temperatura Θ.

3. Quando existem diferenças grandes do tamanho de grandezas características em direções diferentes: em lugar de um único L, muitas vezes é proveitoso utilizar uma dimensão física para cada dimensão do espaço, ou seja: X, Y, Z.

Eu gostaria de ter um método numérico que funcionasse para "qualquer" equação diferencial ordinária (EDO) do tipo

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} &= \operatorname{sen}(x),\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y}{x} + \operatorname{sen}(x).\end{aligned}$$

O que significam os k's

$$\begin{aligned}y_{n+1} &\approx y_n + h\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_n\\ k_1 &= h\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_n = hf(x_n, y_n)\\ y_{n+1} &= y_n + k_1\end{aligned}$$

Um sistema de equações diferenciais ordinárias já colocado na forma "clássica" do método de Runge-Kutta é algo do tipo: um conjunto de incógnitas $y_1, y_2, \ldots, y_n$ que evoluem com uma dinâmica acoplada:

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}x} &= f_1(x, y_1, \ldots, y_n),\\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= f_2(x, y_1, \ldots, y_n),\\ &\vdots\\ \frac{\mathrm{d}y_n}{\mathrm{d}x} &= f_n(x, y_1, \ldots, y_n).\end{aligned}$$

Vetores, convenções, etc.

$$\begin{aligned}\boldsymbol{i} &= (1, 0, 0),\\ \boldsymbol{j} &= (0, 1, 0),\\ \boldsymbol{k} &= (0, 0, 1).\end{aligned}$$

$$\begin{aligned}\boldsymbol{u} &= (1, 2, 3),\\ \boldsymbol{u} &= 1\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}.\end{aligned}$$

$$
\begin{aligned}
c &= (d+e) \\
c^2 &= (d+e)^2 \\
c^2 &= d^2 + e^2 + 2de \ \blacksquare
\end{aligned}
$$

Notação indicial ou de Einstein

$$y_i = A_{ij} x_j$$

quer dizer

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \qquad \forall i = 1, \dots n.$$

$$
\begin{aligned}
\delta_{ii} &= \sum_{i=1}^{3} \delta_{(i)(i)} \\
&= \delta_{11} + \delta_{22} + \delta_{33} \\
&= 1 + 1 + 1 \\
&= 3.
\end{aligned}
$$

$$
\begin{aligned}
[\boldsymbol{\delta}][\boldsymbol{x}] &= [\delta_{ij}][x_j] \\
&= \delta_{ij} x_j = x_i \\
&= [x_i] = [\boldsymbol{x}].
\end{aligned}
$$

No caso unidimensional,

$$[\boldsymbol{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\boldsymbol{x}]^\top = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

O significado de $[\boldsymbol{y}]^\top [\boldsymbol{\delta}]$ é

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

Em notação indicial:

$$y_i \delta_{ij} = y_j$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

O que são dois vetores ortogonais no $\mathbb{R}^3$?

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\left(\frac{\pi}{2}\right) = 0.$$

No $\mathbb{R}^n$, nós generalizamos sem poder "ver" os vetores:

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

Uma base ortogonal é uma base $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ tal que, se $i \neq j$,

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0.$$

Se o índice for repetido, obviamente,

$$\boldsymbol{v}_{(i)} \cdot \boldsymbol{v}_{(i)} = |\boldsymbol{v}_{(i)}|^2.$$

Uma base *ortogonal* é tudo isso aí em cima somado a

$$\boldsymbol{v}_{(i)} \cdot \boldsymbol{v}_{(i)} = 1.$$

Dada uma base ortonormal E,

$$\begin{aligned} \boldsymbol{u} &= u_{Ei}\boldsymbol{e}_i, \\ \boldsymbol{v} &= v_{Ej}\boldsymbol{e}_j. \end{aligned}$$

onde

$$\boldsymbol{u} = \begin{bmatrix} u_{E1} \\ u_{E2} \\ \vdots \\ u_{En} \end{bmatrix}_E$$

Lembre-se! $u = (u_1, u_2, \ldots, u_n)$, e em geral $u_1 \neq u_{E1}$, $u_2 \neq u_{E2}$, etc.

É possível usar as coordendas u_{Ei} e v_{Ej} para calcular o produto escalar $\boldsymbol{u} \cdot \boldsymbol{j}$? Sim:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}\boldsymbol{e}_i \cdot v_{Ei}\boldsymbol{e}_i$$

Paro, e recomeço:

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}\boldsymbol{e}_i \cdot v_{Ej}\boldsymbol{e}_j \\ \alpha\boldsymbol{x} \cdot \beta\boldsymbol{y} &= \alpha\beta(\boldsymbol{x} \cdot \boldsymbol{y}); \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}v_{Ej}(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}v_{Ej}\delta_{ij} \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}[v_{Ej}\delta_{ij}] \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}v_{Ei} = \sum_{i=1}^{n} u_{Ei}v_{Ei}; \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_i v_i = \sum_{i=1}^{n} u_i v_i. \end{aligned}$$

Às vezes escrever $\boldsymbol{u} = u_{Ei}\boldsymbol{e}_i$ é demais. Quando não houver motivo de confusão, nós simplicaremos a notação e escreveremos

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_E$$

ou simplesmente $\boldsymbol{u} = u_i\boldsymbol{e}_i$, etc.

Reforçando: elemento de vetor $\times$ coordenada de vetor.

$$\boldsymbol{v} = (3, 4)$$

Se eu escolher a base $((1, 1), (1, -1))$ para representar o $\mathbb{R}^2$, então

$$\boldsymbol{v} = \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}_{((1,1),(1,-1))}$$

$$\begin{aligned} (3, 4) &= 7/2(1, 1) - 1/2(1, -1) \\ &= (7/2, 7/2) + (-1/2, 1/2) \\ &= (6/2, 8/2) \\ &= (3, 4) \blacksquare \end{aligned}$$

Bases!

A $= (\boldsymbol{i}, \boldsymbol{j})$, B $= (\boldsymbol{j}, \boldsymbol{i})$.

O símbolo de permutação de Levi-Civita é calculado sempre com n entradas. Exemplos:

$$\begin{aligned} \epsilon_{1234} &= +1, \\ \epsilon_{1243} &= -1, \\ \epsilon_{1223} &= 0. \end{aligned}$$

Permutações cíclicas (apenas com 3 elementos) de 1, 2, 3:

$$\begin{gathered} 1, 2, 3, \\ 3, 1, 2, \\ 2, 3, 1. \end{gathered}$$

Permutações cíclicas (apenas com 3 elementos) de 3, 2, 1:

$$\begin{gathered} 3, 2, 1, \\ 1, 3, 2, \\ 2, 1, 3. \end{gathered}$$

Para definir o produto vetorial eu parto dos “absolutos”

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3), \\ \boldsymbol{v} &= (v_1, v_2, v_3), \\ \boldsymbol{u} \times \boldsymbol{v} &\equiv (\epsilon_{ij1} u_i v_j, \epsilon_{ij2} u_i v_j, \epsilon_{ij3} u_i v_j). \end{aligned}$$

A forma "antiga" de calcular o produto vetorial é

$$
\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= (u_2 v_3 - u_3 v_2)\boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3)\boldsymbol{e}_2 + (u_1 v_2 - u_2 v_1)\boldsymbol{e}_3
\end{aligned}
$$

A forma "nova" de calcular o produto vetorial é

$$
\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \\
&= (\epsilon_{231} u_2 v_3 + \epsilon_{321} u_3 v_2)\boldsymbol{e_1} + (\epsilon_{132} u_1 v_3 + \epsilon_{312} u_3 v_1)\boldsymbol{e_2} + (\epsilon_{123} u_1 v_2 + \epsilon_{213} u_2 v_1)\boldsymbol{e_3} \\
&= (u_2 v_3 - u_3 v_2)\boldsymbol{e_1} + (u_3 v_1 - u_1 v_3)\boldsymbol{e_2} + (u_1 v_2 - u_2 v_1)\boldsymbol{e_3}
\end{aligned}
$$

Vamos provar agora que $\boldsymbol{u} \times \boldsymbol{v}$ é perpendicular a $\boldsymbol{u}$.

$$
\begin{aligned}
[\boldsymbol{u} \times \boldsymbol{v}] \cdot \boldsymbol{u} &= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \cdot u_l \boldsymbol{e}_l \\
\alpha \boldsymbol{a} \cdot \beta \boldsymbol{b} &= (\alpha\beta)(\boldsymbol{a} \cdot \boldsymbol{b}) \\
&= \epsilon_{ijk} u_i v_j u_l (\boldsymbol{e}_k \cdot \boldsymbol{e}_l) \\
&= \epsilon_{ijk} u_i v_j u_l \delta_{kl} \\
&= \epsilon_{ijk} u_i v_j u_k \\
&= \frac{1}{2} \left[\epsilon_{ijk} u_i v_j u_k + \epsilon_{ijk} u_i v_j u_k \right] \\
&= \frac{1}{2} \left[\epsilon_{ijk} u_i v_j u_k + \epsilon_{kji} u_k v_j u_i \right] \\
&= \frac{1}{2} \left[\epsilon_{ijk} + \epsilon_{kji} \right] u_i v_j u_k \\
&= \frac{1}{2} \left[\epsilon_{ijk} + \epsilon_{jik} \right] u_i v_j u_k \\
&= 0.
\end{aligned}
$$

O produto vetorial é anti-simétrico:

$$
\begin{aligned}
\boldsymbol{v} \times \boldsymbol{u} &= \epsilon_{ijk} v_i u_j \boldsymbol{e}_k \\
&= \epsilon_{ijk} u_j v_i \boldsymbol{e}_k \\
&= -\epsilon_{jik} u_j v_i \boldsymbol{e}_k \\
&= -\boldsymbol{u} \times \boldsymbol{v} \ \blacksquare
\end{aligned}
$$

Em seguida:

$$
\begin{aligned}
|\boldsymbol{u} \times \boldsymbol{v}|^2 &= [\boldsymbol{u} \times \boldsymbol{v}] \cdot [\boldsymbol{u} \times \boldsymbol{v}] \\
&= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \cdot \epsilon_{lmn} u_l v_m \boldsymbol{e}_n \\
\alpha \boldsymbol{a} \cdot \beta \boldsymbol{b} &= \alpha\beta(\boldsymbol{a} \cdot \boldsymbol{b}) \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m (\boldsymbol{e}_k \cdot \boldsymbol{e}_n) \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m \delta_{kn} \\
&= \epsilon_{ijk} u_i v_j (\epsilon_{lmn} \delta_{kn}) u_l v_m \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmk} u_l v_m \\
&= [\epsilon_{ijk} \epsilon_{lmk}] u_i v_j u_l v_m \\
&= [\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}]\, u_i v_j u_l v_m \\
&= \delta_{il} \delta_{jm} u_i v_j u_l v_m - \delta_{im} \delta_{jl} u_i v_j u_l v_m \\
&= u_i v_j u_i v_j - u_i v_j u_j v_i \\
&= (u_i u_i)(v_j v_j) - (u_i v_i)(u_j v_j) \\
&= (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})(\boldsymbol{u} \cdot \boldsymbol{v}) \\
&= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 - (|\boldsymbol{u}||\boldsymbol{v}| \cos(\theta))(|\boldsymbol{u}||\boldsymbol{v}| \cos(\theta)) \\
&= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \left(1 - \cos^2(\theta)\right) \\
&= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \operatorname{sen}^2(\theta);\ \Rightarrow \\
|\boldsymbol{u} \times \boldsymbol{v}| &= |\boldsymbol{u}||\boldsymbol{v}| \operatorname{sen}(\theta)\ \blacksquare
\end{aligned}
$$

1. Um “1-volume” é um comprimento.
2. Um “2-volume” é uma área.
3. Um “3-volume” é um volume usual.
4. Um “4-volume” é um “volume quadridimensional”, ou um hipervolume.

Determinantes e hipervolumes.

Considere n vetores no $\mathbb{R}^n$:

$$\begin{aligned}
\boldsymbol{u}_1 &= (u_{11}, u_{21}, \ldots, u_{n1});\\
\boldsymbol{u}_2 &= (u_{12}, u_{22}, \ldots, u_{n2});\\
&\vdots\\
\boldsymbol{u}_n &= (u_{1n}, u_{2n}, \ldots, u_{nn}).
\end{aligned}$$

O determinante associado a esses n vetores é definido como

$$\begin{aligned}
V^n &\equiv \det(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_n)\\
&\equiv \epsilon_{i_1,i_2,\ldots,i_n} u_{1i_1} u_{2i_2} \ldots u_{ni_n}.
\end{aligned}$$

O determinante está associado à matriz

$$[\boldsymbol{u}] = [u_{ij}] = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n}\\ u_{21} & u_{22} & \ldots & u_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{bmatrix}$$

Suponha que uma função f é linear:

$$\begin{aligned}
f(\alpha x) &= \alpha f(x),\\
f(x_1 + x_2) &= f(x_1) + f(x_2).
\end{aligned}$$

Desejamos provar que, então,

$$f(x) = cx.$$

Prova:

$$\frac{\mathrm{d}f(\alpha x)}{\mathrm{d}x} = \alpha f'(\alpha x),$$

mas f é linear:

$$\begin{aligned}
f(\alpha x) &= \alpha f(x)\\
\alpha f'(\alpha x) &= \alpha f'(x).
\end{aligned}$$

Portanto, no caso de f ser linear, teremos que

$$f'(\alpha x) = f'(x), \qquad \forall \alpha \in \mathbb{R},$$

ou seja: a derivada é constante!

$$\begin{aligned}
f'(x) &= c,\\
f(x) &= cx + b,\\
f(0) &= 0 \Rightarrow b = 0 \blacksquare
\end{aligned}$$

Transformações lineares. Pergunta: será possível definir uma operação entre $\boldsymbol{A}$ e $\boldsymbol{x}$ que nos permita substituir

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{x})$$

por

$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}?$$

$$\begin{aligned}
A_i \delta_{im} &= \sum_{i=1}^{3} A_i \delta_{im} \\
&= A_1 \delta_{1m} + A_2 \delta_{2m} + A_3 \delta_{3m} \\
&= A_m \\
A_{ijkl} B_{ijkp} \delta_{ln} &= A_{ijkn} B_{ijkp}
\end{aligned}$$

Transformações lineares

$$
\begin{aligned}
E &= (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n);\\
\boldsymbol{x} &= \sum_{j=1}^{3} x_j \boldsymbol{e}_j = x_j \boldsymbol{e}_j\\
\boldsymbol{A}(\boldsymbol{x}) &= \boldsymbol{A}(x_j \boldsymbol{e}_j)\\
&= x_j \boldsymbol{A}(\boldsymbol{e}_j)
\end{aligned}
$$

mas

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{e}_j) \in \mathbb{V};&\\
\boldsymbol{A}(\boldsymbol{e}_j) &= A_{ij} \boldsymbol{e}_i
\end{aligned}
$$

portanto

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= x_j A_{ij} \boldsymbol{e}_i\\
&= A_{ij} x_j \boldsymbol{e}_i\\
\boldsymbol{A}(\boldsymbol{x}) &= \boldsymbol{y};\\
\boldsymbol{y} &= y_i \boldsymbol{e}_i = A_{ij} x_j \boldsymbol{e}_i :\\
y_i &= A_{ij} x_j,\\
[\boldsymbol{y}]_E &= [\boldsymbol{A}]_{E,E} [\boldsymbol{x}]_E.
\end{aligned}
$$

Agora, eu mudo um pouco o jogo: a base do domínio continua a ser $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$; e a base do contra-domínio é $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n)$. Agora,

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= x_j \boldsymbol{A}(\boldsymbol{e}_j)\\
\boldsymbol{A}(\boldsymbol{e}_j) &= A_{ij} \boldsymbol{f}_i\\
\boldsymbol{A}(\boldsymbol{x}) &= A_{ij} x_j \boldsymbol{f}_i\\
\boldsymbol{y} &= y_i \boldsymbol{f}_i\\
y_i &= A_{ij} x_j\\
[\boldsymbol{y}]_F &= [\boldsymbol{A}]_{F,E} [\boldsymbol{x}]_E
\end{aligned}
$$

Vamos agora nos lembrar de que um funcional linear em sua forma mais geral é dado por

$$
f(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = c_j x_j = \boldsymbol{c} \cdot \boldsymbol{x}.
$$

Mas

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= A_{ij}x_j\boldsymbol{f}_i,\\
\boldsymbol{A}_i &= \boldsymbol{c};\\
\boldsymbol{A}_i &= A_{ij}\boldsymbol{e}_j\\
\boldsymbol{A}(\boldsymbol{x}) &= [\boldsymbol{c}\cdot\boldsymbol{x}]_i\boldsymbol{f}_i\\
&= [A_{ij}\boldsymbol{e}_j\cdot\boldsymbol{x}]\boldsymbol{f}_i\\
&= [A_{ij}\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k]\boldsymbol{f}_i\\
&= [A_{ij}x_k\boldsymbol{e}_j\cdot\boldsymbol{e}_k]\boldsymbol{f}_i\\
&= [A_{ij}x_k\delta_{jk}]\boldsymbol{f}_i\\
&= [A_{ij}x_j]\boldsymbol{f}_i\\
&= [A_{ij}\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k]\boldsymbol{f}_i\\
&= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k\\
&= [A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j]\cdot[x_k\boldsymbol{e}_k]\\
\boldsymbol{y} &= \boldsymbol{A}\cdot\boldsymbol{x}\\
\boldsymbol{A} &= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j
\end{aligned}
$$

Uma transformação de $\mathbb{R}^3$ em $\mathbb{R}^2$ é representada matricialmente por

$$
\begin{bmatrix} y_1\\ y_2\end{bmatrix}_{2\times 1} = \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\end{bmatrix}_{2\times 3}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}_{3\times 1}
$$

Bases!

$$
\begin{aligned}
E &= (\boldsymbol{e}_1,\boldsymbol{e}_2) = ((1,0),(0,1));\\
\boldsymbol{v} &= (3,4)\\
\boldsymbol{v} &= 3\boldsymbol{e}_1 + 4\boldsymbol{e}_2\\
[\boldsymbol{v}]_E &= \begin{bmatrix} 3\\ 4\end{bmatrix}\\
F &= (\boldsymbol{f}_1,\boldsymbol{f}_2) = ((1,1),(1,-1))\\
\boldsymbol{v} &= (7/2)\boldsymbol{f}_1 - (1/2)\boldsymbol{f}_2\\
[\boldsymbol{v}]_F &= \begin{bmatrix} 7/2\\ -1/2\end{bmatrix}
\end{aligned}
$$

Às vezes é preciso “disciplinar” a notação indicial:

$$
\sum_{i=1}^{3} C_{ij}^2 = \sum_{i=1}^{3} C_{ij}C_{ij} \equiv C_{i(j)}C_{i(j)} = 1
$$

Notação preferida para transformações lineares:

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{A}(\boldsymbol{x}),\\
\boldsymbol{y} &= \boldsymbol{A}\cdot\boldsymbol{x}\\
\boldsymbol{y} &= y_i\boldsymbol{f}_i,\\
\boldsymbol{x} &= x_j\boldsymbol{e}_j,\\
\boldsymbol{A} &= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j\\
\boldsymbol{y} &= \boldsymbol{A}\cdot\boldsymbol{x}\\
&= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k\\
&= A_{ij}x_k\boldsymbol{f}_i(\boldsymbol{e}_j\cdot\boldsymbol{e}_k)\\
&= A_{ij}x_k\boldsymbol{f}_i\delta_{jk}\\
y_i\boldsymbol{f}_i &= A_{ij}x_j\boldsymbol{f}_i
\end{aligned}
$$

Rotações e suas matrizes/transformações

$$
\begin{aligned}
\boldsymbol{u} &= u'_j\boldsymbol{e}'_j = u_i\boldsymbol{e}_i\\
u_i &= (\boldsymbol{u}\cdot\boldsymbol{e}_i)
\end{aligned}
$$

mas

$$
\begin{aligned}
\boldsymbol{e}_i &= C_{ij}\boldsymbol{e}'_j\\
u_i &= (\boldsymbol{u}\cdot C_{ij}\boldsymbol{e}'_j)\\
&= C_{ij}(\boldsymbol{u}\cdot\boldsymbol{e}'_j)\\
u_i &= C_{ij}u'_j\\
[\boldsymbol{u}]_E &= [\boldsymbol{C}][\boldsymbol{u}]_{E'}\\
[\boldsymbol{u}]'_E &= [\boldsymbol{C}]^\top[\boldsymbol{u}]_E
\end{aligned}
$$

Podemos mostrar que $[\boldsymbol{C}]$ e $[\boldsymbol{C}]^\top$ são inversas operando diretamente:

$$
\begin{aligned}
\boldsymbol{u} &= u_k\boldsymbol{e}_k = u_i\boldsymbol{e}_i = u_iC_{ij}\boldsymbol{e}'_j = u_iC_{ij}C_{kj}\boldsymbol{e}_k\\
u_k &= u_iC_{ij}C_{kj} = C_{ij}C^\top_{jk}u_i\\
u_k &= \delta_{ki}u_i,\\
C_{ij}C^\top_{jk} &= \delta_{ki}\\
[\boldsymbol{C}][\boldsymbol{C}]^\top &= [\boldsymbol{\delta}].
\end{aligned}
$$

O determinante de uma rotação é igual a 1:

$$
\begin{aligned}
1 &= [\boldsymbol{e}_1\times\boldsymbol{e}_2]\cdot\boldsymbol{e}_3\\
&= [C_{1l}\boldsymbol{e}'_l\times C_{2m}\boldsymbol{e}'_m]\cdot C_{3n}\boldsymbol{e}'_n\\
&= C_{1l}C_{2m}C_{3n}\underbrace{[\boldsymbol{e}'_l\times\boldsymbol{e}'_m]\cdot\boldsymbol{e}'_n}_{\epsilon_{lmn}}\\
1 &= \epsilon_{lmn}C_{1l}C_{2m}C_{3n} = \det[\boldsymbol{C}].
\end{aligned}
$$

Qual é a regra para a mudança das matrizes de uma transformação $\boldsymbol{A}$ quando eu mudo da base E para a base E' (ou vice-versa)?

$$
\begin{aligned}
\boldsymbol{A} &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j,\\
\boldsymbol{A} &= A'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j,\\
\boldsymbol{A} &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{kl}\boldsymbol{e}'_k\boldsymbol{e}'_l\\
A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= A'_{kl}C_{ik}\boldsymbol{e}_iC_{jl}\boldsymbol{e}_j\\
A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= A'_{kl}C_{ik}C_{jl}\boldsymbol{e}_i\boldsymbol{e}_j\\
A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= C_{ik}A'_{kl}C^{\top}_{lj}\boldsymbol{e}_i\boldsymbol{e}_j\\
[\boldsymbol{A}]_E &= [\boldsymbol{C}][\boldsymbol{A}]_{E'}[\boldsymbol{C}]^{\top},\\
[\boldsymbol{A}]_{E'} &= [\boldsymbol{C}]^{\top}[\boldsymbol{A}]_E[\boldsymbol{C}].
\end{aligned}
$$

Exemplo:

Parto da base canônica $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ e desejo construir uma outra base ortonormal dextrógira $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Os dois primeiros vetores são

$$
\begin{aligned}
\boldsymbol{f}_1 &= \frac{1}{\sqrt{3}}(1,1,1),\\
\boldsymbol{f}_2 &= \frac{1}{\sqrt{6}}(2,-1,-1).
\end{aligned}
$$

Obviamente

$$
\boldsymbol{f}_3 = \boldsymbol{f}_1 \times \boldsymbol{f}_2.
$$

$$
\begin{aligned}
\boldsymbol{f}_3 &= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ 1 & 1 & 1\\ 2 & -1 & -1\end{vmatrix}\\
&= \frac{1}{\sqrt{18}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ 1 & 1 & 1\\ 2 & -1 & -1\end{vmatrix}\\
&= \frac{1}{3\sqrt{3}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ 1 & 1 & 1\\ 2 & -1 & -1\end{vmatrix}\\
&= \frac{1}{3\sqrt{2}}[0\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 3\boldsymbol{e}_3]\\
&= \frac{1}{\sqrt{2}}(0,1,-1).
\end{aligned}
$$

A matriz de rotaçao de E para F é

$$
\begin{aligned}
\boldsymbol{f}_j &= C_{ij}\boldsymbol{e}_i,\\
C_{ij} &= \boldsymbol{f}_j \cdot \boldsymbol{e}_i.
\end{aligned}
$$

Portanto,

$$
[\boldsymbol{C}] = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0\\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2}\\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2}\end{bmatrix}
$$

Se eu quero rodar em torno do eixo 2, devo rodar de 1 para 3 ou de 3 para 1?

$$\epsilon_{132} = \epsilon_{321} = -1,$$
$$\epsilon_{312} = \epsilon_{123} = +1.$$

Resposta: de 3 para 1!

Um sistema de equações lineares é algo da forma

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= y_1, \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= y_2, \\ &\vdots \\ A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{nn}x_n &= y_n. \end{aligned}$$

A transformação

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

projeta qq vetor do $\mathbb{R}^3$ no plano que passa pela origem e cujo normal é $\boldsymbol{k}$. Note que a equação desse plano é

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0.$$

$$\begin{aligned} \boldsymbol{b} &= \boldsymbol{P} \cdot \boldsymbol{a}, \\ b_i &= \left[\delta_{ij} - \frac{k_i k_j}{k^2} \right] a_j \\ b_i &= \delta_{ij} a_j - \frac{k_i k_j}{k^2} a_j \\ &= a_i - \frac{k_i k_j}{k^2} a_j; \\ \boldsymbol{b} \cdot \boldsymbol{k} &= b_i k_i \\ &= \left[a_i - \frac{k_i k_j}{k^2} a_j \right] k_i \\ &= a_i k_i - \left(\frac{k_i k_i}{k^2} \right) k_j a_j \\ &= a_i k_i - a_j k_j = 0 \ \blacksquare \end{aligned}$$

Teorema dos Pis:

$$\begin{aligned} \left(M^A \right)^x &= M^{Ax}. \\ 1 &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{ij}} \right)_j^x \times \ldots \end{aligned}$$

Vamos voltar!

$$
\begin{aligned}
\Pi &= \prod_{j=1}^{n} v_j^{x_j}, \\
[\![\Pi]\!] &= \left[\!\!\left[\prod_{j=1}^{n} v_j^{x_j}\right]\!\!\right] \\
[\![\Pi]\!] &= \prod_{j=1}^{n} [\![v_j^{x_j}]\!] \\
[\![\Pi]\!] &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\mathsf{L}^{A_{2j}}\mathsf{T}^{A_{3j}}\right)^{x_j} \\
[\![\Pi]\!] &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\right)^{x_j} \left(\mathsf{L}^{A_{2j}}\right)^{x_j} \left(\mathsf{T}^{A_{3j}}\right)^{x_j} \\
[\![\Pi]\!] &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\right)^{x_j} \prod_{j=1}^{n} \left(\mathsf{L}^{A_{2j}}\right)^{x_j} \prod_{j=1}^{n} \left(\mathsf{T}^{A_{3j}}\right)^{x_j}
\end{aligned}
$$

Se $n = 4$, teremos por exemplo

$$
\begin{aligned}
\prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\right)^{x_j} &= \left(\mathsf{M}^{A_{11}}\right)^{x_1} \times \left(\mathsf{M}^{A_{12}}\right)^{x_2} \times \left(\mathsf{M}^{A_{13}}\right)^{x_3} \times \left(\mathsf{M}^{A_{14}}\right)^{x_4} = 1 \\
\mathsf{M}^{A_{11}x_1} \times \mathsf{M}^{A_{12}x_2} \times \mathsf{M}^{A_{13}x_3} \times \mathsf{M}^{A_{14}x_4} &= 1 \\
\mathsf{M}^{A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4} &= 1 \\
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 &= 0
\end{aligned}
$$

Autovalores e autovetores

Se $\boldsymbol{v}$ é autovetor de $\boldsymbol{A}$, então $\alpha\boldsymbol{v}$, $\alpha \neq 0$, também é:

$$
\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{v} &= \lambda \boldsymbol{v}; \\
\boldsymbol{A} \cdot [\alpha \boldsymbol{v}] &= \alpha \boldsymbol{A} \cdot \boldsymbol{v} \\
&= \alpha \lambda \boldsymbol{v} \\
&= \lambda [\alpha \boldsymbol{v}].
\end{aligned}
$$

No exemplo em tela, $\lambda = 1$ possui multiplicidade 2 e também está associado a um subespaço de dimensão 2. A equação

$$
x_1 + 2x_2 + x_3 = 0
$$

é a equação de um plano (pela origem), e quaisquer dois vetores desse plano são autovetores. De fato, sejam $\boldsymbol{v}_1, \boldsymbol{v_2}$ dois autovetores LI associados a λ; então

$$
\begin{aligned}
\boldsymbol{A} \cdot [\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2] &= \alpha_1 \boldsymbol{A} \cdot \boldsymbol{v}_1 + \alpha_2 \boldsymbol{A} \cdot \boldsymbol{v}_2 \\
&= \alpha_1 \lambda \boldsymbol{v}_1 + \alpha_2 \lambda \boldsymbol{v}_2 \\
&= \lambda [\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2].
\end{aligned}
$$

No caso univariado,

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 4x, \\
x &= \mathrm{e}^{4t}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 4\mathrm{e}^{4t} = 4x.
\end{aligned}
$$

Transposta de uma transformação linear

$$
\begin{aligned}
\boldsymbol{x} \cdot [\boldsymbol{A}^\top \cdot \boldsymbol{y}] &\equiv \boldsymbol{y} \cdot [\boldsymbol{A} \cdot \boldsymbol{x}] \\
x_k \boldsymbol{e}_k \cdot [A^\top_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \cdot y_l \boldsymbol{e}_l] &= y_k \boldsymbol{e}_k \cdot [A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \cdot x_l \boldsymbol{e}_l] \\
x_k \boldsymbol{e}_k \cdot [A^\top_{ij} \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_l) y_l] &= y_k \boldsymbol{e}_k \cdot [A_{ij} \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_l) x_l] \\
x_k \boldsymbol{e}_k \cdot [A^\top_{ij} \boldsymbol{e}_i \delta_{jl} y_l] &= y_k \boldsymbol{e}_k \cdot [A_{ij} \boldsymbol{e}_i \delta_{jl} x_l] \\
x_k \boldsymbol{e}_k \cdot [A^\top_{ij} y_j \boldsymbol{e}_i] &= y_k \boldsymbol{e}_k \cdot [A_{ij} x_j \boldsymbol{e}_i] \\
x_k A^\top_{ij} y_j (\boldsymbol{e}_k \cdot \boldsymbol{e}_i) &= y_k A_{ij} x_j (\boldsymbol{e}_k \cdot \boldsymbol{e}_i) \\
x_k A^\top_{ij} y_j \delta_{ki} &= y_k A_{ij} x_j \delta_{ki} \\
x_i A^\top_{ij} y_j &= y_i A_{ij} x_j \\
x_i A^\top_{ij} y_j &= y_j A_{ji} x_i \\
x_i A^\top_{ij} y_j &= x_i A_{ji} y_j \\
A^\top_{ij} &= A_{ji} \blacksquare
\end{aligned}
$$

Até agora nós vimos (e demos significado a)

$$
\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{x} &= A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \cdot x_k \boldsymbol{e}_k \\
&= A_{ij} x_k \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) \\
&= A_{ij} x_k \boldsymbol{e}_i \delta_{jk} \\
&= A_{ij} x_j \boldsymbol{e}_i \\
&= [\boldsymbol{A}][\boldsymbol{x}]
\end{aligned}
$$

Eu também posso definir a operação

$$
\begin{aligned}
\boldsymbol{x} \cdot \boldsymbol{A} &= x_k \boldsymbol{e}_k \cdot A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \\
&= x_k A_{ij} (\boldsymbol{e}_k \cdot \boldsymbol{e}_i) \boldsymbol{e}_j \\
&= x_k A_{ij} \delta_{ki} \boldsymbol{e}_j \\
&= x_i A_{ij} \boldsymbol{e}_j \\
&= [x_1 A_{11} + x_2 A_{21} + x_3 A_{31}] \boldsymbol{e}_1 \\
&+ [x_1 A_{12} + x_2 A_{22} + x_3 A_{32}] \boldsymbol{e}_2 \\
&+ [x_1 A_{13} + x_2 A_{23} + x_3 A_{33}] \boldsymbol{e}_3 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \\
&= [\boldsymbol{x}]^\top [\boldsymbol{A}]
\end{aligned}
$$

Portanto eu posso definir a transformação simétrica via

$$
\begin{gathered}
\boldsymbol{x} \cdot \boldsymbol{A}^\top \equiv \boldsymbol{A} \cdot \boldsymbol{x}, \\
\vdots \\
A^\top_{ij} = A_{ji}.
\end{gathered}
$$

Na equação

$$A_{ij} x_j = \lambda x_i$$

Os $A_{ij}s$ são reais. O conjugado da equação é

$$
\begin{aligned}
(A_{ij} x_j)^* &= (\lambda x_i)^* \\
A^*_{ij} x^*_j &= \lambda^* x^*_i \\
A_{ij} x^*_j &= \lambda^* x^*_i
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{e}_1 &= (1, 0, 0), \\
\boldsymbol{f}_1 &= \frac{1}{\sqrt{3}} (1, 1, 1), \\
\boldsymbol{e}_1 \cdot \boldsymbol{f}_1 &= \frac{1}{\sqrt{3}} =
\end{aligned}
$$

$$\boldsymbol{e}_j \qquad \boldsymbol{j}$$

Quando existe uma base de autovetores para a transformação $\boldsymbol{A}$, a matriz de $\boldsymbol{A}$ nessa base é uma matriz diagonal composta pelos autovalores.

Olhe para o caso 3×3

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_F$$

Cada coluna representa a imagem de um dos vetores da base e **se** eu estiver na base dos autovetores:

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{f}_1 &= A_{11}\boldsymbol{f}_1 + A_{21}\boldsymbol{f}_2 + A_{31}\boldsymbol{f}_3 = \lambda_1 \boldsymbol{f}_1 \\ \boldsymbol{A} \cdot \boldsymbol{f}_2 &= A_{12}\boldsymbol{f}_1 + A_{22}\boldsymbol{f}_2 + A_{32}\boldsymbol{f}_3 = \lambda_2 \boldsymbol{f}_2 \\ \boldsymbol{A} \cdot \boldsymbol{f}_3 &= A_{13}\boldsymbol{f}_1 + A_{23}\boldsymbol{f}_2 + A_{33}\boldsymbol{f}_3 = \lambda_3 \boldsymbol{f}_3 \end{aligned}$$

Portanto, concluímos que

$$\boldsymbol{A}_F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_F$$

Exercício 5.31

$$\begin{aligned} \boldsymbol{S} : \boldsymbol{A} &= S_{ij}A_{lm}(\boldsymbol{e}_j \cdot \boldsymbol{e}_l)(\boldsymbol{e}_i \cdot \boldsymbol{e}_m) \\ &= S_{ij}A_{lm}\delta_{jl}\delta_{im} \\ &= S_{ij}A_{ji} \\ &= \frac{1}{2}S_{ij}A_{ji} + \frac{1}{2}S_{ij}A_{ji} \\ &= \frac{1}{2}S_{ij}A_{ji} + \frac{1}{2}S_{ji}A_{ij} \\ &= \frac{1}{2}S_{ij}(A_{ji} + A_{ij}) = 0. \end{aligned}$$

Uma EDO de ordem 2 pode ser escrita como um sistema de duas EDOs acopladas de ordem 1.

$$
\begin{aligned}
\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u &= 0\\
\frac{\mathrm{d}u}{\mathrm{d}t} &= v\\
\frac{\mathrm{d}v}{\mathrm{d}t} + u &= 0\\
\frac{\mathrm{d}u}{\mathrm{d}t} &= v\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= -u
\end{aligned}
$$

$$
\begin{aligned}
u(t) &= \frac{A - \mathrm{i}B}{2}\left[\cos t + \mathrm{i}\,\mathrm{sen}\, t\right] + \frac{A + \mathrm{i}B}{2}\left[\cos t - \mathrm{i}\,\mathrm{sen}\, t\right]\\
&= \frac{A}{2}[\cos t + \mathrm{i}\,\mathrm{sen}\, t] - \frac{\mathrm{i}B}{2}[\cos t + \mathrm{i}\,\mathrm{sen}\, t] + \frac{A}{2}[\cos t - \mathrm{i}\,\mathrm{sen}\, t] + \frac{\mathrm{i}B}{2}[\cos t - \mathrm{i}\,\mathrm{sen}\, t]\\
&= \frac{A}{2}\cos t + \frac{A}{2}\cos t + \frac{-\mathrm{i}^2 B}{2}\,\mathrm{sen}\, t + \frac{-\mathrm{i}^2 B}{2}\,\mathrm{sen}\, t + \mathrm{i}\left[\frac{A}{2}\,\mathrm{sen}\, t - \frac{B}{2}\cos t - \frac{A}{2}\,\mathrm{sen}\, t + \frac{B}{2}\cos t\right]\\
&= A\cos t + B\,\mathrm{sen}\, t \;\blacksquare
\end{aligned}
$$

Ordem de diferenciação (ou de derivação):

$$
\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).
$$

Em geral,

$$
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.
$$

Série de Taylor

$$
f(x) = \frac{1}{0!}f(x_0) + \frac{1}{1!}\frac{\mathrm{d}f(x_0)}{\mathrm{d}x}(x - x_0) + \frac{1}{2!}\frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2}(x - x_0)^2 + \ldots + \frac{1}{n!}\frac{\mathrm{d}^n f(x_0)}{\mathrm{d}x^n}(x - x_0)^n + \ldots
$$

No caso de $n = 2$, e de uma função (também) de 2 variáveis

$$
\begin{aligned}
\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_i \partial x_j}(x_i - x_{0i})(x_j - x_{0j}) &= \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_i \partial x_j}(x_i - x_{0i})(x_j - x_{0j})\\
&= \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_1}(x_1 - x_{01})(x_1 - x_{01}) +\\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_2}(x_1 - x_{01})(x_2 - x_{02}) +\\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2 \partial x_1}(x_2 - x_{02})(x_1 - x_{01}) +\\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2 \partial x_2}(x_2 - x_{02})(x_2 - x_{02}).
\end{aligned}
$$

Verifique o número de índices repetidos em uma expressão

$$\begin{aligned}
\boldsymbol{P}\cdot\boldsymbol{a} &= P_{il}a_l\boldsymbol{e}_i \\
&= \left[\delta_{il} - \frac{k_ik_l}{k^2}\right]a_l\boldsymbol{e}_i \\
[\boldsymbol{P}\cdot\boldsymbol{a}]\times\boldsymbol{k} &= \epsilon_{ijk}\left[\delta_{il} - \frac{k_ik_l}{k^2}\right]a_lk_j\boldsymbol{e}_k
\end{aligned}$$

Teorema da função implícita

$$\begin{aligned}
\frac{\mathrm{d}u(x_0)}{\mathrm{d}x} &= -\frac{\frac{\partial f(x_0,u_0)}{\partial x}}{\frac{\partial f(x_0,u_0)}{\partial u}} \\
u(x) &= u_0 + \frac{\mathrm{d}u(x_0)}{\mathrm{d}x}(x-x_0)+\dots
\end{aligned}$$

$$\begin{aligned}
x &= x(u,v) & u &= u(x,y) \\
y &= y(u,v) & v &= v(x,y)
\end{aligned}$$

A regra de Leibnitz:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x &= \int_a^b \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x, \qquad \text{porém} \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &\neq \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x. \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x + f(b,t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a,t)\frac{\mathrm{d}a}{\mathrm{d}t}; \\
\frac{\mathrm{D}N}{\mathrm{D}t} &= \frac{\partial}{\partial t}\int_{VC}\eta\rho\,\mathrm{d}V + \oint_{SC}(\boldsymbol{n}\cdot\boldsymbol{v})\eta\rho\,\mathrm{d}S
\end{aligned}$$

A idéia é mudar a variável de integração de x para ξ:

$$\begin{aligned}
x &= X(\xi,t) \\
\mathrm{d}x &= \frac{\partial X}{\partial \xi}\mathrm{d}\xi + \frac{\partial X}{\partial t}\mathrm{d}t, \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta f(X(\xi,t),t)\frac{\partial X}{\partial \xi}\mathrm{d}\xi
\end{aligned}$$

Revisite a regra da cadeia!

$$\begin{aligned}
&f(x,u); \\
F(x) &= f(x,u(x)); \\
\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}
\end{aligned}$$

Quero o módulo de $\boldsymbol{v} = (1,\mathrm{i},0)$:

$$
\begin{aligned}
|\boldsymbol{v}| &= \sqrt{\sum_{i=1}^{3} v_i^* v_i} \\
&= \sqrt{(1 \times 1 + (-\mathrm{i} \times \mathrm{i}) + 0 \times 0)} \\
&= \sqrt{1 - \mathrm{i}^2 + 0} \\
&= \sqrt{1 + 1 + 0} \\
&= \sqrt{2}
\end{aligned}
$$

Regra de Leibnitz:
Considere

$$
\begin{aligned}
\frac{\partial}{\partial \xi}\frac{\partial X}{\partial t} &=? \\
\frac{\partial X}{\partial t} &= h(\xi, t) \\
\frac{\partial}{\partial \xi}\frac{\partial X}{\partial t} &= \frac{\partial h}{\partial \xi} = \frac{\partial h}{\partial x}\frac{\partial X}{\partial \xi}
\end{aligned}
$$

Surgimento de uma “velocidade”

$$
\begin{aligned}
x &= X(\xi, t), \\
u &= \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial X}{\partial t}
\end{aligned}
$$

A regra de Leibnitz é

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= f(b,t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a,t)\frac{\mathrm{d}b}{\mathrm{d}t} + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t &= f(x,b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(x,a)\frac{\mathrm{d}a}{\mathrm{d}x} + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\,\mathrm{d}t \\
b(x) &= x, \\
a(x) &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \left[xt + \mathrm{e}^{-t^2}\right]\,\mathrm{d}t &= x^2 + \mathrm{e}^{-x^2} + \int_0^x \frac{\partial}{\partial x}\left[xt + \mathrm{e}^{-t^2}\right]\,\mathrm{d}t \\
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \left[xt + \mathrm{e}^{-t^2}\right]\,\mathrm{d}t &= x^2 + \mathrm{e}^{-x^2} + \int_0^x t\,\mathrm{d}t \\
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \left[xt + \mathrm{e}^{-t^2}\right]\,\mathrm{d}t &= \frac{3x^2}{2} + \mathrm{e}^{-x^2}
\end{aligned}
$$

Quero calcular um comprimento de arco sobre a curva (entre $t = 0$ e $t = \tau$)

$$
\begin{aligned}
\boldsymbol{r}(t) &= \cos(t)[\boldsymbol{e}_x + \boldsymbol{e}_y] - \sqrt{2}\,\mathrm{sen}(t)\boldsymbol{e}_z \\
\ell_{\mathscr{L}} &= \int_{r\in\mathscr{L}} \left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]^{1/2} \mathrm{d}t \\
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= -\,\mathrm{sen}(t)[\boldsymbol{e}_x + \boldsymbol{e}_y] - \sqrt{2}\cos(t)\boldsymbol{e}_z \\
&= (-\,\mathrm{sen}(t), -\,\mathrm{sen}(t), -\sqrt{2}\cos(t)) \\
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \mathrm{sen}^2(t) + \mathrm{sen}^2(t) + 2\cos^2(t) = 2 \\
\ell_{\mathscr{L}} &= \int_0^\tau \sqrt{2}\,\mathrm{d}t = \sqrt{2}\tau.
\end{aligned}
$$

Exemplo 7.11: Calcule a área da superfície externa do parabolóide de revolução $z = x^2 + y^2$, $x^2 + y^2 \leq 1$

Procuro a teoria:

$$
\begin{aligned}
z &= g(x,y) = x^2 + y^2, \\
A_{\mathscr{S}} &= \int_{R_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y \\
\frac{\partial g}{\partial x} &= 2x, \\
\frac{\partial g}{\partial y} &= 2y, \\
A_{\mathscr{S}} &= \int_{R_{xy}} \sqrt{1 + (2x)^2 + (2y)^2}\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1 + 4(x^2 + y^2)}\,r\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r\mathrm{d}\theta \\
&= \left[\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r\right]\left[\int_{\theta=0}^{2\pi} \mathrm{d}\theta\right] \\
&= 2\pi\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r \\
&= 2\pi\frac{5\sqrt{5} - 1}{12} = \frac{5\sqrt{5} - 1}{6}.
\end{aligned}
$$

Virtudes acadêmicas **e** profissionais:

1. Trabalho.

2. Persistência.

3. Iniciativa.

Divergência, gradiente, rotacional

$$
\begin{aligned}
\operatorname{div} \boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
\boldsymbol{\nabla} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= ? \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (u, v, w) \\
&= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.
\end{aligned}
$$

$\boldsymbol{\nabla}$ é o **operador** “nabla”.

Posso repetir, de forma mais resumida, em notação indicial. Suponha que $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ seja uma base **ortonormal** do $\mathbb{R}^3$.

$$
\boldsymbol{\nabla} \equiv \boldsymbol{e}_i \frac{\partial}{\partial x_i},
$$

e não

$$
\begin{aligned}
\boldsymbol{\nabla} &\equiv \frac{\partial}{\partial x_i} \boldsymbol{e}_i, \\
\boldsymbol{u} &= u_j \boldsymbol{e}_j \\
\operatorname{div} \boldsymbol{u} &\equiv \boldsymbol{\nabla} \cdot \boldsymbol{u} \\
&= \boldsymbol{e}_i \frac{\partial}{\partial x_i} \cdot u_j \boldsymbol{e}_j \\
&= (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \frac{\partial u_j}{\partial x_i} \\
&= \delta_{ij} \frac{\partial u_j}{\partial x_i} \\
&= \frac{\partial u_i}{\partial x_i}.
\end{aligned}
$$

Lei de Fourier é

$$
\begin{aligned}
\boldsymbol{q} &= -\rho c_p \alpha \boldsymbol{\nabla} T, \\
\dot{Q} &= -\oint_{\text{casca do ovo}} (\boldsymbol{n} \cdot \boldsymbol{q}) \, \mathrm{d}A
\end{aligned}
$$

Gradiente, e sua “definição” (em coordenadas cartesianas) com o uso de $\boldsymbol{\nabla}$:

$$
\begin{aligned}
&\alpha\boldsymbol{u}, \\
&\boldsymbol{u}\alpha, \\
\mathbf{grad}\, f &= \nabla f, \\
\mathbf{grad}\, f &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f \\
\mathbf{grad}\, f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\mathbf{grad}\, f &= \boldsymbol{e}_i \frac{\partial}{\partial x_i} f \\
&= \boldsymbol{e}_i \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \boldsymbol{e}_i
\end{aligned}
$$

O rotacional:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{u} &= \epsilon_{ijk} \nabla_i u_j \boldsymbol{e}_k \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_i} u_j \boldsymbol{e}_k \\
&= \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \boldsymbol{e}_k
\end{aligned}$$

Em resumo, as operações relacionados a

$$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

são

1. A divergência: $\boldsymbol{u} = (u, v, w)$ e
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z};$$

2. O gradiente: $f(x, y, z)$ e
$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

3. O rotacional:
$$\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}.$$

O laplaciano!

$$\begin{aligned}
\nabla^2 f &\equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f \\
&\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\
&\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial^2 f}{\partial x_i \partial x_i}.
\end{aligned}$$

Por extensão, também podemos tratar do laplaciano de um vetor:

$$\begin{aligned}
\nabla^2 \boldsymbol{u} &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \boldsymbol{i} \\
&+ \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \boldsymbol{j} \\
&+ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \boldsymbol{k} \\
&= \frac{\partial^2 u_i}{\partial x_k \partial x_k} \boldsymbol{e}_i.
\end{aligned}$$

Aplicações em mecânica dos fluidos:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \boldsymbol{\cdot} (\rho \boldsymbol{u}) = 0, \\
\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u} \right] = \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u},
\end{aligned}
$$

Importante: a forma das operações muda quando nós mudamos de sistema de coordenadas. Por exemplo, em coordenadas cilíndricas, **não** é verdade que

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

$$\begin{aligned}
r &= (x^2 + y^2)^{1/2}, \\
\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2 + y^2)^{-1/2} \times 2x \\
&= \frac{x}{(x^2 + y^2)^{1/2}} = \frac{x}{r} = \cos(\theta). \\
\theta &= \operatorname{arctg}\left(\frac{y}{x}\right), \\
\frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x} \operatorname{arctg}\left(\frac{y}{x}\right), \\
\frac{\mathrm{d} \operatorname{arctg} u}{\mathrm{d}u} &= \frac{1}{1+u^2}, \\
\frac{\partial \theta}{\partial x} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \times -\frac{y}{x^2}
\end{aligned}$$

Teorema da divergência, aplicações em Mecânica dos Fluidos.

$$\begin{aligned}
\frac{\partial}{\partial t} \int_{\mathscr{C}} \rho \, \mathrm{d}V + \oint_{\mathscr{S}} \rho (\boldsymbol{n} \cdot \boldsymbol{v}) \, \mathrm{d}A = 0 \\
\frac{\partial}{\partial t} \int_{\mathscr{C}} \rho \, \mathrm{d}V + \oint_{\mathscr{S}} (\boldsymbol{n} \cdot [\rho \boldsymbol{v}]) \, \mathrm{d}A = 0 \\
\int_{\mathscr{C}} \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_{\mathscr{C}} \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] \, \mathrm{d}V = 0 \\
\int_{\mathscr{C}} \left\{ \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] \right\} \mathrm{d}V = 0 \\
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] = 0
\end{aligned}$$

Exercício 7.9

$$\begin{aligned}
x^2 + y^2 = 1 \\
z &= 1 - y^2, \\
z &= u, \\
x &= \cos v, \\
y &= \operatorname{sen} v, \\
\boldsymbol{r} &= (\cos v, \operatorname{sen} v, u), \\
\frac{\partial \boldsymbol{r}}{\partial u} &= (0, 0, 1), \\
\frac{\partial \boldsymbol{r}}{\partial v} &= (-\operatorname{sen} v, \cos v, 0)
\end{aligned}$$

A área da superfície é

$$A = \iint_{R_{uv}} \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| \mathrm{d}u\mathrm{d}v$$

No nosso caso a área é simplesmente

$$\begin{aligned} A &= \iint_{R_{uv}} \mathrm{d}u\mathrm{d}v \\ &= \int_{v=0}^{2\pi} \int_{u=0}^{1-v^2} \mathrm{d}u\mathrm{d}v \end{aligned}$$

Campos irrotacionais

$$\boldsymbol{F} = (F_1, F_2, F_3)$$

A integral de log:

$$
\begin{aligned}
\int \frac{\mathrm{d}u}{u} &= \ln|u|; \\
\int_a^b \frac{\mathrm{d}u}{u} &= \ln|b| - \ln|a| \\
&= \ln\frac{|b|}{|a|}
\end{aligned}
$$

O que acontece se os sinais de a e b forem iguais: teremos somente duas possibilidades:

$$
\begin{aligned}
|a| &= a & \mathrm{e}|b| &= b, \\
|a| &= -a & \mathrm{e}|b| &= -b.
\end{aligned}
$$

Em ambos os casos,

$$
\ln\frac{|b|}{|a|} = \ln\frac{b}{a}.
$$

Ao tentarmos resolver

$$
\mathscr{D}y = f(x)
$$

queremos encontrar a **função** $y(x)$.

$$
\begin{aligned}
\mathscr{D} &= \frac{\mathrm{d}}{\mathrm{d}x}, \\
\mathscr{D}y &= \frac{\mathrm{d}y}{\mathrm{d}x}, \\
\mathscr{D} &= x + y\frac{\mathrm{d}}{\mathrm{d}x}, \\
\mathscr{D}y &= xy + y\frac{\mathrm{d}y}{\mathrm{d}x}, \\
\mathscr{D} &= x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2, \\
\mathscr{D}y &= x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2y, \\
\mathscr{D} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x} + 1, \\
\mathscr{D}y &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y.
\end{aligned}
$$

$$
\mathscr{D}y = f(x)
$$

Uma EDO com coeficientes não constantes é

$$
x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.
$$

EDOs lineares, coeficientes constantes, ordem 2.

$$\begin{aligned}
a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy &= 0,\\
y &= \mathrm{e}^{\lambda x},\\
y' &= \lambda \mathrm{e}^{\lambda x},\\
y'' &= \lambda^2 \mathrm{e}^{\lambda x},\\
a\lambda^2\mathrm{e}^{\lambda x} + b\lambda\mathrm{e}^{\lambda x} + c\mathrm{e}^{\lambda x} &= 0,\\
\mathrm{e}^{\lambda x}\left[a\lambda^2 + b\lambda + c\right] &= 0
\end{aligned}$$

Se

$$ay'' + by' + cy = 0,$$

a equação característica é

$$a\lambda^2 + b\lambda + c = 0.$$

$$\begin{aligned}
\lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};\\
&= -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}\\
&= \alpha \pm \beta\\
y &= C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x}\\
&= C_1\mathrm{e}^{(\alpha+\beta)x} + C_2\mathrm{e}^{(\alpha-\beta)x}\\
&= C_1\mathrm{e}^{\alpha x}\mathrm{e}^{\beta x} + C_2\mathrm{e}^{\alpha x}\mathrm{e}^{-\beta x}\\
&= \mathrm{e}^{\alpha x}\left[C_1\mathrm{e}^{\beta x} + C_2\mathrm{e}^{-\beta x}\right]\\
C_1 &= \frac{1}{2}\left(D_1 + D_2\right)\\
C_2 &= \frac{1}{2}\left(D_1 - D_2\right)\\
y &= \mathrm{e}^{\alpha x}\left[\frac{1}{2}\left(D_1 + D_2\right)\mathrm{e}^{\beta x} + \frac{1}{2}\left(D_1 - D_2\right)\mathrm{e}^{-\beta x}\right]\\
y &= \mathrm{e}^{\alpha x}\left[D_1\frac{\mathrm{e}^{\beta x} + \mathrm{e}^{-\beta x}}{2} + D_2\frac{\mathrm{e}^{\beta x} - \mathrm{e}^{-\beta x}}{2}\right]\\
\cosh(x) &\equiv \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2},\\
\operatorname{senh}(x) &\equiv \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2},\\
y &= \mathrm{e}^{\alpha x}\left[D_1\cosh(\beta x) + D_2\operatorname{senh}(\beta x)\right] \blacksquare
\end{aligned}$$

Problema “real”, raízes “complexas”: temos

$$ay'' + by' + cy = 0$$

e desejamos $y(x) \in \mathbb{R}$. Porém,

$$b^2 - 4ac < 0.$$

A solução geral será a mesma de antes:

$$
\begin{aligned}
y &= C_1 \mathrm{e}^{\lambda_1 x} + C_2 \mathrm{e}^{\lambda_2 x} \\
&= C_1 \mathrm{e}^{(\alpha + \mathrm{i}\beta)x} + C_2 \mathrm{e}^{(\alpha - \mathrm{i}\beta)x} \\
&= C_1 \mathrm{e}^{\alpha x} \mathrm{e}^{\mathrm{i}\beta x} + C_2 \mathrm{e}^{\alpha x} \mathrm{e}^{-\mathrm{i}\beta x} \\
&= \mathrm{e}^{\alpha x} \left[C_1 \mathrm{e}^{\mathrm{i}\beta x} + C_2 \mathrm{e}^{-\mathrm{i}\beta x} \right]; \\
C_1 &= \frac{1}{2}(A - \mathrm{i}B), \\
C_2 &= \frac{1}{2}(A + \mathrm{i}B), \\
y &= \mathrm{e}^{\alpha x} \left[\frac{1}{2}(A - \mathrm{i}B)\mathrm{e}^{\mathrm{i}\beta x} + \frac{1}{2}(A + \mathrm{i}B)\mathrm{e}^{-\mathrm{i}\beta x} \right] \\
y &= \mathrm{e}^{\alpha x} \left[A \left(\frac{\mathrm{e}^{\mathrm{i}\beta x} + \mathrm{e}^{-\mathrm{i}\beta x}}{2} \right) - \mathrm{i}B \left(\frac{\mathrm{e}^{\mathrm{i}\beta x} - \mathrm{e}^{-\mathrm{i}\beta x}}{2} \right) \right] \\
\mathrm{e}^{\mathrm{i}\beta x} &= \cos(\beta x) + \mathrm{i}\,\mathrm{sen}(\beta x), \\
\mathrm{e}^{-\mathrm{i}\beta x} &= \cos(\beta x) - \mathrm{i}\,\mathrm{sen}(\beta x), \\
\cosh(\mathrm{i}\beta x) &= \frac{\mathrm{e}^{\mathrm{i}\beta x} + \mathrm{e}^{-\mathrm{i}\beta x}}{2} = \frac{1}{2}\left[(\cos(\beta x) + \mathrm{i}\,\mathrm{sen}(\beta x)) + (\cos(\beta x) - \mathrm{i}\,\mathrm{sen}(\beta x)) \right] \\
&= \cos(\beta x); \\
\mathrm{senh}(\mathrm{i}\beta x) &= \frac{\mathrm{e}^{\mathrm{i}\beta x} - \mathrm{e}^{-\mathrm{i}\beta x}}{2} = \frac{1}{2}\left[(\cos(\beta x) + \mathrm{i}\,\mathrm{sen}(\beta x)) - (\cos(\beta x) - \mathrm{i}\,\mathrm{sen}(\beta x)) \right] \\
&= \mathrm{i}\,\mathrm{sen}(\beta x); \\
y &= \mathrm{e}^{\alpha x} \left[A \cos(\beta x) - \mathrm{i}B \times \mathrm{i}\,\mathrm{sen}(\beta x) \right] \\
&= \mathrm{e}^{\alpha x} \left[A \cos(\beta x) + B\,\mathrm{sen}(\beta x) \right] \;\blacksquare
\end{aligned}
$$

O método de variação de constantes, equações diferenciais ordinárias de ordem 2.

$$
\begin{aligned}
ay'' + by' + cy &= f(x), \\
ay_h'' + by_h' + cy_h &= 0,
\end{aligned}
$$

ache y_1 e y_2 de tal maneira que

$$
\begin{aligned}
ay_1'' + by_1' + cy_1 &= 0, \\
ay_2'' + by_2' + cy_2 &= 0,
\end{aligned}
$$

Equações diferenciais ordinárias de ordem 2, não-homogêneas: solução pelo método de "variação de constantes".

$$
\begin{aligned}
y'' - 3y' + 2y &= x,\\
y_h'' - 3y_h' + 2y_h &= 0,\\
\lambda^2 - 3\lambda + 2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \frac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2}\\
\lambda &= \frac{3 \pm 1}{2}\\
&= 2, \qquad 1;\\
y_h &= A\mathrm{e}^x + B\mathrm{e}^{2x};
\end{aligned}
$$

Procure a solução na forma

$$
\begin{aligned}
y(x) &= A(x)\mathrm{e}^x + B(x)\mathrm{e}^{2x};\\
y'(x) &= A\mathrm{e}^x + 2B\mathrm{e}^{2x} + \left[A'\mathrm{e}^x + B'\mathrm{e}^{2x}\right]
\end{aligned}
$$

Controle as derivadas de A e B, e impeça derivadas de ordem 2

$$
y'(x) = A\mathrm{e}^x + 2B\mathrm{e}^{2x} + \underbrace{\left[A'\mathrm{e}^x + B'\mathrm{e}^{2x}\right]}_{=0}
$$

$$
\begin{aligned}
A'\mathrm{e}^x + B'\mathrm{e}^{2x} &= 0;\\
y'(x) &= A\mathrm{e}^x + 2B\mathrm{e}^{2x},\\
y''(x) &= A\mathrm{e}^x + 4B\mathrm{e}^{2x} + A'\mathrm{e}^x + 2B'\mathrm{e}^{2x}
\end{aligned}
$$

Substituindo na equação original, obtemos

$$
\begin{aligned}
A\mathrm{e}^x + 4B\mathrm{e}^{2x} + A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} - 3\left[A\mathrm{e}^x + 2B\mathrm{e}^{2x}\right] + 2\left[A\mathrm{e}^x + B\mathrm{e}^{2x}\right] &= x,\\
\mathrm{e}^x\left[A - 3A + 2A\right] + \mathrm{e}^{2x}\left[4B - 6B + 2B\right] + A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x\\
A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x
\end{aligned}
$$

Reúno agora duas EDOs em A e B:

$$
\begin{aligned}
A'\mathrm{e}^x + B'\mathrm{e}^{2x} &= 0;\\
A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x\\
2A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= 0;\\
A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x\\
A'\mathrm{e}^x &= -x\\
\frac{\mathrm{d}A}{\mathrm{d}x} &= -x\mathrm{e}^{-x}
\end{aligned}
$$

Integrando,

$$
\begin{aligned}
A(x) &= (x+1)\mathrm{e}^{-x} + C_1 \\
A'\mathrm{e}^{x} + B'\mathrm{e}^{2x} &= 0; \\
A'\mathrm{e}^{x} + 2B'\mathrm{e}^{2x} &= x \\
B'\mathrm{e}^{2x} &= x, \\
\frac{\mathrm{d}B}{\mathrm{d}x} &= x\mathrm{e}^{-2x}, \\
B(x) &= -\frac{1}{4}(2x+1)\mathrm{e}^{-2x} + C_2, \\
y(x) &= A(x)\mathrm{e}^{x} + B(x)\mathrm{e}^{2x}, \\
&= \left[(x+1)\mathrm{e}^{-x} + C_1\right]\mathrm{e}^{x} + \left[-\frac{1}{4}(2x+1)\mathrm{e}^{-2x} + C_2\right]\mathrm{e}^{2x} \\
&= C_1\mathrm{e}^{x} + C_2\mathrm{e}^{2x} + x + 1 - \frac{1}{4}(2x+1) \\
y(x) &= \underbrace{C_1\mathrm{e}^{x} + C_2\mathrm{e}^{2x}}_{y_h(x)} + \underbrace{\frac{2x+3}{4}}_{y_p(x)} \blacksquare
\end{aligned}
$$

Uma equação de Euler com raízes **complexas**:

$$
\begin{aligned}
x^2 y'' + y &= 0,\\
y &= x^m,\\
y' &= m x^{m-1},\\
y'' &= (m-1) m x^{m-2}\\
x^2 \underbrace{(m-1) m x^{m-2}}_{y''} + \underbrace{x^m}_{y} &= 0\\
\left[(m-1)m + 1\right] x^m &= 0,\\
m^2 - m + 1 &= 0,\\
m &= \frac{1 \pm \mathrm{i}\sqrt{3}}{2}\\
y &= C_1 x^{\frac{1+\mathrm{i}\sqrt{3}}{2}} + C_2 x^{\frac{1-\mathrm{i}\sqrt{3}}{2}}\\
x^a &= \exp(\ln x^a) = \exp(a \ln x);\\
x^{\frac{1\pm\mathrm{i}\sqrt{3}}{2}} &= x^{\frac{1}{2}} x^{\frac{\pm\mathrm{i}\sqrt{3}}{2}}\\
x^{\frac{\pm\mathrm{i}\sqrt{3}}{2}} &= \exp\left(\mathrm{i}\frac{\pm\sqrt{3}}{2}\ln x\right)\\
\exp(\mathrm{i}\theta) &= \cos(\theta) + \mathrm{i}\,\mathrm{sen}(\theta)\\
x^{\frac{\pm\mathrm{i}\sqrt{3}}{2}} &= \cos\left(\frac{\sqrt{3}}{2}\ln x\right) \pm \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\\
y &= C_1 x^{\frac{1+\mathrm{i}\sqrt{3}}{2}} + C_2 x^{\frac{1-\mathrm{i}\sqrt{3}}{2}}\\
y &= C_1 x^{1/2} x^{+\frac{\mathrm{i}\sqrt{3}}{2}} + C_2 x^{1/2} x^{-\frac{\mathrm{i}\sqrt{3}}{2}}\\
y &= C_1 x^{1/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\right]\\
&\quad + C_2 x^{1/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) - \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\right]
\end{aligned}
$$

Agora, escolha C_1 e C_2 conjugados complexos:

$$
\begin{aligned}
C_1 &= \frac{1}{2}\left(D_1 - \mathrm{i}D_2\right),\\
C_2 &= \frac{1}{2}\left(D_1 + \mathrm{i}D_2\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{y}{x^{1/2}} &= \frac{1}{2}\left(D_1 - \mathrm{i}D_2\right)[C + \mathrm{i}S] + \frac{1}{2}\left(D_1 + \mathrm{i}D_2\right)[C - \mathrm{i}S]\\
&= \frac{1}{2}\left\{D_1 C - \mathrm{i}^2 D_2 S + \mathrm{i}(-D_2 C + D_1 S) + D_1 C - \mathrm{i}^2 D_2 S + \mathrm{i}(D_2 C - D_1 S)\right\}\\
&= D_1 C + D_2 S;\\
y &= x^{1/2}\left[D_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + D_2\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\right] \blacksquare
\end{aligned}
$$

Desigualdades importantes com números complexos
Resultado auxiliar:

$$\begin{aligned}
z_1^* z_2 + z_1 z_2^* &= (z_1^* z_2) + (z_1^* z_2)^* = 2\,\mathrm{Re}(z_1^* z_2) \le 2|z_1^* z_2| = 2|z_1^*||z_2| = 2|z_1||z_2| \\
|x + \mathrm{i}y|^2 &= x^2 + y^2 \\
&= (x - \mathrm{i}y)(x + \mathrm{i}y) \\
&= x^2 - (\mathrm{i}y)^2 \\
&= x^2 - \mathrm{i}^2 y^2 \\
&= x^2 + y^2 \;\blacksquare
\end{aligned}$$

A desigualdade do triângulo, agora, é

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)^*(z_1 + z_2) \\
&= z_1^* z_1 + z_2^* z_2 + z_1^* z_2 + z_2^* z_1 \\
&= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1^* z_2) \\
&\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
&= (|z_1| + |z_2|)^2; \;\Rightarrow \\
|z_1 + z_2| &\le |z_1| + |z_2|
\end{aligned}$$

Variáveis complexas têm mil (e uma) utilidades

$$\begin{aligned}
I &= \int_0^\infty \mathrm{e}^{-x}\cos(ax)\,\mathrm{d}x; \\
\mathrm{e}^{\mathrm{i}ax} &= \cos(ax) + \mathrm{i}\,\mathrm{sen}(ax); \\
I &= \int_0^\infty \mathrm{e}^{-x}\,\mathrm{Re}(\mathrm{e}^{\mathrm{i}ax})\,\mathrm{d}x; \\
I &= \mathrm{Re}\int_0^\infty \mathrm{e}^{-x}\mathrm{e}^{\mathrm{i}ax}\,\mathrm{d}x; \\
I &= \mathrm{Re}\int_0^\infty \mathrm{e}^{(\mathrm{i}a-1)x}\,\mathrm{d}x; \\
I &= \mathrm{Re}\,\frac{1}{\mathrm{i}a - 1}\int_0^\infty \mathrm{e}^{(\mathrm{i}a-1)x}\,\mathrm{d}(\mathrm{i}a-1)x; \\
I &= \mathrm{Re}\,\frac{1}{\mathrm{i}a - 1}\mathrm{e}^{(\mathrm{i}a-1)x}\Big|_{x=0}^{x=\infty} \\
I &= \mathrm{Re}\,\frac{1}{\mathrm{i}a - 1}\left[\mathrm{e}^{(\mathrm{i}a-1)\infty} - 1\right]
\end{aligned}$$

Agora avalio o limite cuidadosamente:

$$\begin{aligned}
\lim_{x\to\infty} \mathrm{e}^{(\mathrm{i}a-1)x} &= \lim_{x\to\infty} \mathrm{e}^{-x}(\cos(ax) + \mathrm{i}\,\mathrm{sen}(ax)) \\
\lim_{x\to\infty} |\mathrm{e}^{(\mathrm{i}a-1)x}| &= \lim_{x\to\infty} |\mathrm{e}^{-x}||(\cos(ax) + \mathrm{i}\,\mathrm{sen}(ax))|; \\
|\cos(ax) + \mathrm{i}\,\mathrm{sen}(ax)| &= 1 \\
|x + \mathrm{i}y| &= \sqrt{x^2 + y^2} \\
\lim_{x\to\infty} |\mathrm{e}^{(\mathrm{i}a-1)x}| &= \lim_{x\to\infty} |\mathrm{e}^{-x}| = 0; \\
\lim_{x\to\infty} \mathrm{e}^{(\mathrm{i}a-1)x} &= 0 \;\blacksquare
\end{aligned}$$

Portanto,

$$
\begin{aligned}
I &= \operatorname{Re} \frac{1}{\mathrm{i}a - 1}\,[-1] \\
&= \operatorname{Re} \frac{1}{1 - \mathrm{i}a} \\
&= \operatorname{Re} \frac{1 + \mathrm{i}a}{(1 - \mathrm{i}a)(1 + \mathrm{i}a)} \\
&= \operatorname{Re} \frac{1 + \mathrm{i}a}{1 + a^2} = \frac{1}{1 + a^2} \ \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
\cos(\mathrm{i}x) &= \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2} \\
&= \frac{\cos \mathrm{i}x + \mathrm{i} \operatorname{sen} \mathrm{i}x + \cos \mathrm{i}x - \mathrm{i} \operatorname{sen} \mathrm{i}x}{2} \\
&= \cos \mathrm{i}x. \\
\operatorname{sen}(\mathrm{i}x) &= \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} \\
&= \frac{\cos \mathrm{i}x + \mathrm{i} \operatorname{sen} \mathrm{i}x - (\cos \mathrm{i}x - \mathrm{i} \operatorname{sen} \mathrm{i}x)}{2\mathrm{i}} \\
\cos^2(\mathrm{i}x) + \operatorname{sen}^2(\mathrm{i}x) &= \frac{\mathrm{e}^{2\mathrm{i}x} + 2 + \mathrm{e}^{-2\mathrm{i}x}}{4} + \frac{\mathrm{e}^{2\mathrm{i}x} - 2 + \mathrm{e}^{-2\mathrm{i}x}}{-4} \\
&= \frac{\mathrm{e}^{2\mathrm{i}x} + 2 + \mathrm{e}^{-2\mathrm{i}x}}{4} + \frac{-\mathrm{e}^{2\mathrm{i}x} + 2 - \mathrm{e}^{-2\mathrm{i}x}}{4} \\
&= \frac{4}{4} = 1 \ \blacksquare
\end{aligned}
$$

Variáveis complexas:

$$
\frac{1}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}^2} = \frac{\mathrm{i}}{-1} = -\mathrm{i}.
$$

Uma integral no plano complexo:

$$
\begin{aligned}
\zeta - z &= R\mathrm{e}^{\mathrm{i}\theta}, \\
\mathrm{d}\zeta &= \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta
\end{aligned}
$$

Encontrando coeficientes de Laurent:

$$
\begin{aligned}
f(z) &= \frac{1}{(z - \mathrm{i})^3 (z + \mathrm{i})^2} \\
f(z) &= \sum_{n=-\infty}^{+\infty} c_n (z - \mathrm{i})^n = \sum_{n=-3}^{+\infty} c_n (z - \mathrm{i})^n
\end{aligned}
$$

“Bem perto” de $z = \mathrm{i}$:

$$
\begin{aligned}
f(z) &\sim \frac{1}{(z - \mathrm{i})^3 (2\mathrm{i})^2} = -\frac{1}{4} \frac{1}{(z - \mathrm{i})^3}; \\
f(z) &= \frac{1}{(z - \mathrm{i})^3 (z + \mathrm{i})^2} = c_{-3}(z - \mathrm{i})^{-3} + c_{-2}(z - \mathrm{i})^{-2} + c_{-1}(z - \mathrm{i})^{-1} + \ldots; \\
(z - \mathrm{i})^3 f(z) &= \frac{1}{(z + \mathrm{i})^2} = c_{-3} + c_{-2}(z - \mathrm{i}) + c_{-1}(z - \mathrm{i})^2 + \ldots;
\end{aligned}
$$

Exemplo 9.9

$$f(z) = \frac{z}{(z^2+1)^2}$$

Ache os polos:

$$\begin{aligned} z^2 + 1 &= 0, \\ z^2 &= -1, \\ z &= \pm \mathrm{i}. \end{aligned}$$

Tente colocar $\pm\mathrm{i}$ explicitamente na fórmula da função:

$$\begin{aligned} z^2 + 1 &= z^2 - (-1) \\ &= z^2 - (\mathrm{i}^2) \\ &= (z+\mathrm{i})(z-\mathrm{i}) \\ &= z^2 - (\mathrm{i})^2 \\ &= z^2 - (-1) \end{aligned}$$

Reescrevo agora a função:

$$\begin{aligned} f(z) &= \frac{z}{[(z-\mathrm{i})(z+\mathrm{i})]^2} \\ &= \frac{z}{(z-\mathrm{i})^2(z+\mathrm{i})^2} \end{aligned}$$

Exemplo 9.10:

$$f(z) = \frac{1}{z+\mathrm{i}}$$

É importante nos lembrarmos da série geométrica:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots; \qquad |x| < 1 \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \ldots; \qquad |x| < 1 \end{aligned}$$

Dentro do círculo unitário, $|z/\mathrm{i}| < 1$:

$$\begin{aligned} \frac{1}{z+\mathrm{i}} &= \frac{1}{\mathrm{i}\left(\frac{z}{\mathrm{i}} + 1\right)} \\ &= \frac{1}{\mathrm{i}\left(1 + \frac{z}{\mathrm{i}}\right)} \\ &= \frac{1}{\mathrm{i}\,(1 - (\mathrm{i}z))} \\ &= \frac{1}{\mathrm{i}}\left[1 + (\mathrm{i}z) + (\mathrm{i}z)^2 + (\mathrm{i}z)^3 + \ldots\right] \\ &= -\mathrm{i}\left[1 + \mathrm{i}z - z^2 - \mathrm{i}z^3 + \ldots\right] \\ &= -\mathrm{i} + z + \mathrm{i}z^2 - z^3 + \ldots \end{aligned}$$

Procuramos agora a série de Laurent de $f(z)$ **fora** do círculo unitário; nessa região, $|\mathrm{i}/z| < 1$.

$$\begin{aligned}
\frac{1}{z+\mathrm{i}} &= \frac{1}{z\left(1+\frac{\mathrm{i}}{z}\right)} \\
&= \frac{1}{z}\left[1 - \frac{\mathrm{i}}{z} + \left(\frac{\mathrm{i}}{z}\right)^2 - \left(\frac{\mathrm{i}}{z}\right)^3 + \dots\right] \\
&= \frac{1}{z} - \frac{\mathrm{i}}{z^2} - \frac{1}{z^3} + \frac{\mathrm{i}}{z^4} + \dots
\end{aligned}$$

Próximo exemplo!

$$\begin{aligned}
f(z) &= \frac{1}{(z-2)(z-3)}; \\
|z| < 3 &\Rightarrow \left|\frac{z}{3}\right| < 1; \\
|z| > 2 &\Rightarrow \frac{|z|}{|z|} > \frac{2}{|z|} \Rightarrow \left|\frac{2}{z}\right| < 1. \\
\frac{1}{(z-2)(z-3)} &= \frac{A}{z-2} + \frac{B}{z-3} \\
\frac{1}{(z-2)(z-3)} &= \frac{A(z-3)+B(z-2)}{(z-2)(z-3)} \\
&= \frac{(A+B)z - 3A - 2B}{(z-2)(z-3)} \\
A + B &= 0, \\
-3A - 2B &= 1, \\
\frac{1}{(z-2)(z-3)} &= \frac{1}{z-3} - \frac{1}{z-2}; \\
\frac{1}{z-3} &= \frac{1}{3\left(\frac{z}{3}-1\right)} = \frac{-1/3}{1-\frac{z}{3}} \\
&= -\frac{1}{3}\left[1 + (z/3) + (z/3)^2 + (z/3)^3 + \dots\right] \\
\frac{1}{z-2} &= \frac{1}{z\left(1-\frac{2}{z}\right)} \\
&= \frac{1}{z}\left[1 + (2/z) + (2/z)^2 + (2/z)^3\right]
\end{aligned}$$

Integração de Contorno e o Teorema dos Resíduos:

$$\int_{\mathscr{L}} f(z)\,\mathrm{d}z$$

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta} &= \cos(\theta) + \mathrm{i}\,\mathrm{sen}(\theta);\\
|\mathrm{e}^{\mathrm{i}\theta}| &= \sqrt{\cos^2(\theta) + \mathrm{sen}^2(\theta)} = 1.
\end{aligned}$$

$$\begin{aligned}
\frac{a}{|1+b|} &\leq ?\\
|b| &\leq |1+b|; \Rightarrow\\
\frac{a}{|1+b|} &\leq \frac{a}{|b|}?
\end{aligned}$$

$$\begin{aligned}
\int_{\theta=0}^{\pi} \frac{R}{|R^2\mathrm{e}^{2\mathrm{i}\theta}|}\,\mathrm{d}\theta &= \int_{\theta=0}^{\pi} \frac{R}{|R^2||\mathrm{e}^{2\mathrm{i}\theta}|}\,\mathrm{d}\theta\\
&= \int_{\theta=0}^{\pi} \frac{R}{|R^2|}\,\mathrm{d}\theta\\
&= \int_{\theta=0}^{\pi} \frac{1}{R}\,\mathrm{d}\theta\\
&= \frac{1}{R}\int_{\theta=0}^{\pi} \mathrm{d}\theta = \frac{\pi}{R}
\end{aligned}$$

Como achar as raízes de $z^4 = -1$?

$$\begin{aligned}
z &= r\mathrm{e}^{\mathrm{i}\theta};\\
z^4 &= r^4\mathrm{e}^{4\mathrm{i}\theta_k} = -1 = 1\mathrm{e}^{\mathrm{i}(\pi+2(k-1)\pi)}\\
r &= 1,\\
\theta_k &= \frac{\pi}{4} + \frac{(k-1)\pi}{2}
\end{aligned}$$

Equações de ordem 2 que eu sei resolver:

$$\begin{aligned}
ay'' + by' + cy &= f(x),\\
ax^2y'' + bxy' + cy &= f(x)
\end{aligned}$$

Eu gostaria de saber resolver equações diferenciais lineares de ordem 2 mais gerais, de coeficientes não constantes:

$$y'' + p(x)y' + q(x)y = f(x),$$

Considere a equação diferencial

$$\begin{aligned}
y' + y &= 0;\\
y(x) &= \sum_{n=0}^{\infty} a_n x^n,\\
y'(x) &= \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n
\end{aligned}$$

$$\sum_{n=0}^{\infty} \left[a_n + (n+1)a_{n+1}\right] x^n = 0$$

Método de Frobenius:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0; \\ x^2y'' + x[xp(x)]y' + [x^2q(x)]y &= 0; \end{aligned}$$

Um exemplo de singularidade fora da origem:

$$\begin{aligned} 2(\xi - 1)\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} + \frac{\mathrm{d}y}{\mathrm{d}\xi} + y &= 0 \\ \frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} + \frac{1}{2(\xi-1)}\frac{\mathrm{d}y}{\mathrm{d}\xi} + \frac{1}{2(\xi-1)}y &= 0 \\ x &= \xi - 1; \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\xi}; \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= \frac{\mathrm{d}y}{\mathrm{d}x}; \\ \frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]\frac{\mathrm{d}x}{\mathrm{d}\xi}; \\ \frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\ 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y &= 0 \end{aligned}$$

O método de Frobenius
Caso (i) $r_1 \neq r_2$, e $r_1 - r_2 \notin \mathbb{Z}$.

$$\begin{aligned} 2xy'' + y' + y &= 0, \\ y'' + \frac{1}{2x}y' + \frac{1}{2x}y &= 0, \\ xp(x) &= \frac{1}{2}, \\ x^2q(x) &= \frac{x}{2}. \end{aligned}$$

Portanto, $x = 0$ é um ponto singular regular, e o método de Frobenius aplica-se, ou seja: podemos encontrar uma solução em série em torno de $x = 0$.

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{r+n}, \\ y' &= \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}, \\ y'' &= \sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2}. \end{aligned}$$

$$2x\left[\sum_{n=0}^{\infty}(r+n-1)(r+n)a_nx^{r+n-2}\right]+\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}+\sum_{n=0}^{\infty}a_nx^{r+n}=0,$$
$$\left[\sum_{n=0}^{\infty}2(r+n-1)(r+n)a_nx^{r+n-1}\right]+\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}+\sum_{n=0}^{\infty}a_nx^{r+n}=0$$
$$\sum_{n=0}^{\infty}\left[2(r+n-1)(r+n)+(r+n)\right]a_nx^{r+n-1}+\sum_{n=0}^{\infty}a_nx^{r+n}=0$$

$$r+n-1=r+m,$$
$$n-1=m,$$
$$n=m+1.$$

$$\sum_{m=-1}^{\infty}\left[2(r+m)(r+m+1)+(r+m+1)\right]a_{m+1}x^{r+m}+\sum_{n=0}^{\infty}a_nx^{r+n}=0$$
$$\sum_{n=-1}^{\infty}\left[2(r+n)(r+n+1)+(r+n+1)\right]a_{n+1}x^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n}=0$$
$$[2(r-1)r+r]a_0x^{r-1}+\sum_{n=0}^{\infty}\left[2(r+n)(r+n+1)+(r+n+1)\right]a_{n+1}x^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n}=0$$
$$[2(r-1)r+r]a_0x^{r-1}+\sum_{n=0}^{\infty}\left[(r+n+1)(2(r+n)+1)\right]a_{n+1}x^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n}=0$$
$$[2(r-1)r+r]a_0x^{r-1}+\sum_{n=0}^{\infty}\left\{(r+n+1)(2(r+n)+1)a_{n+1}+a_n\right\}x^{r+n}=0$$

A equação indicial é

$$2(r-1)r+r=0,$$
$$2(r^2-r)+r=0,$$
$$2r^2-2r+r=0,$$
$$2r^2-r=0,$$
$$r(2r-1)=0,$$
$$r_2=0,$$
$$r_1=\frac{1}{2}.$$

A relação de recorrência será

$$
\begin{aligned}
(r+n+1)(2(r+n)+1)a_{n+1} + a_n &= 0, \\
(r+n+1)(2(r+n)+1)a_{n+1} &= -a_n, \\
a_{n+1} &= -\frac{a_n}{(r+n+1)(2(r+n)+1)}, \\
r_1 &= 1/2, \\
a_{n+1} &= -\frac{a_n}{(n+3/2)(2(1/2+n)+1)}, \\
a_{n+1} &= -\frac{a_n}{(n+3/2)(2(1+n))}, \\
a_{n+1} &= -\frac{a_n}{(2n+3)(1+n)}, \\
b_{n+1} &= -\frac{b_n}{(r+n+1)(2(r+n)+1)}, \\
r_2 &= 0, \\
b_{n+1} &= -\frac{b_n}{(n+1)(2n+1)},
\end{aligned}
$$

Agora eu calculo alguns dos termos de cada série:

$$
\begin{aligned}
&a_0, \\
a_1 &= -\frac{a_0}{3}, \\
a_2 &= -\frac{a_1}{10} = +\frac{a_0}{30}, \\
a_3 &= -\frac{a_2}{21} = -\frac{a_0}{630}, \\
y_1(x) &= x^{1/2}\left[1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \dots\right]
\end{aligned}
$$

$$
\begin{aligned}
&b_0, \\
b_1 &= -b_0, \\
b_2 &= -\frac{b_1}{6} = \frac{b_0}{6}, \\
b_3 &= -\frac{b_2}{15} = -\frac{b_0}{90}, \\
b_4 &= -\frac{b_3}{28} = \frac{b_0}{2520}, \\
y_2(x) &= \left[1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \dots\right]
\end{aligned}
$$

A solução geral é da forma

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Caso (ii): $r_1 = r_2$

$$
\begin{aligned}
x^2y'' - (x + x^2)y' + y = 0,& \\
y &= \sum_{n=0}^{\infty} a_n x^{r+n}, \\
y' &= \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}, \\
y'' &= \sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2}
\end{aligned}
$$

$$
x^2\left[\sum_{n=0}^{\infty}(r+n-1)(r+n)a_n x^{r+n-2}\right] - (x+x^2)\left[\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}\right] + \sum_{n=0}^{\infty} a_n x^{r+n} = 0,
$$

$$
\sum_{n=0}^{\infty}\left[(r+n-1)(r+n) - (r+n) + 1\right] a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n+1} = 0,
$$

$$
\begin{aligned}
r + m &= r + n + 1, \\
m &= n + 1, \\
n &= m - 1
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left[(r+n-1)(r+n) - (r+n) + 1\right] a_n x^{r+n} - \sum_{m=1}^{\infty}(r+m-1)a_{m-1} x^{r+m} &= 0, \\
\sum_{n=0}^{\infty}\left[(r+n-1)(r+n) - (r+n) + 1\right] a_n x^{r+n} - \sum_{n=1}^{\infty}(r+n-1)a_{n-1} x^{r+n} &= 0, \\
\left[(r-1)r - r + 1\right] a_0 x^r + \sum_{n=1}^{\infty}\left\{\left[(r+n-1)(r+n) - (r+n) + 1\right] a_n - (r+n-1)a_{n-1}\right\} x^{r+n} &= 0,
\end{aligned}
$$

A equação indicial será

$$
\begin{aligned}
(r-1)r - r + 1 &= 0, \\
r^2 - r - r + 1 &= 0, \\
r^2 - 2r + 1 &= 0, \\
(r-1)^2 &= 0.
\end{aligned}
$$

Portanto, $r = 1$ é uma raiz dupla.

Preciso inicialmente encontrar uma solução. A relação de recorrência é

$$\begin{aligned}
[(r+n-1)(r+n)-(r+n)+1]\, a_n-(r+n-1)a_{n-1}&=0\\
[(r+n-1)(r+n)-(r+n)+1]\, a_n&=(r+n-1)a_{n-1}\\
a_n&=\frac{(r+n-1)a_{n-1}}{(r+n-1)(r+n)-(r+n)+1}\\
a_n&=\frac{na_{n-1}}{n(1+n)-(1+n)+1}\\
a_n&=\frac{na_{n-1}}{n+n^2-1-n+1}\\
a_n&=\frac{na_{n-1}}{n^2}\\
a_n&=\frac{a_{n-1}}{n}.
\end{aligned}$$

$$\begin{aligned}
a_0&=1=\frac{1}{0!},\\
a_1&=\frac{1}{1}=1=\frac{1}{1!},\\
a_2&=\frac{1}{2}=\frac{1}{2!}\\
a_3&=\frac{1/2}{3}=\frac{1}{6}=\frac{1}{3!},\\
a_4&=\frac{1/6}{4}=\frac{1}{24}=\frac{1}{4!},\\
a_n&=\frac{1}{n!}
\end{aligned}$$

A 1ª solução será

$$\begin{aligned}
y_1(x)&=x\left[\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right]\\
&=x\mathrm{e}^x \blacksquare
\end{aligned}$$

A segunda solução, LI da primeira, deve ser procurada na forma

$$\begin{aligned}
y_2&=y_1\ln(x)+\sum_{n=1}^{\infty}c_nx^{n+1},\\
y_2'&=y_1'\ln(x)+y_1\frac{1}{x}+\sum_{n=1}^{\infty}(n+1)c_nx^n,\\
y_2''&=y_1''\ln(x)+y_1'\frac{1}{x}+y_1'\frac{1}{x}-y_1\frac{1}{x^2}+\sum_{n=1}^{\infty}n(n+1)c_nx^{n-1}\\
&=y_1''\ln(x)+\frac{2y_1'}{x}-y_1\frac{1}{x^2}+\sum_{n=1}^{\infty}n(n+1)c_nx^{n-1}
\end{aligned}$$

A equação diferencial é

$$x^2y''-(x+x^2)y'+y=0.$$

Substituindo,

$$x^2\left[y_1''\ln(x)+\frac{2y_1'}{x}-y_1\frac{1}{x^2}+\sum_{n=1}^{\infty}n(n+1)c_nx^{n-1}\right]-$$
$$(x+x^2)\left[y_1'\ln(x)+y_1\frac{1}{x}+\sum_{n=1}^{\infty}(n+1)c_nx^n\right]+y_1\ln(x)+\sum_{n=1}^{\infty}c_nx^{n+1}=0$$

$$\ln(x)\underbrace{\left[x^2y_1''-(x+x^2)y_1'+y_1\right]}_{=0}+x^2\left[\frac{2y_1'}{x}-y_1\frac{1}{x^2}+\sum_{n=1}^{\infty}n(n+1)c_nx^{n-1}\right]-$$
$$(x+x^2)\left[+y_1\frac{1}{x}+\sum_{n=1}^{\infty}(n+1)c_nx^n\right]+\sum_{n=1}^{\infty}c_nx^{n+1}=0$$

$$x^2\left[\frac{2y_1'}{x}-y_1\frac{1}{x^2}+\sum_{n=1}^{\infty}n(n+1)c_nx^{n-1}\right]-$$
$$(x+x^2)\left[+y_1\frac{1}{x}+\sum_{n=1}^{\infty}(n+1)c_nx^n\right]+\sum_{n=1}^{\infty}c_nx^{n+1}=0$$

$$2xy_1'-y_1-y_1-xy_1+x^2\left[\sum_{n=1}^{\infty}n(n+1)c_nx^{n-1}\right]-$$
$$(x+x^2)\left[\sum_{n=1}^{\infty}(n+1)c_nx^n\right]+\sum_{n=1}^{\infty}c_nx^{n+1}=0$$

$$2xy_1'-2y_1-xy_1+$$
$$\sum_{n=1}^{\infty}n(n+1)c_nx^{n+1}-\sum_{n=1}^{\infty}(n+1)c_nx^{n+1}-\sum_{n=1}^{\infty}(n+1)c_nx^{n+2}+\sum_{n=1}^{\infty}c_nx^{n+1}=0$$

$$2xy_1'-2y_1-xy_1+\sum_{n=1}^{\infty}\left[n(n+1)-(n+1)+1\right]c_nx^{n+1}-\sum_{n=1}^{\infty}(n+1)c_nx^{n+2}=0$$

$$2xy_1'-2y_1-xy_1+\sum_{n=1}^{\infty}n^2c_nx^{n+1}-\sum_{n=1}^{\infty}(n+1)c_nx^{n+2}=0,$$
$$\sum_{n=1}^{\infty}n^2c_nx^{n+1}-\sum_{n=1}^{\infty}(n+1)c_nx^{n+2}+2xy_1'-(2+x)y_1=0$$

$$\begin{aligned}m+1&=n+2,\\ m&=n+1,\\ n&=m-1\end{aligned}$$

$$\sum_{n=1}^{\infty}(n+1)c_n x^{n+2} = \sum_{m=2}^{\infty} mc_{m-1}x^{m+1} = \sum_{n=2}^{\infty} nc_{n-1}x^{n+1}.$$

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=2}^{\infty} nc_{n-1}x^{n+1} + 2xy_1' - (2+x)y_1 = 0,$$

Seja criativo e faça $c_0 = 0$:

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} nc_{n-1}x^{n+1} + 2xy_1' - (2+x)y_1 = 0,$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - nc_{n-1}\right] x^{n+1} + 2xy_1' - (2+x)y_1 = 0,$$

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!},$$

$$y_1' = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!},$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - nc_{n-1}\right] x^{n+1} + 2x\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} - (2+x)\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = 0,$$

$$\sum_{n=1}^{\infty} \left[n^2 c_n - nc_{n-1}\right] x^{n+1} = (2+x)\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - 2x\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Concentremo-nos nos somatórios!

$$\begin{aligned}
(2+x)\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - 2x\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} &= \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty} \frac{2nx^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty} \frac{2nx^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=1}^{\infty} \frac{2nx^{n+1}}{n!} \\
n! &= n \times \underbrace{n-1 \times n-2 \times \ldots 1}_{=(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!} \\
n+2 &= m+1, \\
n+1 &= m, \\
n &= m-1, \\
&= \sum_{m=1}^{\infty} \frac{x^{m+1}}{(m-1)!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!} \\
&= -\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!}
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left[n^2c_n - nc_{n-1}\right]x^{n+1} &= -\sum_{n=1}^{\infty}\frac{x^{n+1}}{(n-1)!} \\
n^2c_n - nc_{n-1} &= -\frac{1}{(n-1)!} \\
c_n - \frac{n}{n^2}c_{n-1} &= -\frac{1}{n^2(n-1)!} \\
c_n &= \frac{1}{n^2}\left[nc_{n-1} - \frac{1}{(n-1)!}\right] \\
&= \frac{1}{n^2}\left[\frac{n(n-1)!c_{n-1} - 1}{(n-1)!}\right] \\
&= \frac{1}{n^2}\left[\frac{n!c_{n-1} - 1}{(n-1)!}\right] \\
&= \frac{1}{n}\left[\frac{n!c_{n-1} - 1}{n(n-1)!}\right] \\
&= \frac{1}{n}\left[\frac{n!c_{n-1} - 1}{n!}\right] \\
&= \frac{1}{n}\left[c_{n-1} - \frac{1}{n!}\right]
\end{aligned}
$$

Caso (iii)-1: $r_1 - r_2 \in \mathbb{Z}$, e a menor raiz leva a duas soluções.

$$x^2y'' + xy' + (x^2 - 1/4)y = 0.$$

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{r+n}, \\
y' &= \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}, \\
y'' &= \sum_{n=0}^{\infty}(r+n-1)(r+n)a_n x^{r+n-2}.
\end{aligned}
$$

$$
\begin{aligned}
x^2\sum_{n=0}^{\infty}(r+n-1)(r+n)a_nx^{r+n-2}+x\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}+\left(x^2-\frac{1}{4}\right)\sum_{n=0}^{\infty}a_nx^{r+n}&=0,\\
\sum_{n=0}^{\infty}(r+n-1)(r+n)a_nx^{r+n}+\sum_{n=0}^{\infty}(r+n)a_nx^{r+n}-\frac{1}{4}\sum_{n=0}^{\infty}a_nx^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n+2}&=0,\\
\sum_{n=0}^{\infty}\left[(r+n-1)(r+n)+(r+n)-\frac{1}{4}\right]a_nx^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n+2}&=0,\\
\sum_{n=0}^{\infty}\left[(r+n-1+1)(r+n)-\frac{1}{4}\right]a_nx^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n+2}&=0,\\
\sum_{n=0}^{\infty}\left[(r+n-1+1)(r+n)-\frac{1}{4}\right]a_nx^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n+2}&=0,\\
\sum_{n=0}^{\infty}\left[(r+n)^2-\frac{1}{4}\right]a_nx^{r+n}+\sum_{n=0}^{\infty}a_nx^{r+n+2}&=0,
\end{aligned}
$$

$$
\begin{aligned}
r+n+2&=r+m,\\
n+2&=m,\\
n&=m-2
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left[(r+n)^2-\frac{1}{4}\right]a_nx^{r+n}+\sum_{m=2}^{\infty}a_{m-2}x^{r+m}&=0,\\
\sum_{n=0}^{\infty}\left[(r+n)^2-\frac{1}{4}\right]a_nx^{r+n}+\sum_{n=2}^{\infty}a_{n-2}x^{r+n}&=0,\\
(r^2-1/4)a_0x^r+[(r+1)^2-1/4]a_1x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-\frac{1}{4}\right]a_n+a_{n-2}\right\}x^{r+n}&=0
\end{aligned}
$$

Faça

$$
\begin{aligned}
r^2&=1/4,\\
r_1&=+1/2,\\
r_2&=-1/2.
\end{aligned}
$$

Se $r=-1/2$,

$$
\begin{aligned}
(r+1)^2-1/4&=(-1/2+1)^2-1/4\\
(1/2)^2-1/4&=0.
\end{aligned}
$$

A fórmula de recorrência será

$$
\begin{aligned}
\left[(r+n)^2-\frac{1}{4}\right]a_n+a_{n-2}&=0,\\
[(-1/2+n)^2-1/4]a_n+a_{n-2}&=0,\\
[1/4-n+n^2-1/4]a_n+a_{n-2}&=0,\\
[-n+n^2]a_n+a_{n-2}&=0,\\
a_n&=-\frac{a_{n-2}}{n(n-1)}.
\end{aligned}
$$

A solução geral é

$$
\begin{aligned}
y_1 &= a_0 \left\{ x^{-1/2} \left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots \right] \right\} = x^{-1/2}\cos(x), \\
y_2 &= a_1 \left\{ x^{-1/2} \left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \ldots \right] \right\} = x^{-1/2}\,\mathrm{sen}(x).
\end{aligned}
$$

Na literatura, nós encontramos a notação

$$
\begin{aligned}
J_{-1/2}(x) &\equiv \sqrt{\frac{2}{\pi}} x^{-1/2}\cos(x), \\
J_{+1/2}(x) &\equiv \sqrt{\frac{2}{\pi}} x^{-1/2}\,\mathrm{sen}(x).
\end{aligned}
$$

Finalmente, vamos para o caso iii-b: Duas raízes que diferem por um número inteiro, mas a menor raiz não leva a nenhuma solução.

$$
x^2 y'' + x y' + (x^2 - 1) y = 0.
$$

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{r+n}, \\
y' &= \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \\
y'' &= \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}.
\end{aligned}
$$

$$
\begin{aligned}
x^2 \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2} + x \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + x^2 \sum_{n=0}^{\infty} a_n x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} &= 0, \\
\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) + (r+n) - 1 \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} &= 0
\end{aligned}
$$

$$
\begin{aligned}
r + m &= r + n + 2, \\
m &= n + 2, \\
n &= m - 2.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) + (r+n) - 1 \right] a_n x^{r+n} + \sum_{m=2}^{\infty} a_{m-2} x^{r+m} &= 0 \\
\sum_{n=0}^{\infty} \left[(r+n)(r+n) - 1 \right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} &= 0, \\
(r^2 - 1) a_0 x^r + [(r+1)^2 - 1] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ \left[(r+n)(r+n) - 1 \right] a_n + a_{n-2} \right\} x^{r+n}
\end{aligned}
$$

Fazendo $a_0 \neq 0$:

$$\begin{aligned} r^2 &= 1, \\ r_1 &= 1, \\ r_2 &= -1 \end{aligned}$$

Discuto o que acontece quando $r_2 = -1$:

$$\begin{aligned} [(-1+1)^2 - 1]a_1 x^{r+1} &= -a_1 x^0 = 0; \Rightarrow a_1 = 0. \\ [(-1+n)(-1+n) - 1]\, a_n + a_{n-2} &= 0, \\ \left[n^2 - 2n + 1 - 1\right] a_n + a_{n-2} &= 0, \\ a_n &= -\frac{a_{n-2}}{n(n-2)}, \\ a_2 &= -\frac{a_0}{2 \times 0}. \end{aligned}$$

Portanto, a menor raiz não leva a nenhuma solução. Faço "tudo de novo" com $r = 1$:

$$(r^2 - 1)a_0 x^r + [(r+1)^2 - 1]a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{[(r+n)(r+n) - 1]\, a_n + a_{n-2}\right\} x^{r+n}$$

$$\begin{aligned} (r+1)^2 - 1]a_1 &= 0, \\ [4-1]a_1 &= 0, \\ a_1 &= 0. \end{aligned}$$

A relação de recorrência fica

$$\begin{aligned} [(r+n)(r+n) - 1]\, a_n + a_{n-2} &= 0, \\ [(n+1)^2 - 1]a_n + a_{n-2} &= 0, \\ [n^2 + 2n + 1 - 1]a_n + a_{n-2} &= 0 \\ [n^2 + 2n]a_n + a_{n-2} &= 0, \\ a_n &= -\frac{a_{n-2}}{n(n+2)}. \end{aligned}$$

$$x^2 \left[y_1'' \ln(x) + \frac{2y_1'}{x} - \frac{y_1}{x^2} + v''\right] + x\left[y_1' \ln(x) + \frac{y_1}{x} + v'\right] + (x^2 - 1)\left[y_1 \ln(x) + v\right] = 0$$

Os termos em $\ln(x)$ são

$$\left[x^2 y_1'' + x y_1' + (x^2 - 1) y_1\right] \ln(x)$$

Transformada de Laplace
A transformada de Laplace da derivada de $f(t)$ é

$$
\begin{aligned}
\mathscr{L}\{f'(t)\} &= \int_0^\infty \underbrace{\mathrm{e}^{-st}}_{u}\,\underbrace{f'(t)\,\mathrm{d}t}_{dv}\\
&= \mathrm{e}^{-st} f(t)\Big|_0^\infty - \left[\int_0^\infty f(t)(-s\mathrm{e}^{-st})\,\mathrm{d}t\right]\\
&= -f(0) - \left[\int_0^\infty f(t)(-s\mathrm{e}^{-st})\,\mathrm{d}t\right]\\
&= -f(0) + s\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\\
&= -f(0) + s\overline{f}(s)\\
\mathscr{L}\{f'(t)\} &= s\overline{f}(s) - f(0).\\
\mathscr{L}\{g'(t)\} &= s\overline{g}(s) - g(0).\\
\mathscr{L}\{g'(t)\} &= s\mathscr{L}\{g\} - g(0).\\
\mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\} - f'(0)\\
g(t) &= f'(t)\\
g'(t) &= f''(t)\\
\mathscr{L}\{f''(t)\} &= s\left[s\overline{f}(s) - f(0)\right] - f'(0)\\
\mathscr{L}\{f''(t)\} &= s^2\overline{f}(s) - sf(0) - f'(0)
\end{aligned}
$$

Exemplo:

$$
y'' + 3y' + 4y = x\,\mathrm{sen}(x), \qquad y(0) = 1, \qquad y'(0) = 1.
$$

Faço

$$
\begin{aligned}
\mathscr{L}\{y'' + 3y' + 4y\} &= \mathscr{L}\{x\,\mathrm{sen}(x)\}\\
[s^2\overline{y}(s) - sy(0) - y'(0)] + 3[s\overline{y}(s) - y(0)] + 4\overline{y}(s) &= \mathscr{L}\{x\,\mathrm{sen}(x)\}\\
\mathscr{L}\{x\,\mathrm{sen}(x)\} &= \int_0^\infty \mathrm{e}^{-sx} x\,\mathrm{sen}(x)\,\mathrm{d}x\\
&= \frac{2s}{(s^2+1)^2};\\
[s^2\overline{y}(s) - sy(0) - y'(0)] + 3[s\overline{y}(s) - y(0)] + 4\overline{y}(s) &= \frac{2s}{(s^2+1)^2}\\
[s^2\overline{y}(s) - s - 1] + 3[s\overline{y}(s) - 1] + 4\overline{y}(s) &= \frac{2s}{(s^2+1)^2}\\
[s^2\overline{y} - s - 1] + 3s\overline{y} - 3 + 4\overline{y} &= \frac{2s}{(s^2+1)^2}\\
\overline{y}[s^2 + 3s + 4] - s - 4 &= \frac{2s}{(s^2+1)^2}\\
\overline{y}(s) &= \ldots\\
y(x) &= \ldots
\end{aligned}
$$

Teorema da Convolução
Definição de convolução **no sentido de Laplace**:

$$[f * g]\,(t) \equiv \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau$$

Cuidado com abusos contemporâneos de notação!!!!

$$\begin{aligned}
F &= ma\\
F &= m \cdot a\\
F &= m \times a\\
F &= m * a \qquad \textbf{nããão!}
\end{aligned}$$

Perigo!!!

$$f * g \neq f(t)g(t) \qquad !!!!$$

O ponto fundamental de transformadas de Laplace é o seguinte:

$$f(t) \leftrightarrow \overline{f}(s).$$

O teorema da convolução para transformadas de Laplace:

$$\mathscr{L}\left\{[f * g](t)\right\} = \overline{f}(s)\overline{g}(s).$$

$$\begin{aligned}
\mathscr{L}\left\{[f * g](t)\right\} &= \int_0^\infty \mathrm{e}^{-st}[f * g](t)\,\mathrm{d}t,\\
[f * g](t) &\equiv \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau,\\
\mathscr{L}\left\{[f * g](t)\right\} &= \int_{t=0}^\infty \mathrm{e}^{-st} \int_{\tau=0}^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\
\mathscr{L}\left\{[f * g](t)\right\} &= \int_{\tau=0}^\infty \int_{t=\tau}^\infty \mathrm{e}^{-st} f(\tau)g(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau
\end{aligned}$$

Fazemos agora uma mudança de variáveis:

$$(\tau, t) \leftrightarrow (x, y)$$

$$\begin{aligned}
x &= \tau & \tau &= x,\\
y &= t - \tau & t &= x + y.
\end{aligned}$$

Agora,

$$\begin{aligned}
\iint_{\tau,t} f(\tau,t)\,\mathrm{d}t\,\mathrm{d}\tau &= \iint_{x,y} f(\tau(x,y), t(x,y)) \left|\frac{\partial(\tau,t)}{\partial(x,y)}\right| \mathrm{d}y\,\mathrm{d}x\\
\frac{\partial(\tau,t)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y}\\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y}\end{vmatrix} = \begin{vmatrix} 1 & 0\\ 1 & 1\end{vmatrix} = 1,\\
\left|\frac{\partial(\tau,t)}{\partial(x,y)}\right| &= 1.
\end{aligned}$$

De volta à integral dupla:

$$
\begin{aligned}
\mathscr{L}\left\{[f*g](t)\right\} &= \int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty} \mathrm{e}^{-st} f(\tau)g(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\
\mathscr{L}\left\{[f*g](t)\right\} &= \int_{x=0}^{\infty}\int_{y=0}^{\infty} \mathrm{e}^{-s(x+y)} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\
\mathscr{L}\left\{[f*g](t)\right\} &= \int_{x=0}^{\infty}\int_{y=0}^{\infty} \mathrm{e}^{-sx}\mathrm{e}^{-sy} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\
\mathscr{L}\left\{[f*g](t)\right\} &= \int_{x=0}^{\infty} \mathrm{e}^{-sx} f(x)\left[\int_{y=0}^{\infty} \mathrm{e}^{-sy} g(y)\,\mathrm{d}y\right]\mathrm{d}x \\
\mathscr{L}\left\{[f*g](t)\right\} &= \underbrace{\left[\int_{y=0}^{\infty} \mathrm{e}^{-sy} g(y)\,\mathrm{d}y\right]}_{\overline{g}(s)}\underbrace{\left[\int_{x=0}^{\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x\right]}_{\overline{f}(s)} \\
&= \overline{f}(s)\overline{g}(s) \;\blacksquare
\end{aligned}
$$

Fórmula da inversão, etc.

$$
\begin{aligned}
\left|\frac{\omega R \mathrm{e}^{s_x t}}{(R\mathrm{e}^{\mathrm{i}\theta}+a)^2+\omega^2}\right| &= \left|\frac{\omega R}{(R\mathrm{e}^{\mathrm{i}\theta}+a)^2+\omega^2}\right|\left|\mathrm{e}^{s_x t}\right| \\
&\leq \left|\frac{\omega R}{(R\mathrm{e}^{\mathrm{i}\theta}+a)^2+\omega^2}\right|
\end{aligned}
$$

Existem muitos caminhos! Por exemplo,

$$
\begin{aligned}
\overline{f}(s) &= \frac{\omega}{(s+a)^2+\omega^2}; \\
\mathscr{L}^{-1}\left\{\overline{f}(s)\right\} &= ? \\
\mathscr{L}\left\{\operatorname{sen}(at)\right\} &= \frac{a}{s^2+a^2} \\
\mathscr{L}\left\{\operatorname{sen}(\omega t)\right\} &= \frac{\omega}{s^2+\omega^2} \\
\mathscr{L}\left\{\mathrm{e}^{at} f(t)\right\} &= \overline{f}(s-a) \\
\mathscr{L}^{-1}\left\{\overline{f}(s-a)\right\} &= \mathrm{e}^{at} f(t) \\
\mathscr{L}^{-1}\left\{\overline{f}(s+a)\right\} &= \mathrm{e}^{-at} f(t) \\
\mathscr{L}\left\{\mathrm{e}^{-at}\operatorname{sen}(\omega t)\right\} &= \frac{\omega}{(s+a)^2+\omega^2} \\
\mathscr{L}^{-1}\left\{\overline{f}(s)\right\} &= \mathrm{e}^{-at}\operatorname{sen}(\omega t) \;\blacksquare
\end{aligned}
$$

Exemplos de aplicações com transformadas de Laplace

$$
\begin{aligned}
y'''' - y &= 0, \\
y(0) &= 0, \\
y'(0) &= 0, \\
y''(0) &= 0, \\
y'''(0) &= 1.
\end{aligned}
$$

Precisamos de

$$\mathscr{L}(y'''') = s^4\overline{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$$

Transformando a equação diferencial, obteremos

$$\begin{aligned}
\mathscr{L}\{y'''' - y\} &= \mathscr{L}\{0\}, \\
s^4\overline{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \overline{y} &= 0, \\
s^4\overline{y} - 1 - \overline{y} &= 0, \\
s^4\overline{y} - \overline{y} &= 1, \\
\overline{y}(s^4 - 1) &= 1 \\
\overline{y} &= \frac{1}{s^4 - 1}
\end{aligned}$$

Decomposição em frações parciais

$$\begin{aligned}
s^4 - 1 &= (s^2 - 1)(s^2 + 1) \\
&= (s - 1)(s + 1)(s^2 + 1) \\
\frac{1}{s^4 - 1} &= \frac{1}{(s - 1)(s + 1)(s^2 + 1)} \\
&= \frac{A}{s - 1} + \frac{B}{s + 1} + \frac{Cs + D}{s^2 + 1} \\
\frac{1}{s^4 - 1} &= \frac{A(s + 1)(s^2 + 1) + B(s - 1)(s^2 + 1) + Cs(s - 1)(s + 1) + D(s - 1)(s + 1)}{s^4 - 1} \\
\frac{1}{s^4 - 1} &= \frac{A(s^3 + s^2 + s + 1) + B(s^3 - s^2 + s - 1) + Cs(s^2 - 1) + D(s^2 - 1)}{s^4 - 1} \\
\frac{1}{s^4 - 1} &= \frac{A(s^3 + s^2 + s + 1) + B(s^3 - s^2 + s - 1) + C(s^3 - s) + D(s^2 - 1)}{s^4 - 1} \\
\frac{1}{s^4 - 1} &= \frac{(A + B + C)s^3 + (A - B + D)s^2 + (A + B - C)s + (A - B - D)}{s^4 - 1}
\end{aligned}$$

Obtenho o sistema de equações

$$\begin{aligned}
A + B + C + 0D &= 0, \\
A - B + 0C + D &= 0, \\
A + B - C + 0D &= 0, \\
A - B + 0C - D &= 1
\end{aligned}$$

$$\begin{aligned}
2A + C + D &= 0, \\
2A - C - D &= 1, \\
2C + 2D &= -1,
\end{aligned}$$

$$
\begin{aligned}
2B + C - D &= 0,\\
2B - C + D &= -1\\
2C - 2D &= 1,
\end{aligned}
$$

$$
\begin{aligned}
4C + 0D &= 0,\\
C &= 0.\\
4D &= -2,\\
D &= -\frac{1}{2}\\
A &= \frac{1}{4}\\
1/4 - B + -1/2 &= 0,\\
-B &= 1/2 - 1/4 = 1/4,\\
B &= -\frac{1}{4}.
\end{aligned}
$$

$$
\begin{aligned}
\overline{y}(s) &= \frac{1}{s^4-1} = \frac{1}{4} \times \frac{1}{s-1} - \frac{1}{4} \times \frac{1}{s+1} - \frac{1}{2} \times \frac{1}{s^2+1}\\
\mathscr{L}^{-1}\left\{\overline{y}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{4} \times \frac{1}{s-1} - \frac{1}{4} \times \frac{1}{s+1} - \frac{1}{2} \times \frac{1}{s^2+1}\right\}\\
y(t) &= \mathscr{L}^{-1}\left\{\frac{1}{4} \times \frac{1}{s-1} - \frac{1}{4} \times \frac{1}{s+1} - \frac{1}{2} \times \frac{1}{s^2+1}\right\}\\
y(t) &= \frac{1}{4}\mathrm{e}^{t} - \frac{1}{4}\mathrm{e}^{-t} - \frac{1}{2}\operatorname{sen}(t) \ \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
y''' + y &= \mathrm{e}^{-t},\\
y(0) &= 0,\\
y'(0) &= 0,\\
y''(0) &= 0.
\end{aligned}
$$

$$
\mathscr{L}\left\{y'''\right\} = s^3\overline{y} - s^2 y(0) - s y'(0) - y''(0).
$$

Transformo a equação diferencial:

$$
\begin{aligned}
\mathscr{L}\left\{y''' + y\right\} &= \mathscr{L}\left\{\mathrm{e}^{-t}\right\},\\
s^3\overline{y} - s^2 y(0) - s y'(0) - y''(0) + \overline{y} &= \frac{1}{s+1},\\
s^3\overline{y} + \overline{y} &= \frac{1}{s+1},\\
\overline{y}(s^3+1) &= \frac{1}{s+1}\\
\overline{y} &= \frac{1}{(s+1)(s^3+1)}
\end{aligned}
$$

Preciso fatorar!!!

$$\begin{aligned}
s^3+1 &= (s+1)(s^2+as+b);\\
s^3+1 &= s^3+as^2+bs+s^2+as+b\\
1 &= as^2+bs+s^2+as+b\\
1 &= (a+1)s^2+(a+b)s+b\\
a &= -1,\\
b &= 1,\\
s^3+1 &= (s+1)(s^2-s+1).
\end{aligned}$$

$$\begin{aligned}
\overline{y}(s^3+1) &= \frac{1}{s+1}\\
\overline{y} &= \frac{1}{(s+1)(s^3+1)}\\
\overline{y} &= \frac{1}{(s+1)(s+1)(s^2-s+1)}\\
\overline{y} &= \frac{1}{(s+1)(s^3+1)} = \frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2-s+1}
\end{aligned}$$

Prossigo para obter A, B, C e D:

$$1 = A(s+1)(s^2-s+1)+B(s^2-s+1)+Cs(s+1)^2+D(s+1)^2$$

$$\begin{aligned}
\frac{A}{s+1} &= \frac{A(s+1)(s^2-s+1)}{(s+1)(s+1)(s^2-s+1)}\\
\frac{B}{(s+1)^2} &= \frac{B(s^2-s+1)}{(s+1)^2(s^2-s+1)}\\
\frac{Cs+D}{s^2-s+1} &= \frac{(Cs+D)(s+1)^2}{(s^2-s+1)(s+1)^2}
\end{aligned}$$

$$\begin{aligned}
1 &= A(s^3+1)+B(s^2-s+1)+Cs(s+1)^2+D(s+1)^2\\
1 &= A(s^3+1)+B(s^2-s+1)+Cs(s^2+2s+1)+D(s^2+2s+1)\\
1 &= A(s^3+1)+B(s^2-s+1)+C(s^3+2s^2+s)+D(s^2+2s+1)\\
1 &= (A+C)s^3+(B+2C+D)s^2+(-B+C+2D)s+(A+B+D)
\end{aligned}$$

O sistema de equações resultante é

$$\begin{aligned}
A+C &= 0,\\
B+2C+D &= 0,\\
-B+C+2D &= 0,\\
A+B+D &= 1.
\end{aligned}$$

$$
\begin{aligned}
B + 2C + D &= 0,\\
-B + C + 2D &= 0,\\
B - C + D &= 1.
\end{aligned}
$$

$$
\begin{aligned}
3C + 3D &= 0,\\
3D &= 1,\\
D &= 1/3,\\
C &= -1/3,\\
A &= 1/3,\\
B - C + D &= 1,\\
B + 1/3 + 1/3 &= 1,\\
B &= 1 - 2/3 = 1/3.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{(s+1)^2(s^2-s+1)} &= \frac{1}{3}\times\frac{1}{s+1} + \frac{1}{3}\times\frac{1}{(s+1)^2} - \frac{1}{3}\times\frac{s}{s^2-s+1} + \frac{1}{3}\times\frac{1}{s^2-s+1}\\
\frac{1}{(s+1)^2(s^2-s+1)} &= \frac{1}{3}\times\frac{1}{s+1} + \frac{1}{3}\times\frac{1}{(s+1)^2} - \frac{1}{3}\times\frac{s-1}{s^2-s+1}
\end{aligned}
$$

Agora procuro uma tabela de transformadas de Laplace:

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} &= \mathrm{e}^{-t};\\
\mathscr{L}\left\{\mathrm{e}^{at}f(t)\right\} &= \overline{f}(s-a),\\
\mathscr{L}\left\{t^n\right\} &= \frac{n!}{s^{n+1}},\\
\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} &= t\mathrm{e}^{-t}
\end{aligned}
$$

Se eu fizer

$$
s^2 - s + 1 = (s-a)^2 + b^2,
$$

O último termo fica

$$
\mathscr{L}^{-1}\left\{\frac{s-1}{(s-a)^2+b^2}\right\}
$$

Prossigo:

$$\begin{aligned}
s^2 - s + 1 &= (s-a)^2 + b^2, \\
s^2 - s + 1 &= s^2 - 2as + a^2 + b^2, \\
-2as &= -s, \\
2a &= 1, \\
a &= \frac{1}{2}, \\
a^2 + b^2 &= 1, \\
\frac{1}{4} + b^2 &= 1, \\
b^2 &= \frac{3}{4}, \\
b &= \frac{\sqrt{3}}{2}.
\end{aligned}$$

Retorno ao último termo que desejo inverter:

$$\begin{aligned}
\frac{s-1}{s^2-s+1} &= \frac{s-1}{(s-1/2)^2 + (\sqrt{3}/2)^2} \\
\mathscr{L}\left\{\cos(bt)\right\} &= \frac{s}{s^2+b^2}, \\
\mathscr{L}\left\{\operatorname{sen}(bt)\right\} &= \frac{b}{s^2+b^2}, \\
\mathscr{L}\left\{\mathrm{e}^{at} f(t)\right\} &= \overline{f}(s-a), \\
\mathscr{L}\left\{\mathrm{e}^{at} \cos(bt)\right\} &= \frac{s-a}{(s-a)^2+b^2}, \\
\mathscr{L}\left\{\mathrm{e}^{at} \operatorname{sen}(bt)\right\} &= \frac{b}{(s-a)^2+b^2}, \\
\frac{s-1}{s^2-s+1} &= \frac{s-1/2}{(s-1/2)^2 + (\sqrt{3}/2)^2} - \frac{1/2}{(s-1/2)^2 + (\sqrt{3}/2)^2} \\
\frac{s-1}{s^2-s+1} &= \frac{s-1/2}{(s-1/2)^2 + (\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}/2}{(s-1/2)^2 + (\sqrt{3}/2)^2}
\end{aligned}$$

Agora eu obtenho cada uma das transformadas inversas:

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s-1/2}{(s-1/2)^2 + (\sqrt{3}/2)^2}\right\} &= \mathrm{e}^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right), \\
\mathscr{L}^{-1}\left\{\frac{\sqrt{3}/2}{(s-1/2)^2 + (\sqrt{3}/2)^2}\right\} &= \mathrm{e}^{t/2} \operatorname{sen}\left(\frac{\sqrt{3}}{2} t\right)
\end{aligned}$$

Agora eu obtenho cada inversa e junto tudo

$$\begin{aligned}
\frac{1}{(s+1)^2(s^2-s+1)} &= \frac{1}{3} \times \frac{1}{s+1} + \frac{1}{3} \times \frac{1}{(s+1)^2} - \frac{1}{3} \times \frac{s-1}{s^2-s+1} \\
y(t) &= \frac{\mathrm{e}^{-t}}{3} + \frac{t\mathrm{e}^{-t}}{3} - \frac{1}{3}\mathrm{e}^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{3\sqrt{3}} \mathrm{e}^{t/2} \operatorname{sen}\left(\frac{\sqrt{3}}{2} t\right)
\end{aligned}$$

Mais um exemplo!

$$\begin{aligned} y''' + y' &= t, \\ y(0) &= y'(0) = 0, \\ y''(0) &= 1. \end{aligned}$$

Solução:

$$\begin{aligned} \mathscr{L}\left\{y''' + y'\right\} &= \mathscr{L}\left\{t\right\}, \\ \mathscr{L}\left\{y'''\right\} + \mathscr{L}\left\{y'\right\} &= \mathscr{L}\left\{t\right\}, \\ s^3\overline{y} - s^2y(0) - sy'(0) - y''(0) + s\overline{y} - y(0) &= \frac{1}{s^2} \\ s^3\overline{y} - 1 + s\overline{y} &= \frac{1}{s^2} \\ s^3\overline{y} + s\overline{y} &= 1 + \frac{1}{s^2} \\ \overline{y}(s^3 + s) &= \frac{s^2+1}{s^2} \\ \overline{y}s(s^2+1) &= \frac{s^2+1}{s^2} \\ \overline{y} &= \frac{1}{s^3} \\ \mathscr{L}\left\{t^n\right\} &= \frac{n!}{s^{n+1}}, \\ \mathscr{L}\left\{t^2\right\} &= \frac{2!}{s^3}, \\ \overline{y} &= \frac{1}{2}\frac{2}{s^3} \\ y(t) &= \frac{t^2}{2}. \end{aligned}$$