TEA013 Matemática Aplicada II
Curso de Engenharia Ambiental
Departamento de Engenharia Ambiental, UFPR
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Declaro que segui o código de ética do Curso de Engenharia Ambiental ao realizar esta prova

NOME: GABARITO Assinatura: ____________________

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1 [20] Dada a equação diferencial

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} - k\phi,$$

onde $D > 0$ e $k > 0$, a sua discretização com um esquema de diferenças finitas totalmente implícito, progressivo no tempo e centrado no espaço, produz uma equação geral do tipo

$$A\phi_{i-1}^{n+1} + B\phi_i^{n+1} + C\phi_{i+1}^{n+1} = \phi_i^n,$$

onde como sempre ϕ_i^n é a aproximação em grade de $\phi(i\Delta x, n\Delta t)$. Obtenha A, B e C em função dos parâmetros adimensionais

$$\begin{aligned}\mathrm{Fo} &= \frac{D\Delta t}{\Delta x^2},\\ \mathrm{Kt} &= k\Delta t.\end{aligned}$$

SOLUÇÃO DA QUESTÃO

$$\begin{aligned}
\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} &= D\frac{\phi_{i+1}^{n+1} - 2\phi_i^n + \phi_{i-1}^{n+1}}{\Delta x^2} - k\phi_i^n\\
\phi_i^{n+1} - \phi_i^n &= \frac{D\delta t}{\Delta x^2}\left(\phi_{i+1}^{n+1} - 2\phi_i^n + \phi_{i-1}^{n+1}\right) - (k\Delta t)\phi_i^{n+1}\\
\phi_i^{n+1} - \phi_i^n &= \mathrm{Fo}\left(\phi_{i+1}^{n+1} - 2\phi_i^n + \phi_{i-1}^{n+1}\right) - \mathrm{Kt}\phi_i^{n+1}\\
\phi_i^{n+1} - \mathrm{Fo}\left(\phi_{i+1}^{n+1} - 2\phi_i^n + \phi_{i-1}^{n+1}\right) + \mathrm{Kt}\phi_i^{n+1} &= \phi_i^n\\
-\mathrm{Fo}\phi_{i-1}^{n+1} + (1 + 2\mathrm{Fo} + \mathrm{Kt})\phi_i^{n+1} - \mathrm{Fo}\phi_{i+1}^{n+1} &= \phi_i^n \;\Rightarrow\\
A &= -\mathrm{Fo},\\
B &= (1 + 2\mathrm{Fo} + \mathrm{Kt}),\\
C &= -\mathrm{Fo}\ \blacksquare
\end{aligned}$$

Continue a solução no verso ⟹

2 [20] Dada a matriz

$$[A] = \begin{bmatrix} 2 & 1+\mathrm{i} \\ 1-\mathrm{i} & 2 \end{bmatrix},$$

a) [5] Sem fazer nenhum cálculo, o que você pode dizer sobre seus autovalores e autovetores?

b) [5] Calcule os autovalores. Confirme sua resposta sobre (a).

c) [10] Calcule os autovetores. Confirme, com cálculos, sua resposta sobre (b).

SOLUÇÃO DA QUESTÃO

a) $[A]$ é auto-adjunta; logo, os autovalores são reais, e os autovetores são ortogonais.

b) e c) Com Maxima,

```
(%i1) A : matrix([2,1+%i],[1-%i,2]);
                              [   2     %i + 1 ]
(%o1)                         [                ]
                              [ 1 - %i    2    ]
(%i2) eigenvectors(A);
(%o2) [[[2 - sqrt(2), sqrt(2) + 2], [1, 1]],
                    sqrt(2) %i - sqrt(2)           sqrt(2) %i - sqrt(2)
              [[[1, --------------------]], [[1, - --------------------]]]]
                             2                              2
```

b) Os autovalores são

$$\lambda_{\mathrm{i}} = 2 - \sqrt{2},$$
$$\lambda_{\mathrm{ii}} = 2 + \sqrt{2}.$$

c) Os autovetores são

$$\boldsymbol{v}^{\mathrm{i}} = \left(1, \frac{-\sqrt{2} + \mathrm{i}\sqrt{2}}{2}\right),$$
$$\boldsymbol{v}^{\mathrm{ii}} = \left(1, \frac{\sqrt{2} - \mathrm{i}\sqrt{2}}{2}\right).$$

Os autovalores são reais, de fato. Os autovetores são ortogonais:

$$\begin{aligned}
\left\langle \boldsymbol{v}^{\mathrm{i}}, \boldsymbol{v}^{\mathrm{ii}} \right\rangle &= [\boldsymbol{v}^{\mathrm{i}}_1]^* \boldsymbol{v}^{\mathrm{ii}}_1 + [\boldsymbol{v}^{\mathrm{i}}_2]^* \boldsymbol{v}^{\mathrm{ii}}_2 \\
&= 1 \times 1 + \left(\frac{-\sqrt{2} + \mathrm{i}\sqrt{2}}{2}\right)^* \times \left(\frac{\sqrt{2} - \mathrm{i}\sqrt{2}}{2}\right) \\
&= 1 + \left(\frac{-\sqrt{2} - \mathrm{i}\sqrt{2}}{2}\right)^* \left(\frac{\sqrt{2} - \mathrm{i}\sqrt{2}}{2}\right) \\
&= 1 + \frac{-2 + 2\mathrm{i} - 2\mathrm{i} + 2\mathrm{i}^2}{4} \\
&= 1 - \frac{4}{4} = 0 \blacksquare
\end{aligned}$$

Continue a solução no verso ⟹

3 [20] Encontre a função de Green da equação diferencial ordinária

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} + \operatorname{sen}(x)\phi(x) = f(x); \qquad \phi(0) = \phi_0.$$

SOLUÇÃO DA QUESTÃO:

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \operatorname{sen}(\xi)\phi(\xi) &= f(\xi)\\
G(x,\xi)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + G(x,\xi)\operatorname{sen}(\xi)\phi(\xi) &= G(x,\xi)f(\xi)\\
\int_0^\infty G(x,\xi)\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\,\mathrm{d}\xi + \int_0^\infty G(x,\xi)\operatorname{sen}(\xi)\phi(\xi)\,\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi\\
G(x,\xi)\phi(\xi)\Big|_0^\infty - \int_0^\infty \phi(\xi)\frac{\mathrm{d}G}{\mathrm{d}\xi}\,\mathrm{d}\xi + \int_0^\infty G(x,\xi)\operatorname{sen}(\xi)\phi(\xi)\,\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi\\
\underbrace{G(x,\infty)}_{=0}\phi(\infty) - G(x,0)\phi_0 + \int_0^\infty \phi(\xi)\underbrace{\left[-\frac{\mathrm{d}G}{\mathrm{d}\xi} + G(x,\xi)\operatorname{sen}(\xi)\right]}_{=\delta(\xi-x)}\mathrm{d}\xi &= \int_0^\infty G(x,\xi)f(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Resolvamos, portanto

$$-\frac{\mathrm{d}G}{\mathrm{d}\xi} + G(x,\xi)\operatorname{sen}(\xi) = \delta(\xi - x); \qquad G(x,\infty) = 0:$$

$$\begin{aligned}
G(x,\xi) &= u(x,\xi)v(x,\xi);\\
-\left[u\frac{\mathrm{d}v}{\mathrm{d}\xi} + v\frac{\mathrm{d}u}{\mathrm{d}\xi}\right] + uv\operatorname{sen}(\xi) &= \delta(\xi - x);\\
u\left[-\frac{\mathrm{d}v}{\mathrm{d}\xi} + v\operatorname{sen}(\xi)\right] - v\frac{\mathrm{d}u}{\mathrm{d}\xi} &= \delta(\xi - x);\\
-\frac{\mathrm{d}v}{\mathrm{d}\xi} + v\operatorname{sen}(\xi) &= 0;\\
\frac{\mathrm{d}v}{\mathrm{d}\xi} &= v(x,\xi)\operatorname{sen}(\xi);\\
\frac{\mathrm{d}v}{v(x,\xi)} &= \operatorname{sen}(\xi)\mathrm{d}\xi\\
\int_0^\xi \frac{\mathrm{d}v}{v} &= \int_0^\xi \operatorname{sen}(\eta)\,\mathrm{d}\eta\\
\ln\frac{v(x,\xi)}{v(x,0)} &= -\cos(\eta)\Big|_0^\xi = 1 - \cos(\xi)\\
\frac{v(x,\xi)}{v(x,0)} &= \exp\left[1 - \cos(\xi)\right]\\
v(x,\xi) &= v(x,0)\exp\left[1 - \cos(\xi)\right].
\end{aligned}$$

Obtemos agora u:

$$\begin{aligned}
-v\frac{\mathrm{d}u}{\mathrm{d}\xi} &= \delta(\xi - x);\\
-v(x,0)\exp\left[1-\cos(\xi)\right]\frac{\mathrm{d}u}{\mathrm{d}\xi} &= \delta(\xi - x)\\
\frac{\mathrm{d}u}{\mathrm{d}\xi} &= -\frac{1}{v(x,0)}\exp\left[-(1-\cos(\xi))\right]\delta(\xi - x)\\
u(x,\xi) &= u(x,0) - \frac{1}{v(x,0)}\int_{\eta=0}^{\xi}\exp\left[-(1-\cos(\eta))\right]\delta(\eta - x)\,\mathrm{d}\eta\\
&= u(x,0) - \frac{H(\xi - x)}{v(x,0)}\exp\left[-(1-\cos(x))\right].
\end{aligned}$$

Continue a solução no verso ⟹

A função de Green, portanto, é

$$
\begin{aligned}
G(x,\xi) &= u(x,\xi)v(x,\xi) \\
&= \left\{u(x,0) - \frac{H(\xi-x)}{v(x,0)}\exp\left[-(1-\cos(x))\right]\right\} v(x,0)\exp\left[1-\cos(\xi)\right] \\
&= u(x,0)v(x,0) - H(\xi-x)\exp\left[-1+\cos(x)+1-\cos(\xi)\right] \\
&= G(x,0)\exp[1-\cos(\xi)] - H(\xi-x)\exp\left[-1+\cos(x)+(1-\cos(\xi))\right]; \\
&= \{G(x,0) - H(\xi-x)\exp[-(1-\cos(x))]\}\exp[1-\cos(\xi)]
\end{aligned}
$$

aplicando a condição de contorno no infinito,

$$
\begin{aligned}
G(x,\infty) &= 0 \;\Rightarrow \\
0 &= [G(x,0) - \exp[-(1-\cos(x))]]\exp[1-\cos(\infty)] \;\Rightarrow \\
G(x,0) &= \exp[-(1-\cos(x))].
\end{aligned}
$$

Finalmente,

$$
\begin{aligned}
G(x,\xi) &= [1-H(\xi-x)]\exp[-(1-\cos(x))]\exp[1-\cos(\xi)] \\
&= [1-H(\xi-x)]\exp[\cos(x)-\cos(\xi)] \;\blacksquare
\end{aligned}
$$

Continue a solução no verso $\Longrightarrow$

4 [20] Usando obrigatoriamente o método das características, resolva

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x^{1/2}, \qquad u(x,0) = f(x).$$

SOLUÇÃO DA QUESTÃO:
Sejam

$$\begin{aligned}
x &= X(s),\\
T &= T(s),\\
u(x,t) &= u(X(s),T(s)) = U(s);\\
\frac{\mathrm{d}U}{\mathrm{d}s} &= \frac{\partial u}{\partial t}\frac{\mathrm{d}T}{\mathrm{d}s} + \frac{\partial u}{\partial x}\frac{\mathrm{d}X}{\mathrm{d}s}; \ \Rightarrow\\
\frac{\mathrm{d}T}{\mathrm{d}s} &= 1 \ \Rightarrow T(s) = \cancelto{0}{T(0)} + s;\\
\frac{\mathrm{d}X}{\mathrm{d}s} &= X(s) \ \Rightarrow X(s) = X(0)\mathrm{e}^{s}.
\end{aligned}$$

A EDO em $U(s)$ será

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}s} &= X^{1/2}(s)\\
&= [X(0)\mathrm{e}^{s}]^{1/2}\\
&= (X(0))^{1/2}\mathrm{e}^{s/2} \ \Rightarrow\\
U(s) - U(0) &= 2(X(0))^{1/2}[\mathrm{e}^{s/2} - 1].
\end{aligned}$$

Mas $s = t$, e:

$$\begin{aligned}
X &= X(0)\mathrm{e}^{s};\\
X(0) &= X\mathrm{e}^{-s} = x\mathrm{e}^{-t};\\
u(x,0) &= u(X(0),0) = U(0) = f(X(0)).
\end{aligned}$$

Portanto,

$$\begin{aligned}
U(s) &= f(X(0)) + 2(X(0))^{1/2}[\mathrm{e}^{s/2} - 1];\\
u(x,t) &= f(x\mathrm{e}^{-t}) + 2(x\mathrm{e}^{-t})^{1/2}[\mathrm{e}^{t/2} - 1] \ \blacksquare
\end{aligned}$$

Continue a solução no verso $\Longrightarrow$

5 [20] Encontre os autovalores e as autofunções do problema de Sturm-Liouville

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \lambda y = 0,\\
y(0) = y'(1) = 0.
\end{aligned}
$$

SOLUÇÃO DA QUESTÃO:
Discuta os valores de λ:
$\lambda = -k^2 < 0$:

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - k^2 y &= 0;\\
r^2 - k^2 &= 0;\\
r &= \pm k;\\
y(x) &= A\cosh(kx) + B\,\mathrm{senh}(kx);\\
y'(x) &= A\,\mathrm{senh}(kx) + B\cosh(kx).\\
y(0) = 0 \;\Rightarrow\; A\cosh(0) + B\,\mathrm{senh}(0) &= 0,\\
A &= 0.\\
y'(1) = 0 \;\Rightarrow B\cosh(1) &= 0,\\
B &= 0.
\end{aligned}
$$

Portanto não há autovalores para esse caso.
$\lambda = 0$:

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 0;\\
y(x) &= Ax + B.\\
y(0) = 0 \;&\Rightarrow B = 0.\\
y'(1) = 0 \;&\Rightarrow A = 0.
\end{aligned}
$$

Portanto não há autovalores para esse caso.
$\lambda = k^2 > 0$:

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + k^2 y &= 0;\\
r^2 + k^2 &= 0;\\
r &= \pm \mathrm{i}\sqrt{k};\\
y(x) &= A\cos(kx) + B\,\mathrm{sen}(kx).\\
y'(x) &= -A\,\mathrm{sen}(kx) + B\cos(kx)\\
y(0) = 0 \;&\Rightarrow A = 0.\\
y'(1) = B\cos(k) = 0 &\Rightarrow\\
\cos(k) = 0 \;&\Rightarrow\\
k_n &= \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi, \qquad n = 0, 1, \ldots
\end{aligned}
$$

Portanto os autovalores e as autofunções são

$$
\begin{aligned}
\lambda_n &= \left[\frac{2n+1}{2}\pi\right]^2, \qquad n = 0, 1, \ldots\\
y_n(x) &= \mathrm{sen}\left(\frac{(2n+1)\pi x}{2}\right), \qquad n = 0, 1, \ldots \;\blacksquare
\end{aligned}
$$

Continue a solução no verso ⟹