Como denotar uma dimensão "à mão livre":

$$
\begin{aligned}
[\![T]\!] &= \mathbb{T},\\
[\![L]\!] &= \mathbb{L},\\
[\![M]\!] &= \mathbb{M}
\end{aligned}
$$

$$
\begin{aligned}
1 &= \mathsf{M}^{1+c}\mathsf{L}^{1+a+b-3c}\mathsf{T}^{-2-b},\\
0 &= 1+c,\\
0 &= 1+a+b-3c,\\
0 &= -2-b
\end{aligned}
$$

$$
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}
$$

Um parâmetro adimensional elevado a qualquer potência continua a ser (é óbvio) adimensional:

$$
\begin{aligned}
\left[\!\!\left[\frac{F}{\rho U^2 D^2}\right]\!\!\right] &= 1,\\
\left[\!\!\left[\left[\frac{F}{\rho U^2 D^2}\right]^p\right]\!\!\right] &= 1,\\
\left[\!\!\left[\frac{\nu}{UD}\right]\!\!\right] &= 1,\\
\left[\!\!\left[\left[\frac{\nu}{UD}\right]^p\right]\!\!\right] &= 1,\\
\left[\!\!\left[\frac{UD}{\nu}\right]\!\!\right] &= 1,
\end{aligned}
$$

Refaço agora o Exemplo 1.1 escolhendo como variáveis comuns a todos os parâmetros (grupos) adimensionais ρ, ν, e D.

$$
\begin{aligned}
\Pi_1 &= F^1\rho^a\nu^b D^c,\\
[\![\Pi_1]\!] &= \mathsf{MLT}^{-2}\left[\mathsf{ML}^{-3}\right]^a\left[\mathsf{L}^2\mathsf{T}^{-1}\right]^b[\mathsf{L}]^c\\
1 &= \mathsf{M}^{1+a}\mathsf{L}^{1-3a+2b+c}\mathsf{T}^{-2-b}
\end{aligned}
$$

Isso produz o sistema de equações lineares

$$
\begin{aligned}
a &= -1,\\
-3a+2b+c &= -1,\\
-b &= 2
\end{aligned}
$$

Com solução

$$a = -1, \qquad b = -2, c = 0.$$

$$\Pi_1 = \frac{F}{\rho\nu^2},$$
$$[\![\Pi_1]\!] = \frac{\mathsf{MLT}^{-2}}{\mathsf{ML}^{-3}(\mathsf{L}^2\mathsf{T}^{-1})^2} = \frac{\mathsf{MLT}^{-2}}{\mathsf{ML}^{-3}\mathsf{L}^4\mathsf{T}^{-2}} = 1 \blacksquare$$

Procuramos agora o Π_2:

$$\Pi_2 = U^1\rho^a\nu^b D^c,$$
$$[\![\Pi_1]\!] = \mathsf{LT}^{-1}\left[\mathsf{ML}^{-3}\right]^a\left[\mathsf{L}^2\mathsf{T}^{-1}\right]^b[\mathsf{L}]^c$$
$$1 = \mathsf{M}^a\mathsf{L}^{1-3a+2b+c}\mathsf{T}^{-1-b}$$

Isso produz o sistema de equações lineares

$$a = 0,$$
$$-3a + 2b + c = -1,$$
$$-b = 1.$$

A solução é

$$a = 0, \qquad b = -1, \qquad c = 1.$$

Isso produz o segundo grupo adimensional

$$\Pi_2 = \frac{UD}{\nu},$$

que é o número de Reynolds! Antes eu tinha, agora eu tenho:

$$\Pi_1 = \frac{F}{\rho U^2 D^2}, \qquad \Pi_1' = \frac{F}{\rho\nu^2},$$
$$\Pi_2 = \frac{\nu}{UD}, \qquad \Pi_2' = \frac{UD}{\nu}$$

Faço

$$\Pi_1 \times \Pi_2^{-2} = \frac{F}{\rho U^2 D^2} \times \left(\frac{UD}{\nu}\right)^2 = \frac{F}{\rho\nu^2} = \Pi_1',$$
$$\Pi_2^{-1} = \Pi_2'.$$

Extensões naturais de “dimensões fundamentais” para problemas específicos:

1. Quando há espécies químicas diferentes *sem* reações químicas entre elas: M é a massa total, M_v é a massa de vapor d’água, M_c é a massa de CO_2, etc..

2. Quando há transferência de *calor sem* que haja conversão significativa de energia mecânica em energia interna. Neste caso, além de energia mecânica, $\mathsf{ML}^2\mathsf{T}^2$, “calor” deve se expresso em termos de uma dimensão extra correspondente, em geral temperatura Θ.

3. Quando existem diferenças grandes do tamanho de grandezas características em direções diferentes: em lugar de um único L, muitas vezes é proveitoso utilizar uma dimensão física para cada dimensão do espaço, ou seja: X, Y, Z.

Eu gostaria de ter um método numérico que funcionasse para "qualquer" equação diferencial ordinária (EDO) do tipo

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} &= \operatorname{sen}(x),\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y}{x} + \operatorname{sen}(x).\end{aligned}$$

O que significam os k's

$$\begin{aligned}y_{n+1} &\approx y_n + h\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_n\\ k_1 &= h\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_n = hf(x_n, y_n)\\ y_{n+1} &= y_n + k_1\end{aligned}$$

Um sistema de equações diferenciais ordinárias já colocado na forma "clássica" do método de Runge-Kutta é algo do tipo: um conjunto de incógnitas $y_1, y_2, \ldots, y_n$ que evoluem com uma dinâmica acoplada:

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}x} &= f_1(x, y_1, \ldots, y_n),\\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= f_2(x, y_1, \ldots, y_n),\\ &\vdots\\ \frac{\mathrm{d}y_n}{\mathrm{d}x} &= f_n(x, y_1, \ldots, y_n).\end{aligned}$$

Vetores, convenções, etc.

$$\begin{aligned}\boldsymbol{i} &= (1, 0, 0),\\ \boldsymbol{j} &= (0, 1, 0),\\ \boldsymbol{k} &= (0, 0, 1).\end{aligned}$$

$$\begin{aligned}\boldsymbol{u} &= (1, 2, 3),\\ \boldsymbol{u} &= 1\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}.\end{aligned}$$

$$
\begin{aligned}
c &= (d+e) \\
c^2 &= (d+e)^2 \\
c^2 &= d^2 + e^2 + 2de \;\blacksquare
\end{aligned}
$$

Notação indicial ou de Einstein

$$y_i = A_{ij} x_j$$

quer dizer

$$y_i = \sum_{j=1}^{n} A_{ij} x_j, \qquad \forall i = 1, \dots n.$$

$$
\begin{aligned}
\delta_{ii} &= \sum_{i=1}^{3} \delta_{(i)(i)} \\
&= \delta_{11} + \delta_{22} + \delta_{33} \\
&= 1 + 1 + 1 \\
&= 3.
\end{aligned}
$$

$$
\begin{aligned}
[\boldsymbol{\delta}][\boldsymbol{x}] &= [\delta_{ij}][x_j] \\
&= \delta_{ij} x_j = x_i \\
&= [x_i] = [\boldsymbol{x}].
\end{aligned}
$$

No caso unidimensional,

$$[\boldsymbol{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\boldsymbol{x}]^\top = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

O significado de $[\boldsymbol{y}]^\top [\boldsymbol{\delta}]$ é

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

Em notação indicial:

$$y_i \delta_{ij} = y_j$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

O que são dois vetores ortogonais no $\mathbb{R}^3$?

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\left(\frac{\pi}{2}\right) = 0.$$

No $\mathbb{R}^n$, nós generalizamos sem poder "ver" os vetores:

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

Uma base ortogonal é uma base $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ tal que, se $i \neq j$,

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0.$$

Se o índice for repetido, obviamente,

$$\boldsymbol{v}_{(i)} \cdot \boldsymbol{v}_{(i)} = |\boldsymbol{v}_{(i)}|^2.$$

Uma base *ortogonal* é tudo isso aí em cima somado a

$$\boldsymbol{v}_{(i)} \cdot \boldsymbol{v}_{(i)} = 1.$$

Dada uma base ortonormal E,

$$\begin{aligned} \boldsymbol{u} &= u_{Ei}\boldsymbol{e}_i, \\ \boldsymbol{v} &= v_{Ej}\boldsymbol{e}_j. \end{aligned}$$

onde

$$\boldsymbol{u} = \begin{bmatrix} u_{E1} \\ u_{E2} \\ \vdots \\ u_{En} \end{bmatrix}_E$$

Lembre-se! $u = (u_1, u_2, \ldots, u_n)$, e em geral $u_1 \neq u_{E1}$, $u_2 \neq u_{E2}$, etc.

É possível usar as coordendas u_{Ei} e v_{Ej} para calcular o produto escalar $\boldsymbol{u} \cdot \boldsymbol{j}$? Sim:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_{Ei}\boldsymbol{e}_i \cdot v_{Ei}\boldsymbol{e}_i$$

Paro, e recomeço:

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}\boldsymbol{e}_i \cdot v_{Ej}\boldsymbol{e}_j \\ \alpha\boldsymbol{x} \cdot \beta\boldsymbol{y} &= \alpha\beta(\boldsymbol{x} \cdot \boldsymbol{y}); \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}v_{Ej}(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}v_{Ej}\delta_{ij} \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}[v_{Ej}\delta_{ij}] \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_{Ei}v_{Ei} = \sum_{i=1}^{n} u_{Ei}v_{Ei}; \\ \boldsymbol{u} \cdot \boldsymbol{v} &= u_i v_i = \sum_{i=1}^{n} u_i v_i. \end{aligned}$$

Às vezes escrever $\boldsymbol{u} = u_{Ei}\boldsymbol{e}_i$ é demais. Quando não houver motivo de confusão, nós simplicaremos a notação e escreveremos

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_E$$

ou simplesmente $\boldsymbol{u} = u_i\boldsymbol{e}_i$, etc.

Reforçando: elemento de vetor $\times$ coordenada de vetor.

$$\boldsymbol{v} = (3, 4)$$

Se eu escolher a base $((1,1),(1,-1))$ para representar o $\mathbb{R}^2$, então

$$\boldsymbol{v} = \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}_{((1,1),(1,-1))}$$

$$\begin{aligned} (3,4) &= 7/2(1,1) - 1/2(1,-1) \\ &= (7/2, 7/2) + (-1/2, 1/2) \\ &= (6/2, 8/2) \\ &= (3,4) \blacksquare \end{aligned}$$

Bases!

A $= (\boldsymbol{i}, \boldsymbol{j})$, B $= (\boldsymbol{j}, \boldsymbol{i})$.

O símbolo de permutação de Levi-Civita é calculado sempre com n entradas. Exemplos:

$$\begin{aligned} \epsilon_{1234} &= +1, \\ \epsilon_{1243} &= -1, \\ \epsilon_{1223} &= 0. \end{aligned}$$

Permutações cíclicas (apenas com 3 elementos) de 1, 2, 3:

$$\begin{gathered} 1,2,3, \\ 3,1,2, \\ 2,3,1. \end{gathered}$$

Permutações cíclicas (apenas com 3 elementos) de 3, 2, 1:

$$\begin{gathered} 3,2,1, \\ 1,3,2, \\ 2,1,3. \end{gathered}$$

Para definir o produto vetorial eu parto dos "absolutos"

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3), \\ \boldsymbol{v} &= (v_1, v_2, v_3), \\ \boldsymbol{u} \times \boldsymbol{v} &\equiv (\epsilon_{ij1} u_i v_j, \epsilon_{ij2} u_i v_j, \epsilon_{ij3} u_i v_j). \end{aligned}$$

A forma "antiga" de calcular o produto vetorial é

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= (u_2 v_3 - u_3 v_2)\boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3)\boldsymbol{e}_2 + (u_1 v_2 - u_2 v_1)\boldsymbol{e}_3
\end{aligned}$$

A forma "nova" de calcular o produto vetorial é

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \\
&= (\epsilon_{231} u_2 v_3 + \epsilon_{321} u_3 v_2)\boldsymbol{e_1} + (\epsilon_{132} u_1 v_3 + \epsilon_{312} u_3 v_1)\boldsymbol{e_2} + (\epsilon_{123} u_1 v_2 + \epsilon_{213} u_2 v_1)\boldsymbol{e_3} \\
&= (u_2 v_3 - u_3 v_2)\boldsymbol{e_1} + (u_3 v_1 - u_1 v_3)\boldsymbol{e_2} + (u_1 v_2 - u_2 v_1)\boldsymbol{e_3}
\end{aligned}$$

Vamos provar agora que $\boldsymbol{u} \times \boldsymbol{v}$ é perpendicular a $\boldsymbol{u}$.

$$\begin{aligned}
[\boldsymbol{u} \times \boldsymbol{v}] \cdot \boldsymbol{u} &= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \cdot u_l \boldsymbol{e}_l \\
\alpha \boldsymbol{a} \cdot \beta \boldsymbol{b} &= (\alpha\beta)(\boldsymbol{a} \cdot \boldsymbol{b}) \\
&= \epsilon_{ijk} u_i v_j u_l (\boldsymbol{e}_k \cdot \boldsymbol{e}_l) \\
&= \epsilon_{ijk} u_i v_j u_l \delta_{kl} \\
&= \epsilon_{ijk} u_i v_j u_k \\
&= \frac{1}{2} \left[\epsilon_{ijk} u_i v_j u_k + \epsilon_{ijk} u_i v_j u_k\right] \\
&= \frac{1}{2} \left[\epsilon_{ijk} u_i v_j u_k + \epsilon_{kji} u_k v_j u_i\right] \\
&= \frac{1}{2} \left[\epsilon_{ijk} + \epsilon_{kji}\right] u_i v_j u_k \\
&= \frac{1}{2} \left[\epsilon_{ijk} + \epsilon_{jik}\right] u_i v_j u_k \\
&= 0.
\end{aligned}$$

O produto vetorial é anti-simétrico:

$$\begin{aligned}
\boldsymbol{v} \times \boldsymbol{u} &= \epsilon_{ijk} v_i u_j \boldsymbol{e}_k \\
&= \epsilon_{ijk} u_j v_i \boldsymbol{e}_k \\
&= -\epsilon_{jik} u_j v_i \boldsymbol{e}_k \\
&= -\boldsymbol{u} \times \boldsymbol{v} \;\blacksquare
\end{aligned}$$

Em seguida:

$$
\begin{aligned}
|\boldsymbol{u} \times \boldsymbol{v}|^2 &= [\boldsymbol{u} \times \boldsymbol{v}] \cdot [\boldsymbol{u} \times \boldsymbol{v}] \\
&= \epsilon_{ijk} u_i v_j \boldsymbol{e}_k \cdot \epsilon_{lmn} u_l v_m \boldsymbol{e}_n \\
\alpha \boldsymbol{a} \cdot \beta \boldsymbol{b} &= \alpha\beta(\boldsymbol{a} \cdot \boldsymbol{b}) \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m (\boldsymbol{e}_k \cdot \boldsymbol{e}_n) \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m \delta_{kn} \\
&= \epsilon_{ijk} u_i v_j (\epsilon_{lmn} \delta_{kn}) u_l v_m \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmk} u_l v_m \\
&= [\epsilon_{ijk} \epsilon_{lmk}] u_i v_j u_l v_m \\
&= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]\, u_i v_j u_l v_m \\
&= \delta_{il}\delta_{jm} u_i v_j u_l v_m - \delta_{im}\delta_{jl} u_i v_j u_l v_m \\
&= u_i v_j u_i v_j - u_i v_j u_j v_i \\
&= (u_i u_i)(v_j v_j) - (u_i v_i)(u_j v_j) \\
&= (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})(\boldsymbol{u} \cdot \boldsymbol{v}) \\
&= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 - (|\boldsymbol{u}||\boldsymbol{v}| \cos(\theta))(|\boldsymbol{u}||\boldsymbol{v}| \cos(\theta)) \\
&= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \left(1 - \cos^2(\theta)\right) \\
&= |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \operatorname{sen}^2(\theta); \; \Rightarrow \\
|\boldsymbol{u} \times \boldsymbol{v}| &= |\boldsymbol{u}||\boldsymbol{v}| \operatorname{sen}(\theta) \; \blacksquare
\end{aligned}
$$

1. Um “1-volume” é um comprimento.
2. Um “2-volume” é uma área.
3. Um “3-volume” é um volume usual.
4. Um “4-volume” é um “volume quadridimensional”, ou um hipervolume.

Determinantes e hipervolumes.

Considere n vetores no $\mathbb{R}^n$:

$$\begin{aligned}
\boldsymbol{u}_1 &= (u_{11}, u_{21}, \ldots, u_{n1});\\
\boldsymbol{u}_2 &= (u_{12}, u_{22}, \ldots, u_{n2});\\
&\vdots\\
\boldsymbol{u}_n &= (u_{1n}, u_{2n}, \ldots, u_{nn}).
\end{aligned}$$

O determinante associado a esses n vetores é definido como

$$\begin{aligned}
V^n &\equiv \det(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_n)\\
&\equiv \epsilon_{i_1, i_2, \ldots, i_n} u_{1i_1} u_{2i_2} \ldots u_{ni_n}.
\end{aligned}$$

O determinante está associado à matriz

$$[\boldsymbol{u}] = [u_{ij}] = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n}\\ u_{21} & u_{22} & \ldots & u_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{bmatrix}$$

Suponha que uma função f é linear:

$$\begin{aligned}
f(\alpha x) &= \alpha f(x),\\
f(x_1 + x_2) &= f(x_1) + f(x_2).
\end{aligned}$$

Desejamos provar que, então,

$$f(x) = cx.$$

Prova:

$$\frac{\mathrm{d} f(\alpha x)}{\mathrm{d} x} = \alpha f'(\alpha x),$$

mas f é linear:

$$\begin{aligned}
f(\alpha x) &= \alpha f(x)\\
\alpha f'(\alpha x) &= \alpha f'(x).
\end{aligned}$$

Portanto, no caso de f ser linear, teremos que

$$f'(\alpha x) = f'(x), \qquad \forall \alpha \in \mathbb{R},$$

ou seja: a derivada é constante!

$$\begin{aligned}
f'(x) &= c,\\
f(x) &= cx + b,\\
f(0) &= 0 \Rightarrow b = 0 \;\blacksquare
\end{aligned}$$

Transformações lineares. Pergunta: será possível definir uma operação entre $\boldsymbol{A}$ e $\boldsymbol{x}$ que nos permita substituir

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{x})$$

por

$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}?$$

$$\begin{aligned}
A_i \delta_{im} &= \sum_{i=1}^{3} A_i \delta_{im} \\
&= A_1 \delta_{1m} + A_2 \delta_{2m} + A_3 \delta_{3m} \\
&= A_m \\
A_{ijkl} B_{ijkp} \delta_{ln} &= A_{ijkn} B_{ijkp}
\end{aligned}$$

Transformações lineares

$$\begin{aligned}
E &= (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n);\\
\boldsymbol{x} &= \sum_{j=1}^{3} x_j \boldsymbol{e}_j = x_j \boldsymbol{e}_j\\
\boldsymbol{A}(\boldsymbol{x}) &= \boldsymbol{A}(x_j \boldsymbol{e}_j)\\
&= x_j \boldsymbol{A}(\boldsymbol{e}_j)
\end{aligned}$$

mas

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{e}_j) &\in \mathbb{V};\\
\boldsymbol{A}(\boldsymbol{e}_j) &= A_{ij} \boldsymbol{e}_i
\end{aligned}$$

portanto

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= x_j A_{ij} \boldsymbol{e}_i\\
&= A_{ij} x_j \boldsymbol{e}_i\\
\boldsymbol{A}(\boldsymbol{x}) &= \boldsymbol{y};\\
\boldsymbol{y} &= y_i \boldsymbol{e}_i = A_{ij} x_j \boldsymbol{e}_i :\\
y_i &= A_{ij} x_j,\\
[\boldsymbol{y}]_E &= [\boldsymbol{A}]_{E,E} [\boldsymbol{x}]_E.
\end{aligned}$$

Agora, eu mudo um pouco o jogo: a base do domínio continua a ser $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$; e a base do contra-domínio é $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n)$. Agora,

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= x_j \boldsymbol{A}(\boldsymbol{e}_j)\\
\boldsymbol{A}(\boldsymbol{e}_j) &= A_{ij} \boldsymbol{f}_i\\
\boldsymbol{A}(\boldsymbol{x}) &= A_{ij} x_j \boldsymbol{f}_i\\
\boldsymbol{y} &= y_i \boldsymbol{f}_i\\
y_i &= A_{ij} x_j\\
[\boldsymbol{y}]_F &= [\boldsymbol{A}]_{F,E} [\boldsymbol{x}]_E
\end{aligned}$$

Vamos agora nos lembrar de que um funcional linear em sua forma mais geral é dado por

$$f(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = c_j x_j = \boldsymbol{c} \cdot \boldsymbol{x}.$$

Mas

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= A_{ij} x_j \boldsymbol{f}_i, \\
\boldsymbol{A}_i &= \boldsymbol{c}; \\
\boldsymbol{A}_i &= A_{ij} \boldsymbol{e}_j \\
\boldsymbol{A}(\boldsymbol{x}) &= [\boldsymbol{c} \cdot \boldsymbol{x}]_i \boldsymbol{f}_i \\
&= [A_{ij} \boldsymbol{e}_j \cdot \boldsymbol{x}] \boldsymbol{f}_i \\
&= [A_{ij} \boldsymbol{e}_j \cdot x_k \boldsymbol{e}_k] \boldsymbol{f}_i \\
&= [A_{ij} x_k \boldsymbol{e}_j \cdot \boldsymbol{e}_k] \boldsymbol{f}_i \\
&= [A_{ij} x_k \delta_{jk}] \boldsymbol{f}_i \\
&= [A_{ij} x_j] \boldsymbol{f}_i \\
&= [A_{ij} \boldsymbol{e}_j \cdot x_k \boldsymbol{e}_k] \boldsymbol{f}_i \\
&= A_{ij} \boldsymbol{f}_i \boldsymbol{e}_j \cdot x_k \boldsymbol{e}_k \\
&= [A_{ij} \boldsymbol{f}_i \boldsymbol{e}_j] \cdot [x_k \boldsymbol{e}_k] \\
\boldsymbol{y} &= \boldsymbol{A} \cdot \boldsymbol{x} \\
\boldsymbol{A} &= A_{ij} \boldsymbol{f}_i \boldsymbol{e}_j
\end{aligned}
$$

Uma transformação de $\mathbb{R}^3$ em $\mathbb{R}^2$ é representada matricialmente por

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}_{2\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}
$$

Bases!

$$
\begin{aligned}
E &= (\boldsymbol{e}_1, \boldsymbol{e}_2) = ((1,0),(0,1)); \\
\boldsymbol{v} &= (3,4) \\
\boldsymbol{v} &= 3\boldsymbol{e}_1 + 4\boldsymbol{e}_2 \\
[\boldsymbol{v}]_E &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
F &= (\boldsymbol{f}_1, \boldsymbol{f}_2) = ((1,1),(1,-1)) \\
\boldsymbol{v} &= (7/2)\boldsymbol{f}_1 - (1/2)\boldsymbol{f}_2 \\
[\boldsymbol{v}]_F &= \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}
\end{aligned}
$$

Às vezes é preciso “disciplinar” a notação indicial:

$$
\sum_{i=1}^{3} C_{ij}^2 = \sum_{i=1}^{3} C_{ij} C_{ij} \equiv C_{i(j)} C_{i(j)} = 1
$$

Notação preferida para transformações lineares:

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{A}(\boldsymbol{x}),\\
\boldsymbol{y} &= \boldsymbol{A}\cdot\boldsymbol{x}\\
\boldsymbol{y} &= y_i\boldsymbol{f}_i,\\
\boldsymbol{x} &= x_j\boldsymbol{e}_j,\\
\boldsymbol{A} &= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j\\
\boldsymbol{y} &= \boldsymbol{A}\cdot\boldsymbol{x}\\
&= A_{ij}\boldsymbol{f}_i\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k\\
&= A_{ij}x_k\boldsymbol{f}_i(\boldsymbol{e}_j\cdot\boldsymbol{e}_k)\\
&= A_{ij}x_k\boldsymbol{f}_i\delta_{jk}\\
y_i\boldsymbol{f}_i &= A_{ij}x_j\boldsymbol{f}_i
\end{aligned}
$$

Rotações e suas matrizes/transformações

$$
\begin{aligned}
\boldsymbol{u} &= u'_j\boldsymbol{e}'_j = u_i\boldsymbol{e}_i\\
u_i &= (\boldsymbol{u}\cdot\boldsymbol{e}_i)
\end{aligned}
$$

mas

$$
\begin{aligned}
\boldsymbol{e}_i &= C_{ij}\boldsymbol{e}'_j\\
u_i &= (\boldsymbol{u}\cdot C_{ij}\boldsymbol{e}'_j)\\
&= C_{ij}(\boldsymbol{u}\cdot\boldsymbol{e}'_j)\\
u_i &= C_{ij}u'_j\\
[\boldsymbol{u}]_E &= [\boldsymbol{C}][\boldsymbol{u}]_{E'}\\
[\boldsymbol{u}]'_E &= [\boldsymbol{C}]^\top[\boldsymbol{u}]_E
\end{aligned}
$$

Podemos mostrar que $[\boldsymbol{C}]$ e $[\boldsymbol{C}]^\top$ são inversas operando diretamente:

$$
\begin{aligned}
\boldsymbol{u} &= u_k\boldsymbol{e}_k = u_i\boldsymbol{e}_i = u_iC_{ij}\boldsymbol{e}'_j = u_iC_{ij}C_{kj}\boldsymbol{e}_k\\
u_k &= u_iC_{ij}C_{kj} = C_{ij}C^\top_{jk}u_i\\
u_k &= \delta_{ki}u_i,\\
C_{ij}C^\top_{jk} &= \delta_{ki}\\
[\boldsymbol{C}][\boldsymbol{C}]^\top &= [\boldsymbol{\delta}].
\end{aligned}
$$

O determinante de uma rotação é igual a 1:

$$
\begin{aligned}
1 &= [\boldsymbol{e}_1\times\boldsymbol{e}_2]\cdot\boldsymbol{e}_3\\
&= [C_{1l}\boldsymbol{e}'_l\times C_{2m}\boldsymbol{e}'_m]\cdot C_{3n}\boldsymbol{e}'_n\\
&= C_{1l}C_{2m}C_{3n}\underbrace{[\boldsymbol{e}'_l\times\boldsymbol{e}'_m]\cdot\boldsymbol{e}'_n}_{\epsilon_{lmn}}\\
1 &= \epsilon_{lmn}C_{1l}C_{2m}C_{3n} = \det[\boldsymbol{C}].
\end{aligned}
$$

Qual é a regra para a mudança das matrizes de uma transformação $\boldsymbol{A}$ quando eu mudo da base E para a base E' (ou vice-versa)?

$$
\begin{aligned}
\boldsymbol{A} &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j, \\
\boldsymbol{A} &= A'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j, \\
\boldsymbol{A} &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{kl}\boldsymbol{e}'_k\boldsymbol{e}'_l \\
A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= A'_{kl}C_{ik}\boldsymbol{e}_i C_{jl}\boldsymbol{e}_j \\
A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= A'_{kl}C_{ik}C_{jl}\boldsymbol{e}_i\boldsymbol{e}_j \\
A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j &= C_{ik}A'_{kl}C^{\top}_{lj}\boldsymbol{e}_i\boldsymbol{e}_j \\
[\boldsymbol{A}]_E &= [\boldsymbol{C}][\boldsymbol{A}]_{E'}[\boldsymbol{C}]^{\top}, \\
[\boldsymbol{A}]_{E'} &= [\boldsymbol{C}]^{\top}[\boldsymbol{A}]_E[\boldsymbol{C}].
\end{aligned}
$$

Exemplo:

Parto da base canônica $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ e desejo construir uma outra base ortonormal dextrógira $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Os dois primeiros vetores são

$$
\begin{aligned}
\boldsymbol{f}_1 &= \frac{1}{\sqrt{3}}(1, 1, 1), \\
\boldsymbol{f}_2 &= \frac{1}{\sqrt{6}}(2, -1, -1).
\end{aligned}
$$

Obviamente

$$
\boldsymbol{f}_3 = \boldsymbol{f}_1 \times \boldsymbol{f}_2.
$$

$$
\begin{aligned}
\boldsymbol{f}_3 &= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\
&= \frac{1}{\sqrt{18}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\
&= \frac{1}{3\sqrt{3}}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\
&= \frac{1}{3\sqrt{2}}[0\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 3\boldsymbol{e}_3] \\
&= \frac{1}{\sqrt{2}}(0, 1, -1).
\end{aligned}
$$

A matriz de rotaçao de E para F é

$$
\begin{aligned}
\boldsymbol{f}_j &= C_{ij}\boldsymbol{e}_i, \\
C_{ij} &= \boldsymbol{f}_j \cdot \boldsymbol{e}_i.
\end{aligned}
$$

Portanto,

$$
[\boldsymbol{C}] = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}
$$

Se eu quero rodar em torno do eixo 2, devo rodar de 1 para 3 ou de 3 para 1?

$$\epsilon_{132} = \epsilon_{321} = -1,$$
$$\epsilon_{312} = \epsilon_{123} = +1.$$

Resposta: de 3 para 1!

Um sistema de equações lineares é algo da forma

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= y_1, \\
A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= y_2, \\
&\vdots \\
A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{nn}x_n &= y_n.
\end{aligned}
$$

A transformação

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

projeta qq vetor do $\mathbb{R}^3$ no plano que passa pela origem e cujo normal é $\boldsymbol{k}$. Note que a equação desse plano é

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0.$$

$$
\begin{aligned}
\boldsymbol{b} &= \boldsymbol{P} \cdot \boldsymbol{a}, \\
b_i &= \left[\delta_{ij} - \frac{k_i k_j}{k^2} \right] a_j \\
b_i &= \delta_{ij} a_j - \frac{k_i k_j}{k^2} a_j \\
&= a_i - \frac{k_i k_j}{k^2} a_j; \\
\boldsymbol{b} \cdot \boldsymbol{k} &= b_i k_i \\
&= \left[a_i - \frac{k_i k_j}{k^2} a_j \right] k_i \\
&= a_i k_i - \left(\frac{k_i k_i}{k^2} \right) k_j a_j \\
&= a_i k_i - a_j k_j = 0 \;\blacksquare
\end{aligned}
$$

Teorema dos Pis:

$$
\begin{aligned}
\left(M^A \right)^x &= M^{Ax}. \\
1 &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{ij}} \right)_j^x \times \ldots
\end{aligned}
$$

Vamos voltar!

$$
\begin{aligned}
\Pi &= \prod_{j=1}^{n} v_j^{x_j}, \\
[\![\Pi]\!] &= \left[\!\!\left[\prod_{j=1}^{n} v_j^{x_j}\right]\!\!\right] \\
[\![\Pi]\!] &= \prod_{j=1}^{n} [\![v_j^{x_j}]\!] \\
[\![\Pi]\!] &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\mathsf{L}^{A_{2j}}\mathsf{T}^{A_{3j}}\right)^{x_j} \\
[\![\Pi]\!] &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\right)^{x_j} \left(\mathsf{L}^{A_{2j}}\right)^{x_j} \left(\mathsf{T}^{A_{3j}}\right)^{x_j} \\
[\![\Pi]\!] &= \prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\right)^{x_j} \prod_{j=1}^{n} \left(\mathsf{L}^{A_{2j}}\right)^{x_j} \prod_{j=1}^{n} \left(\mathsf{T}^{A_{3j}}\right)^{x_j}
\end{aligned}
$$

Se $n = 4$, teremos por exemplo

$$
\begin{aligned}
\prod_{j=1}^{n} \left(\mathsf{M}^{A_{1j}}\right)^{x_j} &= \left(\mathsf{M}^{A_{11}}\right)^{x_1} \times \left(\mathsf{M}^{A_{12}}\right)^{x_2} \times \left(\mathsf{M}^{A_{13}}\right)^{x_3} \times \left(\mathsf{M}^{A_{14}}\right)^{x_4} = 1 \\
\mathsf{M}^{A_{11}x_1} \times \mathsf{M}^{A_{12}x_2} \times \mathsf{M}^{A_{13}x_3} \times \mathsf{M}^{A_{14}x_4} &= 1 \\
\mathsf{M}^{A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4} &= 1 \\
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 &= 0
\end{aligned}
$$

Autovalores e autovetores
Se $\boldsymbol{v}$ é autovetor de $\boldsymbol{A}$, então $\alpha\boldsymbol{v}$, $\alpha \neq 0$, também é:

$$
\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{v} &= \lambda\boldsymbol{v};\\
\boldsymbol{A}\cdot[\alpha\boldsymbol{v}] &= \alpha\boldsymbol{A}\cdot\boldsymbol{v}\\
&= \alpha\lambda\boldsymbol{v}\\
&= \lambda[\alpha\boldsymbol{v}].
\end{aligned}
$$

No exemplo em tela, $\lambda = 1$ possui multiplicidade 2 e também está associado a um subespaço de dimensão 2. A equação

$$
x_1 + 2x_2 + x_3 = 0
$$

é a equação de um plano (pela origem), e quaisquer dois vetores desse plano são autovetores. De fato, sejam $\boldsymbol{v}_1, \boldsymbol{v_2}$ dois autovetores LI associados a λ; então

$$
\begin{aligned}
\boldsymbol{A}\cdot[\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2] &= \alpha_1\boldsymbol{A}\cdot\boldsymbol{v}_1 + \alpha_2\boldsymbol{A}\cdot\boldsymbol{v}_2\\
&= \alpha_1\lambda\boldsymbol{v}_1 + \alpha_2\lambda\boldsymbol{v}_2\\
&= \lambda[\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2].
\end{aligned}
$$

No caso univariado,

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 4x,\\
x &= \mathrm{e}^{4t},\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 4\mathrm{e}^{4t} = 4x.
\end{aligned}
$$

Transposta de uma transformação linear

$$
\begin{aligned}
\boldsymbol{x}\cdot[\boldsymbol{A}^\top\cdot\boldsymbol{y}] &\equiv \boldsymbol{y}\cdot[\boldsymbol{A}\cdot\boldsymbol{x}]\\
x_k\boldsymbol{e}_k\cdot[A^\top_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\cdot y_l\boldsymbol{e}_l] &= y_k\boldsymbol{e}_k\cdot[A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\cdot x_l\boldsymbol{e}_l]\\
x_k\boldsymbol{e}_k\cdot[A^\top_{ij}\boldsymbol{e}_i(\boldsymbol{e}_j\cdot\boldsymbol{e}_l)y_l] &= y_k\boldsymbol{e}_k\cdot[A_{ij}\boldsymbol{e}_i(\boldsymbol{e}_j\cdot\boldsymbol{e}_l)x_l]\\
x_k\boldsymbol{e}_k\cdot[A^\top_{ij}\boldsymbol{e}_i\delta_{jl}y_l] &= y_k\boldsymbol{e}_k\cdot[A_{ij}\boldsymbol{e}_i\delta_{jl}x_l]\\
x_k\boldsymbol{e}_k\cdot[A^\top_{ij}y_j\boldsymbol{e}_i] &= y_k\boldsymbol{e}_k\cdot[A_{ij}x_j\boldsymbol{e}_i]\\
x_kA^\top_{ij}y_j(\boldsymbol{e}_k\cdot\boldsymbol{e}_i) &= y_kA_{ij}x_j(\boldsymbol{e}_k\cdot\boldsymbol{e}_i)\\
x_kA^\top_{ij}y_j\delta_{ki} &= y_kA_{ij}x_j\delta_{ki}\\
x_iA^\top_{ij}y_j &= y_iA_{ij}x_j\\
x_iA^\top_{ij}y_j &= y_jA_{ji}x_i\\
x_iA^\top_{ij}y_j &= x_iA_{ji}y_j\\
A^\top_{ij} &= A_{ji} \blacksquare
\end{aligned}
$$

Até agora nós vimos (e demos significado a)

$$
\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{x} &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\cdot x_k\boldsymbol{e}_k\\
&= A_{ij}x_k\boldsymbol{e}_i(\boldsymbol{e}_j\cdot\boldsymbol{e}_k)\\
&= A_{ij}x_k\boldsymbol{e}_i\delta_{jk}\\
&= A_{ij}x_j\boldsymbol{e}_i\\
&= [\boldsymbol{A}][\boldsymbol{x}]
\end{aligned}
$$

Eu também posso definir a operação

$$
\begin{aligned}
\boldsymbol{x} \cdot \boldsymbol{A} &= x_k \boldsymbol{e}_k \cdot A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \\
&= x_k A_{ij} (\boldsymbol{e}_k \cdot \boldsymbol{e}_i) \boldsymbol{e}_j \\
&= x_k A_{ij} \delta_{ki} \boldsymbol{e}_j \\
&= x_i A_{ij} \boldsymbol{e}_j \\
&= [x_1 A_{11} + x_2 A_{21} + x_3 A_{31}] \boldsymbol{e}_1 \\
&+ [x_1 A_{12} + x_2 A_{22} + x_3 A_{32}] \boldsymbol{e}_2 \\
&+ [x_1 A_{13} + x_2 A_{23} + x_3 A_{33}] \boldsymbol{e}_3 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \\
&= [\boldsymbol{x}]^\top [\boldsymbol{A}]
\end{aligned}
$$

Portanto eu posso definir a transformação simétrica via

$$
\begin{gathered}
\boldsymbol{x} \cdot \boldsymbol{A}^\top \equiv \boldsymbol{A} \cdot \boldsymbol{x}, \\
\vdots \\
A_{ij}^\top = A_{ji}.
\end{gathered}
$$

Na equação

$$A_{ij} x_j = \lambda x_i$$

Os $A_{ij}s$ são reais. O conjugado da equação é

$$
\begin{aligned}
(A_{ij} x_j)^* &= (\lambda x_i)^* \\
A_{ij}^* x_j^* &= \lambda^* x_i^* \\
A_{ij} x_j^* &= \lambda^* x_i^*
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{e}_1 &= (1, 0, 0), \\
\boldsymbol{f}_1 &= \frac{1}{\sqrt{3}} (1, 1, 1), \\
\boldsymbol{e}_1 \cdot \boldsymbol{f}_1 &= \frac{1}{\sqrt{3}} =
\end{aligned}
$$

$$\boldsymbol{e}_j \qquad \boldsymbol{j}$$

Quando existe uma base de autovetores para a transformação $\boldsymbol{A}$, a matriz de $\boldsymbol{A}$ nessa base é uma matriz diagonal composta pelos autovalores.

Olhe para o caso 3×3

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_F$$

Cada coluna representa a imagem de um dos vetores da base e **se** eu estiver na base dos autovetores:

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{f}_1 &= A_{11}\boldsymbol{f}_1 + A_{21}\boldsymbol{f}_2 + A_{31}\boldsymbol{f}_3 = \lambda_1 \boldsymbol{f}_1 \\ \boldsymbol{A} \cdot \boldsymbol{f}_2 &= A_{12}\boldsymbol{f}_1 + A_{22}\boldsymbol{f}_2 + A_{32}\boldsymbol{f}_3 = \lambda_2 \boldsymbol{f}_2 \\ \boldsymbol{A} \cdot \boldsymbol{f}_3 &= A_{13}\boldsymbol{f}_1 + A_{23}\boldsymbol{f}_2 + A_{33}\boldsymbol{f}_3 = \lambda_3 \boldsymbol{f}_3 \end{aligned}$$

Portanto, concluímos que

$$\boldsymbol{A}_F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_F$$

Exercício 5.31

$$\begin{aligned} \boldsymbol{S} : \boldsymbol{A} &= S_{ij} A_{lm} (\boldsymbol{e}_j \cdot \boldsymbol{e}_l)(\boldsymbol{e}_i \cdot \boldsymbol{e}_m) \\ &= S_{ij} A_{lm} \delta_{jl} \delta_{im} \\ &= S_{ij} A_{ji} \\ &= \frac{1}{2} S_{ij} A_{ji} + \frac{1}{2} S_{ij} A_{ji} \\ &= \frac{1}{2} S_{ij} A_{ji} + \frac{1}{2} S_{ji} A_{ij} \\ &= \frac{1}{2} S_{ij} (A_{ji} + A_{ij}) = 0. \end{aligned}$$

Uma EDO de ordem 2 pode ser escrita como um sistema de duas EDOs acopladas de ordem 1.

$$
\begin{aligned}
\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + u &= 0\\
\frac{\mathrm{d}u}{\mathrm{d}t} &= v\\
\frac{\mathrm{d}v}{\mathrm{d}t} + u &= 0\\
\frac{\mathrm{d}u}{\mathrm{d}t} &= v\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= -u
\end{aligned}
$$

$$
\begin{aligned}
u(t) &= \frac{A-\mathrm{i}B}{2}\left[\cos t + \mathrm{i}\,\mathrm{sen}\, t\right] + \frac{A+\mathrm{i}B}{2}\left[\cos t - \mathrm{i}\,\mathrm{sen}\, t\right]\\
&= \frac{A}{2}[\cos t + \mathrm{i}\,\mathrm{sen}\, t] - \frac{\mathrm{i}B}{2}[\cos t + \mathrm{i}\,\mathrm{sen}\, t] + \frac{A}{2}[\cos t - \mathrm{i}\,\mathrm{sen}\, t] + \frac{\mathrm{i}B}{2}[\cos t - \mathrm{i}\,\mathrm{sen}\, t]\\
&= \frac{A}{2}\cos t + \frac{A}{2}\cos t + \frac{-\mathrm{i}^2B}{2}\,\mathrm{sen}\, t + \frac{-\mathrm{i}^2B}{2}\,\mathrm{sen}\, t + \mathrm{i}\left[\frac{A}{2}\,\mathrm{sen}\, t - \frac{B}{2}\cos t - \frac{A}{2}\,\mathrm{sen}\, t + \frac{B}{2}\cos t\right]\\
&= A\cos t + B\,\mathrm{sen}\, t \;\blacksquare
\end{aligned}
$$

Ordem de diferenciação (ou de derivação):

$$
\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).
$$

Em geral,

$$
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.
$$

Série de Taylor

$$
f(x) = \frac{1}{0!}f(x_0) + \frac{1}{1!}\frac{\mathrm{d}f(x_0)}{\mathrm{d}x}(x-x_0) + \frac{1}{2!}\frac{\mathrm{d}^2f(x_0)}{\mathrm{d}x^2}(x-x_0)^2 + \ldots + \frac{1}{n!}\frac{\mathrm{d}^nf(x_0)}{\mathrm{d}x^n}(x-x_0)^n + \ldots
$$

No caso de $n=2$, e de uma função (também) de 2 variáveis

$$
\begin{aligned}
\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_i \partial x_j}(x_i - x_{0i})(x_j - x_{0j}) &= \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_i \partial x_j}(x_i - x_{0i})(x_j - x_{0j})\\
&= \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_1}(x_1 - x_{01})(x_1 - x_{01}) +\\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_1 \partial x_2}(x_1 - x_{01})(x_2 - x_{02}) +\\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2 \partial x_1}(x_2 - x_{02})(x_1 - x_{01}) +\\
&\quad \frac{1}{2!}\frac{\partial^2 f(\boldsymbol{x}_0)}{\partial x_2 \partial x_2}(x_2 - x_{02})(x_2 - x_{02}).
\end{aligned}
$$

Verifique o número de índices repetidos em uma expressão

$$\begin{aligned}\boldsymbol{P}\cdot\boldsymbol{a} &= P_{il}a_l\boldsymbol{e}_i\\ &= \left[\delta_{il}-\frac{k_ik_l}{k^2}\right]a_l\boldsymbol{e}_i\\ [\boldsymbol{P}\cdot\boldsymbol{a}]\times\boldsymbol{k} &= \epsilon_{ijk}\left[\delta_{il}-\frac{k_ik_l}{k^2}\right]a_lk_j\boldsymbol{e}_k\end{aligned}$$

Teorema da função implícita

$$\begin{aligned}\frac{\mathrm{d}u(x_0)}{\mathrm{d}x} &= -\frac{\frac{\partial f(x_0,u_0)}{\partial x}}{\frac{\partial f(x_0,u_0)}{\partial u}}\\ u(x) &= u_0+\frac{\mathrm{d}u(x_0)}{\mathrm{d}x}(x-x_0)+\dots\end{aligned}$$

$$\begin{aligned}x &= x(u,v) & u &= u(x,y)\\ y &= y(u,v) & v &= v(x,y)\end{aligned}$$

A regra de Leibnitz:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x &= \int_a^b\frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x, \qquad \text{porém}\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &\neq \int_{a(t)}^{b(t)}\frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x.\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= \int_{a(t)}^{b(t)}\frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x+f(b,t)\frac{\mathrm{d}b}{\mathrm{d}t}-f(a,t)\frac{\mathrm{d}a}{\mathrm{d}t};\\ \frac{\mathrm{D}N}{\mathrm{D}t} &= \frac{\partial}{\partial t}\int_{VC}\eta\rho\,\mathrm{d}V+\oint_{SC}(\boldsymbol{n}\cdot\boldsymbol{v})\eta\rho\,\mathrm{d}S\end{aligned}$$

A idéia é mudar a variável de integração de x para ξ:

$$\begin{aligned}x &= X(\xi,t)\\ \mathrm{d}x &= \frac{\partial X}{\partial \xi}\mathrm{d}\xi+\frac{\partial X}{\partial t}\mathrm{d}t,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta f(X(\xi,t),t)\frac{\partial X}{\partial\xi}\mathrm{d}\xi\end{aligned}$$

Revisite a regra da cadeia!

$$\begin{aligned}&f(x,u);\\ F(x) &= f(x,u(x));\\ \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\end{aligned}$$

Quero o módulo de $\boldsymbol{v}=(1,\mathrm{i},0)$:

$$
\begin{aligned}
|\boldsymbol{v}| &= \sqrt{\sum_{i=1}^{3} v_i^* v_i} \\
&= \sqrt{(1 \times 1 + (-\mathrm{i} \times \mathrm{i}) + 0 \times 0)} \\
&= \sqrt{1 - \mathrm{i}^2 + 0} \\
&= \sqrt{1 + 1 + 0} \\
&= \sqrt{2}
\end{aligned}
$$

Regra de Leibnitz:
Considere

$$
\begin{aligned}
\frac{\partial}{\partial \xi} \frac{\partial X}{\partial t} &=? \\
\frac{\partial X}{\partial t} &= h(\xi, t) \\
\frac{\partial}{\partial \xi} \frac{\partial X}{\partial t} &= \frac{\partial h}{\partial \xi} = \frac{\partial h}{\partial x} \frac{\partial X}{\partial \xi}
\end{aligned}
$$

Surgimento de uma "velocidade"

$$
\begin{aligned}
x &= X(\xi, t), \\
u &= \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial X}{\partial t}
\end{aligned}
$$

A regra de Leibnitz é

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x &= f(b,t)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a,t)\frac{\mathrm{d}b}{\mathrm{d}t} + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t &= f(x,b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(x,a)\frac{\mathrm{d}a}{\mathrm{d}x} + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\,\mathrm{d}t \\
b(x) &= x, \\
a(x) &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \left[xt + \mathrm{e}^{-t^2}\right] \mathrm{d}t &= x^2 + \mathrm{e}^{-x^2} + \int_0^x \frac{\partial}{\partial x} \left[xt + \mathrm{e}^{-t^2}\right] \mathrm{d}t \\
\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \left[xt + \mathrm{e}^{-t^2}\right] \mathrm{d}t &= x^2 + \mathrm{e}^{-x^2} + \int_0^x t\,\mathrm{d}t \\
\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \left[xt + \mathrm{e}^{-t^2}\right] \mathrm{d}t &= \frac{3x^2}{2} + \mathrm{e}^{-x^2}
\end{aligned}
$$

Quero calcular um comprimento de arco sobre a curva (entre $t = 0$ e $t = \tau$)

$$
\begin{aligned}
\boldsymbol{r}(t) &= \cos(t)[\boldsymbol{e}_x + \boldsymbol{e}_y] - \sqrt{2}\,\mathrm{sen}(t)\boldsymbol{e}_z \\
\ell_{\mathscr{L}} &= \int_{r\in\mathscr{L}} \left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]^{1/2} \mathrm{d}t \\
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= -\,\mathrm{sen}(t)[\boldsymbol{e}_x + \boldsymbol{e}_y] - \sqrt{2}\cos(t)\boldsymbol{e}_z \\
&= (-\,\mathrm{sen}(t), -\,\mathrm{sen}(t), -\sqrt{2}\cos(t)) \\
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \mathrm{sen}^2(t) + \mathrm{sen}^2(t) + 2\cos^2(t) = 2 \\
\ell_{\mathscr{L}} &= \int_0^\tau \sqrt{2}\,\mathrm{d}t = \sqrt{2}\tau.
\end{aligned}
$$

Exemplo 7.11: Calcule a área da superfície externa do parabolóide de revolução $z = x^2 + y^2$, $x^2 + y^2 \leq 1$

Procuro a teoria:

$$
\begin{aligned}
z &= g(x, y) = x^2 + y^2, \\
A_{\mathscr{S}} &= \int_{R_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y \\
\frac{\partial g}{\partial x} &= 2x, \\
\frac{\partial g}{\partial y} &= 2y, \\
A_{\mathscr{S}} &= \int_{R_{xy}} \sqrt{1 + (2x)^2 + (2y)^2}\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1 + 4(x^2 + y^2)}\,r\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r\mathrm{d}\theta \\
&= \left[\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r\right]\left[\int_{\theta=0}^{2\pi} \mathrm{d}\theta\right] \\
&= 2\pi\int_{r=0}^{1} \sqrt{1 + 4r^2}\,r\mathrm{d}r \\
&= 2\pi\frac{5\sqrt{5} - 1}{12} = \frac{5\sqrt{5} - 1}{6}.
\end{aligned}
$$

Virtudes acadêmicas **e** profissionais:

1. Trabalho.
2. Persistência.
3. Iniciativa.

Divergência, gradiente, rotacional

$$
\begin{aligned}
\operatorname{div}\boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
\boldsymbol{\nabla} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= ? \\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(u, v, w) \\
&= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.
\end{aligned}
$$

$\boldsymbol{\nabla}$ é o **operador** “nabla”.

Posso repetir, de forma mais resumida, em notação indicial. Suponha que $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ seja uma base **ortonormal** do $\mathbb{R}^3$.

$$
\boldsymbol{\nabla} \equiv \boldsymbol{e}_i \frac{\partial}{\partial x_i},
$$

e não

$$
\begin{aligned}
\boldsymbol{\nabla} &\equiv \frac{\partial}{\partial x_i}\boldsymbol{e}_i, \\
\boldsymbol{u} &= u_j \boldsymbol{e}_j \\
\operatorname{div}\boldsymbol{u} &\equiv \boldsymbol{\nabla}\cdot\boldsymbol{u} \\
&= \boldsymbol{e}_i \frac{\partial}{\partial x_i}\cdot u_j\boldsymbol{e}_j \\
&= (\boldsymbol{e}_i\cdot\boldsymbol{e}_j)\frac{\partial u_j}{\partial x_i} \\
&= \delta_{ij}\frac{\partial u_j}{\partial x_i} \\
&= \frac{\partial u_i}{\partial x_i}.
\end{aligned}
$$

Lei de Fourier é

$$
\begin{aligned}
\boldsymbol{q} &= -\rho c_p \alpha \boldsymbol{\nabla} T, \\
\dot{Q} &= -\oint_{\text{casca do ovo}} (\boldsymbol{n}\cdot\boldsymbol{q})\,\mathrm{d}A
\end{aligned}
$$

Gradiente, e sua "definição" (em coordenadas cartesianas) com o uso de $\boldsymbol{\nabla}$:

$$
\begin{aligned}
&\alpha \boldsymbol{u}, \\
&\boldsymbol{u}\alpha, \\
\mathbf{grad}\, f &= \nabla f, \\
\mathbf{grad}\, f &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) f \\
\mathbf{grad}\, f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\
\mathbf{grad}\, f &= \boldsymbol{e}_i \frac{\partial}{\partial x_i} f \\
&= \boldsymbol{e}_i \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \boldsymbol{e}_i
\end{aligned}
$$

O rotacional:

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{u} &= \epsilon_{ijk} \nabla_i u_j \boldsymbol{e}_k \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_i} u_j \boldsymbol{e}_k \\
&= \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \boldsymbol{e}_k
\end{aligned}
$$

Em resumo, as operações relacionados a

$$
\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
$$

são

1. A divergência: $\boldsymbol{u} = (u, v, w)$ e
$$
\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z};
$$
2. O gradiente: $f(x, y, z)$ e
$$
\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)
$$
3. O rotacional:
$$
\boldsymbol{\nabla} \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}.
$$

O laplaciano!

$$
\begin{aligned}
\nabla^2 f &\equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f \\
&\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\
&\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial^2 f}{\partial x_i \partial x_i}.
\end{aligned}
$$

Por extensão, também podemos tratar do laplaciano de um vetor:

$$
\begin{aligned}
\nabla^2 \boldsymbol{u} &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \boldsymbol{i} \\
&+ \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \boldsymbol{j} \\
&+ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \boldsymbol{k} \\
&= \frac{\partial^2 u_i}{\partial x_k \partial x_k} \boldsymbol{e}_i.
\end{aligned}
$$

Aplicações em mecânica dos fluidos:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \boldsymbol{\cdot} (\rho \boldsymbol{u}) = 0, \\
\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u} \right] = \rho \boldsymbol{g} - \boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u},
\end{aligned}$$

Importante: a forma das operações muda quando nós mudamos de sistema de coordenadas. Por exemplo, em coordenadas cilíndricas, **não** é verdade que

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

$$\begin{aligned}
r &= (x^2+y^2)^{1/2},\\
\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2+y^2)^{-1/2}\times 2x\\
&= \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r} = \cos(\theta).\\
\theta &= \operatorname{arctg}\left(\frac{y}{x}\right),\\
\frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}\operatorname{arctg}\left(\frac{y}{x}\right),\\
\frac{\mathrm{d}\operatorname{arctg} u}{\mathrm{d}u} &= \frac{1}{1+u^2},\\
\frac{\partial \theta}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\times -\frac{y}{x^2}
\end{aligned}$$

Teorema da divergência, aplicações em Mecânica dos Fluidos.

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathscr{C}}\rho\,\mathrm{d}V + \oint_{\mathscr{S}}\rho(\boldsymbol{n}\cdot\boldsymbol{v})\,\mathrm{d}A &= 0\\
\frac{\partial}{\partial t}\int_{\mathscr{C}}\rho\,\mathrm{d}V + \oint_{\mathscr{S}}(\boldsymbol{n}\cdot[\rho\boldsymbol{v}])\,\mathrm{d}A &= 0\\
\int_{\mathscr{C}}\frac{\partial \rho}{\partial t}\,\mathrm{d}V + \int_{\mathscr{C}}\boldsymbol{\nabla}\cdot[\rho\boldsymbol{v}]\,\mathrm{d}V &= 0\\
\int_{\mathscr{C}}\left\{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot[\rho\boldsymbol{v}]\right\}\mathrm{d}V &= 0\\
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot[\rho\boldsymbol{v}] &= 0
\end{aligned}$$

Exercício 7.9

$$\begin{aligned}
x^2+y^2 &= 1\\
z &= 1-y^2,\\
z &= u,\\
x &= \cos v,\\
y &= \operatorname{sen} v,\\
\boldsymbol{r} &= (\cos v, \operatorname{sen} v, u),\\
\frac{\partial \boldsymbol{r}}{\partial u} &= (0,0,1),\\
\frac{\partial \boldsymbol{r}}{\partial v} &= (-\operatorname{sen} v, \cos v, 0)
\end{aligned}$$

A área da superfície é

$$A = \iint_{R_{uv}} \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| \mathrm{d}u\mathrm{d}v$$

No nosso caso a área é simplesmente

$$\begin{aligned} A &= \iint_{R_{uv}} \mathrm{d}u\mathrm{d}v \\ &= \int_{v=0}^{2\pi} \int_{u=0}^{1-v^2} \mathrm{d}u\mathrm{d}v \end{aligned}$$

Campos irrotacionais

$$\boldsymbol{F} = (F_1, F_2, F_3)$$

A integral de log:

$$
\begin{aligned}
\int \frac{\mathrm{d}u}{u} &= \ln|u|;\\
\int_a^b \frac{\mathrm{d}u}{u} &= \ln|b| - \ln|a|\\
&= \ln\frac{|b|}{|a|}
\end{aligned}
$$

O que acontece se os sinais de a e b forem iguais: teremos somente duas possibilidades:

$$
\begin{aligned}
|a| &= a & \mathrm{e}|b| &= b,\\
|a| &= -a & \mathrm{e}|b| &= -b.
\end{aligned}
$$

Em ambos os casos,

$$
\ln\frac{|b|}{|a|} = \ln\frac{b}{a}.
$$

Ao tentarmos resolver

$$
\mathscr{D}y = f(x)
$$

queremos encontrar a **função** $y(x)$.

$$
\begin{aligned}
\mathscr{D} &= \frac{\mathrm{d}}{\mathrm{d}x},\\
\mathscr{D}y &= \frac{\mathrm{d}y}{\mathrm{d}x},\\
\mathscr{D} &= x + y\frac{\mathrm{d}}{\mathrm{d}x},\\
\mathscr{D}y &= xy + y\frac{\mathrm{d}y}{\mathrm{d}x},\\
\mathscr{D} &= x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2,\\
\mathscr{D}y &= x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2y,\\
\mathscr{D} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x} + 1,\\
\mathscr{D}y &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y.
\end{aligned}
$$

$$
\mathscr{D}y = f(x)
$$

Uma EDO com coeficientes não constantes é

$$
x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.
$$

EDOs lineares, coeficientes constantes, ordem 2.

$$
\begin{aligned}
a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,\\
y = \mathrm{e}^{\lambda x},\\
y' = \lambda \mathrm{e}^{\lambda x},\\
y'' = \lambda^2 \mathrm{e}^{\lambda x},\\
a\lambda^2 \mathrm{e}^{\lambda x} + b\lambda \mathrm{e}^{\lambda x} + c\mathrm{e}^{\lambda x} = 0,\\
\mathrm{e}^{\lambda x}\left[a\lambda^2 + b\lambda + c\right] = 0
\end{aligned}
$$

Se

$$
ay'' + by' + cy = 0,
$$

a equação característica é

$$
a\lambda^2 + b\lambda + c = 0.
$$

$$
\begin{aligned}
\lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};\\
&= -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}\\
&= \alpha \pm \beta\\
y &= C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x}\\
&= C_1\mathrm{e}^{(\alpha+\beta)x} + C_2\mathrm{e}^{(\alpha-\beta)x}\\
&= C_1\mathrm{e}^{\alpha x}\mathrm{e}^{\beta x} + C_2\mathrm{e}^{\alpha x}\mathrm{e}^{-\beta x}\\
&= \mathrm{e}^{\alpha x}\left[C_1\mathrm{e}^{\beta x} + C_2\mathrm{e}^{-\beta x}\right]\\
C_1 &= \frac{1}{2}\left(D_1 + D_2\right)\\
C_2 &= \frac{1}{2}\left(D_1 - D_2\right)\\
y &= \mathrm{e}^{\alpha x}\left[\frac{1}{2}\left(D_1 + D_2\right)\mathrm{e}^{\beta x} + \frac{1}{2}\left(D_1 - D_2\right)\mathrm{e}^{-\beta x}\right]\\
y &= \mathrm{e}^{\alpha x}\left[D_1\frac{\mathrm{e}^{\beta x} + \mathrm{e}^{-\beta x}}{2} + D_2\frac{\mathrm{e}^{\beta x} - \mathrm{e}^{-\beta x}}{2}\right]\\
\cosh(x) &\equiv \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2},\\
\operatorname{senh}(x) &\equiv \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2},\\
y &= \mathrm{e}^{\alpha x}\left[D_1\cosh(\beta x) + D_2\operatorname{senh}(\beta x)\right] \blacksquare
\end{aligned}
$$

Problema “real”, raízes “complexas”: temos

$$
ay'' + by' + cy = 0
$$

e desejamos $y(x) \in \mathbb{R}$. Porém,

$$
b^2 - 4ac < 0.
$$

A solução geral será a mesma de antes:

$$
\begin{aligned}
y &= C_1 \mathrm{e}^{\lambda_1 x} + C_2 \mathrm{e}^{\lambda_2 x} \\
&= C_1 \mathrm{e}^{(\alpha + \mathrm{i}\beta)x} + C_2 \mathrm{e}^{(\alpha - \mathrm{i}\beta)x} \\
&= C_1 \mathrm{e}^{\alpha x} \mathrm{e}^{\mathrm{i}\beta x} + C_2 \mathrm{e}^{\alpha x} \mathrm{e}^{-\mathrm{i}\beta x} \\
&= \mathrm{e}^{\alpha x} \left[C_1 \mathrm{e}^{\mathrm{i}\beta x} + C_2 \mathrm{e}^{-\mathrm{i}\beta x} \right]; \\
C_1 &= \frac{1}{2}(A - \mathrm{i}B), \\
C_2 &= \frac{1}{2}(A + \mathrm{i}B), \\
y &= \mathrm{e}^{\alpha x} \left[\frac{1}{2}(A - \mathrm{i}B) \mathrm{e}^{\mathrm{i}\beta x} + \frac{1}{2}(A + \mathrm{i}B) \mathrm{e}^{-\mathrm{i}\beta x} \right] \\
y &= \mathrm{e}^{\alpha x} \left[A \left(\frac{\mathrm{e}^{\mathrm{i}\beta x} + \mathrm{e}^{-\mathrm{i}\beta x}}{2} \right) - \mathrm{i}B \left(\frac{\mathrm{e}^{\mathrm{i}\beta x} - \mathrm{e}^{-\mathrm{i}\beta x}}{2} \right) \right] \\
\mathrm{e}^{\mathrm{i}\beta x} &= \cos(\beta x) + \mathrm{i}\,\mathrm{sen}(\beta x), \\
\mathrm{e}^{-\mathrm{i}\beta x} &= \cos(\beta x) - \mathrm{i}\,\mathrm{sen}(\beta x), \\
\cosh(\mathrm{i}\beta x) &= \frac{\mathrm{e}^{\mathrm{i}\beta x} + \mathrm{e}^{-\mathrm{i}\beta x}}{2} = \frac{1}{2} \left[(\cos(\beta x) + \mathrm{i}\,\mathrm{sen}(\beta x)) + (\cos(\beta x) - \mathrm{i}\,\mathrm{sen}(\beta x)) \right] \\
&= \cos(\beta x); \\
\mathrm{senh}(\mathrm{i}\beta x) &= \frac{\mathrm{e}^{\mathrm{i}\beta x} - \mathrm{e}^{-\mathrm{i}\beta x}}{2} = \frac{1}{2} \left[(\cos(\beta x) + \mathrm{i}\,\mathrm{sen}(\beta x)) - (\cos(\beta x) - \mathrm{i}\,\mathrm{sen}(\beta x)) \right] \\
&= \mathrm{i}\,\mathrm{sen}(\beta x); \\
y &= \mathrm{e}^{\alpha x} \left[A \cos(\beta x) - \mathrm{i}B \times \mathrm{i}\,\mathrm{sen}(\beta x) \right] \\
&= \mathrm{e}^{\alpha x} \left[A \cos(\beta x) + B\,\mathrm{sen}(\beta x) \right] \blacksquare
\end{aligned}
$$

O método de variação de constantes, equações diferenciais ordinárias de ordem 2.

$$
\begin{aligned}
ay'' + by' + cy &= f(x), \\
ay_h'' + by_h' + cy_h &= 0,
\end{aligned}
$$

ache y_1 e y_2 de tal maneira que

$$
\begin{aligned}
ay_1'' + by_1' + cy_1 &= 0, \\
ay_2'' + by_2' + cy_2 &= 0,
\end{aligned}
$$

Equações diferenciais ordinárias de ordem 2, não-homogêneas: solução pelo método de “variação de constantes”.

$$
\begin{aligned}
y'' - 3y' + 2y &= x,\\
y_h'' - 3y_h' + 2y_h &= 0,\\
\lambda^2 - 3\lambda + 2 &= 0\\
\lambda &= \frac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2}\\
\lambda &= \frac{3 \pm 1}{2}\\
&= 2, \qquad 1;\\
y_h &= A\mathrm{e}^x + B\mathrm{e}^{2x};
\end{aligned}
$$

Procure a solução na forma

$$
\begin{aligned}
y(x) &= A(x)\mathrm{e}^x + B(x)\mathrm{e}^{2x};\\
y'(x) &= A\mathrm{e}^x + 2B\mathrm{e}^{2x} + \left[A'\mathrm{e}^x + B'\mathrm{e}^{2x}\right]
\end{aligned}
$$

Controle as derivadas de A e B, e impeça derivadas de ordem 2

$$
y'(x) = A\mathrm{e}^x + 2B\mathrm{e}^{2x} + \underbrace{\left[A'\mathrm{e}^x + B'\mathrm{e}^{2x}\right]}_{=0}
$$

$$
\begin{aligned}
A'\mathrm{e}^x + B'\mathrm{e}^{2x} &= 0;\\
y'(x) &= A\mathrm{e}^x + 2B\mathrm{e}^{2x},\\
y''(x) &= A\mathrm{e}^x + 4B\mathrm{e}^{2x} + A'\mathrm{e}^x + 2B'\mathrm{e}^{2x}
\end{aligned}
$$

Substituindo na equação original, obtemos

$$
\begin{aligned}
A\mathrm{e}^x + 4B\mathrm{e}^{2x} + A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} - 3\left[A\mathrm{e}^x + 2B\mathrm{e}^{2x}\right] + 2\left[A\mathrm{e}^x + B\mathrm{e}^{2x}\right] &= x,\\
\mathrm{e}^x\left[A - 3A + 2A\right] + \mathrm{e}^{2x}\left[4B - 6B + 2B\right] + A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x\\
A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x
\end{aligned}
$$

Reúno agora duas EDOs em A e B:

$$
\begin{aligned}
A'\mathrm{e}^x + B'\mathrm{e}^{2x} &= 0;\\
A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x\\
2A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= 0;\\
A'\mathrm{e}^x + 2B'\mathrm{e}^{2x} &= x\\
A'\mathrm{e}^x &= -x\\
\frac{\mathrm{d}A}{\mathrm{d}x} &= -x\mathrm{e}^{-x}
\end{aligned}
$$

Integrando,

$$
\begin{aligned}
A(x) &= (x+1)\mathrm{e}^{-x} + C_1 \\
A'\mathrm{e}^{x} + B'\mathrm{e}^{2x} &= 0; \\
A'\mathrm{e}^{x} + 2B'\mathrm{e}^{2x} &= x \\
B'\mathrm{e}^{2x} &= x, \\
\frac{\mathrm{d}B}{\mathrm{d}x} &= x\mathrm{e}^{-2x}, \\
B(x) &= -\frac{1}{4}(2x+1)\mathrm{e}^{-2x} + C_2, \\
y(x) &= A(x)\mathrm{e}^{x} + B(x)\mathrm{e}^{2x}, \\
&= \left[(x+1)\mathrm{e}^{-x} + C_1\right]\mathrm{e}^{x} + \left[-\frac{1}{4}(2x+1)\mathrm{e}^{-2x} + C_2\right]\mathrm{e}^{2x} \\
&= C_1\mathrm{e}^{x} + C_2\mathrm{e}^{2x} + x + 1 - \frac{1}{4}(2x+1) \\
y(x) &= \underbrace{C_1\mathrm{e}^{x} + C_2\mathrm{e}^{2x}}_{y_h(x)} + \underbrace{\frac{2x+3}{4}}_{y_p(x)} \blacksquare
\end{aligned}
$$

Uma equação de Euler com raízes **complexas**:

$$
\begin{aligned}
x^2y'' + y &= 0,\\
y &= x^m,\\
y' &= mx^{m-1},\\
y'' &= (m-1)mx^{m-2}\\
x^2\underbrace{(m-1)mx^{m-2}}_{y''} + \underbrace{x^m}_{y} &= 0\\
\left[(m-1)m+1\right]x^m &= 0,\\
m^2 - m + 1 &= 0,\\
m &= \frac{1 \pm \mathrm{i}\sqrt{3}}{2}\\
y &= C_1x^{\frac{1+\mathrm{i}\sqrt{3}}{2}} + C_2x^{\frac{1-\mathrm{i}\sqrt{3}}{2}}\\
x^a &= \exp(\ln x^a) = \exp(a\ln x);\\
x^{\frac{1\pm\mathrm{i}\sqrt{3}}{2}} &= x^{\frac{1}{2}}x^{\frac{\pm\mathrm{i}\sqrt{3}}{2}}\\
x^{\frac{\pm\mathrm{i}\sqrt{3}}{2}} &= \exp\left(\mathrm{i}\frac{\pm\sqrt{3}}{2}\ln x\right)\\
\exp(\mathrm{i}\theta) &= \cos(\theta) + \mathrm{i}\,\mathrm{sen}(\theta)\\
x^{\frac{\pm\mathrm{i}\sqrt{3}}{2}} &= \cos\left(\frac{\sqrt{3}}{2}\ln x\right) \pm \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\\
y &= C_1x^{\frac{1+\mathrm{i}\sqrt{3}}{2}} + C_2x^{\frac{1-\mathrm{i}\sqrt{3}}{2}}\\
y &= C_1x^{1/2}x^{+\frac{\mathrm{i}\sqrt{3}}{2}} + C_2x^{1/2}x^{-\frac{\mathrm{i}\sqrt{3}}{2}}\\
y &= C_1x^{1/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\right]\\
&\quad + C_2x^{1/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) - \mathrm{i}\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\right]
\end{aligned}
$$

Agora, escolha C_1 e C_2 conjugados complexos:

$$
\begin{aligned}
C_1 &= \frac{1}{2}\left(D_1 - \mathrm{i}D_2\right),\\
C_2 &= \frac{1}{2}\left(D_1 + \mathrm{i}D_2\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{y}{x^{1/2}} &= \frac{1}{2}\left(D_1 - \mathrm{i}D_2\right)\left[C + \mathrm{i}S\right] + \frac{1}{2}\left(D_1 + \mathrm{i}D_2\right)\left[C - \mathrm{i}S\right]\\
&= \frac{1}{2}\left\{D_1C - \mathrm{i}^2D_2S + \mathrm{i}(-D_2C + D_1S) + D_1C - \mathrm{i}^2D_2S + \mathrm{i}(D_2C - D_1S)\right\}\\
&= D_1C + D_2S;\\
y &= x^{1/2}\left[D_1\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + D_2\,\mathrm{sen}\left(\frac{\sqrt{3}}{2}\ln x\right)\right] \blacksquare
\end{aligned}
$$

Desigualdades importantes com números complexos
Resultado auxiliar:

$$\begin{aligned}
z_1^* z_2 + z_1 z_2^* &= (z_1^* z_2) + (z_1^* z_2)^* = 2\,\mathrm{Re}(z_1^* z_2) \le 2|z_1^* z_2| = 2|z_1^*||z_2| = 2|z_1||z_2| \\
|x + \mathrm{i}y|^2 &= x^2 + y^2 \\
&= (x - \mathrm{i}y)(x + \mathrm{i}y) \\
&= x^2 - (\mathrm{i}y)^2 \\
&= x^2 - \mathrm{i}^2 y^2 \\
&= x^2 + y^2 \;\blacksquare
\end{aligned}$$

A desigualdade do triângulo, agora, é

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)^*(z_1 + z_2) \\
&= z_1^* z_1 + z_2^* z_2 + z_1^* z_2 + z_2^* z_1 \\
&= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1^* z_2) \\
&\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
&= (|z_1| + |z_2|)^2; \;\Rightarrow \\
|z_1 + z_2| &\le |z_1| + |z_2|
\end{aligned}$$

Variáveis complexas têm mil (e uma) utilidades

$$\begin{aligned}
I &= \int_0^\infty \mathrm{e}^{-x} \cos(ax)\,\mathrm{d}x; \\
\mathrm{e}^{\mathrm{i}ax} &= \cos(ax) + \mathrm{i}\,\mathrm{sen}(ax); \\
I &= \int_0^\infty \mathrm{e}^{-x}\,\mathrm{Re}(\mathrm{e}^{\mathrm{i}ax})\,\mathrm{d}x; \\
I &= \mathrm{Re}\int_0^\infty \mathrm{e}^{-x}\mathrm{e}^{\mathrm{i}ax}\,\mathrm{d}x; \\
I &= \mathrm{Re}\int_0^\infty \mathrm{e}^{(\mathrm{i}a-1)x}\,\mathrm{d}x; \\
I &= \mathrm{Re}\,\frac{1}{\mathrm{i}a - 1}\int_0^\infty \mathrm{e}^{(\mathrm{i}a-1)x}\,\mathrm{d}(\mathrm{i}a - 1)x; \\
I &= \mathrm{Re}\,\frac{1}{\mathrm{i}a - 1}\mathrm{e}^{(\mathrm{i}a-1)x}\Big|_{x=0}^{x=\infty} \\
I &= \mathrm{Re}\,\frac{1}{\mathrm{i}a - 1}\left[\mathrm{e}^{(\mathrm{i}a-1)\infty} - 1\right]
\end{aligned}$$

Agora avalio o limite cuidadosamente:

$$\begin{aligned}
\lim_{x\to\infty} \mathrm{e}^{(\mathrm{i}a-1)x} &= \lim_{x\to\infty} \mathrm{e}^{-x}(\cos(ax) + \mathrm{i}\,\mathrm{sen}(ax)) \\
\lim_{x\to\infty} |\mathrm{e}^{(\mathrm{i}a-1)x}| &= \lim_{x\to\infty} |\mathrm{e}^{-x}||(\cos(ax) + \mathrm{i}\,\mathrm{sen}(ax))|; \\
|\cos(ax) + \mathrm{i}\,\mathrm{sen}(ax)| &= 1 \\
|x + \mathrm{i}y| &= \sqrt{x^2 + y^2} \\
\lim_{x\to\infty} |\mathrm{e}^{(\mathrm{i}a-1)x}| &= \lim_{x\to\infty} |\mathrm{e}^{-x}| = 0; \\
\lim_{x\to\infty} \mathrm{e}^{(\mathrm{i}a-1)x} &= 0 \;\blacksquare
\end{aligned}$$

Portanto,

$$\begin{aligned}
I &= \operatorname{Re}\frac{1}{\mathrm{i}a-1}\,[-1] \\
&= \operatorname{Re}\frac{1}{1-\mathrm{i}a} \\
&= \operatorname{Re}\frac{1+\mathrm{i}a}{(1-\mathrm{i}a)(1+\mathrm{i}a)} \\
&= \operatorname{Re}\frac{1+\mathrm{i}a}{1+a^2} = \frac{1}{1+a^2}\ \blacksquare
\end{aligned}$$

$$\begin{aligned}
\cos(\mathrm{i}x) &= \frac{\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x}}{2} \\
&= \frac{\cos\mathrm{i}x+\mathrm{i}\,\mathrm{sen}\,\mathrm{i}x+\cos\mathrm{i}x-\mathrm{i}\,\mathrm{sen}\,\mathrm{i}x}{2} \\
&= \cos\mathrm{i}x. \\
\mathrm{sen}(\mathrm{i}x) &= \frac{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} \\
&= \frac{\cos\mathrm{i}x+\mathrm{i}\,\mathrm{sen}\,\mathrm{i}x-(\cos\mathrm{i}x-\mathrm{i}\,\mathrm{sen}\,\mathrm{i}x)}{2\mathrm{i}} \\
\cos^2(\mathrm{i}x)+\mathrm{sen}^2(\mathrm{i}x) &= \frac{\mathrm{e}^{2\mathrm{i}x}+2+\mathrm{e}^{-2\mathrm{i}x}}{4}+\frac{\mathrm{e}^{2\mathrm{i}x}-2+\mathrm{e}^{-2\mathrm{i}x}}{-4} \\
&= \frac{\mathrm{e}^{2\mathrm{i}x}+2+\mathrm{e}^{-2\mathrm{i}x}}{4}+\frac{-\mathrm{e}^{2\mathrm{i}x}+2-\mathrm{e}^{-2\mathrm{i}x}}{4} \\
&= \frac{4}{4} = 1\ \blacksquare
\end{aligned}$$

Variáveis complexas:

$$\frac{1}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}^2} = \frac{\mathrm{i}}{-1} = -\mathrm{i}.$$