University College London, March 5, 2007

IS TURBULENCE ERGODIC?

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One of a series of "Lectures on Conceptual Aspects of Turbulence" as a part of the activity within the frame of the *Marie Curie* Chair "Fundamental and Conceptual Aspects of Turbulent Flows".

We absolutely must leave room for doubt or there is no progress and no learning. There is no learning without posing a question. And a question requires doubt...Now the freedom of doubt, which is absolutely essential for the development of science, was born from a struggle with constituted authorities... FEYNMANN, 1964

TAKING A STRICTLY RIGOROUS (MATHEMATICALLY) POSITION THE ANSWER SEEMS TO BE NEGATIVE!

HOWEVER, THE ARE SEVERAL IMPORTANT "HOWEVERS", WHICH ARE THE MAIN THEME OF THIS TALK.

- For statistically stationary flows ERGODICITY is (roughly) equivalence of `true' statistical properties (not only means/averages, but `almost' all statistical properties) of an ensemble to those obtained using time series in one very long realization.
- A similar property is defined in space by replacing time by space coordinate(s) in which the flow domain has an infinite extension, at least in one direction.

Turbulent flows are believed/known (empirically) to be ergodic.

There is no way to confirm that those turbulence data used in analysis represent typical properties of turbulence. VAN VEEN, KIDA, & KAWAHARA (2006) Periodic motion representing isotropic turbulence, *Fluid Dynamics Research*, **38,** 19–46.

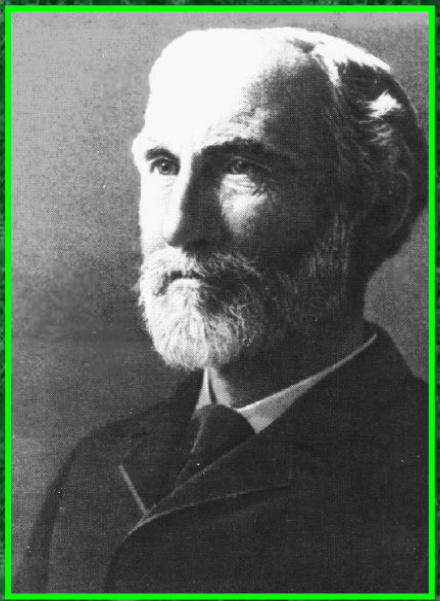
The ergodicity of turbulence sounds to me as an assumption which is hard to avoid or test

Is it possible to do anything or should we stay with the belief ? ? ?

FOUNDING FATHERS OF STATISTICAL MECHANICS

LUDWIG BOLTZMANN

JAMES MAXWELL



JOSIA GIBBS

ALBERT EINSTEIN

Introduced concepts such as *ensembles, ergodicity and coarse graining*. BOLTZMANN introduced the *ergodic hypothesis* in 1871

RAISED SERIOUS MATHEMATICAL PROBLEMS $\Rightarrow$ ERGODIC THEORY :
There is much more to the mathematical study of Gibbs ensembles than that the question whether or not time averages and ensemble averages are equal.

On 'Analogy' between statistical mechanics and turbulence

The approach, which treats the fields of hydrodynamic variables of a turbulent flow as random fields, was initiated by the works of Kolmogorov and his school [see, e.g., Millionshchikov (1939)] and the work of Kampé de Fériet (1939), see Monin and Yaglom, 1971.

Adopting the assumption of the existence of probability distributions for all fluid dynamic fields, we may further make wide use of the mathematical techniques of modern probability theory; the operation of averaging is then defined uniquely and has all the properties naturally required of it.

In discussing the statistical nature of the theory of turbulence, comparison is often made with the kinetic theory of gases, which investigates many-particle systems of interacting molecules. This comparison is reasonable in the sense that in both these theories an exact description of the evolution of the individual mechanical system is theoretically impossible and practically useless. However, it must be remembered that between the statistical mechanics of the ensembles of molecules investigated by Gibbs, Boltzmann and others, and the statistical fluid mechanics of a viscous fluid, there is a fundamental difference. This is connected primarily with the fact that, the total kinetic energy of a molecular ensemble is time-invariant (in any case, under the simplified assumptions on molecular interactions adopted in the kinetic theory of gases) while the kinetic energy of real fluid flow is always dissipated to heat due to the action of viscosity. Of less importance, but still significant, is the fact that molecular ensembles are discrete by nature and their time evolution is described by systems of ordinary differential equations, while in fluid mechanics we are dealing with the motion of a continuous medium, which is described by partial differential equations. Thus the analogy with the kinetic theory of gases is of relatively little help in the formulation of the theory of turbulence, and is useful only for a preliminary understanding of the concept of a statistical approach to physical theory.

MONIN AND YAGLOM 1971

RECENT REFERENCES – (S)NSE

DA PRATO, G. AND DEBUSSCHE, A. (2003) Ergodicity for the 3D stochasitc Navier-Stokes equations, J. de Math. Pures et Appl., 82, 877-947.4.

ROMITO, M. (2003) Ergodicity of the Finite Dimensional Approximation of the 3D Navier--Stokes Equations Forced by a Degenerate Noise, *J. Stat. Phys.*, **114**, 155-177.

MATTINGLY, J. C. On Recent Progress for the Stochastic Navier–Stokes Equations. *Journ´ees "Equations aux D´eriv´ees Partielles" (Forges-les-Eaux, 2003)*, **XV**, Summer 2003, Art. No. 11, 52 p.

WAYMIREY, E.C. (2005) Probability & incompressible Navier-Stokes equations: An overview of some recent developments, *Probability Surveys*, **2**, 1-32.

FLANDOLI, F.(2005) An Introduction to 3D Stochastic Fluid Dynamics, *CIME lectures*, 108 pp.

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Note that this is all SNSE, i. e. stochastic forcing both in 3D and 2D. In the latter case it is unlikely that with a deterministic forcing one can expect anything like ergodicity

AN IMPORTANT ASPECT

Note that the above is all SNSE, i. e. stochastic forcing. A typical statement is as follows

One of the oldest open problems in theoretical physics is that of describing fully developed turbulence on the basis of a mi(a)croscopic model. The latter is usually taken to be the stochastic Navier - Stokes (NS) equation subject to an external random force that models the energy injection by the large-scale modes

ADZHEMYAN ET AL 2003

AN IMPORTANT ASPECT

However, what about a great variety of turbulent flows in which the 'forcing' is not random and in many cases is even not time dependent - juts constant in time? Such flows at large enough Reynolds numbers become turbulent due to what can be called intrinsic stochasticity (nobody seems to know what it is), and, e.g. statistically stationary turbulent flows are massively studied using temporal statistics instead of the 'true' one based on ensembles or probability measures (which are anyhow not accessible). All observed so far statistical (not only average) properties of many such turbulent flows (but not all) are remarkably reproducible (statistical stabilty) and – as mentioned – are believed to be ergodic in spite of the fact that, say, a constant in time large scale, i.e. deterministic forcing breaks the ergodicity 'on large scales'

- In statistically stationary situations the time statistics obtained in experiments is believed to correspond to a probability measure invariant under time evolution. This comprises the essence of the ergodic hypothesis, which is usually expressed in terms of ensemble and is widely used in experiments.
- A similar statement is made for situations with at least one homogeneous spatial coordinate.
- ! In dynamical systems the equivalence of two is used as a definiton of egrodicity : ***Defintion 7.1 : An abstract dynamical system is ergodic if for every complex-valued μ-summable function the time mean is equal to the space mean.*** ARNOLD AND AVEZ (1968)

Ergodicity. Let x denote a point in R and let $G(x)$ denote a function defined on R. Then the time average of $G(x)$ along the orbit of x, denoted $G^*(x)$, is defined by

$$G^*(x) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} G(S^n(x)).$$

The space average of $G(x)$, denoted $\bar{G}$, is defined by

$$\bar{G} = \int_D G(x)\, \mathrm{d}\mu$$

(recall that we have normalized the measure of the volume of R so that $\mu(R) = 1$). The transformation S is said to *ergodic* if, for a sufficiently large class of functions, denoted by the set of functions $G(x)$, we have $G^*(x) = \bar{G}$ (except, possibly, on certain sets of zero volume). In other words, the time average of functions along an orbit (with the possible exception of a set of orbits of zero volume, or *measure zero*) is equal to the space average. This is significant because clearly the space average of a function is just a number. From its definition, the time average may vary from point to point, but *not* if the transformation S is ergodic. We remark that even if the transformation is not ergodic on all of R, it may be ergodic on invariant subsets of R. In fact, there is a mathematically rigorous way of partitioning R into invariant sets, where the transformation is ergodic on each invariant set. This *ergodic partition* (an idea originating in the work of von Neumann, Halmos and Rokhlin) is described in Mezić & Wiggins (1999)

WIGGINS, S. AND OTTINO J.M. (2004) Foundations of chaotic mixing, *Phil. Trans. R. Soc. Lond.* A **362**, 937–970

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Is it possible to do anything or should we stay with the belief ? ? ?

FOIAS ET AL. (2001) have shown that there are measures on function space that are time invariant. However, invariance under time evolution is not enough to specify a unique measure which would describe turbulence. Another problem is that it is not clear how the objects that the authors have constructed and used in their proofs are relevant/related or even have anything to do with turbulence.

WE REMIND AGAIN THAT THIS IS ALL SNSE, I. E. STOCHASTIC FORCING BOTH IN THE RHS OF NSE

Thus, one deals with two different aspects: one statistical analysis over the entire flow field at a certain moment in time, and another one for one position in space over a very long period of time. The first one may not be representative for a longer period of time, while the second one may not be representative for all the points in space.

The point is that if the flow is ergodic the two types of statistics should give the same result.

Many ensembles, (like the human populations), are not ergodic.

A NUMERICAL EXPERIMENT

Galanti and Tsinober (2004) Is turbulence ergodic?, *Physics Letters* A 330 , 173–180

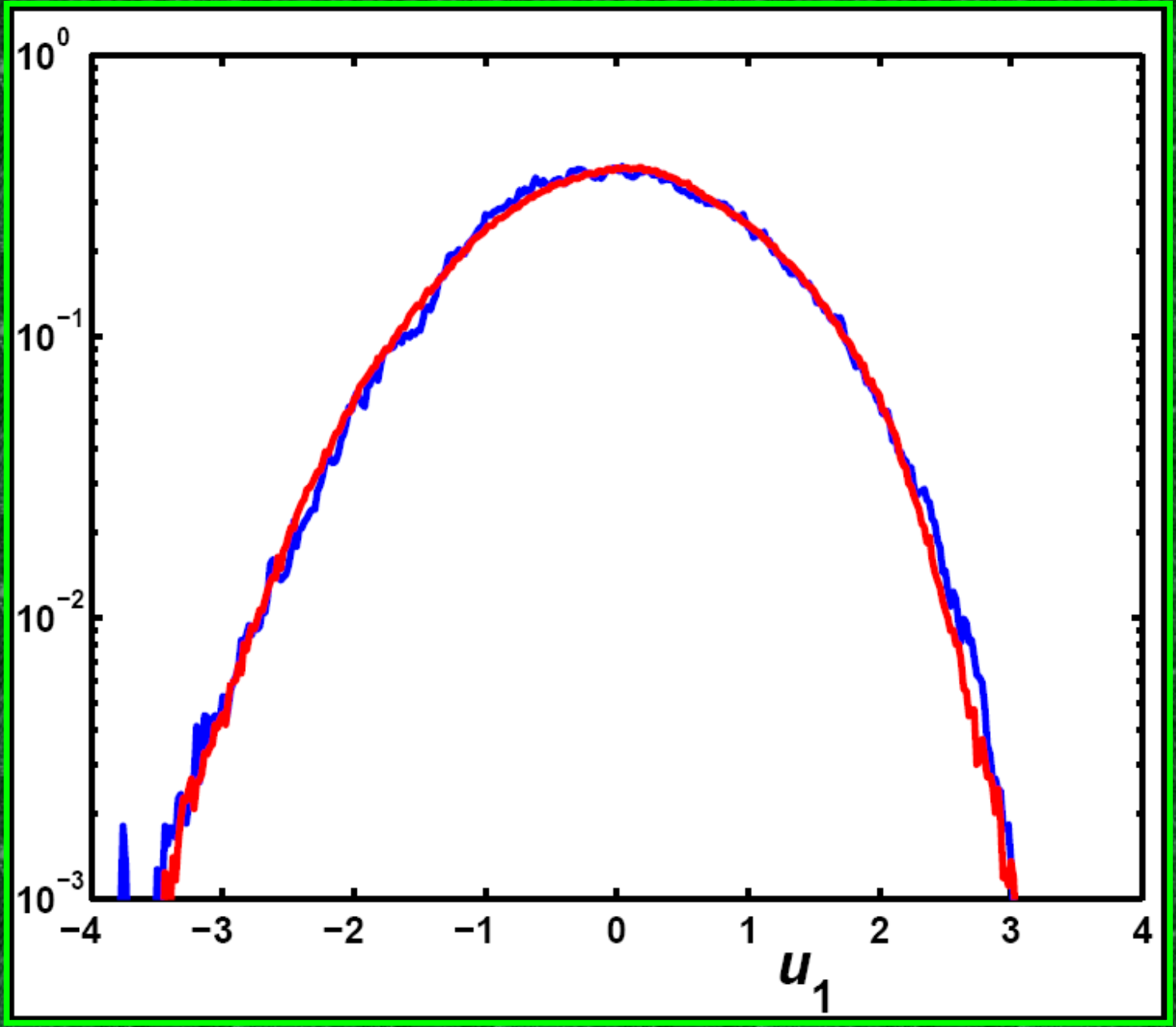
- There seems to exist no direct evidence regarding the validity of the ergodicity hypothesis in turbulent flows.
- We made an attempt to obtain such evidence via direct numerical simulations of the Navier–Stokes equations without (!) performing a large number of simulations at different initial conditions representing the members of an ensemble.
- The main idea is simple and is based on the fact that if a turbulent flow is both statistically stationary in time and homogeneous in space than its temporal and spatial statistical properties should be the same if the ergodic hypothesis is correct.

An important consequence is that it is not necessary to perform a large number of time/labor consuming "brutal force" experiments with different initial conditions in order to compare the time-averaged value of a given observable against the "ensemble" averaged value at a given time (as suggested by our referee).

Galanti and Tsinober (2004) Is turbulence ergodic?, *Phys. Lett.*, A 330 , 173–180

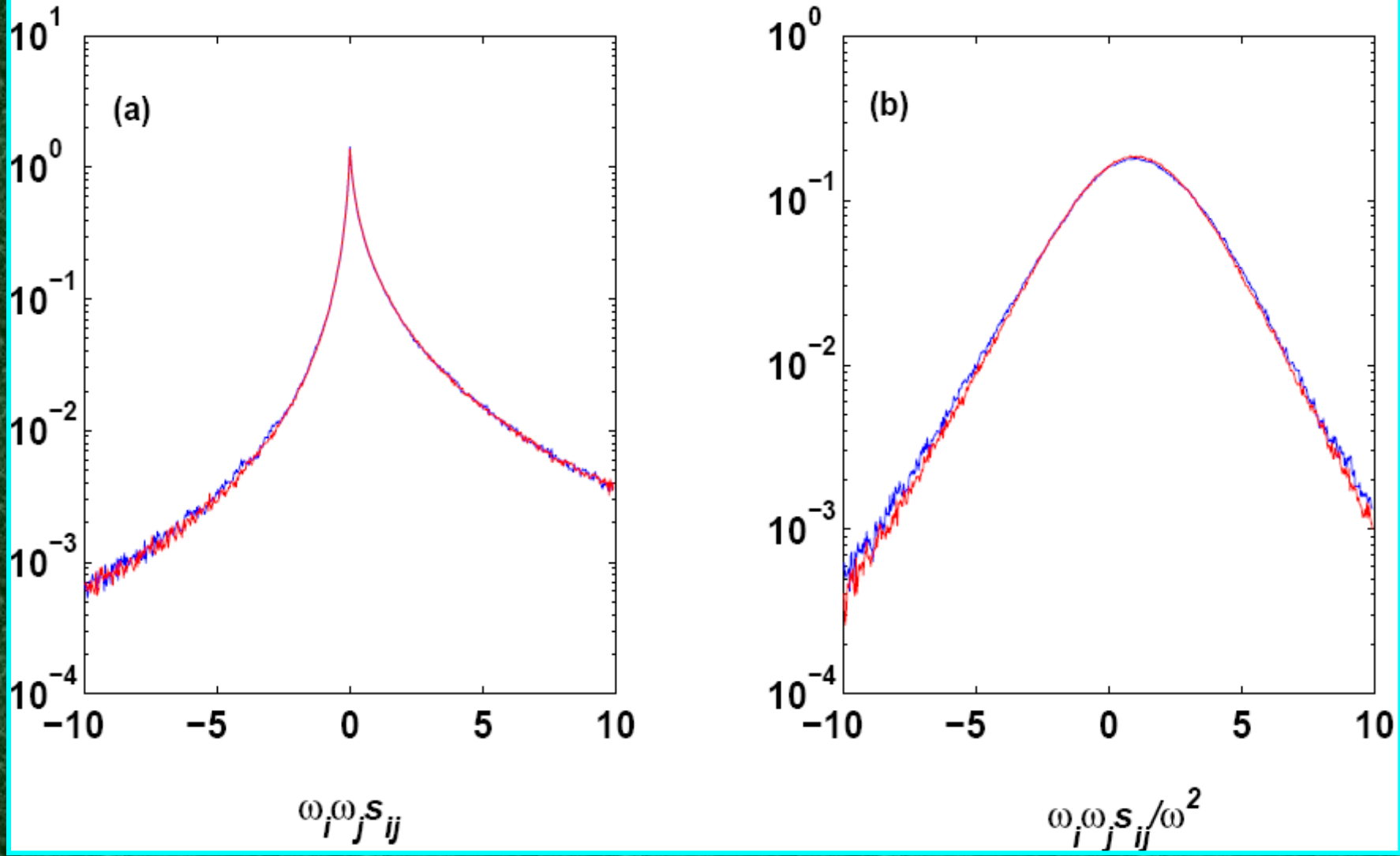
We used for this purpose a flow in the simplest geometry of a cubic domain with periodic boundary conditions and performed direct numerical simulations of the Navier–Stokes equations [5] with resolution 128^3 uniformly distributed grid points. In order to have comparable information for the time statistics the equations were run for 2 200 000 time steps (cf. with $128^3 = 2\,097\,152$) for 84 points in different locations in the flow domain. We used a simple deterministic forcing in large scales in the form $\mathbf{f} = A \cos z \cos y,\ B \cos x \cos z,\ C \cos y \cos x,\ A = B = C$. The Taylor microscale Reynolds number, $Re_\lambda \approx 145$.

VELOCITY



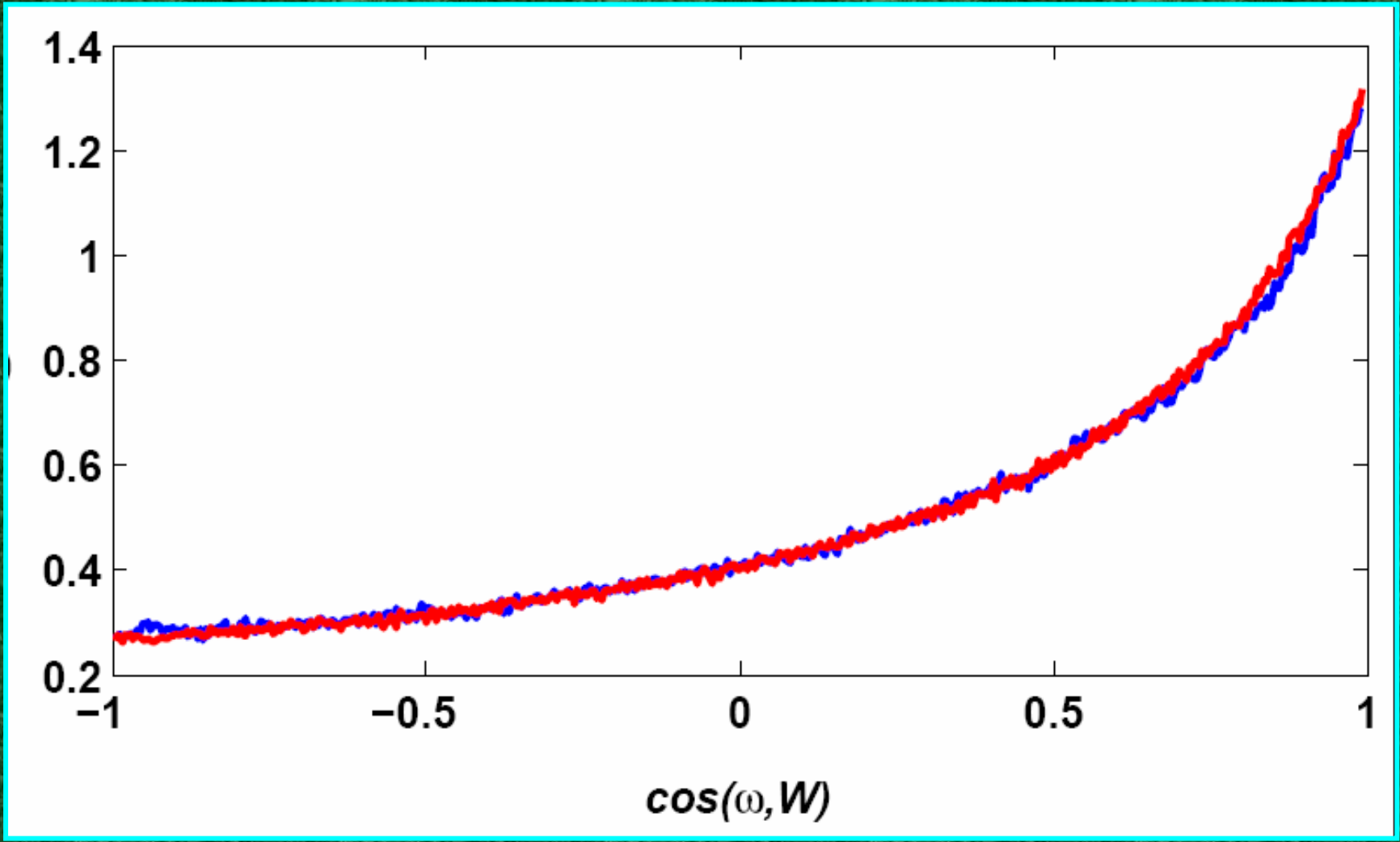
FLUCTUATIONS

ENSTROPHY PRODUCTION AND ITS RATE

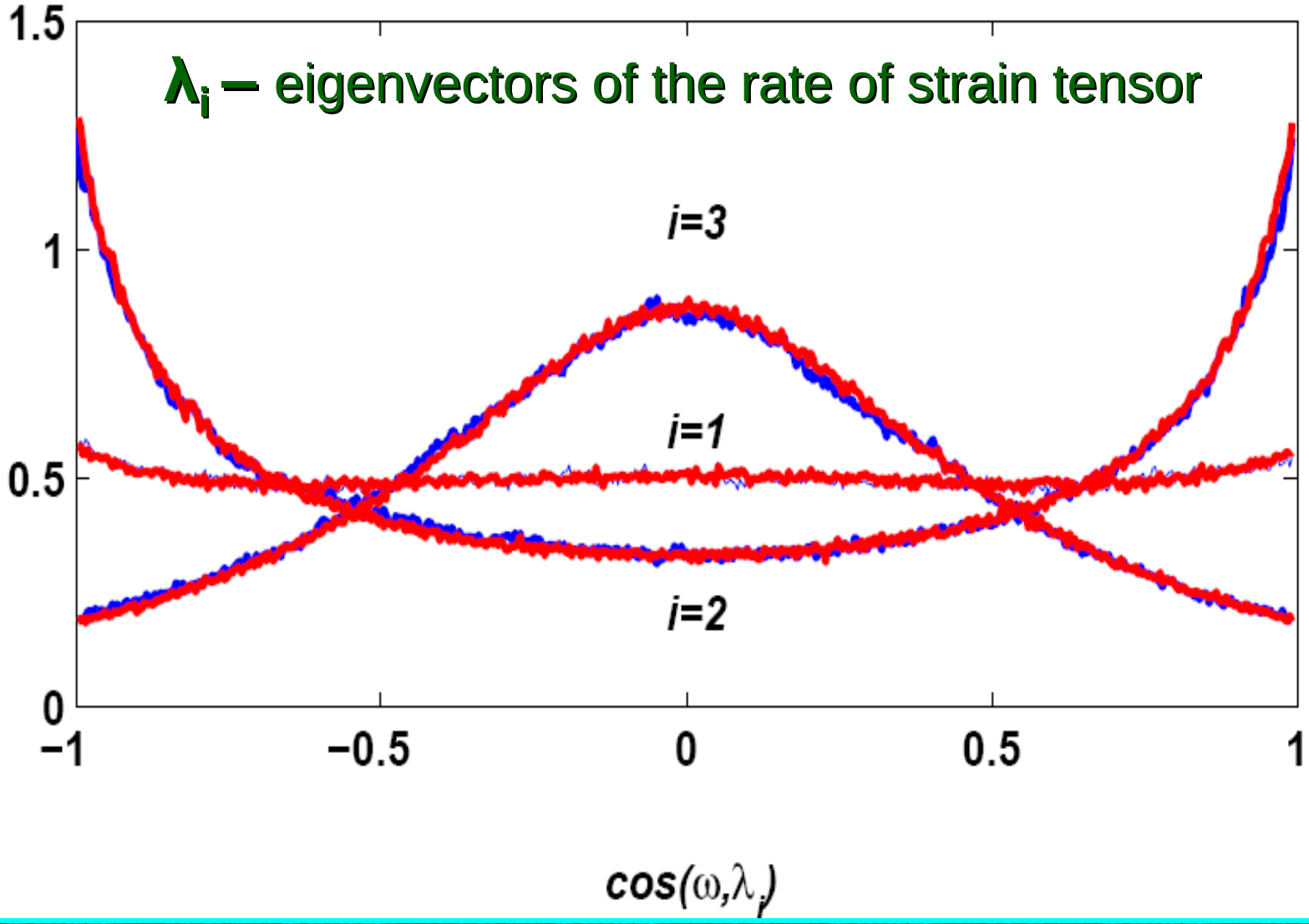


PDFs OF cos{ω,W}

$W_i = \omega_i s_{ij}$ – vortex stretching vector

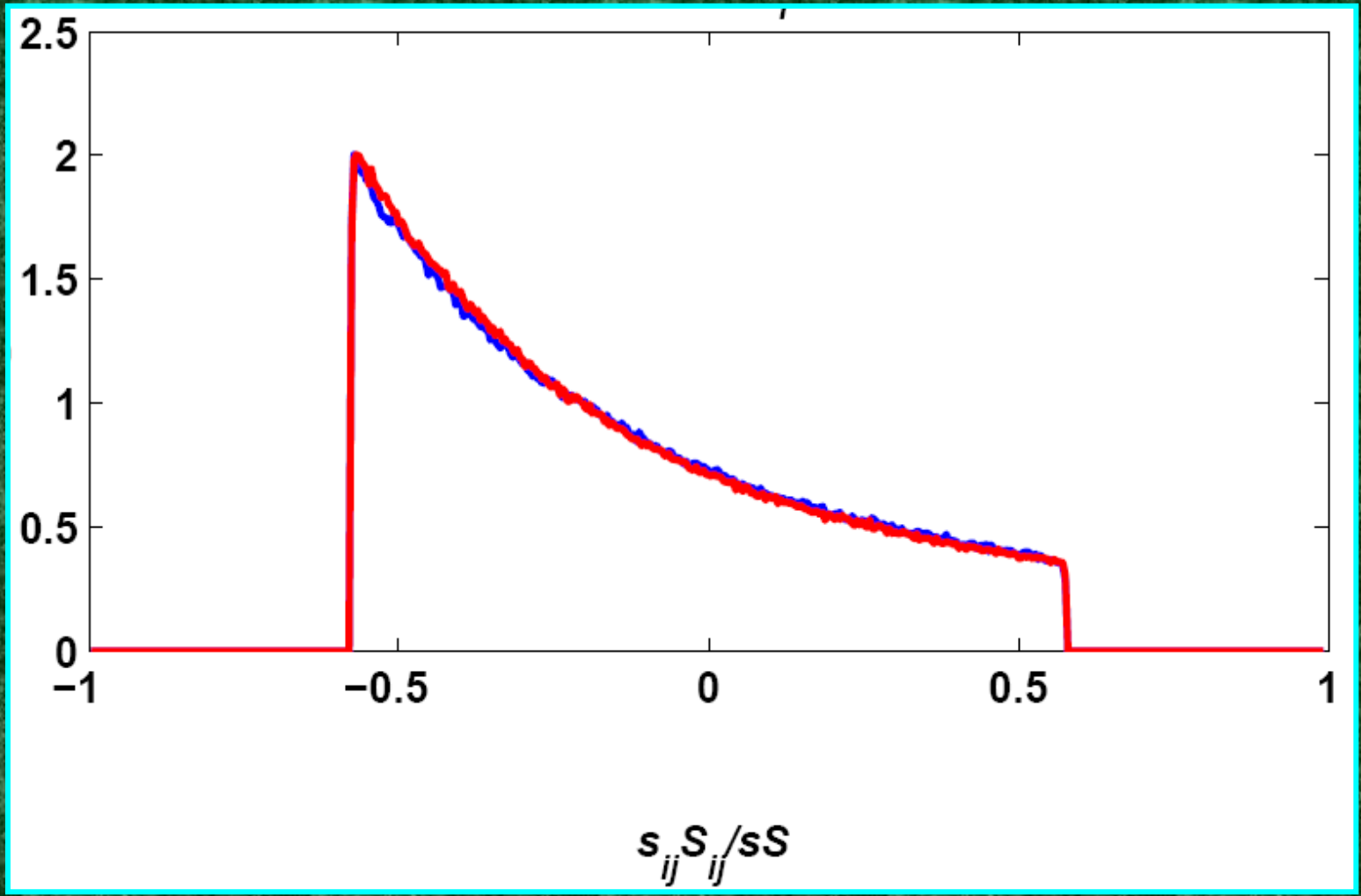


PDFs OF $\cos\{\omega,\lambda_i\}$



PDFs of cos{s,S}

$\mathbf{s} = s_{ij}, \mathbf{S} = s_{ik}s_{kj}$



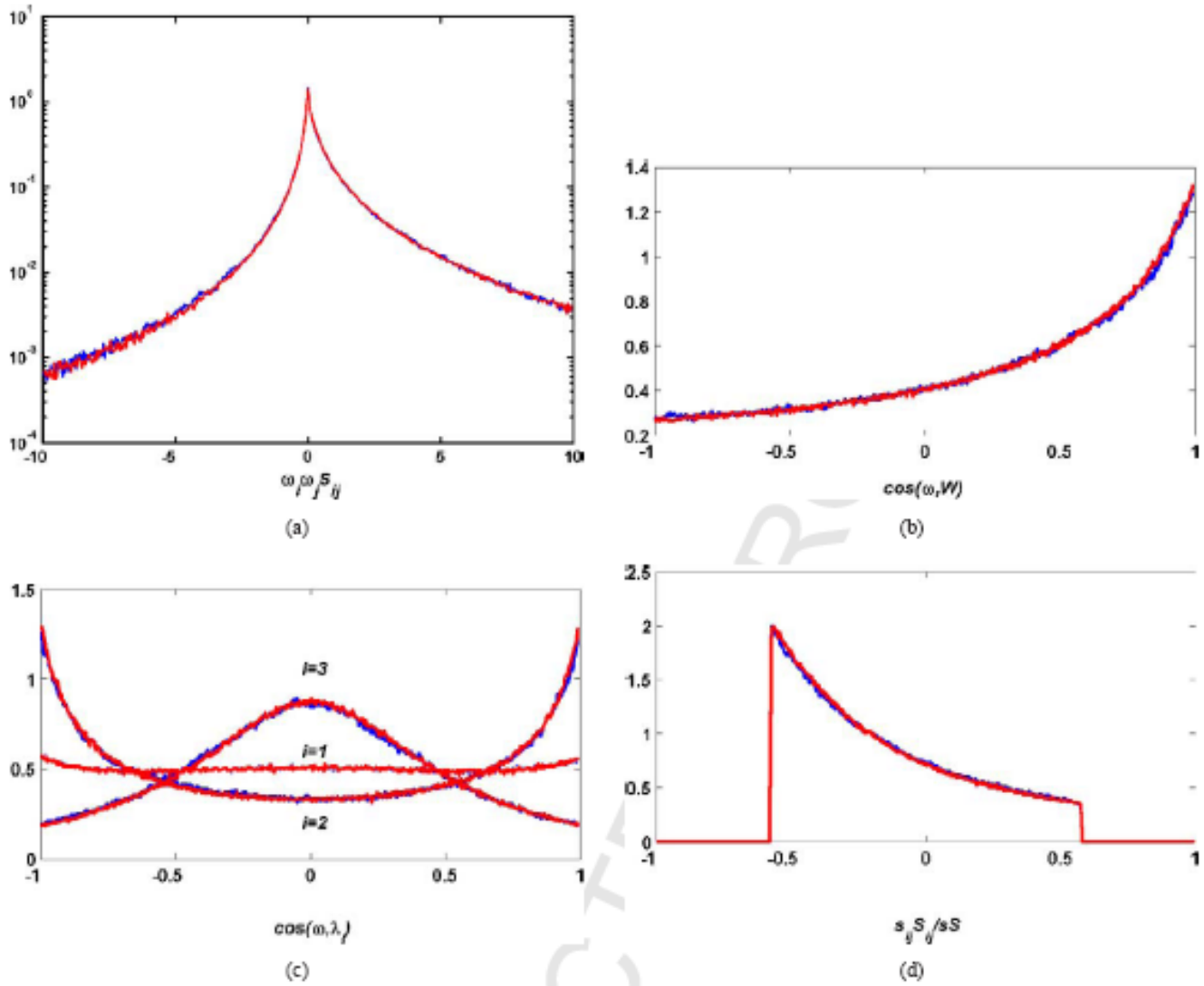
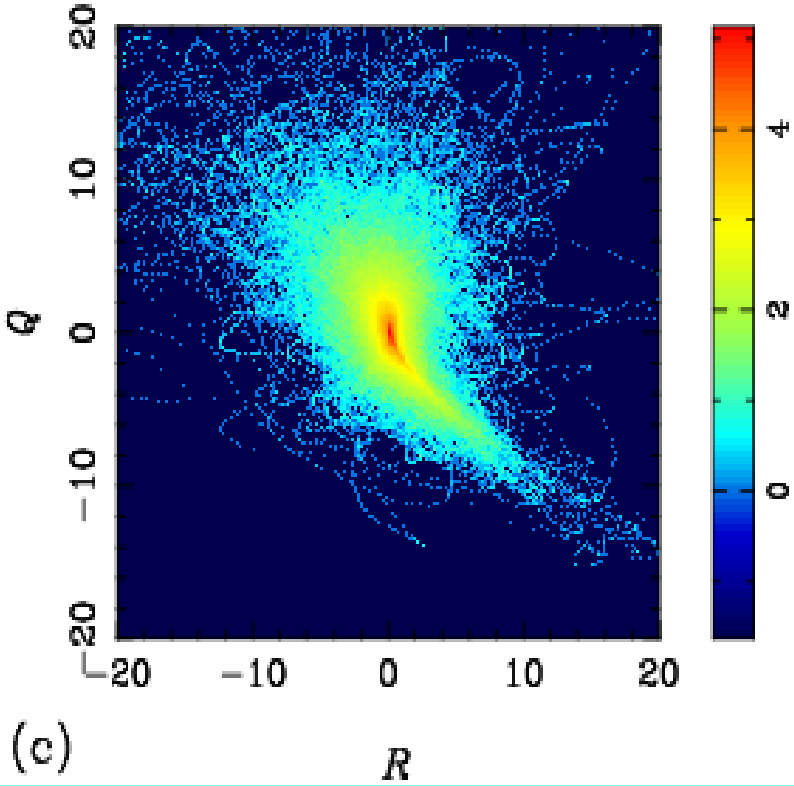
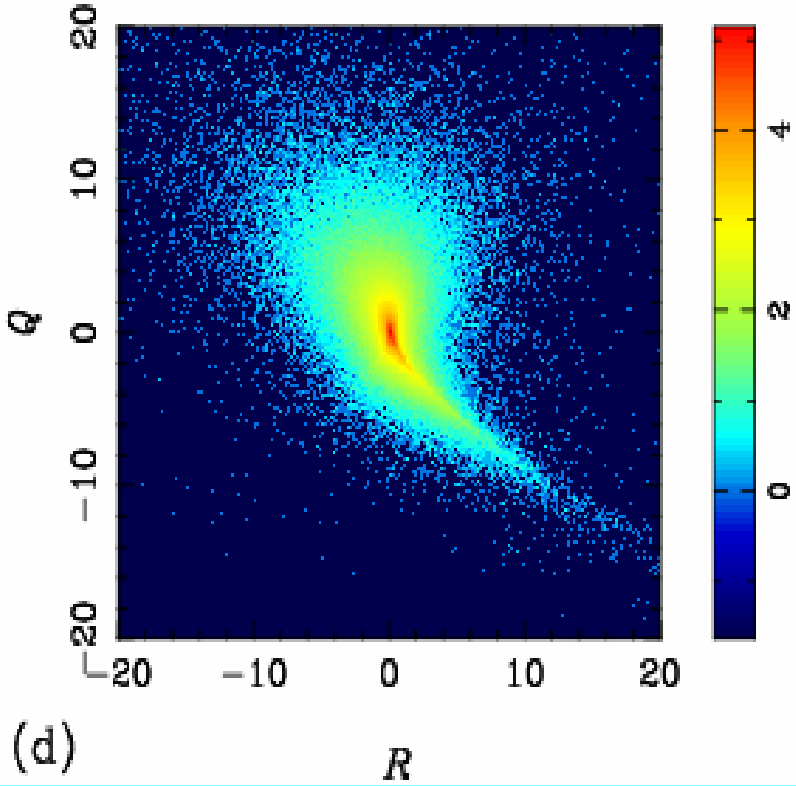


Fig. 3. Univariate probability density functions (PDFs) of dynamically significant quantities associated with the field of velocity derivatives–vorticity vector, ω_i and rate of strain tensor, s_{i_k}: (a) entrophy production $\omega_i \omega_k s_{i_k}$; (b) cosine $\cos(\omega, \mathrm{W})$ of the angle between vorticity ω_i and vortex stretching vector $W_i = \omega_k s_{ik}$; (c) cosine $\cos(\omega, \lambda_i)$ of the angle between vorticity ω_i and the eigenframe λ_i of the rate of strain tensor s_{ij}; (d) cosine of the "angle" between the rate of strain tensor s_{ij} and its stretching tensor $S_{ij} = s_{jk} s_{kj}$; this angle is defined via the relation $\cos(s, S) = s_{ij} S_{ij} / sS; s^2 = s_{ij} s_{ij}, S^2 = S_{ij} S_{ij}$.

JOINT STATISTICS I

Temporal **Spatial**





$Q = (1/4)\{\omega^2 - 2s^2\}$, $R = -(1/3)\{s_{ij}s_{jk}s_{ki} + (3/4)\omega_i\omega_j s_{ij}\}$

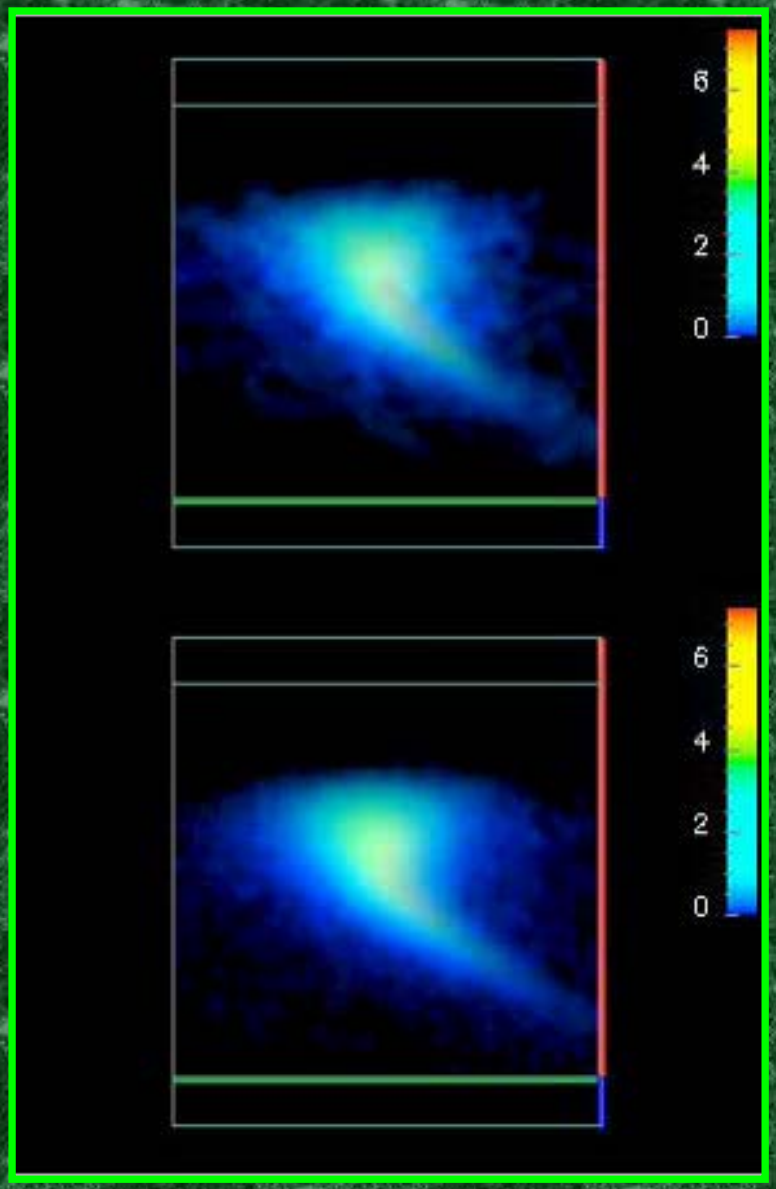
$Q = (1/4)\{\omega^2 - 2s^2\}$

$R = -(1/3)\{s_{ij}s_{jk}s_{ki} + (3/4)\omega_i\omega_j s_{ij}\}$

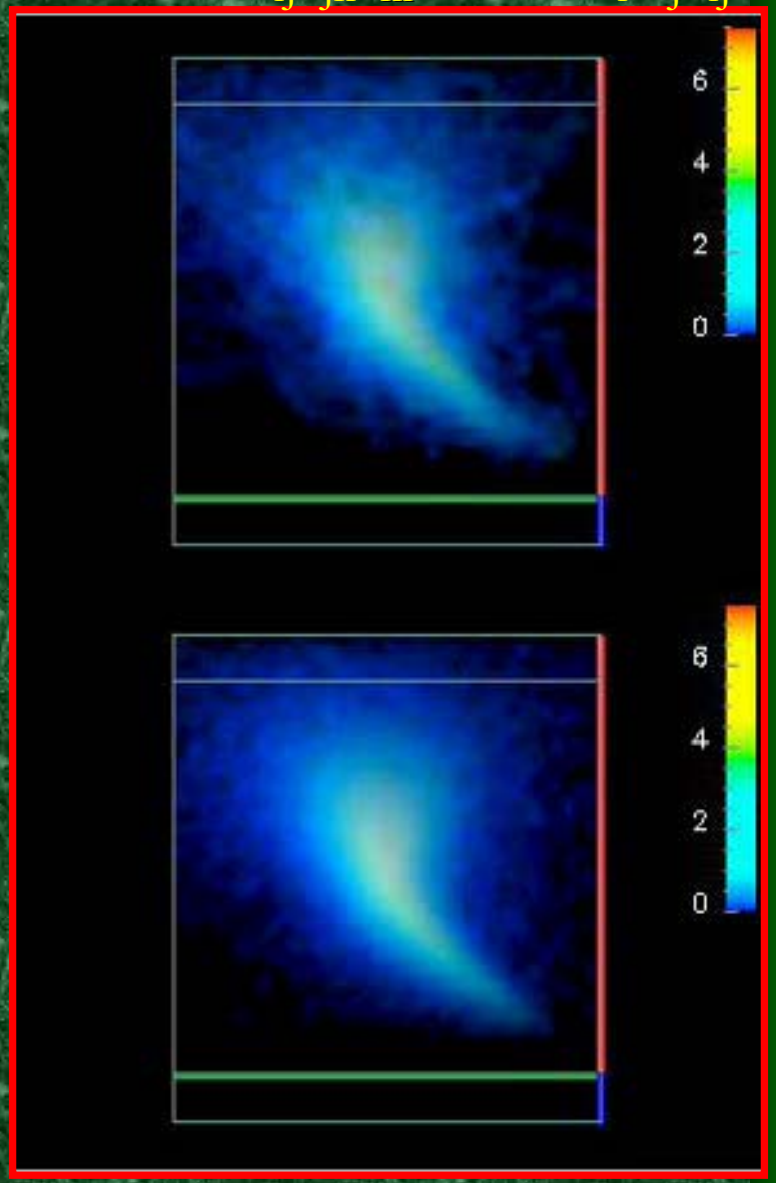
Temporal

Q

Spatial



R



Third axis ω^2

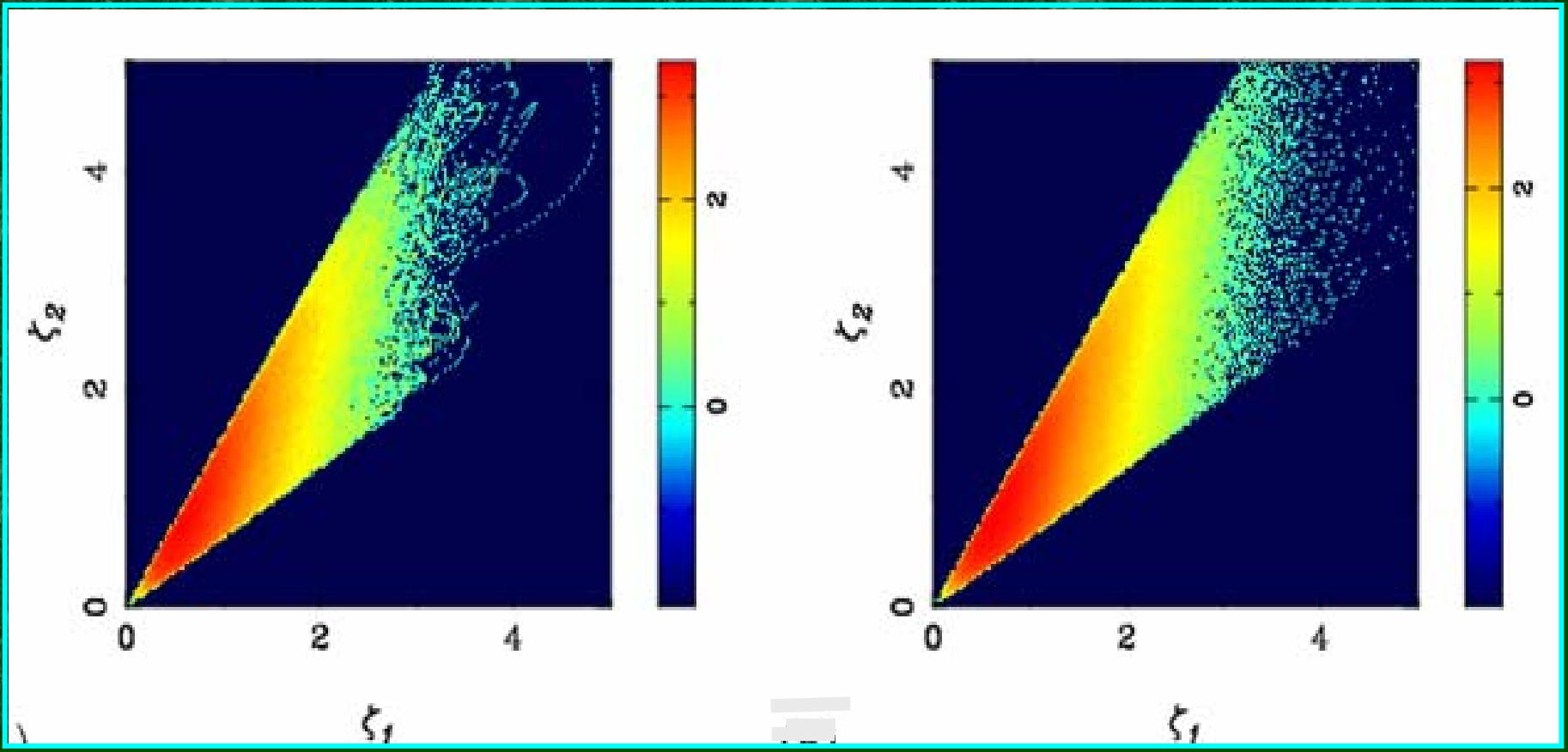
Third axis s^2

JOINT STATISTICS II

EIGEVALUES Λ_i OF THE RATE OF STRAIN TENSOR

IN THE PLANE $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$

Temporal **Spatial**

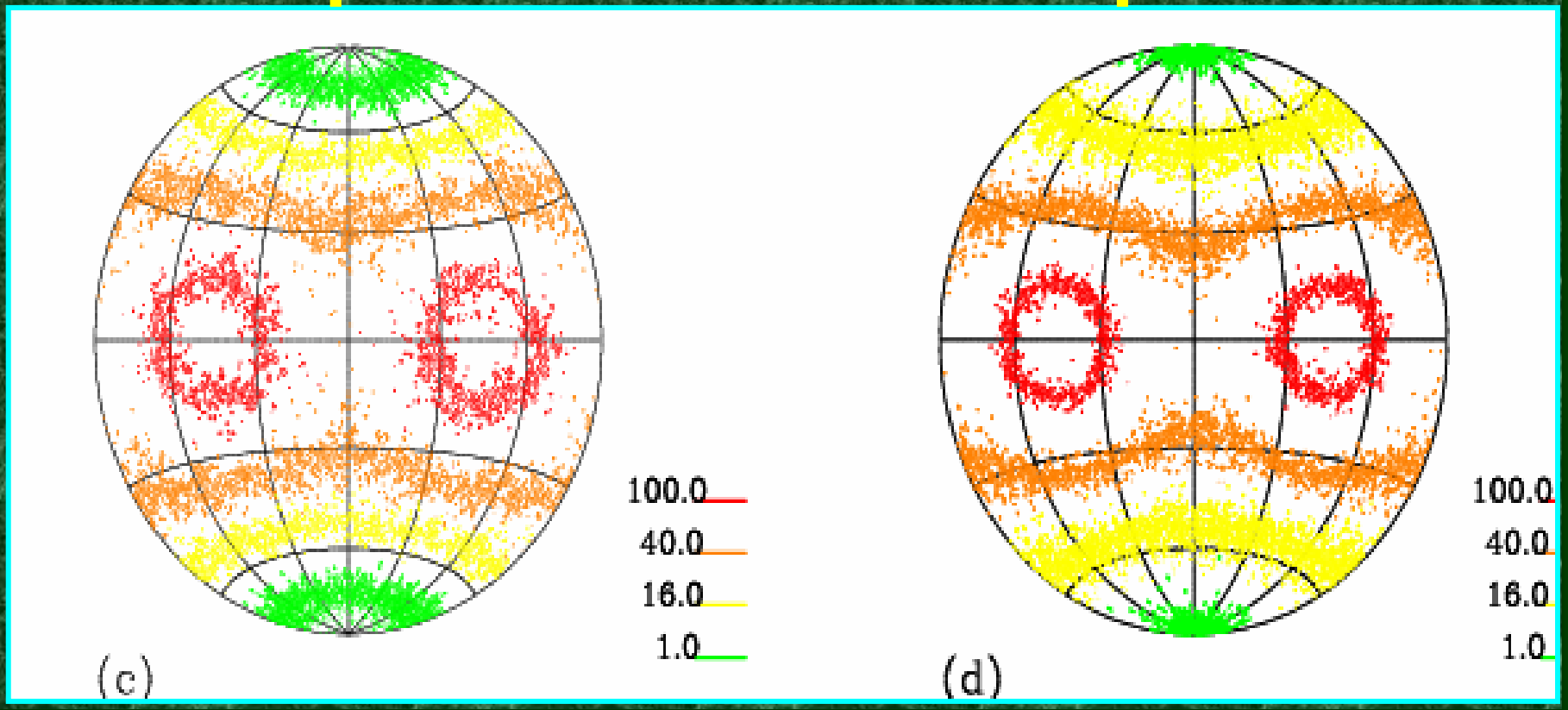


JOINT STATISTICS III

Joint PDFS of $\cos\{\omega,\lambda_i\}$ in Hummer-Aitoff projection

$$\cos^2\{\omega,\lambda_1\}+\cos^2\{\omega,\lambda_2\}+\cos^2\{\omega,\lambda_3\}=1$$

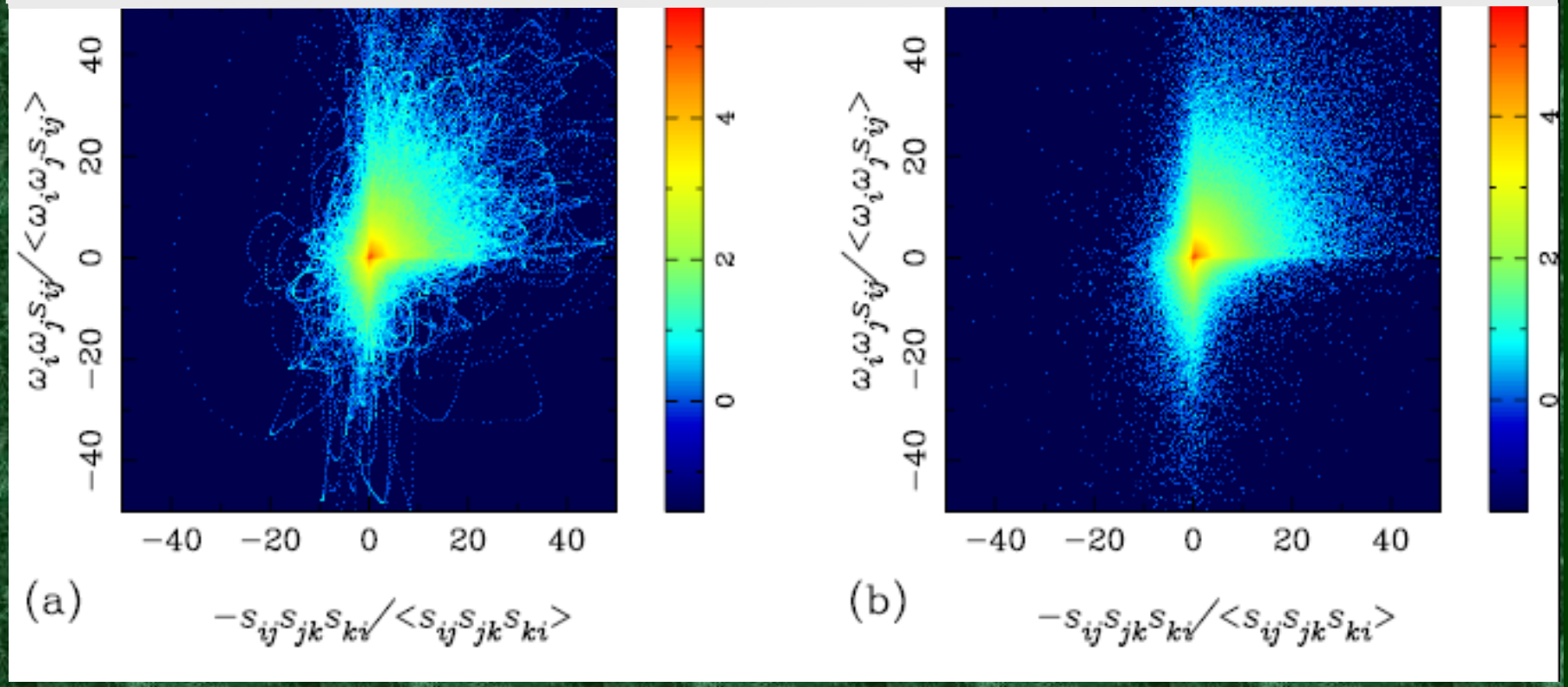
Temporal **Spatial**



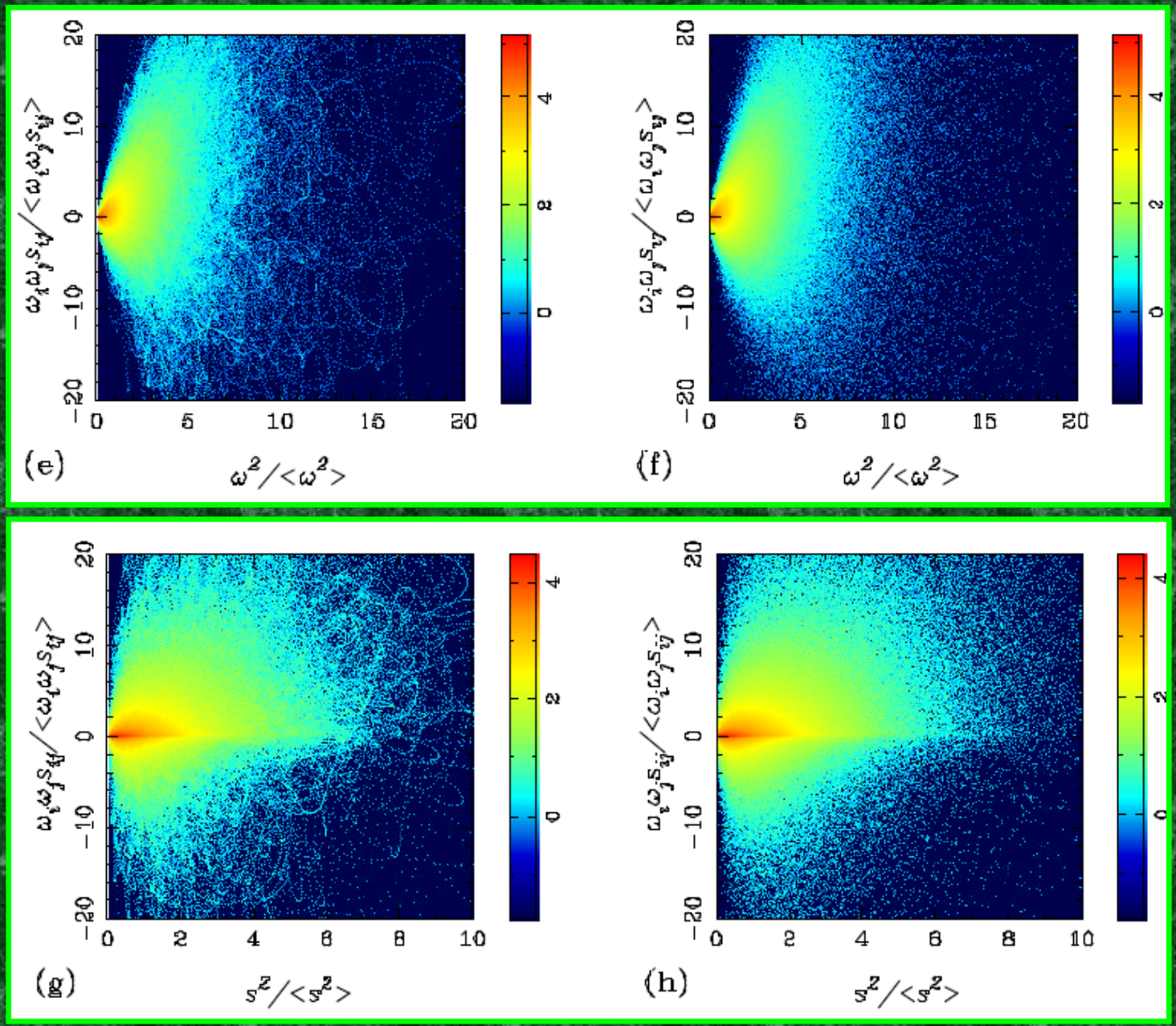
JOINT STATISTICS IV

Temporal

Spatial



Temporal



Spatial

Two-point statistics I

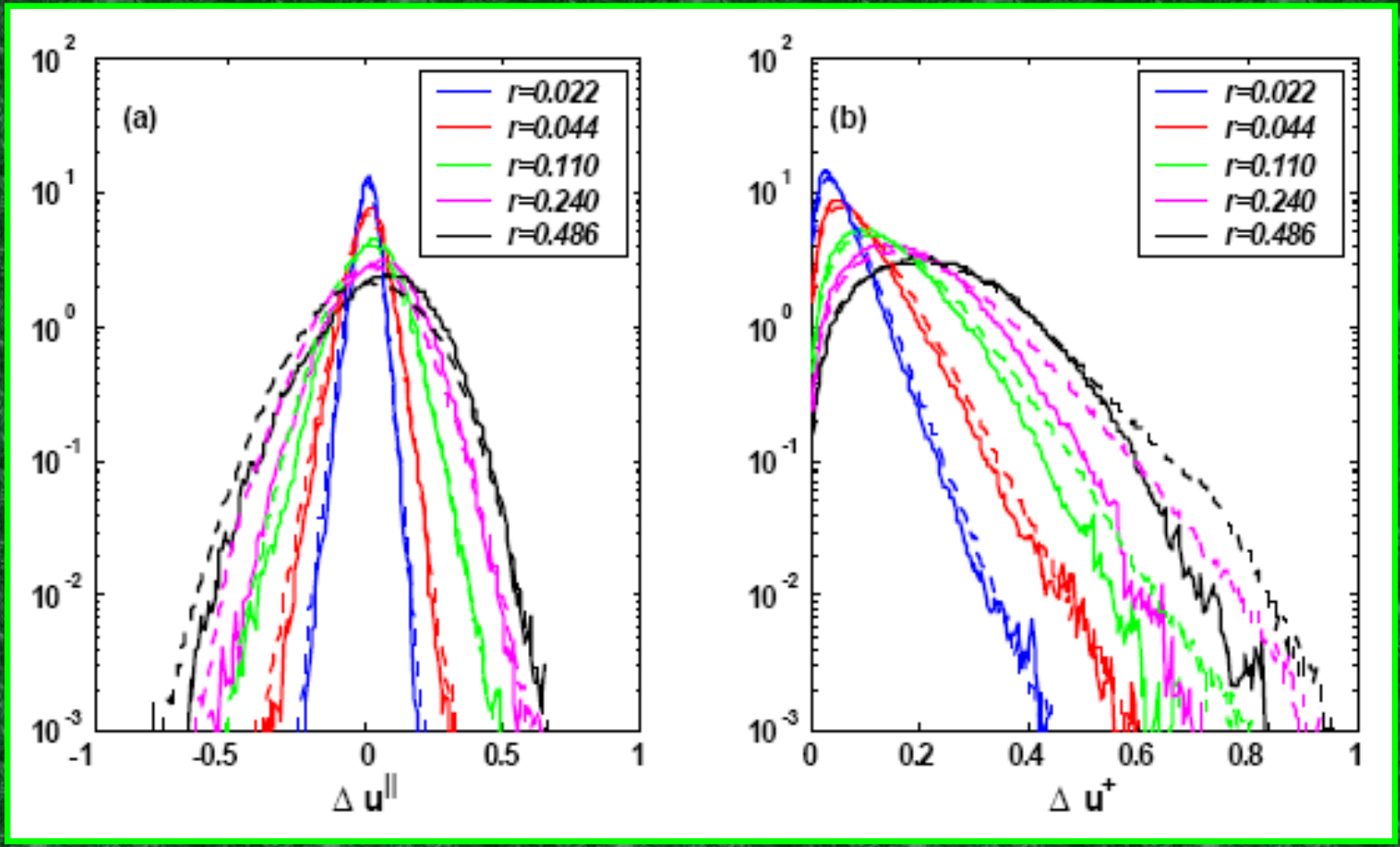


FIG. 2. Two-point statistics I. PDFs of longitudinal $\Delta u_n^{\parallel}$ (a), and transversal $\Delta u_n^{\perp}$ (b) velocity increments for several distances between the points. The temporal statistics is shown by continuous lines.

TWO-POINT STATISTICS II

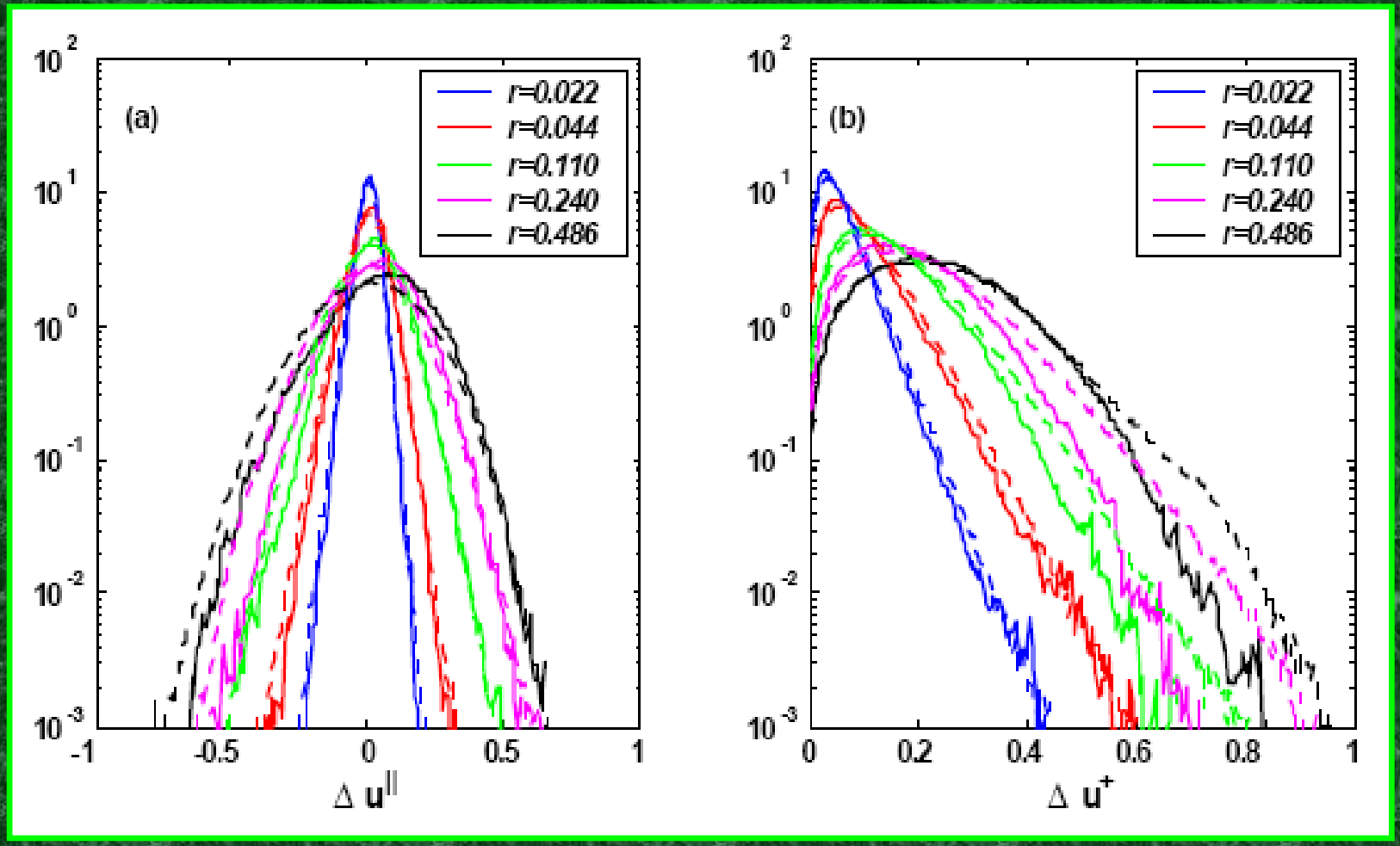


FIG. 2. Two-point statistics I. PDFs of longitudinal $\Delta u_n^{\parallel}$ (a), and transversal $\Delta u_n^{\perp}$ (b) velocity increments for several distances between the points. The temporal statistics is shown by continuous lines.

THREE-POINT STATISTICS

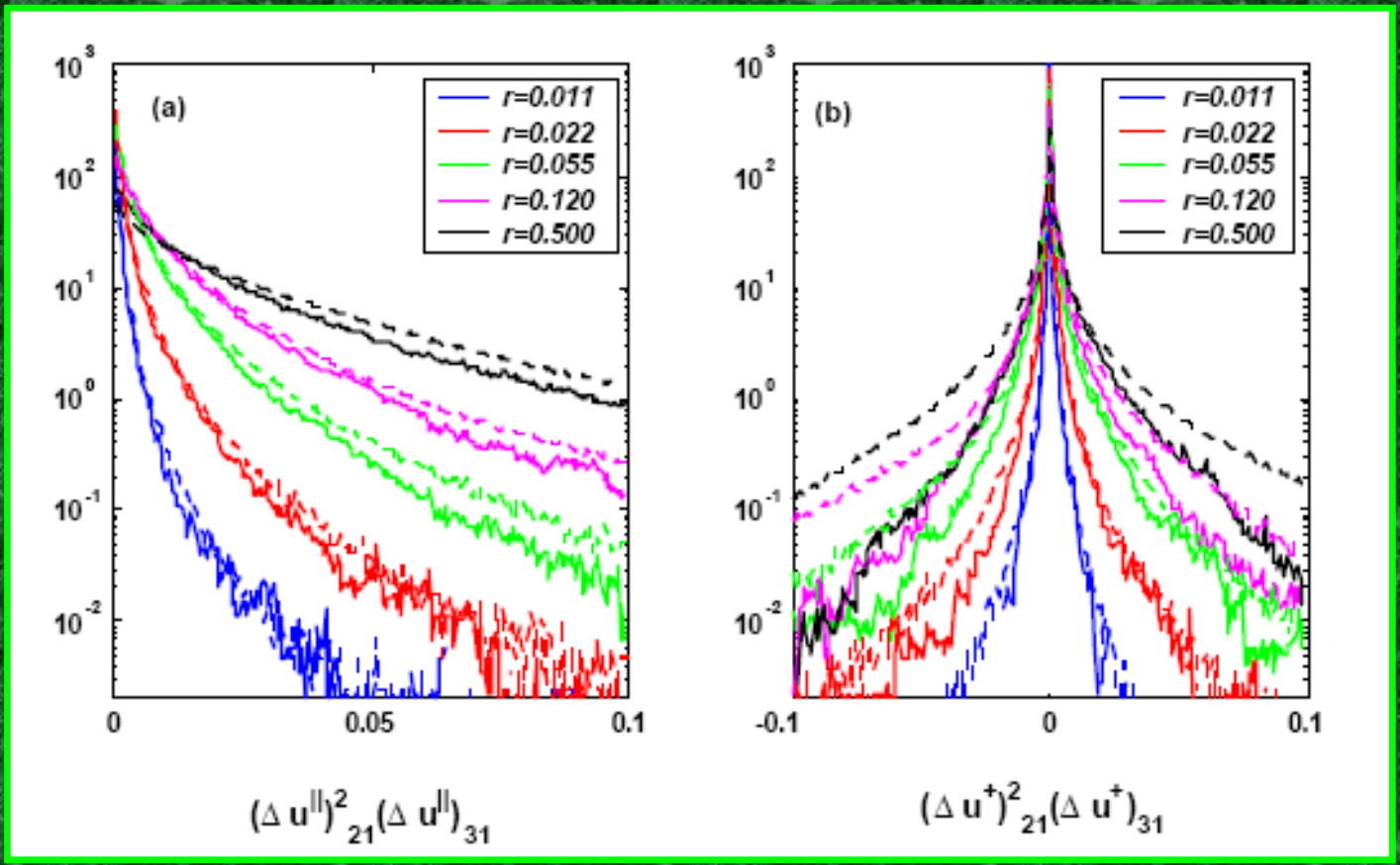


FIG. 9. Three-point statistics I. Velocity increments. PDFs of longitudinal $\Delta u_n^{\parallel}$ (a) and transversal $\Delta u_n^{\perp}$ (b) velocity increments for several distances between the points. The temporal statistics is shown by continuous lines.

IN LIEU OF CONCLUSION

The reported results from a long enough in time numerical simulation provides clear evidence that if a turbulent flow is both statistically stationary in time and homogeneous in space than its temporal and spatial statistical properties are the same. This can be seen as evidence in favor of validity of the ergodic hypothesis in turbulence.

Is this really the sace for all turbulent flows. Can one claim more than that ?

A natural question concerns the **inhomogeneous** flows. One can expect similar results as obtained above for flows with homogeneous coordinates, such as the flow in a plane channel. An obvious conjecture is that the temporal and spatial statistical properties of such a flow will be the same for fixed values of the distance from the wall.

A positive addition to the answer on the question *(when) do simulations reproduce statistics?* At least in some cases one time snapshot is pretty representative

IS THIS ALL?

A ‘CONFESSION’

Whereas it is naturally to expect that non-linear systems driven by a random force should be ergodic, our simulation was made *with purely deterministic and constant in time nonhelical forcing*. Nevertheless, the flow clearly exhibited strong similarity between its temporal and spatial statistical properties with the exception of the largest scales. A possible explanation is that this happens due to the property of self-randomization of fluid-dynamical turbulence (intrinsic stochasticity)

Chaotic behaviour versus ergodicity

One of our referees wrote "*The point in the conclusions stressing that the forcing is deterministic is very weak as it is very well-known that deterministic forcing can yield a random dynamics even for a few degrees of freedom, let alone for a turbulent flow*". Our referee is right, but seems to be not aware that most of low-dimensional chaotic systems are not ergodic!!!

Moreover, the issue is broader and is a part of that on differences between ergodicity and randomness. The story goes back to the general belief that any kind of nonlinearity in a system with large number of degrees of freedom would give rise to ergodicity, see, e.g. FERMI, E. (1923), Beweis dass ein mechanisches normal system im allgemeinen quasi-periodisch ist, *Phys. Z.*, 24, 261 sd.

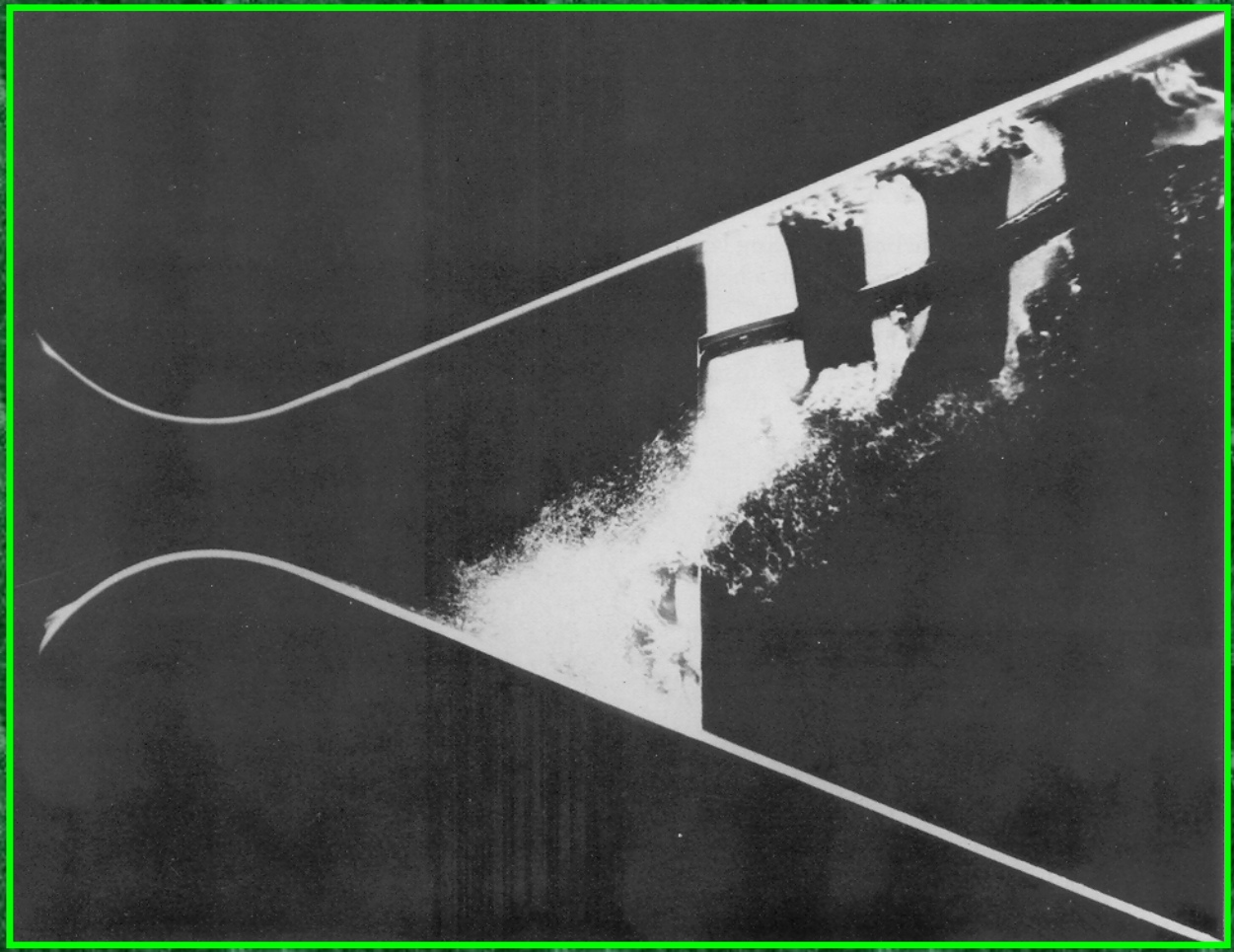
There is another important and very difficult issue. Since the large scale deterministic forcing breaks the ergodicity on large scales the authors removed the mean velocity before comparing the temporal and spatial statistics of the velocity field. So one may put forward an objection that ergodicity is a global property of the dynamical system represented by the Navier-Stokes equations and there cannot be small-scale ergodicity.

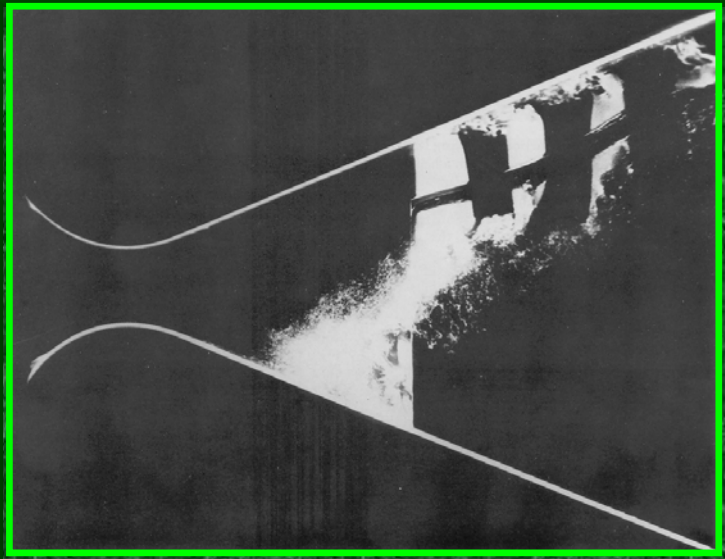
Is it possible to speak about 'approximate' ergodicity or 'modified' ergodicity?

ARE THERE NON-ERGODIC STATISTICALLY STATIONARY TURBULENT FLOWS?

There are many flows that cannot be easily qualified as 'cleanly' ergodic: Flows in diffusers with separation. All partly turbulent flows (mixing layers, jets, wakes past , bodies, boundary layers) which properties depend strongly on the conditions at the entrance (small oscillations of the body, acoustic excitation, etc.) and on the level of disturbances in the quasi-potential flows outside. Minute changes in the latter (i.e. in the entrance conditions in the quasi-potential flows outside) often result in dramatic ones in flows like mentioned above. Sometimes this is considered as 'long memory' of such flows, but there seems to be much more than that as minute changes produce dramatic ones in the statistical properties of the flows.

175. Separated flow in a diffuser. A pulsed hydrogen-bubble wire spanning the diffuser in water shows a turbulent boundary layer attached to the upper wall, but detachment and recirculation along the lower wall. *Kline 1963*





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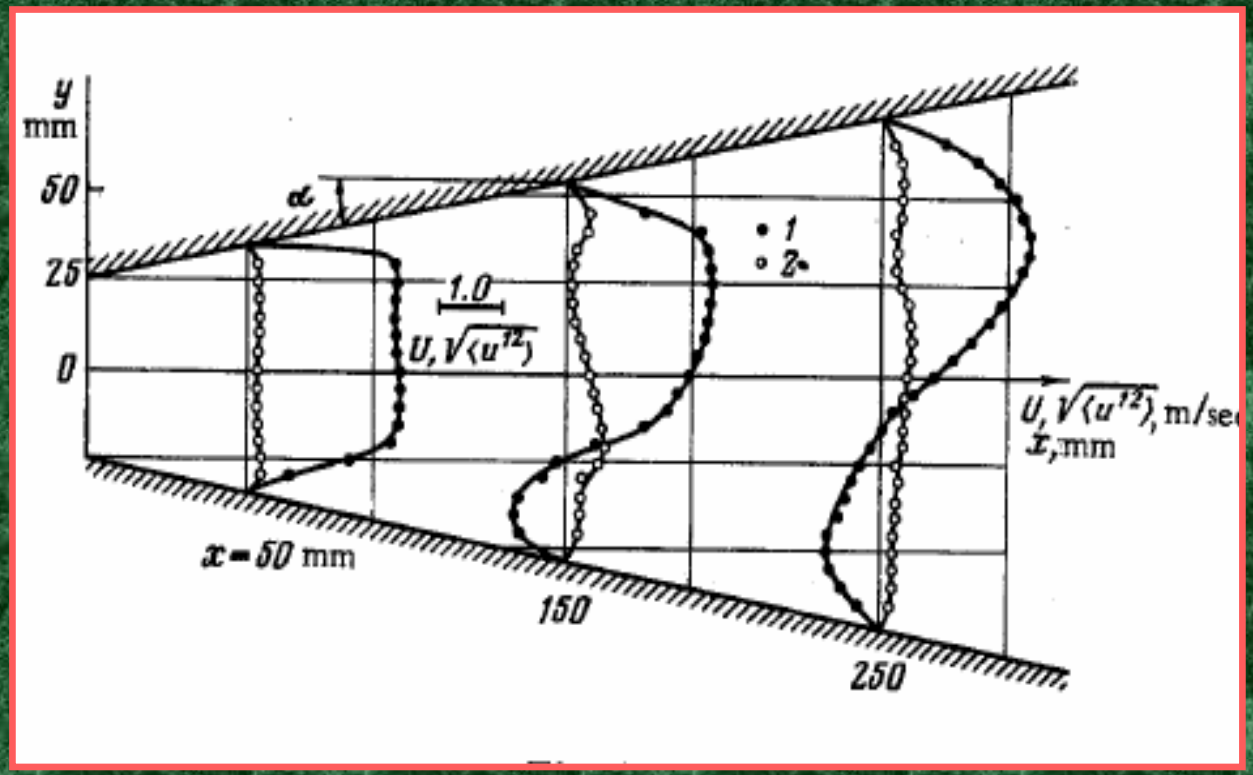

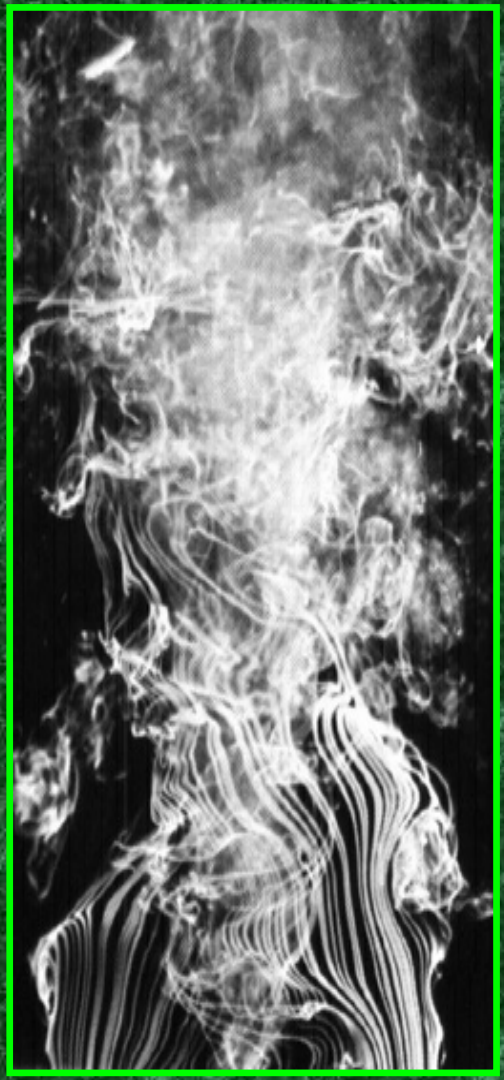

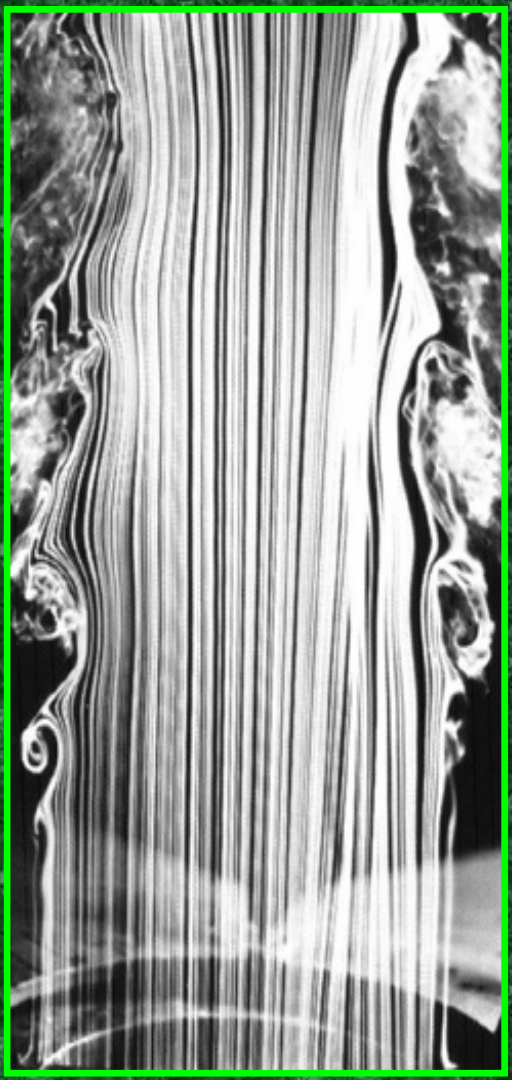

Flows in axisymmetric geometries, e.g. (spontaneously swirling) turbulent flows in jets and pipes

‘Wind’ in turbulent convection

VOLUME 93, NUMBER 16 PHYSICAL REVIEW LETTERS week ending 15 OCTOBER 2004

Multistability and Memory Effect in a Highly Turbulent Flow: Experimental Evidence for a Global Bifurcation

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We report experimental evidence of a global bifurcation on a highly turbulent von Kármán flow. The mean flow presents multiple solutions: the canonical symmetric solution becomes marginally unstable towards a flow which breaks the basic symmetry of the driving apparatus even at very large Reynolds numbers. The global bifurcation between these states is highly subcritical and the system thus keeps a memory of its history. The transition recalls low-dimension dynamical system transitions and exhibits very peculiar statistics. We discuss the role of turbulence in two ways: the multiplicity of hydrodynamical solutions and the effect of fluctuations on the nature of transitions.

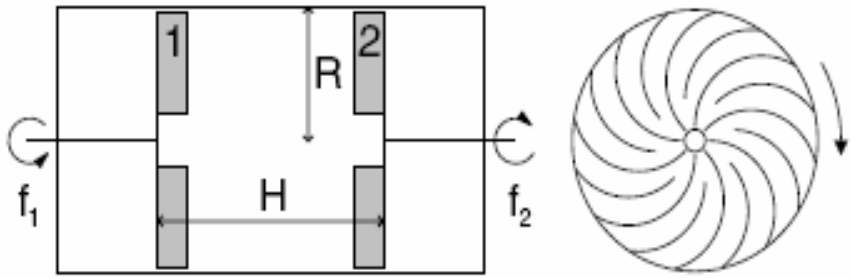

FIG. 1. Sketch of the experimental setup and of the impellers blade profile. The arrow indicates the positive rotation sense.

ARE THERE NON-ERGODIC STATISTICALLY STATIONARY TURBULENT FLOWS?

Related issues

- Memory of turbulence: role of initial conditions, conditions at the entrance etc.
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