TEA010 Matemática Aplicada I
Curso de Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
P03B, 16 Jul 2021
Entrega em 17 Jul 2021, 09:30.
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ATENÇÃO: PROVA SEM CONSULTA, E SEM USO DE CALCULADORAS, ETC..
Declaro que segui o código de ética do Curso de Engenharia Ambiental ao realizar esta prova

NOME: Assinatura: ______________________

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1 [25] Seja

$$f(x,y,z) = z^2 + e^{-y}\cos(x).$$

Calcule a derivada direcional $\frac{df}{ds}$ ao longo da curva

$$\begin{aligned} x &= s,\\ y &= 2s+1,\\ z &= \operatorname{sen}(s), \end{aligned}$$

no ponto $(1, 3, \operatorname{sen}(1))$.

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned} \frac{df}{ds} &= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}\\ &= \boldsymbol{\nabla} f \cdot \frac{d\boldsymbol{r}}{ds}, \qquad \boldsymbol{r} = (x,y,z), \end{aligned}$$

com

$$\begin{aligned} \boldsymbol{\nabla} f &= (-e^{-y}\operatorname{sen}(x), -e^{-y}\cos(x), 2z);\\ \frac{d\boldsymbol{r}}{ds} &= (1, 2, \cos(s)). \end{aligned}$$

Em $P = (1, 3, \operatorname{sen}(1))$,

$$\begin{aligned} \boldsymbol{\nabla} f &= (-e^{-3}\operatorname{sen}(1), -e^{-3}\cos(1), 2\operatorname{sen}(1));\\ \frac{d\boldsymbol{r}}{ds} &= (1, 2, \cos(1));\\ \frac{df}{ds} &= -e^{-3}\operatorname{sen}(1) - 2e^{-3}\cos(1) + 2\operatorname{sen}(1)\cos(1) \blacksquare \end{aligned}$$

Continue a solução no verso ⟹

2 [25] Em coordenadas cartesianas, se $u(x,y)$ é uma função no $\mathbb{R}^2$,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Obtenha $\nabla^2 u$ em coordenadas polares:

$$\begin{aligned} x &= r\cos(\theta), \\ y &= r\,\mathrm{sen}(\theta). \end{aligned}$$

SOLUÇÃO DA QUESTÃO:
Este é um exercício de aplicação sistemática (e cuidadosa) da regra da cadeia. Primeiro, obtemos r, θ em função de x, y:

$$\begin{aligned} r &= (x^2+y^2)^{1/2}, \\ \theta &= \operatorname{arctg}\left(\frac{y}{x}\right). \end{aligned}$$

Em seguida, todas as derivadas possíveis são (omitindo os detalhes):

$$\begin{aligned} \frac{\partial r}{\partial x} &= \cos(\theta), \\ \frac{\partial r}{\partial y} &= \mathrm{sen}(\theta), \\ \frac{\partial \theta}{\partial x} &= -\frac{\mathrm{sen}(\theta)}{r}, \\ \frac{\partial \theta}{\partial y} &= +\frac{\cos(\theta)}{r}. \end{aligned}$$

Agora,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\ &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right] \\ &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta}\frac{\mathrm{sen}(\theta)}{r}\right] \\ &= \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta}\frac{\mathrm{sen}(\theta)}{r}\right]\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta}\frac{\mathrm{sen}(\theta)}{r}\right]\frac{\partial \theta}{\partial x} \\ &= \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta}\frac{\mathrm{sen}(\theta)}{r}\right]\cos(\theta) - \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\cos(\theta) - \frac{\partial u}{\partial \theta}\frac{\mathrm{sen}(\theta)}{r}\right]\frac{\mathrm{sen}(\theta)}{r} \\ &= \frac{\partial^2 u}{\partial r^2}\cos^2(\theta) + \frac{\mathrm{sen}(\theta)\cos(\theta)}{r^2}\frac{\partial u}{\partial \theta} - \frac{\mathrm{sen}(\theta)\cos(\theta)}{r}\frac{\partial^2 u}{\partial r \partial \theta} \\ &\quad - \frac{\mathrm{sen}(\theta)\cos(\theta)}{r}\frac{\partial^2 u}{\partial \theta \partial r} + \frac{\mathrm{sen}^2(\theta)}{r}\frac{\partial u}{\partial r} + \frac{\mathrm{sen}^2\,\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\mathrm{sen}(\theta)\cos(\theta)}{r^2}\frac{\partial u}{\partial \theta}. \end{aligned}$$

e

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\ &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}\right] \\ &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\mathrm{sen}(\theta) + \frac{\partial u}{\partial \theta}\frac{\cos(\theta)}{r}\right] \\ &= \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\mathrm{sen}(\theta) + \frac{\partial u}{\partial \theta}\frac{\cos(\theta)}{r}\right]\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\mathrm{sen}(\theta) + \frac{\partial u}{\partial \theta}\frac{\cos(\theta)}{r}\right]\frac{\partial \theta}{\partial y} \\ &= \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\mathrm{sen}(\theta) + \frac{\partial u}{\partial \theta}\frac{\cos(\theta)}{r}\right]\mathrm{sen}(\theta) + \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\mathrm{sen}(\theta) + \frac{\partial u}{\partial \theta}\frac{\cos(\theta)}{r}\right]\frac{\cos(\theta)}{r} \\ &= \frac{\partial^2 u}{\partial r^2}\mathrm{sen}^2(\theta) - \frac{\mathrm{sen}(\theta)\cos(\theta)}{r^2}\frac{\partial u}{\partial \theta} + \frac{\mathrm{sen}(\theta)\cos(\theta)}{r}\frac{\partial^2 u}{\partial r \partial \theta} \\ &\quad + \frac{\mathrm{sen}(\theta)\cos(\theta)}{r}\frac{\partial^2 u}{\partial \theta \partial r} + \frac{\cos^2(\theta)}{r}\frac{\partial u}{\partial r} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\mathrm{sen}(\theta)\cos(\theta)}{r^2}\frac{\partial u}{\partial \theta}. \end{aligned}$$

Continue a solução no verso ⟹

Somando,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \blacksquare$$

Continue a solução no verso $\Longrightarrow$

3 [25] Calcule $\boldsymbol{\nabla}\cdot[\boldsymbol{u}\times\boldsymbol{v}]$. Use notação indicial e a base canônica para os cálculos, **mas exprima o resultado final vetorialmente**, utilizando o operador $\boldsymbol{\nabla}$.

SOLUÇÃO DA QUESTÃO:

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot[\boldsymbol{u}\times\boldsymbol{v}] &= \boldsymbol{e}_l\frac{\partial}{\partial x_l}\cdot\epsilon_{ijk}u_iv_j\boldsymbol{e}_k\\
&= \epsilon_{ijk}\frac{\partial(u_iv_j)}{\partial x_l}(\boldsymbol{e}_l\cdot\boldsymbol{e}_k)\\
&= \epsilon_{ijk}\frac{\partial(u_iv_j)}{\partial x_l}\delta_{lk}\\
&= \epsilon_{ijk}\frac{\partial(u_iv_j)}{\partial x_k}\\
&= \epsilon_{ijk}u_i\frac{\partial v_j}{\partial x_k}+\epsilon_{ijk}v_j\frac{\partial u_i}{\partial x_k}\\
&= \epsilon_{kij}u_i\frac{\partial v_j}{\partial x_k}+\epsilon_{kij}v_j\frac{\partial u_i}{\partial x_k}\\
&= -\epsilon_{kji}u_i\frac{\partial v_j}{\partial x_k}+\epsilon_{kij}v_j\frac{\partial u_i}{\partial x_k}\\
&= -u_i\epsilon_{kji}\frac{\partial v_j}{\partial x_k}+v_j\epsilon_{kij}\frac{\partial u_i}{\partial x_k}\\
&= -\boldsymbol{u}\cdot\boldsymbol{\nabla}\times\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}\ \blacksquare
\end{aligned}
$$

Continue a solução no verso $\Longrightarrow$

4 [25] Calcule a área da superfície

$$g(x,y) = \sqrt{3}x + y^2$$

no domínio $0 \le x \le 1$, $0 \le y \le 1$, sabendo que

$$\int_0^1 \sqrt{1+u^2}\,\mathrm{d}u = \frac{\operatorname{arcsenh}(1) + \sqrt{2}}{2}.$$

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
A &= \int_{R_{xy}} \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y;\\
\frac{\partial g}{\partial x} &= \sqrt{3},\\
\frac{\partial g}{\partial y} &= 2y,\\
A &= \int_{y=0}^1 \int_{x=0}^1 \sqrt{1+3+4y^2}\,\mathrm{d}x\mathrm{d}y\\
&= \int_{y=0}^1 \sqrt{1+3+4y^2}\left[\int_{x=0}^1 \mathrm{d}x\right]\mathrm{d}y\\
&= \int_{y=0}^1 \sqrt{1+3+4y^2}\,\mathrm{d}y\\
&= \int_{y=0}^1 \sqrt{4+4y^2}\,\mathrm{d}y\\
&= 2\int_{y=0}^1 \sqrt{1+y^2}\,\mathrm{d}y\\
&= \operatorname{arcsenh}(1) + \sqrt{2} \;\blacksquare
\end{aligned}$$

Continue a solução no verso ⟹