TEA013 Matemática Aplicada II
Curso de Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
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Declaro que segui o código de ética do Curso de Engenharia Ambiental ao realizar esta prova.

NOME: GABARITO Assinatura: ____________________

AO REALIZAR ESTA PROVA, VOCÊ DEVE JUSTIFICAR TODAS AS PASSAGENS. EVITE "PULAR" PARTES IMPORTANTES DO DESENVOLVIMENTO DE CADA QUESTÃO. JUSTIFIQUE CADA PASSO IMPORTANTE. SIMPLIFIQUE AO MÁXIMO SUAS RESPOSTAS.

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1 [25] Generalize a identidade de Parseval. Se $f(x)$ e $g(x)$ são duas funções complexas de uma variável real x no intervalo $[a,b]$, e se

$$
\begin{aligned}
f(x) &= \sum_{m=-\infty}^{+\infty} c_m \mathrm{e}^{\frac{2m\pi \mathrm{i} x}{L}},\\
g(x) &= \sum_{n=-\infty}^{+\infty} d_n \mathrm{e}^{\frac{2n\pi \mathrm{i} x}{L}},
\end{aligned}
$$

obtenha

$$
\frac{1}{L}\int_a^b f^*(x) g(x)\,\mathrm{d}x
$$

como uma soma sobre os $c_n s$ e $d_n s$. **Sugestão:** escreva o produto $f^*(x)g(x)$ em termos das séries de Fourier acima, e integre de a até b. Observe que

$$
\int_a^b \mathrm{e}^{-\frac{2m\pi \mathrm{i} x}{L}} \mathrm{e}^{\frac{2n\pi \mathrm{i} x}{L}}\,\mathrm{d}x = \delta_{mn} L.
$$

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
f^*(x)g(x) &= \left[\sum_{m=-\infty}^{+\infty} c_m^* \mathrm{e}^{-\frac{2m\pi \mathrm{i} x}{L}}\right]\left[\sum_{n=-\infty}^{+\infty} d_n \mathrm{e}^{\frac{2n\pi \mathrm{i} x}{L}}\right]\\
&= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_m^* d_n \mathrm{e}^{-\frac{2m\pi \mathrm{i} x}{L}} \mathrm{e}^{\frac{2n\pi \mathrm{i} x}{L}};\\
\int_a^b f^*(x)g(x)\,\mathrm{d}x &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_m^* d_n \int_a^b \mathrm{e}^{-\frac{2m\pi \mathrm{i} x}{L}} \mathrm{e}^{\frac{2n\pi \mathrm{i} x}{L}}\,\mathrm{d}x\\
&= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_m^* d_n \delta_{mn} L \Rightarrow\\
\frac{1}{L}\int_a^b f^*(x)g(x)\,\mathrm{d}x &= \sum_{n=-\infty}^{+\infty} c_n^* d_n \blacksquare
\end{aligned}
$$

Continue a solução no verso ⟹

2 [25] **Usando obrigatoriamente funções de Green**, resolva

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} + tx &= f(t),\\
x(0) &= 0.
\end{aligned}
$$

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
G(t,\tau)\frac{\mathrm{d}x}{\mathrm{d}\tau} + \tau G(t,\tau)x(\tau) &= G(t,\tau)f(\tau),\\
\int_{\tau=0}^{\infty} G(t,\tau)\frac{\mathrm{d}x}{\mathrm{d}\tau}\,\mathrm{d}\tau + \int_{\tau=0}^{\infty} \tau G(t,\tau)x(\tau)\,\mathrm{d}\tau &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau\\
G(t,\tau)x(\tau)\Big|_{\tau=0}^{\infty} - \int_{\tau=0}^{\infty} x(\tau)\frac{\mathrm{d}G}{\mathrm{d}\tau}\,\mathrm{d}\tau + \int_{\tau=0}^{\infty} \tau G(t,\tau)x(\tau)\,\mathrm{d}\tau &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau,
\end{aligned}
$$

Faça $G(t,\infty) = 0$;

$$
\begin{aligned}
\int_{\tau=0}^{\infty} x(\tau)\left[-\frac{\mathrm{d}G}{\mathrm{d}\tau} + \tau G\right]\mathrm{d}\tau &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau;\\
-\frac{\mathrm{d}G}{\mathrm{d}\tau} + \tau G = \delta(\tau - t) &\;\Rightarrow\\
x(t) &= \int_{\tau=0}^{\infty} G(t,\tau)f(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Agora resolvemos a EDO para $G(t,\tau)$:

$$
\begin{aligned}
-\frac{\mathrm{d}G}{\mathrm{d}\tau} + \tau G &= \delta(\tau - t),\\
G(t,\tau) &= u(t,\tau)v(t,\tau),\\
-u\frac{\mathrm{d}v}{\mathrm{d}\tau} - v\frac{\mathrm{d}u}{\mathrm{d}\tau} + \tau uv &= \delta(\tau - t),\\
u\left[-\frac{\mathrm{d}v}{\mathrm{d}\tau} + \tau v\right] - v\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \delta(\tau - t),\\
-\frac{\mathrm{d}v}{\mathrm{d}\tau} + \tau v &= 0,\\
\frac{\mathrm{d}v}{\mathrm{d}\tau} &= \tau v,\\
\frac{\mathrm{d}v}{v} &= \tau \mathrm{d}\tau,\\
\int_{v(t,0)}^{v(t,\tau)} \frac{\mathrm{d}v}{v} &= \int_{\xi=0}^{\tau} \xi \mathrm{d}\xi,\\
\ln\frac{v(t,\tau)}{v(t,0)} &= \frac{\tau^2}{2},\\
v(t,\tau) &= v(t,0)\mathrm{e}^{\frac{\tau^2}{2}};\\
-v(t,0)\mathrm{e}^{\frac{\tau^2}{2}}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \delta(\tau - t),\\
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= -\frac{1}{v(t,0)}\mathrm{e}^{-\frac{\tau^2}{2}}\delta(\tau - t),\\
\int_{u(t,0)}^{u(t,\tau)} \mathrm{d}u &= -\frac{1}{v(t,0)}\int_{\xi=0}^{\tau} \mathrm{e}^{-\frac{\xi^2}{2}}\delta(\xi - t)\,\mathrm{d}\xi,\\
u(t,\tau) - u(t,0) &= -\frac{1}{v(t,0)}\mathrm{e}^{-\frac{t^2}{2}}H(\tau - t),\\
u(t,\tau) &= u(t,0) - \frac{1}{v(t,0)}\mathrm{e}^{-\frac{t^2}{2}}H(\tau - t).
\end{aligned}
$$

Continue a solução no verso ⟹

Substituímos agora na função de Green:

$$
\begin{aligned}
G(t,\tau) &= u(t,\tau)v(t,\tau) \\
&= \left[u(t,0) - \frac{1}{v(t,0)}\mathrm{e}^{-\frac{t^2}{2}}H(\tau - t)\right] v(t,0)\mathrm{e}^{\frac{\tau^2}{2}} \\
&= G(t,0)\mathrm{e}^{\tau^2/2} - H(\tau - t)\mathrm{e}^{(\tau^2-t^2)/2} \\
&= \left[G(t,0) - H(\tau - t)\mathrm{e}^{-t^2/2}\right]\mathrm{e}^{\tau^2/2}.
\end{aligned}
$$

mas

$$
\begin{aligned}
&\lim_{\tau\to\infty} G(t,\tau) = 0 \;\Rightarrow \\
&\quad G(t,0) = \mathrm{e}^{-t^2/2}, \\
&\quad G(t,\tau) = \left[\mathrm{e}^{-t^2/2} - H(\tau - t)\mathrm{e}^{-t^2/2}\right]\mathrm{e}^{\tau^2/2} \\
&\qquad\quad = \left[1 - H(\tau - t)\right]\mathrm{e}^{(\tau^2-t^2)/2}.
\end{aligned}
$$

Finalmente,

$$
\begin{aligned}
x(t) &= \int_{\tau=0}^{\infty} \left[1 - H(\tau - t)\right]\mathrm{e}^{(\tau^2-t^2)/2} f(\tau)\,\mathrm{d}\tau \\
&= \int_{\tau=0}^{t} \mathrm{e}^{(\tau^2-t^2)/2} f(\tau)\,\mathrm{d}\tau \;\blacksquare
\end{aligned}
$$

Continue a solução no verso ⟹

Continue a solução no verso $\Longrightarrow$

3

a) [12.5] Calcule

$$\widehat{\delta}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Se $\delta(x) = \frac{\mathrm{d}H(x)}{\mathrm{d}x}$, onde

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0, \end{cases}$$

mostre que

$$\widehat{H}(k) = \frac{1}{2\pi\mathrm{i}k}.$$

b) [12.5] Prove que

$$\mathscr{F}\{f(x-a)\} = \mathrm{e}^{-\mathrm{i}ka}\widehat{f}(k) \Leftrightarrow f(x-a) = \int_{k=-\infty}^{+\infty} \widehat{f}(k)\mathrm{e}^{\mathrm{i}k(x-a)}\,\mathrm{d}k.$$

Sugestão: faça $\xi = x - a$ e substitua na definição de $\mathscr{F}\{f(x-a)\}$.

SOLUÇÃO DA QUESTÃO:

a)

$$\begin{aligned}\widehat{\delta}(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\mathrm{e}^{0} = \frac{1}{2\pi}.\end{aligned}$$

$$\begin{aligned}\delta(x) &= \frac{\mathrm{d}H(x)}{\mathrm{d}x}, \\ \widehat{\delta}(k) &= \mathrm{i}k\widehat{H}(k), \\ \widehat{H}(k) &= \frac{1}{2\pi\mathrm{i}k}.\end{aligned}$$

b)

$$\begin{aligned}\mathscr{F}\{f(x-a)\} &= \frac{1}{2\pi}\int_{x=-\infty}^{+\infty} f(x-a)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{\xi=-\infty}^{+\infty} f(\xi)\mathrm{e}^{-\mathrm{i}k(\xi+a)}\,\mathrm{d}\xi \\ &= \mathrm{e}^{-\mathrm{i}ka}\frac{1}{2\pi}\int_{\xi=-\infty}^{+\infty} f(\xi)\mathrm{e}^{-\mathrm{i}k\xi}\,\mathrm{d}\xi \\ &= \mathrm{e}^{-\mathrm{i}ka}\widehat{f}(k) \;\blacksquare\end{aligned}$$

Continue a solução no verso ⟹

4 [25] Desenhe a função $H(x+L)-H(x-L)$, onde $H(x)$ é a função de Heaviside. Utilizando obrigatoriamente a transformada de Fourier, e os resultados da questão **3** (mesmo que você não tenha conseguido fazê-la), resolva a EDP

$$
\begin{aligned}
\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} &= 0,\\
u &= \text{constante},\\
c(x,0) &= c_0\left[H(x+L)-H(x-L)\right],\\
c_0 &= \text{constante}.
\end{aligned}
$$

Sugestão: não use $\mathrm{e}^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\,\mathrm{sen}(\theta)$; deixe todas as exponenciais complexas intactas e interprete, utilizando os resultados da questão **3**.

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
\frac{\mathrm{d}\widehat{c}}{\mathrm{d}t} + u\mathrm{i}k\widehat{c} &= 0,\\
\frac{\mathrm{d}\widehat{c}}{\widehat{c}} &= -u\mathrm{i}k\mathrm{d}t,\\
\ln\frac{\widehat{c}(k,t)}{\widehat{c}(k,0)} &= -u\mathrm{i}kt,\\
\widehat{c}(k,t) &= \widehat{c}(k,0)\mathrm{e}^{-\mathrm{i}kut}.
\end{aligned}
$$

Agora encontramos $\widehat{c}(k,0)$:

$$
\begin{aligned}
\widehat{c}(k,0) &= \mathscr{F}\{c(x,0)\}\\
&= \mathscr{F}\{c_0\left[H(x+L)-H(x-L)\right]\}\\
&= \frac{c_0}{2\pi\mathrm{i}k}\left[\mathrm{e}^{\mathrm{i}kL}-\mathrm{e}^{-\mathrm{i}kL}\right].
\end{aligned}
$$

Voltando ao problema original,

$$
\begin{aligned}
c(x,t) &= \int_{k=-\infty}^{+\infty}\widehat{c}(k,t)\mathrm{e}^{+\mathrm{i}kx}\,\mathrm{d}k\\
&= \int_{k=-\infty}^{+\infty}\frac{c_0}{2\pi\mathrm{i}k}\left[\mathrm{e}^{\mathrm{i}kL}-\mathrm{e}^{-\mathrm{i}kL}\right]\mathrm{e}^{-\mathrm{i}kut}\mathrm{e}^{+\mathrm{i}kx}\,\mathrm{d}k\\
&= \int_{k=-\infty}^{+\infty}\frac{c_0}{2\pi\mathrm{i}k}\mathrm{e}^{\mathrm{i}kL}\mathrm{e}^{-\mathrm{i}kut}\mathrm{e}^{+\mathrm{i}kx}\,\mathrm{d}k - \int_{k=-\infty}^{+\infty}\frac{c_0}{2\pi\mathrm{i}k}\mathrm{e}^{-\mathrm{i}kL}\mathrm{e}^{-\mathrm{i}kut}\mathrm{e}^{+\mathrm{i}kx}\,\mathrm{d}k\\
&= \int_{k=-\infty}^{+\infty}\frac{c_0}{2\pi\mathrm{i}k}\mathrm{e}^{\mathrm{i}k(x-ut+L)}\,\mathrm{d}k - \int_{k=-\infty}^{+\infty}\frac{c_0}{2\pi\mathrm{i}k}\mathrm{e}^{-\mathrm{i}k(x-ut-L)}\,\mathrm{d}k\\
&= c_0H(x-ut+L) - c_0H(x-ut-L)\\
&= c_0\left[H(x-ut+L)-H(x-ut-L)\right] \blacksquare
\end{aligned}
$$

Continue a solução no verso ⟹