TEA013 Matemática Aplicada II
Curso de Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
P03B, 08 Abr 2022
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Declaro que segui o código de ética do Curso de Engenharia Ambiental ao realizar esta prova.

NOME: GABARITO Assinatura: ____________________

AO REALIZAR ESTA PROVA, VOCÊ DEVE JUSTIFICAR TODAS AS PASSAGENS. EVITE "PULAR" PARTES IMPORTANTES DO DESENVOLVIMENTO DE CADA QUESTÃO. JUSTIFIQUE CADA PASSO IMPORTANTE. SIMPLIFIQUE AO MÁXIMO SUAS RESPOSTAS.

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1 [25] Se

$$
f(x) = \begin{cases} 1 - \left(\frac{x}{a}\right)^2, & |x| \le a, \\ 0, & |x| > a, \end{cases}
$$

obtenha a sua transformada de Fourier $\widehat{f}(k)$, sabendo que

$$
\int_0^a \left[1 - \left(\frac{x}{a}\right)^2\right] \cos(kx)\,\mathrm{d}x = \frac{2\,\mathrm{sen}(ak) - 2ak\cos(ak)}{a^2k^3}.
$$

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
\widehat{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-a}^{+a} \left[1 - \left(\frac{x}{a}\right)^2\right] \left[\cos(kx) - \mathrm{i}\,\mathrm{sen}(kx)\right]\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-a}^{+a} \left[1 - \left(\frac{x}{a}\right)^2\right] \cos(kx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{+a} \left[1 - \left(\frac{x}{a}\right)^2\right] \cos(kx)\,\mathrm{d}x \\
&= \frac{2\,\mathrm{sen}(ak) - 2ak\cos(ak)}{\pi a^2k^3} \blacksquare
\end{aligned}
$$

Continue a solução no verso ⟹

2 [25] **Usando obrigatoriamente funções de Green**, resolva

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} + tx &= f(t),\\
x(0) &= 0.
\end{aligned}
$$

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
G(t,\tau)\frac{\mathrm{d}x}{\mathrm{d}\tau} + \tau G(t,\tau)x(\tau) &= G(t,\tau)f(\tau),\\
\int_{\tau=0}^{\infty} G(t,\tau)\frac{\mathrm{d}x}{\mathrm{d}\tau}\,\mathrm{d}\tau + \int_{\tau=0}^{\infty} \tau G(t,\tau)x(\tau)\,\mathrm{d}\tau &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau\\
G(t,\tau)x(\tau)\Big|_{\tau=0}^{\infty} - \int_{\tau=0}^{\infty} x(\tau)\frac{\mathrm{d}G}{\mathrm{d}\tau}\,\mathrm{d}\tau + \int_{\tau=0}^{\infty} \tau G(t,\tau)x(\tau)\,\mathrm{d}\tau &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau,
\end{aligned}
$$

Faça $G(t,\infty) = 0$;

$$
\begin{aligned}
\int_{\tau=0}^{\infty} x(\tau)\left[-\frac{\mathrm{d}G}{\mathrm{d}\tau} + \tau G\right]\mathrm{d}\tau &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau;\\
-\frac{\mathrm{d}G}{\mathrm{d}\tau} + \tau G = \delta(\tau - t) \;\;&\Rightarrow\\
x(t) &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Agora resolvemos a EDO para $G(t,\tau)$:

$$
\begin{aligned}
-\frac{\mathrm{d}G}{\mathrm{d}\tau} + \tau G &= \delta(\tau - t),\\
G(t,\tau) &= u(t,\tau)v(t,\tau),\\
-u\frac{\mathrm{d}v}{\mathrm{d}\tau} - v\frac{\mathrm{d}u}{\mathrm{d}\tau} + \tau uv &= \delta(\tau - t),\\
u\left[-\frac{\mathrm{d}v}{\mathrm{d}\tau} + \tau v\right] - v\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \delta(\tau - t),\\
-\frac{\mathrm{d}v}{\mathrm{d}\tau} + \tau v &= 0,\\
\frac{\mathrm{d}v}{\mathrm{d}\tau} &= \tau v,\\
\frac{\mathrm{d}v}{v} &= \tau \mathrm{d}\tau,\\
\int_{v(t,0)}^{v(t,\tau)} \frac{\mathrm{d}v}{v} &= \int_{\xi=0}^{\tau} \xi \mathrm{d}\xi,\\
\ln\frac{v(t,\tau)}{v(t,0)} &= \frac{\tau^2}{2},\\
v(t,\tau) &= v(t,0)\mathrm{e}^{\frac{\tau^2}{2}};\\
-v(t,0)\mathrm{e}^{\frac{\tau^2}{2}}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \delta(\tau - t),\\
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= -\frac{1}{v(t,0)}\mathrm{e}^{-\frac{\tau^2}{2}}\delta(\tau - t),\\
\int_{u(t,0)}^{u(t,\tau)} \mathrm{d}u &= -\frac{1}{v(t,0)}\int_{\xi=0}^{\tau} \mathrm{e}^{-\frac{\xi^2}{2}}\delta(\xi - t)\,\mathrm{d}\xi,\\
u(t,\tau) - u(t,0) &= -\frac{1}{v(t,0)}\mathrm{e}^{-\frac{t^2}{2}}H(\tau - t),\\
u(t,\tau) &= u(t,0) - \frac{1}{v(t,0)}\mathrm{e}^{-\frac{t^2}{2}}H(\tau - t).
\end{aligned}
$$

Continue a solução no verso ⟹

Substituímos agora na função de Green:

$$
\begin{aligned}
G(t,\tau) &= u(t,\tau)v(t,\tau) \\
&= \left[u(t,0) - \frac{1}{v(t,0)}\mathrm{e}^{-\frac{t^2}{2}}H(\tau - t)\right] v(t,0)\mathrm{e}^{\frac{\tau^2}{2}} \\
&= G(t,0)\mathrm{e}^{\tau^2/2} - H(\tau - t)\mathrm{e}^{(\tau^2-t^2)/2} \\
&= \left[G(t,0) - H(\tau - t)\mathrm{e}^{-t^2/2}\right]\mathrm{e}^{\tau^2/2}.
\end{aligned}
$$

mas

$$
\begin{aligned}
\lim_{\tau\to\infty} G(t,\tau) = 0 \;\Rightarrow \\
G(t,0) &= \mathrm{e}^{-t^2/2}, \\
G(t,\tau) &= \left[\mathrm{e}^{-t^2/2} - H(\tau - t)\mathrm{e}^{-t^2/2}\right]\mathrm{e}^{\tau^2/2} \\
&= \left[1 - H(\tau - t)\right]\mathrm{e}^{(\tau^2-t^2)/2}.
\end{aligned}
$$

Finalmente,

$$
\begin{aligned}
x(t) &= \int_{\tau=0}^{\infty} \left[1 - H(\tau - t)\right]\mathrm{e}^{(\tau^2-t^2)/2} f(\tau)\,\mathrm{d}\tau \\
&= \int_{\tau=0}^{t} \mathrm{e}^{(\tau^2-t^2)/2} f(\tau)\,\mathrm{d}\tau \;\blacksquare
\end{aligned}
$$

Continue a solução no verso $\Longrightarrow$

3 [25] Se

$$f(x) = x,$$
$$g(x) = \mathrm{e}^{-|x|},$$

Calcule a integral de convolução (no sentido de Fourier) $[f * g](x)$.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
[f * g](x) &= \int_{\xi=-\infty}^{+\infty} f(x-\xi) g(\xi)\,\mathrm{d}\xi \\
&= \int_{\xi=-\infty}^{0} (x-\xi)\mathrm{e}^{-|\xi|}\,\mathrm{d}\xi + \int_{\xi=0}^{+\infty} (x-\xi)\mathrm{e}^{-|\xi|}\,\mathrm{d}\xi \\
&= \int_{\xi=-\infty}^{0} (x-\xi)\mathrm{e}^{\xi}\,\mathrm{d}\xi + \int_{\xi=0}^{+\infty} (x-\xi)\mathrm{e}^{-\xi}\,\mathrm{d}\xi \\
&= (x+1) + (x-1) = 2x \blacksquare
\end{aligned}$$

Continue a solução no verso $\Longrightarrow$

4 [25] Dada a equação diferencial parcial

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} + E\frac{\partial^4 \phi}{\partial x^4},$$

onde D e E são constantes reais e $\phi = \phi(x,t)$, obtenha a equação diferencial ordinária em $\widehat{\phi}(k,t)$, onde

$$\widehat{\phi}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(x,t)\mathrm{e}^{-ikx}\,\mathrm{d}x$$

é a transformada de Fourier de $\phi(x,t)$ (em relação a x). **Não tente resolver a equação resultante**.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
\frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}t} &= D(\mathrm{i}k)^2\widehat{\phi} + E(\mathrm{i}k)^4\widehat{\phi},\\
\frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}t} &= -Dk^2\widehat{\phi} + Ek^4\widehat{\phi},\\
\frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}t} &= \left[-Dk^2 + Ek^4\right]\widehat{\phi} \blacksquare
\end{aligned}$$

Continue a solução no verso ⟹