EAMB 7004 Camadas-Limite Naturais e Dispersão de Poluentes
Programa de Pós-Graduação em Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
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Prova com consulta exclusivamente ao material didático da disciplina.

NOME: Assinatura: ______________________________

Um método de "perturbação" para as equações governantes médias sob a aproximação de Boussinesq

A notação desta prova é a mesma adotada no curso. Por isso, os símbolos usuais não serão definidos.

1 [25] **Uma equação de estado consistente para as flutuações de Boussinesq e de Reynolds**
Suponha que vale a equação de estado linearizada

$$\frac{\rho_\delta}{\rho_r} = -\beta_p T_\delta,$$

onde $\beta_p = \beta_p(T_r, p_r)$ é calculado no estado hidrostático de referência. Usando as decomposições de Boussinesq e de Reynolds para ρ e T apresentadas em aula, mostre que ela se desdobra em duas equações de estado "naturais" para as médias das flutuações de Boussinesq, e para as flutuações de Reynolds:

$$\begin{aligned}\frac{\overline{\rho_\delta}}{\rho_r} &= -\beta_p \overline{T_\delta},\\ \frac{\rho'}{\rho_r} &= -\beta_p T'.\end{aligned}$$

Além disso, utilizando diretamente

$$\frac{\mathrm{d}\rho}{\rho} = -\beta_p \mathrm{d}T + \kappa_T \mathrm{d}p,$$

mostre que

$$\frac{\partial \rho_r}{\partial x_i} = -\rho_r \beta_p \frac{\partial T_r}{\partial x_i} + \rho_r \kappa_T \frac{\partial p_r}{\partial x_i}.$$

Note que a partir da linearlização, β_p pode ser considerado um número fixo que não varia nem com t, nem com x_i.

SOLUÇÃO DA QUESTÃO:
Basta fazer uma decomposição de Reynolds:

$$\begin{aligned}\frac{\rho_\delta}{\rho_r} &= -\beta_p T_\delta,\\ \rho_\delta &= \overline{\rho_\delta} + \rho',\\ T_\delta &= \overline{T_\delta} + T',\\ \frac{\overline{\rho_\delta} + \rho'}{\rho_r} &= -\beta_p(\overline{T_\delta} + T'),\\ \frac{\overline{\rho_\delta}}{\rho_r} &= -\beta_p \overline{T_\delta},\\ \frac{\rho'}{\rho_r} &= -\beta_p T'.\end{aligned}$$

Em seguida,

$$\begin{aligned}\frac{\mathrm{d}\rho}{\rho} &= -\beta_p \mathrm{d}T + \kappa_T \mathrm{d}p,\\ \frac{\mathrm{d}\rho_r}{\rho_r} &= -\beta_p \mathrm{d}T_r + \kappa_T \mathrm{d}p_r,\\ \frac{\partial \rho_r}{\partial x_i} &= -\beta_p \rho_r \frac{\partial T_r}{\partial x_i} + \kappa_T \rho_r \frac{\partial p_r}{\partial x_i} \ \blacksquare\end{aligned}$$

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2 [25] **O termo de 2ª ordem da equação da continuidade**

Considere a equação da continuidade e suas ordens de grandeza:

$$\underbrace{\frac{\partial\overline{\rho_\delta}}{\partial t}}_{\text{I}}+\underbrace{\overline{u_i}\frac{\partial\rho_r}{\partial x_i}}_{\text{II}}+\underbrace{\overline{u_i}\frac{\partial\overline{\rho_\delta}}{\partial x_i}}_{\text{III}}+\underbrace{\rho_r\frac{\partial\overline{u_i}}{\partial x_i}}_{\text{IV}}+\underbrace{\overline{\rho_\delta}\frac{\partial\overline{u_i}}{\partial x_i}}_{\text{V}}+\underbrace{\frac{\partial\overline{\rho' u_i'}}{\partial x_i}}_{\text{VI}}=0.$$

Das aulas, sabemos que cada termo de IV tem ordem de grandeza $\rho_0\tilde{u}/\ell$, e que todos os outros os termos acima (sem considerar somas) têm ordem de grandeza muito menor, $\tilde{\rho}\tilde{u}/\ell$. Reescreva portanto:

$$\underbrace{\frac{\partial\overline{\rho_\delta}}{\partial t}}_{\text{I}}+\underbrace{\overline{u_i}\frac{\partial\rho_r}{\partial x_i}}_{\text{II}}+\underbrace{\overline{u_i}\frac{\partial\overline{\rho_\delta}}{\partial x_i}}_{\text{III}}+\underbrace{\overline{\rho_\delta}\frac{\partial\overline{u_i}}{\partial x_i}}_{\text{V}}+\underbrace{\frac{\partial\overline{\rho' u_i'}}{\partial x_i}}_{\text{VI}}=-\underbrace{\rho_r\frac{\partial\overline{u_i}}{\partial x_i}}_{\text{IV}}=0.$$

Suponha que o campo médio de velocidade é, em 1ª aproximação, solenoidal: $\partial\overline{u_i}/\partial x_i=0$. Mostre que

$$\frac{\partial\overline{\rho_\delta}}{\partial t}+\overline{u_i}\frac{\partial\overline{\rho_\delta}}{\partial x_i}+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}+\frac{\partial\overline{\rho' u_i'}}{\partial x_i}=0.$$

SOLUÇÃO DA QUESTÃO:

$$\frac{\partial\overline{\rho_\delta}}{\partial t}+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}+\overline{u_i}\frac{\partial\overline{\rho_\delta}}{\partial x_i}+\overline{\rho_\delta}\underbrace{\frac{\partial\overline{u_i}}{\partial x_i}}_{=0}+\frac{\partial\overline{\rho' u_i'}}{\partial x_i}=-\rho_r\frac{\partial\overline{u_i}}{\partial x_i}=0,$$

$$\frac{\partial\overline{\rho_\delta}}{\partial t}+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}+\overline{u_i}\frac{\partial\overline{\rho_\delta}}{\partial x_i}+\frac{\partial\overline{\rho' u_i'}}{\partial x_i}=0\ \blacksquare$$

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3 [50] Combinando os resultados das Questões **1** e **2**, deduza uma equação para o campo médio de temperatura:

$$\frac{\partial(T_r+\overline{T_\delta})}{\partial t}+\overline{u_i}\frac{\partial(T_r+\overline{T_\delta})}{\partial x_i}+\frac{\partial\overline{T'u_i'}}{\partial x_i}-\overline{u_i}\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}+\overline{T'u_i'}\left(-\beta_p\frac{\partial T_r}{\partial x_i}+\kappa_T\frac{\partial p_r}{\partial x_i}\right)=0.$$

Atenção: Na dedução da equação acima, você deve desprezar um termo considerando que $\frac{\overline{\rho_\delta}}{\rho_r}\ll 1$.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
\frac{\partial\overline{\rho_\delta}}{\partial t}+\overline{u_i}\frac{\partial\overline{\rho_\delta}}{\partial x_i}+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}+\frac{\partial\overline{\rho' u_i'}}{\partial x_i}&=0;\\
\rho_\delta&=-\beta_p\rho_r\overline{T_\delta};\\
\rho'&=-\beta_p\rho_r T';\\
\frac{\partial\rho_r}{\partial x_i}&=-\rho_r\beta_p\frac{\partial T_r}{\partial x_i}+\rho_r\kappa_T\frac{\partial p_r}{\partial x_i},\\
\frac{\partial\left[-\beta_p\rho_r\overline{T_\delta}\right]}{\partial t}+\overline{u_i}\frac{\partial\left[-\beta_p\rho_r\overline{T_\delta}\right]}{\partial x_i}+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}-\beta_p\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
-\beta_p\rho_r\frac{\partial\overline{T_\delta}}{\partial t}-\beta_p\overline{u_i}\left[\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{T_\delta}\frac{\partial\rho_r}{\partial x_i}\right]+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}-\beta_p\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
-\beta_p\rho_r\frac{\partial\overline{T_\delta}}{\partial t}-\beta_p\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[(-\beta_p\overline{T_\delta})\frac{\partial\rho_r}{\partial x_i}+\frac{\partial\rho_r}{\partial x_i}\right]-\beta_p\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
-\beta_p\rho_r\frac{\partial\overline{T_\delta}}{\partial t}-\beta_p\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[\underbrace{\frac{\overline{\rho_\delta}}{\rho_r}}_{\ll 1}+1\right]\frac{\partial\rho_r}{\partial x_i}-\beta_p\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
-\beta_p\rho_r\frac{\partial\overline{T_\delta}}{\partial t}-\beta_p\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\frac{\partial\rho_r}{\partial x_i}-\beta_p\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
-\beta_p\rho_r\frac{\partial\overline{T_\delta}}{\partial t}-\beta_p\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[-\rho_r\beta_p\frac{\partial T_r}{\partial x_i}+\rho_r\kappa_T\frac{\partial p_r}{\partial x_i}\right]-\beta_p\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
\rho_r\frac{\partial\overline{T_\delta}}{\partial t}+\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[\rho_r\frac{\partial T_r}{\partial x_i}-\rho_r\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}\right]+\frac{\partial\overline{\left[\rho_r T' u_i'\right]}}{\partial x_i}&=0;\\
\rho_r\frac{\partial\overline{T_\delta}}{\partial t}+\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[\rho_r\frac{\partial T_r}{\partial x_i}-\rho_r\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}\right]+\left[\rho_r\frac{\partial\overline{T'u_i'}}{\partial x_i}+\overline{T'u_i'}\frac{\partial\rho_r}{\partial x_i}\right]&=0;\\
\rho_r\frac{\partial\overline{T_\delta}}{\partial t}+\overline{u_i}\rho_r\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[\rho_r\frac{\partial T_r}{\partial x_i}-\rho_r\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}\right]+\left[\rho_r\frac{\partial\overline{T'u_i'}}{\partial x_i}+\overline{T'u_i'}\left(-\beta_p\rho_r\frac{\partial T_r}{\partial x_i}+\rho_r\kappa_T\frac{\partial p_r}{\partial x_i}\right)\right]&=0;\\
\frac{\partial\overline{T_\delta}}{\partial t}+\overline{u_i}\frac{\partial\overline{T_\delta}}{\partial x_i}+\overline{u_i}\left[\frac{\partial T_r}{\partial x_i}-\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}\right]+\left[\frac{\partial\overline{T'u_i'}}{\partial x_i}+\overline{T'u_i'}\left(-\beta_p\frac{\partial T_r}{\partial x_i}+\kappa_T\frac{\partial p_r}{\partial x_i}\right)\right]&=0;\\
\frac{\partial(T_r+\overline{T_\delta})}{\partial t}+\overline{u_i}\frac{\partial(T_r+\overline{T_\delta})}{\partial x_i}-\overline{u_i}\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}+\left[\frac{\partial\overline{T'u_i'}}{\partial x_i}+\overline{T'u_i'}\left(-\beta_p\frac{\partial T_r}{\partial x_i}+\kappa_T\frac{\partial p_r}{\partial x_i}\right)\right]&=0;\\
\frac{\partial(T_r+\overline{T_\delta})}{\partial t}+\overline{u_i}\frac{\partial(T_r+\overline{T_\delta})}{\partial x_i}+\frac{\partial\overline{T'u_i'}}{\partial x_i}-\overline{u_i}\frac{\kappa_T}{\beta_p}\frac{\partial p_r}{\partial x_i}+\overline{T'u_i'}\left(-\beta_p\frac{\partial T_r}{\partial x_i}+\kappa_T\frac{\partial p_r}{\partial x_i}\right)&=0\ \blacksquare
\end{aligned}$$

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