

Como denotar uma dimensão “à mão livre”:

$$\begin{aligned}\llbracket T \rrbracket &= \mathbb{T}, \\ \llbracket L \rrbracket &= \mathbb{L}, \\ \llbracket M \rrbracket &= \mathbb{M}\end{aligned}$$

$$\begin{aligned}1 &= \mathbf{M}^{1+c} \mathbf{L}^{1+a+b-3c} \mathbf{T}^{-2-b}, \\ 0 &= 1 + c, \\ 0 &= 1 + a + b - 3c, \\ 0 &= -2 - b\end{aligned}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Um parâmetro adimensional elevado a qualquer potência continua a ser (é óbvio) adimensional:

$$\begin{aligned}\left[\left[\frac{F}{\rho U^2 D^2} \right] \right] &= 1, \\ \left[\left[\left[\frac{F}{\rho U^2 D^2} \right]^p \right] \right] &= 1, \\ \left[\left[\frac{\nu}{UD} \right] \right] &= 1, \\ \left[\left[\left[\frac{\nu}{UD} \right]^p \right] \right] &= 1, \\ \left[\left[\frac{UD}{\nu} \right] \right] &= 1,\end{aligned}$$

Refaço agora o Exemplo 1.1 escolhendo como variáveis comuns a todos os parâmetros (grupos) adimensionais ρ , ν , e D .

$$\begin{aligned}\Pi_1 &= F^1 \rho^a \nu^b D^c, \\ \llbracket \Pi_1 \rrbracket &= \mathbf{M} \mathbf{L} \mathbf{T}^{-2} \left[\mathbf{M} \mathbf{L}^{-3} \right]^a \left[\mathbf{L}^2 \mathbf{T}^{-1} \right]^b \left[\mathbf{L} \right]^c \\ 1 &= \mathbf{M}^{1+a} \mathbf{L}^{1-3a+2b+c} \mathbf{T}^{-2-b}\end{aligned}$$

Isso produz o sistema de equações lineares

$$\begin{aligned}a &= -1, \\ -3a + 2b + c &= -1, \\ -b &= 2\end{aligned}$$

Com solução

$$a = -1, \quad b = -2, c = 0.$$

$$\Pi_1 = \frac{F}{\rho \nu^2},$$

$$[\Pi_1] = \frac{\text{MLT}^{-2}}{\text{ML}^{-3}(\text{L}^2\text{T}^{-1})^2} = \frac{\text{MLT}^{-2}}{\text{ML}^{-3}\text{L}^4\text{T}^{-2}} = 1 \blacksquare$$

Procuramos agora o Π_2 :

$$\Pi_2 = U^1 \rho^a \nu^b D^c,$$

$$[\Pi_2] = \text{LT}^{-1} [\text{ML}^{-3}]^a [\text{L}^2\text{T}^{-1}]^b [\text{L}]^c$$

$$1 = \text{M}^a \text{L}^{1-3a+2b+c} \text{T}^{-1-b}$$

Isso produz o sistema de equações lineares

$$\begin{aligned} a &= 0, \\ -3a + 2b + c &= -1, \\ -b &= 1. \end{aligned}$$

A solução é

$$a = 0, \quad b = -1, \quad c = 1.$$

Isso produz o segundo grupo adimensional

$$\Pi_2 = \frac{UD}{\nu},$$

que é o número de Reynolds! Antes eu tinha, agora eu tenho:

$$\begin{aligned} \Pi_1 &= \frac{F}{\rho U^2 D^2}, & \Pi'_1 &= \frac{F}{\rho \nu^2}, \\ \Pi_2 &= \frac{\nu}{UD}, & \Pi'_2 &= \frac{UD}{\nu} \end{aligned}$$

Faço

$$\begin{aligned} \Pi_1 \times \Pi_2^{-2} &= \frac{F}{\rho U^2 D^2} \times \left(\frac{UD}{\nu} \right)^2 = \frac{F}{\rho \nu^2} = \Pi'_1, \\ \Pi_2^{-1} &= \Pi'_2. \end{aligned}$$

Extensões naturais de “dimensões fundamentais” para problemas específicos:

1. Quando há espécies químicas diferentes *sem* reações químicas entre elas: M é a massa total, M_v é a massa de vapor d’água, M_c é a massa de CO_2 , etc..
2. Quando há transferência de *calor sem* que haja conversão significativa de energia mecânica em energia interna. Neste caso, além de energia mecânica, ML^2T^2 , “calor” deve se expresso em termos de uma dimensão extra correspondente, em geral temperatura Θ .

3. Quando existem diferenças grandes do tamanho de grandezas características em direções diferentes: em lugar de um único L , muitas vezes é proveitoso utilizar uma dimensão física para cada dimensão do espaço, ou seja: X, Y, Z .

Eu gostaria de ter um método numérico que funcionasse para “qualquer” equação diferencial ordinária (EDO) do tipo

$$\frac{dy}{dx} = f(x, y).$$

$$\begin{aligned}\frac{dy}{dx} + \frac{y}{x} &= \text{sen}(x), \\ \frac{dy}{dx} &= -\frac{y}{x} + \text{sen}(x).\end{aligned}$$

O que significam os k 's

$$\begin{aligned}y_{n+1} &\approx y_n + h \left. \frac{dy}{dx} \right|_n \\ k_1 &= h \left. \frac{dy}{dx} \right|_n = hf(x_n, y_n) \\ y_{n+1} &= y_n + k_1\end{aligned}$$

Um sistema de equações diferenciais ordinárias já colocado na forma “clássica” do método de Runge-Kutta é algo do tipo: um conjunto de incógnitas y_1, y_2, \dots, y_n que evoluem com uma dinâmica acoplada:

$$\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y_1, \dots, y_n), \\ \frac{dy_2}{dx} &= f_2(x, y_1, \dots, y_n), \\ &\vdots \\ \frac{dy_n}{dx} &= f_n(x, y_1, \dots, y_n).\end{aligned}$$

Vetores, convenções, etc.

$$\begin{aligned}\mathbf{i} &= (1, 0, 0), \\ \mathbf{j} &= (0, 1, 0), \\ \mathbf{k} &= (0, 0, 1).\end{aligned}$$

$$\begin{aligned}\mathbf{u} &= (1, 2, 3), \\ \mathbf{u} &= 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}.\end{aligned}$$

$$\begin{aligned}c &= (d + e) \\c^2 &= (d + e)^2 \\c^2 &= d^2 + e^2 + 2de \blacksquare\end{aligned}$$

Notação indicial ou de Einstein

$$y_i = A_{ij}x_j$$

quer dizer

$$y_i = \sum_{j=1}^n A_{ij}x_j, \quad \forall i = 1, \dots, n.$$

$$\begin{aligned}\delta_{ii} &= \sum_{i=1}^3 \delta_{(i)(i)} \\&= \delta_{11} + \delta_{22} + \delta_{33} \\&= 1 + 1 + 1 \\&= 3.\end{aligned}$$

$$\begin{aligned}[\boldsymbol{\delta}][\boldsymbol{x}] &= [\delta_{ij}][x_j] \\&= \delta_{ij}x_j = x_i \\&= [x_i] = [\boldsymbol{x}].\end{aligned}$$

No caso unidimensional,

$$[\boldsymbol{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\boldsymbol{x}]^\top = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

O significado de $[\boldsymbol{y}]^\top [\boldsymbol{\delta}]$ é

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

Em notação indicial:

$$y_i \delta_{ij} = y_j$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

O que são dois vetores ortogonais no \mathbb{R}^3 ?

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}| |\boldsymbol{v}| \cos\left(\frac{\pi}{2}\right) = 0.$$

No \mathbb{R}^n , nós generalizamos sem poder “ver” os vetores:

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Uma base ortogonal é uma base $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ tal que, se $i \neq j$,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0.$$

Se o índice for repetido, obviamente,

$$\mathbf{v}_{(i)} \cdot \mathbf{v}_{(i)} = |\mathbf{v}_{(i)}|^2.$$

Uma base *ortogonal* é tudo isso aí em cima somado a

$$\mathbf{v}_{(i)} \cdot \mathbf{v}_{(i)} = 1.$$

Dada uma base ortonormal E ,

$$\mathbf{u} = u_{Ei} \mathbf{e}_i,$$

$$\mathbf{v} = v_{Ej} \mathbf{e}_j.$$

onde

$$\mathbf{u} = \begin{bmatrix} u_{E1} \\ u_{E2} \\ \vdots \\ u_{En} \end{bmatrix}_E$$

Lembre-se! $u = (u_1, u_2, \dots, u_n)$, e em geral $u_1 \neq u_{E1}$, $u_2 \neq u_{E2}$, etc.

É possível usar as coordenadas u_{Ei} e v_{Ej} para calcular o produto escalar $\mathbf{u} \cdot \mathbf{v}$? Sim:

$$\mathbf{u} \cdot \mathbf{v} = u_{Ei} \mathbf{e}_i \cdot v_{Ei} \mathbf{e}_i$$

Paro, e recomeço:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_{Ei} \mathbf{e}_i \cdot v_{Ej} \mathbf{e}_j \\ \alpha \mathbf{x} \cdot \beta \mathbf{y} &= \alpha \beta (\mathbf{x} \cdot \mathbf{y}); \\ \mathbf{u} \cdot \mathbf{v} &= u_{Ei} v_{Ej} (\mathbf{e}_i \cdot \mathbf{e}_j) \\ \mathbf{u} \cdot \mathbf{v} &= u_{Ei} v_{Ej} \delta_{ij} \\ \mathbf{u} \cdot \mathbf{v} &= u_{Ei} [v_{Ej} \delta_{ij}] \\ \mathbf{u} \cdot \mathbf{v} &= u_{Ei} v_{Ei} = \sum_{i=1}^n u_{Ei} v_{Ei}; \\ \mathbf{u} \cdot \mathbf{v} &= u_i v_i = \sum_{i=1}^n u_i v_i. \end{aligned}$$

Às vezes escrever $\mathbf{u} = u_{Ei} \mathbf{e}_i$ é demais. Quando não houver motivo de confusão, nós simplificaremos a notação e escreveremos

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_E$$

ou simplesmente $\mathbf{u} = u_i \mathbf{e}_i$, etc.

Reforçando: elemento de vetor \times coordenada de vetor.

$$\mathbf{v} = (3, 4)$$

Se eu escolher a base $((1, 1), (1, -1))$ para representar o \mathbb{R}^2 , então

$$\mathbf{v} = \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}_{((1,1),(1,-1))}$$

$$\begin{aligned} (3, 4) &= 7/2(1, 1) - 1/2(1, -1) \\ &= (7/2, 7/2) + (-1/2, 1/2) \\ &= (6/2, 8/2) \\ &= (3, 4) \blacksquare \end{aligned}$$

Bases!

$$\mathbf{A} = (\mathbf{i}, \mathbf{j}), \mathbf{B} = (\mathbf{j}, \mathbf{i}).$$

O símbolo de permutação de Levi-Civita é calculado sempre com n entradas. Exemplos:

$$\begin{aligned} \epsilon_{1234} &= +1, \\ \epsilon_{1243} &= -1, \\ \epsilon_{1223} &= 0. \end{aligned}$$

Permutações cíclicas (apenas com 3 elementos) de 1, 2, 3:

$$\begin{aligned} 1, 2, 3, \\ 3, 1, 2, \\ 2, 3, 1. \end{aligned}$$

Permutações cíclicas (apenas com 3 elementos) de 3, 2, 1:

$$\begin{aligned} 3, 2, 1, \\ 1, 3, 2, \\ 2, 1, 3. \end{aligned}$$

Para definir o produto vetorial eu parto dos “absolutos”

$$\begin{aligned} \mathbf{u} &= (u_1, u_2, u_3), \\ \mathbf{v} &= (v_1, v_2, v_3), \\ \mathbf{u} \times \mathbf{v} &\equiv (\epsilon_{ij1}u_iv_j, \epsilon_{ij2}u_iv_j, \epsilon_{ij3}u_iv_j). \end{aligned}$$

A forma “antiga” de calcular o produto vetorial é

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3\end{aligned}$$

A forma “nova” de calcular o produto vetorial é

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \epsilon_{ijk}u_iv_j\mathbf{e}_k \\ &= (\epsilon_{231}u_2v_3 + \epsilon_{321}u_3v_2)\mathbf{e}_1 + (\epsilon_{132}u_1v_3 + \epsilon_{312}u_3v_1)\mathbf{e}_2 + (\epsilon_{123}u_1v_2 + \epsilon_{213}u_2v_1)\mathbf{e}_3 \\ &= (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3\end{aligned}$$

Vamos provar agora que $\mathbf{u} \times \mathbf{v}$ é perpendicular a \mathbf{u} .

$$\begin{aligned}[\mathbf{u} \times \mathbf{v}] \cdot \mathbf{u} &= \epsilon_{ijk}u_iv_j\mathbf{e}_k \cdot u_l\mathbf{e}_l \\ \alpha\mathbf{a} \cdot \beta\mathbf{b} &= (\alpha\beta)(\mathbf{a} \cdot \mathbf{b}) \\ &= \epsilon_{ijk}u_iv_ju_l(\mathbf{e}_k \cdot \mathbf{e}_l) \\ &= \epsilon_{ijk}u_iv_ju_l\delta_{kl} \\ &= \epsilon_{ijk}u_iv_ju_k \\ &= \frac{1}{2}[\epsilon_{ijk}u_iv_ju_k + \epsilon_{ijk}u_iv_ju_k] \\ &= \frac{1}{2}[\epsilon_{ijk}u_iv_ju_k + \epsilon_{kji}u_ku_ju_i] \\ &= \frac{1}{2}[\epsilon_{ijk} + \epsilon_{kji}]u_iv_ju_k \\ &= \frac{1}{2}[\epsilon_{ijk} + \epsilon_{jik}]u_iv_ju_k \\ &= 0.\end{aligned}$$

O produto vetorial é anti-simétrico:

$$\begin{aligned}\mathbf{v} \times \mathbf{u} &= \epsilon_{ijk}v_iv_j\mathbf{e}_k \\ &= \epsilon_{ijk}u_jv_i\mathbf{e}_k \\ &= -\epsilon_{jik}u_jv_i\mathbf{e}_k \\ &= -\mathbf{u} \times \mathbf{v} \blacksquare\end{aligned}$$

Em seguida:

$$\begin{aligned}
|\mathbf{u} \times \mathbf{v}|^2 &= [\mathbf{u} \times \mathbf{v}] \cdot [\mathbf{u} \times \mathbf{v}] \\
&= \epsilon_{ijk} u_i v_j \mathbf{e}_k \cdot \epsilon_{lmn} u_l v_m \mathbf{e}_n \\
\alpha \mathbf{a} \cdot \beta \mathbf{b} &= \alpha \beta (\mathbf{a} \cdot \mathbf{b}) \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m (\mathbf{e}_k \cdot \mathbf{e}_n) \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmn} u_l v_m \delta_{kn} \\
&= \epsilon_{ijk} u_i v_j (\epsilon_{lmn} \delta_{kn}) u_l v_m \\
&= \epsilon_{ijk} u_i v_j \epsilon_{lmk} u_l v_m \\
&= [\epsilon_{ijk} \epsilon_{lmk}] u_i v_j u_l v_m \\
&= [\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}] u_i v_j u_l v_m \\
&= \delta_{il} \delta_{jm} u_i v_j u_l v_m - \delta_{im} \delta_{jl} u_i v_j u_l v_m \\
&= u_i v_j u_i v_j - u_i v_j u_j v_i \\
&= (u_i u_i)(v_j v_j) - (u_i v_i)(u_j v_j) \\
&= (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v}) \\
&= |\mathbf{u}|^2 |\mathbf{v}|^2 - (|\mathbf{u}| |\mathbf{v}| \cos(\theta)) (|\mathbf{u}| |\mathbf{v}| \cos(\theta)) \\
&= |\mathbf{u}|^2 |\mathbf{v}|^2 (1 - \cos^2(\theta)) \\
&= |\mathbf{u}|^2 |\mathbf{v}|^2 \sin^2(\theta); \Rightarrow \\
|\mathbf{u} \times \mathbf{v}| &= |\mathbf{u}| |\mathbf{v}| \sin(\theta) \blacksquare
\end{aligned}$$

1. Um “1-volume” é um comprimento.
2. Um “2-volume” é uma área.
3. Um “3-volume” é um volume usual.
4. Um “4-volume” é um “volume quadridimensional”, ou um hipervolume.

Determinantes e hipervolumes.

Considere n vetores no \mathbb{R}^n :

$$\begin{aligned}\mathbf{u}_1 &= (u_{11}, u_{21}, \dots, u_{n1}); \\ \mathbf{u}_2 &= (u_{12}, u_{22}, \dots, u_{n2}); \\ &\vdots \\ \mathbf{u}_n &= (u_{1n}, u_{2n}, \dots, u_{nn}).\end{aligned}$$

O determinante associado a esses n vetores é definido como

$$\begin{aligned}V^n &\equiv \det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_n) \\ &\equiv \epsilon_{i_1, i_2, \dots, i_n} u_{1i_1} u_{2i_2} \dots u_{ni_n}.\end{aligned}$$

O determinante está associado à matriz

$$[\mathbf{u}] = [u_{ij}] = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix}$$

Suponha que uma função f é linear:

$$\begin{aligned}f(\alpha x) &= \alpha f(x), \\ f(x_1 + x_2) &= f(x_1) + f(x_2).\end{aligned}$$

Desejamos provar que, então,

$$f(x) = cx.$$

Prova:

$$\frac{df(\alpha x)}{dx} = \alpha f'(\alpha x),$$

mas f é linear:

$$\begin{aligned}f(\alpha x) &= \alpha f(x) \\ \alpha f'(\alpha x) &= \alpha f'(x).\end{aligned}$$

Portanto, no caso de f ser linear, teremos que

$$f'(\alpha x) = f'(x), \quad \forall \alpha \in \mathbb{R},$$

ou seja: a derivada é constante!

$$\begin{aligned}f'(x) &= c, \\ f(x) &= cx + b, \\ f(0) &= 0 \Rightarrow b = 0 \blacksquare\end{aligned}$$

Transformações lineares. Pergunta: será possível definir uma operação entre \mathbf{A} e \mathbf{x} que nos permita substituir

$$\mathbf{y} = \mathbf{A}(\mathbf{x})$$

por

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}?$$

$$\begin{aligned} A_i \delta_{im} &= \sum_{i=1}^3 A_i \delta_{im} \\ &= A_1 \delta_{1m} + A_2 \delta_{2m} + A_3 \delta_{3m} \\ &= A_m \\ A_{ijkl} B_{ijkp} \delta_{ln} &= A_{ijkn} B_{ijkp} \end{aligned}$$

Transformações lineares

$$\begin{aligned} E &= (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n); \\ \mathbf{x} &= \sum_{j=1}^3 x_j \mathbf{e}_j = x_j \mathbf{e}_j \\ \mathbf{A}(\mathbf{x}) &= \mathbf{A}(x_j \mathbf{e}_j) \\ &= x_j \mathbf{A}(\mathbf{e}_j) \end{aligned}$$

mas

$$\begin{aligned} \mathbf{A}(\mathbf{e}_j) &\in \mathbb{V}; \\ \mathbf{A}(\mathbf{e}_j) &= A_{ij} \mathbf{e}_i \end{aligned}$$

portanto

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= x_j A_{ij} \mathbf{e}_i \\ &= A_{ij} x_j \mathbf{e}_i \\ \mathbf{A}(\mathbf{x}) &= \mathbf{y}; \\ \mathbf{y} &= y_i \mathbf{e}_i = A_{ij} x_j \mathbf{e}_i : \\ y_i &= A_{ij} x_j, \\ [\mathbf{y}]_E &= [\mathbf{A}]_{E,E} [\mathbf{x}]_E. \end{aligned}$$

Agora, eu mudo um pouco o jogo: a base do domínio continua a ser $E = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$; e a base do contra-domínio é $F = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n)$. Agora,

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= x_j \mathbf{A}(\mathbf{e}_j) \\ \mathbf{A}(\mathbf{e}_j) &= A_{ij} \mathbf{f}_i \\ \mathbf{A}(\mathbf{x}) &= A_{ij} x_j \mathbf{f}_i \\ \mathbf{y} &= y_i \mathbf{f}_i \\ y_i &= A_{ij} x_j \\ [\mathbf{y}]_F &= [\mathbf{A}]_{F,E} [\mathbf{x}]_E \end{aligned}$$

Vamos agora nos lembrar de que um funcional linear em sua forma mais geral é dado por

$$f(\mathbf{x}) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n = c_j x_j = \mathbf{c} \cdot \mathbf{x}.$$

Mas

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= A_{ij}x_j\mathbf{f}_i, \\
\mathbf{A}_i &= \mathbf{c}; \\
\mathbf{A}_i &= A_{ij}\mathbf{e}_j \\
\mathbf{A}(\mathbf{x}) &= [\mathbf{c} \cdot \mathbf{x}]_i \mathbf{f}_i \\
&= [A_{ij}\mathbf{e}_j \cdot \mathbf{x}] \mathbf{f}_i \\
&= [A_{ij}\mathbf{e}_j \cdot x_k \mathbf{e}_k] \mathbf{f}_i \\
&= [A_{ij}x_k \mathbf{e}_j \cdot \mathbf{e}_k] \mathbf{f}_i \\
&= [A_{ij}x_k \delta_{jk}] \mathbf{f}_i \\
&= [A_{ij}x_j] \mathbf{f}_i \\
&= [A_{ij}\mathbf{e}_j \cdot x_k \mathbf{e}_k] \mathbf{f}_i \\
&= A_{ij}\mathbf{f}_i \mathbf{e}_j \cdot x_k \mathbf{e}_k \\
&= [A_{ij}\mathbf{f}_i \mathbf{e}_j] \cdot [x_k \mathbf{e}_k] \\
\mathbf{y} &= \mathbf{A} \cdot \mathbf{x} \\
\mathbf{A} &= A_{ij}\mathbf{f}_i \mathbf{e}_j
\end{aligned}$$

Uma transformação de \mathbb{R}^3 em \mathbb{R}^2 é representada matricialmente por

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}_{2 \times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1}$$

Bases!

$$\begin{aligned}
E &= (\mathbf{e}_1, \mathbf{e}_2) = ((1, 0), (0, 1)); \\
\mathbf{v} &= (3, 4) \\
\mathbf{v} &= 3\mathbf{e}_1 + 4\mathbf{e}_2 \\
[\mathbf{v}]_E &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
F &= (\mathbf{f}_1, \mathbf{f}_2) = ((1, 1), (1, -1)) \\
\mathbf{v} &= (7/2)\mathbf{f}_1 - (1/2)\mathbf{f}_2 \\
[\mathbf{v}]_F &= \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}
\end{aligned}$$

Às vezes é preciso “disciplinar” a notação indicial:

$$\sum_{i=1}^3 C_{ij}^2 = \sum_{i=1}^3 C_{ij}C_{ij} \equiv C_{i(j)}C_{i(j)} = 1$$

Notação preferida para transformações lineares:

$$\begin{aligned}
\mathbf{y} &= \mathbf{A}(\mathbf{x}), \\
\mathbf{y} &= \mathbf{A} \cdot \mathbf{x} \\
\mathbf{y} &= y_i \mathbf{f}_i, \\
\mathbf{x} &= x_j \mathbf{e}_j, \\
\mathbf{A} &= A_{ij} \mathbf{f}_i \mathbf{e}_j \\
\mathbf{y} &= \mathbf{A} \cdot \mathbf{x} \\
&= A_{ij} \mathbf{f}_i \mathbf{e}_j \cdot x_k \mathbf{e}_k \\
&= A_{ij} x_k \mathbf{f}_i (\mathbf{e}_j \cdot \mathbf{e}_k) \\
&= A_{ij} x_k \mathbf{f}_i \delta_{jk} \\
y_i \mathbf{f}_i &= A_{ij} x_j \mathbf{f}_i
\end{aligned}$$

Rotações e suas matrizes/transformações

$$\begin{aligned}
\mathbf{u} &= u'_j \mathbf{e}'_j = u_i \mathbf{e}_i \\
u_i &= (\mathbf{u} \cdot \mathbf{e}_i)
\end{aligned}$$

mas

$$\begin{aligned}
\mathbf{e}_i &= C_{ij} \mathbf{e}'_j \\
u_i &= (\mathbf{u} \cdot C_{ij} \mathbf{e}'_j) \\
&= C_{ij} (\mathbf{u} \cdot \mathbf{e}'_j) \\
u_i &= C_{ij} u'_j \\
[\mathbf{u}]_E &= [\mathbf{C}][\mathbf{u}]_{E'} \\
[\mathbf{u}]'_E &= [\mathbf{C}]^\top [\mathbf{u}]_E
\end{aligned}$$

Podemos mostrar que $[\mathbf{C}]$ e $[\mathbf{C}]^\top$ são inversas operando diretamente:

$$\begin{aligned}
\mathbf{u} &= u_k \mathbf{e}_k = u_i \mathbf{e}_i = u_i C_{ij} \mathbf{e}'_j = u_i C_{ij} C_{kj} \mathbf{e}_k \\
u_k &= u_i C_{ij} C_{kj} = C_{ij} C_{jk}^\top u_i \\
u_k &= \delta_{ki} u_i, \\
C_{ij} C_{jk}^\top &= \delta_{ki} \\
[\mathbf{C}][\mathbf{C}]^\top &= [\delta].
\end{aligned}$$

O determinante de uma rotação é igual a 1:

$$\begin{aligned}
1 &= [\mathbf{e}_1 \times \mathbf{e}_2] \cdot \mathbf{e}_3 \\
&= [C_{1l} \mathbf{e}'_l \times C_{2m} \mathbf{e}'_m] \cdot C_{3n} \mathbf{e}'_n \\
&= C_{1l} C_{2m} C_{3n} \underbrace{[\mathbf{e}'_l \times \mathbf{e}'_m] \cdot \mathbf{e}'_n}_{\epsilon_{lmn}} \\
1 &= \epsilon_{lmn} C_{1l} C_{2m} C_{3n} = \det [\mathbf{C}].
\end{aligned}$$

Qual é a regra para a mudança das matrizes de uma transformação \mathbf{A} quando eu mudo da base E para a base E' (ou vice-versa)?

$$\begin{aligned}\mathbf{A} &= A_{ij} \mathbf{e}_i \mathbf{e}_j, \\ \mathbf{A} &= A'_{ij} \mathbf{e}'_i \mathbf{e}'_j, \\ \mathbf{A} &= A_{ij} \mathbf{e}_i \mathbf{e}_j = A'_{kl} \mathbf{e}'_k \mathbf{e}'_l \\ A_{ij} \mathbf{e}_i \mathbf{e}_j &= A'_{kl} C_{ik} \mathbf{e}_i C_{jl} \mathbf{e}_j \\ A_{ij} \mathbf{e}_i \mathbf{e}_j &= A'_{kl} C_{ik} C_{jl} \mathbf{e}_i \mathbf{e}_j \\ A_{ij} \mathbf{e}_i \mathbf{e}_j &= C_{ik} A'_{kl} C_{lj}^\top \mathbf{e}_i \mathbf{e}_j \\ [\mathbf{A}]_E &= [\mathbf{C}][\mathbf{A}]_{E'}[\mathbf{C}]^\top, \\ [\mathbf{A}]_{E'} &= [\mathbf{C}]^\top [\mathbf{A}]_E [\mathbf{C}].\end{aligned}$$

Exemplo:

Parto da base canônica $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ e desejo construir uma outra base ortonormal dextrógira $F = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. Os dois primeiros vetores são

$$\begin{aligned}\mathbf{f}_1 &= \frac{1}{\sqrt{3}}(1, 1, 1), \\ \mathbf{f}_2 &= \frac{1}{\sqrt{6}}(2, -1, -1).\end{aligned}$$

Obviamente

$$\mathbf{f}_3 = \mathbf{f}_1 \times \mathbf{f}_2.$$

$$\begin{aligned}\mathbf{f}_3 &= \frac{1}{\sqrt{3}} \frac{1}{\sqrt{6}} \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\ &= \frac{1}{\sqrt{18}} \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\ &= \frac{1}{3\sqrt{3}} \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\ &= \frac{1}{3\sqrt{2}} [0\mathbf{e}_1 + 3\mathbf{e}_2 - 3\mathbf{e}_3] \\ &= \frac{1}{\sqrt{2}} (0, 1, -1).\end{aligned}$$

A matriz de rotação de E para F é

$$\begin{aligned}\mathbf{f}_j &= C_{ij} \mathbf{e}_i, \\ C_{ij} &= \mathbf{f}_j \cdot \mathbf{e}_i.\end{aligned}$$

Portanto,

$$[\mathbf{C}] = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$$

Se eu quero rodar em torno do eixo 2, devo rodar de 1 para 3 ou de 3 para 1?

$$\epsilon_{132} = \epsilon_{321} = -1,$$

$$\epsilon_{312} = \epsilon_{123} = +1.$$

Resposta: de 3 para 1!

Um sistema de equações lineares é algo da forma

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n &= y_1, \\ A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n &= y_2, \\ &\vdots \\ A_{n1}x_1 + A_{n2}x_2 + \dots + A_{nn}x_n &= y_n. \end{aligned}$$

A transformação

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

projeta qq vetor do \mathbb{R}^3 no plano que passa pela origem e cujo normal é \mathbf{k} . Note que a equação desse plano é

$$k_1x_1 + k_2x_2 + k_3x_3 = 0.$$

$$\begin{aligned} \mathbf{b} &= \mathbf{P} \cdot \mathbf{a}, \\ b_i &= \left[\delta_{ij} - \frac{k_i k_j}{k^2} \right] a_j \\ b_i &= \delta_{ij} a_j - \frac{k_i k_j}{k^2} a_j \\ &= a_i - \frac{k_i k_j}{k^2} a_j; \\ \mathbf{b} \cdot \mathbf{k} &= b_i k_i \\ &= \left[a_i - \frac{k_i k_j}{k^2} a_j \right] k_i \\ &= a_i k_i - \left(\frac{k_i k_i}{k^2} \right) k_j a_j \\ &= a_i k_i - a_j k_j = 0 \blacksquare \end{aligned}$$

Teorema dos Pis:

$$\begin{aligned} (M^A)^x &= M^{Ax}. \\ 1 &= \prod_{j=1}^n (M^{A_{ij}})^x_j \times \dots \end{aligned}$$

Vamos voltar!

$$\begin{aligned}
\Pi &= \prod_{j=1}^n v_j^{x_j}, \\
\llbracket \Pi \rrbracket &= \left\llbracket \prod_{j=1}^n v_j^{x_j} \right\rrbracket \\
\llbracket \Pi \rrbracket &= \prod_{j=1}^n \llbracket v_j^{x_j} \rrbracket \\
\llbracket \Pi \rrbracket &= \prod_{j=1}^n \left(\mathsf{M}^{A_{1j}} \mathsf{L}^{A_{2j}} \mathsf{T}^{A_{3j}} \right)^{x_j} \\
\llbracket \Pi \rrbracket &= \prod_{j=1}^n \left(\mathsf{M}^{A_{1j}} \right)^{x_j} \left(\mathsf{L}^{A_{2j}} \right)^{x_j} \left(\mathsf{T}^{A_{3j}} \right)^{x_j} \\
\llbracket \Pi \rrbracket &= \prod_{j=1}^n \left(\mathsf{M}^{A_{1j}} \right)^{x_j} \prod_{j=1}^n \left(\mathsf{L}^{A_{2j}} \right)^{x_j} \prod_{j=1}^n \left(\mathsf{T}^{A_{3j}} \right)^{x_j}
\end{aligned}$$

Se $n = 4$, teremos por exemplo

$$\begin{aligned}
\prod_{j=1}^n \left(\mathsf{M}^{A_{1j}} \right)^{x_j} &= \left(\mathsf{M}^{A_{11}} \right)^{x_1} \times \left(\mathsf{M}^{A_{12}} \right)^{x_2} \times \left(\mathsf{M}^{A_{13}} \right)^{x_3} \times \left(\mathsf{M}^{A_{14}} \right)^{x_4} = 1 \\
\mathsf{M}^{A_{11}x_1} \times \mathsf{M}^{A_{12}x_2} \times \mathsf{M}^{A_{13}x_3} \times \mathsf{M}^{A_{14}x_4} &= 1 \\
\mathsf{M}^{A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4} &= 1 \\
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 &= 0
\end{aligned}$$

Autovalores e autovetores

Se \mathbf{v} é autovetor de \mathbf{A} , então $\alpha\mathbf{v}$, $\alpha \neq 0$, também é:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{v} &= \lambda\mathbf{v}; \\ \mathbf{A} \cdot [\alpha\mathbf{v}] &= \alpha\mathbf{A} \cdot \mathbf{v} \\ &= \alpha\lambda\mathbf{v} \\ &= \lambda[\alpha\mathbf{v}].\end{aligned}$$

No exemplo em tela, $\lambda = 1$ possui multiplicidade 2 e também está associado a um subespaço de dimensão 2. A equação

$$x_1 + 2x_2 + x_3 = 0$$

é a equação de um plano (pela origem), e quaisquer dois vetores desse plano são autovetores. De fato, sejam $\mathbf{v}_1, \mathbf{v}_2$ dois autovetores LI associados a λ ; então

$$\begin{aligned}\mathbf{A} \cdot [\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2] &= \alpha_1\mathbf{A} \cdot \mathbf{v}_1 + \alpha_2\mathbf{A} \cdot \mathbf{v}_2 \\ &= \alpha_1\lambda\mathbf{v}_1 + \alpha_2\lambda\mathbf{v}_2 \\ &= \lambda[\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2].\end{aligned}$$

No caso univariado,

$$\begin{aligned}\frac{dx}{dt} &= 4x, \\ x &= e^{4t}, \\ \frac{dx}{dt} &= 4e^{4t} = 4x.\end{aligned}$$

Transposta de uma transformação linear

$$\begin{aligned}\mathbf{x} \cdot [\mathbf{A}^\top \cdot \mathbf{y}] &\equiv \mathbf{y} \cdot [\mathbf{A} \cdot \mathbf{x}] \\ x_k \mathbf{e}_k \cdot [A_{ij}^\top \mathbf{e}_i \mathbf{e}_j \cdot y_l \mathbf{e}_l] &= y_k \mathbf{e}_k \cdot [A_{ij} \mathbf{e}_i \mathbf{e}_j \cdot x_l \mathbf{e}_l] \\ x_k \mathbf{e}_k \cdot [A_{ij}^\top \mathbf{e}_i (\mathbf{e}_j \cdot \mathbf{e}_l) y_l] &= y_k \mathbf{e}_k \cdot [A_{ij} \mathbf{e}_i (\mathbf{e}_j \cdot \mathbf{e}_l) x_l] \\ x_k \mathbf{e}_k \cdot [A_{ij}^\top \mathbf{e}_i \delta_{jl} y_l] &= y_k \mathbf{e}_k \cdot [A_{ij} \mathbf{e}_i \delta_{jl} x_l] \\ x_k \mathbf{e}_k \cdot [A_{ij}^\top y_j \mathbf{e}_i] &= y_k \mathbf{e}_k \cdot [A_{ij} x_j \mathbf{e}_i] \\ x_k A_{ij}^\top y_j (\mathbf{e}_k \cdot \mathbf{e}_i) &= y_k A_{ij} x_j (\mathbf{e}_k \cdot \mathbf{e}_i) \\ x_k A_{ij}^\top y_j \delta_{ki} &= y_k A_{ij} x_j \delta_{ki} \\ x_i A_{ij}^\top y_j &= y_i A_{ij} x_j \\ x_i A_{ij}^\top y_j &= y_j A_{ji} x_i \\ x_i A_{ij}^\top y_j &= x_i A_{ji} y_j \\ A_{ij}^\top &= A_{ji} \blacksquare\end{aligned}$$

Até agora nós vimos (e demos significado a)

$$\begin{aligned}\mathbf{A} \cdot \mathbf{x} &= A_{ij} \mathbf{e}_i \mathbf{e}_j \cdot x_k \mathbf{e}_k \\ &= A_{ij} x_k \mathbf{e}_i (\mathbf{e}_j \cdot \mathbf{e}_k) \\ &= A_{ij} x_k \mathbf{e}_i \delta_{jk} \\ &= A_{ij} x_j \mathbf{e}_i \\ &= [\mathbf{A}][\mathbf{x}]\end{aligned}$$

Eu também posso definir a operação

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{A} &= x_k \mathbf{e}_k \cdot A_{ij} \mathbf{e}_i \mathbf{e}_j \\
&= x_k A_{ij} (\mathbf{e}_k \cdot \mathbf{e}_i) \mathbf{e}_j \\
&= x_k A_{ij} \delta_{ki} \mathbf{e}_j \\
&= x_i A_{ij} \mathbf{e}_j \\
&= [x_1 A_{11} + x_2 A_{21} + x_3 A_{31}] \mathbf{e}_1 \\
&\quad + [x_1 A_{12} + x_2 A_{22} + x_3 A_{32}] \mathbf{e}_2 \\
&\quad + [x_1 A_{13} + x_2 A_{23} + x_3 A_{33}] \mathbf{e}_3 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \\
&= [\mathbf{x}]^\top [\mathbf{A}]
\end{aligned}$$

Portanto eu posso definir a transformação simétrica via

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{A}^\top &\equiv \mathbf{A} \cdot \mathbf{x}, \\
&\vdots \\
A_{ij}^\top &= A_{ji}.
\end{aligned}$$

Na equação

$$A_{ij} x_j = \lambda x_i$$

Os A_{ij} s são reais. O conjugado da equação é

$$\begin{aligned}
(A_{ij} x_j)^* &= (\lambda x_i)^* \\
A_{ij}^* x_j^* &= \lambda^* x_i^* \\
A_{ij} x_j^* &= \lambda^* x_i^*
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0), \\
\mathbf{f}_1 &= \frac{1}{\sqrt{3}}(1, 1, 1), \\
\mathbf{e}_1 \cdot \mathbf{f}_1 &= \frac{1}{\sqrt{3}} = \\
\mathbf{e}_j &\quad \mathbf{f}_j
\end{aligned}$$

Quando existe uma base de autovetores para a transformação \mathbf{A} , a matriz de \mathbf{A} nessa base é uma matriz diagonal composta pelos autovalores.

Olhe para o caso 3×3

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_F$$

Cada coluna representa a imagem de um dos vetores da base e **se** eu estiver na base dos autovetores:

$$\mathbf{A} \cdot \mathbf{f}_1 = A_{11}\mathbf{f}_1 + A_{21}\mathbf{f}_2 + A_{31}\mathbf{f}_3 = \lambda_1\mathbf{f}_1$$

$$\mathbf{A} \cdot \mathbf{f}_2 = A_{12}\mathbf{f}_1 + A_{22}\mathbf{f}_2 + A_{32}\mathbf{f}_3 = \lambda_2\mathbf{f}_2$$

$$\mathbf{A} \cdot \mathbf{f}_3 = A_{13}\mathbf{f}_1 + A_{23}\mathbf{f}_2 + A_{33}\mathbf{f}_3 = \lambda_3\mathbf{f}_3$$

Portanto, concluímos que

$$\mathbf{A}_F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_F$$

Exercício 5.31

$$\begin{aligned} \mathbf{S} : \mathbf{A} &= S_{ij}A_{lm}(\mathbf{e}_j \cdot \mathbf{e}_l)(\mathbf{e}_i \cdot \mathbf{e}_m) \\ &= S_{ij}A_{lm}\delta_{jl}\delta_{im} \\ &= S_{ij}A_{ji} \\ &= \frac{1}{2}S_{ij}A_{ji} + \frac{1}{2}S_{ij}A_{ji} \\ &= \frac{1}{2}S_{ij}A_{ji} + \frac{1}{2}S_{ji}A_{ij} \\ &= \frac{1}{2}S_{ij}(A_{ji} + A_{ij}) = 0. \end{aligned}$$

Uma EDO de ordem 2 pode ser escrita como um sistema de duas EDOs acopladas de ordem 1.

$$\begin{aligned}\frac{d^2u}{dt^2} + u &= 0 \\ \frac{du}{dt} &= v \\ \frac{dv}{dt} + u &= 0 \\ \frac{du}{dt} &= v \\ \frac{dv}{dt} &= -u\end{aligned}$$

$$\begin{aligned}u(t) &= \frac{A - iB}{2} [\cos t + i \sin t] + \frac{A + iB}{2} [\cos t - i \sin t] \\ &= \frac{A}{2} [\cos t + i \sin t] - \frac{iB}{2} [\cos t + i \sin t] + \frac{A}{2} [\cos t - i \sin t] + \frac{iB}{2} [\cos t - i \sin t] \\ &= \frac{A}{2} \cos t + \frac{A}{2} \cos t + \frac{-i^2 B}{2} \sin t + \frac{-i^2 B}{2} \sin t + i \left[\frac{A}{2} \sin t - \frac{B}{2} \cos t - \frac{A}{2} \sin t + \frac{B}{2} \cos t \right] \\ &= A \cos t + B \sin t \blacksquare\end{aligned}$$

Ordem de diferenciação (ou de derivação):

$$\frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right).$$

Em geral,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Série de Taylor

$$f(x) = \frac{1}{0!} f(x_0) + \frac{1}{1!} \frac{df(x_0)}{dx} (x - x_0) + \frac{1}{2!} \frac{d^2 f(x_0)}{dx^2} (x - x_0)^2 + \dots + \frac{1}{n!} \frac{d^n f(x_0)}{dx^n} (x - x_0)^n + \dots$$

No caso de $n = 2$, e de uma função (também) de 2 variáveis

$$\begin{aligned}\frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_i \partial x_j} (x_i - x_{0i})(x_j - x_{0j}) &= \frac{1}{2!} \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_i \partial x_j} (x_i - x_{0i})(x_j - x_{0j}) \\ &= \frac{1}{2!} \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_1 \partial x_1} (x_1 - x_{01})(x_1 - x_{01}) + \\ &\quad \frac{1}{2!} \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_1 \partial x_2} (x_1 - x_{01})(x_2 - x_{02}) + \\ &\quad \frac{1}{2!} \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_2 \partial x_1} (x_2 - x_{02})(x_1 - x_{01}) + \\ &\quad \frac{1}{2!} \frac{\partial^2 f(\mathbf{x}_0)}{\partial x_2 \partial x_2} (x_2 - x_{02})(x_2 - x_{02}).\end{aligned}$$

Verifique o número de índices repetidos em uma expressão

$$\begin{aligned}\mathbf{P} \cdot \mathbf{a} &= P_{il} a_l \mathbf{e}_i \\ &= \left[\delta_{il} - \frac{k_i k_l}{k^2} \right] a_l \mathbf{e}_i \\ [\mathbf{P} \cdot \mathbf{a}] \times \mathbf{k} &= \epsilon_{ijk} \left[\delta_{il} - \frac{k_i k_l}{k^2} \right] a_l k_j \mathbf{e}_k\end{aligned}$$

Teorema da função implícita

$$\begin{aligned}\frac{du(x_0)}{dx} &= - \frac{\frac{\partial f(x_0, u_0)}{\partial x}}{\frac{\partial f(x_0, u_0)}{\partial u}} \\ u(x) &= u_0 + \frac{du(x_0)}{dx} (x - x_0) + \dots\end{aligned}$$

$$\begin{aligned}x &= x(u, v) & u &= u(x, y) \\ y &= y(u, v) & v &= v(x, y)\end{aligned}$$

A regra de Leibnitz:

$$\begin{aligned}\frac{d}{dt} \int_a^b f(x, t) dx &= \int_a^b \frac{\partial f(x, t)}{\partial t} dx, \quad \text{porém} \\ \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx &\neq \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx. \\ \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx &= \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx + f(b, t) \frac{db}{dt} - f(a, t) \frac{da}{dt}; \\ \frac{DN}{Dt} &= \frac{\partial}{\partial t} \int_{VC} \eta \rho dV + \oint_{SC} (\mathbf{n} \cdot \mathbf{v}) \eta \rho dS\end{aligned}$$

A idéia é mudar a variável de integração de x para ξ :

$$\begin{aligned}x &= X(\xi, t) \\ dx &= \frac{\partial X}{\partial \xi} d\xi + \frac{\partial X}{\partial t} dt, \\ \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx &= \frac{d}{dt} \int_{\alpha}^{\beta} f(X(\xi, t), t) \frac{\partial X}{\partial \xi} d\xi\end{aligned}$$

Revisite a regra da cadeia!

$$\begin{aligned}f(x, u); \\ F(x) &= f(x, u(x)); \\ \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial u}{\partial x}\end{aligned}$$

Quero o módulo de $\mathbf{v} = (1, i, 0)$:

$$\begin{aligned}
|\mathbf{v}| &= \sqrt{\sum_{i=1}^3 v_i^* v_i} \\
&= \sqrt{(1 \times 1 + (-i \times i) + 0 \times 0)} \\
&= \sqrt{1 - i^2 + 0} \\
&= \sqrt{1 + 1 + 0} \\
&= \sqrt{2}
\end{aligned}$$

Regra de Leibnitz:

Considere

$$\begin{aligned}
\frac{\partial}{\partial \xi} \frac{\partial X}{\partial t} &=? \\
\frac{\partial X}{\partial t} &= h(\xi, t) \\
\frac{\partial}{\partial \xi} \frac{\partial X}{\partial t} &= \frac{\partial h}{\partial \xi} = \frac{\partial h}{\partial x} \frac{\partial X}{\partial \xi}
\end{aligned}$$

Surgimento de uma “velocidade”

$$\begin{aligned}
x &= X(\xi, t), \\
u &= \frac{dx}{dt} = \frac{\partial X}{\partial t}
\end{aligned}$$

A regra de Leibnitz é

$$\begin{aligned}
\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx &= f(b, t) \frac{db}{dt} - f(a, t) \frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx \\
\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt &= f(x, b) \frac{db}{dx} - f(x, a) \frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x, t)}{\partial x} dt \\
b(x) &= x, \\
a(x) &= 0, \\
\frac{d}{dx} \int_0^x [xt + e^{-t^2}] dt &= x^2 + e^{-x^2} + \int_0^x \frac{\partial}{\partial x} [xt + e^{-t^2}] dt \\
\frac{d}{dx} \int_0^x [xt + e^{-t^2}] dt &= x^2 + e^{-x^2} + \int_0^x t dt \\
\frac{d}{dx} \int_0^x [xt + e^{-t^2}] dt &= \frac{3x^2}{2} + e^{-x^2}
\end{aligned}$$

Quero calcular um comprimento de arco sobre a curva (entre $t = 0$ e $t = \tau$)

$$\begin{aligned}\mathbf{r}(t) &= \cos(t)[\mathbf{e}_x + \mathbf{e}_y] - \sqrt{2}\sin(t)\mathbf{e}_z \\ \ell_{\mathcal{L}} &= \int_{\mathbf{r} \in \mathcal{L}} \left[\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} \right]^{1/2} dt \\ \frac{d\mathbf{r}}{dt} &= -\sin(t)[\mathbf{e}_x + \mathbf{e}_y] - \sqrt{2}\cos(t)\mathbf{e}_z \\ &= (-\sin(t), -\sin(t), -\sqrt{2}\cos(t)) \\ \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} &= \sin^2(t) + \sin^2(t) + 2\cos^2(t) = 2 \\ \ell_{\mathcal{L}} &= \int_0^\tau \sqrt{2} dt = \sqrt{2}\tau.\end{aligned}$$

Exemplo 7.11: Calcule a área da superfície externa do parabolóide de revolução $z = x^2 + y^2$, $x^2 + y^2 \leq 1$

Procuro a teoria:

$$\begin{aligned}z &= g(x, y) = x^2 + y^2, \\ A_{\mathcal{S}} &= \int_{R_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dx dy \\ \frac{\partial g}{\partial x} &= 2x, \\ \frac{\partial g}{\partial y} &= 2y, \\ A_{\mathcal{S}} &= \int_{R_{xy}} \sqrt{1 + (2x)^2 + (2y)^2} dx dy \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^1 \sqrt{1 + 4(x^2 + y^2)} r dr d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^1 \sqrt{1 + 4r^2} r dr d\theta \\ &= \left[\int_{r=0}^1 \sqrt{1 + 4r^2} r dr \right] \left[\int_{\theta=0}^{2\pi} d\theta \right] \\ &= 2\pi \int_{r=0}^1 \sqrt{1 + 4r^2} r dr \\ &= 2\pi \frac{5\sqrt{5} - 1}{12} = \frac{5\sqrt{5} - 1}{6}.\end{aligned}$$

Virtudes acadêmicas e profissionais:

1. Trabalho.
2. Persistência.
3. Iniciativa.

Divergência, gradiente, rotacional

$$\operatorname{div} \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

$$\nabla \cdot \mathbf{u} = ?$$

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (u, v, w) \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \end{aligned}$$

∇ é o **operador** “nabla”.

Posso repetir, de forma mais resumida, em notação indicial. Suponha que $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ seja uma base **ortonormal** do \mathbb{R}^3 .

$$\nabla \equiv \mathbf{e}_i \frac{\partial}{\partial x_i},$$

e não

$$\nabla \equiv \frac{\partial}{\partial x_i} \mathbf{e}_i,$$

$$\mathbf{u} = u_j \mathbf{e}_j$$

$$\operatorname{div} \mathbf{u} \equiv \nabla \cdot \mathbf{u}$$

$$= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot u_j \mathbf{e}_j$$

$$= (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial u_j}{\partial x_i}$$

$$= \delta_{ij} \frac{\partial u_j}{\partial x_i}$$

$$= \frac{\partial u_i}{\partial x_i}.$$

Lei de Fourier é

$$\mathbf{q} = -\rho c_p \alpha \nabla T,$$

$$\dot{Q} = - \oint_{\text{casca do ovo}} (\mathbf{n} \cdot \mathbf{q}) \, dA$$

Gradiente, e sua “definição” (em coordenadas cartesianas) com o uso de ∇ :

$$\alpha \mathbf{u},$$

$$\mathbf{u}\alpha,$$

$$\mathbf{grad} f = \nabla f,$$

$$\mathbf{grad} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) f$$

$$\mathbf{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\begin{aligned} \mathbf{grad} f &= \mathbf{e}_i \frac{\partial}{\partial x_i} f \\ &= \mathbf{e}_i \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \mathbf{e}_i \end{aligned}$$

O rotacional:

$$\begin{aligned}\nabla \times \mathbf{u} &= \epsilon_{ijk} \nabla_i u_j \mathbf{e}_k \\ &= \epsilon_{ijk} \frac{\partial}{\partial x_i} u_j \mathbf{e}_k \\ &= \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \mathbf{e}_k\end{aligned}$$

Em resumo, as operações relacionados a

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

são

1. A divergência: $\mathbf{u} = (u, v, w)$ e

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z};$$

2. O gradiente: $f(x, y, z)$ e

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

3. O rotacional:

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}.$$

O laplaciano!

$$\begin{aligned}\nabla^2 f &\equiv \nabla \cdot \nabla f \\ &\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{\partial^2 f}{\partial x_i \partial x_i}.\end{aligned}$$

Por extensão, também podemos tratar do laplaciano de um vetor:

$$\begin{aligned}\nabla^2 \mathbf{u} &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \mathbf{i} \\ &+ \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \mathbf{j} \\ &+ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \mathbf{k} \\ &= \frac{\partial^2 u_i}{\partial x_k \partial x_k} \mathbf{e}_i.\end{aligned}$$

Aplicações em mecânica dos fluidos:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] &= \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u},\end{aligned}$$

Importante: a forma das operações muda quando nós mudamos de sistema de coordenadas. Por exemplo, em coordenadas cilíndricas, **não** é verdade que

$$\nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

$$\begin{aligned} r &= (x^2 + y^2)^{1/2}, \\ \frac{\partial r}{\partial x} &= \frac{1}{2}(x^2 + y^2)^{-1/2} \times 2x \\ &= \frac{x}{(x^2 + y^2)^{1/2}} = \frac{x}{r} = \cos(\theta). \\ \theta &= \arctg\left(\frac{y}{x}\right), \\ \frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x} \arctg\left(\frac{y}{x}\right), \\ \frac{d \arctg u}{du} &= \frac{1}{1 + u^2}, \\ \frac{\partial \theta}{\partial x} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \times -\frac{y}{x^2} \end{aligned}$$

Teorema da divergência, aplicações em Mecânica dos Fluidos.

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho dV + \oint_{\mathcal{S}} \rho(\mathbf{n} \cdot \mathbf{v}) dA &= 0 \\ \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho dV + \oint_{\mathcal{S}} (\mathbf{n} \cdot [\rho \mathbf{v}]) dA &= 0 \\ \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV + \int_{\mathcal{V}} \nabla \cdot [\rho \mathbf{v}] dV &= 0 \\ \int_{\mathcal{V}} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] \right\} dV &= 0 \\ \frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] &= 0 \end{aligned}$$

Exercício 7.9

$$\begin{aligned} x^2 + y^2 &= 1 \\ z &= 1 - y^2, \\ z &= u, \\ x &= \cos v, \\ y &= \sin v, \\ \mathbf{r} &= (\cos v, \sin v, u), \\ \frac{\partial \mathbf{r}}{\partial u} &= (0, 0, 1), \\ \frac{\partial \mathbf{r}}{\partial v} &= (-\sin v, \cos v, 0) \end{aligned}$$

A área da superfície é

$$A = \iint_{R_{uv}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

No nosso caso a área é simplesmente

$$\begin{aligned} A &= \iint_{R_{uv}} du dv \\ &= \int_{v=0}^{2\pi} \int_{u=0}^{1-v^2} du dv \end{aligned}$$

Campos irrotacionais

$$\mathbf{F} = (F_1, F_2, F_3)$$

A integral de log:

$$\begin{aligned}\int \frac{du}{u} &= \ln |u|; \\ \int_a^b \frac{du}{u} &= \ln |b| - \ln |a| \\ &= \ln \frac{|b|}{|a|}\end{aligned}$$

O que acontece se os sinais de a e b forem iguais: teremos somente duas possibilidades:

$$\begin{array}{ll} |a| = a & e|b| = b, \\ |a| = -a & e|b| = -b. \end{array}$$

Em ambos os casos,

$$\ln \frac{|b|}{|a|} = \ln \frac{b}{a}.$$

Ao tentarmos resolver

$$\mathcal{D}y = f(x)$$

queremos encontrar a **função** $y(x)$.

$$\mathcal{D} = \frac{d}{dx},$$

$$\mathcal{D}y = \frac{dy}{dx},$$

$$\mathcal{D} = x + y \frac{d}{dx},$$

$$\mathcal{D}y = xy + y \frac{dy}{dx},$$

$$\mathcal{D} = x^2 \frac{d^2}{dx^2} + x^2,$$

$$\mathcal{D}y = x^2 \frac{d^2 y}{dx^2} + x^2 y,$$

$$\mathcal{D} = \frac{d^2}{dx^2} + 2 \frac{d}{dx} + 1,$$

$$\mathcal{D}y = \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y.$$

$$\mathcal{D}y = f(x)$$

Uma EDO com coeficientes não constantes é

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

EDOs lineares, coeficientes constantes, ordem 2.

$$\begin{aligned}
 a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy &= 0, \\
 y &= e^{\lambda x}, \\
 y' &= \lambda e^{\lambda x}, \\
 y'' &= \lambda^2 e^{\lambda x}, \\
 a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} &= 0, \\
 e^{\lambda x} [a\lambda^2 + b\lambda + c] &= 0
 \end{aligned}$$

Se

$$ay'' + by' + cy = 0,$$

a equação característica é

$$a\lambda^2 + b\lambda + c = 0.$$

$$\begin{aligned}
 \lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}; \\
 &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
 &= \alpha \pm \beta
 \end{aligned}$$

$$\begin{aligned}
 y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\
 &= C_1 e^{(\alpha+\beta)x} + C_2 e^{(\alpha-\beta)x} \\
 &= C_1 e^{\alpha x} e^{\beta x} + C_2 e^{\alpha x} e^{-\beta x} \\
 &= e^{\alpha x} [C_1 e^{\beta x} + C_2 e^{-\beta x}]
 \end{aligned}$$

$$C_1 = \frac{1}{2} (D_1 + D_2)$$

$$C_2 = \frac{1}{2} (D_1 - D_2)$$

$$y = e^{\alpha x} \left[\frac{1}{2} (D_1 + D_2) e^{\beta x} + \frac{1}{2} (D_1 - D_2) e^{-\beta x} \right]$$

$$y = e^{\alpha x} \left[D_1 \frac{e^{\beta x} + e^{-\beta x}}{2} + D_2 \frac{e^{\beta x} - e^{-\beta x}}{2} \right]$$

$$\cosh(x) \equiv \frac{e^x + e^{-x}}{2},$$

$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2},$$

$$y = e^{\alpha x} [D_1 \cosh(\beta x) + D_2 \sinh(\beta x)] \blacksquare$$

Problema “real”, raízes “complexas”: temos

$$ay'' + by' + cy = 0$$

e desejamos $y(x) \in \mathbb{R}$. Porém,

$$b^2 - 4ac < 0.$$

A solução geral será a mesma de antes:

$$\begin{aligned}
y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\
&= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \\
&= C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} \\
&= e^{\alpha x} [C_1 e^{i\beta x} + C_2 e^{-i\beta x}]; \\
C_1 &= \frac{1}{2}(A - iB), \\
C_2 &= \frac{1}{2}(A + iB), \\
y &= e^{\alpha x} \left[\frac{1}{2}(A - iB) e^{i\beta x} + \frac{1}{2}(A + iB) e^{-i\beta x} \right] \\
y &= e^{\alpha x} \left[A \left(\frac{e^{i\beta x} + e^{-i\beta x}}{2} \right) - iB \left(\frac{e^{i\beta x} - e^{-i\beta x}}{2} \right) \right] \\
e^{i\beta x} &= \cos(\beta x) + i \operatorname{sen}(\beta x), \\
e^{-i\beta x} &= \cos(\beta x) - i \operatorname{sen}(\beta x), \\
\cosh(i\beta x) &= \frac{e^{i\beta x} + e^{-i\beta x}}{2} = \frac{1}{2} [(\cos(\beta x) + i \operatorname{sen}(\beta x)) + (\cos(\beta x) - i \operatorname{sen}(\beta x))] \\
&= \cos(\beta x); \\
\sinh(i\beta x) &= \frac{e^{i\beta x} - e^{-i\beta x}}{2} = \frac{1}{2} [(\cos(\beta x) + i \operatorname{sen}(\beta x)) - (\cos(\beta x) - i \operatorname{sen}(\beta x))] \\
&= i \operatorname{sen}(\beta x); \\
y &= e^{\alpha x} [A \cos(\beta x) - iB \times i \operatorname{sen}(\beta x)] \\
&= e^{\alpha x} [A \cos(\beta x) + B \operatorname{sen}(\beta x)] \blacksquare
\end{aligned}$$

O método de variação de constantes, equações diferenciais ordinárias de ordem 2.

$$\begin{aligned}
ay'' + by' + cy &= f(x), \\
ay_h'' + by_h' + cy_h &= 0,
\end{aligned}$$

ache y_1 e y_2 de tal maneira que

$$\begin{aligned}
ay_1'' + by_1' + cy_1 &= 0, \\
ay_2'' + by_2' + cy_2 &= 0,
\end{aligned}$$

Equações diferenciais ordinárias de ordem 2, não-homogêneas: solução pelo método de “variação de constantes”.

$$\begin{aligned}
y'' - 3y' + 2y &= x, \\
y_h'' - 3y_h' + 2y_h &= 0, \\
\lambda^2 - 3\lambda + 2 &= 0 \\
\lambda &= \frac{3 \pm \sqrt{9 - 4 \times 1 \times 2}}{2} \\
\lambda &= \frac{3 \pm 1}{2} \\
&= 2, \quad 1; \\
y_h &= Ae^x + Be^{2x};
\end{aligned}$$

Procure a solução na forma

$$\begin{aligned}
y(x) &= A(x)e^x + B(x)e^{2x}; \\
y'(x) &= Ae^x + 2Be^{2x} + [A'e^x + B'e^{2x}]
\end{aligned}$$

Controle as derivadas de A e B , e impeça derivadas de ordem 2

$$y'(x) = Ae^x + 2Be^{2x} + \underbrace{[A'e^x + B'e^{2x}]}_{=0}$$

$$\begin{aligned}
A'e^x + B'e^{2x} &= 0; \\
y'(x) &= Ae^x + 2Be^{2x}, \\
y''(x) &= Ae^x + 4Be^{2x} + A'e^x + 2B'e^{2x}
\end{aligned}$$

Substituindo na equação original, obtemos

$$\begin{aligned}
Ae^x + 4Be^{2x} + A'e^x + 2B'e^{2x} - 3[Ae^x + 2Be^{2x}] + 2[Ae^x + Be^{2x}] &= x, \\
e^x [A - 3A + 2A] + e^{2x} [4B - 6B + 2B] + A'e^x + 2B'e^{2x} &= x \\
A'e^x + 2B'e^{2x} &= x
\end{aligned}$$

Reúno agora duas EDOs em A e B :

$$\begin{aligned}
A'e^x + B'e^{2x} &= 0; \\
A'e^x + 2B'e^{2x} &= x \\
2A'e^x + 2B'e^{2x} &= 0; \\
A'e^x + 2B'e^{2x} &= x \\
A'e^x &= -x \\
\frac{dA}{dx} &= -xe^{-x}
\end{aligned}$$

Integrando,

$$\begin{aligned}
 A(x) &= (x+1)e^{-x} + C_1 \\
 A'e^x + B'e^{2x} &= 0; \\
 A'e^x + 2B'e^{2x} &= x \\
 B'e^{2x} &= x, \\
 \frac{dB}{dx} &= xe^{-2x}, \\
 B(x) &= -\frac{1}{4}(2x+1)e^{-2x} + C_2, \\
 y(x) &= A(x)e^x + B(x)e^{2x}, \\
 &= \left[(x+1)e^{-x} + C_1\right]e^x + \left[-\frac{1}{4}(2x+1)e^{-2x} + C_2\right]e^{2x} \\
 &= C_1e^x + C_2e^{2x} + x + 1 - \frac{1}{4}(2x+1) \\
 y(x) &= \underbrace{C_1e^x + C_2e^{2x}}_{y_h(x)} + \underbrace{\frac{2x+3}{4}}_{y_p(x)} \blacksquare
 \end{aligned}$$

Uma equação de Euler com raízes **complexas**:

$$\begin{aligned}
x^2 y'' + y &= 0, \\
y &= x^m, \\
y' &= m x^{m-1}, \\
y'' &= (m-1) m x^{m-2} \\
x^2 \underbrace{(m-1) m x^{m-2}}_{y''} + \underbrace{x^m}_y &= 0 \\
[(m-1)m + 1] x^m &= 0, \\
m^2 - m + 1 &= 0, \\
m &= \frac{1 \pm i\sqrt{3}}{2} \\
y &= C_1 x^{\frac{1+i\sqrt{3}}{2}} + C_2 x^{\frac{1-i\sqrt{3}}{2}} \\
x^a &= \exp(\ln x^a) = \exp(a \ln x); \\
x^{\frac{1 \pm i\sqrt{3}}{2}} &= x^{\frac{1}{2}} x^{\frac{\pm i\sqrt{3}}{2}} \\
x^{\frac{\pm i\sqrt{3}}{2}} &= \exp\left(i \frac{\pm\sqrt{3}}{2} \ln x\right) \\
\exp(i\theta) &= \cos(\theta) + i \sin(\theta) \\
x^{\frac{\pm i\sqrt{3}}{2}} &= \cos\left(\frac{\sqrt{3}}{2} \ln x\right) \pm i \sin\left(\frac{\sqrt{3}}{2} \ln x\right) \\
y &= C_1 x^{\frac{1+i\sqrt{3}}{2}} + C_2 x^{\frac{1-i\sqrt{3}}{2}} \\
y &= C_1 x^{1/2} x^{+\frac{i\sqrt{3}}{2}} + C_2 x^{1/2} x^{-\frac{i\sqrt{3}}{2}} \\
y &= C_1 x^{1/2} \left[\cos\left(\frac{\sqrt{3}}{2} \ln x\right) + i \sin\left(\frac{\sqrt{3}}{2} \ln x\right) \right] \\
&\quad + C_2 x^{1/2} \left[\cos\left(\frac{\sqrt{3}}{2} \ln x\right) - i \sin\left(\frac{\sqrt{3}}{2} \ln x\right) \right]
\end{aligned}$$

Agora, escolha C_1 e C_2 conjugados complexos:

$$\begin{aligned}
C_1 &= \frac{1}{2} (D_1 - iD_2), \\
C_2 &= \frac{1}{2} (D_1 + iD_2),
\end{aligned}$$

$$\begin{aligned}
\frac{y}{x^{1/2}} &= \frac{1}{2} (D_1 - iD_2) [C + iS] + \frac{1}{2} (D_1 + iD_2) [C - iS] \\
&= \frac{1}{2} \{ D_1 C - i^2 D_2 S + i(-D_2 C + D_1 S) + D_1 C - i^2 D_2 S + i(D_2 C - D_1 S) \} \\
&= D_1 C + D_2 S; \\
y &= x^{1/2} \left[D_1 \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + D_2 \sin\left(\frac{\sqrt{3}}{2} \ln x\right) \right] \blacksquare
\end{aligned}$$

Desigualdades importantes com números complexos

Resultado auxiliar:

$$\begin{aligned}
 z_1^* z_2 + z_1 z_2^* &= (z_1^* z_2) + (z_1^* z_2)^* = 2 \operatorname{Re}(z_1^* z_2) \leq 2|z_1^* z_2| = 2|z_1^*||z_2| = 2|z_1||z_2| \\
 |x + iy|^2 &= x^2 + y^2 \\
 &= (x - iy)(x + iy) \\
 &= x^2 - (iy)^2 \\
 &= x^2 - i^2 y^2 \\
 &= x^2 + y^2 \blacksquare
 \end{aligned}$$

A desigualdade do triângulo, agora, é

$$\begin{aligned}
 |z_1 + z_2|^2 &= (z_1 + z_2)^*(z_1 + z_2) \\
 &= z_1^* z_1 + z_2^* z_2 + z_1^* z_2 + z_2^* z_1 \\
 &= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1^* z_2) \\
 &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
 &= (|z_1| + |z_2|)^2; \Rightarrow \\
 |z_1 + z_2| &\leq |z_1| + |z_2|
 \end{aligned}$$

Variáveis complexas têm mil (e uma) utilidades

$$\begin{aligned}
 I &= \int_0^\infty e^{-x} \cos(ax) \, dx; \\
 e^{iax} &= \cos(ax) + i \sin(ax); \\
 I &= \int_0^\infty e^{-x} \operatorname{Re}(e^{iax}) \, dx; \\
 I &= \operatorname{Re} \int_0^\infty e^{-x} e^{iax} \, dx; \\
 I &= \operatorname{Re} \int_0^\infty e^{(ia-1)x} \, dx; \\
 I &= \operatorname{Re} \frac{1}{ia-1} \int_0^\infty e^{(ia-1)x} \, d(ia-1)x; \\
 I &= \operatorname{Re} \frac{1}{ia-1} e^{(ia-1)x} \Big|_{x=0}^{x=\infty} \\
 I &= \operatorname{Re} \frac{1}{ia-1} [e^{(ia-1)\infty} - 1]
 \end{aligned}$$

Agora avalio o limite cuidadosamente:

$$\begin{aligned}
 \lim_{x \rightarrow \infty} e^{(ia-1)x} &= \lim_{x \rightarrow \infty} e^{-x} (\cos(ax) + i \sin(ax)) \\
 \lim_{x \rightarrow \infty} |e^{(ia-1)x}| &= \lim_{x \rightarrow \infty} |e^{-x}| |(\cos(ax) + i \sin(ax))|; \\
 |\cos(ax) + i \sin(ax)| &= 1 \\
 |x + iy| &= \sqrt{x^2 + y^2} \\
 \lim_{x \rightarrow \infty} |e^{(ia-1)x}| &= \lim_{x \rightarrow \infty} |e^{-x}| = 0; \\
 \lim_{x \rightarrow \infty} e^{(ia-1)x} &= 0 \blacksquare
 \end{aligned}$$

Portanto,

$$\begin{aligned}
 I &= \operatorname{Re} \frac{1}{ia - 1} [-1] \\
 &= \operatorname{Re} \frac{1}{1 - ia} \\
 &= \operatorname{Re} \frac{1 + ia}{(1 - ia)(1 + ia)} \\
 &= \operatorname{Re} \frac{1 + ia}{1 + a^2} = \frac{1}{1 + a^2} \blacksquare
 \end{aligned}$$

$$\begin{aligned}
 \cos(ix) &= \frac{e^{ix} + e^{-ix}}{2} \\
 &= \frac{\cos ix + i \operatorname{sen} ix + \cos ix - i \operatorname{sen} ix}{2} \\
 &= \cos ix.
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{sen}(ix) &= \frac{e^{ix} - e^{-ix}}{2i} \\
 &= \frac{\cos ix + i \operatorname{sen} ix - (\cos ix - i \operatorname{sen} ix)}{2i}
 \end{aligned}$$

$$\begin{aligned}
 \cos^2(ix) + \operatorname{sen}^2(ix) &= \frac{e^{2ix} + 2 + e^{-2ix}}{4} + \frac{e^{2ix} - 2 + e^{-2ix}}{-4} \\
 &= \frac{e^{2ix} + 2 + e^{-2ix}}{4} + \frac{-e^{2ix} + 2 - e^{-2ix}}{4} \\
 &= \frac{4}{4} = 1 \blacksquare
 \end{aligned}$$

Variáveis complexas:

$$\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i.$$

Uma integral no plano complexo:

$$\begin{aligned}
 \zeta - z &= Re^{i\theta}, \\
 d\zeta &= iRe^{i\theta} d\theta
 \end{aligned}$$

Encontrando coeficientes de Laurent:

$$\begin{aligned}
 f(z) &= \frac{1}{(z - i)^3(z + i)^2} \\
 f(z) &= \sum_{n=-\infty}^{+\infty} c_n(z - i)^n = \sum_{n=-3}^{+\infty} c_n(z - i)^n
 \end{aligned}$$

“Bem perto” de $z = i$:

$$\begin{aligned}
 f(z) &\sim \frac{1}{(z - i)^3(2i)^2} = -\frac{1}{4} \frac{1}{(z - i)^3}; \\
 f(z) &= \frac{1}{(z - i)^3(z + i)^2} = c_{-3}(z - i)^{-3} + c_{-2}(z - i)^{-2} + c_{-1}(z - i)^{-1} + \dots; \\
 (z - i)^3 f(z) &= \frac{1}{(z + i)^2} = c_{-3} + c_{-2}(z - i) + c_{-1}(z - i)^2 + \dots;
 \end{aligned}$$

Exemplo 9.9

$$f(z) = \frac{z}{(z^2 + 1)^2}$$

Ache os polos:

$$\begin{aligned} z^2 + 1 &= 0, \\ z^2 &= -1, \\ z &= \pm i. \end{aligned}$$

Tente colocar $\pm i$ explicitamente na fórmula da função:

$$\begin{aligned} z^2 + 1 &= z^2 - (-1) \\ &= z^2 - (i^2) \\ &= (z + i)(z - i) \\ &= z^2 - (i)^2 \\ &= z^2 - (-1) \end{aligned}$$

Reescrevo agora a função:

$$\begin{aligned} f(z) &= \frac{z}{[(z - i)(z + i)]^2} \\ &= \frac{z}{(z - i)^2(z + i)^2} \end{aligned}$$

Exemplo 9.10:

$$f(z) = \frac{1}{z + i}$$

É importante nos lembrarmos da série geométrica:

$$\begin{aligned} \frac{1}{1 - x} &= 1 + x + x^2 + x^3 + \dots; & |x| < 1 \\ \frac{1}{1 + x} &= 1 - x + x^2 - x^3 + \dots; & |x| < 1 \end{aligned}$$

Dentro do círculo unitário, $|z/i| < 1$:

$$\begin{aligned} \frac{1}{z + i} &= \frac{1}{i \left(\frac{z}{i} + 1 \right)} \\ &= \frac{1}{i \left(1 + \frac{z}{i} \right)} \\ &= \frac{1}{i (1 - (iz))} \\ &= \frac{1}{i} \left[1 + (iz) + (iz)^2 + (iz)^3 + \dots \right] \\ &= -i \left[1 + iz - z^2 - iz^3 + \dots \right] \\ &= -i + z + iz^2 - z^3 + \dots \end{aligned}$$

Procuramos agora a série de Laurent de $f(z)$ **fora** do círculo unitário; nessa região, $|i/z| < 1$.

$$\begin{aligned}\frac{1}{z+i} &= \frac{1}{z\left(1+\frac{i}{z}\right)} \\ &= \frac{1}{z} \left[1 - \frac{i}{z} + \left(\frac{i}{z}\right)^2 - \left(\frac{i}{z}\right)^3 + \dots \right] \\ &= \frac{1}{z} - \frac{i}{z^2} - \frac{1}{z^3} + \frac{i}{z^4} + \dots\end{aligned}$$

Próximo exemplo!

$$\begin{aligned}f(z) &= \frac{1}{(z-2)(z-3)}; \\ |z| < 3 &\Rightarrow \left|\frac{z}{3}\right| < 1; \\ |z| > 2 &\Rightarrow \frac{|z|}{|z|} > \frac{2}{|z|} \Rightarrow \left|\frac{2}{z}\right| < 1. \\ \frac{1}{(z-2)(z-3)} &= \frac{A}{z-2} + \frac{B}{z-3} \\ \frac{1}{(z-2)(z-3)} &= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)} \\ &= \frac{(A+B)z - 3A - 2B}{(z-2)(z-3)} \\ A+B &= 0, \\ -3A-2B &= 1, \\ \frac{1}{(z-2)(z-3)} &= \frac{1}{z-3} - \frac{1}{z-2}; \\ \frac{1}{z-3} &= \frac{1}{3\left(\frac{z}{3}-1\right)} = \frac{-1/3}{1-\frac{z}{3}} \\ &= -\frac{1}{3} \left[1 + (z/3) + (z/3)^2 + (z/3)^3 + \dots \right] \\ \frac{1}{z-2} &= \frac{1}{z\left(1-\frac{2}{z}\right)} \\ &= \frac{1}{z} \left[1 + (2/z) + (2/z)^2 + (2/z)^3 \right]\end{aligned}$$

Integração de Contorno e o Teorema dos Resíduos:

$$\int_{\mathcal{L}} f(z) \, dz$$

$$e^{i\theta} = \cos(\theta) + i \operatorname{sen}(\theta);$$

$$|e^{i\theta}| = \sqrt{\cos^2(\theta) + \operatorname{sen}^2(\theta)} = 1.$$

$$\frac{a}{|1+b|} \leq ?$$

$$|b| \leq |1+b|; \Rightarrow$$

$$\frac{a}{|1+b|} \leq \frac{a}{|b|}?$$

$$\begin{aligned} \int_{\theta=0}^{\pi} \frac{R}{|R^2 e^{2i\theta}|} \, d\theta &= \int_{\theta=0}^{\pi} \frac{R}{|R^2| |e^{2i\theta}|} \, d\theta \\ &= \int_{\theta=0}^{\pi} \frac{R}{|R^2|} \, d\theta \\ &= \int_{\theta=0}^{\pi} \frac{1}{R} \, d\theta \\ &= \frac{1}{R} \int_{\theta=0}^{\pi} d\theta = \frac{\pi}{R} \end{aligned}$$

Como achar as raízes de $z^4 = -1$?

$$z = r e^{i\theta};$$

$$z^4 = r^4 e^{4i\theta_k} = -1 = 1 e^{i(\pi+2(k-1)\pi)}$$

$$r = 1,$$

$$\theta_k = \frac{\pi}{4} + \frac{(k-1)\pi}{2}$$

Equações de ordem 2 que eu sei resolver:

$$ay'' + by' + cy = f(x),$$

$$ax^2y'' + bxy' + cy = f(x)$$

Eu gostaria de saber resolver equações diferenciais lineares de ordem 2 mais gerais, de coeficientes não constantes:

$$y'' + p(x)y' + q(x)y = f(x),$$

Considere a equação diferencial

$$y' + y = 0;$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$y'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\sum_{n=0}^{\infty} [a_n + (n+1)a_{n+1}] x^n = 0$$

Método de Frobenius:

$$\begin{aligned}y'' + p(x)y' + q(x)y &= 0; \\x^2y'' + x[xp(x)]y' + [x^2q(x)]y &= 0;\end{aligned}$$

Um exemplo de singularidade fora da origem:

$$\begin{aligned}2(\xi - 1)\frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} + y &= 0 \\ \frac{d^2y}{d\xi^2} + \frac{1}{2(\xi - 1)}\frac{dy}{d\xi} + \frac{1}{2(\xi - 1)}y &= 0 \\ x &= \xi - 1; \\ \frac{dy}{d\xi} &= \frac{dy}{dx} \frac{dx}{d\xi}; \\ \frac{dy}{d\xi} &= \frac{dy}{dx}; \\ \frac{d^2y}{d\xi^2} &= \frac{d}{dx} \left[\frac{dy}{dx} \right] \frac{dx}{d\xi}; \\ \frac{d^2y}{d\xi^2} &= \frac{d^2y}{dx^2} \\ 2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y &= 0\end{aligned}$$

O método de Frobenius

Caso (i) $r_1 \neq r_2$, e $r_1 - r_2 \notin \mathbb{Z}$.

$$\begin{aligned}2xy'' + y' + y &= 0, \\ y'' + \frac{1}{2x}y' + \frac{1}{2x}y &= 0, \\ xp(x) &= \frac{1}{2}, \\ x^2q(x) &= \frac{x}{2}.\end{aligned}$$

Portanto, $x = 0$ é um ponto singular regular, e o método de Frobenius aplica-se, ou seja: podemos encontrar uma solução em série em torno de $x = 0$.

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^{r+n}, \\ y' &= \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \\ y'' &= \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}.\end{aligned}$$

$$\begin{aligned}
2x \left[\sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2} \right] + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0, \\
\left[\sum_{n=0}^{\infty} 2(r+n-1)(r+n)a_n x^{r+n-1} \right] + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0 \\
\sum_{n=0}^{\infty} [2(r+n-1)(r+n) + (r+n)] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0
\end{aligned}$$

$$\begin{aligned}
r+n-1 &= r+m, \\
n-1 &= m, \\
n &= m+1.
\end{aligned}$$

$$\begin{aligned}
\sum_{m=-1}^{\infty} [2(r+m)(r+m+1) + (r+m+1)] a_{m+1} x^{r+m} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0 \\
\sum_{n=-1}^{\infty} [2(r+n)(r+n+1) + (r+n+1)] a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0 \\
[2(r-1)r+r]a_0 x^{r-1} + \sum_{n=0}^{\infty} [2(r+n)(r+n+1) + (r+n+1)] a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0 \\
[2(r-1)r+r]a_0 x^{r-1} + \sum_{n=0}^{\infty} [(r+n+1)(2(r+n)+1)] a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} &= 0 \\
[2(r-1)r+r]a_0 x^{r-1} + \sum_{n=0}^{\infty} \{(r+n+1)(2(r+n)+1)a_{n+1} + a_n\} x^{r+n} &= 0
\end{aligned}$$

A equação indicial é

$$\begin{aligned}
2(r-1)r+r &= 0, \\
2(r^2-r)+r &= 0, \\
2r^2-2r+r &= 0, \\
2r^2-r &= 0, \\
r(2r-1) &= 0, \\
r_2 &= 0, \\
r_1 &= \frac{1}{2}.
\end{aligned}$$

A relação de recorrência será

$$\begin{aligned}
(r+n+1)(2(r+n)+1)a_{n+1}+a_n &= 0, \\
(r+n+1)(2(r+n)+1)a_{n+1} &= -a_n, \\
a_{n+1} &= -\frac{a_n}{(r+n+1)(2(r+n)+1)}, \\
r_1 &= 1/2, \\
a_{n+1} &= -\frac{a_n}{(n+3/2)(2(1/2+n)+1)}, \\
a_{n+1} &= -\frac{a_n}{(n+3/2)(2(1+n))}, \\
a_{n+1} &= -\frac{a_n}{(2n+3)(1+n)}, \\
b_{n+1} &= -\frac{b_n}{(r+n+1)(2(r+n)+1)}, \\
r_2 &= 0, \\
b_{n+1} &= -\frac{b_n}{(n+1)(2n+1)},
\end{aligned}$$

Agora eu calculo alguns dos termos de cada série:

$$\begin{aligned}
a_0, \\
a_1 &= -\frac{a_0}{3}, \\
a_2 &= -\frac{a_1}{10} = +\frac{a_0}{30}, \\
a_3 &= -\frac{a_2}{21} = -\frac{a_0}{630}, \\
y_1(x) &= x^{1/2} \left[1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \dots \right]
\end{aligned}$$

$$\begin{aligned}
b_0, \\
b_1 &= -b_0, \\
b_2 &= -\frac{b_1}{6} = \frac{b_0}{6}, \\
b_3 &= -\frac{b_2}{15} = -\frac{b_0}{90}, \\
b_4 &= -\frac{b_3}{28} = \frac{b_0}{2520}, \\
y_2(x) &= \left[1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \dots \right]
\end{aligned}$$

A solução geral é da forma

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Caso (ii): $r_1 = r_2$

$$x^2 y'' - (x + x^2) y' + y = 0,$$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n},$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1},$$

$$y'' = \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}$$

$$x^2 \left[\sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2} \right] - (x + x^2) \left[\sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \right] + \sum_{n=0}^{\infty} a_n x^{r+n} = 0,$$

$$\sum_{n=0}^{\infty} [(r+n-1)(r+n) - (r+n) + 1] a_n x^{r+n} - \sum_{n=0}^{\infty} (r+n) a_n x^{r+n+1} = 0,$$

$$r+m = r+n+1,$$

$$m = n+1,$$

$$n = m-1$$

$$\sum_{n=0}^{\infty} [(r+n-1)(r+n) - (r+n) + 1] a_n x^{r+n} - \sum_{m=1}^{\infty} (r+m-1) a_{m-1} x^{r+m} = 0,$$

$$\sum_{n=0}^{\infty} [(r+n-1)(r+n) - (r+n) + 1] a_n x^{r+n} - \sum_{n=1}^{\infty} (r+n-1) a_{n-1} x^{r+n} = 0,$$

$$[(r-1)r - r + 1] a_0 x^r + \sum_{n=1}^{\infty} \{[(r+n-1)(r+n) - (r+n) + 1] a_n - (r+n-1) a_{n-1}\} x^{r+n} = 0,$$

A equação indicial será

$$(r-1)r - r + 1 = 0,$$

$$r^2 - r - r + 1 = 0,$$

$$r^2 - 2r + 1 = 0,$$

$$(r-1)^2 = 0.$$

Portanto, $r = 1$ é uma raiz dupla.

Preciso inicialmente encontrar uma solução. A relação de recorrência é

$$\begin{aligned}
 [(r+n-1)(r+n) - (r+n) + 1] a_n - (r+n-1)a_{n-1} &= 0 \\
 [(r+n-1)(r+n) - (r+n) + 1] a_n &= (r+n-1)a_{n-1} \\
 a_n &= \frac{(r+n-1)a_{n-1}}{(r+n-1)(r+n) - (r+n) + 1} \\
 a_n &= \frac{na_{n-1}}{n(1+n) - (1+n) + 1} \\
 a_n &= \frac{na_{n-1}}{n + n^2 - 1 - n + 1} \\
 a_n &= \frac{na_{n-1}}{n^2} \\
 a_n &= \frac{a_{n-1}}{n}.
 \end{aligned}$$

$$\begin{aligned}
 a_0 &= 1 = \frac{1}{0!}, \\
 a_1 &= \frac{1}{1} = 1 = \frac{1}{1!}, \\
 a_2 &= \frac{1}{2} = \frac{1}{2!} \\
 a_3 &= \frac{1/2}{3} = \frac{1}{6} = \frac{1}{3!}, \\
 a_4 &= \frac{1/6}{4} = \frac{1}{24} = \frac{1}{4!}, \\
 a_n &= \frac{1}{n!}
 \end{aligned}$$

A 1ª solução será

$$\begin{aligned}
 y_1(x) &= x \left[\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right] \\
 &= xe^x \blacksquare
 \end{aligned}$$

A segunda solução, LI da primeira, deve ser procurada na forma

$$\begin{aligned}
 y_2 &= y_1 \ln(x) + \sum_{n=1}^{\infty} c_n x^{n+1}, \\
 y_2' &= y_1' \ln(x) + y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n, \\
 y_2'' &= y_1'' \ln(x) + y_1' \frac{1}{x} + y_1' \frac{1}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \\
 &= y_1'' \ln(x) + \frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1}
 \end{aligned}$$

A equação diferencial é

$$x^2 y'' - (x + x^2) y' + y = 0.$$

Substituindo,

$$x^2 \left[y_1'' \ln(x) + \frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] - \\ (x+x^2) \left[y_1' \ln(x) + y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n \right] + y_1 \ln(x) + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$\ln(x) \underbrace{\left[x^2 y_1'' - (x+x^2)y_1' + y_1 \right]}_{=0} + x^2 \left[\frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] - \\ (x+x^2) \left[y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n \right] + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$x^2 \left[\frac{2y_1'}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] - \\ (x+x^2) \left[y_1 \frac{1}{x} + \sum_{n=1}^{\infty} (n+1)c_n x^n \right] + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2xy_1' - y_1 - y_1 - xy_1 + x^2 \left[\sum_{n=1}^{\infty} n(n+1)c_n x^{n-1} \right] - \\ (x+x^2) \left[\sum_{n=1}^{\infty} (n+1)c_n x^n \right] + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2xy_1' - 2y_1 - xy_1 + \\ \sum_{n=1}^{\infty} n(n+1)c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2xy_1' - 2y_1 - xy_1 + \sum_{n=1}^{\infty} [n(n+1) - (n+1) + 1] c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} = 0$$

$$2xy_1' - 2y_1 - xy_1 + \sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} = 0, \\ \sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} (n+1)c_n x^{n+2} + 2xy_1' - (2+x)y_1 = 0$$

$$m+1 = n+2, \\ m = n+1, \\ n = m-1$$

$$\sum_{n=1}^{\infty} (n+1)c_n x^{n+2} = \sum_{m=2}^{\infty} m c_{m-1} x^{m+1} = \sum_{n=2}^{\infty} n c_{n-1} x^{n+1}.$$

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=2}^{\infty} n c_{n-1} x^{n+1} + 2x y_1' - (2+x)y_1 = 0,$$

Seja criativo e faça $c_0 = 0$:

$$\sum_{n=1}^{\infty} n^2 c_n x^{n+1} - \sum_{n=1}^{\infty} n c_{n-1} x^{n+1} + 2x y_1' - (2+x)y_1 = 0,$$

$$\sum_{n=1}^{\infty} [n^2 c_n - n c_{n-1}] x^{n+1} + 2x y_1' - (2+x)y_1 = 0,$$

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!},$$

$$y_1' = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!},$$

$$\sum_{n=1}^{\infty} [n^2 c_n - n c_{n-1}] x^{n+1} + 2x \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} - (2+x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = 0,$$

$$\sum_{n=1}^{\infty} [n^2 c_n - n c_{n-1}] x^{n+1} = (2+x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - 2x \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Concentremo-nos nos somatórios!

$$(2+x) \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - 2x \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!} = \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty} \frac{2nx^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty} \frac{2nx^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=1}^{\infty} \frac{2nx^{n+1}}{n!}$$

$$n! = n \times \underbrace{n-1 \times n-2 \times \dots 1}_{=(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!}$$

$$n+2 = m+1,$$

$$n+1 = m,$$

$$n = m-1,$$

$$= \sum_{m=1}^{\infty} \frac{x^{m+1}}{(m-1)!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} - \sum_{n=1}^{\infty} \frac{2x^{n+1}}{(n-1)!}$$

$$= - \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} [n^2 c_n - n c_{n-1}] x^{n+1} &= - \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} \\
n^2 c_n - n c_{n-1} &= - \frac{1}{(n-1)!} \\
c_n - \frac{n}{n^2} c_{n-1} &= - \frac{1}{n^2 (n-1)!} \\
c_n &= \frac{1}{n^2} \left[n c_{n-1} - \frac{1}{(n-1)!} \right] \\
&= \frac{1}{n^2} \left[\frac{n(n-1)! c_{n-1} - 1}{(n-1)!} \right] \\
&= \frac{1}{n^2} \left[\frac{n! c_{n-1} - 1}{(n-1)!} \right] \\
&= \frac{1}{n} \left[\frac{n! c_{n-1} - 1}{n(n-1)!} \right] \\
&= \frac{1}{n} \left[\frac{n! c_{n-1} - 1}{n!} \right] \\
&= \frac{1}{n} \left[c_{n-1} - \frac{1}{n!} \right]
\end{aligned}$$

Caso (iii)-1: $r_1 - r_2 \in \mathbb{Z}$, e a menor raiz leva a duas soluções.

$$x^2 y'' + x y' + (x^2 - 1/4) y = 0.$$

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{r+n}, \\
y' &= \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \\
y'' &= \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2}.
\end{aligned}$$

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n-2} + x \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} + \left(x^2 - \frac{1}{4}\right) \sum_{n=0}^{\infty} a_n x^{r+n} &= 0, \\
\sum_{n=0}^{\infty} (r+n-1)(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n x^{r+n} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} &= 0, \\
\sum_{n=0}^{\infty} \left[(r+n-1)(r+n) + (r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} &= 0, \\
\sum_{n=0}^{\infty} \left[(r+n-1+1)(r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} &= 0, \\
\sum_{n=0}^{\infty} \left[(r+n-1+1)(r+n) - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} &= 0, \\
\sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} &= 0,
\end{aligned}$$

$$r+n+2 = r+m,$$

$$n+2 = m,$$

$$n = m-2$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{1}{4} \right] a_n x^{r+n} + \sum_{m=2}^{\infty} a_{m-2} x^{r+m} &= 0, \\
\sum_{n=0}^{\infty} \left[(r+n)^2 - \frac{1}{4} \right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} &= 0.
\end{aligned}$$