Dynamical Systems and Stochastic Processes.

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Preface.

This course is devoted to the study of dynamical systems from a probabilistic/statistical point of view. We will first give the basic definitions and show several examples with complicated (chaotic behaviour). It has long been recognized that the study of complicated dynamics requires a statistical approach (see the work of Boltzmann for example).

A rigorous link with the theory of stochastic processes is made through the notion of invariant measure and ergodic theory. We can then adapt the ideas/concepts of probability and statistics to the study of dynamical systems (correlations, CLT, large deviations etc).

There are however some peculiarities in the study of dynamical systems coming from the fact that the phase space (probability space) has in general more structure (distance, geometry) than in probability theory. We will in particular discuss the chaotic aspects of dynamical systems: s.d.i.c., Lyapunov exponents, entropy.

Measurement techniques have been developed for the analysis of dynamical systems from a statistical point of view. We will discuss several classical techniques and also reconstruction, dimension etc.

In the last section we will mention other probabilistic results concerning dynamical systems: entrance time, quasi stationary distributions, stochastic perturbations etc.

There are many exercises in these lectures. Most of them are very easy. They were built in order that the reader manipulates in simple situations the concepts and the results. Some exercises also describe simple extensions/complements.

Do The Exercises!

The photos are from the Saint Andrews University History of Mathematics archive `http://www-history.mcs.st-and.ac.uk/`.

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1 Introduction: definitions, examples, attractors.

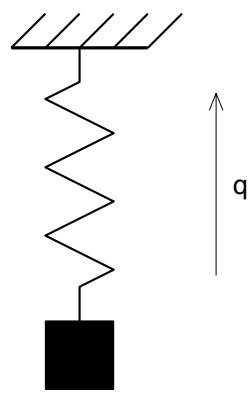
A (classical physical) system is described by the set Ω of all its possible states, often called the phase space of the system.

At a given time, all the properties of the system can be recovered from the knowledge of the instantaneous state $x \in \Omega$. We will see several examples below.

The system is observed using the so called observables which are real valued functions on Ω.

Most often the space of states Ω is a metric space (so we can speak of nearby states). In many physical situations there are even more structures on Ω: $\mathbb{R}^d$, Borel structure, metric space, Riemannian manifolds, Banach or Hilbert spaces etc.

As a simple example one can consider a mechanical system with one degree of freedom like a mass suspended to a (massless) spring and oscillating vertically The state of the system at a given time is given by two real numbers: the



position q and momentum p ($p = mv$, m the mass and $v = dq/dt$ the velocity). The state space is therefore $\mathbb{R}^2$.

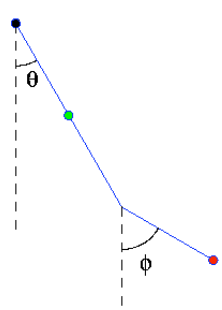


Figure 1: The double pendulum.

For the double pendulum, the state is completely described by the two angles

θ and ϕ and the two angular velocities $\dot\theta$, $\dot\phi$. The phase space is therefore $\Omega = \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{R}^2$.

A continuous material (solid, fluid etc) is characterized by the fields of local velocities, pressure, density, temperature etc. In that case one often uses phase spaces which are Banach spaces.

As time goes on, the instantaneous state of the system changes (unless the system is in a situation of equilibrium).

The double pendulum gives examples of complicated time evolutions. The time evolution is a rule giving the change of the state with time. It comes in several flavors and descriptions.

Discrete time evolution.

This is a map T from the state space Ω into itself producing the new state from the old one after one unit of time. If x_0 is the state of the system at time zero, the state at time one is $x_1 = T(x_0)$ and more generally the state at time n is given recursively by $x_n = T(x_{n-1})$. This is often written $x_n = T^n(x_0)$ with $T^n = T \circ T \circ \cdots \circ T$ (n-times). x_n is often called the nth iterate of x_0 by the map T. The sequence $\big(T^n(x_0)\big)$ is called the trajectory or the orbit of the initial condition x_0.

Continuous time semi-flow.

A semi-flow is a family $\big(\varphi_t\big)_{t\in\mathbb{R}^+}$ of maps of Ω satisfying

$$\varphi_0 = \mathrm{Id}\ , \qquad \varphi_s \circ \varphi_t = \varphi_{s+t}\ .$$

The set $\big(\varphi_t(x_0)\big)_{t\in\mathbb{R}^+}$ is called the trajectory (orbit) of the initial condition x_0. Note that if we fix a time step $\tau > 0$, and observe only the state at times $n\,\tau$ ($n \in \mathbb{N}$), we obtain a discrete time dynamical system given by the map $T = \varphi_\tau$.

A differential equation.

The phase space is a differential manifold and the dynamics associated to a vector field $\vec F$

$$\frac{d\vec x}{dt} = \vec F(\vec x)\ .$$

This is for example the case of a mechanical system in the Hamiltonian formalism. Under regularity conditions on $\vec F$, the integration of this equation leads to a semi-flow (and even a flow): the family φ_t of transformations which map the initial state x_0 at time 0 to the state x_t at time t.

Exercise 1. *Compute the kinetic energy, the potential energy and the Lagrangian for the double pendulum. Use Euler-Lagrange equations to derive the evolution equations.*

There are other more complicated situations like parabolic and hyperbolic non linear partial differential equations, non-autonomous systems where the time evolution depends explicitly on time (for example periodically or stochastically forced systems), systems with memory, systems with delay, etc. but we will mostly not consider them in these lectures.

A dynamical system is a set of states Ω equipped with a time evolution.

If there is more structure on Ω, one can put more structure on the time evolution itself. For example in the case of discrete time, the map T may be measurable, continuous, differentiable etc.

Needless to say, dynamical systems abound in all domains of science. We already mentioned mechanical systems with a finite number of degrees of freedom and continuum mechanics. Mechanical systems with many degrees of freedom are at the root of thermodynamics (and a lot of works have been devoted to the study of billiards which are related with simple models of gases). One can also mention chemical reactions, biological and ecological systems, etc., even many random number generators turn out to be dynamical systems (we will see some examples below). We refer to [104], [113], and [123] for more on the subject and for quality tests.

A useful general notion is that of equivalence (conjugation) of dynamical systems. Assume we have two dynamical systems (Ω_1, T_1) and (Ω_2, T_2). We say that they are conjugated if there is a bijection Φ from Ω_1 to Ω_2 such that

$$\Phi \circ T_1 = T_2 \circ \Phi \ .$$

It follows immediately that $\Phi \circ T_1^n = T_2^n \circ \Phi$, and a similar definition holds for (semi-)flows. Of course the map Φ may have supplementary properties: measurable, continuous etc.

The case of flows of vector fields is left as an exercise.

Exercise 2. *Write the definition of conjugation of two vector fields by differentiating the conjugation of the two associated semi flows.*

Exercise 3. *Let X be a continuous vector field on the circle $\mathbb{T}^1$. Show that if $X \neq 0$ on $\mathbb{T}^1$, then X is conjugated to a constant vector field. Show that this is not true if X is non identically zero and vanishes in at least one point.*

1.1 Examples of dynamical systems.

We now give a list of examples of dynamical systems to which we will refer many times later on. In each of them we first specify the phase space and then the time evolution. These examples appear often in the literature in equivalent (conjugated) forms.

These examples were selected along the history of the subject because they are either simple although showing clearly important concepts, or they were the first systems where one of these important concepts was discovered. As such they are often used in courses (in particular here) to illustrate the concepts and the Theorems.

Example 1. *The phase space is the interval $\Omega = [0, 1]$, the time evolution is the map*

$$T(x) = 2x \pmod 1 \ .$$

This is one of the simplest but, as we will see later on, it is one of the most chaotic dynamical system. Note that this map is not invertible.

Drawback: for hardware reasons this dynamical system is difficult to simulate on a computer.

We now recall the geometric construction of the first iterate for maps of an interval.

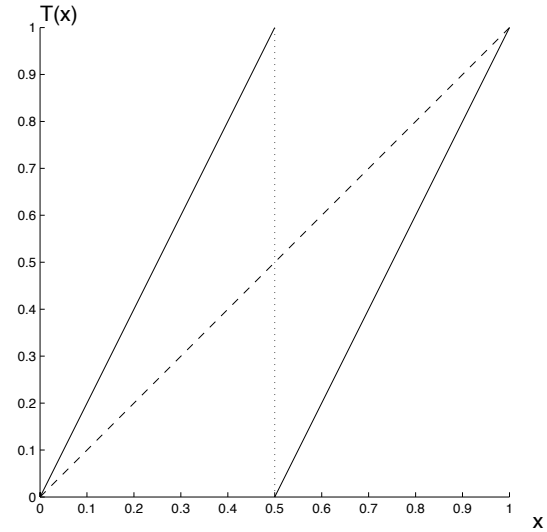


Figure 2: Graph of the map $T(x) = 2x \pmod 1$.

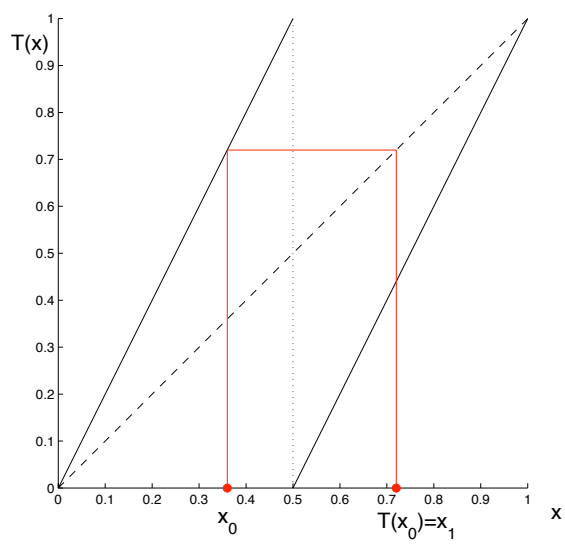


Figure 3: Geometric construction of the first iterate

A generalisation of the above example is the class of piecewise expanding maps of the interval. The phase space is again the interval $\Omega = [0, 1]$. There is a finite sequence $0 = a_0 < a_1 < \ldots < a_k = 1$ and for each interval $]a_j, a_{j+1}[$ $(0 \leq j \leq k-1)$ a monotone map T_j from $]a_j, a_{j+1}[$ to $[0, 1]$ which extends to a C^2 map on $[a_j, a_{j+1}]$.

The time evolution T is given by

$$T(x) = T_j(x) \quad \text{if} \quad x \in]a_j, a_{j+1}[\ .$$

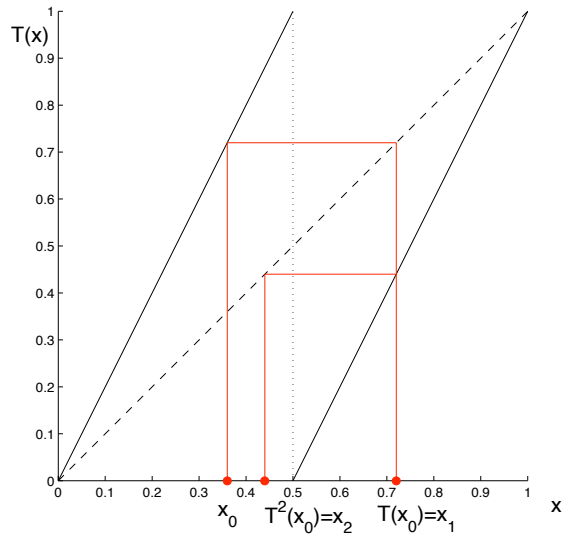


Figure 4: Geometric construction of the first and second iterate

One still has to decide the images of the points a_j, and to impose a (uniform) expanding property.

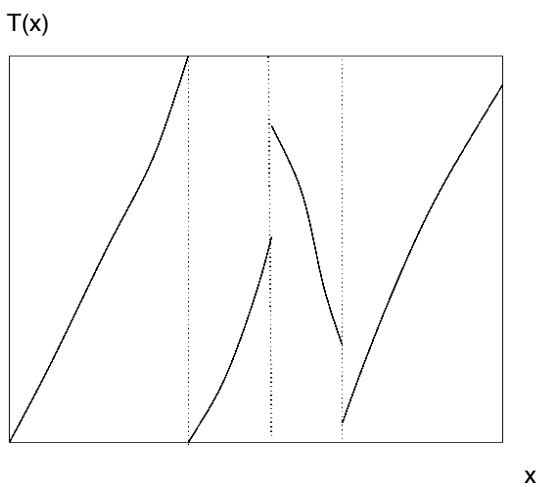


Figure 5: The graph of a piecewise expanding map of the interval.

Maps with slope of modulus larger than one have strong chaotic properties as we will see later on. This leads to the following definition.

Definition 1. *A map of the interval has the (uniform) expanding property if there is an integer $m > 0$ and a constant $c > 1$ such that at any point where T^m is differentiable we have*

$$\left|T^{m\prime}\right| \geq c\,.$$

The map $2x \pmod 1$ is of course an example of piecewise expanding maps of the interval.

A piecewise expanding map of the interval is said to have the Markov property if the closure of the image of any defining interval $]a_j, a_{j+1}[$ is a union of closures of such intervals. This is of course a topological notion but we will see later on that it has some connection with the notion of Markov chains in probability theory.

Graph of a (piecewise expanding) Markov map of the interval: $\overline{T(]a_0, a_1[)} = [a_1, a_2] \cup [a_2, a_3]$ etc. The map $2x \pmod 1$ is of course an example of piecewise

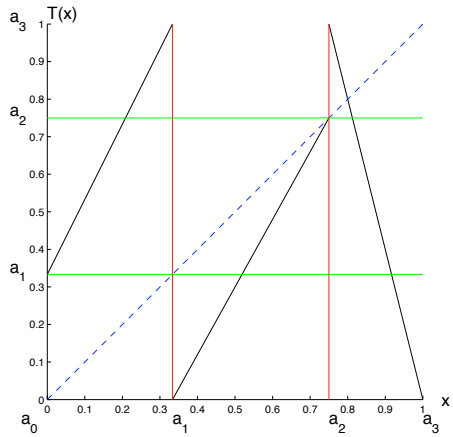


Figure 6: Graph of a (piecewise expanding) Markov map of the interval: $\overline{T(]a_0, a_1[)} = [a_1, a_2] \cup [a_2, a_3]$ etc.

expanding Markov maps of the interval.

Most random number generators turn out to be dynamical systems given by simple expressions. They generate a sequence of numbers which is meant to be a typical realisation of a sequence of independent drawings of a random variable. Since computers act on a finite set of rational numbers, the theory of random number generators is mostly devoted to the study of the arithmetic properties of the maps (for example large minimal period, uniform distribution).

Some popular examples are maps of the interval used in random number generators include $\left[0, 2^{31}-1\right]$ given by $T(x) = 16807x \mod \left(2^{31}-1\right)$ or $T(x) = 48271x \mod \left(2^{31}-1\right)$ which are among the best generators for 32 bits machines.

Example 2. *The previous dynamical systems have discontinuities, and many examples have been studied which are more regular, but often more difficult to analyse. One of the intensively studied example is the one parameter family of quadratic maps. The phase space is the interval $\Omega = [-1, 1]$ and for a value of the parameter $\mu \in [0, 2]$, the time evolution is given by the map T_μ*

$$T_\mu(x) = 1 - \mu x^2 \ .$$

There is a vast literature on these maps and on the more general case of unimodal maps.

Example 3. *We now describe a more abstract example which is in some sense the general model for uniformly chaotic systems.*

Let $\mathscr{A}$ be a finite set (often called a finite alphabet). Let $\mathcal{M}$ be a matrix of zeros and ones of size $|\mathscr{A}| \times |\mathscr{A}|$ ($|\mathscr{A}|$ denotes the cardinality of $\mathscr{A}$). The matrix $\mathcal{M}$ is often called the incidence matrix. Recall that $\mathscr{A}^{\mathbb{Z}}$ is the set of all bi-infinite sequences $\underline{x} = \big(x_p\big)_{p\in\mathbb{Z}}$ of elements of $\mathscr{A}$.

The phase space Ω is defined by

$$\Omega = \big\{\underline{x} \in \mathcal{A}^{\mathbb{Z}} \,\big|\, \mathcal{M}_{x_j,x_{j+1}} = 1 \;\forall\, j \in \mathbb{Z}\big\} \ .$$

In other words, the matrix $\mathcal{M}$ tells which letter can follow a given one.

The time evolution is the shift map $\mathcal{S}$ given by

$$\mathcal{S}\big(\underline{x}\big)_j = x_{j+1} \ .$$

Exercise 4. *Show that $\mathcal{S}$ is an invertible map from Ω to itself (in particular describe the inverse map).*

This dynamical system $\big(\Omega, \mathcal{S}\big)$ is called a sub-shift of finite type. When all the entries of the matrix $\mathcal{M}$ are equal to one, the dynamical system is called the full shift. If one uses $\mathbb{N}$ instead of $\mathbb{Z}$ in the definition of Ω, one speaks of a unilateral shift (which is not invertible).

Consider again the simple case of the map $2x \pmod 1$. A point of the interval $[0,1]$ can be coded by its dyadic decomposition, namely

$$x = \sum_{j=1}^{\infty} \epsilon_j 2^{-j}$$

with $\epsilon_j = 0$ or 1. The decomposition is unique except for a countable set of points.

Exercise 5. *Describe the countable set of points for which the decomposition is not unique.*

It is easy to verify that the map $2x \pmod 1$ acts by shifting the dyadic sequence $\big(\epsilon_j\big)$. Therefore if we consider the alphabet with two symbols $\mathcal{A} = \{0,1\}$, we have a conjugation between the dynamical system given by the map $2x \pmod 1$ on the unit interval and the unilateral full shift over two symbols except for a countable set.

A similar conjugation can be constructed in the case of piecewise expanding Markov maps of the interval. We will use this conjugation later.

Exercise 6. *Consider a piecewise expanding Markov map T of the interval with monotonicity and regularity intervals $]a_j, a_{j+1}[$, $0 \le j < k$. To each point x of the interval we associate the following code $\underline{\omega} \in \mathscr{A}^{\mathbb{N}}$ on the alphabet of k symbols $\mathscr{A} = \{0, \dots, k-1\}$ defined for $n = 0, 1, \dots$ by*

$$\omega_n = j \quad \text{if} \quad T^n(x) \in [a_j, a_{j+1}] \ .$$

Show that this gives a well defined and invertible coding except for a countable number of exceptional points (hint: use the expansivity property). Prove that this coding defines a conjugacy between the piecewise expanding Markov map T and a unilateral sub-shift of finite type (start by constructing the transition matrix $\mathcal{M}$ of zeros and ones).

Up to now, all the examples were one dimensional. We now give examples in two dimension.

Example 4. *An historically important example in two dimension is the so called baker map.*

The phase space is the square $[0,1]^2$ *and the map* T *is defined by*

$$T(x,y) = \begin{cases} (2x, y/2) & \text{if } x \leq 1/2 \\ (2x-1, (1+y)/2) & \text{if } x > 1/2 \end{cases} \tag{1}$$

Exercise 7. *Show that the baker map is area preserving. Hint: compute the Jacobian where the map is differentiable.*

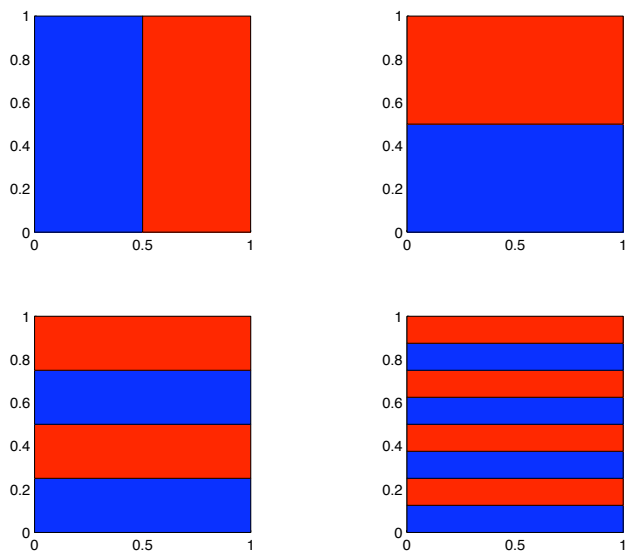
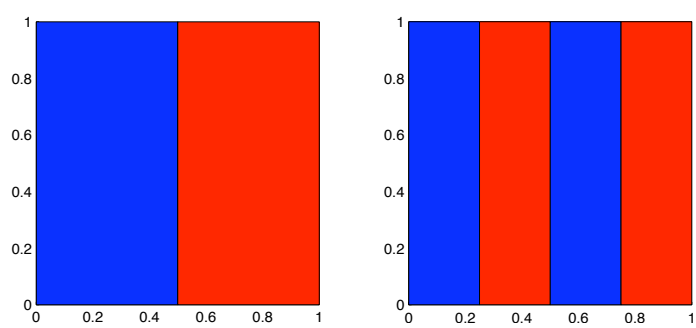


Figure 7: Three iterates of the baker map.

Exercise 8. *Show that except for a "small" set of exceptional points the baker map is invertible.*



Exercise 9. *To each point* (x,y) *of the phase space* $[0,1]^2$*, we can associate a bi-infinite sequence* $\underline{\omega} \in \{0,1\}^{\mathbb{Z}}$ *by*

$$\omega_n = \begin{cases} 0 & \text{if } 0 \leq T^n(x,y)_1 \leq 1/2 \\ 1 & \text{otherwise.} \end{cases} .$$

Here $T^n(x,y)_1$ denotes the first coordinate of the point $T^n(x,y) \in [0,1]^2$. Using this coding show that the baker map is conjugated (except for a "small" set of points) to the full (bilateral) shift over two symbols.

There are of course many variants of this construction which is a particular case of a skew-product system. A dynamical system is called a skew-product if the phase space is a product space $\Omega = \Omega_1 \times \Omega_2$, and the time evolution T has the special form

$$T(x,y) = \big(T_1(x), T_2(x,y)\big) \ .$$

The dynamical system $\big(\Omega_1, T_1\big)$ is often called the base of the skew product, and the map T_2 the action in the fibers (Ω_2).

Another example of skew-product is the dissipative baker map. The phase space is again the unit square $[0,1] \times [0,1]$. The map is for example given by

$$T(x,y) = \begin{cases} (3x, y/4) & \text{if } 0 \le x < 1/3 \\ (3(x-1)/2, (2+y)/3) & \text{if } 1/3 \le x \le 1. \end{cases} \tag{2}$$

There are many variants of this example.

Exercise 10. *Show that T is not invertible on Ω. Show that except for a "small" set, T is a bijection between Ω and $T(\Omega)$. Show that T is area contracting.*

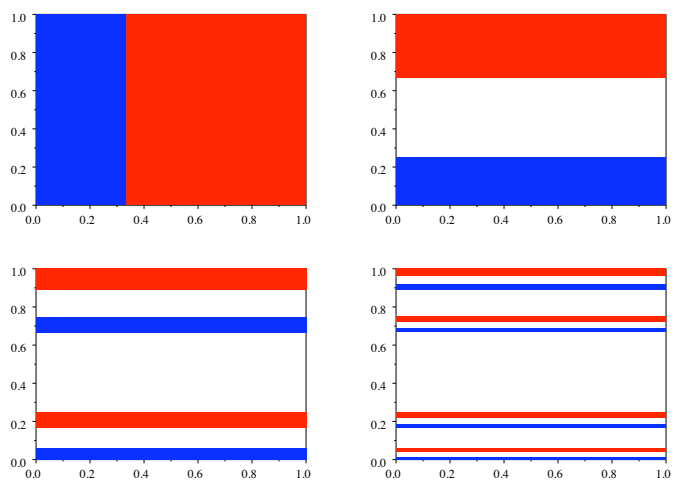


Figure 8: Several iterations of the square under the dissipative baker map (2).

Example 5. *An historically important example of flow in dimension three is the Lorenz system (Lorenz flow).*

The phase space is $\Omega = \mathbb{R}^3$ and the vector field (depending on three parameters σ, r and b) defining the time evolution is given by

$$\vec{X}(x,y,z) = \begin{pmatrix} \sigma(y-x) \\ rx - y - xz \\ xy - bz \end{pmatrix} \ . \tag{3}$$

The historical values of the parameters are $\sigma = 10$, $r = 28$, $b = 8/3$.

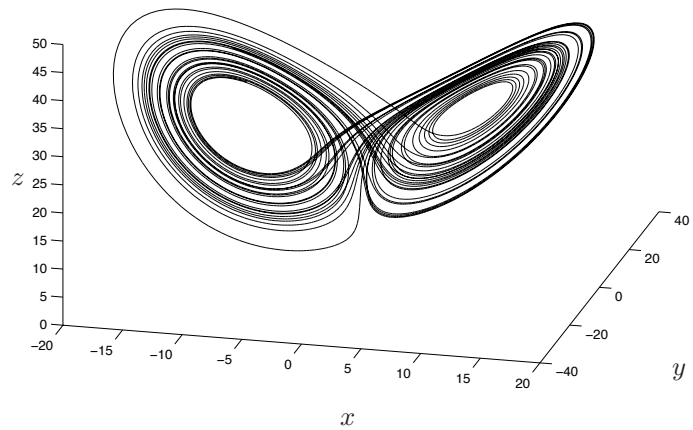


Figure 9: One trajectory of the Lorenz flow

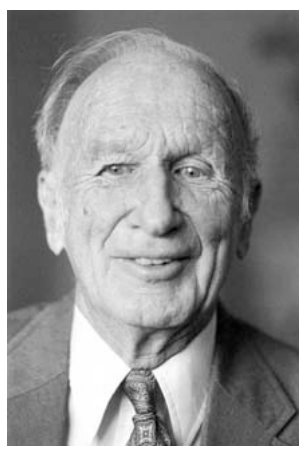


Figure 10: Edward Lorenz (1917-2008).

We now give some more examples of differentiable dynamical systems.

Example 6. *The phase space is the two torus* $\Omega = \mathbb{T}^2$*, and the map is given by*

$$T(x,y) = \begin{pmatrix} x+y & \pmod{2\pi} \\ x+2y & \pmod{2\pi} \end{pmatrix}.$$

This map is often called the cat map.

Exercise 11. *Prove that this map is a* C^∞ *map of the torus with a* C^∞ *inverse. Prove that it is area preserving.*

The dynamics is often illustrated using the picture of a cat. The first stretching can be viewed on the plane. One then identifies the corresponding points

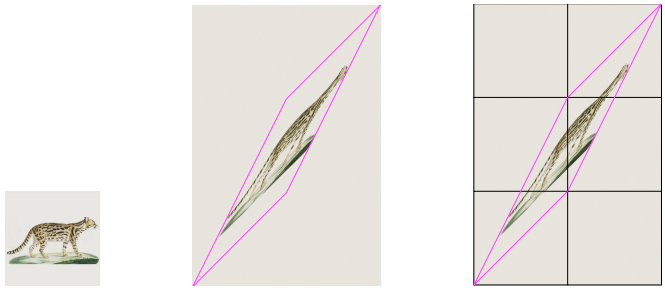


Figure 11: Geometrical construction of the iteration of the cat map.

on the torus. The polygon in red is the image of the original rectangle.

One then identifies the corresponding points on the torus.

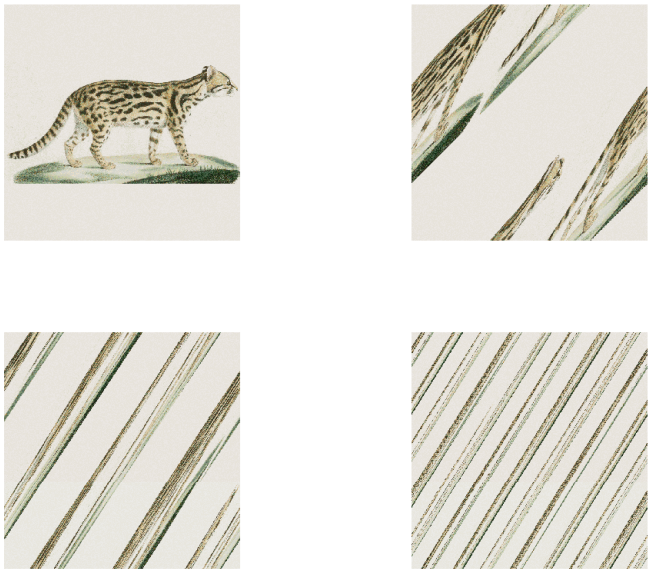


Figure 12: Three iterations of the cat map.

Example 7. *A more sophisticated example is the so called standard map of the torus*

$$T(x,y) = \begin{pmatrix} 2x - y + a\sin(2\pi x) \pmod 1 \\ x \end{pmatrix}. \tag{4}$$

This is a one parameter family of maps with parameter a. These maps appear as simple models for other more complicated and interesting area preserving systems.

Exercise 12. *Prove that the standard map of the torus is C^∞ with a C^∞ inverse and area preserving.*

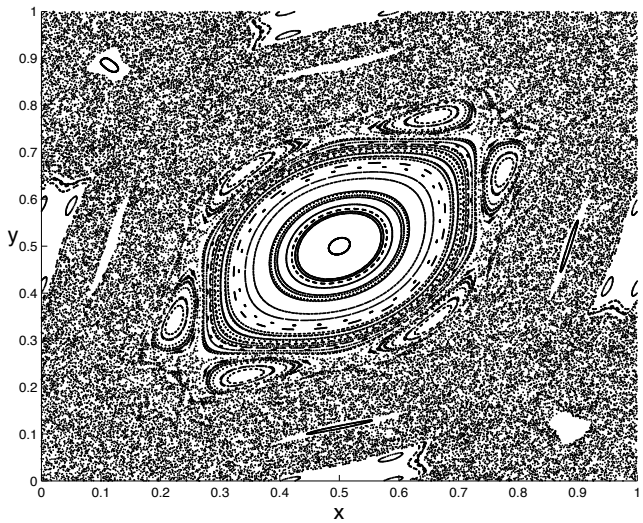


Figure 13: Several trajectories of the standard map with parameter $a = .2387$.

Example 8. *Another historically important example is the Hénon map [80].*

The phase space is $\Omega = \mathbb{R}^2$ *and the two parameters (a and b) family of maps is given by*

$$T_{a,b}(x,y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} . \tag{5}$$

The "historical" values of the parameters are $a = 1.4$, $b = .3$.

Exercise 13. *Prove that the Hénon map is invertible with a regular inverse if* $b \neq 0$ *(give the formula for the inverse). Prove that it is area contracting.*

Example 9. *Another important example in three dimensions is given by the so-called solenoid map.*

The phase space is $\Omega = \mathbb{R}^3$. *In cylindrical coordinates* (ρ, θ, z) *the map is given by*

$$T(\rho,\theta,z) = \begin{pmatrix} R + r\cos(\theta) + \lambda\rho \\ 2\theta \pmod{2\pi} \\ r\sin(\theta) + \lambda z \end{pmatrix}$$

where $0 < \lambda < 1/2$ *and* $R > r/\lambda > 0$.

Exercise 14. *Show that for any* $r' \in]r/(1-\lambda), r/\lambda[$, *the solid torus* $(\rho - R)^2 + z^2 < r'^2$ *is mapped injectively into itself by* T.

For the case of continuous time dynamical systems, given by vector fields, important examples are mechanical systems. We recall that the phase space is $\Omega = \mathbb{R}^{2d}$ (or some even dimensional manifold). The first d coordinates $q_1, \ldots, q_d$ are position coordinates, and the remaining d coordinates $p_1, \ldots, p_d$ are the corresponding momentum coordinates. A real valued function $H(\vec{q}, \vec{p})$ is given on the phase space and is called the Hamiltonian of the mechanical system. The time evolution is given by Hamilton's system of equations

$$\frac{d\,q_j}{dt} = \partial_{p_j} H \qquad \frac{d\,p_j}{dt} = -\partial_{q_j} H \qquad (j = 1 \ldots n) . \tag{6}$$

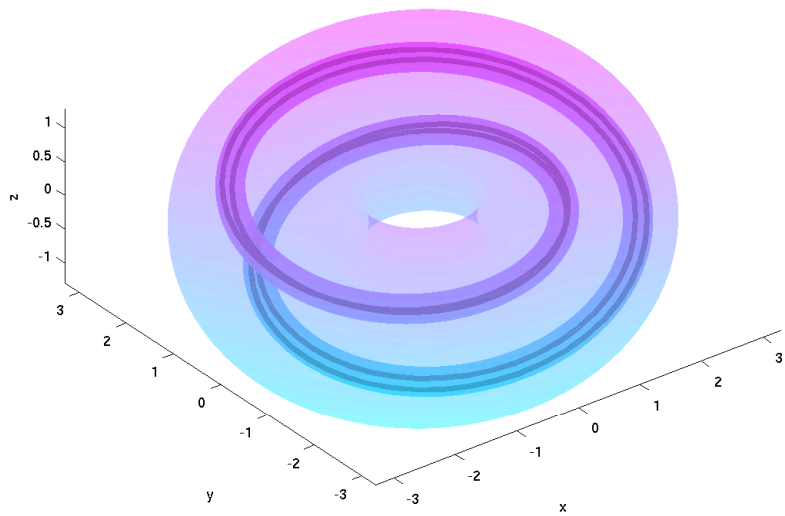


Figure 14: Three iterations of the torus under the solenoid map.

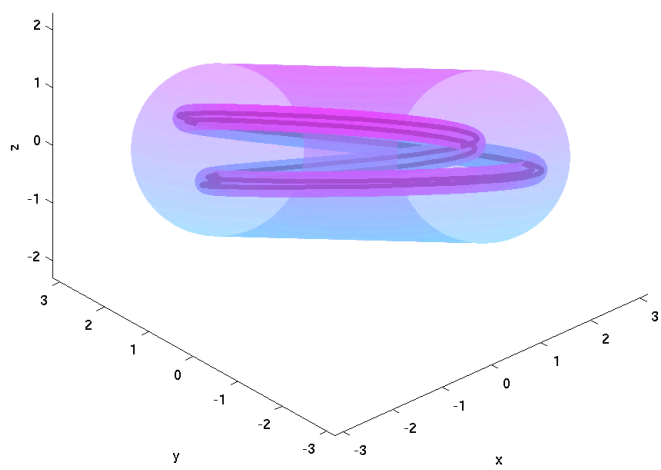


Figure 15: Three iterations of the torus under the solenoid map, lateral view.

One can also add friction terms. Note that systems with time dependent forcing (non autonomous systems) do not fall in our present description, unless one can give a dynamical description of the forcing (this often leads to a skew

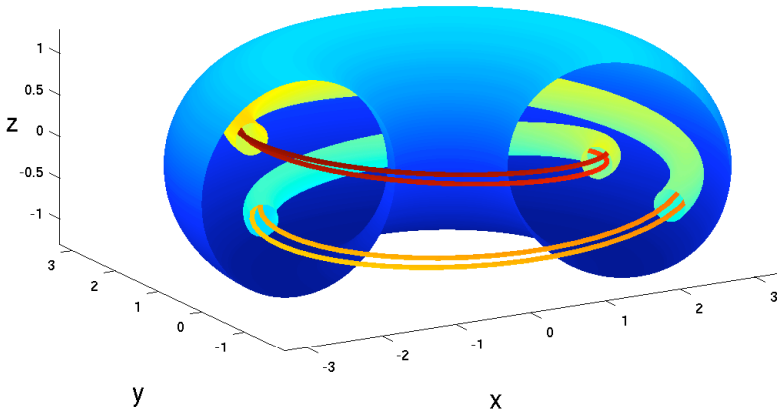


Figure 16: Inside the solenoid.

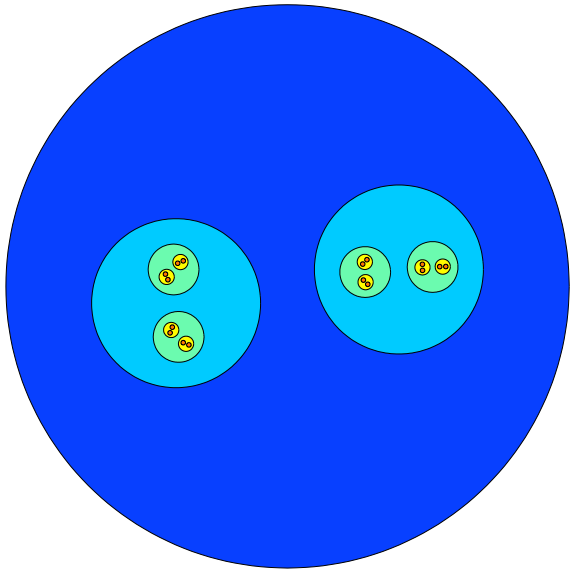


Figure 17: A section of the solenoid.

product system).

There are many other continuous time dynamical systems from physical origin like for example the Navier-Stokes equation. We will not attempt to list them here.

Many works have been devoted to the study of particularly important families of dynamical systems. I will just mention some of these but we will not have time to go into the details.

Hamiltonian systems (KAM theory), diffeomorphisms of the circle, holomor-

phic dynamics, maps of the interval, systems with rapid forcing (averaging), etc.

1.2 A geometric concept: Attractor.

We will see later on that there is a nice interplay in the study of dynamical systems between the geometric and ergodic/stochastic approach.

One of the first example of this interplay comes from the notion of attractor in dissipative systems. There are several possible definitions for an attractor and we will adopt the following one formulated for discrete time evolution. There is an analogous notion for continuous time evolution.

Definition 2. *We say that a subset A of the phase space is invariant by the map T if $T(A) \subset A$.*

Definition 3. *A (compact) invariant subset $\mathscr{A}$ of the phase space is an attracting set if there is a neighborhood V of $\mathscr{A}$ such that for any neighborhood U of $\mathscr{A}$, there is an integer n_U such that for any $n > n_U$, $T^n(V) \subset U$.*

In particular, all orbits with initial condition in V accumulate on $\mathscr{A}$.

We will illustrate this concept using the Hénon map

$$T_{a,b}(x,y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} .$$

using parameter values for which the attractor is composed of a (stable) periodic orbit of period 2.

The Hénon map with $a = .64$, $b = .3$ has a stable periodic orbit of period 2.

Definition 4. *We will say that a (compact) invariant subset $\mathscr{A}$ of the phase space is an attractor if it is an attracting set containing a dense orbit (this ensure irreducibility in a topological sense).*

The basin of attraction of an attracting set $\mathscr{A}$ is the set of initial conditions whose orbit accumulate on $\mathscr{A}$.

It follows immediately from this definition that if we have an attracting set $\mathscr{A}$,

any invariant measure with support contained in the basin of this attracting set should have support in $\mathscr{A}$.

Attractors may have complicated geometry in which case they are called strange attractors (see the solenoid above for example). We now give some more pictures of attractors.

The next figure is a plot of the "attractor" for the Hénon map

$$T(x,y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} ,$$

with the original values of the parameters $a = 1.4$, $b = .3$ of Michel Hénon. This picture was obtained by starting with an initial condition at the origin $(0,0)$, iterating a large number of times to be as near as possible to the attractor,

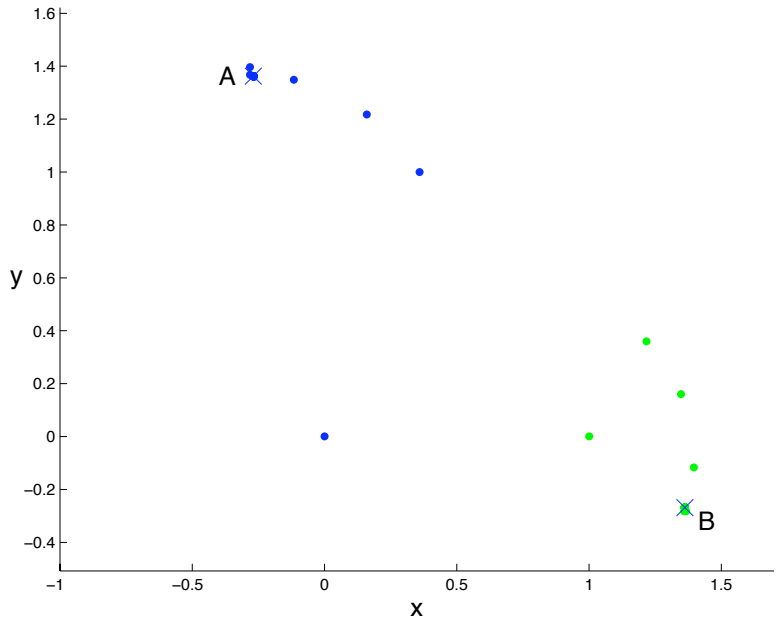


Figure 18: Iteration of the Hénon map with $a = .64$, $b = .3$, initial condition at the origin. Even iterates are in blue, odd iterates in green.

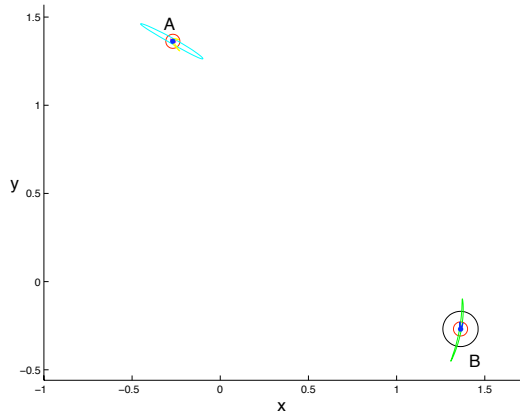


Figure 19: Iteration of the Hénon map with $a = .64$, $b = .3$. The neighborhood V is in black, (only half of it is represented), its iterates are represented using different colors. The neighborhood U is in red.

and then plotting a certain number of the following iterations supposed to visit densely the attractor.

Recall the formula for the Lorenz differential system [118]

$$\frac{dx}{dt} = \sigma(y - x)$$

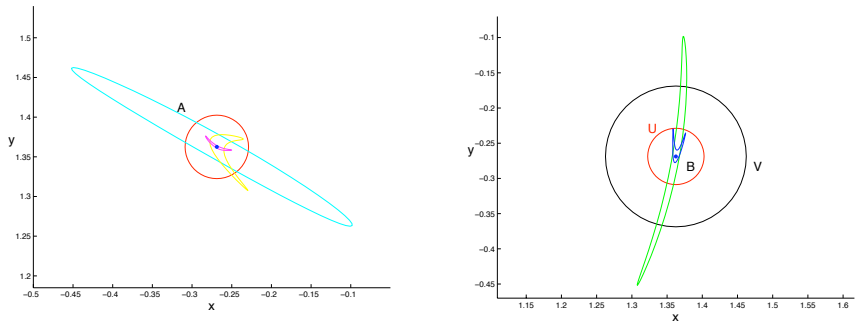


Figure 20: Zoom of the previous picture.

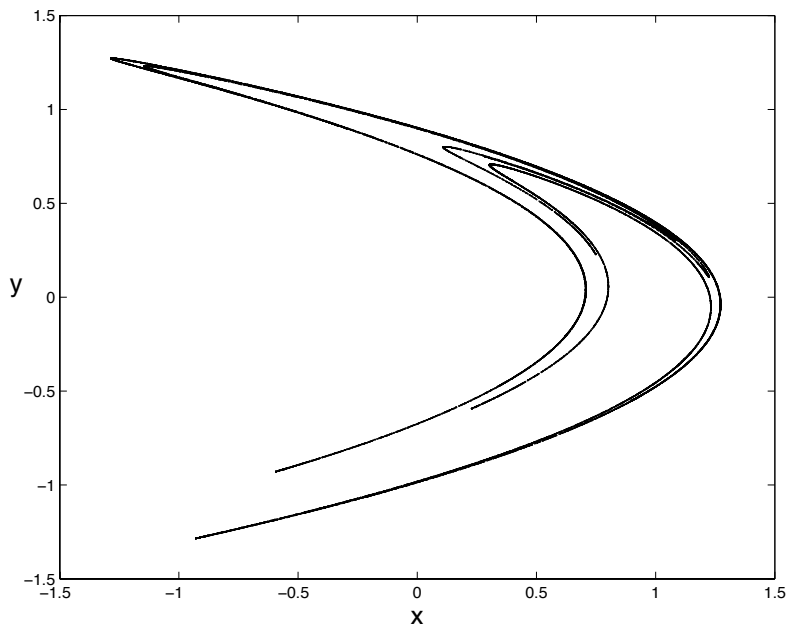


Figure 21: Attractor of the Hénon map.

$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - bz$$

We now make some general remarks.

1. The simplest attractors are the stable fixed points (stationary solutions) and the stable periodic orbits (stable invariant cycles in the case of flows). See the appendix of section 5 for the notion of stability of a fixed point or periodic orbit.

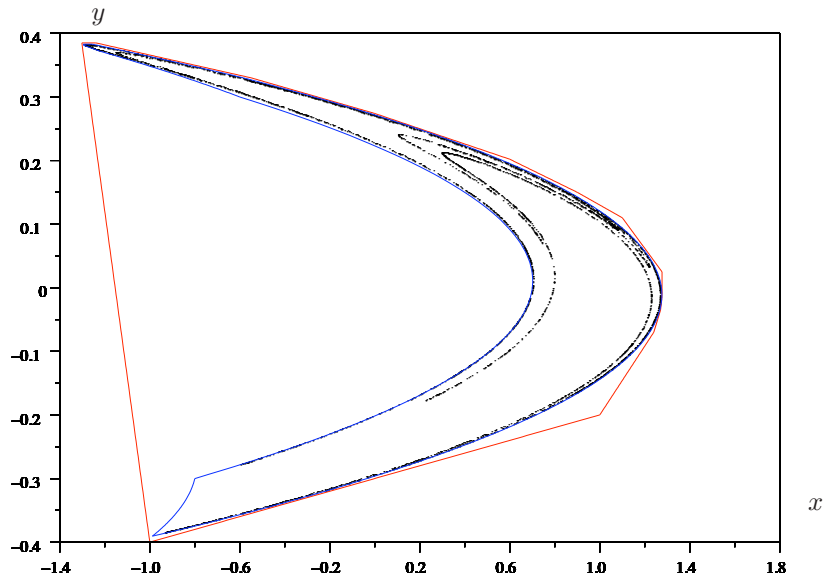


Figure 22: A subset V for checking that we have an attracting set for the Hénon map, and its first iterate.

2. If the dynamical system depends on a parameter, the attractor will also in general change with the parameter, not only quantitatively but also qualitatively as occurs for example in bifurcations. In the next figure is drawn the attractor of the quadratic family $x \mapsto 1 - \mu x^2$ as a function of the parameter μ. One sees in particular the famous sequence of period doubling bifurcations.

3. A dynamical system may have several attractors in the same phase space.

4. Each attractor has its own basin: the set of initial conditions attracted by this attractor. The interior of the basins of attraction of different attractors are disjoint. The boundaries of the basins of attraction of different attractors are disjoint may not be disjoint.

5. The boundaries of the basins are in general complicated invariant sets, repelling transversally (toward the attractors). A well known example is provided by the Newton method applied to the equation $z^3 = 1$. The method consists in iterating the map T given by $z_{n+1} = T(z_n) = (z_n + 2/z_n^2)/3$. There are three stable fixed points (attractors): $1, j, \overline{j}$. The next figure is a drawing of the common boundary of the basins of attraction of these three fixed points.

The concept of attractor is suited for the study of dissipative systems.

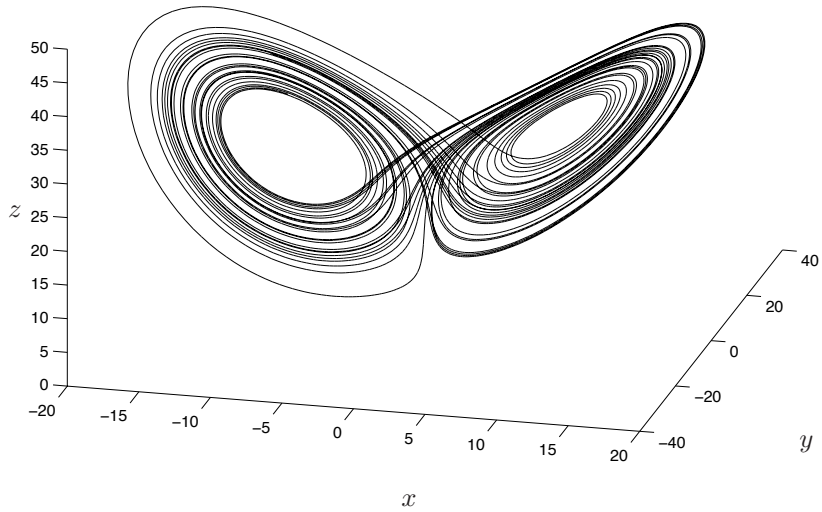


Figure 23: The attractor of the Lorenz system (3). Parameter values $\sigma = 10$, $r = 28$, $b = 8/3$.

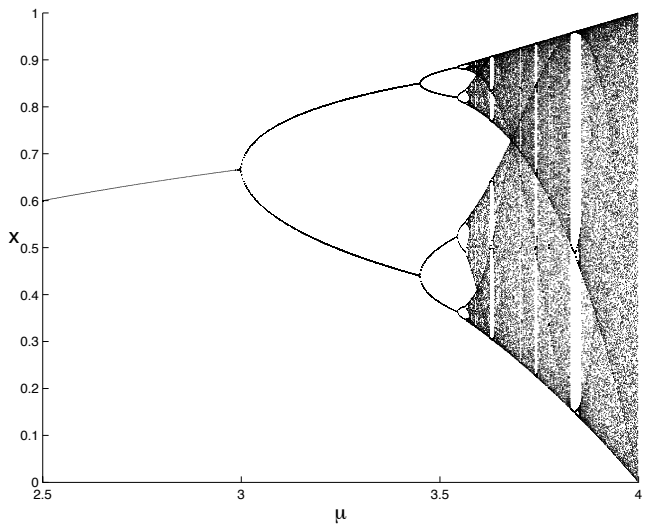


Figure 24: Attractor of the quadratic family as a function of the parameter.

Volume preserving systems do not have attractors, like the mechanical systems without friction.

One can think in that case that the whole phase space is an attractor if there

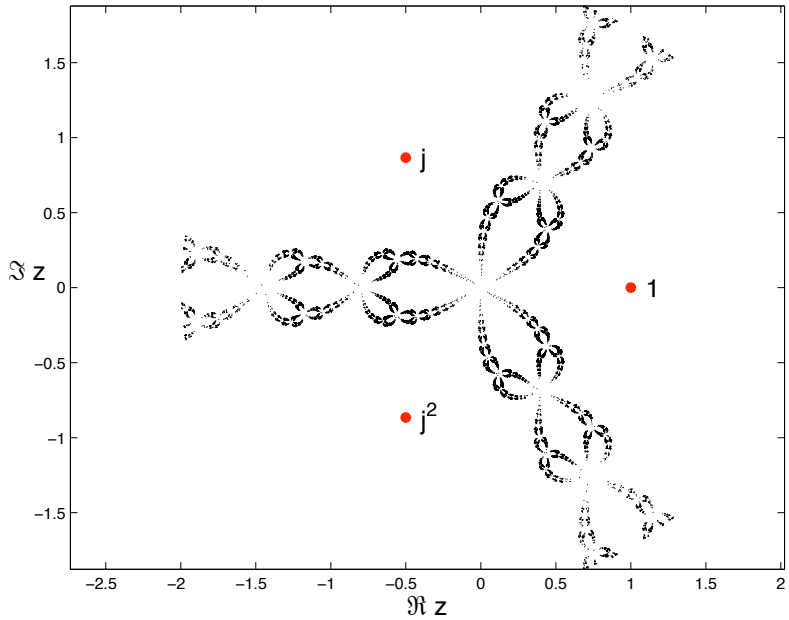


Figure 25: Intersection with the disk of radius 2 of the common boundary of the basins of attraction of the three fixed points of the Newton method applied to $z^3 = 1$.

is a dense orbit.

Exercise 15. *Show that the attractor of the dissipative baker map (2) is the product of a Cantor set by a segment.*

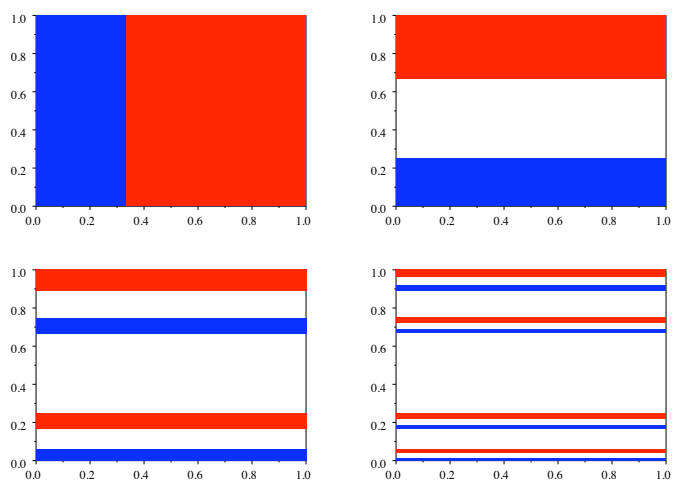


Figure 26: Several iterations of the square under the dissipative baker map (2).

1.3 Appendix: some simple topological notions.

The first step in a topological study of dynamical systems concerns the simplest aspects of time evolution.

Definition 5. *For a discrete time evolution T, a fixed point is a point x in the phase space such that $T(x) = x$. This is a state of rest.*

This generalises immediately to periodic orbits.

Definition 6. *For a discrete time evolution T, a point x in the phase space is periodic of period $k \geq 1$ (minimal period) if $T^k(x) = x$ and $T^\ell(x) \neq x$ for $\ell = 1, \ldots, k-1$.*

A fixed point is periodic of period one.

The set of points $\{x, T(x), \ldots, T^{k-1}(x)\}$ is called a periodic orbit.

Exercise 16. *Show that if x is a periodic point of period k, any point in the orbit of x is also a periodic point of period k. Show that any point in the orbit of x is a fixed point of T^k.*

Definition 7. *For a continuous time dynamical system given by a vector field X or a flow (φ_t), a point x in the phase space is a fixed point if $X(x) = 0$ or equivalently $\varphi_t(x) = x$ for any $t \in \mathbb{R}$.*

Definition 8. *For a continuous time dynamical system given by a vector field X or a flow (φ_t), a point x is periodic of (minimal) period $\tau > 0$ if $\varphi_\tau(x) = x$, and $\varphi_t(x) \neq x$ for any $0 < t < \tau$. The orbit of x is called a closed orbit or a cycle.*

Any point of the orbit of x is of course periodic with period τ.

When a flow has a closed orbit, one can reduce (near the closed orbit) the continuous time dynamical system to a discrete time map by the technique of Poincaré map. Let π be a small enough piece of regular manifold, intersecting

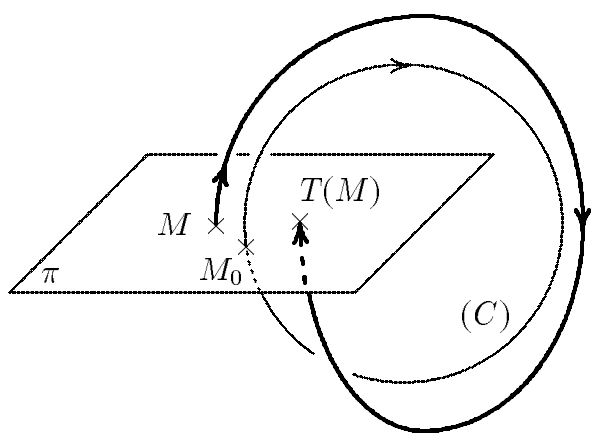


Figure 27: Poincaré Section.

transversally the closed invariant curve (C) in a (unique) point M_0. For M near

M_0 in π, the trajectory of M hits π for the first time in a point $T(M)$. The map T is the Poincaré map of the invariant curve (C). The piece of regular manifold π has to be small enough so that the map T is well defined and the return crossing happens in "the same direction".

Figure 28: Henri Poincaré (1854-1912)

.

We also mention that one can construct a continuous time dynamical system using the technique of suspension of a discrete time system, see [94].

2 Invariant measures.

The basic goal of the study of dynamical systems is to understand the effect of time evolution on the state of the system, and primarily the long time behaviour of the state (although transient behaviours are sometimes very interesting and important in applications). There are basically two approaches to this study.

The first one can be called topological or geometric and studies objects like periodic orbits, attractors, bifurcations etc. We will barely touch this aspect of dynamical systems in these notes. We refer the reader to [38], [79], [94], [138], and references therein for the large literature on this subject.

The second approach can be called ergodic and studies the behavior of trajectories from a measure theoretic (probabilistic/statistical) point of view. This is mostly interesting when there is a large variety of complex dynamical behaviours depending on the initial condition. One can for example discuss average quantities whose interest often come from the geometrical approach.

In this course we will mostly deal with this second more statistical approach. Nevertheless, as we will often see below the interplay between the two approaches (geometric and ergodic) is particularly important and fruitful.

The main contact between dynamical systems and probabilities is through measures. We will consider normalised (probability) measures on the phase space Ω of the dynamical system. Let μ be such a measure, we can think of it as a distribution on initial conditions, or some kind of weight on the phase space.

Recall that there are various sorts of measures.

- Absolutely continuous measures which are given by a density ρ with respect to the Lebesgue measure dx. The measure (mass/weigth/probability) of a (measurable set) A is given by

  $$\mu(A) = \int_A \rho(x)\, dx = \int \chi_A(x)\rho(x)\, dx$$

  where χ_A denotes the characteristic function of the set A.

- Atomic measures which are linear combinations of Dirac point masses. A Dirac mass at x is a measure δ_x given by

  $$\delta_x(A) = \{ \ 1 \text{ if } x \in A 0 \text{ otherwise}$$

  For an atomic measure ν with finitely many atoms $(x_1, \ldots, x_N)$, the measure of a set A is given by

  $$\nu(A) = \sum_{m=1}^{N} \alpha_m \delta_m(A)$$

  where $(\alpha_1, \ldots, \alpha_N)$ are numbers (integers, reals, complex). One can think of Dirac mass as the limit of a normalised measure with positive density whose width tend to zero.

- None of the above (like the Cantor measure).

- Linear combinations of the above.

Recall that a positive measure of total mass unity is called a probability measure. The support of a positive measure is the smallest closed set whose complement has measure zero.

It is natural to ask what is the distribution of the points in phase space after n time steps if they were distributed according to a measure μ at time zero. This distribution at time n is the image measure or the push-forward of μ by T^n, namely

$$T^n_* \mu(A) = \mu(T^{-n}(A)) \ .$$

In this formula, when T is not invertible, T^{-1} denotes the set inverse map, namely

$$T^{-1}(A) = \big\{ \omega \in \Omega \,\big|\, T(\omega) \in A \big\} \ .$$

If μ is a probability measure on Ω, ${T^n}_* \mu = \mu \circ T^{-n}$ is also a probability measure on Ω and we can 'test' this distribution by using random variables (observables). For example, the probability that at time n a trajectory is in the subset A of Ω is

$$\begin{aligned} \int_A d{T^n}_* \mu = \int_\Omega \chi_A d{T^n}_* \mu &= \int_\Omega \chi_A \circ T^n(\omega) \, d\mu(\omega) \\ &= \int_\Omega \chi_A \big(T^n(\omega)\big) \, d\mu(\omega) \ . \end{aligned}$$

Given a probability measure μ on the phase space Ω and the time evolution map T (or a (semi)-flow), we are exactly in the setting of stochastic processes. For example, if we have a preferred observable g, namely a real measurable function g on Ω (for example the characteristic function of a measurable subset), we can look at the stochastic process (X_n) given by

$$X_n = g \circ T^n \ .$$

This is indeed a sequence of real random variables on the probability space (Ω, μ), namely a real valued stochastic process.

When a probability measure μ is selected on Ω, we will use the standard probabilistic notation $\mathbb{P}$, $\mathbb{E}$ for the probability and expectation with respect to μ. For example with the above random variables (X_n) we have

$$\begin{aligned} \mathbb{P}\big(X_0 \in A_0, \dots, X_n \in A_n\big) &= \mathbb{P}\big(g(x) \in A_0, \dots, g\big(T^n(x)\big) \in A_n\big) \\ &= \mathbb{P}\big(x \in g^{-1}\big(A_0\big), \dots, T^n(x) \in g^{-1}\big(A_n\big)\big) \\ &= \mathbb{P}\big(x \in g^{-1}\big(A_0\big), x \in T^{-1}\big(g^{-1}\big(A_1\big)\big) \dots, x \in T^{-n}\big(g^{-1}\big(A_n\big)\big)\big) \\ &= \int_{\cap_{j=0}^n T^{-j}\big(g^{-1}\big(A_n\big)\big)} d\mu(x) = \int \prod_{j=0}^n \chi_{g^{-1}(A)} \circ T^j(x) \, d\mu(x) \ . \end{aligned}$$

Exercise 17. *Prove this important formula and invent a few other equivalent ones.*

The fact which may look a little unusual for a $\mathbb{P}$robabilist is that the probability is given on the initial condition, and there is no randomness appearing in the time evolution. However the points on the phase space completely characterize the orbits, and we can think of μ as a probability measure on the orbits, the time evolution being the shift.

In that sense, any stochastic process is a dynamical system (a not very useful remark in practice).

Recall that we will be mostly interested in the long time behaviour of the orbits, and in particular in the statistical description of complicated long time evolution. The tools for this kind of approach were developed under the name of ergodic theory starting from the works of Boltzmann.

Figure 29: Ludwig Boltzmann (1844-1906).

To introduce in more details the ergodic approach, let us consider the following question which is historically one of the first motivation of ergodic theory. Let A be subset of the phase space Ω (describing the states with a property of interest, for example that all the molecules in this room are in the left half). One would like to know for example in a long time interval $[0, N]$ (N large) how much time the system has spent in A, namely how often the state has the property described by A? Assume for simplicity we have a discrete time evolution.

If χ_A denotes the characteristic function of the set A, the average time the system has spent in A over an interval $[0, N]$ starting in the initial state x_0 is given by

$$\mathcal{A}_N(x_0, A) = \frac{1}{N+1} \sum_{j=0}^{N} \chi_A\big(T^j(x_0)\big) \ . \tag{7}$$

It is natural to ask if this quantity has a limit when N tends to infinity. The answer may of course depend on A and x_0, but we can already make two important remarks. Assume the limit exists and denote it by $\mu_{x_0}(A)$.

First remark: it is easy to check that the limit also exists for $T(x_0)$ and also for any $y \in T^{-1}(x_0) = \big\{z \,|\, T(z) = x_0\big\}$. Moreover we have

$$\mu_{T(x_0)}(A) = \mu_y(A) = \mu_{x_0}(A) \ . \tag{8}$$

Second remark: the limit also exists if A is replaced by $T^{-1}(A)$ $(= \{\omega \,|\, T(\omega) \in$

$A\}$) and has the same value, namely

$$\mu_{x_0}(T^{-1}(A)) = \mu_{x_0}(A) \ . \tag{9}$$

If one assumes that $\mu_{x_0}(A)$ does not depend on x_0 at least for Borel sets A (or some other sigma algebra but we will mostly consider the Borel sigma algebra below), one is immediately lead to the notion of invariant measure.

Definition 9. *A measure μ on a sigma-algebra $\mathscr{B}$ is invariant by the measurable map T if for any measurable set A*

$$\mu\big(T^{-1}(A)\big) = \mu(A) \ . \tag{10}$$

A similar definition holds for (semi-)flows.

Unless otherwise stated, when speaking below of an invariant measure we will assume it is a probability measure. For simplicity we will also assume that the phase space is compact, although this condition is in many cases not necessary. We will denote by $(\Omega, T, \mathscr{B}, \mu)$ the dynamical system with state space Ω, discrete time evolution T, $\mathscr{B}$ is a sigma-algebra on Ω such that T is measurable with respect to $\mathscr{B}$ and μ is a measure on $\mathscr{B}$ invariant by T. As said above $\mathscr{B}$ will most often be the Borel sigma-algebra and we will often not mention it.

Exercise 18. *Let (Ω, T) be a dynamical system and assume Ω is a metric space. Assume T is continuous, and let μ be a Borel measure. Show that μ is invariant if and only if*

$$\int g \circ T \, d\mu = \int g \, d\mu \ ,$$

for any bounded continuous function g.

The goal of ergodic theory is to study systems $(\Omega, T, \mathscr{B}, \mu)$ and in particular their large time evolution. Assume now we have an observable g which we recall is a measurable function on the phase space Ω. We have seen above one such observable, namely the function χ_A which takes the value one if the state is in A (has the property described by A) and takes the value zero otherwise. $(\Omega, \mathscr{B}, \mu)$ is a probability space and therefore g is a random variable on this probability space.

More generally $(X_n) = \big(g \circ T^n\big)$ is a discrete time, stationary, stochastic process. Indeed by the invariance of the measure

$$\mathbb{P}\big(X_p \in A_0, \ldots, (X_{p+n} \in A_n\big) = \int \prod_{m=0}^{n} \chi_{g^{-1}(A_m)}\big(T^{n+j}(x)\big) \, d\mu(x)$$

$$= \int \prod_{m=0}^{n} \chi_{g^{-1}(A_m)}\big(T^j\big(T^n(x)\big)\big) \, d\mu(x) = \int \prod_{m=0}^{n} \chi_{g^{-1}(A_m)}\big(T^j(y)\big) \, d\mu(y)$$

$$= \mathbb{P}\big(X_0 \in A_0, \ldots, (X_n \in A_n\big) \ .$$

If $g = \chi_A$, this process describes the occurrences of a certain event (property) as times goes on.

Therefore we can apply all the ideas and results of the theory of stochastic process to dynamical systems equipped with an invariant measure. The goal of

the statistical approach to dynamical systems is to study these "particular" processes. As mentioned above and as we will see in more details below, this will be particularly interesting when done in conjunction with questions and concepts coming from the geometric approach. In other words, dynamical systems are stochastic processes with more structure(s) on the space of trajectories.

Although any (stationary) stochastic process can be considered as a dynamical system (with phase space the set of all possible trajectories and transformation given by the time shift), there are however some important differences to mention.

An important one is that it is often the case that a dynamical system has many invariant measures. In other words, given a dynamical system (Ω, T), there may (and often will) exist many invariant measures. This raises the question of whether there is a more natural (most beautiful) invariant measure. This is indeed the case from a Physical point of view (the so called Physical measure when it exists, will be defined later). However other invariant measures can be interesting from different point of views (measure of maximal entropy for example).

In other words, in dynamical system theory, a set of measure zero for an invariant measure may be important from some other point of view (for example of full measure for another invariant measure). There are other concepts like Hausdorff dimension and Hausdorff measure which can be interesting but may involve sets of measure zero. It is also important to mention that some sets of measure zero can indeed be "observed" (like for example in the multifractal formalism). It is worth recalling (see the beginning of this lecture) that the choice of a particular invariant measure is related to the choice of an initial condition for a trajectory. We will come back to this point when discussing the ergodic theorem.

The conclusion is that in dynamical systems, sets of measure zero should not be disregarded as systematically as in probability theory.

Exercise 19. *If μ_1 is an invariant measure for the dynamical system (Ω_1, T_1) and the map Φ is measurable, then the measure $\mu_2 = \mu_1 \circ \Phi^{-1}$ is an invariant measure for the dynamical system (Ω_2, T_2) conjugated by Φ.*

We now discuss some simple examples of invariant measures. The simplest situation is when a system has a fixed point, namely a point in phase space which does not move through time evolution. If we have a discrete time system given by a map T, this is a point ω of phase space such that $T(\omega) = \omega$.

If we have a continuous time system given by a vector field $\vec{X}$, a fixed point (also called a stationary state) is a point ω of phase space such that $\vec{X}(\omega) = \vec{0}$. Such a point satisfies $\varphi_t(\omega) = \omega$ for any $t \in \mathbb{R}$ where φ_t is the flow integrating $\vec{X}$. It is easy to verify that if a system has a fixed point, the Dirac mass in this fixed point is an invariant measure, namely this measure satisfies equation (10) $\mu\big(T^{-1}(A)\big) = \mu(A)$ (one can use also exercise 18).

More generally, if we have a periodic orbit for a discrete time evolution, namely a (finite) sequence of points in phase space $(\omega_0, \ldots, \omega_{n-1})$ such that $T(\omega_k) = \omega_{k+1 \pmod n}$, the average of the Dirac masses at the points of the orbit is an invariant measure.

Exercise 20. *Find an infinite number of different periodic orbits for the map $x \mapsto 2x \pmod 1$, hence an infinite number of different invariant measures.*

The Lebesgue measure is also invariant by the map $x \mapsto 2x \pmod 1$. To prove this we compute $T^{-1}(A)$ for each measurable set $A \subset [0,1]$. It is easy to

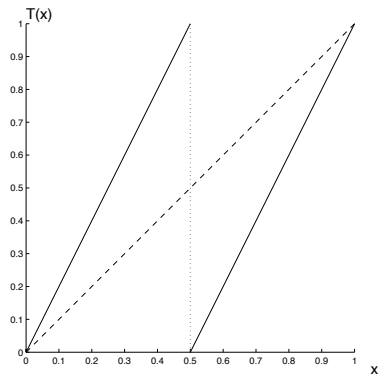


Figure 30: The map $x \mapsto 2x \pmod 1$.

verify that (if $1 \notin A$)

$$T^{-1}(A) = \big(A/2\big) \cup \big(1/2 + A/2\big)$$

where

$$\big(A/2\big) = \big\{x \in [0,1/2] \,\big|\, 2x \in A\big\} \ ,$$

and

$$\big(1/2 + A/2\big) = \big\{x \in [1/2,1] \,\big|\, 2x - 1 \in A\big\} \ .$$

If λ denotes the Lebesgue measure, we have

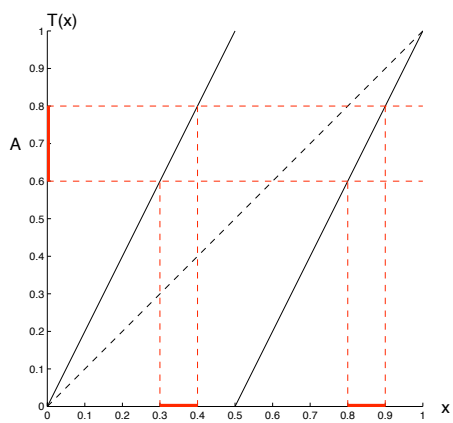


Figure 31: Geometric construction of the preimages of the segment for the map $x \mapsto 2x \pmod 1$.

$$\lambda\big(A/2\big) = \lambda\big(1/2 + A/2\big) = \lambda(A)/2$$

and since the intersection of the two sets $\big(A/2\big)$ and $\big(1/2 + A/2\big)$ is empty or reduced to the point $1/2$, equality (10) follows for $\mu = \lambda$ and any measurable set A. We will see below a generalisation of this idea.

Exercise 21. *Prove that the Lebesgue measure on* $[-1, 1]$ *is invariant for the map* $T(x) = 1 - 2|x|$. *Use exercise 19 to prove that the measure* $dx/\sqrt{1 - x^2}$ *is invariant by the map* $T_2(x) = 1 - 2x^2$.

Exercise 22. *Consider the middle third triadic Cantor set* K. *Recall that* $K = \cap K_n$ *where each* K_n *is a disjoint union of* 2^n *intervals, the intervals of* K_{n+1} *being obtained from those of* K_n *by dropping the (open) middle third subinterval. Show that this is the set of points whose triadic representation does not contain any* 1. *Define a measure* ν *by giving the weight* 2^{-n} *to any interval in* K_n. *Show that* ν *is invariant by the map* $3x \pmod 1$.

Figure 32: Recursive construction of a Cantor set.

Let p be a Markov transition matrix on a finite alphabet $\mathcal{A}$. Recall that this is a $\big|\mathcal{A}\big| \times \big|\mathcal{A}\big|$ matrix with non negative elements and such that for any $a \in \mathcal{A}$ we have

$$\sum_{b \in \mathcal{A}} p_{b,a} = 1 \ .$$

Such matrices are often denoted by $p(b|a)$.

Let q be an eigenvector with non negative entries and eigenvalue one, namely for any $b \in \mathcal{A}$

$$q_b = \sum_{a \in \mathcal{A}} p_{b,a} q_a \ . \tag{11}$$

Let $\Omega = \mathcal{A}^{\mathbb{Z}}$ and consider the dynamical system on this phase space given by the shift. Given a finite sequence $x_r, \ldots, x_p$ $(r \leq p \in \mathbb{Z})$ of elements of $\mathcal{A}$, often denoted by the short hand notation x_r^p, we denote by $C(x_r^p)$ the cylinder subset of Ω given by

$$C(x_r^p) = \left\{ \underline{y} \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, y_j = x_j \ \ r \leq j \leq p \right\} \ .$$

We now define a measure μ on Ω by its value on any (finite) cylinder set:

$$\mu\big(C(x_r^p)\big) = q_{x_r} \prod_{j=r}^{p-1} p_{x_{j+1}, x_j} \ .$$

It is easy to verify that this defines a measure on Ω, which is invariant by the shift. This is of course nothing but a stationary Markov chain with transition probability p.

In the particular case where for any $b \in \mathcal{A}$ the numbers $p_{b,a}$ do not depend on a we get a sequence of i.i.d. random variables on $\mathcal{A}$. The shift equipped with this invariant measure is called a Bernoulli shift (or a Bernoulli scheme). If $\mathcal{A}$ is a set of two symbols this is the game of head or tales.

This construction can be generalised in various ways and in particular to the construction of Gibbs measures on sub-shifts of finite type (see [140]) .

Andrei Markov (1856-1922).

Exercise 23. *Let $p \in]0,1[$ and $q = 1-p$.*

Consider the phase space $\Omega = \{0,1\}^N$ equipped with the shift operator $\mathcal{S}$.

Let μ_p be the infinite product measure with $\mu_p(0) = p$ (coin flipping).

Using the conjugation of the shift and the map $T(x) = 2x \pmod 1$ (except for a countable set), show that one obtains an invariant measure (also denoted μ_p) for the map T.

Show that the measure of any dyadic interval of length 2^{-n} is of the form $p^r q^{n-r}$ where $0 \le r \le n$ is given by the dyadic coding of the interval.

Show that for $p = q = 1/2$ one gets the Lebesgue measure.

Exercise 24. *Prove that the two dimensional Lebesgue measure is invariant for the baker map, the cat map and the standard map defined in the preceding chapter. Hint: compute the Jacobian. Show that there is an invariant measure for the dissipative baker map which can be identified as the product of the Lebesgue measure (in the horizontal direction) with the Bernoulli (product) measure of parameters $(1/2, 1/2)$ (in the vertical direction).*

2.1 The Perron-Frobenius operator.

It is natural at this point to ask how one finds the invariant measures of a dynamical system. Solving the invariance condition (equation 10)

$$\mu\big(T^{-1}(A)\big) = \mu(A)$$

is in general a non trivial task, in particular it is often the case that dynamical systems have an uncountable number of (inequivalent) invariant measures.

One can then try to determine the invariant measures with some special properties. Consider the particular case of a map T of the unit interval $[0,1]$ and look for the invariant probability measures which are absolutely continuous with respect to the Lebesgue measure (a.c.i.p.m. for short). In other words, we are looking for measures $d\mu = hdx$, with h a non negative integrable function (of integral one) satisfying the invariance condition (equation 10). This is equivalent to

$$\int_0^1 g(x)h(x)dx = \int_0^1 g(T(x))h(x)dx \ , \tag{12}$$

for any continuous function g.

Assume T is piecewise monotone with finitely many pieces, namely there is a sequence $a_0 = 0 < a_1 < \ldots < a_k = 1$ such that on each interval $[a_j, a_{j+1}]$ the map T is monotone and continuous. We have seen the case of piecewise expanding maps of the interval, but this is also the case for the quadratic family $x \mapsto 1 - \mu x^2$ and more generally for unimodal maps and for regular maps of the interval with finitely many critical points.

Let ψ_j denote the inverse of the map $T_{\big|[a_j,a_{j+1}]}$. This inverse exists since T is monotone on $[a_j, a_{j+1}]$, it is also a monotone map from $[T(a_j), T(a_{j+1})]$ to $[a_j, a_{j+1}]$. If the inverse branches ψ_j are differentiable, we can perform a change

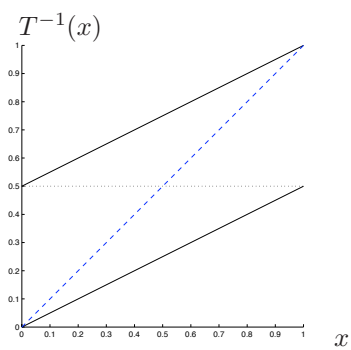


Figure 33: The two inverse branches ψ_1 and ψ_2 of the map $T(x) = 2x \mod 1$.

of variables $y = T(x)$ in the right hand side of the above identity (12) for the density h of an invariant measure.

More precisely

$$\int_0^1 g\big(T(x)\big)h(x)dx = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} g(T(x))h(x)dx$$

$$= \sum_{j=0}^{k-1} \int_{T(a_j)}^{T(a_{j+1})} g(y)\big|\psi_j'(y)\big|h\big(\psi_j(y)\big)dy \ . \tag{13}$$

Since the invariance relation

$$\int_0^1 g(x)h(x)dx = \int_0^1 g(T(x))h(x)dx \ ,$$

should hold for any bounded and measurable function g, we conclude that h is the density of an a.c.i.p.m. for the map f if and only if $Ph = h$ where P is the so called Perron-Frobenius operator given by

$$Pg(x) = \sum_j \chi_{[T(a_j),T(a^{j+1})]}(x) g\big(\psi_j(x)\big)\big|\psi_j'(x)\big| \ . \tag{14}$$

In other words, h is the density of an a.c.i.p.m. for the map T if and only if it is a nonnegative eigenvector of eigenvalue one for the operator P (in other words $Ph = h$). Using the fact that the functions ψ_j are local inverses of T, the Perron-Frobenius operator is also given by

$$Pg(x) = \sum_{y,\, T(y)=x} \frac{g(y)}{\big|T'(y)\big|} \tag{15}$$

An immediate consequence of this formula is that for any measurable functions u and g,

$$P\big(u\ g \circ T\big) = g\ P\big(u\big) \ . \tag{16}$$

The above calculation can be generalised to the problem of finding an invariant measure absolutely continuous with respect to (a non invariant) reference measure. In the case where the map is invertible (diffeomorphism for example) the above ideas do not help. If there is an attractor one can project along the transverse stable manifolds and work with a non invertible transformation on the unstable manifolds (see below for these notions).

Perron and Frobenius studied the simple case of iteration of matrices with nonnegative entries. The above operator is a vast generalisation of this case (see below for Markov chains).

Figure 34: Oskar Perron (1880-1975), Georg Frobenius (1849-1917).

We mention without proof an important result for piecewise expanding maps of the interval. Before we state this result we recall that a function g on the interval is of bounded variation if there is a finite number $C > 0$ such that for any strictly increasing sequence $0 \le b_0 < b_1 < \ldots < b_n \le 1$ we have

$$\sum_{j=0}^{n-1} \big|g(b_j) - g(b_{j+1})\big| \le C \ .$$

The space of functions of bounded variations equipped with a norm given by the supremum of all these numbers C plus the sup norm is a Banach space denoted below by **BV**.

Theorem 2.1. *Any piecewise expanding map of the interval has at least one absolutely continuous invariant probability measure with density in the space of functions of bounded variation.*

Recall that a piecewise expanding map T of the interval satisfies $|T'| > \alpha > 1$ (or more generally there is an integer n such that $|T^{n\prime}| > \alpha > 1$).

This Theorem due to Lasota and Yorke is proved by investigating the spectral properties of the Perron-Frobenius operator. We refer to [87] for the details, references and consequences (see also [36]). We will also give later some complements to this result.

The equation $Ph = h$ for the density of an a.c.i.p.m. where P is the Perron-Frobenius operator is very reminiscent of equation (11) for the invariant measures of a Markov chain

$$q_b = \sum_{a \in \mathcal{A}} p_{b,a} q_a \ .$$

The next exercise develops this analogy.

Exercise 25. *Consider a piecewise expanding Markov map of the interval with affine pieces. Show that there are a.c.i.p.m. with piecewise constant densities. Using the coding of exercise I.7, show that this system is conjugated to a stationary Markov chain. Show that any stationary Markov chain with a finite number of states can be realised by a piecewise expanding Markov map of the interval with affine pieces.*

Exercise 26. *Let $p_1,\ p_2, \ldots$ be a sequence of positive numbers summing to one. Define an infinite sequence of intervals $I_1 = [a_2, a_1]$, $I_2 = [a_3, a_2]$ $\ldots$, $I_j = [a_{j+1}, a_j]$ $\ldots$ where $a_1 = 1$ and*

$$a_j = \sum_{l=j}^{\infty} p_l \ .$$

Define a map of the unit interval into itself by

$$T(x) = \begin{cases} \frac{x-a_1}{1-a_1} & \text{if } x \in I_1 \ , \\ a_j + (a_{j-1} - a_j)\frac{x-a_{j+1}}{a_j - a_{j+1}} & \text{if } x \in I_j \quad \text{for} \quad j > 1 \ . \end{cases}$$

Choose an initial condition at random with respect to the Lebesgue measure on the interval I_1 and generate the trajectory. Show that this dynamical system can be interpreted as a renewal process.

There is another version of the invariance relation (12) $\int g \circ T \ d\mu = \int g \ d\mu$ that will be useful later on. Let U be the Koopman operator defined on measurable functions by

$$Ug(x) = g(T(x)) \ . \tag{17}$$

It is easy to verify that if μ is an invariant measure, U is an isometry of $L^2(d\mu)$.

Equation (12) can now be written in the following form

$$\int_0^1 g_2 \, U\big(g_1\big) dx = \int_0^1 P\big(g_2\big) \ g_1 dx \tag{18}$$

for any pair g_1, g_2 of square integrable functions.

2.2 Gibbs measures.

Although we have worked here explicitly with the Lebesgue measure, we mentioned already that similar relations can be obtained for other reference measures. An important case is the case of Gibbs states on sub-shift of finite type. Recall that for a sub-shift of finite type, the phase space Ω is a shift invariant subset of $\Theta = \mathcal{A}^{\mathbb{Z}}$ where $\mathcal{A}$ is a finite alphabet.

For two elements $\underline{x}$ and $\underline{y}$ of Θ, denote by $\delta\big(\underline{x}, \underline{y}\big)$ the nearest position to the origin where these two sequences differ, namely

$$\delta\big(\underline{x}, \underline{y}\big) = \min \left\{ |q| \,\middle|\, x_q \neq y_q \right\} .$$

For a given number $0 < \zeta < 1$ we define a distance d_ζ (denoted simply by d when there is no ambiguity in the choice of ζ) by

$$d_\zeta = \zeta^{\delta\left(\underline{x}, \underline{y}\right)} . \tag{19}$$

Exercise 27. *Prove that d_ζ is a distance (even ultrametric) and that Θ is compact in this topology. Show that the topology does not depend on ζ. Show that cylinder sets are open closed. Show that characteristic functions of cylinder sets are continuous. Show that the phase space Ω of any sub-shift of finite type with alphabet $\mathcal{A}$ is closed in Θ. Prove that the shift map $\mathcal{S}$ on Ω is continuous (and even Hölder), with a continuous inverse.*

Theorem 2.2. *Let ϕ be a real valued Hölder continuous function on Ω the phase space of sub-shift of finite type. Assume the incidence matrix $\mathcal{M}$ of the sub-shift is irreducible and aperiodic (in other words, there is an integer r such that all the entries of the matrix $\mathcal{M}^r$ are non zero).*

Then there is a unique probability measure μ invariant by the shift and a positive constant $\Gamma > 1$ such that for any cylinder set $C\big(x_q^p\big)$ $(q \leq p)$ and for any $\underline{y} \in C\big(x_q^p\big)$

$$\Gamma^{-1} \leq \frac{\mu\big(C\big(x_q^p\big)\big)}{e^{-(p-q+1)P_\phi} e^{\sum_{j=q}^{p} \phi\left(\mathcal{S}^j(\underline{y})\right)}} \leq \Gamma , \tag{20}$$

where P_ϕ is the pressure defined by

$$P_\phi = \lim_{n\to\infty} \frac{1}{2n+1} \log \left(\sum_{x_{-n}^n \,,\, M_{x_j, x_{j+1}}=1} e^{\sum_{j=-n}^{n} \phi\left(\mathcal{S}^j(\underline{x})\right)} \right) . \tag{21}$$

In this formula, $\underline{x}$ denote any point of Ω belonging to the cylinder set $C\big(x_{-n}^n\big)$.

Exercise 28. *Prove that the pressure does not depend on the choice of the point $\underline{x}$ in the cylinder set $C\big(x_q^p\big)$ (use the Hölder continuity of ϕ).*

This measure (often called a Gibbs measure is a particular case of a (one deimensional) Gibbs state of statistical mechanics with potential ϕ.

We refer to [21] or [140] for a detailed proof of this result using a Perron-Frobenius operator and the relation with Gibbs states in statistical mechanics.

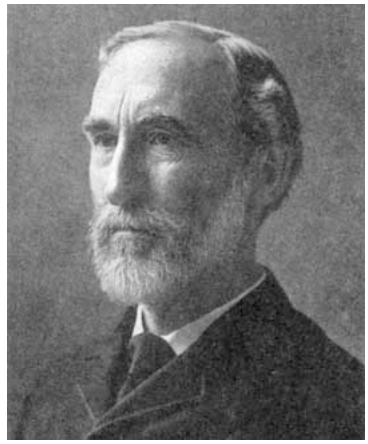


Figure 35: Josiah Willard Gibbs, 1839 - 1903.

We just give here the expression of the Perron-Frobenius operator in this case (assuming $\phi(\underline{x})$ depends only on (x_0^∞))

$$P\psi(x_0^\infty) = \lambda^{-1} \sum_{y \in \mathscr{A}\,,\, \mathcal{M}_{y,x_0}=1} e^{\phi(yx_0^\infty)} \psi(yx_0^\infty)$$

where yx_0^∞ denotes the sequence $y, x_0, x_1, \ldots$ and λ is a suitable positive constant ($\lambda = \exp(-P_\phi)$).

Exercise 29. *Show that when $\phi\big(\underline{x}\big)$ depends only on x_0 and x_1 one gets all the Markov chains with finite states (start by constructing the incidence matrix).*

3 The ergodic theorem.

We now come back to the question of the asymptotic limit of the ergodic (or time) average

$$\mathcal{A}_N(x_0, A) = \frac{1}{N+1} \sum_{j=0}^{N} \chi_A\big(T^j(x_0)\big) \ .$$

This is settled by the ergodic theorems.

There are two main ergodic theorems. The ergodic theorem of [160]Von-Neumann applies in an L^2 context, while the [15]Birkhoff ergodic theorem applies almost surely (we refer to [107], [92] [100] and [132] for proofs and extensions from other points of views (see [12]).

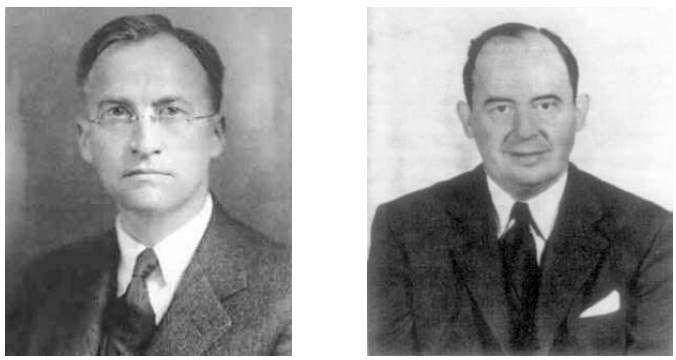


Figure 36: George Birkhoff (1884-1944), John von Neumann (1903-1957).

We now state the Birkhoff ergodic theorem. In a sense it is a generalisation of the law of large numbers.

Theorem 3.1 (Birkhoff)**.** *Let $(\Omega, T, \mathscr{B}, \mu)$ be a dynamical system (recall that T is measurable for the sigma algebra $\mathcal{B}$ and the measure μ on $\mathcal{B}$ is T invariant). Then for any $f \in L^1(\Omega, \mathcal{B}, d\mu)$*

$$\mathcal{A}_N(x, f) = \frac{1}{N+1} \sum_{j=0}^{N} f\big(T^j(x)\big) \ .$$

converges when N tends to infinity for μ almost every x.

We now make several remarks about this fundamental result.

- It follows from what we have seen previously that the set of points where the limit exists is invariant (and of full measure by Birkhoff's Theorem). Moreover, if we denote by $g(x) = \lim_{N\to\infty} \mathcal{A}_N(x, f)$ the limiting function (which exists for μ almost every x), it is invariant, namely $g(T(x)) = g(x)$.

- The set of μ measure zero where nothing is claimed depends on f and μ. One can often use a set independent of f (for example if $L^1(d\mu)$ is separable). We will comment below on its dependence on μ. Although small from the point of view of the measure μ, this set may be big from other points of views (for another measure for example).

The most interesting case is of course when the limit in the ergodic theorem is independent of the initial condition (except for a set of μ measure zero). This leads to the definition of ergodicity.

Definition 10. *A measure μ invariant for a dynamical system $(\Omega, \mathcal{A}, T)$ is ergodic if any invariant function (i.e. any measurable function f such that $f \circ T = f$, μ almost surely) is μ almost surely constant.*

Note that this definition involves the measure and the transformation.

There are two useful conditions equivalent to the definition of ergodicity. The first one is in terms of invariant sets. An invariant (probability) measure is ergodic if and only if

$$\mu\big(A \,\Delta T^{-1}(A)\big) = 0 \iff \mu(A) = 0 \quad \text{or} \quad \mu(A) = 1 \,. \tag{22}$$

Exercise 30. *Prove this assertion*

The second equivalent condition is in terms of the Birkhoff average. An invariant (probability) measure is ergodic if and only if for any $f \in L^1(d\mu)$

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} f\big(T^j(x)\big) = \int f \, d\mu \tag{23}$$

μ almost surely.

Remark 1.

- *In the above ergodicity condition (23), the limit exists μ almost surely by the Birkhoff ergodic Theorem. It is enough to require that the limit is μ almost surely constant since the constant has to be equal to the integral by the invariance of the measure:*

$$\int \frac{1}{N+1} \sum_{j=0}^{N} f\big(T^j(x)\big) d\mu(x) = \int f \, d\mu \,.$$

- *In formula (23), $\lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} f\big(T^j(x)\big) = \int f \, d\mu$, the state x does not appear on the right hand side,*

  but this information is hidden in the fact that the formula is only true outside a set of measure zero.

- *Ergodicity is often sloppily stated as saying that the time average is equal to the space average. This has to be taken with a grain of salt. This is only true almost surely. As we have already emphasized, changing the measure may change drastically the exceptional set of measure zero where the ergodic theorem does not claim anything, and this can lead to completely different results for the limit.*

- *It often happens that a dynamical system $(\Omega, \mathcal{A}, T)$ has several ergodic invariant (probability) measures. Let μ and ν be two different ones. It is easy to verify that they are disjoint, namely one can find a set of measure one which is of measure zero for the other and vice versa. This explains why the ergodic theorem applies to both measures leading in general to different time averages.*

- *For non ergodic measures, one can use an ergodic decomposition (disintegration). However in concrete cases this may lead to rather complicated sets. We refer to [107] for more information.*

Exercise 31. *Let μ and ν be two different probability measures on a phase space Ω, invariant and ergodic with respect to the same transformation T. Show that there are two measurable subsets A and B of Ω such that $\mu(A) = \nu(B) = 1$ and $\mu(B) = \nu(A) = 0$. Hint: apply the Birkhoff ergodic theorem to a function f such that $\int f\, d\mu \neq \int f\, d\nu$. Show that the probability measure $(\mu + \nu)/2$ is not ergodic.*

A word of caution on the terminology.

Birkhoff's ergodic theorem does not prove ergodicity.

It may be a very useful tool in proving ergodicity but as such it does not claim it. In particular it applies in non-ergodic cases. Birkhoff's ergodic applies as soon as you have an invariant measure. You have to work more to decide whether or not the measure is ergodic.

Note also that it follows immediately from the invariance of the measure μ (ergodic or not) that

$$\int \left(\frac{1}{n} \sum_{j=0}^{n-1} g\big(T^j(x)\big) \right) d\mu = \int g\, d\mu \ .$$

The strong law of large numbers for a sequence of i.i.d. random variables (ξ_n) with $\mathbb{E}(|\xi_1|) < \infty$ follows directly from the Birkhoff ergodic theorem. We will see later an easy proof of ergodicity in this case using mixing. This can be extended to many other non independent processes.

Birkhoff's ergodic theorem holds for semi flows (continuous time average). It also holds of course for the map obtained by sampling the semi flow uniformly in time. However non uniform sampling may spoil the result (see [142] and references therein)s.

Simple cases of non ergodicity come from Hamiltonian systems with the invariant Liouville measure. First of all since the energy is conserved (the Hamiltonian is a non trivial invariant function), the system is not ergodic. One has to restrict the dynamics to each energy surface.

More generally if there are other independent constants of the motion one should restrict oneself to lower dimensional manifolds. Even then the system may be non ergodic because of the presence of separatrices, invariant islands, tori etc. Roughly speaking, if you have a measure spread everywhere like the Lebesgue (Liouville) measure, the dynamical system has better to explore the whole phase space to have a chance to be ergodic, this can be illustrated using the standard map.

Recall the definition of the standard map of the torus

$$T(x, y) = \begin{pmatrix} 2x - y + a \sin(2\pi x) \pmod 1 \\ x \end{pmatrix} .$$

It is conjectured that the closure of this set contains an invariant set of positive Lebesgue measure but not of full measure. Hence the Lebesgue measure which is invariant would not be ergodic.

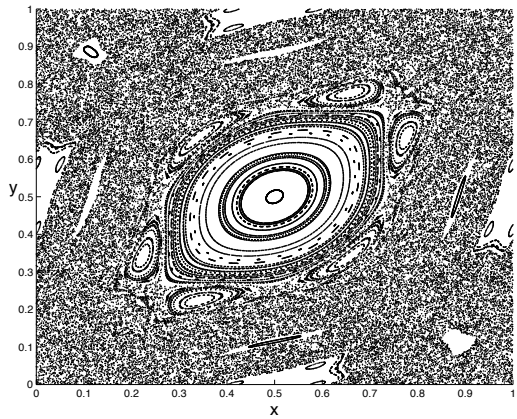


Figure 37: Picture of several (150) orbits of the standard map for $a = .2387$

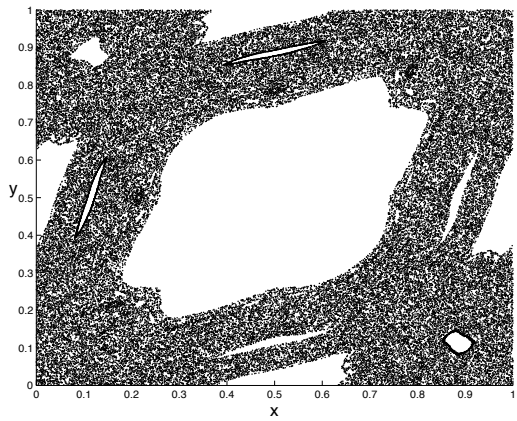


Figure 38: Orbit (length 75000) under the standard map with $a = .2387$ of the initial point $(.05, .04)$.

For completely integrable systems, one is reduced to a constant flow on a torus which is ergodic if the frequencies are incommensurable. It is also known that generic Hamiltonian systems are neither integrable nor ergodic (on the energy surfaces), see [122].

Proving ergodicity of a measure is sometimes a very hard problem. Note that it is enough to prove the condition (23)

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} f\big(T^j(x)\big) = \int f \, d\mu$$

for a dense set of functions in $L^1(d\mu)$, for example the continuous functions.

To illustrate the ergodic theorem, we will use the map $T(x) = 3x \pmod 1$ of the unit interval and as function f the characteristic function of the interval $[1/2, 1]$. The next figure shows S_n/n as a function of n for two initial conditions,

where

$$S_n(x) = \sum_{\ell=0}^{n-1} f\big(T^\ell(x)\big) \ .$$

The Lebesgue measure is invariant and ergodic (prove this statement) and

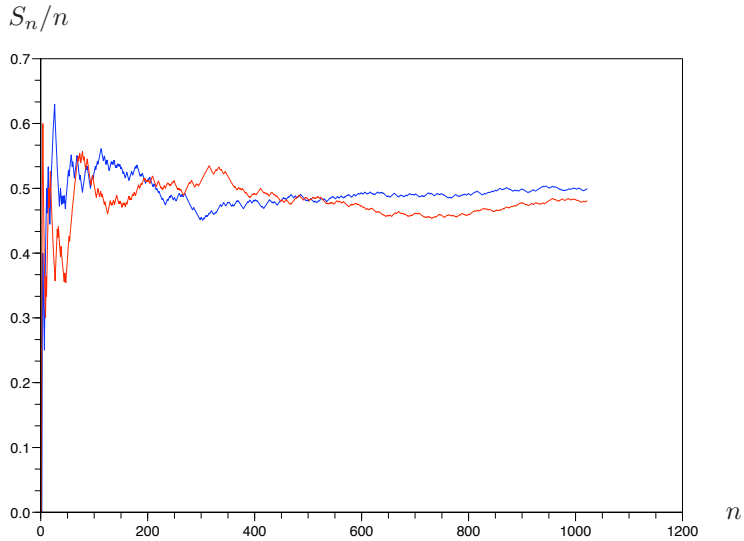


Figure 39: Evolution of S_n/n for two randomly chosen initial conditions. The map is $T(x) = 3x \pmod 1$ and f is the characteristic function of the interval $[1/2, 1]$.

we expect that if we choose an initial condition uniformly at random on $[0, 1]$, then S_n/n should converge to $1/2$ (the average of f with respect to the Lebesgue measure). This is what we see in the figure where two different initial conditions were chosen at random according to the Lebesgue measure. Note that the convergence to the limit $1/2$ is not very fast. We will come back to this question later on.

We now consider the evolution of S_n/n for a periodic initial condition. The map is still $T(x) = 3x \pmod 1$ and f is still the characteristic function of the interval $[1/2, 1]$.

The point $x_0 = 7/80$ is periodic of period four, and its orbit has three points in the interval $[1/2, 1]$ (exercise: verify this statement). The invariant measure (the sum of the Dirac measures on the four points of the orbit) is ergodic and therefore, S_n/n converges to $3/4$ (the average of f with respect to the discrete invariant measure supported by the orbit).

Exercise 32. *Prove these statements.*

The next figure was drawn using an atypical initial condition for which S_n/n will oscillate forever between two extreme values, although the oscillations will

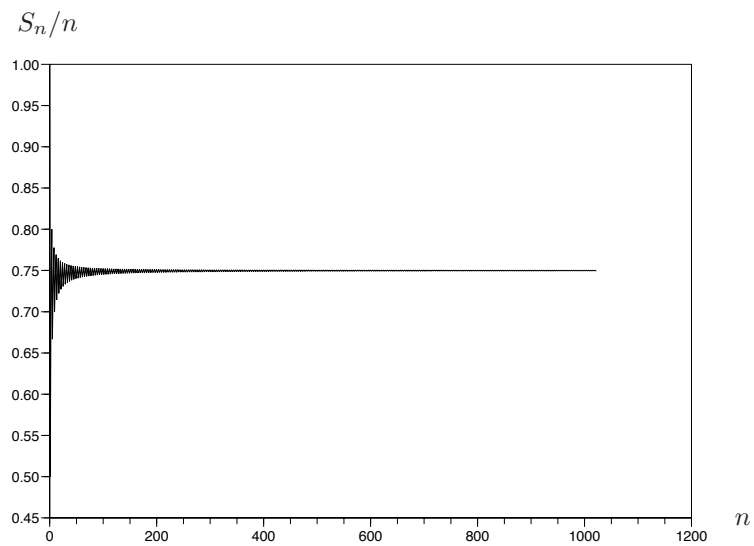


Figure 40: S_n/n as a function of n for the initial condition $x_0 = 7/80$.

take longer and longer time. The initial condition is atypical for any ergodic invariant measure!

An extension of the ergodic theorem is the so called sub-additive ergodic theorem which has many useful applications.

Theorem 3.2. *Let $(\Omega, \mathscr{B}, T)$ a measurable dynamical system and μ an ergodic invariant measure. Let (g_n) be a sequence of integrable functions satisfying for each $x \in \Omega$ and for any pair of integers m and n the inequality*

$$g_{n+m}(x) \leq g_n(x) + g_m\big(T^n(x)\big) \ , \tag{24}$$

and such that

$$\liminf_{n\to\infty} \frac{1}{n} \int g_n d\mu > -\infty \ .$$

Then μ almost surely

$$\lim_{n\to\infty} \frac{g_n(x)}{n} = \lim_{n\to\infty} \frac{1}{n} \int g_n d\mu \ .$$

We refer to [107] for a proof. The ergodic theorem corresponds to the case where

$$g_n(x) = S_n(x) = \sum_{j=0}^{n-1} g\big(T^j(x)\big)$$

and we obtain immediately

$$S_{n+m}(x) = S_n(x) + S_m\big(T^n(x)\big) \ .$$

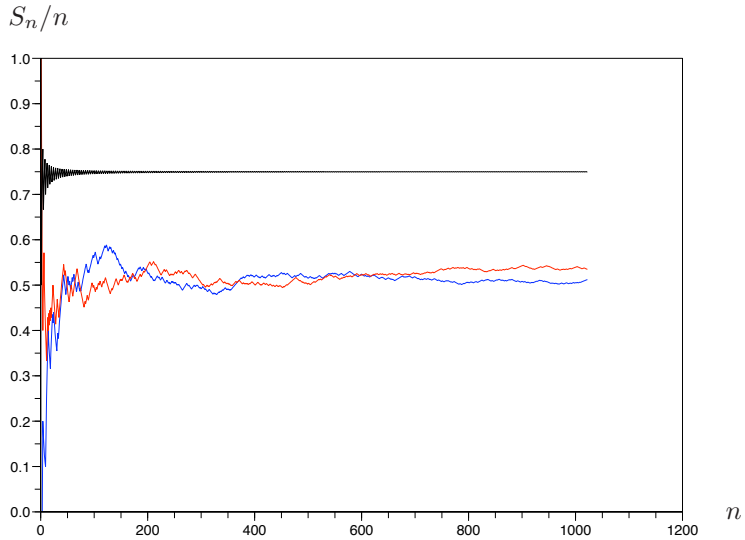


Figure 41: Comparison for two initial condition at random (Lebesgue) and one on the periodic orbit.

3.1 Physical measure.

As we have seen already several times, it is often the case that a dynamical system has different ergodic invariant measures.

It is therefore natural to ask if there is one of these measures which is in some sense more important than the others.

From a physical point of view that we will not discuss here, the Lebesgue measure of the phase space is singled out (the most harmonious one). In general it is not an invariant measure.

Even worse, if the dynamical system has a rather small attractor, for example a set of Lebesgue measure zero as often occurs in dissipative systems, it does not seem very interesting to look at the Lebesgue measure.

In such cases there is no invariant measure absolutely continuous with respect to the Lebesgue measure.

The study of a particular class of dynamical systems called axiom A (see later 17 for a definition) by Sinai, Ruelle and Bowen led however to a generalisation of the notion of ergodicity which nevertheless makes use of the Lebesgue measure on the whole phase space.

Definition 11. *Consider a dynamical system given by a map T on a phase space Ω which is a compact Riemannian manifold. An ergodic invariant measure μ for a map T is called a Physical measure if there is a subset U of Ω of positive Lebesgue measure such that for any continuous function g on Ω and any initial*

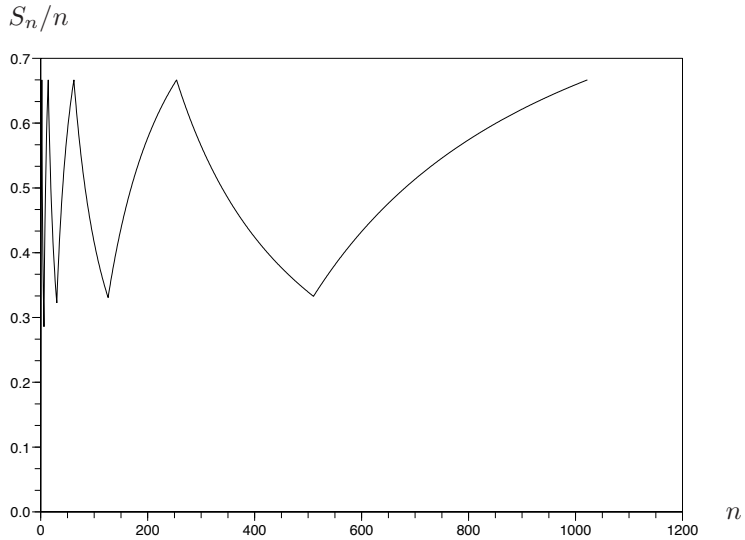


Figure 42: The Ergodic Theorem does not work for this initial condition.

point $x \in U$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g\big(T^j(x)\big) = \int g \, d\mu \; .$$

In this definition, if there are several attractors, one may reduce Ω to a basin of attraction. We emphasize again that the measure μ may be supported by an attractor of zero Lebesgue measure.

The notion of Physical measure extends (and is slightly different) from the ergodic theorem because the measure determining the exceptional set of initial conditions is not the invariant measure.

Physical measures do not always exist.

For axiom A systems 17 it is known that such measures always exit.

Physical measures have some more properties and are particular cases of SRB measures (see [68] and below for the definition and results).

The idea behind the definition is that if the dynamical system has an attractor, this set should be transversally attracting (an easy situation is a stable periodic orbit as we discussed for the Hénon map). An orbit will therefore rapidly approach the attractor, and a continuous function will take almost the same value for a point on the attractor and a point nearby. One would then be able to apply the ergodic theorem for the ergodic measure μ (supported by the attractor) provided exceptional sets of μ measure zero would in some sense be projections of sets of Lebesgue measure zero in the phase space. This requires an additional technical property (the so-called absolute continuity of the projection along the stable manifolds).

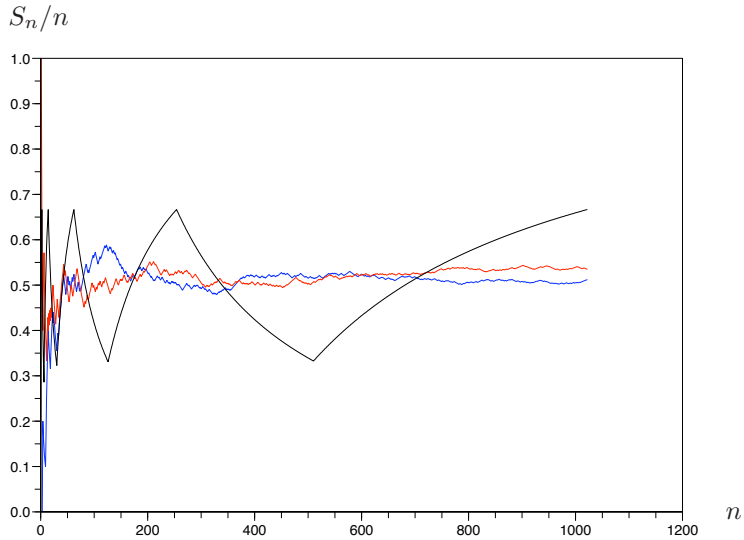


Figure 43: Comparison for two initial condition at random (Lebesgue) and an atypical one.

Figure 44: Rufus Bowen (1947-1978).

We have already seen examples of Physical measures. The simplest case is of course when the system has an invariant measure absolutely continuous with respect to the Lebesgue measure.

For the quadratic family of maps of the interval [-1,1] given by $T_\mu(x) = 1 - \mu x^2$ it is known that for any parameter $\mu \in [0, 2]$, there is a unique Physical measure. Moreover, there is a subset of positive measure of the parameter space

(the interval $[0, 2]$) for which the Physical measure is absolutely continuous. We recall the picture of the attractor as a function of the parameter From this

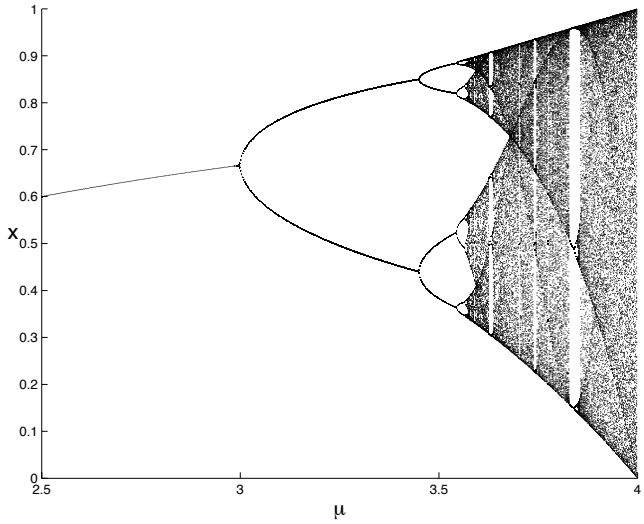


Figure 45: Attractor as a function of the parameter for the quadratic map.

picture it seems plausible (and this has been proved) that for many parameter values, the attractor has positive Lebesgue measure. It even supports an absolutely continuous ergodic invariant measure. We refer to [5] for recent works and references.

Similarly, for the Hénon map

$$T(x, y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} ,$$

it has been shown that for a set of positive (two dimensional) Lebesgue measure of parameter values (a,b), Physical measures exist. However the attractor has two dimensional Lebesgue measure zero (see [10], [11]). We refer to [68] for a general discussion.

Exercise 33. *Show that the invariant measure for the dissipative baker map constructed in exercise II.7 is a Physical measure.*

We emphasise that there are dynamical systems which do not possess Physical measures. We refer to [163] for more information.

4 Statistical Properties of Dynamical Systems.

From now on we will always assume that we have a dynamical system (Ω, T) and an invariant ergodic probability measure μ.

As explained before, this induces a stationary stochastic process (Y_n) on Ω given by

$$Y_n(\omega) = T^n(\omega) \ .$$

If we have a preferred observable g, for example a measurable real valued function g on Ω, we can also look at the image process (X_n) on $\mathbb{R}$ given by

$$X_n(\omega) = g\big(T^n(\omega)\big) \ .$$

Since μ is ergodic, it follows from Birkhoff's ergodic theorem that this process satisfies the strong law of large numbers (provided f is integrable). It is also stationary.

There however some differences with the usual theory of stochastic processes, and we will not in general be able to blindly apply the results. For example as we will see below the decay of the time correlations requires some particular study. We will also see that supplementary structures on the phase space (the probability space) bring interesting concepts and questions.

4.1 Convergence rate in the ergodic Theorem.

The first natural question is to ask how fast is the convergence of the ergodic average to its limit in Birkhoff's ergodic theorem (strong law of large numbers). Recall that

$$\mathbb{E}\left(\frac{1}{n}\sum_{j=0}^{n-1} f\circ T^j\right) = \int f\, d\mu \ .$$

Therefore we want to look at the fluctuations of the Birkhoff sums $S_n = \sum_{j=0}^{n-1} f\circ T^j$, or the time averages S_n/n.

We already observed numerically that this convergence can be slow. At this level of generality any kind of velocity above $1/n$ can occur. Indeed Halasz and Krengel have proved the following results (see [91] for a review).

Theorem 4.1. *Consider a (measurable) automorphism T of the unit interval $\Omega = [0,1]$, leaving the Lebesgue measure $d\lambda = dx$ invariant.*

- *For any increasing diverging sequence $a_1, a_2, \cdots$, with $a_1 \geq 2$, and for any number $\alpha \in]0,1[$, there is a measurable subset $A \in \Omega$ such that $\lambda(A) = \alpha$, and*

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\big(\chi_A\circ T^j - \lambda(A)\big)\right| \leq \frac{a_n}{n}$$

  λ (Lebesgue) almost surely, for all n.

- *For any sequence $b_1, b_2, \cdots$, of positive numbers converging to zero, there is a measurable subset $B \in \Omega$ with $\lambda(B) \in]0,1[$ such that λ (Lebesgue) almost surely*

$$\lim_{n\to\infty}\left|\frac{1}{nb_n}\sum_{j=0}^{n-1}\big(\chi_B\circ T^j - \lambda(B)\big)\right| = \infty \ .$$

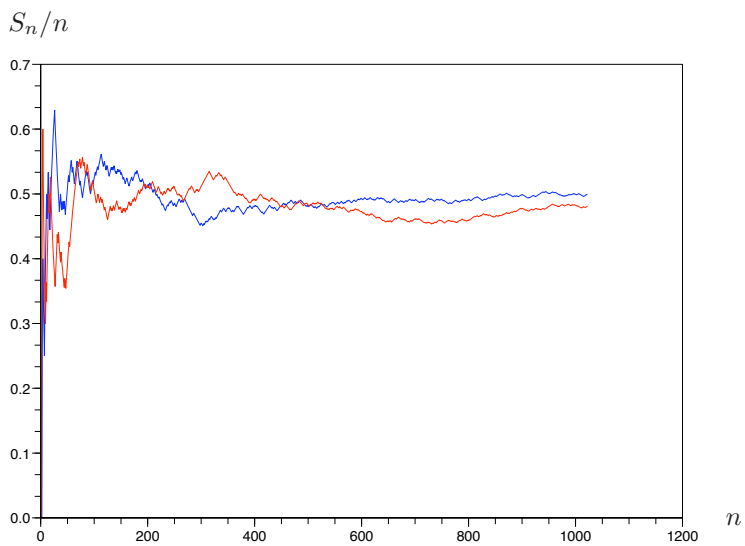


Figure 46: Evolution of S_n/n for two random initial conditions.

The first result says that convergence can be almost as fast as $1/n$, the second that it can be as slow as one likes. Indeed, there is an integer valued random variable (function) $N(x)$, Lebesgue (λ) almost surely finite such that for any $n > N(x)$ we have

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \big(\chi_B\big(T^j(x)\big) - \lambda(B) \big) \right| \geq b_n \ .$$

In spite of these negative results, there is however an interesting and somewhat surprising general theorem by Ivanov dealing with a slightly different question. We refer to the end of this chapter for a statement of Ivanov's Theorem.

In order to get more precise information on the rate of convergence in the ergodic theorem, one has to make some hypothesis on the dynamical system and on the observable.

4.2 Mixing and decay of correlations

If one considers the numerator of the ergodic average, namely the ergodic sum (also called Birkhoff sum) for an observable (real function) f

$$S_n(f)(x) = \sum_{j=0}^{n-1} f\big(T^j(x)\big) \tag{25}$$

we have seen that this can be considered as a sum of random variables, although in general not independent. It is then natural to ask if there is something similar

to the central limit theorem in probability theory. To have such a theorem, one has first to obtain the limiting variance.

Assuming for simplicity that the average of f is zero, we are faced with the question of convergence of the sequence

$$\mathbb{E}\left(\left(\frac{S_n}{\sqrt{n}}\right)^2\right) = \frac{1}{n}\int \Big(S_n(f)(x)\Big)^2 \, d\mu(x)$$

$$= \int f^2(x)\, d\mu(x) + 2\sum_{j=1}^{n-1} \frac{n-j}{n} \int f(x)\, f\big(T^j(x)\big)\, d\mu(x) \ ,$$

which follows from the invariance of μ. Here we restrict of course the discussion to observables which are square integrable.

This sequence may diverge when n tends to infinity.

It may also tend to zero.

This is for example the case if $f = u - u \circ T$ with $u \in L^2(d\mu)$. Indeed, in that case $S_n = u - u \circ T^n$ is of order one in L^2 and not of order $\sqrt{n}$.

In probability theory, when dealing with a sequence of non-trivial (non constant) i.i.d. random variables (square integrable), this phenomenon cannot occur.

However, let (Y_n) be a sequence of square integrable i.i.d random variables with non zero variance. Define a new sequence (X_n) by $X_n = Y_{n+1} - Y_n$. These random variables are no longer independent but they are square integrable and have zero mean. We get

$$S_n = \sum_{m=1}^{n} X_m = Y_{n+1} - Y_1$$

and $\mathbb{E}(S_n^2) = \mathcal{O}(1)$. Therefore $n^{-1}\mathbb{E}(S_n^2)$ tends to zero when n tends to infinity. Surely the central limit theorem does not apply in this case.

Recall the expression we found for the variance

$$\mathbb{E}\left(\left(\frac{S_n}{\sqrt{n}}\right)^2\right) = \int f^2(x)\, d\mu(x) + 2\sum_{j=1}^{n-1} \frac{n-j}{n} \int f(x)\, f\big(T^j(x)\big)\, d\mu(x) \ .$$

A quantity which comes out naturally from this expression is the auto-correlation function $C_{f,f}$ of the observable f. This is the sequence defined by

$$C_{f,f}(j) = \int f(x)\, f\big(T^j(x)\big)\, d\mu(x) = \mathbb{E}\big(X_0\, X_j\big) \ . \tag{26}$$

where $X_j = f \circ T^j$. If this sequence belongs to l^1, the limiting variance exists and is given by

$$\sigma_f^2 = C_{f,f}(0) + 2\sum_{j=1}^{\infty} C_{f,f}(j) \ . \tag{27}$$

This shows that the decay of the auto-correlation function (26) is an important quantity to study.

A natural generalization of the auto-correlation is the cross correlation (often called simply the correlation) between two square integrable observables f and g. This function (sequence) is defined by

$$C_{f,g}(j) = \int f(x)\, g\big(T^j(x)\big)\, d\mu(x) - \int f\, d\mu \int g\, d\mu\ . \tag{28}$$

The second term appears in the general case when neither f nor g has zero average.

In a probabilistic notation

$$C_{f,g}(j) = \mathbb{E}\big(X_0\, Y_j\big) - \mathbb{E}\big(X_0\big)\, \mathbb{E}\big(Y_0\big)\ .$$

where $X_k = f \circ T^k$ and $Y_k = g \circ T^k$.

Similar definitions hold for (semi-) flows.

The case where f and g are characteristic functions is particularly interesting (by linear combinations and closure one may get "all function"). If $f = \chi_A$ and $g = \chi_B$ (assuming $\mu(A)\mu(B) \neq 0$ otherwise $C_{\chi_A,\chi_B}(n) = 0$), we have

$$\begin{aligned} C_{\chi_A,\chi_B}(n) &= \int_A \chi_B \circ T^n d\mu = \mu\big(A \cap T^{-n}(B)\big) \\ &= \mu\big(T^n(x) \in B \big| x \in A\big)\ \mu(A)\ . \end{aligned}$$

If for large time the system looses memory of its initial condition, it is natural to expect that $\mu\big(T^n(x) \in B \big| x \in A\big)$ converges to $\mu(B)$. This leads to the definition of mixing.

Definition 12. *We say that for a dynamical system $(\Omega, \mathscr{B}, T)$ the T invariant measure μ is mixing if for any measurable subsets A and B of Ω, we have*

$$\lim_{n\to\infty} \mu\big(A \cap T^{-n}(B)\big) = \mu(A)\ \mu(B)\ . \tag{29}$$

Note that this can also be written (for $\mu(B) > 0$)

$$\lim_{n\to\infty} \mu\big(T^n(x) \in B \big| x \in A\big) = \mu(B)\ ,$$

and if the map T is invertible

$$\lim_{n\to\infty} \mu\big(T^n(A) \big| B\big) = \mu(A)\ ,$$

In other words, if we have the information that at some time the state of the system is in the subset A, then if we wait long enough (more precisely asymptotically in time), this information is lost in the sense that we get the same statistics as if we had not imposed this condition.

There are many other mixing conditions including various aspects and velocity of convergence. We refer to [62] for details.

Exercise 34. *Show that mixing implies ergodicity.*

Exercise 35. *Let T be a map with a finite periodic orbit $x_0, \ldots, x_{k-1}$ ($T(x_r) = x_{r+1 \pmod k}$ for $0 \leq r < k$). Consider the invariant probability measure obtained by putting Dirac masses with weight $1/k$ at the points of the orbit. Show that this measure is ergodic but not mixing.*

Exercise 36. *Consider the map* $x \mapsto 2x \pmod 1$ *and the ergodic invariant measure* μ_p *defined before. Show that this dynamical system is mixing (hint: show formula (29) for any cylinder set and conclude by approximation).*

Exercise 37. *Let* $p \in]0,1[$ *and* $q = 1-p$*. Consider the phase space* $\Omega = \{0,1\}^N$ *equipped with the shift operator* $\mathcal{S}$ *(coin flipping). Let* μ_p *be the infinite product measure with* $\mu_p(0) = p$ *(coin flipping). Show that the measure* μ_p *is mixing. Hint: consider first cylinder sets in the mixing condition, and then approximate any set by a finite union of cylinder sets. Recall that for* $x_r^p \in \{0,1\}^{p-r+1}$ *the cylinder set* $C(x_r^p)$ *is defined by* $C(x_r^p) = \{\underline{y} \in \Omega \,|\, y_j = x_j \;\; r \le j \le p\}$.

Deduce that the measure μ_p *is ergodic.*

The dynamical system $(\{0,1\}^N, \mathcal{S}, \mu_p)$ is called a Bernoulli scheme (the general case involves product measures on a finite alphabet).

Figure 47: Jacob Bernoulli (1654-1705).

As explained before, in order to show that the ergodic sum has a (normalized) limiting variance, we need to estimate the convergence rate in the auto-correlation function (28)

$$C_{f,f}(j) = \int f(x)\, f\big(T^j(x)\big)\, d\mu(x) = \mathbb{E}\big(X_0\, X_j\big)\ .$$

For a general square integrable function (or even for the characteristic function of a general measurable set), this is often a hopeless task.

It is often the case that the rate of convergence depends on the class of functions one considers. Namely one may get useful estimates by restricting the class of functions (observables). Note that the correlation function can also be written

$$C_{f,g}(j) = \int f(x)\, U^j g(x)\, d\mu(x) - \int f\, d\mu \int g\, d\mu\ ,$$

where U is the Koopman operator defined by $Ug(x) = g(T(x))$. In the case of maps of the interval, when the invariant probability measure μ is absolutely continuous with respect to the Lebesgue measure with density h the correlations

can be expressed in term of the Perron-Frobenius operator, namely

$$C_{f,g}(j) = \int \mathcal{L}^j f(x)\, g(x)\, d\mu(x) - \int f\, d\mu \int g\, d\mu\ , \tag{30}$$

with

$$\mathcal{L}f = \frac{1}{h} P\big(hf\big)\ . \tag{31}$$

Note that since $P\big(h\big) = h$ this formula is well defined Lebesgue almost everywhere even if h vanishes by setting $\mathcal{L}f = 0$ where $h = 0$. It is now obvious from the above expression (30) for the correlation function that information about the spectrum of $\mathcal{L}$ translates into information on the decay of correlations.

Recall that the spectrum of an operator may depend of the function space where this operator acts, i.e. of the function space to which the observable belongs. For example, if $\mathcal{L}$ has a simple eigenvalue 1, and the rest of the spectrum is contained in a disk of the complex plane of radius $\sigma < 1$, then for any $\epsilon > 0$ there is a constant $C_{\epsilon>0}$ such that for any integer n

$$\big|C_{f,g}(n)\big| \leq C_\epsilon(\sigma + \epsilon)^n\ .$$

We have also $\mathcal{L}1 = 1$, and the corresponding eigen-form is $f \mapsto \int f d\mu$. This corresponds to the subtraction of the average in the formula for the correlation function. Since $\mathcal{L}$ is conjugated to the Perron-Frobenius operator P, it is enough to study the spectrum of this last operator.

We recall that the relation between decay of correlations and spectrum of the transition matrix is also a well known fact in the theory of Markov chains.

To illustrate these ideas, consider again the map $T(x) = 2x \pmod 1$ of the interval $[0, 1]$ with the Lebesgue measure as invariant measure. If f and g belong to L^2, one can use their Fourier series $\big(f_p\big)$ and $\big(g_p\big)$, namely

$$f(x) = \sum_{p\in\mathbb{Z}} f_p e^{2\pi i p x}$$

where

$$f_p = \int_0^1 e^{-2\pi i p y} f(y) dy\ . \tag{32}$$

One can express the correlation function in terms of the Fourier coefficients. Replacing f and g by their Fourier expansion in the formula for the correlation function and using $T^n(x) = 2^n x \pmod 1$, we get

$$C_{f,g}(j) = \sum_{p,q} f_p g_q \int_0^1 e^{2\pi i p x} e^{2\pi i q 2^j x} dx - f_0 g_0 = \sum_{q\neq 0} f_{-q2^j}\, g_q\ . \tag{33}$$

A first brutal estimate using Schwarz inequality gives

$$\big|C_{f,g}(j)\big| \leq \|g\|_2 \left(\sum_{q\neq 0} \big|f_{q2^j}\big|^2\right)^{1/2}\ . \tag{34}$$

It follows easily from $f \in L^2$ using Parseval's inequality that the last term tends to zero when j tends to infinity.

In other words, we have just proved that for the Lebesgue invariant measure the map $2x \pmod 1$ is mixing. Note that it is crucial in this argument that the sum extends on $q \neq 0$, this is exactly the role of the subtraction of the product of the averages in the definition of the correlation function

$$C_{f,g}(j) = \int f(x)\, g\big(T^j(x)\big)\, d\mu(x) - \int f\, d\mu \int g\, d\mu\ .$$

Recall that this also amounts to replace f (respectively g) by $f - \int f\, d\mu$ (respectively $g - \int g\, d\mu$) which has zero average. If f can be extended to an analytic function in a domain containing the interval $[0,1]$, its Fourier coefficients decay exponentially fast. In this case, for any $g \in L^2$ we have a doubly exponential decay of the correlations.

Exercise 38. *Prove these two statements, the first one using formula (32), the second one using formula (34).*

If we only have a power law bound on the decay of the Fourier coefficients of f, we can only conclude that the correlations have a power law decay.

Recall that the behaviour of the Fourier coefficients is related to the regularity properties of the function. For example if $f \in C^1$ we have for any $p \neq 0$

$$\big|f_p\big| \leq \frac{1}{2\pi|p|} \|f'\|_2\ .$$

Exercise 39. *Prove this estimate using integration by parts in formula (32).*

Using formula (33) for the correlation function, we conclude that for any $g \in L^2$ and any $f \in C^1$

$$\big|C_{f,g}(j)\big| = \left|\sum_{q\neq 0} f_{q2^j}\, g_q\right| \leq \sum_{q\neq 0} \frac{1}{2\pi\, |q|\, 2^j} |g_q|\, \|f'\|_2 \leq \Gamma\ 2^{-j}\ \|g\|_2\, \|f'\|_2 \tag{35}$$

with

$$\Gamma = \frac{1}{2\pi} \left(\sum_{q\neq 0} \frac{1}{q^2}\right)^{1/2}\ .$$

By comparing the case of f analytic and $f \in C^1$ we see that the bounds we have obtained depend on the regularity properties of the function f.

One may wonder if this is an artifact of our crude estimation method or if this is indeed the case. To see this, we can consider a particular example, namely, let for $p \neq 0$

$$f_p = \frac{1}{|p|^\alpha}$$

for some fixed $1/2 < \alpha < 1$.

Using Parseval's inequality, it follows at once that the function f whose Fourier series is (f_p) belongs to L^2. If we take $g_p = 0$ for any p except that $g_1 = g_{-1} = 1$. We get from (33)

$$C_{f,g}(j) = 2\ 2^{-j\alpha}\ .$$

We conclude by comparison with (35) that the function f does not belong to C^1 and its correlation decays indeed more slowly.

Exercise 40. *Compute the autocorrelation function $C_{f,f}(j)$ and discuss its decay rate as a function of α. Same question with the function f whose Fourier series is given by $f_0 = 0$ and for $q \neq 0$*

$$f_q = \frac{1}{|q|^{1/2} \log |q|} \ .$$

For this function f and the previous function g, compare the behaviour for large j of $C_{f,g}(j)$ and $C_{g,f}(j)$.

Exercise 41. *Use the same technique of Fourier series to discuss the decay of correlations for the cat map.*

The situation described above relating the rate of decay of correlations with some kind of regularity properties of the observables is quite frequent in the theory of dynamical systems. One looks for two Banach spaces of functions on the phase space $\mathcal{B}_1$ and $\mathcal{B}_2$ and a function $C_{\mathcal{B}_1,\mathcal{B}_2}(n)$ tending to zero when n tends to infinity such that for any $f \in \mathcal{B}_1$, for any $g \in \mathcal{B}_2$ and for any integer n

$$\left| \int f(x)\, g\big(T^n(x)\big)\ d\mu(x) - \int f\ d\mu \int g\ d\mu \right| = \big|C_{f,g}(n)\big|$$
$$\leq C_{\mathcal{B}_1,\mathcal{B}_2}(n)\ \|f\|_{\mathcal{B}_1}\ \|g\|_{\mathcal{B}_2}\ . \tag{36}$$

Note that it is not necessary for $\mathcal{B}_1$ and $\mathcal{B}_2$ to be contained in $L^2(d\mu)$, they only need to be in duality in some sense (see [116]).

As explained above, this kind of estimate may follow from adequate information on the spectral theory of a Perron-Frobenius operator. We mention here without proof the case of piecewise expanding maps of the interval.

Theorem 4.2. *Let T be a piecewise expanding map of the interval. The associated Perron-Frobenius operator P maps the space $\mathbf{BV}$ of functions of bounded variations into itself. In this space, the spectrum is composed of the eigenvalue one with finite multiplicity, a finite number (may be there are none) of eigenvalues of modulus one with finite multiplicity, and the rest of the spectrum is contained into a disk centered at the origin and of radius $\sigma < 1$. Any eigenvector of eigenvalue one is a linear combination of a finite number of non negative eigenvectors with support of disjoint interiors, which are finite union of intervals permuted by the map. The eigenvalues of modulus one different from one if they exist are rational multiples of 2π and therefore do not occur in the spectrum of a sufficiently high power of P.*

We refer to [87] for a proof of this theorem, to [45] [6] for more results and extension. We can now consider some iterate of T and choose one of the absolutely continuous invariant measure with positive density h and support an interval $[a, b]$ (recall that a finite iterate of a piecewise expanding map of the interval is also a piecewise expanding map of the interval). In that case, in the space $\mathbf{BV}([a, b])$ of functions of bounded variation with support equal to the support of h, the Perron-Frobenius operator P has one as a simple eigenvalue (with eigenvector h), and the rest of the spectrum contained in a disk centered at the origin and of radius $\sigma < 1$.

In particular, for any $1 > \rho > \sigma$, there is a constant $\Gamma_\rho > 0$ such that for any $f \in \mathbf{BV}([a, b])$ and any integer n, we have

$$\big\|P^n f\big\|_{\mathbf{BV}([a,b])} \leq \Gamma_\rho\ \rho^n\ \big\|f\big\|_{\mathbf{BV}([a,b])}\ . \tag{37}$$

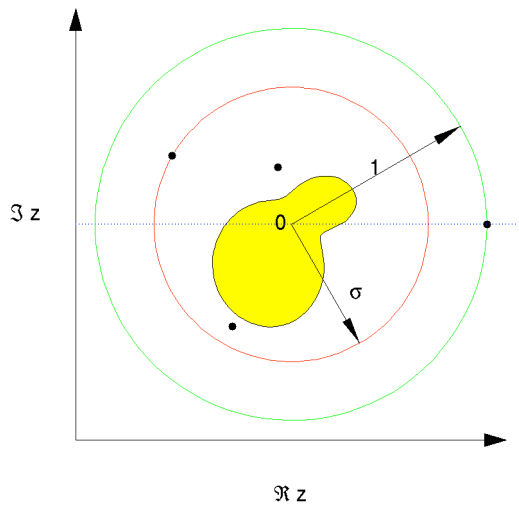


Figure 48: An example of spectrum with a dominating eigenvalue 1. Isolated eigenvalues are in black, the essential spectrum is in yellow.

This immediately implies that for any $f \in \mathbf{BV}([a,b])$ and $g \in L^1(dx)$, we have

$$\left|C_{f,g}(n)\right| \leq \Gamma_\rho \, \rho^n \, \|f\|_{\mathbf{BV}([a,b])} \, \|g\|_{L^1} \, . \tag{38}$$

In other words, we have the estimate (36)

$$\left|C_{f,g}(n)\right| \leq C_{\mathcal{B}_1,\mathcal{B}_2}(n) \, \|f\|_{\mathcal{B}_1} \, \|g\|_{\mathcal{B}_2}$$

with $\mathcal{B}_1 = \mathbf{BV}([a,b])$, $\mathcal{B}_2 = L^1$, and $C_{\mathcal{B}_1,\mathcal{B}_2}(n) = \Gamma_\rho \, \rho^n$.

The method of Perron-Frobenius operators has also been used to prove the decay of correlations for Gibbs states on sub-shifts of finite type, and for some non uniformly hyperbolic dynamical systems. (see [21] and [140]), and for some non uniformly hyperbolic dynamical systems (see [164] and references therein).

The relation between the rate of decay of correlations and other dynamical quantities (described below like entropy, Lyapunov exponents etc.) is in general not simple. We refer to [53] for some results.

4.3 Coupling

Another important method to investigate the decay of correlations (and other properties of dynamical systems) is the method of coupling borrowed from probability theory. These ideas were used in particular by L.S.Young to investigate the decay of correlations in some non uniformly hyperbolic systems. We refer to [7] for the case of piecewise expanding maps of the interval, [25] for the case of chains with complete connexions which generalize the case of Gibbs states on sub-shifts of finite type, to [26], and to [162] for non uniformly hyperbolic dynamical systems.

We give here some ideas for the case of uniformly expanding maps of the interval.

We first recall that if we have two measures μ on a space X and ν on a space Y, a coupling between μ and ν is a measure on the product space such that its marginals are μ and ν respectively. Let T is a piecewise expanding map on the interval with an a.c.i.p.m. μ with a non-vanishing density h. In this situation we can define a Markov chain on the interval by the following formula for its transition probability

$$p\big(y|x\big) = \frac{h(y)}{h(x)}\delta\big(x - T(y)\big) = \frac{1}{h(x)} \sum_{z,\, T(z)=x} \frac{h(z)}{\big|T'(z)\big|}\delta(y-z)\ . \tag{39}$$

Exercise 42. *Show that the integral over y is equal to one.*

This transition probability is nothing but the kernel of the operator $\mathcal{L}$ defined in formula (31). This is a rather irregular kernel, it only charges a finite number of points, namely the points y such that $T(y) = x$. These are called the preimages of x under the map T. Nevertheless, for piecewise expanding maps of the interval the following result has been proved in [7].

Theorem 4.3. *For any piecewise expanding map on the interval with an a.c.i.p.m. μ with a non-vanishing density, there are three positive constants C, $\rho_1 < 1$ and $\rho_2 < 1$, and for any pair of points x_1 and x_2 in the interval, there is a coupling $\mathbb{P}_{x_1,x_2}$ between the two Markov chains $\big(X_n^1\big)$ and $\big(X_n^2\big)$ defined by the kernel (39) starting in x_1 and x_2 respectively and such that for any $n \in \mathbf{n}$*

$$\mathbb{P}_{x_1,x_2}\left(\big|X_n^1 - X_n^2\ \big| > \rho_1^n\right) \le C\rho_2^n\ .$$

The statement in the original paper [7] is in fact stronger but we will only use the above form.

Note that the constants C, ρ_1 and ρ_2 do not depend on x_1 and x_2. Note also the difference with the coupling for Markov chains on finite state space. Here because of the singularity of the kernel, the realisations of $\big(X_n^1\big)$ and $\big(X_n^2\big)$ will in general not meet. However they can approach each other fast enough. The condition that h does not vanish is not restrictive. We have seen already that this is first connected with ergodicity (see Theorem 4.2). Once ergodicity is ensured, the vanishing of h can be discussed in details. We refer to [7] and [30] for the details.

We now show the application of this theorem to the proof of the decay of correlations. Le g be a function of bounded variation. We have from the definition of the Markov chains $\big(X_n^1\big)$ and $\big(X_n^2\big)$ and from the definition of the coupling

$$\big(\mathcal{L}^n g\big)(x_1) - \big(\mathcal{L}^n g\big)(x_2) = \mathbb{E}\big(g\big(X_n^1\big)\big) - \mathbb{E}\big(g\big(X_n^2\big)\big) = \mathbb{E}_{x_1,x_2}\Big(\big(g\big(X_n^1\big)\big) - \big(g\big(X_n^2\big)\big)\Big)\ .$$

Assume now g is a Lipschitz function with Lipschitz constant L_g, then we get by applying Theorem 4.3

$$\begin{aligned}
&\Big|\mathbb{E}_{x_1,x_2}\Big(\big(g\big(X_n^1\big)\big) - \big(g\big(X_n^2\big)\big)\Big)\Big| \\
&\le \mathbb{E}_{x_1,x_2}\Big(\chi_{\left|X_n^1 - X_n^2\right| \le \rho_1^n}\Big|\big(g\big(X_n^1\big)\big) - \big(g\big(X_n^2\big)\big)\Big|\Big) \\
&+\mathbb{E}_{x_1,x_2}\Big(\chi_{\left|X_n^1 - X_n^2\right| > \rho_1^n}\Big|\big(g\big(X_n^1\big)\big) - \big(g\big(X_n^2\big)\big)\Big|\Big)
\end{aligned}$$

$$\leq L_g \rho_1^n + 2\ C\ L_g \rho_2^n\ .$$

We now consider a correlation $C_{u,v}(n)$ between a Lipschitz function u and an integrable function v. We have easily using formula (30) and the invariance of μ

$$C_{u,v}(n) = \int\int \Big(\mathcal{L}^n u(x_1) v(x_1) - \mathcal{L}^n u(x_1) v(x_2)\Big)\ d\mu(x_1)\ d\mu(x_2)\ ,$$

We can now use the above estimate to get

$$|C_{u,v}(n)| \leq \big(L_u \rho_1^n + 2\ C\ L_u \rho_2^n\big) \int |v|\ d\mu\ ,$$

which is the exponential decay of correlations as in formulas (36) and (38).

We explain here very briefly the idea of coupling on the particularly simple case of the (Markov) map $T(x) = 2x \pmod 1$. In that case, the transition probability is given by

$$p(y|x) = \frac{1}{2} \sum_{z\ ,\ 2z=x \pmod 1} \delta(y-z) = \delta(2y-x)\ . \tag{40}$$

Let $\mathcal{P}$ be the partition of the interval $\mathcal{P} = \big\{[0,1/2],]1/2,1[$. It is easy to verify that, except for a countable number of points x, any of the 2^n atoms of the partition $\vee_{j=1}^n T^{-j}\mathcal{P}$ contains one and only one preimage of order n of a point x (the points y such that $T^n(y) = x$).

Exercise 43. *Use the coding with the full unilateral shift on two symbols to prove this assertion (see example 6).*

In other words for two points x and x' we have a coupling at the topological level of their preimages by putting together the preimages which are in the same atom of $\vee_{j=1}^n T^{-j}\mathcal{P}$. We can now write the coupling on the trajectories of the Markov chain. For any pair of points x and x' in $[0,1]$, for any integer n, and for any pair of points $\big(y_1,\ldots,y_n\big)$ and $\big(y'_1,\ldots,y'_n\big)$ of $[0,1]^2$ we define

$$\mu_{x,x'}\ \big((y_1,\ldots,y_n),(y'_1,\ldots,y'_n)\big)$$

$$= \sum_{I\in\vee_{j=1}^n T^{-j}\mathcal{P}} \chi_I(y_n)\chi_I(y'_n) \prod_{l=1}^n \delta\big(2y_l - y_{l-1}\big) \prod_{l=1}^n \delta\big(2y'_l - y'_{l-1}\big)$$

where for convenience of notations we have defined $y_0 = x$ and $y_{0'} = x'$.

Exercise 44. *Show that the above formula defines a measure on* $\big([0,1]\times[0,1]\big)^{\mathbf{N}}$ *(use Kolmogorof's Theorem, see for example [76]). Show that except for a countable number of points x and x' of the interval, this measure is a coupling between of the two Markov chains with transition kernel (40) and initial points x and x'. Show that this coupling satisfies the conclusions of Theorem 4.3.*

The more general cases use basically the same ideas with the complication that the transition probability is not constant. We refer to the above mentioned references for the details.

4.4 Central limit Theorem.

Recall that

$$S_n(f) = \sum_{\ell=0}^{n-1} f \circ T^\ell \ .$$

In the case where the asymptotic standard deviation σ_f given by formula (27)

$$\sigma_f^2 = \int f^2(x)\ d\mu(x) + 2\sum_{j=1}^{\infty} \int f(x)\, f\big(T^j(x)\big)\ d\mu(x)\ .$$

is non zero, we say that the central limit theorem holds if

$$\lim_{n\to\infty} \mu\left(\left\{x \ \middle|\ \frac{S_n(f)(x)}{\sigma_f\sqrt{n}} \le t\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du\ . \tag{41}$$

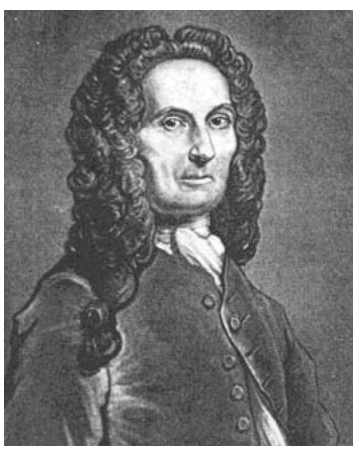


Figure 49: Abraham de Moivre (1667 - 1754), Pierre-Simon Laplace (1749-1827).

We emphasize that this kind of result has been only established for certain classes of dynamical systems and observables. We refer to [57] and [119] for reviews and [162] for recent results.

There are two methods which were mostly used up to now for proving this result. One is based on a result of Gordin (see [77]) which reduces the problem to the known central limit theorem for martingales. The other method can be used when a Perron-Frobenius operator with adequate spectral properties is available. We illustrate this idea for the case of piecewise expanding maps of the interval.

Let f be a real function on the unit interval with bounded variations and zero average with respect to an ergodic and mixing a.c.i.p.m. with density h.

According to a theorem of Paul Levy (see for example [14] or [72]), if $\sigma_f > 0$, in order to prove the central limit theorem (41), it is enough to prove convergence of the Fourier transforms of the laws (characteristic function), namely that for any real θ

$$\lim_{n\to\infty} \int e^{i\theta S_n(f)(x)/\sqrt{n}} h(x) dx = e^{-\sigma_f^2\theta^2/2}\ . \tag{42}$$

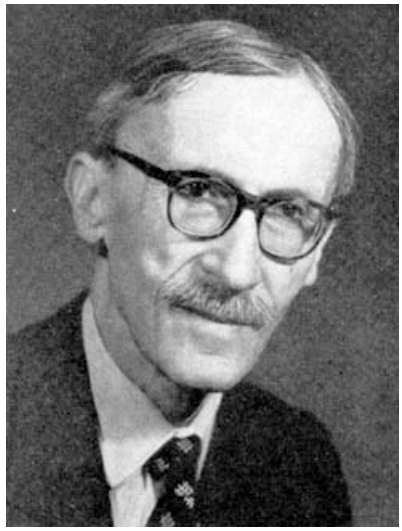


Figure 50: Paul Lévy (1886 - 1971).

To prove this convergence, we first define a family of operators P_v ($v \in \mathbf{BV}$) by

$$P_v u(x) = P\left(e^v u\right) \ . \tag{43}$$

Using several times the elementary properties of the Perron-Frobenius operator, it follows that the characteristic function of the random variable $S_n(f)/\sqrt{n}$ is equal to

$$\int e^{i\theta S_n(f)(x)/\sqrt{n}} h(x) dx = \int P^n_{(if\theta/\sqrt{n})}\, h dx \ .$$

For large n, the function $i\theta f/\sqrt{n}$ is small and therefore we expect the operator $P_{if\theta/\sqrt{n}}$ to be in some sense near the operator $P = P_0$. This is indeed the case in the following sense.

In the space of functions of bounded variations, it is easy to verify that for any complex number z of modulus smaller than one

$$\left\|P - P_{zf}\right\|_{\mathbf{BV}} \leq \mathcal{O}(1)|z| \ .$$

Therefore, if $|z|$ is small enough, using Theorem 4.2 and the analytic perturbation theory around simple eigenvalues [96] one can establish the following result.

Theorem 4.4. *Under the hypothesis that* 1 *is a simple eigenvalue of* P *and the unique point of the spectrum of modulus one, there is a positive number* ζ *and two positive numbers* Γ_1 *and* $\rho_1 < 1$ *such that for any complex valued function* v *of bounded variation satisfying* $\|v\|_{\mathbf{BV}} < \zeta$*, the operator* P_v *has a unique simple eigenvalue* λ_v *of modulus larger than* ρ_1 *with eigenvector* h_v *and eigenform* α_v *(satisfying* $\alpha_v\big(h_v\big) = 1$*).* λ_v*,* h_v *and* α_v *are analytic in* v *in the ball of functions of bounded variation centered at the origin and of radius* ζ*. Moreover,* $\lambda_0 = 1$*,* $h_0 = h$*, and* α_0 *is the integration against the Lebesgue measure. The rest of the*

spectrum of P_v (in the space of functions of bounded variations) is contained in the disk of radius ρ_1 and we have for any positive integer m the estimate

$$\left\|P_v^m \Pi_v\right\| \leq \Gamma_1 \rho_1^m \ ,$$

where Π_v is the spectral projection on λ_v given by

$$\Pi_v(g) = \alpha_v(g)\ h_v \ .$$

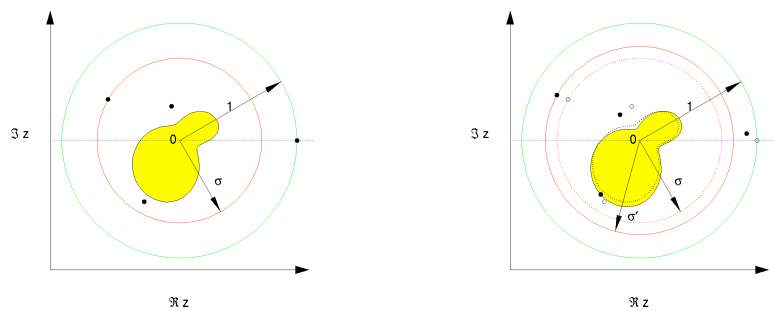


Figure 51: On the left, the spectrum of an unperturbed Perron-Frobenius operator. On the right, the spectrum under a small perturbation.

In other words,

$$P_v^m g = \lambda_v^m\ \alpha_v(g)\ h_v + \mathscr{O}\big(\rho_1^m\big) \ .$$

Since for v small we have $|\lambda_v| > \rho_1$, the first term dominates for large m (unless $\alpha_v(g) = 0$). We can now apply this result with $v = i\theta f/\sqrt{n}$ and $m = n$ because the estimates are uniform. We get for n large enough such that $|\theta|/\sqrt{n} < \zeta$

$$P^n_{i\theta f/\sqrt{n}}\ h = \lambda^n_{i\theta f/\sqrt{n}}\ h_{i\theta f/\sqrt{n}}\ \alpha_{i\theta/\sqrt{n}}(h) + \mathcal{O}(1)\ \rho_1^n \ .$$

A simple perturbation theory computation (see [96] or [36]) shows that since f has average zero

$$\lambda_{zf} = 1 + \frac{z^2}{2\sigma_f^2} + \mathcal{O}\big(|z|^3\big) \ ,$$

and the result follows using Levy's theorem by letting n tend to infinity since

$$\lambda^n_{i\theta f/\sqrt{n}} = e^{-\theta^2/(2\sigma_f^2) + \mathscr{O}(n^{-1/2})} \ .$$

Exercise 45. *Show that for large n, $\alpha_{i\theta/\sqrt{n}}(h) \neq 0$, and complete the details of the proof*

For some particular classes of dynamical systems some more precise results have been established. For Gibbs states over sub-shifts of finite type, a Berry-Esseen inequality has been obtained by Coelho and Parry [35]. They proved that for such dynamical systems, for any Hölder continuous observable f (of zero average and with $\sigma_f > 0$), there is a constant $C > 0$ such that for any integer $n > 0$

$$\left|\mu\left(\left\{x \ \middle|\ \frac{S_n(f)(x)}{\sigma_f\sqrt{n}} \leq t\right\}\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-u^2/2} du\right| \leq \frac{C}{\sqrt{n}} \ .$$

Analogous results were also obtained in non uniformly hyperbolic systems by S.Gouezel [78]. Local limit theorems have also been proved (see for example [28]).

One can also study more globally the sequence of random variables $S_n(f)/\sqrt{n}$ (with f of zero average and $\sigma_f > 0$). Recall that the Brownian motion $B_\cdot$ is the unique continuous time Gaussian process with independent increments, defined for for $t \geq 0$, with $B_0 = 0$, with B_t of zero average for any $t > 0$ and such that

$$\mathbb{E}\big(B_t\, B_s\big) = \min\big\{t, s\big\}\ .$$

Let $\big(\xi_n\big)$ be the sequence of random functions defined by

$$\xi_n(t) = \frac{1}{\sigma_f\sqrt{n}}\sum_{j=0}^{[nt]} f\circ T^j + \frac{1}{\sigma_f\sqrt{n}}\big(nt - [nt]\big) f\circ T^{[nt]+1}\ , \tag{44}$$

where [ ] denotes the integer part. This is a random sequence of continuous functions (recall that the randomness comes from the choice of the initial condition with respect to the invariant measure). If we drop the last term, we obtain a random sequence of piecewise constant functions.

Exercise 46. *Prove that for any fixed integer n, the function ξ_n defined in (44) is continuous.*

There are several classes of dynamical systems where one can show that this process converges weakly to a Brownian motion (even without the last term). See for example [14] or [76] for the definition of weak convergence of stochastic processes. We state here the case of piecewise expanding map of the interval. We refer to [57] for more results and references.

Theorem 4.5. *Let T be a piecewise expanding map of the interval and μ a mixing absolutely continuous invariant probability measure with a non vanishing density h. Let f be a Lipschitz continuous function on the interval with zero μ average and with nonzero standard deviation σ_f. Then the sequence of processes $\big(\xi_n(\,\cdot\,)\big)$ defined in (44) converges weakly to the Brownian motion.*

This theorem also holds without the last term in equation (44). We will see later on a proof of this theorem using exponential estimates.

There are many other results around the central limit theorem, we will just mention here a few. They have been proved under various hypothesis, and we refer the reader to the literature for the details.

We consider as before the ergodic sum S_n of an observable f with zero average and finite (non zero) variance σ_f, and satisfying some adequate hypothesis. The law of iterated logarithm says that almost surely,

$$\limsup_{n\to\infty} \frac{S_n}{\sigma_f\sqrt{2n\log\log n}} = 1\ . \tag{45}$$

We refer to Denker and Broise ([28], [57]) for precise hypothesis and proofs, for example in the case of piecewise expanding maps of the interval.

A more precise comparison with Brownian motion is provided by the almost sure invariance principle. One says that the almost sure invariance principle holds if there is another probability space with a Brownian motion $B_\cdot$ and a

sequence of random variables $\big(\tilde{S}_n\big)$ with the same joint distributions as (S_n) such there is an integer valued random variable N almost surely finite and a constant $\delta > 0$ such that for any $n > N$

$$\big|\tilde{S}_n - \sigma_f B_n\big| \leq n^{-\delta+1/2} \ .$$

See [133] for a general approach. This has been proved for piecewise expanding maps of the interval and for some non uniformly hyperbolic dynamical systems, see [57], [167], [28], [125] for the hypothesis and the proofs.

One can also prove almost sure central limit theorems, for example that for any real s, the sequence of random variables

$$\frac{1}{D_n}\sum_{k=1}^{n}\frac{1}{k}\,\theta\left(s - \frac{S_k}{\sigma_f\sqrt{k}}\right) = \frac{1}{D_n}\sum_{k=1}^{n}\frac{1}{k}\,\mathbb{1}_{\frac{S_k}{\sigma_f\sqrt{k}}<s}$$

with

$$D_n = \sum_{k=1}^{n}\frac{1}{k}$$

converges almost surely to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-u^2/2}du \ .$$

Here θ is the Heaviside function, namely $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x \geq 0$. This has been established for piecewise expanding maps of the interval. We refer to [31] for the hypothesis, proofs and references. An analog of the Erdös-Renyi asymptotic is also proved in [31]. Moderate deviations have been treated in [56].

4.5 Large deviations.

Beyond the central limit theorem and its variants and corrections, one can wonder about the probability that an ergodic sum deviates appreciably from its limit, namely S_n/n differs from $\mathbb{E}(S_n)/n$ at order one. In other words, if μ is an ergodic invariant measure and f an integrable observable with zero average, one can ask for the probability of a large deviation, namely for $t > 0$ this is the quantity

$$\begin{aligned}\mathbb{P}\big(S_n(f) > nt\big) &= \mu\left(\left\{x \,\middle|\, S_n(f) > nt\right\}\right)\\ &= \mu\left(\left\{x \,\middle|\, \sum_{l=0}^{n-1} f\big(T^l(x)\big) > nt\right\}\right) \ . \end{aligned} \tag{46}$$

One can look for a similar quantity for $t < 0$, namely $\mathbb{P}\big(S_n(f) < nt\big)$ (or equivalently change f to $-f$). This quantity $\mathbb{P}\big(S_n(f) > nt\big)$ often decays exponentially fast with n, and it is therefore convenient to define the large deviation function $\varphi(t)$ by

$$\begin{aligned}\varphi(t) &= -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(S_n(f) > nt\big)\\ &= -\lim_{n\to\infty}\frac{1}{n}\log\mu\left(\left\{x \,\middle|\, S_n(f) > nt\right\}\right) \ . \end{aligned} \tag{47}$$

Of course the limit may not exist and one can take the limsup or the liminf. In order to investigate this quantity, one can use an analogy with statistical mechanics (see [111] and [140]). There one first computes another quantity called the pressure function defined in the present context (for $z \in \mathbb{C}$) by

$$\Xi(z) = \lim_{n\to\infty} \frac{1}{n} \log \left(\int e^{zS_n(f)} d\mu \right) . \tag{48}$$

As we will see below it is often convenient to take for z a complex number. Of course the limit may not exist. We now explain informally the relation between the large deviation function φ and the pressure Ξ.

From the definition of the pressure (48), we have

$$\int e^{zS_n(f)} d\mu \sim e^{n\Xi(z)} .$$

On the other hand, if z is real we have from the definition of φ (47) and Chebyshev's inequality

$$\int e^{zS_n(f)} d\mu \geq e^{nzt} \mu \left(\left\{ x \,\middle|\, S_n(f) > nt \right\} \right) \sim e^{nzt} e^{-n\varphi(t)} .$$

We conclude immediately from these two relations that for any real z

$$\varphi(t) \geq zt - \Xi(z) .$$

Since this should hold for any real z, we conclude that

$$\varphi(t) \geq \sup_{z\in\mathbb{R}} \big(zt - \Xi(z) \big) .$$

The function

Figure 52: Pafnuty Chebyshev (1821-1894).

$$\psi(t) = \sup_{z\in\mathbb{R}} \big(zt - \Xi(z)\big) \ .$$

is called the Legendre transform of the function Ξ.

In the other direction,

$$e^{n\Xi(z)} \sim \int e^{zS_n(f)} d\mu \sim \int e^{znt} d\mathbb{P}(S_n(f) > nt) \sim \int e^{znt} e^{-n\varphi(t)} dt \ ,$$

using the definition of $\varphi(t)$ (in a slightly sloppy way). Using steepest descent (see for example [70]) we get

$$e^{n\Xi(z)} \sim \int e^{znt} e^{-n\varphi(t)} dt \sim e^{n(zt_z^* - \varphi(t_z^*)}$$

where $\varphi'(t_z^*) = z$.

Summarising, we have obtained the relation

$$\Xi(z) = zt_z^* - \varphi(t_z^*)$$

where t_z^* is the solution of $\varphi'(t_z^*) = z$ (assumed to be unique). It now remains to invert the Legendre transform to conclude that

$$\varphi(t) = tz_t^\sharp - \Xi(z_t^\sharp)$$

where $z_t^\sharp$ is the solution of $\Xi'(z_t^\sharp) = t$.

Exercise 47. *Compute the derivative with respect to z of the equality $\varphi(t_z^*) = zt_z^* - \Xi(z)$ (with $\varphi'(t_z^*) = z$). Derive that $\Xi'(z) = t_z^*$, $t^*_{z_t^\sharp} = t$ (assuming uniqueness of the solution of the various equations) and conclude that $\varphi(t) = tz_t^\sharp - \Xi(z_t^\sharp)$ with $\Xi'(z_t^\sharp) = t$.*

This argument can be made rigorous under certain hypothesis on the pressure (basically differentiability which in statistical mechanics correspond to the absence of phase transition). We refer to [134], [111] and [69] for the general theory.

We now explain briefly how to prove the existence of the pressure in the simple case of piecewise expanding maps of the interval. We consider the case of μ the a.c.i.p.m. with density h and an observable f of bounded variation. Using the properties of the Perron-Frobenius operator, we get

$$\int e^{zS_n(f)} d\mu = \int e^{zS_n(f)} h dx = \int P_{zf}^n(h) \tag{49}$$

where P_{zf} is the operator defined in (43).

Exercise 48. *Prove this equality.*

Intuitively the integral should behave like the eigenvalue λ_z of largest modulus of the operator P_{zf} (this argument was already used for the proof of the central limit theorem). For values of z of modulus not too large, Theorem 4.4 shows that the peripheral spectrum is reduced to a simple eigenvalue and we get immediately

$$\Xi(z) = \log \lambda_z \ .$$

Differentiability of the pressure can be derived similarly, and as explained before this allows to obtain the large deviation function in a neighborhood of the origin (even analyticity of the pressure follows from the analyticity properties of λ_z and the fact that this function does not vanish near the origin). In the case of (piecewise regular) Markov maps of the interval and for Gibbs states over sub-shifts of finite type (with Hölder potentials), one can show that for Hölder continuous functions f, the limit in (48) exists for any real z and is regular. In this case one can construct the large deviation function for all t (more precisely between the supremum and the infimum of the observable f). We refer to [21] and [140] for a detailed study of these cases. We refer to [166] for a general approach to the large deviations of dynamical systems.

4.6 Exponential and related estimates.

The large deviation results are very precise but as we saw above they require proving that the pressure exists. This is often a difficult problem and one would prefer sometimes to obtain less precise estimates at a lower cost.

The related concentration phenomena has been known for a long time (in some sense already to Gibbs and others) and was revived about ten years ago by Talagrand mostly in the context of independent random variables (see [155], and [114]). Several results in the non independent case have been obtained since and it turns out that one can prove such results for some classes of dynamical systems. To be more precise I will now state a definition.

We consider in this section a discrete time dynamical system given by a map T in a phase space Ω, equipped with a metric d, and with an ergodic invariant measure μ.

Definition 13. *A real valued function K on Ω^n will be said componentwise Lipschitz if for any $1 \leq j \leq n$, the constant $L_j(K)$ defined by*

$$L_j(K) = \\ sup_{x_1,\dots,x_n,y} \frac{\left|K\big(x_1^{j-1}, x_j, x_{j+1}^n\big) - K\big(x_1^{j-1}, y, x_{j+1}^n\big)\right|}{d(x_j, y)} \tag{50}$$

is finite (notation: $x_p^q = (x_p, \dots, x_q)$) .

In other words, we assume the function to be Lipschitz in each component with a uniformly bounded Lipschitz constant.

Exercise 49. *Show that a function K is componentwise Lipschitz if and only if it is Lipschitz. Compare the numbers $L_j(K)$ to the global Lipschitz constant.*

Definition 14. *We say that the measure μ satisfies the exponential inequality for the map T if there are two constants $C_1 > 0$ and $C_2 > 0$ such that for any integer $n > 0$, and for any componentwise Lipschitz function K of n variables we have*

$$\int e^{K\left(x, T(x), \dots T^{n-1}(x)\right) - \mathbb{E}(K)} d\mu(x) \leq C_1\, e^{C_2 \sum_{j=1}^n L_j(K)^2} \tag{51}$$

where

$$\mathbb{E}\big(K\big) = \int K\big(x, T(x), \dots T^{n-1}(x)\big) d\mu(x) \ .$$

There are several important facts to emphasize about this definition.

Remark 2.

i) Comparing with the definition of the pressure (48) one sees that we have on the left hand side the same kind of integral with $K = z\, S_n(f)$.

ii) However we require here an estimation for a larger class of functions, not only ergodic sums.

iii) On the other hand, contrary to the case of the pressure, we require only an upper bound, not the existence of a precise limit.

iv) It is also worse emphasizing that the estimate is assumed for any n, not only as a limit.

v) One often exploits the exponential inequality by using a Chebyshev inequality (see examples below).

A weaker version of the exponential inequality is the Devroye inequality on the variance.

Definition 15. *We say that the measure μ satisfies the Devroye inequality for the map T if there is a constant $C > 0$ such that for any integer $n > 0$, and for any componentwise Lipschitz function K of n variables we have*

$$\mathrm{Var}(K) = \int \big(K\big(x, T(x), \dots T^{n-1}(x)\big) - \mathbb{E}(K)\big)^2 \, d\mu(x)$$

$$\leq C \sum_{j=1}^{n} L_j(K)^2 \, . \tag{52}$$

As was mentioned before, there is a relation between the exponential estimate and concentration. This relation is somewhat analogous to the relation between pressure and large deviation function explained in the previous section, although less precise. It follows immediately from the Chebyshev exponential inequality applied to the exponential estimate for the functions sK and $-sK$, with an optimal choice for the number s. Namely

$$\mathbb{P}\big(\big|K\big(x, T(x), \dots T^{n-1}(x)\big) - \mathbb{E}(K)\big| > t\big) \leq 2\, C_1\, e^{-t^2/(4C_2 \sum_{j=1}^{n} L_j(K)^2)}$$

Exercise 50. *Prove this estimate and the analogous estimate based on the variance inequality.*

As an example we consider a sub-shift of finite type on the finite alphabet $\mathcal{A}$.

We consider on the phase space a metric d_ζ defined in 19 denoted below by d. For a fixed integer $p > 0$, let A be a measurable subset of Ω^p such that

$$\alpha = \mu\left(\left\{x \,\middle|\, \big(x, T(x), \dots, T^{p-1}(x)\big) \in A\right\}\right) > 0.$$

For a given $\epsilon > 0$, denote by B_ϵ the neighborhood of A, given by

$$B_\epsilon = \left\{ \big(x_1, \dots, x_p\big) \,\middle|\, \exists \big(y_1, \dots, y_p\big) \in A \text{ such that } \frac{1}{p} \sum_{j=1}^{p} d\big(x_j, y_j\big) \leq \epsilon \right\} .$$

Let K_A be the function defined on Ω^p by

$$K_A\big(x_1,\dots,x_p\big) = \inf_{(y_1,\dots,y_p)\in A} \frac{1}{p}\sum_{j=1}^{p} d\big(x_j,y_j\big)\ .$$

This function measures the average distance to A of the different components of the vector $\big(x_1,\dots,x_p\big)$. Note that with this definition, we have $B_\epsilon = \{K_A \le \epsilon\}$. It is easy to verify that K_A is a componentwise Lipschitz function with all the Lipschitz constants L_j equal to $1/p$.

Exercise 51. *Prove this assertion.*

Assume the measure μ satisfies the exponential inequality (51).

Then we get for any real number β

$$\int e^{\beta K_A\left(x,T(x),\dots,T^{p-1}(x)\right)} d\mu(x) \le C_1 e^{\beta\mathbb{E}(K_A)} e^{\beta^2 C_2/p}\ .$$

By Chebyshev's inequality, we immediately derive

$$\mu\big(\big\{x \,\big|\, K_A\big(x,T(x),\dots,T^{p-1}(x)\big) > \epsilon\big\}\big) \le C_1 e^{-\beta\epsilon} e^{\beta\mathbb{E}(K_A)} e^{\beta^2 C_2/p}\ . \tag{53}$$

It remains to estimate $\mathbb{E}\big(K_A\big)$. For the particular case at hand this can be done easily.

Observe that $-\beta K_A$ is also componentwise Lipschitz, and we can apply the exponential inequality to this function. Equivalently we can change β into $-\beta$. We get

$$\int e^{-\beta K_A\left(x,T(x),\dots,T^{p-1}(x)\right)} d\mu(x) \le C_1 e^{-\beta\mathbb{E}(K_A)} e^{\beta^2 C_2/p}\ .$$

We now observe that if $\big(x,T(x),\dots,T^{p-1}(x)\big)$ belongs to A, then

$$K_A\big(x,T(x),\dots,T^{p-1}(x)\big) = 0\ .$$

Therefore we have

$$\alpha \le \int e^{-\beta K_A\left(x,T(x),\dots,T^{p-1}(x)\right)} d\mu(x) \le C_1 e^{-\beta\mathbb{E}(K_A)} e^{\beta^2 C_2/p}\ ,$$

which implies

$$e^{\beta\mathbb{E}(K_A)} \le C_1 \alpha^{-1} e^{\beta^2 C_2/p}\ .$$

Combining with equation (53) we get

$$\mu\big(\big\{x \,\big|\, K_A\big(x,T(x),\dots,T^{p-1}(x)\big) > \epsilon\big\}\big) \le C_1^2 \alpha^{-1} e^{2\beta^2 C_2/p} e^{-\beta\epsilon}\ .$$

since this is true for any real β we can take (the optimal value of β in the bound) $\beta = \frac{\epsilon p}{4C_2}$ and obtain the bound

$$\begin{aligned}\mu\big(B_\epsilon^c\big) &= \mu\big(\big\{x \,\big|\, K_A\big(x,T(x),\dots,T^{p-1}(x)\big) > \epsilon\big\}\big)\\ &\le C_1^2 \alpha^{-1} e^{-\epsilon^2 p/(8C_2)}\ .\end{aligned}$$

We can now see the concentration phenomena. Recall that

$$\mu\big(B_\epsilon^c\big) = \mu\big(\big\{x \,\big|\, K_A\big(x,T(x),\dots,T^{p-1}(x)\big) > \epsilon\big\}\big)\ .$$

Even if α is small (for example equal to 1/2), and for a fixed (small) ϵ, if p is large enough ($p \gg \epsilon^{-2} \log \alpha^{-1}$), the set B_ϵ^c has a very small measure.

This was phrased by Talagrand by saying that in large dimension (p large), sets of measure 1/2 (here A) are big (in the sense that a small neighborhood B_ϵ has almost full measure).

We give another useful application to the supremum of ergodic sums (25). Let g be a Lipschitz function on the phase space with zero average, and consider the sequence of its ergodic sums $\big(S_n\big)$. Recall that

$$S_n = \sum_{j=0}^{n-1} g \circ T^j \ .$$

One is often interested in a bound for the probability that $\sup_{1 \le j \le n} S_j$ is large. This can be obtained from an exponential estimate.

We first use again Pisier inequality, namely for any fixed number β (to be chosen adequately later on) we have

$$\mathbb{E}\left(e^{\beta \sup_{1 \le j \le n} S_j}\right) \le \sum_{j=1}^{n} \mathbb{E}\left(e^{\beta S_j}\right) \ .$$

We now use the exponential estimate to bound the right hand side.

For this purpose consider the function

$$K\big(x_1, \dots, x_j\big) = \sum_{m=1}^{j} g(x_m) \ .$$

If $\big(y_1, \dots, y_j\big)$ is another sequence of points in the phase space with $y_l = x_l$ for any $l \neq m$, we have

$$\left| \sum_{m=1}^{j} g(x_m) - \sum_{m=1}^{j} g(y_m) \right| \le L_g \big| x_j - y_j \big|$$

where L_g is the Lipschitz constant of g. Therefore, for any $j \le n$ we have from the exponential inequality (51)

$$\mathbb{E}\left(e^{\beta S_j}\right) \le C_1 e^{C_2 j \beta^2 L_g^2} \ .$$

We conclude that for any $\beta > 0$

$$\mathbb{E}\left(e^{\beta \sup_{1 \le j \le n} S_j}\right) \le n C_1 e^{C_2 n \beta^2 L_g^2} \ .$$

From Chebyshev's inequality, we get

$$\mathbb{P}\Big(\sup_{1 \le j \le n} S_j > s \Big) \le e^{-\beta s} \mathbb{E}\left(e^{\beta \sup_{1 \le j \le n} S_j}\right) \le n C_1 e^{-\beta s} e^{C_2 n \beta^2 L_g^2} \ .$$

We can now optimise the upper bound over β to get

$$\mathbb{P}\Big(\sup_{1 \le j \le n} S_j > s \Big) \le n C_1 e^{-s^2/(4 n C_2 L_g^2)} \ .$$

Note that this implies that $\sup_{1\le j\le n} S_j$ is unlikely to be much larger than $\sqrt{n\log n}$ which is essentially the scale of the law of iterated logarithm (45). We refer to ([31]) and references therein for more results in this direction.

As another application of exponential estimates we give a proof of Theorem 4.5. Namely if f is a Lipschitz continuous function on the phase space with zero average and with nonzero standard deviation σ_f, then the sequence of processes $\big(\xi_n(\,\cdot\,)\big)$ defined by

$$\xi_n(t) = \frac{1}{\sigma_f\sqrt{n}}\sum_{j=0}^{[nt]} f\circ T^j + \frac{1}{\sigma_f\sqrt{n}}\big(nt-[nt]\big) f\circ T^{[nt]+1}\ ,$$

converges weakly to a Brownian motion.

Proof of Theorem 4.5. We first recall that to prove that the sequence of processes $\big(\xi_n\big)$ converges to the Brownian motion, we have to prove two things.

i) All the marginal distributions $\mathbb{P}\big(\xi_n(t_1)>s_1,\dots,\xi_n(t_k)>s_k\big)$ converge to those of the Brownian motion. This can be proved using the same technique as for the proof of the central limit theorem. We leave it as an exercise to the reader.

ii) The sequence $\big(\xi_n\big)$ is tight which is a kind of compactness. See [76] or [14] for the definition and main results about tightness.

Recall (see [76] or [14]) that tightness follows if we can prove that there exists three positive numbers α, γ and H such that for any $t_1\ge 0$ and $t_2\ge 0$ we have for any integer n

$$\mathbb{E}\Big(\big|\xi_n(t)-\xi_n(t')\big|^\alpha\Big)\le H\ |t-t'|^{1+\gamma}\ .$$

We can of course assume $t'>t$, and we observe that if $t'-t>1/n$,

$$\big|\xi_n(t)-\xi_n(t')\big|\le\left|\frac{1}{\sigma_f\sqrt{n}}\sum_{[nt]\le j\le[nt']} f\circ T^j\right| + \frac{4|t-t'|^{1/2}L_f}{\sigma_f}$$

where L_f is the Lipschitz constant of f.

On the other hand, it is left as an exercise to the reader to check that if $0<t'-t\le 1/n$, then

$$\big|\xi_n(t)-\xi_n(t')\big|\le\frac{4|t-t'|^{1/2}L_f}{\sigma_f}\ .$$

Therefore it is enough to prove that for some $\alpha>2$, $\gamma>0$ and $H>0$, we have

$$\mathbb{E}\left(\left|\frac{1}{\sigma_f\sqrt{n}}\sum_{[nt]\le j\le[nt']} f\circ T^j\right|^\alpha\right)\le H\ |t-t'|^{1+\gamma}\ . \tag{54}$$

This is where we will use the exponential inequality.

For a fixed integer $p\ge 0$, define the function K_p of p variables $y_1,\dots,y_p$, by

$$K_p\big(y_1,\dots,y_p\big)=\frac{1}{\sigma_f\sqrt{n}}\sum_{j=1}^{p} f\big(y_j\big)\ .$$

It is left as an exercise to verify that for any integer $1 \leq j \leq p$, the Lipschitz constant of the function K_p satisfies

$$L_j\big(K_p\big) \leq \frac{L_f}{\sigma_f \sqrt{n}} \ .$$

We now observe that (recall that $t' > t$)

$$\frac{1}{\sigma_f \sqrt{n}} \sum_{[nt] \leq j \leq [nt']} f\big(T^j(x)\big) = K_{[nt']-[nt]+1}\big(T^{[nt]}(x), \ldots, T^{[nt']}(x)\big) \ .$$

We can now apply the exponential inequality (51) to conclude that for any real β we have for $t' - t > 1/n$

$$\mathbb{E}\left(e^{\beta\left(\sum_{[nt] \leq j \leq [nt']} f \circ T^j\right)}\right) \leq C_1 \ e^{C_2 \beta^2 L_f^2([nt']-[nt])/(n\sigma_f^2)}$$

$$\leq C_1 \ e^{2C_2 \beta^2 L_f^2 (t'-t)/\sigma_f^2} \ .$$

Since the same inequality holds with β changed into $-\beta$ we get

$$\mathbb{E}\left(\cosh\left(\beta \sum_{[nt] \leq j \leq [nt']} f \circ T^j\right)\right) \leq C_1 \ e^{2C_2 \beta^2 L_f^2 (t'-t)/\sigma_f^2} \ .$$

Since the Taylor series of the hyperbolic cosine has only positive terms we get for any integer k

$$\frac{\beta^{2k}}{2k!} \mathbb{E}\left(\left(\sum_{[nt] \leq j \leq [nt']} f \circ T^j\right)^{2k}\right) \leq C_1 \ e^{2C_2 \beta^2 L_f^2 (t'-t)/\sigma_f^2} \ ,$$

or in other words

$$\mathbb{E}\left(\left(\sum_{[nt] \leq j \leq [nt']} f \circ T^j\right)^{2k}\right) \leq C_1 \ 2k! \ \beta^{-2k} \ e^{2C_2 \beta^2 L_f^2 (t'-t)/\sigma_f^2} \ .$$

If we optimise the upper bound over β we get

$$\mathbb{E}\left(\left(\sum_{[nt] \leq j \leq [nt']} f \circ T^j\right)^{2k}\right) \leq (t'-t)^k M_k$$

where

$$M_k = C_1 \ 2k! \ \inf_{s>0} s^{-2k} \ e^{2C_2 s^2 L_f^2/\sigma_f^2} \ .$$

We can take for example $k = 2$, and this proves inequality (54) for $\alpha = 4$ and $\gamma = 1$. □

The same technique can be used to prove Theorem 4.5 without the last term in equation (44). One has to use the Skorokhod topology (see [14] or [76]). We leave the details to the interested reader. We will see later on some other important consequences of the exponential inequality (51).

4.7 Appendix. Proof of an exponential inequality.

Different methods have been used in the non independent case to prove concentration or exponential inequalities like information inequalities, coupling and extensions of the Perron-Frobenius operator. We now give some idea how it can be proved for piecewise expanding maps of the interval using coupling.

Theorem 4.6. *For a piecewise expanding map T of an interval with a mixing a.c.i.p.m. with non vanishing density h, the exponential inequality (51) holds.*

The first step in the proof is a doubling argument which is often used in the case of the variance. From Jensen inequality, we have

$$e^{-\mathbb{E}(K)} \leq \int e^{-K(y,T(y),\dots T^{n-1}(y))} h(y)\, dy \ ,$$

therefore

$$\int e^{K(x,T(x),\dots T^{n-1}(x))-\mathbb{E}(K)} h(x)\, dx$$

$$\leq \int\int e^{K(x,T(x),\dots T^{n-1}(x))-K(y,T(y),\dots T^{n-1}(y))} h(x)\, h(y)\, dx\, dy \ .$$

We now perform a change of variables and use $T^{n-1}(x)$ and $T^{n-1}(y)$ as the integration variables.

We define the measure

$$dp\big(y\big|x\big) = \frac{h(y)}{h(x)}\delta(x-T(y)) = \frac{1}{h(x)} \sum_{z\,,\,T(z)=x} \frac{h(z)}{\big|T'(z)\big|}\delta(y-z) \ .$$

We now have

$$\int\int e^{K\left(x,T(x),\dots T^{n-1}(x)\right)-K\left(y,T(y),\dots T^{n-1}(y)\right)} h(x)\, h(y)\, dx\, dy \ =$$

$$\int\int e^{K\left(x_1,\dots,x_n\right)-K\left(y_1,\dots y_n\right)} \prod_{j=1}^{n-1} dp\big(x_j\big|x_{j+1}\big)$$

$$\times \prod_{j=1}^{n-1} dp\big(y_j\big|y_{j+1}\big)\ h\big(x_n\big)\ h\big(y_n\big)\ dx_n\ dy_n \ . \tag{55}$$

We will now make use of the following general inequality due to Hoeffding (see for example [60]). Let ν be a probability measure on a space Ω and g a bounded function. Then

$$\int_\Omega e^{g-\mathbb{E}(g)} d\nu \leq e^{\mathrm{Osc}(g)^2/8} \tag{56}$$

where the oscillation of g is defined by

$$\mathrm{Osc}(g) = \max_{x\in\Omega} g(x) - \min_{y\in\Omega} g(y) \ .$$

Exercise 52. *Prove the Hoeffding inequality with the exponent* $1/2$ *instead of* $1/8$. *Hint: prove and use that the function* $l(t) = \log \int \exp\big(t(g-\mathbb{E}(g)\big) d\nu$ *is convex, and estimate from above its second derivative. The constant* $1/8$ *is optimal [60]) .*

We will now use recursively Hoeffding's inequality in the right hand side of equation (55). We first use for ν the measure

$$d\nu\big(x_1, y_1\big) = dp\big(x_1\big|x_2\big)\ dp\big(y_1\big|y_2\big)\ .$$

We get immediately for fixed $x_2, \ldots, x_n$ and $y_2, \ldots, y_n$ using Hoeffding's inequality

$$\int e^{K\big(x_1,\ldots,x_n\big)-K\big(y_1,\ldots y_n\big)} d\nu\big(x_1, y_1\big)$$

$$\leq e^{K_1\big(x_2,\ldots,x_n\big)-K_1\big(y_2,\ldots,y_n\big)} e^{Osc_1(K)^2/4}$$

where

$$K_1\big(y_2, \ldots, y_n\big) = \int K\big(x_1, \ldots, x_n\big) p\big(dx_1\big|x_2\big)$$

and

$$\mathrm{Osc}_1(K) = \sup_{x_2,\ldots,x_n} \left(\max_{x_1\in\Omega} K\big(x_1, \ldots, x_n\big) - \min_{x_1\in\Omega} K\big(x_1, \ldots, x_n\big) \right)\ .$$

We leave to the reader to apply iteratively this argument and show that

$$\int e^{K\big(x,T(x),\ldots T^{n-1}(x)\big)-\mathbb{E}(K)} h(x)\ dx \leq e^{\sum_{j=1}^n \mathrm{Osc}_j(K_j)^2/4}\ ,$$

where $K_1 = K$, for $2 \leq l \leq n-1$ the function K_l is defined by

$$K_l\big(x_l, \ldots, x_n\big) = \int K\big(x_1, \ldots, x_n\big) \prod_{r=1}^{l-1} dp\big(x_r\big|x_{r+1}\big)\ ,$$

and Osc_j denotes the oscillation with respect to x_j, namely

$$\mathrm{Osc}_j(K_j) = \sup_{x_{j+1},\ldots,x_n} \left(\max_{x_j\in\Omega} K_j\big(x_j, \ldots, x_n\big) - \min_{x_j\in\Omega} K_j\big(x_j, \ldots, x_n\big) \right)\ .$$

In the last step, one uses Hoeffding's inequality with respect to the measure $h\big(x_n\big)\ h\big(y_n\big)\ dx_n\ dy_n$. This sequence of estimates is similar to the proof of Azuma's inequality [60].

We now estimate each term in the exponential using a coupling result of Barbour, Gerrard and Reinert (see the last section). We have for any u, v and $x_{l+1}, \ldots, x_n$ in the interval

$$K_l\big(u, x_{l+1}, \ldots, x_n\big) - K_l\big(v, x_{l+1}, \ldots, x_n\big)$$

$$= \int d\mu_{u,v}\Big(\big(x_1, \ldots, x_{l-1}\big), \big(y_1, \ldots, y_{l-1}\big)\Big)$$

$$\Big[K\big(x_1, \ldots, x_{l-1}, u, x_{l+1}, \ldots, x_n\big) - K\big(y_1, \ldots, y_{l-1}, v, x_{l+1}, \ldots, x_n\big)\Big]$$

$$= \int d\mu_{u,v}\Big(\big(x_1, \ldots, x_{l-1}\big), \big(y_1, \ldots, y_{l-1}\big)\Big)$$

$$\sum_{r=0}^{l-2} \Big[\ K\big(x_1, \ldots, x_r, x_{r+1}, y_{r+2}, \ldots, y_{l-1}, u, x_{l+1}, \ldots, x_n\big)$$
$$-\quad K\big(x_1, \ldots, x_r, y_{r+1}, y_{r+2}, \ldots, y_{l-1}, v, x_{l+1}, \ldots, x_n\big)\ \Big]$$

where in this last expression the undies out of range should be left out. This immediately implies

$$\big|K_l\big(u, x_{l+1}, \dots, x_n\big) - K_l\big(v, x_{l+1}, \dots, x_n\big)\big|$$

$$\leq \sum_{r=0}^{l-2} L_{r+1}(K) \int\int \big|x_{r+1} - y_{r+1}\big| d\mu_{u,v}\Big(\big(x_1, \dots, x_{l-1}\big), \big(y_1, \dots, y_{l-1}\big)\Big) \ .$$

Using the coupling (see the last section) we get that there are two positive constants C' and $0 < \rho < 1$ such that for any $0 \leq r \leq l-2$, and for any u and v in the interval we have

$$\int\int \big|x_{r+1} - y_{r+1}\big| d\mu_{u,v}\Big(\big(x_1, \dots, x_{l-1}\big), \big(y_1, \dots, y_{l-1}\big)\Big) \leq C' \rho^{l-r} \ ,$$

and this implies for any $1 \leq j \leq n$

$$\mathrm{Osc}_j(K_j) \leq C' \sum_{q=1}^{j} L_q \rho^{j-q} \ .$$

Therefore by Schwarz inequality

$$\sum_{j=1}^{n} \mathrm{Osc}_j(K_j)^2 \leq C'^2 \sum_{j=1}^{n} \left(\sum_{q=1}^{j} L_q \rho^{j-q} \right)^2$$

$$\leq C'^2 \sum_{j=1}^{n} \sum_{q=1}^{j} L_q^2 \rho^{j-q} \sum_{p=1}^{j} \rho^{j-p} \leq C'^2 (1-\rho)^{-2} \sum_{q=1}^{n} L_q^2 \ .$$

4.8 Appendix. Ivanov's Theorem.

We now formulate Ivanov's Theorem about the convergence in the ergodic Theorem. To formulate the result we first define the sequence of down-crossings for a non-negative sequence $(u_n)_{n \in \mathbb{N}}$.

Let a and b be two numbers such that $0 < a < b$.

For an integer $k \geq 0$ such that $u_k \leq a$, we define the first down crossing from b to a after k as the smallest integer $n_d > k$ (if it exists) such that

i) $u_{n_d} \leq a$,

ii) There exists at least one integer $k < j < n_d$ such that $u_j \geq b$.

Let (n_l) be the sequence of successive down-crossings from a to b (this sequence may be finite and even empty).

We denote by $N\big(a, b, p, (u_n)\big)$ the number of successive down-crossings from b to a occurring before time p for the sequence (u_n), namely

$$N(a, b, p, (u_n)) = \sup\big\{ l \,\big|\, n_l \leq p \big\} \ .$$

Theorem 4.7. *Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system. Let f be a non negative observable with $\mu(f) > 0$. Let a and b be two positive real numbers such that $0 < a < \mu(f) < b$, then for any integer r*

$$\mu\left(\left\{ x \,\middle|\, N\left(a, b, \infty, \Big(f\big(T^n(x)\big)\Big)\right) > r \right\} \right) \leq \left(\frac{a}{b}\right)^r \ .$$

The interesting fact about this Theorem is that the bound on the right hand side is explicit and independent of the map T and of the observable f. We refer to the literature for the proof. We refer to [89], [40], [93] for proofs and extensions.

5 s.d.i.c., Lyapunov exponents, entropy.

We have seen that dynamical systems may have very different asymptotic time behaviours. For example the trajectories may end on a (stable) periodic orbit (attractor composed of a finite number of points). This is a very regular (non chaotic) behaviour. We have also seen many examples of chaotic behaviours (Hénon map, Lorenz flow etc.). What is the origin of chaos? Complex dynamical behaviours are often called chaotic, but one would like to have a precise definition of this notion.

It is often observed in these systems that the trajectories depend in a

very sensitive way on the initial conditions.

If one starts with two very nearby generic initial conditions, it is often observed in chaotic systems that as time evolves, the distance between the orbits grow exponentially fast. Note that this does not contradict the continuous (Lipschitz) dependence on initial conditions in (regular) flows. The Lipschitz constant in the estimate grows exponentially fast with the time interval considered.

On the contrary, the reader can compute what happens if one considers a simple harmonic oscillator which is an integrable (non chaotic) system (Hamiltonian $H = p^2/2 + \omega^2 q^2/2$). The evolution equations in the phase space $\mathbb{R}^2$ are

$$\frac{dq}{dt} = p \ , \qquad \frac{dp}{dt} = -\omega^2 \, p \ .$$

It is easy to check that for two nearby initial conditions with the same energy, the distance does not grow at all (the energy surfaces are ellipses). For general nearby points (slightly different energy) the distance grows linearly in time.

The phenomenon of rapid growth of little errors in the initial condition is called sensitive dependence on initial conditions (abbreviated s.d.i.c.).

It is at the origin of chaotic behaviours when coupled to some kind of recurrence, mainly the fact that the phase space is compact. The distance between two trajectories grows for example exponentially fast with time, but if the phase space is compact this cannot continue forever. At some time some kind of (non-linear) folding has to occur, re-injecting the trajectories eventually near to one another.

This phenomenon of sensitive dependence on initial conditions is illustrated in the next figure for the Lorenz model.

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 10\,(y-x) \\ 28\,x - y - xz \\ xy - \frac{8\,z}{3} \end{pmatrix} \ .$$

In the next picture, one sees a plot of the x component as a function of time for two 'very' nearby initial conditions (the two trajectories are in red and black). After having stayed together for a while, they suddenly separate macroscopically.

The time evolution of these two trajectories can be looked at in the phase space. The current points are marked by two little balls of different colors. At the beginning, these two little balls overlap and we see only one of them. After a while, they separate and live their own life more and more independently as time goes on.

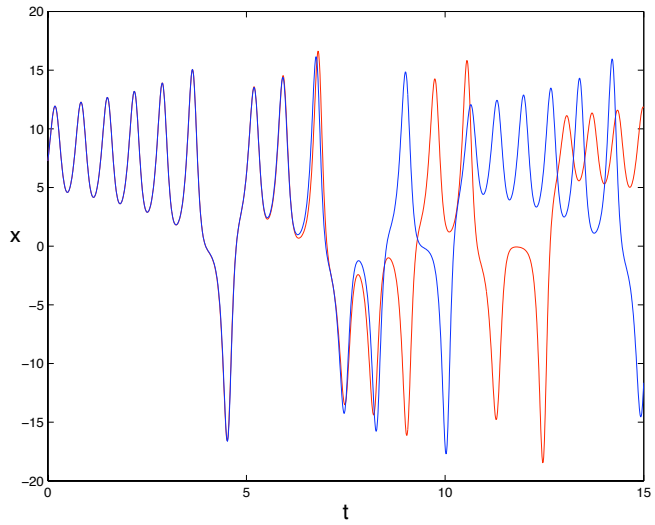


Figure 53: Sensitive dependence to the initial condition in the Lorenz model.

To get some more quantitative idea of the s.d.i.c., one can plot the distance δ between the two trajectories as a function of time. A lin-log plot is more adequate to test the exponential growth.

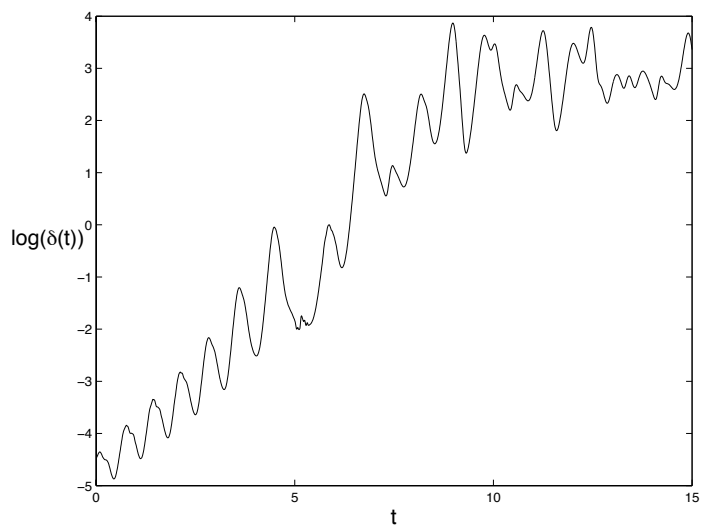


Figure 54: Logarithm of the distance between the two orbits as a function of time.

5.1 Lyapunov exponents.

Since it is so important and so common in chaotic systems, one would like to have a quantitative measurement of the sensitive dependence on initial conditions (s.d.i.c).

We will see here the interplay between ergodic theory, and the differentiable structure on the phase space, namely between geometry and stochastic processes.

An important application of the sub-additive ergodic theorem seen before is the proof of the Oseledec Theorem about the existence of Lyapunov exponents which quantify the notion of s.d.i.c. We first explain the ideas in an informal way for discrete time evolutions.

Assume that the time evolution map T is differentiable (and the phase space Ω is a differentiable manifold, we will assume $\Omega = \mathbb{R}^d$ for simplicity). If we start with two nearby initial conditions x and $y = x + \vec{h}$ with $\vec{h}$ small, we would like to estimate the size of

$$T^n\big(y\big) - T^n\big(x\big) = T^n\big(x + \vec{h}\big) - T^n\big(x\big) \ ,$$

namely how the initially small error $\vec{h}$ evolves in time. Recall that $T^n = T \circ T \cdots \circ T$ (n times). As long as the error has not grown too much, we can expect to have a reasonable answer from the first order Taylor formula, namely

$$T^n\big(x + \vec{h}\big) - T^n\big(x\big) = DT^n_x \vec{h} + \mathscr{O}(\vec{h}^2) \ ,$$

Where DT^n_x denotes the differential of the map T^n at the point x.

We recall that from the chain rule we have

$$DT^n_x = DT_{T^{n-1}(x)} \cdots DT_x$$

where in general the matrices in this product do not commute. The fact that we have a product of n terms suggests that we look (test) for an exponential growth, namely that we take the logarithm, divide by n and take the limit $n \to \infty$. Since the matrices do not commute, the formulation of the result (Oseledec theorem) for the general case will be more sophisticated than for products of numbers. We will start with some simple warming up examples.

First (easy) case: dimension one, the phase space is an interval ([-1,1]), the dynamics is given by a regular map T. One gets by Taylor's formula

$$T^n(x + h) = T^n(x) + T^{n\prime}(x)h + \mathcal{O}(h^2) \ .$$

By the chain rule ($T^n = T \circ T \circ \cdots \circ T$, n times), we have

$$T^{n\prime}(x) = \prod_{j=0}^{n-1} T'\big(T^j(x)\big) \ .$$

As explained above, this naturally suggests an exponential growth in n, and to look for the exponential growth rate per iteration step (unit of time), namely

$$\frac{1}{n} \log \big|T^{n\prime}(x)\big| = \frac{1}{n} \sum_{j=0}^{n-1} \log \big|T'\big(T^j(x)\big)\big| \ .$$

We see appearing a temporal average and we can apply Birkhoff's ergodic Theorem.

Let μ be an ergodic invariant measure such that the function $\log\left|T'\right|$ is integrable. Then except on a set of μ measure zero, we have convergence of the temporal average, and moreover

$$\lim_{n\to\infty}\frac{1}{n}\log\left|T^{n\prime}(x)\right| = \int \log\left|T'(\cdot)\right| d\mu\ .$$

This number is called the Lyapunov exponent of the measure μ for the transformation T.

Here one should stress again the importance of the initial condition. There are many initial conditions for which the limit does not exist. For many other initial conditions, the limit exists, but takes another value. For example take an initial conditions typical of a different ergodic measure.

Exercise 53. *On the phase space* $\Omega = [0,1]$*, consider the map*

$$T(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/3 \\ 3(1-x)/2 & \text{if } 1/3 \le x \le 1. \end{cases} \tag{57}$$

Show that the Lebesgue measure is T invariant and ergodic, and that its Lya-

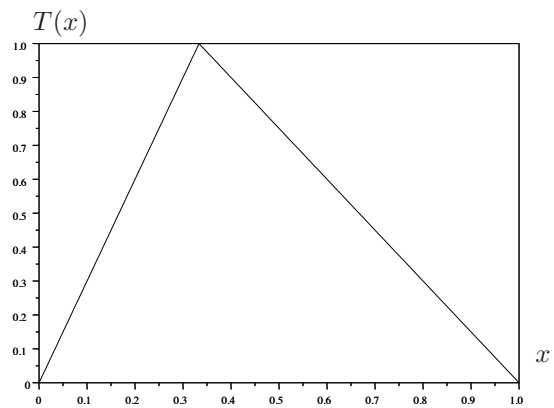


Figure 55: The map $x \mapsto T(x)$

punov exponent is equal to $\frac{1}{3}\log 3 + \frac{2}{3}\log(3/2)$.

Show that the transformation T has a fixed point $x = 3/5$, that the Dirac measure in this point is also invariant and ergodic, and its Lyapunov exponent is equal to $\log(3/2)$.

Show that for any ergodic invariant measure the Lyapunov exponent belongs to the interval $[\log(3/2), \log 3]$.

Show that there is an uncountable set of invariant ergodic measures, all with different Lyapunov exponents (taking all the values between $\log 3/2$ *and* $\log 3$*).*

Hint: use a conjugacy (coding) to the unilateral shift on two symbols, and then consider the product measures.

We now consider the next level of difficulty: dimension of the phase space larger than one.

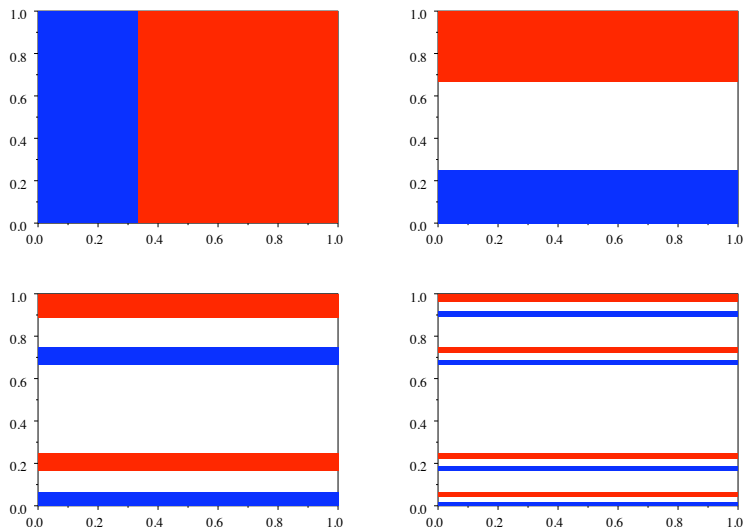
As a simple example we will use the dissipative baker's map T, whose attractor has been studied in a previous exercise.

$$T(x,y) = \begin{cases} (3x, y/4) & \text{if } 0 \le x < 1/3 \\ (3(x-1)/2, (2+y)/3) & \text{if } 1/3 \le x \le 1 \end{cases}$$

The differential of the dissipative baker's map (except at the discontinuities) is given by

$$DT_{(x,y)} = \begin{cases} \begin{pmatrix} 3 & 0 \\ 0 & 1/4 \end{pmatrix} & \text{if } 0 \le x < 1/3 \\ \begin{pmatrix} 3/2 & 0 \\ 0 & -1/3 \end{pmatrix} & \text{if } 1/3 \le x \le 1. \end{cases}$$

To estimate $DT^n_{(x,y)}$ we now have to perform a product of matrices. Recall the picture of the iteration of the dissipative baker's map: If we start from the initial



point (x,y), with an initial error $\vec{h}$, we have after n iteration steps (using the chain rule) an error given (up to order $\mathscr{O}(\vec{h}^2)$) by

$$DT^n_{(x,y)}\vec{h} = DT_{T^{n-1}(x,y)} DT_{T^{n-2}(x,y)} \cdots DT_{(x,y)}\vec{h}\ .$$

In general matrices do not commute

hence one should be careful with the order in the product.

However here we are lucky, they do commute (and they are even diagonal, of course I use this example on purpose). Therefore, to obtain the product matrix, it is enough to take the product of the diagonal elements:

$$DT_{T^{n-1}(x,y)} DT_{T^{n-2}(x,y)} \cdots DT_{(x,y)}$$
$$= \begin{pmatrix} \prod_{j=0}^{n-1} u\big(T^j(x,y)\big) & 0 \\ 0 & \prod_{j=0}^{n-1} v\big(T^j(x,y)\big) \end{pmatrix}$$

where

$$u(x,y) = 3\chi_{[0,1/3]}(x) + (3/2)\chi_{[1/3,1]}(x)$$

and

$$v(x,y) = (1/4)\chi_{[0,1/3]}(x) - (1/3)\chi_{[1/3,1]}(x) \ .$$

We can now take the log of the absolute value of each diagonal entry of the product and apply Birkhoff's ergodic theorem as in the one dimensional case.

Since the functions u and v do not depend on y, the integral with respect to the Physical measure reduces to the integration with respect to the one dimensional Lebesgue measure. Therefore, for Lebesgue almost any (x,y) (two dimensional Lebesgue measure, recall the definition of Physical measure)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| u\big(T^j(x,y)\big| = (1/3)\log 3 + (2/3)\log(3/2) \right.$$

$$= \log\left((27/4)^{1/3}\right) \ .$$

Similarly

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| v\big(T^j(x,y)\big| \right.$$

$$= -(1/3)\log 4 - (2/3)\log(3) = \log\left((36)^{-1/3}\right) \ .$$

We conclude that

$$DT_{T^{n-1}(x,y)} DT_{T^{n-2}(x,y)} \cdots DT_{(x,y)} \approx \begin{pmatrix} (27/4)^{n/3} & 0 \\ 0 & (36)^{-n/3} \end{pmatrix} \ ,$$

and we now have to interpret these results.

This interpretation will be the new fact with respect to the previous case in dimension one. If we take a vector $\vec{h} = \big(h_1, h_2\big)$, we have

$$DT^n_{(x,y)} \vec{h} = \begin{pmatrix} (27/4)^{n/3} h_1 \\ (36)^{-n/3} h_2 \end{pmatrix} \ ,$$

If $h_1 \neq 0$, we see that the first component of $DT^n_{(x,y)}\vec{h}$ grows exponentially fast with exponential growth rate $\log\left((27/4)^{1/3}\right)$, and this component dominates the other one which decreases exponentially fast at the rate $\log\left(36^{-1/3}\right)$ (< 0). In other words, almost any error grows exponentially fast with rate $\log\left((27/4)^{1/3}\right)$, this is the maximal Lyapunov exponent.

But there is another Lyapunov exponent $\log\left(36^{-1/3}\right)$ (< 0) corresponding to special initial errors satisfying $h_1 = 0$ (and $h_2 \neq 0$). These errors do not grow but decay exponentially fast.

This is similar to the diagonalisation of matrices but the interpretation is slightly more involved. Namely, we have to distinguish two subspaces. The first one is the entire space $E_0 = \mathbb{R}^2$. The second one is the subspace of codimension one $E_1 = \big\{(0,h_2)\big\} \subset E_0$.

If $\vec{h} \in E_0 \backslash E_1$, the initial error grows exponentially fast with rate $\log\left(\big(27/4\big)^{1/3}\right)$. If $\vec{h} \in E_1 \backslash \{\vec{0}\}$, the initial error decreases exponentially fast with rate $\log\left(36^{-1/3}\right)$. Note the important fact that $E_0 \backslash E_1$ is not a vector subspace!

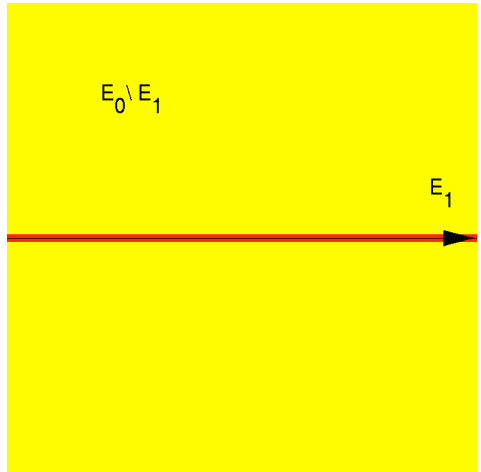


Figure 56: The set $E_0 \backslash E_1$ in yellow and the vector subspace E_1 in red.

A "typical", "generic", "Lebesgue almost sure" vector $\vec{h}$ has $h_1 \neq 0$, namely it belongs to $h \in E_0 \backslash E_1$.

In hyperbolic systems, one has a uniform control on s.d.i.c.

Definition 16. *An invariant set Λ for a diffeomorphims T on a phase space Ω is called hyperbolic if the tangent space $E(x)$ at any point $x \in \Lambda$ has a stable-unstable splitting*

$$E(x) = E^u(x) \oplus E^s(x)$$

where $E^u(x)$ and $E^s(x)$ have the following properties.

i) The fields of subspaces E^u and E^s are invariant, namely for any $x \in \Lambda$

$$DT_x E^s(x) = E^s(T(x)) \quad \text{and} \quad DT_x^{-1} E^u(x) = E^u(T^{-1}(x)) \ .$$

ii) The splitting is uniformly hyperbolic. Namely, there are two positive constants C and $\rho < 1$ such that for any integer n and any $x \in \Lambda$ we have

$$\left\| DT_x^n \big| E^s(x) \right\| \leq C\rho^n \quad \text{and} \quad \left\| DT_x^{-n} \big| E^u(x) \right\| \leq C\rho^n \ .$$

This implies of course that any ergodic invariant measure supported by Λ will have at least one positive Lyapunov exponents.

Definition 17. *An attractor Λ of a dynamical system is said to be Axiom A if it is hyperbolic.*

The are many important results known in this situation. See for example [21], [138], [94], [39].

We now come to the study of Lyapunov exponents in the general case of dynamical systems. This general case combines the two preceding ideas (product of matrices and ergodic theorem) together with two new facts: the matrices in the product do not commute and the subspaces E_0, E_1, etc. depend on the initial condition, they vary from point to point.

Let us first consider the behaviour of the norm of the product of matrices and define

$$f_n(x) = \log \left\| DT_x^n \right\| .$$

It is easy to verify that this sequence of functions satisfies the sub-additive inequality, namely (use $DT_x^{n+m} = DT_{T^n(x)}^m DT_x^n$)

$$f_{n+m}(x) \leq f_n(x) + f_m\big(T^n(x)\big) .$$

Assume now that we have chosen an ergodic invariant measure μ for T, then we can apply the sub-additive ergodic Theorem 3.2 (provided we can check the second assumption of this theorem: f_n is integrable) and we conclude that

$$\frac{1}{n} \log \left\| DT_x^n \right\|$$

converges μ almost surely.

If this quantity is positive, we know that some initially small errors are amplified exponentially fast (μ almost surely). If this quantity is negative or zero, we know that initial errors cannot grow very rapidly. Note that it may still grow for example linearly in time.

More generally, if we have a fixed vector $\vec{h}$, recall that (in a Hilbert space)

$$\left\| DT_x^n \vec{h} \right\|^2 = \left\langle \vec{h} \,\middle|\, \left(DT_x^n\right)^t DT_x^n \vec{h} \right\rangle$$

where $\langle \quad | \quad \rangle$ denotes the scalar product in $\mathbb{R}^d$.

This suggests to study the asymptotic exponential growth of the matrix $\left(DT_x^n\right)^t DT_x^n$. The answer is provided by the following theorem due initially to Oseledec.

Theorem 5.1 (Oseledec)**.** *Let μ be an ergodic invariant measure for a diffeomorphism T of a compact manifold Ω of dimension d.*

Then for μ almost every initial condition x, the sequence of symmetric non-negative matrices

$$\left(\left(DT_x^n\right)^t DT_x^n\right)^{1/2n}$$

converges to a symmetric non-negative matrix Λ (independent of x).

Denote by $\lambda_0 > \lambda_1 > \ldots > \lambda_k$ the strictly decreasing sequence of the logarithms of the eigenvalues of the matrix Λ (some of them may have non trivial multiplicity, hence $k \leq d$).

These numbers are called the Lyapunov exponents of the map T for the ergodic invariant measure μ.

Except for a subset of the phase space of μ measure zero, for every point x there is a decreasing sequence of subspaces (whose dimension does not depend on x)

$$\Omega = E_0(x) \supsetneq E_1(x) \supsetneq \cdots \supsetneq E_k(x) \supsetneq E_{k+1}(x) = \{\vec{0}\}$$

satisfying (μ almost surely) the following properties

$$DT_x E_j(x) = E_j\big(T(x)\big)$$

and for any $\vec{h} \in E_j(x) \backslash E_{j+1}(x)$ for some $0 \leq j \leq k$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| DT_x^n \vec{h} \right\| = \lambda_j .$$

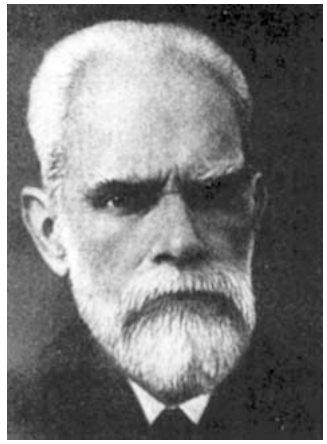


Figure 57: Aleksandr Lyapunov (1857-1918).

We now make some remarks.

- Note that the theorem says that for $\vec{h} \in E_j(x) \backslash E_{j+1}(x)$

$$\left\| DT_x^n \vec{h} \right\| \sim e^{n\lambda_j} \ .$$

  This only holds in a logarithmic sense (and for large n). There may be large (sub-exponential) prefactors which can depend on $\vec{h}$ and x.

- Positive Lyapunov exponents are obviously responsible for the Sensitive Dependence on Initial Conditions. Their corresponding "eigen" directions are tangent to the attractor.

- Transversally to the attractor one gets contracting directions, namely negative Lyapunov exponents.

Oseledec's theorem applies to product of random matrices with stationary distributions (i.i.d. or not) provided ergodicity (and the integrability condition) is satisfied.

Exercise 54. *Prove that if we conjugate the map T to a map T' by a diffeomorphism Φ ($T' = \Phi^{-1} \circ T \circ \Phi$), and consider an image measure, then the Lyapunov exponents are the same.*

If the map depends on a parameter, the Lyapunov exponents depend also in general on the parameter (one has also to decide on an ergodic invariant measure for each parameter). This is illustrated in the next figure where the Lyapunov exponent of the quadratic family of $x \mapsto 1 - \mu x^2$ is drawn as a function of the parameter for the Physical measure.

Exercise 55. *Assume $\Omega = \mathbb{R}^d$, assume Oseledec's theorem holds and* $\mathrm{Codim} E_1 = 1$. *Prove that μ almost surely*

$$\lim_{n \to \infty} \frac{1}{n} \log \left\| DT_x^n \right\| = \lambda_0 \ .$$

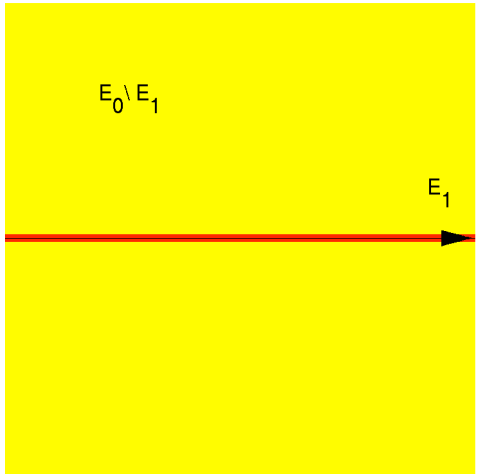


Figure 58: The set $E_0 \backslash E_1$ in yellow and the vector subspace E_1 in red. Keep in mind this picture.

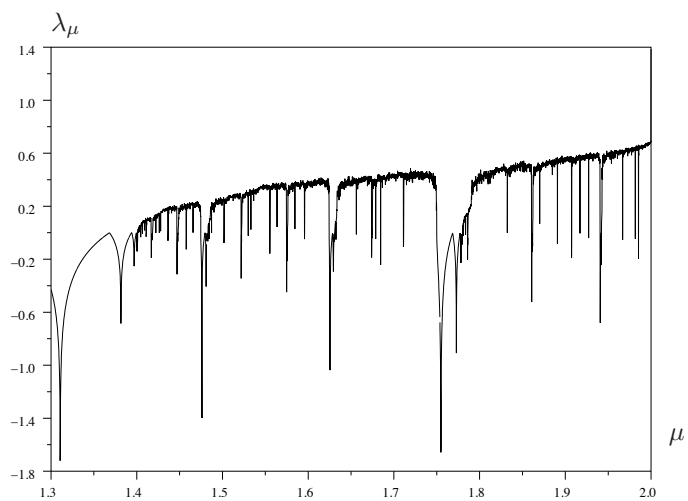


Figure 59: Lyapunov exponent λ_μ as a function of the parameter for the quadratic family $x \mapsto 1 - \mu x^2$ (Physical measure).

Hint: Consider a fixed orthogonal basis $\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_d$ of $\mathbb{R}^d$ with $\vec{e}_1, \ldots, \vec{e}_d \in E_1(x)$. Let $\vec{v}_n$ be such that $\|\vec{v}_n\| = 1$ and $\left\|DT_x^n\right\| = \left\|DT_x^n \vec{v}_n\right\|$. Use the decomposition of $\vec{v}_n$ on the basis. Modify this argument to obtain the same result in the general case (Ω a compact finite dimensional manifold and $\mathrm{Codim} E_1 \geq 1$*).*

We refer to [8], [94], [141], [68] and [9] for proofs and references.

5.2 Partitions, coding and entropy.

Often one does not have access to the points of phase space but only to some fuzzy approximation. For example if one uses a real apparatus which always has a finite precision. There are several ways to formalise this idea.

- If the phase space Ω is a metric space with metric d, for a given precision $\epsilon > 0$, two points at distance less than ϵ are not distinguishable.
- One gives a (measurable) partition $\mathcal{P}$ of the phase space, namely a set of subsets $\mathcal{P} = \{A_1, \cdots, A_k\}$ (k finite or not), satisfying $A_j \cap A_l = \emptyset$ for $j \neq l$ and
$$\Omega = \bigcup_{j=1}^{k} A_j \ .$$
Two points in the same atom of the partition $\mathcal{P}$ are considered indistinguishable. If there is a given measure μ on the phase space it is often useful to use partitions modulo sets of μ measure zero.

The particularly important and very useful notion of partition leads naturally to a coding which links concrete and abstract dynamical systems. This is a map Φ from Ω to $\{1, \cdots, k\}^{\mathbb{N}}$ given by

$$\Phi_n(x) = \ell \qquad \text{if} \qquad T^n(x) \in A_\ell \ .$$

If the map T is invertible, one can also use a bilateral coding. If $\mathcal{S}$ denotes the shift on sequences, it is easy to verify that in all cases $\Phi \circ T = \mathcal{S} \circ \Phi$. In general $\Phi(\Omega)$ is a complicated subset of $\{1, \cdots, k\}^{\mathbb{N}}$, *i.e.* it is difficult to say which codes are admissible.

There are however some examples of very nice (Markovian) codings like for Axiom A attractors (see [21] [100] and [140]). We have seen already the coding of the map of the interval $T(x) = 2x \mod 1$ by the dyadic decomposition. The partition of the interval is made of two atoms $[0,1] = [0,1/2] \cup]1/2,1]$. We therefore have an alphabet of two atoms $\{0,1\}$. The coding is well defined and unique except for a countable number of points which are the preimages of the point $1/2$. Indeed, with our choice of partition and of the value of the map at $1/2$ ($T(1/2) = 0$), the points $x = 0$ and $x = 1/2$ have the same code $0000000\ldots$.

This is not a problem in general. In a measure theoretic setting one has to check that the boundaries of the atoms of the partition have measure zero. In a topological setting one has to show that these boundaries are also negligible in some sense (for example here we get a countable set inside a continuum). Neglecting this effect, in the example of the map $T(x) = 2x \mod 1$ we get a coded dynamical system which is the full shift on two symbols.

The general idea is that two orbits with the same code will be considered indistinguishable (from the point of view of the given partition). However two orbits may stay together for a long time and then separate macroscopically. This is what happens if there is sensitive dependence to the initial condition. Then the codes of the two orbits will share a large block but will ultimately differ.

In chaotic systems, due to the s.d.i.c. one expects to have a large number of (initial conditions leading to) different orbits of a fixed length n. For example

in the case of the full shift on two symbols the number of different trajectories of length n is obviously 2^n.

Let $\mathcal{P}$ and $\mathcal{P}'$ be two partitions, the partition $\mathcal{P} \vee \mathcal{P}'$ is defined by

$$\mathcal{P} \vee \mathcal{P}' = \big\{ A \cap B \ , \ A \in \mathcal{P} \ , \ B \in \mathcal{P}' \big\} \ .$$

If $\mathcal{P}$ is a partition,

$$T^{-1}\mathcal{P} = \big\{ T^{-1}(A) \big\}$$

is also a partition. Recall that (even in the non invertible case) $T^{-1}(A) = \big\{ x \ , \ T(x) \in A \big\}$. A partition $\mathcal{P}$ is said to be generating if (modulo sets of measure zero)

$$\bigvee_{n=0}^{\infty} T^{-n}\mathcal{P} = \epsilon$$

with ϵ the partition into points. In this case the coding is injective (modulo sets of measure zero).

Exercise 56. *For the map of the interval* $T(x) = 3x \pmod 1$*, show that*

$$\mathcal{P} = \big\{ [0, 1/3], [1/3, 1] \big\}$$

is a generating partition (modulo sets of Lebesgue measure zero).

As mentioned before, we can suspect that s.d.i.c. results in a large number of different orbits (we have to say precisely what we mean by different). It is natural to ask if there is a systematic way to count the number of different orbits of length n. This leads to the definition of entropy.

In the study of dynamical systems one mostly uses two entropies, the topological entropy and the so called Kolmogorov-Sinai entropy. Both entropies measure how many different orbits one can detect through a fuzzy observation but from a different point of view for the fuzziness.

The topological entropy is defined independently of a measure. We will only consider here the case of a metric phase space. The topological entropy counts all the orbits modulo metric (or more generally topological) fuzziness, we first make this notion precise. We assume a precision of measurement $\epsilon > 0$ is given.

Definition 18. *We say that two orbits of initial condition x and y respectively are ϵ different before time n (with respect to the metric d) if*

$$\sup_{0 \le k \le n} d\big(T^k(x), T^k(y)\big) > \epsilon.$$

Intuitively the apparatus with precision ϵ has detected their difference. Let $N_n(\epsilon)$ be the maximum number of pairwise ϵ different orbits up to time n. In other words this is the maximum number of pairwise different films the dynamics can generate up to time n if two images differing at most by ϵ are considered identical. The topological entropy is a way of coding the asymptotic growth in time of this number.

Definition 19. *The topological entropy is the number*

$$h_{\text{top}} = \lim_{\epsilon \searrow 0} \lim_{n \to \infty} \frac{1}{n} \log N_n(\epsilon) \ .$$

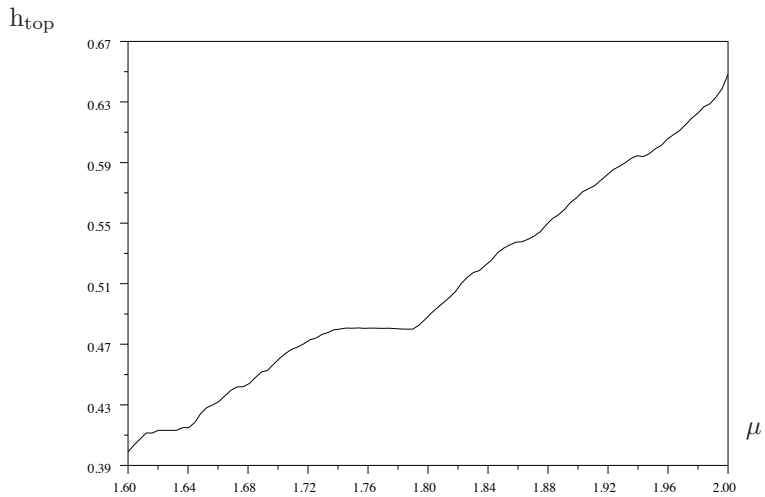


Figure 60: Topological entropy as a function of the parameter for the quadratic family $x \mapsto 1 - \mu x^2$.

One can prove that the limits in the above formula always exist. We show in the next figure the topological entropy as a function of the parameter for the one parameter family T_μ of quadratic maps $x \mapsto 1 - \mu x^2$.

Exercise 57. *Recall that for a (unilateral) shift, we defined the quantity*

$$\delta\big(\underline{x}, \underline{y}\big) = \inf\big\{k \geq 1 \,\big|\, x_k \neq y_k\big\} \ .$$

In other words, $\delta\big(\underline{x}, \underline{y}\big)$ *is the first position where the two sequences differ. For any number* $0 < \rho < 1$*, define the quantify*

$$d_\rho\big(\underline{x}, \underline{y}\big) = \rho^{\delta\left(\underline{x}, \underline{y}\right)} \ .$$

Show that d_ρ *is a metric (and moreover an ultra-metric). Show that* Ω *is a compact complete space, and that the shift is continuous. Show that for any* $1 > \rho > 0$*,* $h_{top} = \log 2$*.*

Let us see what happens for a simple non chaotic system: an irrational rotation on the circle by an angle φ. The map is given by

$$\mathscr{R}_\varphi \theta = \theta + \varphi \pmod 1 \ .$$

Now we count the maximal number of trajectories at precision ϵ. If two initial conditions θ_1 and θ_2 are at a distance less than ϵ on the circle, their orbits will have this property at all time, i.e. they travel together. Therefore $N(n, \epsilon) = 1/\epsilon$ (integer part). We conclude that $\lim_{n\to\infty} n^{-1} \log N(n, \epsilon) = 0$ which implies $\mathrm{h}_{\mathrm{top}}(\mathscr{R}_\varphi) = 0$. It is easy to check that the topological entropy does not change by using equivalent metrics.

Exercise 58. *We have seen that the map* $2x \pmod 1$ *of the unit interval is conjugated to the unilateral shift on two symbols by the dyadic decomposition (except for a countable set of points).*

Show that this conjugacy is bi-Lipschitz when using on the shift the metric $d_{1/2}$.

Prove that for the map $2x \pmod 1$ *of the unit interval* $h_{top} = \log 2$.

If the phase space Ω is a (compact) differentiable manifold, and the transformation T is differentiable, an easy upper bounds can be derived for the topological entropy.

Exercise 59. *Show that*

$$h_{top}(T) \leq \max \left\{ 0 \, , \, \sup_{x \in \Omega} \log \|DT_x\| \right\} \, .$$

A positive topological entropy characterises a large number of qualitatively different trajectories. It is an indication of chaos.

But this chaos may occur in a very small, irrelevant, part of the phase space.

For example one can have a stable periodic orbit whose basin is almost all the phase space and a fractal repeller of small dimension, zero volume, supporting all the positive topological entropy. This is what happens for example for (unimodal) maps of the interval with a stable periodic orbit of period 3, and has led to the famous statement that "period three implies chaos" (see [38]). The map $T(x) = 1 - 1.7549\, x^2$ of the interval $[-1, 1]$ has the origin at a periodic orbit of period three. This periodic orbit is stable, hence its basin does not produce any topological entropy (show that if $N_n(\epsilon)$ denotes the number of different trajectories of length n at precision ϵ with initial condition in the basin of attraction of a stable periodic orbit, then $n^{-1} \log N_n(\epsilon)$ tends to zero when n tends to infinity).

On the other hand if one looks at the previous graph 60 (topological entropy as a function of μ)), one sees that the topological entropy is positive ($\mathrm{h_{top}} \simeq 0.48$). This was rigorously proved by Li and Yorke.

As we have already mentioned, in general one does not observe all the trajectories but only the typical trajectories with respect to a measure (a Physical measure for example). In the preceding example, if one starts with an initial condition chosen at random with respect to the Lebesgue measure, it will almost surely converge to the stable orbit of period three. From that point of view we would like to say that the entropy is zero. However there is in the interval, besides the stable periodic orbit of period 3 and its basin of attraction, an invariant Cantor set supporting all the topological entropy (and with s.d.i.c.).

When an ergodic invariant measure μ is considered, the disadvantage of the topological entropy is that it measures the total number of (distinguishable) trajectories, including trajectories which have an anomalously small probability to be chosen by μ. It even often happens that these trajectories are much more numerous than the ones "felt" by μ. We will see this more quantitatively below on a simple example. The Kolmogorov-Sinai entropy is then more adequate.

If $\mathcal{P}$ is a (measurable) partition, its entropy $H_\mu(\mathcal{P})$ with respect to the measure μ is defined by

$$H_\mu(\mathcal{P}) = -\sum_{A\in\mathcal{P}} \mu(A)\log\mu(A) \ .$$

This is Shannon's formula, and in communication theory one often uses the logarithm base 2 (see [146]). We will see later on a simple example a basic intuition leading to this formula.

Another motivation is from information theory. If one observes an event with large probability (say almost one), one does not gain much information, this event was in some sense almost predictable. On the contrary if one observes an event with very small probability, we are surprised, a lot of information is obtained because we know that the orbit was in a very small subset of the phase space. Shannon's formula provides a measure for the gain of information (amount of surprise) per unit time. The reason for the log will become clear later.

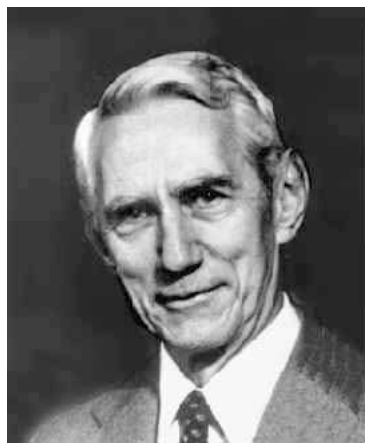


Figure 61: Claude Shannon (1916-2001).

The entropy of the dynamical system with respect to the partition $\mathcal{P}$ and the (invariant, ergodic, probability) measure μ is defined by

$$H_\mu(T,\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\vee_{j=0}^{n-1} T^{-j}\mathcal{P}) \ .$$

One can prove that the limit always exists. Finally the Kolmogorov-Sinai entropy is defined by

$$h_\mu(T) = \sup_{\mathcal{P}\,\text{finite or countable}} H_\mu(T,\mathcal{P}) \ .$$

If $\mathcal{P}$ is generating, and $h_\mu(T) < \infty$, then

$$h_\mu(T) = H_\mu(T,\mathcal{P}) \ .$$

This simplifies a lot the computation of entropy. It is enough to compute the entropy with respect to a generating partition. One has also the general inequality

$$h_\mu(T) \le h_{\text{top}}(T) \ , \tag{58}$$

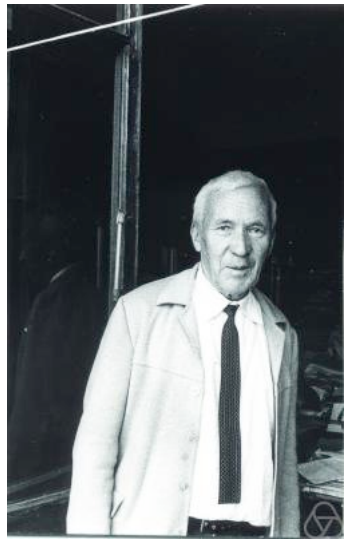


Figure 62: Andrey Kolmogorov (1903-1987).

and moreover

$$h_{\text{top}}(T) = \sup_{\mu,\, T \text{ ergodic}} h_\mu(T) \ .$$

However the maximum may not be reached. We refer to [147] or [132] for proofs and more results on the Kolmogorov Sinai entropy.

An important application of the entropy is Ornstein's isomorphism theorem for Bernoulli schemes [127]. We recall that a Bernoulli scheme is a product measure on a finite alphabet. More precisely, let $\mathscr{A} = \{1, \ldots, k\}$ and let $\vec{p} = (p_1, \ldots, p_k)$ be a non trivial probability vector ($0 < p_\ell < 1$ for $1 \leq \ell \leq k$ and $p_1 + \ldots + p_k = 1$). This vector induces a probability measure $\nu_{\vec{p}}$ on $\mathscr{A}$ given by

$$\nu_{\vec{p}}(\ell) = p_\ell \ .$$

The shift map defines a dynamical system on $\Omega = \mathscr{A}^{\mathbb{Z}}$. The infinite product measure $\mu_{\vec{p}}$ of the $\nu_{\vec{p}}$ is invariant by the shift. We have seen before that it is ergodic and mixing. The partition by cylinder sets of length one (fixing the first symbol of a sequence) is generating (exercise), and the entropy of $\mu_{\vec{p}}$ is given by (exercise)

$$h_{\mu_{\vec{p}}}(\mathcal{S}) = -\sum_{\ell=1}^{k} p_\ell \ \log p_\ell \ .$$

The celebrated Ornstein isomorphism theorem states that two Bernoulli schemes with the same entropy are isomorphic.

They are not required to have the same alphabet! (they do not need to have the same k). The isomorphism is in a measure theoretic sense, hence the map may be very complicated.

There are several useful conditions for the isomorphism of a dynamical system to a Bernoulli scheme. In general the isomorphism is as before in a measure theoretic sense and in general not known explicitly. The case of the map $2x$ mod 1 discussed before (dyadic expansion) is somewhat of an exception.

Another important result about the entropy is the Shannon-McMillan - Breiman theorem which in a sense counts the number of typical orbits for the measure μ. A majority of applications of the notion of entropy are based on this general result which we now state and comment.

Theorem 5.2. *Let μ be an ergodic invariant measure for the map T on the phase space Ω, and let $\mathcal{P}$ be a finite generating partition of Ω. For any $\epsilon > 0$ there is an integer $N(\epsilon)$ such that for any $n > N(\epsilon)$ one can separate the atoms of the partition $\mathcal{P}_n = \vee_{j=0}^{n-1} T^{-j}\mathcal{P}$ into two disjoint subsets*

$$\mathcal{P}_n = \mathcal{B}_n \cup \mathcal{G}_n$$

such that

$$\mu\left(\bigcup_{A\in\mathcal{B}_n} A\right) \leq \epsilon$$

and for any $A \in \mathcal{G}_n$

$$e^{-n(h_\mu(T)+\epsilon)} \leq \mu(A) \leq e^{-n(h_\mu(T)-\epsilon)}$$

In other words, the atoms in $\mathcal{B}_n$ are in some sense the "bad" atoms (for the measure μ), but the good news is that their total mass is small.

On the other hand, the good atoms (those in $\mathcal{G}_n$) have almost the same measure, and of course their union gives almost the total mass of the measure. An immediate consequence is that

$$\left|\mathcal{G}_n\right| \asymp e^{n\, h_\mu(T)} \ ,$$

where $\left|\mathcal{G}_n\right|$ denotes the cardinality of the set $\mathcal{G}_n$.

This formula gives the number of trajectories of length n which are "seen" by the measure μ and are different (distinguished) from the point of view of the partition $\mathcal{P}$. This is similar to the well known formula of Boltzmann in statistical mechanics relating the entropy to the logarithm of the (relevant) volume in phase space. There is also an obvious connection with the equivalence of ensembles in statistical mechanics. We refer to [107] and [111] for more information. As mentioned before, it often happens that $\left|\mathcal{G}_n\right| \ll \left|\mathcal{B}_n\right|$.

A simple example of this phenomenon is given by the Bernoulli shift on two symbols. Let $p \in]0,1[$ with $p \neq 1/2$. Consider on $\Omega = \{0,1\}^{\mathbb{N}}$ the probability measure μ_p which is the infinite product of the measure on $\{0,1\}$ giving probability p to the symbol 1. This is equivalent to flipping a coin, with probability p to get head (symbol 1) and probability $q = 1-p$ to get tail (symbol 0). We have seen in exercise 37 that the measure μ_p is invariant and ergodic for the (unilateral) shift $\mathcal{S}$. Recall that we have defined a distance d on Ω (and in fact a one parameter family of distances) see 19. It is easy to prove that $h_{\text{top}}(\mathcal{S}) = \log 2$. The partition

$$\mathcal{P} = \Big\{ \{x_0 = 0\}, \{x_0 = 1\} \Big\}$$

is generating. For the entropy one has $h_{\mu_p}(\mathcal{S}) = -p\log p - q\log q$. However for n large we get (since $p \neq 1/2$)

$$\left|\mathcal{G}_n\right| \asymp e^{n\, h_{\mu_p}(\mathcal{S})} \ll 2^n \asymp \left|\mathcal{B}_n\right| \ .$$

One can work this classical example in more details, and ask for a more precise asymptotic characterisation of the set of typical sequences among the sequences of zeros (tail) and ones (head) of length n. To count them we will classify them according the number m of ones (head). If m is given, there are $\binom{n}{m}$ sequences of length n with m ones and $n-m$ zeros. Of course

$$2^n = \sum_{m=0}^{n} \binom{n}{m}$$

but also

$$1 = \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} \ .$$

Let us look more carefully at this well known identity. If a sum of $(n+1)$ positive terms is equal to 1, there are two extreme cases: all the terms are equal (to $1/(n+1)$), or one is equal to one and the others are zero. There are of course many other intermediate situations.

The present case is more like the second extreme situation. Indeed, by Stirling's formula

$$\log\left(\binom{n}{m} p^m q^{n-m}\right) \approx$$

$$n\log n - m\log m - (n-m)\log(n-m) + m\log p + (n-m)\log q$$

and this quantity is maximal for $m = \overline{m} = p\ n$. It decreases rather fast when m deviates from $\overline{m}$. For example, for $\left|m - \overline{m}\right| > 2.6\sqrt{pqn}$ we have

$$\sum_{|m-\overline{m}|>2.6\sqrt{pqn}} \binom{n}{m} p^m q^{n-m} \leq .01$$

Hence there is an extremely small probability to observe a sequence with an m (the number of ones) differing form $\overline{m}$ by more than $2.6\sqrt{pqn}$. The important point is that ($p \notin \{0,1\}$)

$$\overline{m} = pn \gg 2.6\ \sqrt{pqn}$$

for large n. This has of course to do with large deviations. When $m = \overline{m}$, $\log\left(p^m q^{n-m}\right)$ is equal to e^{-nh} with

$$h = -p\log p - q\log q\ ,$$

the entropy per flip.

For m near $\overline{m}$

$$\binom{n}{m} p^m q^{n-m} \approx \frac{1}{2\pi\sqrt{pqn}} e^{-(m-\overline{m})^2/(2pqn)} \ .$$

Note that $p = 1/2$ gives the maximal entropy which is equal to $\log 2$. In this case all the sequences of flips have the same probability. This measure is called the measure of maximal entropy, its entropy is equal to the topological entropy and this is the maximum as follows from inequality (58). All the trajectories are equally typical. This is why we assumed before $p \neq 1/2$.

Let us summarise (for $p \notin \{0, 1/2, 1\}$).

- Out of the 2^n possible outcomes of n coin flippings, the observed ones belong with overwhelming probability to a much smaller subset ($m \simeq \overline{m}$).
- In this subset, all sequences have about the same probability $p^{\overline{m}} q^{n-\overline{m}} = e^{-nh}$.
- Since these are the only sequences which weight in probability, their number is $N_p(n) \approx e^{nh}$ (the total probability is one).
- For the Kolmogorov-Sinai entropy of the independent coin flipping we have

$$\lim_{n\to\infty} \frac{1}{n} \log N_p(n) = h = -p \log p - q \log q \ . \tag{59}$$

  and

$$N_p(n) \approx e^{nh} \ll 2^n$$

  for $p \neq 1/2$.
- Said differently, one can separate the possible outcomes of length n in two categories: the good ones and the bad ones. This discrimination depends heavily on the measure (namely on p).
- All bad outcomes together form a set of small measure (not only each one separately).
- The good outcomes have all about the same probability e^{-nh} and together account for almost all the weight (one). Their number is about e^{nh}.
- This is similar to the equivalence of ensembles in statistical mechanics (see [111]).

Another way to formulate the Shannon-McMillan-Breiman theorem is to look at the measure of cylinder sets. For a point $x \in \Omega$, let $C_n(x)$ be the atom of $\vee_{j=0}^{n-1} T^{-j}\mathcal{P}$ which contains x. In other words, $C_n(x)$ is the set of $y \in \Omega$ such that for $0 \leq j \leq n-1$, $T^j(x)$ and $T^j(y)$ belong to the same atom of $\mathcal{P}$ (the trajectories are indistinguishable up to time $n-1$ from the fuzzy observation defined by $\mathcal{P}$, they have the same code). The Shannon-McMillan-Breiman Theorem states that for μ almost every x we have

$$h_\mu(T) = -\lim_{n\to\infty} \frac{1}{n} \log \mu\big(C_n(x)\big) \ .$$

A result similar to the Shannon-McMillan-Breiman theorem holds using a metric instead of a partition. It is due to Brin and Katok, and uses the so called Bowen balls defined for $x \in \Omega$, the transformation T, $\delta > 0$ and an integer n by

$$B(x, T, \delta, n) = \left\{ y \, , \, d\big(T^j(x), T^j(y)\big) < \delta \text{ for } j = 0, \cdots, n-1 \right\} .$$

These are again the initial conditions leading to trajectories indistinguishable (at precision δ) from that of x up to time $n-1$.

Theorem 5.3. *If μ is T ergodic, we have for μ almost any initial condition*

$$h_\mu(T) = \lim_{\delta \searrow 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu\big(B(x, T, \delta, n)\big) \ .$$

We refer to [27] for proofs and related results.

Another way to get the entropy was obtained by Katok [98] in the case of a phase space Ω with a metric d and a continuous transformation T. The formula is analogous to the formula for the topological entropy except that it refers only to typical orbits for a measure. Let A be a set, we denote by $N_n(A,\epsilon)$ the maximal number of ϵ different orbits before time n with initial condition in A. Recall that two orbits of initial condition x and y respectively are ϵ different before time n if

$$\sup_{0\le k\le n-1} d\big(T^k(x),T^k(y)\big) > \epsilon \ .$$

Note that the orbits are not required to stay in A but only to start in A. Let μ be an ergodic invariant probability measure for T. We define for $\delta \in]0,1[$ the sequence of numbers $\tilde{N}_n(\mu,\delta,\epsilon)$ by

$$\tilde{N}_n(\mu,\delta,\epsilon) = \inf_{A,\ \mu(A)\ge 1-\delta} N_n(A,\epsilon) \ .$$

Theorem 5.4. *For any $\delta > 0$ we have*

$$h\big(\mu\big) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log \tilde{N}_n(\mu,\delta,\epsilon)}{n} = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log \tilde{N}_n(\mu,\delta,\epsilon)}{n}$$

We refer to [98] for the proof. We refer to [100], [94], [106], and references therein for more results.

A way to measure the entropy in coded systems was discovered by Ornstein and Weiss using return times to the first cylinder. The result was motivated by the investigation of the asymptotic optimality of Ziv's compression algorithms which has very popular implementations (gzip, zip etc.) Let k be a finite integer, and assume the phase space of a dynamical system is a shift invariant subset of $\{1,\cdots,k\}^{\mathbb{N}}$. As before, we denote the shift by $\mathcal{S}$. Let μ be an ergodic invariant measure. Let n be an integer and for $x\in\Omega$, define $R_n(x)$ as the smallest integer such that the first n symbols of x and $\mathcal{S}^{R_n(x)}(x)$ are identical.

Figure 63: An illustration of the definition of $R_n(x)$.

Theorem 5.5. *For μ almost every x we have*

$$\lim_{n\to\infty}\frac{1}{n}\log R_n(x) = h_\mu(\mathcal{S}) \ .$$

We refer to [128] for the proof.

A metric version of the Ornstein-Weiss theorem was recently obtained by Downarowicz and Weiss, using again Bowen balls, in the context of dynamical systems with a metric phase space. Define for a number $\delta > 0$, an integer n, and a point $x\in\Omega$ the integer

$$R(x,T,\delta,n) = \inf\Big\{ l > 0 \ ,\ d\big(T^{l+j}(x),T^j(x)\big) < \delta \text{ for } j = 0,\cdots,n-1\Big\}.$$

Theorem 5.6. *For μ almost every x, we have*

$$\lim_{\delta \searrow 0} \limsup_{n \to \infty} \frac{1}{n} \log R(x, T, \delta, n) = h_\mu(T) \ .$$

We refer to [61] for the proof and related results.

5.3 SRB measures.

It is natural to ask if there is a connection between entropy and Lyapunov exponents. The first relation is an inequality due to D.Ruelle.

Theorem 5.7. *Let μ be an ergodic invariant measure. The entropy of μ is less than or equal to the sum of the positive Lyapunov exponents of μ.*

We will only give a rough idea of the proof. Take a ball B of radius ϵ (small) such that $\mu(B) > 0$. Iterate k times (k not too large such that $T^k(B)$ is still small). One gets an ellipsoid $T^k(B)$ since some directions are stretched (positive Lyapunov exponents), and others are contracted. We cover this ellipsoid (optimally) with balls of the same radius ϵ. In order to cover $T^k(B)$ by balls of radius ϵ we need at most $N = e^{k\Theta}$ balls, where Θ is the sum of the positive exponents. Indeed, in the direction of exponent j with $\lambda_j > 0$, we stretched from a size ϵ to a size $e^{k\lambda_j}\epsilon$, and we need now $e^{k\lambda_j}$ balls of radius ϵ to cover "in this direction".

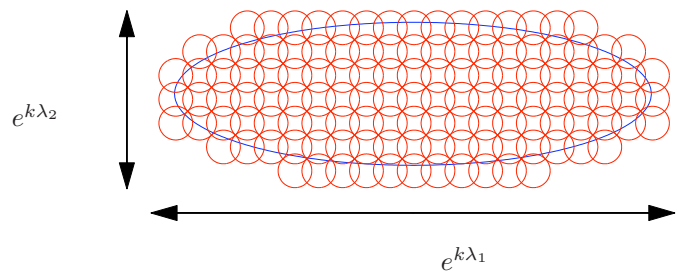


Figure 64: Covering of an ellipsoid by balls.

We now interpret the construction. Let $b_1, \cdots, b_N$ be the balls of radius ϵ used to cover $T^k(B)$. If two initial conditions in B manage to ϵ separate before time k, they will land in two different balls b_j and b_l (forget re-encounters). If they do not separate they will land in the same ball b_k. Therefore

$$N \leq e^{k\Theta} \ ,$$

and using the result of Katok (Theorem 5.4) connecting the entropy with the minimal number of balls covering a set of measure almost one, we get

$$h(\mu) \leq \frac{\log N}{k} \leq \Theta = \sum_{\lambda_i(\mu) > 0} \lambda_i(\mu) \ .$$

One is then eager to know when equality holds in Ruelle's result. In order to explain the answer to this question, we first need some definitions. Consider

a negative Lyapunov exponent for an ergodic invariant measure μ and let x be a typical point for this measure. Then from Oseledec's theorem we know that at the linear level, there are directions at x which are contracted exponentially fast.

One may wonder if such a fact also holds for the complete dynamics and not only for the linearisation. This leads to the definition of the local stable manifold (there are many, one uses local stable manifolds in which x is "at the center").

Definition 20. *An (open) sub-manifold W of the phase space Ω is called a local stable manifold with exponent $\gamma > 0$ if there is a constant $C > 0$ such that for any x and y in W and any integer n we have*

$$d\big(T^n(x), T^n(y)\big) \leq Ce^{-\gamma n} \ .$$

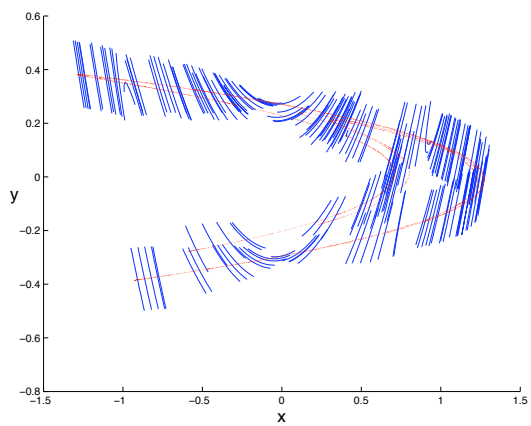


Figure 65: Several local stable manifolds for the Hénon map (in blue). In red the attractor.

Note that it follows immediately from the definition that if W is a local stable manifold with exponent γ, then for any integer p, $T^p(W)$ is also a local stable manifold with exponent γ. When T is invertible, one can define the local unstable manifolds as the local stable manifolds of the inverse.

A local stable manifold containing a point x is called a local stable manifold of x. We refer to [85] for the general case and more details.

When we have a measure μ we can consider local stable manifolds except on sets of measure zero. The idea being that the size of the local stable manifold depends on the point x (or more interestingly the distance of x to the boundary of the manifold). The nice picture is when points without local stable manifolds (interpreted as local stable manifolds of width zero) are of measure zero, and when there is some coherence among the local stable manifolds through the dynamics. This leads naturally to the following definitions.

Definition 21. *Given a (probability) measure μ on an attractor $\mathcal{A}$, invariant under the map T, a coherent field of local stable manifolds for μ with exponent $\gamma > 0$ is a collection $\big(W(x)\big)_{x\in\tilde{\mathcal{A}}}$ of local stable manifolds with exponent γ indexed by a subset $\tilde{\mathcal{A}}$ of $\mathcal{A}$ of full μ measure and such that*

1) For any x in $\tilde{\mathcal{A}}$, $x \in W(x)$.

2) For any x in $\tilde{\mathcal{A}}$, $W(x)$ is tangent at x to one of the Lyapunov subspace $E_j(x)$.

3) For any x in $\tilde{\mathcal{A}}$, $T\big(W(x)\big) \subset W\big(T(x)\big)$.

4) $W(x)$ is not too large in some technical sense.

We now state an "integrated version" of the Oseledec theorem 5.1.

Theorem 5.8. *Let μ be an ergodic invariant measure for a diffeomorphism of a compact manifold T. Let $\lambda_0 > \ldots > \lambda_k > 0$, be the collection of positive Lyapunov exponents. Then there is a set B of full μ measure such that for any ϵ satisfying*

$$0 < \epsilon < \inf_{1 \le j \le k-1} \big(\lambda_j - \lambda_{j+1} \vee 0\big) \ ,$$

and for each $x \in B$, there exists k nested sub-manifolds

$$W_0(x) \supsetneq W_2(x) \supsetneq \ldots \supsetneq W_k(x)$$

where for any j $(0 \le j \le k)$, $W_j(x)$ is a $\lambda_j - \epsilon$ local unstable manifold of x. If y and z belong to $W_j(x) \backslash W_{j+1}(x)$, then

$$d\big(T^{-n}(y), T^{-n}(z)\big) \sim e^{-n\lambda_j} \ .$$

A similar result holds for the negative exponents.

One of the big difference with the case of uniformly hyperbolic systems like axiom A systems is that in non uniformly hyperbolic systems there is in general no uniform lower bound on the size of the local stable or unstable manifolds. At some point, they may turn very rapidly (having large curvature), and even they may not exist at some points (of total measure zero). See figure 65 for the case of the Hénon map. We refer to [131] and [141] for the proof and extension.

We now state the definition of SRB measures.

Definition 22. *Let the phase space Ω of a dynamical system be a compact Riemannian manifold. Let the map T be a C^2 diffeomorphism. An ergodic invariant measure μ is said to be an SRB measure if it has a positive Lyapunov exponent and the conditional measures on the local unstable manifolds are absolutely continuous.*

In other words, one can disintegrate μ as follows, where g is any continuous function

$$\int g(x) d\mu(x) = \int_{\mathcal{W}^u} d\nu(w) \int_w g(y) \rho_w(y) d_w y \tag{60}$$

where $\mathcal{W}^u$ is the set of local unstable manifolds, $d_w y$ is a Lebesgue measure on the local unstable manifold w, $\rho_w(y)$ is a non negative density supported by w, and ν is called the transverse measure. An example is given by the dissipative baker map, see exercise 24.

We can now state an important theorem due initially to Pesin and generalised by Ruelle which complements the inequality in Theorem 5.7.

Theorem 5.9. *Let the phase space Ω of a dynamical system be a compact Riemannian manifold. Let the map T be a C^2 diffeomorphism. Let μ be an ergodic invariant measure. Then μ is an SRB measure if and only if the entropy of μ is equal to the sum of its positive Lyapunov exponents.*

In particular, in dimension one, an ergodic invariant measure is absolutely continuous if and only if it has a positive exponent and its entropy is equal to this exponent. In axiom A systems, Physical and SRB measures are the same. They have another property connected with stochastic perturbations that will be mentioned later on. We mention that there are dynamical systems which do not possess SRB measures. We refer to [163] for more information.

5.4 Appendix: hyperbolicity and chaos.

5.4.1 Hyperbolic fixed points.

Part of the following is analogous (and simpler) to the discussion introducing Lyapunov exponents. However the results are stronger.

To understand hyperbolicity, we start with the simplest asymptotic dynamics: a fixed point M_0 for a map T (recall that $T(M_0) = M_0$). The map T will be assumed regular (as regular as needed). What happens if we perturb a little bit the initial conditions, namely we start with $M = M_0 + \delta M$, with $\|\delta M\| \ll 1$?

For the orbit we can write using Taylor's formula

$$T^n\big(M_0 + \delta M\big) = T^n\big(M_0\big) + DT^n_{M_0}\delta M + \mathcal{O}\big(\|\delta M\|^2\big) \ .$$

Note that the remainder term depends on n. However for n not too large we can neglect it.

That is to say, to first approximation we have

$$T^n\big(M_0 + \delta M\big) = T^n\big(M_0\big) + DT^n_{M_0}\delta M \ .$$

Recall that $T^n(M_0) = M_0$, and by the chain rule,

$$DT^n_{M_0} = \big(DT_{M_0}\big)^n \ .$$

Therefore the time behaviour of the "error" $\|T^n(M_0+\delta M)-M_0\|$ is $\|(DT_{M_0})^n\delta M\|$ for n not too large. In other words we have to study the behaviour of $(DT_{M_0})^n$ as a function of n.

The behaviour of $(DT_{M_0})^n$ depends of course of the spectrum of DT_{M_0}. If all the eigenvalues of DT_{M_0} have modulus less than one, the matrix $(DT_{M_0})^n$ decays exponentially fast with n.

The fixed point is said to be stable. Small initial errors fade away, and the system relaxes to the fixed point.

If there is an eigenvalue of DT_{M_0} of modulus larger than one, one may expect (rightly) exponential growth of most initial errors. The fixed point is said to be unstable. The linear argument holds until n becomes so large that the correction term in Taylor's formula cannot be neglected anymore. Intuitively (and this can be made rigorous) this holds when n is of order $-\log\|\delta M\|$.

If DT_{M_0} has no eigenvalue of modulus one, the fixed point M_0 is said to be hyperbolic.

The above terminology can be extended to periodic orbits and to fixed points and closed cycles in the case of continuous time (one can use in this case the Poincaré section defined in the first chapter), see below.

5.4.2 Linear behaviour.

Since there are some subtleties that will often show up in these questions of hyperbolicity, let us spend some time on a very simple linear example which carries all these subtleties.

The phase space Ω is the plane, and the map T is given by a diagonal matrix A

$$T(\vec{x}) = A\vec{x} \qquad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $\lambda > 1 > \mu > 0$. The origin $M_0 = (0,0)$ is a hyperbolic fixed point. We now start with an initial condition $M = M_0 + \delta M = (x_1, x_2)$ which is a slight perturbation of the origin. A simple calculation leads to

$$T^n(M) = T^n \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda^n x_1 \\ \mu^n x_2 \end{pmatrix}$$

We see that if $x_1 \neq 0$, the error indeed grows exponentially fast, while if $x_1 = 0$ the vector decays exponentially fast.

A typical vector will have a non zero first component, therefore a typical initial error will grow exponentially fast. Note that the horizontal axis is invariant, and there errors grow exponentially fast. This is called the unstable direction.

The vertical axis is also invariant, there the errors decay exponentially fast. This is called the stable direction.

Since the map is linear, the general case can be recovered from these two particular cases. The map T^{-1} is given by $T^{-1}(\vec{x}) = A^{-1}\vec{x}$. The same discussion holds. The unstable manifold is the vertical axis, the stable manifold the horizontal axis. This idea is used very often to avoid blow up: large go forward in time along stable directions and backward in time along unstable directions.

For a general linear map in dimension two with the origin as a hyperbolic fixed point, the orbits are arranged along hyperbola like curves.

In the general case for a fixed point M_0 of a map T, one has to look at the spectral decomposition of DT_{M_0}. If the linear map DT_{M_0} can be diagonalised, the discussion is the same.

Note that for eigenvalues of modulus one, the vectors in the corresponding spectral directions do not grow or decay exponentially fast. They may at most grow polynomially fast.

In the general case, one has to use Jordan decomposition, and polynomial corrections in time may appear.

Very often the matrix DT_{M_0} has real entries (real dynamical system). In that case the spectrum is symmetric by complex conjugation. Other symmetries can appear (symplectic map for mechanical systems).

In all the above discussion we neglected the effect of the remainder term in Taylor's formula. What is its influence? For simplicity we will assume that the spectrum of DT_{M_0} has no eigenvalue of modulus one. In that case the fixed point M_0 is called hyperbolic. The interesting case is when some eigenvalues are of modulus larger than one while some others have modulus smaller than one.

The unstable subspace $\mathbf{E}^u(M_0)$ is defined as the spectral subspace of DT_{M_0} corresponding to all unstable eigenvalues (eigenvalues of modulus larger than one). The stable subspace $\mathbf{E}^s(M_0)$ is defined as the spectral subspace corresponding to all stable eigenvalues (eigenvalues of modulus smaller than one).

Note that $\mathbf{E}^u(M_0) \oplus \mathbf{E}^s(M_0)$ is the tangent space of the phase space at M_0.

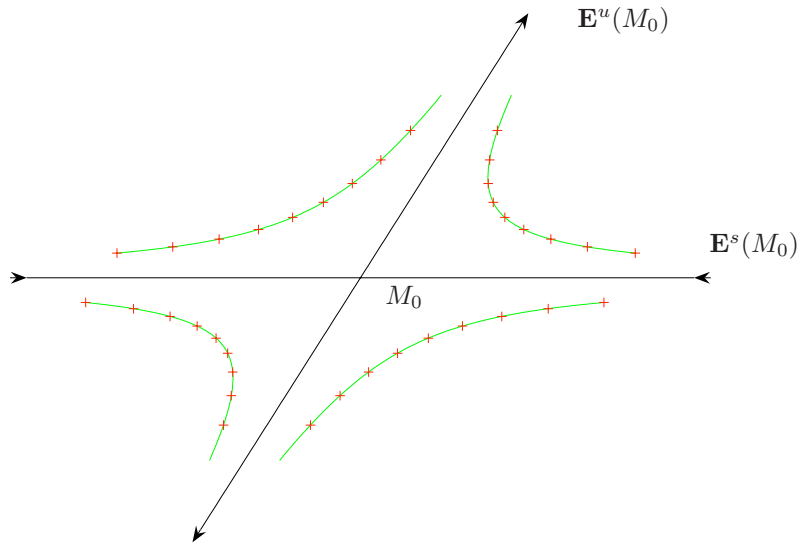


Figure 66: Linearised time evolution around a hyperbolic fixed point.

5.4.3 Local stable/unstable manifolds of a hyperbolic fixed point.

The idea is to generalise the case of the invariant subspaces of linear maps.

Theorem 5.10. *Let M_0 be a hyperbolic fixed point for the regular map T. Then there is a piece of regular manifold $\mathbf{W}^s_{\text{loc}}(M_0)$ (the local stable manifold) tangent to $\mathbf{E}^s(M_0)$ in M_0 and with the same dimension, invariant by T and such that if $M \in \mathbf{W}^s_{\text{loc}}(M_0)$, then $T^n(M)$ converges to M_0 when n tends to infinity. Moreover, there is a number $r > 0$ such that any point in $B_r(M_0)$ whose orbit stays forever in this ball belongs to $\mathbf{W}^s_{\text{loc}}(M_0)$.*

Theorem 5.11. *Let M_0 be a hyperbolic fixed point for the regular invertible map T. Then there is a piece of regular manifold $\mathbf{W}^u_{\text{loc}}(M_0)$ (the local unstable manifold) tangent to $\mathbf{E}^u(M_0)$ in M_0 and with the same dimension, invariant by T^{-1} and such that if $M \in \mathbf{W}^u_{\text{loc}}(M_0)$, then $T^{-n}(M)$ converges to M_0 when n tends to infinity. Moreover, there is a number $r > 0$ such that any point in $B_r(M_0)$ whose backward orbit stays forever in this ball belongs to $\mathbf{W}^u_{\text{loc}}(M_0)$.*

The previous theorem can be suitably generalised to the non invertible case.

One way to construct local stable and unstable manifolds (and also a way to prove the previous theorem) is to construct them as graphs. We assume the map as differentiable as needed. For example, to construct the local unstable manifold (assume the map is invertible), one uses the coordinate system $\mathbf{E}^u(M_0)$, $\mathbf{E}^s(M_0)$ with origin M_0. One looks at $\mathbf{W}^u_{\text{loc}}(M_0)$ as a graph $(x, g(x))$ with $x \in \mathbf{E}^u(M_0)$ and $g(x) \in \mathbf{E}^s(M_0)$. Since $\mathbf{W}^u_{\text{loc}}(M_0)$ passes through M_0, we have $g(0) = 0$. Since $\mathbf{W}^u_{\text{loc}}(M_0)$ is tangent to $\mathbf{E}^u(M_0)$ in M_0, we must have $g'(0) = 0$ (more generally in dimension larger than one $Dg(0) = 0$).

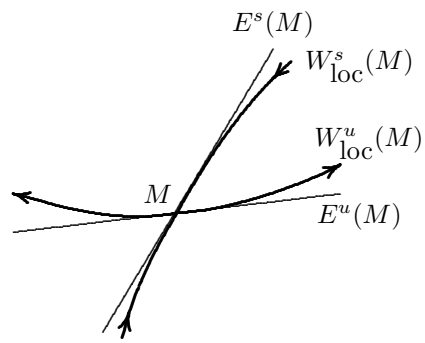


Figure 67: The local manifolds around a hyperbolic fixed point in dimension two.

There is an equation for g, which says that the manifold is invariant.

$$T\big(x, g(x)\big) = \big(x', g(x')\big)$$

for some $x' \in \mathbf{E}^u(M_0)$. The trick is to find at the same time the function g and the map $x \mapsto u(x) = x'$ (which is just the restriction of T to $\mathbf{W}^u_{\text{loc}}(M_0)$ but this manifold is yet unknown). For regular maps, this can be done by finding the Taylor expansion of g.

The Taylor expansion of the above equation yields a recurrence relation for the Taylor's coefficients of g and u.

Exercise 60. *Find the first non trivial order of g for the fixed point of the Hénon map.*

The notion of locality in the previous Theorems is not quantitatively precise. It really means in a small ball around the fixed point, and the radius of this ball depends on the map. Intuitively one can think of these local manifolds as graphs above $\mathbf{E}^s(M_0)$ and $\mathbf{E}^u(M_0)$ respectively.

5.4.4 Global manifolds.

Global manifolds can be defined by suitable iterations of the local ones.

The global stable manifold of M_0 is defined (in the invertible case) by

$$\mathbf{W}^s(M_0) = \bigcup_{n\geq 0} T^{-n}\big(\mathbf{W}^s_{\text{loc}}(M_0)\big)\ .$$

Similarly the global unstable manifold of M_0 is defined by

$$\mathbf{W}^u(M_0) = \bigcup_{n\geq 0} T^{n}\big(\mathbf{W}^u_{\text{loc}}(M_0)\big)\ .$$

The global stable and unstable manifolds are invariant by T and T^{-1}. These global manifolds can do all sorts of wild things, for example they can accumulate on M_0.

The (global) stable manifold cannot intersect itself (or the (global) stable manifold of another fixed point). The (global) unstable manifold cannot intersect itself (or the (global) unstable manifold of another fixed point).

The (global) stable manifold can intersect the (global) unstable manifold at another point than M_0. This is called a homoclinic point (heteroclinic point in the case of two different fixed points). This leads to a complicated (chaotic) time evolution of some trajectories. We will come back to this point later on.

Our previous linear approximation is in fact adequate for the study of the dynamics in a neighbourhood of a hyperbolic fixed point due to a Theorem of Hartman and Grobman (see [79]).

Theorem 5.12. *Let T be a C^2 map of $\mathbb{R}^d$ with a hyperbolic fixed point M_0. Assume DT_{M_0} is invertible. Then there is a homeomorphism Φ of $\mathbb{R}^d$, fixing M_0 ($\Phi(M_0) = M_0$) and a neighbourhood V of M_0 such that in V*

$$T = \Phi^{-1} \circ DT_{M_0} \circ \Phi \ .$$

The map Φ may not be differentiable due to resonances.

The notions of stable and unstable manifolds extend immediately to the case of periodic orbits. Let $x_1 = T(x_0)$, $x_2 = T(x_1)$, ..., $x_0 = T(x_{k-1})$ be a periodic orbit of period k. Then x_0 is a fixed point of T^k (so are $x_1, \ldots, x_{k-1}$), and we can decide the hyperbolicity by looking at the spectrum of $DT^k_{x_0}$.

Exercise 61. *If T is invertible, show that all the matrices $DT^k_{x_j}$ ($j = 0, \ldots, k-1$) are conjugated and therefore have the same spectrum.*

We can therefore speak of a hyperbolic periodic orbit. It is easy to verify that

$$DT_{x_0}\mathbf{E}^s\big(x_0\big) = \mathbf{E}^s\big(x_1\big) \quad \text{and} \quad DT^{-1}_{x_0}\mathbf{E}^u\big(x_1\big) = \mathbf{E}^u\big(x_0\big) \ ,$$

and all the corresponding relations starting in the other points of the orbit. For the local stable manifolds one has

$$T\big(\mathbf{W}^s_{\rm loc}\big(x_0\big)\big) \subset \mathbf{W}^s_{\rm loc}\big(x_1\big) \quad \text{and} \quad T^{-1}\big(\mathbf{W}^u_{\rm loc}\big(x_1\big)\big) = \mathbf{W}^u_{\rm loc}\big(x_0\big) \ .$$

In other words, one has to follow the trajectory to get the correct inclusions.

The stable manifolds being contracted forward, the unstable manifolds being contracted backward.

These ideas can be further generalised. Let $\mathscr{B}$ be an invariant set for a map T on a phase space Ω. Namely $T(\mathscr{B}) \subset \mathscr{B}$.

Assume T regular and invertible. The subset $\mathscr{B}$ is called (uniformly) hyperbolic if for any point $M \in \mathscr{B}$ the tangent space at M can be decomposed in a direct sum $\mathbf{E}^u(M) \oplus \mathbf{E}^s(M)$ such that

1) $DT_M\mathbf{E}^s(M) = \mathbf{E}^s\big(T(M)\big)$,

2) $DT^{-1}_M\mathbf{E}^u(M) = \mathbf{E}^u\big(T^{-1}(M)\big)$,

3) there are two constants $C > 0$ and $\lambda < 1$ independent of M such that for any $n \geq 0$ and any $v \in \mathbf{E}^s(M)$, respectively $w \in \mathbf{E}^u(M)$

$$\big\|DT^n_M v\big\| \leq C\,\lambda^n \|v\| \quad \text{respectively} \big\|DT^{-n}_M w\big\| \leq C\,\lambda^n \|w\| \ .$$

4) In other words, we follow the contraction (respectively expansion) along the orbit.

$\mathbf{E}^u(\,\cdot\,)$ and $\mathbf{E}^s(\,\cdot\,)$ are called the fields of unstable and stable directions respectively.

For regular maps of uniformly hyperbolic invariant sets, one can show the existence of fields of local stable and unstable manifolds. They satisfy the obvious coherence relations

$$T\big(\mathbf{W}^s_{\mathrm{loc}}(M)\big) \subset \mathbf{W}^s_{\mathrm{loc}}\big(T(M)\big)$$

etc. and exponential contraction of distances (at the same rate λ) along the local stable manifolds and "dilation" along the unstable manifolds.

The fields $\mathbf{E}^s(M)$ and $\mathbf{E}^u(M)$ depend continuously on M but in general not differentiably. Similarly each manifold $\mathbf{W}^s_{\mathrm{loc}}(M)$ and $\mathbf{W}^u_{\mathrm{loc}}(M)$ is regular, continuous in M but in general not differentiable in M.

Exercise 62. *Construct the fields $\mathbf{E}^s$ and $\mathbf{E}^u$ for the cat map of the torus. Deduce the fields $\mathbf{W}^s_{\mathrm{loc}}$ and $\mathbf{W}^u_{\mathrm{loc}}$.*

Dynamical systems for which the whole phase space is uniformly hyperbolic are called Anosov systems. This is the case for the cat map of the torus.

Although the Hénon map is not uniformly hyperbolic, one can look at the field of local stable manifolds as we have seen before (see figure 65).

Hyperbolicity plus compactness of the phase space leads to folding, because in a bounded space nothing can grow exponentially fast for ever. This is also the manifestation of the non-linearities. This leads eventually to complicated, chaotic behaviours.

For a continuous time dynamical system given by a vector field X, recall that a fixed point M_0 satisfies $X(M_0) = 0$. One can look at the time τ map φ_τ of the flow integrating X and discuss the hyperbolicity of M_0 for this map.

Exercise 63. *Show that $D\varphi_{\tau\,M_0} = \exp\big(\tau DX_{M_0}\big)$. Conclude that the hyperbolicity criteria does not depend on τ.*

For a vector field X, there is another simple kind of trajectories: the closed cycles. These are closed (regular) curves C invariant by the flow. We will only consider the case where such curves do not contain a fixed point (i.e. the vector field X does not vanish on C). Note that since the curve is invariant, the vector field X is tangent to the curve. The curve must have a finite period, namely there is a $T > 0$ such that for any $x \in C$, $\varphi_T(x) = x$ (hence the name cycle). We can now discuss the stability of a point x of the curve with respect to the map φ_T.

Exercise 64. *Show that if x and y are two points on C, the matrices $D\varphi_T(x)$ and $D\varphi_T(y)$ are conjugated.*

Show that 1 is an eigenvalue of $D\varphi_T(x)$ (apply this matrix to the vector $X(x)$).

In this context one can also use a Poincaré section (see Figure 27).

5.4.5 Transverse homoclinic points.

This phenomenon was discovered by Poincaré in the search for conditions ensuring that a mechanical system is not completely integrable.

Let M_0 be a hyperbolic fixed point of a (regular) map T in a phase space Ω. A point $H \neq M_0$ is called a homoclinic point if it belongs to $\mathbf{W}^u(M_0) \cap \mathbf{W}^s(M_0)$.

There may not be any, namely $\mathbf{W}^u(M_0)$ and $\mathbf{W}^s(M_0)$ may not intersect except in M_0. Note that $\mathbf{W}^u(M_0)$ cannot self intersect except at M_0 (same for $\mathbf{W}^s(M_0)$). A homoclinic fixed point H is transversal (transverse homoclinic point) if the two manifolds $\mathbf{W}^u(M_0)$and $\mathbf{W}^s(M_0)$ are transversal at H. This means that the tangent spaces to these two sub-manifolds span together the tangent space of the whole phase space. Because of the respective dimensions this is equivalent here to saying that the intersection of the two tangent spaces reduces to the zero vector.

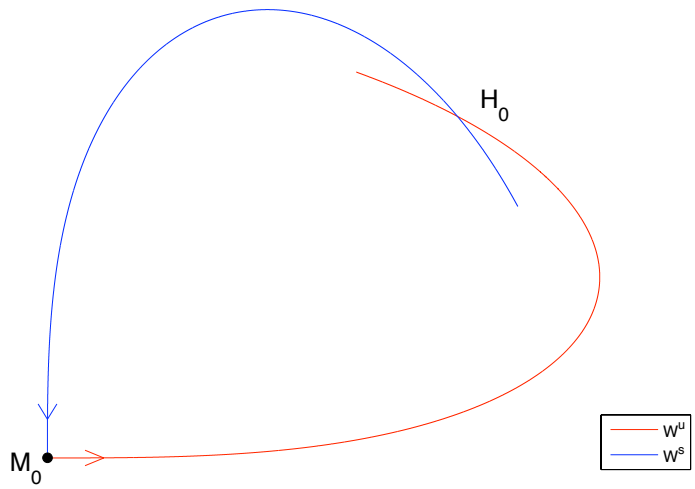


Figure 68: A transverse homoclinic point, also called transverse homoclinic crossing (half of the manifolds).

The complexity of the dynamics when there is a transverse homoclinic point comes from the following fact. Since $H \in \mathbf{W}^u(M_0) \cap \mathbf{W}^s(M_0)$, $T(H)$ also belongs to $\mathbf{W}^u(M_0) \cap \mathbf{W}^s(M_0)$. In other words, $T(H)$ is also a homoclinic point. If DT_H is invertible, $T(H)$ is also a transverse homoclinic point.

Exercise 65. *Prove this statement.*

Therefore if we have one transverse homoclinic point, we have at least a countable number. If the map T is invertible, we can apply the same argument to T^{-1}.

We are now going to describe part of the complexity of this picture.

5.4.6 Geometrical consequences: Smale's horseshoe.

Smale's horseshoe is a geometrical object which systematises what happens when there is a transverse homoclinic point. We start the geometric construction using a transverse homoclinic crossing.

We introduce a tubular neighbourhood $\mathcal{D}$ of a piece of $\mathbf{W}^s(M_0)$ containing M_0 and H (the green rectangle) If $\mathcal{D}$ is thin enough, $T(\mathcal{D})$ (or if necessary $T^k(\mathcal{D})$ for some integer $k > 1$) "intersects transversally" $\mathcal{D}$ at H.

Exercise 66. *Formulate this statement more precisely and prove it.*

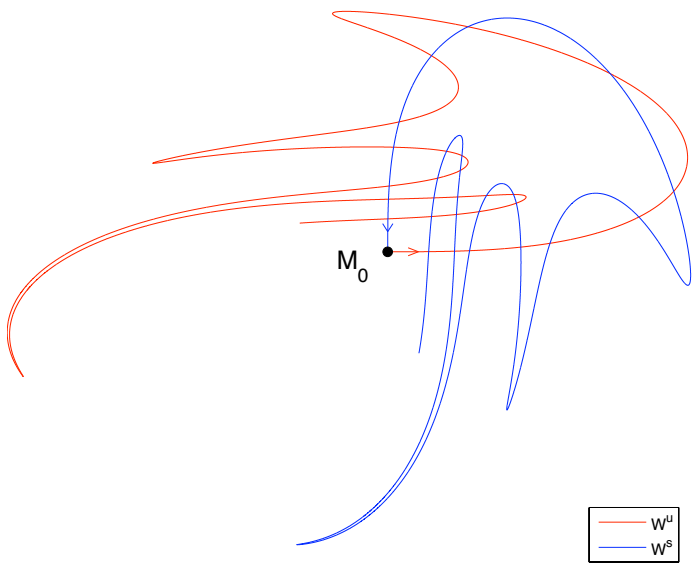


Figure 69: Parts of the (regular) manifolds $\mathbf{W}^u(M_0)$ and $\mathbf{W}^s(M_0)$.

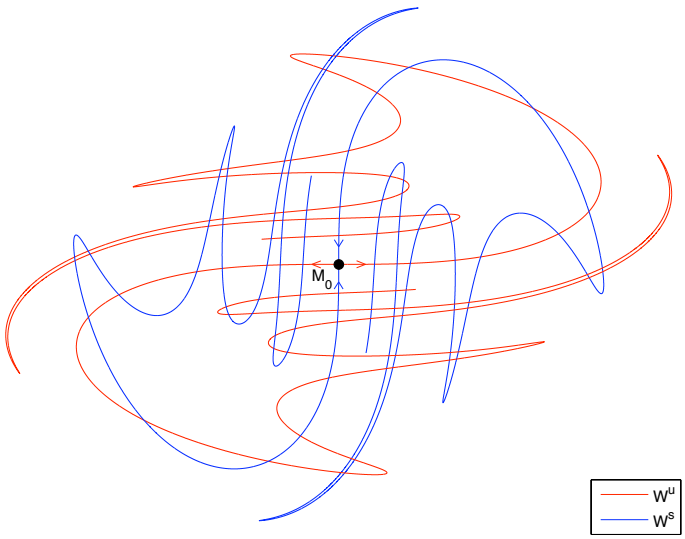


Figure 70: More of the (regular) manifolds $\mathbf{W}^u(M_0)$ and $\mathbf{W}^s(M_0)$.

For simplicity of notation we are going to assume $k = 1$. This is not really a restriction since one can consider the map T^k instead of T. We see that $\mathcal{D} \cap T(\mathcal{D})$ is composed of two essentially horizontal rectangles. Call these rectangles R_1 and R_2, R_1 being the rectangle containing the fixed point M_0.

Schematically (and not only schematically), the dynamics on the rectangle $\mathcal{D}$ stretches and folds as follows The inverse dynamics on the rectangles R_1 and R_2 unfolds and unstretches (contracts horizontally, expands vertically). In the

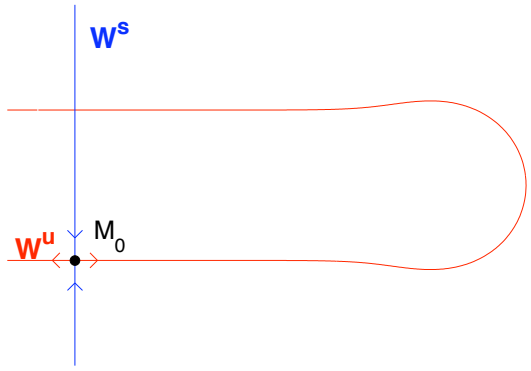


Figure 71: A transverse homoclinic crossing.

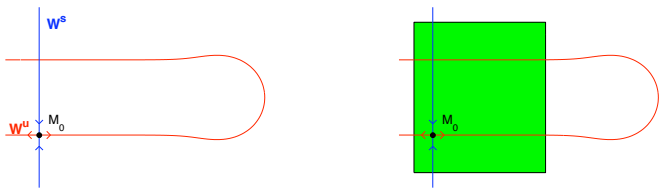


Figure 72: In green the tubular neighbourhood $\mathcal{D}$.

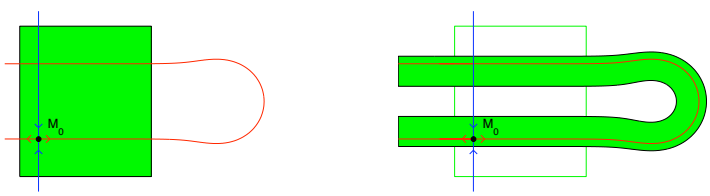


Figure 73: Left: $\mathcal{D}$, right: $T(\mathcal{D})$.

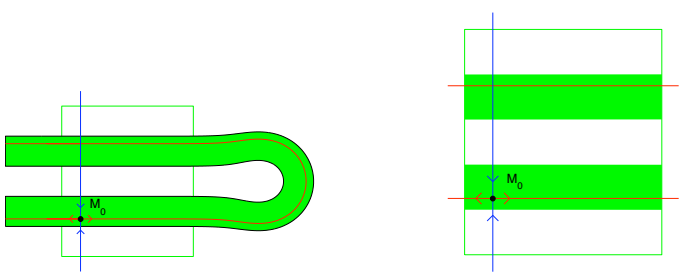


Figure 74: Left: $T(\mathcal{D})$, right: $\mathcal{D} \cap T(\mathcal{D}) = R_1 \cup R_2$

case where T is invertible (or at least invertible on $R_1 \cup R_2$), it is even more interesting to look at $T^{-1}(R_1)$ and $T^{-1}(R_2)$.

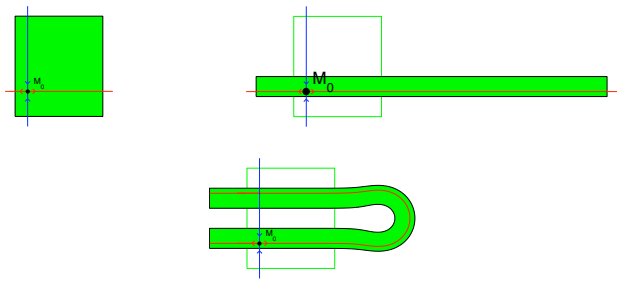


Figure 75: The dynamics of $\mathcal{D}$ decomposed.

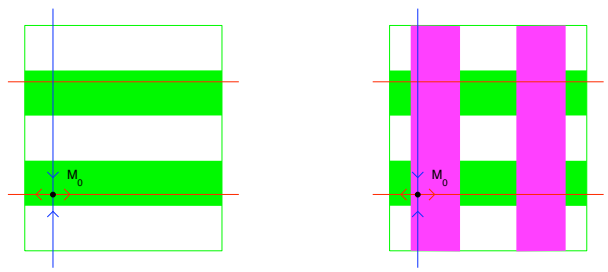


Figure 76: Right: R_1, R_2, $T^{-1}(R_1)$, $T^{-1}(R_2)$.

$T^{-1}(R_1)$ is a "vertical rectangle" containing M_0 and extending from top to bottom of $\mathcal{D}$. $T^{-1}(R_2)$ is similar and "parallel". Let us consider $R_1 \cap T^{-1}(R_1)$.

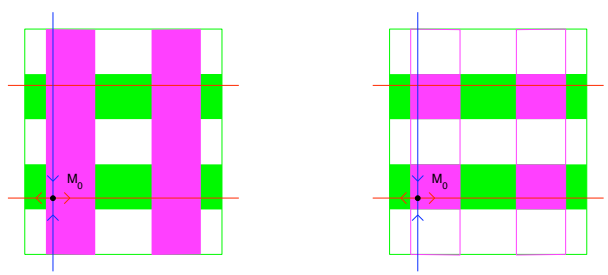


Figure 77: Right: $R_i \cap T^{-1}(R_j),\ i, j = 1, 2$.

Note that by "transversality" $R_1 \cap T^{-1}(R_1) \neq \emptyset$.

A point $P \in R_1 \cap T^{-1}(R_1)$ is such that $P \in R_1$ and $T(P) \in R_1$.

Similarly a point $P \in R_1 \cap T^{-1}(R_2)$ is such that $P \in R_1$ and $T(P) \in R_2$ and so on.

If we attribute the symbols 0 to R_1 and 1 to R_2, we see that for any of the four sequences of two symbols 0 and 1 there is at least one point whose orbit produces this sequence.

We now go on. $T^{-1}(R_1 \cap T^{-1}(R_1))$ is a "vertical rectangle" and similarly

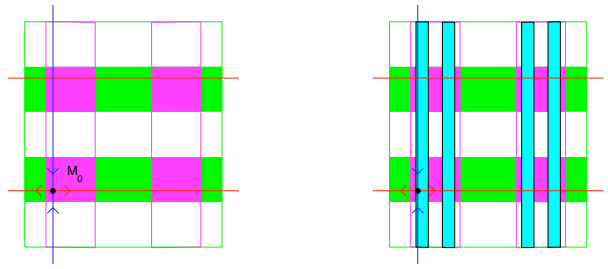


Figure 78: One more iteration of the construction.

for the three other cases. We now have four "vertical rectangles" contained in the previous two.

Continuing the iteration, at step n we have 2^n "parallel" and disjoint "vertical rectangles" going from the top to the bottom of $\mathcal{D}$. We get a decreasing sequence of non empty compact sets. Therefore the intersection $\mathscr{B}$ is compact. The set $\mathscr{B}$ is the set of points whose forward orbit never leaves $\mathcal{D}$.

Exercise 67. *Prove it.*

The set $\mathscr{B}$ is topologically equal to $\mathcal{K} \times I$ where $\mathcal{K}$ is a cantor set and I is a segment. This is easily seen by taking a horizontal section. see Figure 32. The set $\mathscr{B}$ is invariant by T Any (even infinite) sequence of 0 and 1 is realised by a point of $\mathscr{B}$ (in fact at least one vertical segment).

In case of an infinite sequence, the vertical segment is unique. In other words, any outcome of the game of head or tail corresponds to a point in $\mathscr{B}$.

The topological entropy of T on $\mathscr{B}$ is $\log 2$ (or $(\log 2)/k$ if $k > 1$).

Exercise 68. *Prove it.*

If T is invertible, one can repeat the argument and define similarly a set $\mathscr{C}$ invariant by T^{-1}. This set is also topologically the product of a segment by a Cantor set. The set

$$\mathscr{E} = \mathscr{C} \bigcap \mathscr{B}$$

is invariant by T and T^{-1}. Any bi-infinite sequence of zeros and ones correspond to one and only one point in $\mathscr{E}$. This is somewhat equivalent to the case of the dissipative baker map for which the picture is slightly simpler.

WARNING: $\mathscr{E}$ is not an attractor

We refer to [138] and [94] fore more results and references.

6 Measurements.

In this section we will explain some of the methods which have been developed for the measurement of various interesting quantities for dynamical systems and more generally for stochastic processes. These methods can be applied to the output of numerical simulations or experimental data. They are usually targeted to the use of a (long enough) unique realisation of the process.

More precisely, the system is observed through an observable g (a real or vector valued function on the phase space) and for an initial condition x one has a recording of the sequence

$$\big(X_n\big)_{0\le n\le N} = \big(g\big(T^n(x)\big)\big)_{0\le n\le N} \,. \tag{61}$$

This is somewhat different from the usual statistical setting where one assumes to have a sequence of independent observations.

Hopefully, the decay of correlations will remedy to this situation, where ideally we could think of the recorded signal as a collection of somewhat independent chunks. In the sequel we will always assume that an (unknown) ergodic and invariant measure μ has been selected (by nature, numerically or otherwise) and that the initial condition x is typical for this measure. This is of course hard to verify in a numerical situation and even impossible in an experimental setting, this is why the assumption of a Physical measure is relevant.

In any case, we are in the situation of having the recording of the observation of a finite piece of one trajectory of a stochastic process $(X_n)_{0\le n\le N}$. Therefore all the techniques of analysis of stochastic processes are relevant in this context. But there are other questions/techniques dealing more properly with dynamical systems.

It is worth mentioning again that applying the techniques discussed below to an experimental signal, we are assuming that this signal was produced by a dynamical system. In a sense this would the first statistical question one would like to answer.

Although this is probably the hardest problem, we will see later some statistical results which allow in principle to distinguish between a signal produced by a dynamical system with finite dimensional phase space (or more generally finite dimensional attractor), and a truly random signal (if this exists in nature).

We will assume for simplicity that the phase space is $\mathbb{R}^d$ although the methods can be easily extended to the case of differentiable manifolds.

Note that even if we have reasons to believe that the original process is defined for continuous times, experimental recordings by electronic acquisition devices can only be performed at discrete times (not speaking of numerical simulations).

6.1 Correlation function and power spectrum.

One of the easiest thing to determine from the data is the (auto) correlation function. Knowing the decay of correlations is important to get an idea if the other statistics are really meaningful (recall we have only one sample of the process, and we want to argue that well separated chunks are almost independent).

It follows immediately from the ergodic Theorem and the definition of the correlation function that for μ almost every x and for any square observable g (with zero average)

$$C_{g,g}(k) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g\big(T^j(x)\big) g\big(T^{j+k}(x)\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} X_j \; X_{j+k} \; . \qquad (62)$$

In the next figure we show the correlation function computed with this formula for the map $3x \pmod 1$ of the unit interval, and the observable $x - 1/2$.

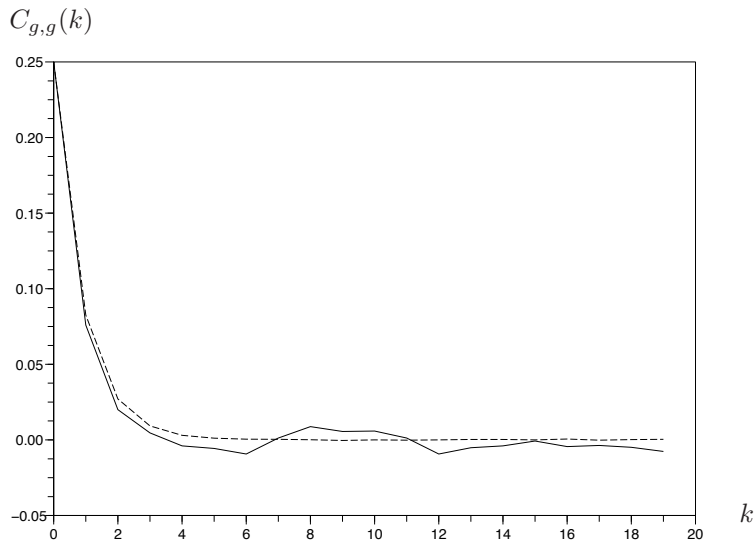


Figure 79: Correlation function for the Lebesgue measure, the map $3x \pmod 1$, and the observable $x - 1/2$. The solid line is for $n = 1024$ and the dash line for $n = 262144$ in the above formula (62).

In practice, the measure μ is in most cases unknown and one cannot subtract to g its average in order to get a function with zero average.

However one can use again the ergodic theorem to estimate the average of g, namely

$$\int g d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g\big(T^j(x)\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} X_j \; .$$

For particular classes of dynamical systems, the convergence rate can be estimated using various techniques.

For example we have obtained recently with J.R.Chazottes and B.Schmitt [32] for some classes of non uniformly hyperbolic systems (and Lipschitz observ-

ables) a bound of the form

$$\mathbb{E}\left(\left[\frac{1}{n}\sum_{j=0}^{n-1} X_j X_{j+k} - C_{g,g}(k)\right]^2\right) =$$

$$\mathbb{E}\left(\left[\frac{1}{n}\sum_{j=0}^{n-1} g\big(T^j(x)\big)g\big(T^{j+k}(x)\big) - C_{g,g}(k)\right]^2\right) \leq A\frac{k}{n^2} + \frac{B}{n} ,$$

valid for any n and k (A and B are some constants depending on the dynamical system, the observable and the invariant measure).

6.2 Power spectrum.

Another quantity often used in practice is the power spectrum W_g (see [99]). This is the Fourier transform of the correlation function, namely

$$W_g(u) = \sum_{k\geq 0} C_{g,g}(k)e^{iku} .$$

With $\mathbb{E}(g)$ the average of g, using the ergodic theorem one is lead to guess that the so-called periodogram $I_n(u,x)$ defined by

$$I_n(u,x) = \frac{1}{n}\left|\sum_{k=0}^{n-1}\big(g\big(T^k(x)\big) - \mathbb{E}(g)\big)e^{iku}\right|^2 = \frac{1}{n}\left|\sum_{k=0}^{n-1}\big(X_k - \mathbb{E}(X_0)\big)e^{iku}\right|^2 \quad (63)$$

should be a good approximation of the power spectrum.

Unfortunately, although it is well known that under some general hypothesis on the system and on the observable g we have

$$\lim_{n\to\infty} \mathbb{E}\big(I_n(u,\,\cdot\,)\big) = \lim_{n\to\infty}\int I_n(u,x)d\mu(x) = W_g(u) ,$$

in general the periodogram does not converge μ almost everywhere in x (it converges to a distribution valued process).

This is illustrated in the next figure which shows the periodogram (63) of the function $\chi_{[0,1/2]} - 1/2$ for the map $3x \pmod 1$ of the unit interval (and an initial condition chosen at random from the Lebesgue measure).

One can prove under suitable conditions that a better quantity to look at is the integrated periodogram J_n given by

$$J_n(u,x) = \int_0^u \frac{1}{n}\left|\sum_{k=0}^{n-1}\big(g\big(T^k(x)\big) - \mathbb{E}(g)\big)e^{iks}\right|^2 ds = \int_0^u \frac{1}{n}\left|\sum_{k=0}^{n-1}\big(X_k - \mathbb{E}(X_0)\big)e^{iks}\right|^2 ds . \quad (64)$$

Namely, for any u, and for μ almost every x we have

$$\lim_{n\to\infty} J_n(u,x) = \int_0^u W_g(s)ds .$$

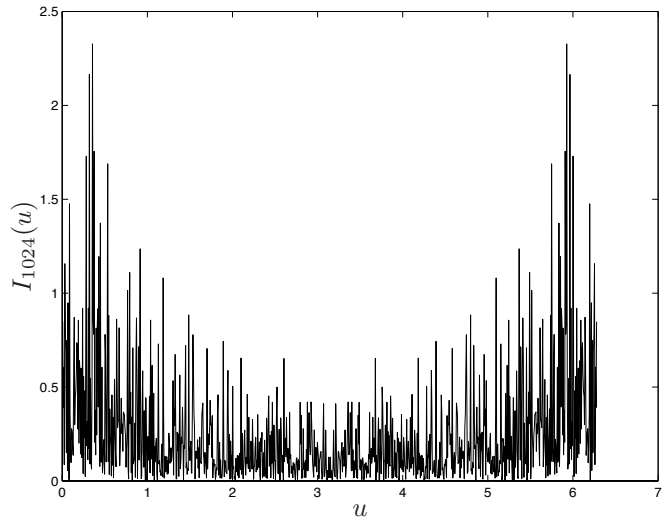


Figure 80: Periodogram of the function $\chi_{[0,1/2]} - 1/2$, with $n = 1024$ in formula (63).

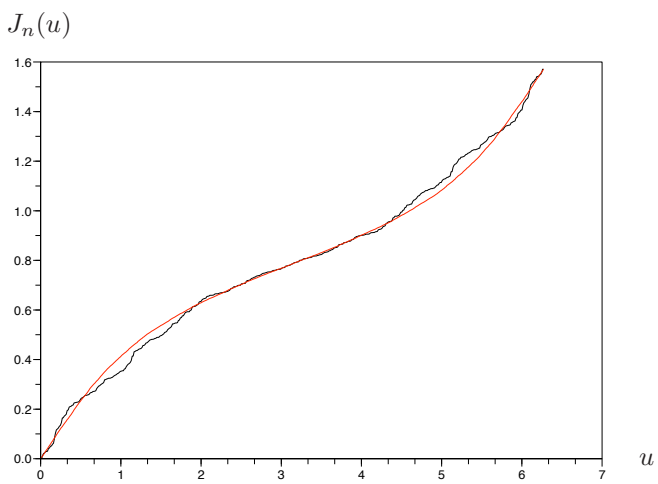


Figure 81: Integrated periodogram of the function $\chi_{[0,1/2]} - 1/2$ for the map $3x \pmod 1$. The red curve corresponds to $n = 32768$ in formula (64) and the black curve to $n = 512$.

The next figure shows the integrated periodogram for the function $\chi_{[0,1/2]} - 1/2$ for the map $3x \pmod 1$ of the unit interval (and random initial condition for the Lebesgue measure).

With J.R.Chazottes and B.Schmitt [32] we have obtained a bound on the velocity of convergence in some non uniformly hyperbolic systems and for Lip-

schitz observables, namely

$$\int d\mu(x) \left(\left[\sup_{u\in[0,2\pi]} \left| J_n(u,x) - \int_0^u W_g(s)ds \right| \right]^2 \right) \leq A \frac{(1+\log n)^{4/3}}{n^{2/3}} \ ,$$

A is a constant depending on the dynamical system, the observable, and the invariant measure. The estimate is valid for all $n \geq 1$. For piecewise expanding maps of the interval (a **BV** observable and an a.c.i.p.m.) the bound is $A(\log n)/n$.

We refer to [32] for the case where the average of g has to be estimated from the data and for (uniform) estimates on the convergence rate under some assumptions on the dynamical system using an exponential or a Devroye inequality.

Other algorithms use local averages of the periodogram instead of the integral form. There is also an algorithm of Burg based on the maximum entropy principle. We refer to [117] for other results.

One may observe in the power spectrum peaks of periodic components and fat parts from noise of dynamical (chaotic) or experimental origin. Using the correlation function and the periodogram also helps often in understanding the basic time scales of the system and to try to answer the basic question whether the available recording is long enough. For example for the map $x \to 1 - 1.7x^2$ of the interval $[-1, 1]$, the next figures shows the correlation function (not much information) and the power spectrum (something appears).

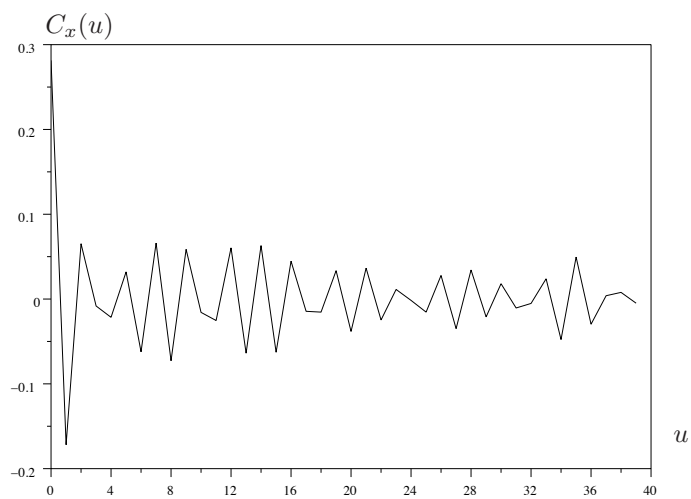


Figure 82: Correlation function for the map $1 - 1.7x^2$.

The information correlation functions are also useful in analysing time series of dynamical systems. If U and V are two independent random variables we have obviously

$$\mathbf{P}\big(U = a,\ V = b\big) = \mathbf{P}\big(U = a\big)\ \mathbf{P}\big(V = b\big)\ .$$

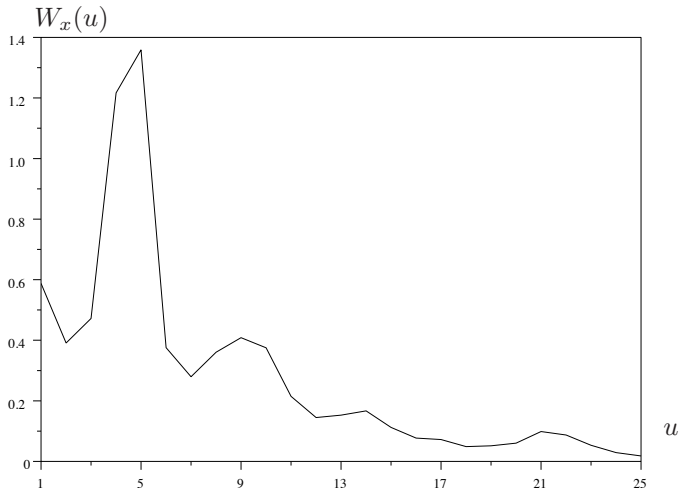


Figure 83: Power spectrum for the map $1 - 1.7x^2$.

To test if the probability $\mathbf{P}\big(U = a,\ V = b\big)$ is a product one uses the relative entropy between this measure and the product measure

$$D = \sum_{a,b} \mathbf{P}\big(U = a\big)\mathbf{P}\big(V = b\big) \log \frac{\mathbf{P}\big(U = a,\ V = b\big)}{\mathbf{P}\big(U = a\big)\mathbf{P}\big(V = b\big)} \ .$$

One can also use the symmetrical expression

$$\sum_{a,b} \mathbb{P}\big(U = a,\ V = b\big) \log \frac{\mathbb{P}\big(U = a\big)\mathbb{P}\big(V = b\big)}{\mathbb{P}\big(U = a,\ V = b\big)} \ .$$

In practice, U is taken as the value of an observable at the initial time, and V its value at a later time $k : V = U \circ T^k$. The ergodic theorem is used to compute the probabilities (measure), it is simpler to use observables taking only finitely many values. For the map $x \to 1.8x^2$ on the phase space $[-1, 1]$ we take for example the characteristic function of the interval $[0, 1] : U = \chi_{[0,1]}$. Therefore

$$\mathbb{P}(U = 1) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \chi_{[0,1]}\big(T^j(x)\big) \ ,$$

and if $V = U \circ T^k$

$$\mathbb{P}(U = 1, V = 1) = \lim_{N\to\infty} \frac{1}{N-k} \sum_{j=0}^{N-k-1} \chi_{[0,1]}\big(T^j(x)\big)\chi_{[0,1]}\big(T^{j+k}(x)\big) \ .$$

Figure 84 shows the information correlation for the map $x \to 1 - 1.8x^2$ of the interval $[-1, 1]$.

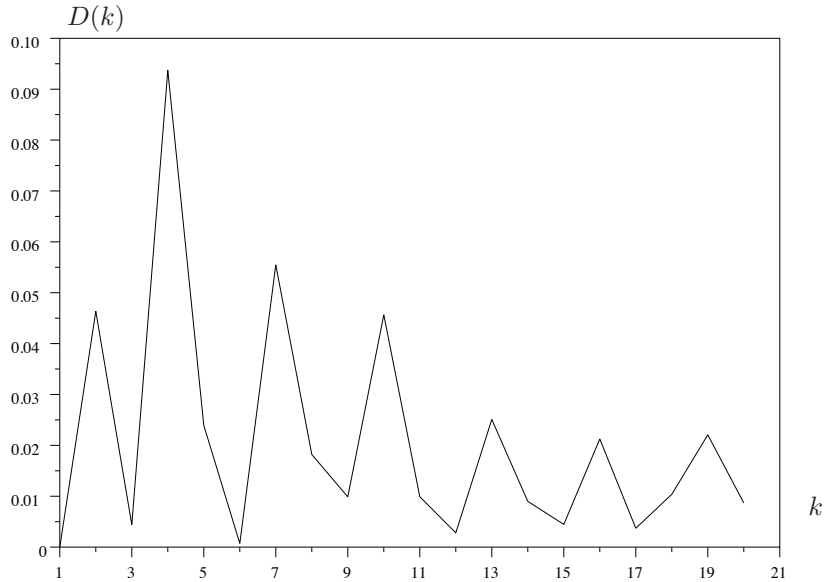


Figure 84: Information correlation for the map $1 - 1.8x^2$.

6.3 Lyapunov exponents

To test for the sensitive dependence to initial conditions, it is natural to try to measure the Lyapunov exponents. There are three main difficulties in the measurement of Lyapunov exponents. Recall that from Oseledec's Theorem, we have to compute $\left(M_n^t M_n\right)^{1/2n}$ where

$$M_n = A_n \dots A_1 \qquad A_j = DT_{T^{j-1}(x)},\ j = 1 \dots n\ .$$

We also recall that in general the order matters in this product. The three main difficulties are

i) For n large, one may wonder what is the most efficient way to compute M_n given that this matrix will in general grow exponentially fast with n (if there is a positive Lyapunov exponent): danger of overflow.

ii) If the phase space is known but not the transformation, how to estimate $A_n = DT_{x_{n-1}}$?

iii) Reconstruct the information in the phase space if it is unknown (for example in experimental situations).

We will deal with the three difficulties one after the other, but we first explain a simpler algorithm for the determination of the maximal exponent λ_0. The idea is that two typical nearby initial conditions x and y should separate

exponentially fast with that rate (namely for typical initial conditions x and y we should have $x - y \notin E_1(x)$). Recall also that in practice we want to use the (finite) orbit of a point x, namely

$$\mathcal{O}_n(x) = \left\{x, T(x), \ldots, T^{n-1}(x)\right\} = \{x_0, \ldots, x_{n-1}\} \, .$$

The algorithm consists in computing for a small fixed $\epsilon > 0$ and several integers s the quantity

$$L_N(s, \epsilon, x) =$$

$$\frac{1}{N-s} \sum_{j=0}^{N-s} \log \left(\frac{1}{\left|\mathcal{U}_\epsilon\big(T^j(x)\big)\right|} \sum_{y \in \mathcal{U}_\epsilon\left(T^j(x)\right)} \left\|T^{j+s}(x) - T^s(y)\right\| \right)$$

$$= \frac{1}{N-s} \sum_{j=0}^{N-s} \log \left(\frac{1}{\left|\mathcal{U}_\epsilon(x_j)\right|} \sum_{y_\ell \in \mathcal{U}_\epsilon(x_j)} \left\|x_{j+s} - y_{\ell+s}\right\| \right)$$

where $\mathcal{U}_\epsilon\big(T^j(x)\big)$ $(\mathcal{U}_\epsilon(x_j))$ is the set of points in the orbit $\mathcal{O}_n(x)$ at a distance less than ϵ of $T^j(x)$ (i.e. x_j). The idea is that each term in the sum should be of order $\epsilon \, e^{s\lambda_0}$: hopefully all points are typical.

The idea to measure Lyapunov exponents will be to measure the local expanding properties.

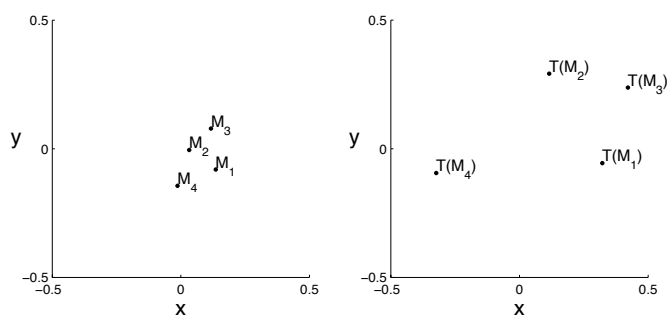


Figure 85: Image of a quadruple of points under the map T. In practice the points M_j will be on a same orbit but not necessarily consecutive in time ($M_j = T^{n_j}(x_0)$).

Hence for s not too small, one expects to see an exponential growth of the distances as a function of s with a slope giving a good approximation of λ_0. However if s is too large, $\epsilon \, e^{s\lambda_0}$ is larger than the size of the attractor, and we should observe a saturation.

The summation in the formula serves to eliminate fluctuations, and it is advisable to compare the results for different values of ϵ. The next figure shows the graph of $L_{2000}(s, \epsilon, x)$ as a function of s for the Hénon map of the plane

$$T(x, y) = \begin{pmatrix} 1 - 1.4\, x^2 + y \\ .3\, x \end{pmatrix} \, .$$

and an initial condition chosen at random according to the Lebesgue measure of the plane.

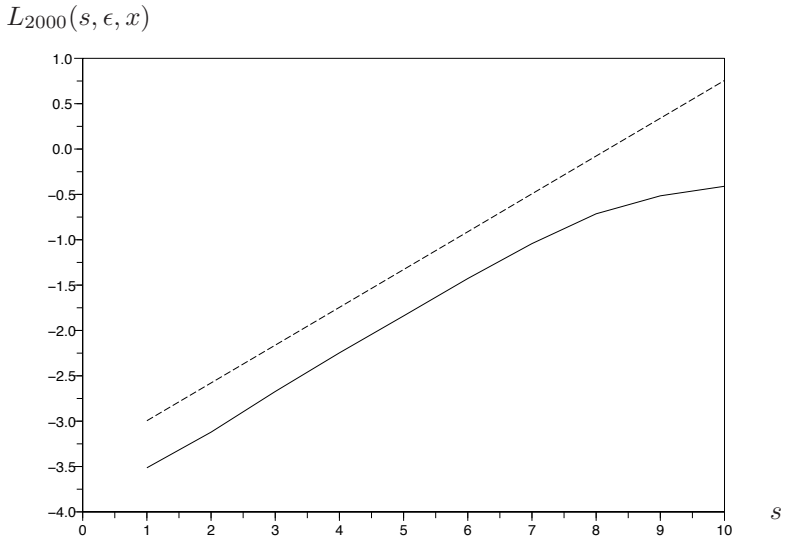


Figure 86: The function $s \mapsto L_{2000}(s, \epsilon, x)$ for the Hénon map (full line) and the theoretical reference slope (dash line).

We now discuss the general case where one would like to compute several (all) Lyapunov exponents. This is often numerically difficult, in particular if there are many exponents. One is often limited by the restricted length of the recording. Several methods have been proposed, we will briefly explain one of them based on the **QR** decomposition.

Recall that any real matrix M can be written as a product $M = QR$ with Q a real orthogonal matrix and R a real upper triangular matrix (see [86]). This leads to a numerically convenient recursive computation of the matrices $M_n = A_n \cdots A_1$. Assume we have the **QR** decomposition of the matrix M_n $(= A_n \cdots A_1)$, namely $M_n = Q_n R_n$ (Q_n orthogonal, R_n upper triangular). It follows from Oseledec's Theorem that the logarithms of the diagonal elements of R_n (the eigenvalues) divided by n converge to the Lyapunov exponents (we have $M_n^t M_n = R_n^t R_n$). Since

$$M_{n+1} = A_{n+1} M_n \ ,$$

we have

$$M_{n+1} = A_{n+1} Q_n R_n \ .$$

We perform now the **QR** decomposition of the matrix $A_{n+1} Q_n$, namely

$$A_{n+1} Q_n = Q_{n+1} W_n$$

with Q_{n+1} orthogonal and W_n upper triangular.

We then have

$$M_{n+1} = Q_{n+1} W_n R_n$$

and since $W_n R_n$ is upper triangular, we can take it as R_{n+1}, and obtain a **QR** decomposition of M_{n+1}.

In practice it is therefore enough to keep at each step the matrix Q_n and to accumulate the logarithm of the absolute values of the diagonal elements (eigenvalues) of the matrices W_n ($R_{n+1} = W_n R_n$). Note that since one accumulates logarithms instead of multiplying numbers, overflows (or underflows) are less likely. Since the matrix Q_n is orthogonal it has no risk of overflow.

This method is illustrated in the next figure for the Hénon map with the parameters $a = 1.4$, $b = .3$. The figure shows how the result evolves (hopefully converges) as a function of the number of iterations.

The top curve corresponds to the positive exponent, the bottom curve to the negative one.

In the case of the Hénon map, the determinant of the differential of the map is constant and equal to $-b$.

Therefore, the sum of the exponents should be equal to $\log|b|$. The sum of the computed exponents minus $\log|b|$ is the dash curve in the figure.

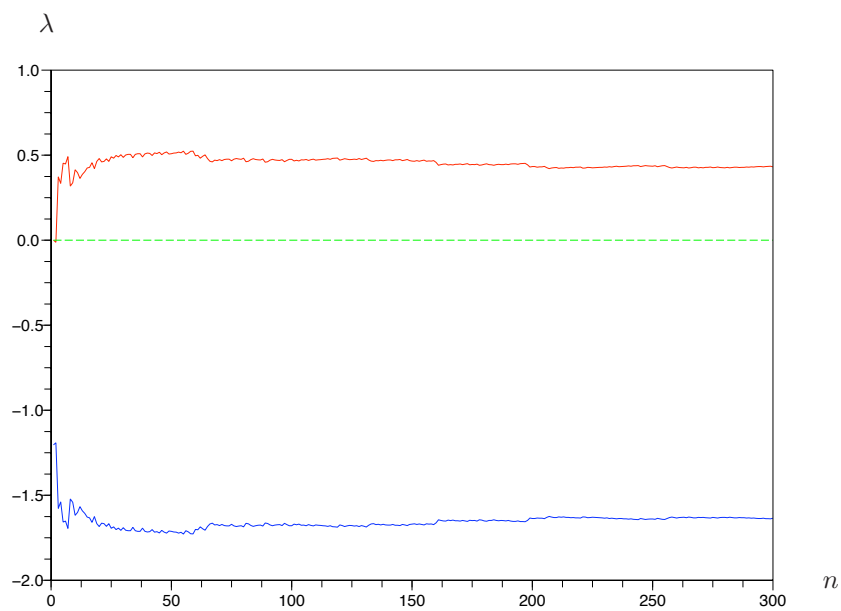


Figure 87: The two Lyapunov exponents for the Hénon map (solid lines) and their sum minus $\log|b|$, as a function of the number of iterations used in the algorithm.

Remark 3.

Some simple facts help sometime in the computation or in checking the numerical accuracy.

i) If the map is invertible, the inverse map has the opposite exponents (for the same measure).

ii) If the volume element of the phase space is preserved by the dynamics (Jacobian of the map is of modulus one, or the vector field is divergence free), the sum of the exponents is zero. This generalises immediately to the case of constant Jacobian as in the Hénon map.

iii) For symplectic maps, if λ is a Lyapunov exponent, $-\lambda$ is also a Lyapunov exponent.

For continuous time dynamical systems one can make a discrete time sampling and use the above technique. There is also a direct **QR** method for continuous time.

In practice, it often happens that the differential of the map DT_x is not known explicitly. In that case one can try to use Taylor's formula to obtain this differential. There are however two difficulties.

We already emphasized that we assumed we only have available one finite orbit $\mathcal{O}_n = \{z, T(z), \ldots, T^{n-1}(z)\}$. The other difficulty is that in practice the data are often corrupted by some noise (numerical or experimental).

In other words, instead of having a sequence $x_1, \ldots, x_n$ with $T\big(x_j\big) = x_{j+1}$, one observes (records) $y_1, \ldots, y_n$ with

$$y_j = x_j + \epsilon_j \ . \tag{65}$$

It is often assumed that the ϵ_j form a sequence of independent Gaussian random variables. This is a convenient assumption but not always verified.

The idea of one of the algorithm to estimate the differential is as follows, assuming first there is no noise.

Let $\mathcal{O}_n$ be an orbit of length n. Let x be a point where we want to estimate DT_x (x need not belong to $\mathcal{O}_n$). Let $\mathscr{U}_x$ be a small neighborhood of x. Assume we have a point x_l of $\mathcal{O}_n$ in $\mathscr{U}_x$, this means that x_l is near x. By Taylor's formula we have

$$\begin{aligned} x_{l+1} &= T\big(x_l\big) = T\big(x + (x_l - x)\big) \\ &= T(x) + DT_x\big(x_l - x\big) + \mathscr{O}\big(\|x_l - x\|^2\big) \ . \end{aligned}$$

Neglecting the quadratic term, this can be written

$$x_{l+1} = c + A\big(x_l - x\big) \tag{66}$$

where $c = T(x)$ is a vector in the phase space $\mathbb{R}^d$ and $A = DT_x$ is a $d \times d$ matrix with real entries.

If we know c (if x is in the orbit $\mathcal{O}_n$), we only need to recover A and therefore we need d^2 independent equations for the entries. These equations come from relations (66) ($x_{l+1} = c + A\big(x_l - x\big)$ recall the local picture 85) if we have at least d points of the orbit $\mathcal{O}_n$ inside the neighborhood $\mathscr{U}_x$, since each relation (66) leads to d linear equations (we may need more points if some of these equations are not independent, or numerically almost dependent).

If c is unknown (if x is not in the orbit $\mathcal{O}_n$), we can at the same time recover $c = T(x)$ and $A = DT_x$ but we need at least $d + 1$ points of the orbit inside the neighborhood $\mathscr{U}_x$.

In the presence of noise, we obtain from equation (65) ($y_j = x_j + \epsilon_j$)

$$y_{l+1} = \epsilon_{l+1} + T\big(x_l\big) = \epsilon_{l+1} + T\big(y_l - \epsilon_l\big)$$
$$= \epsilon_{l+1} + T(x) + DT_x\big(y_l - x\big) - DT_x\epsilon_l + \mathscr{O}\big(\|y_l - x\|^2\big) + \mathscr{O}\big(\|\epsilon_l\|^2\big).$$

Neglecting the quadratic terms, we have with the previous notations

$$y_{l+1} = c + A\big(y_l - x\big) + \epsilon_{l+1} - DT_x\epsilon_l \ .$$

We are now in the classical problem of estimating an affine transformation in the presence of noise. Note that from our assumptions, ϵ_{l+1} and $DT_x\epsilon_l$ are independent and Gaussian with zero average.

To solve this estimation problem, one can for example use a least square algorithm. In the case where $T(x)$ is known this leads to

$$A = \mathrm{argmin}_B \sum_{y_j \in \mathscr{U}_x} \big\|y_{j+1} - T(x) - B\big(y_j - x\big)\big\|^2 \ .$$

If $T(x)$ is not known, one can use the algorithm

$$(c, A) = \mathrm{argmin}_{(b,B)} \sum_{y_j \in \mathscr{U}_x} \big\|y_{j+1} - b - B\big(y_j - x\big)\big\|^2 \ ,$$

to estimate $T(x)$ and DT_x at the same time.

Remark 4.

i) This and related methods have been proposed to perform noise reduction since we recover the deterministic quantity $T(x)$ out of the noisy data $y_1, \ldots, y_n$.

ii) This algorithm performs a prediction of the future orbit of the initial condition x. Having predicted $T(x)$ one can repeat the algorithm at this new point. We refer to [97] for more on this method and references.

Remark 5.

The choice of the size of the neighborhood $\mathscr{U}_x$ is important and relates to two competing constraints.

- *The neighborhood $\mathscr{U}_x$ should be small enough so that quadratic terms are indeed negligible. Some authors have proposed to take the quadratic corrections into account but the algorithm becomes of course much more complex.*
- *If the neighborhood $\mathscr{U}_x$ is too large the non linearities are not negligible and fake exponents may appear. These spurious exponents are often multiples of other ones. If one observes this phenomenon of having an exponent multiple of another one, it is advisable to diminish the size of the neighborhood and see if this relation persists.*
- *If the neighborhood $\mathscr{U}_x$ is too small, the statistics in the least square may become poor, there may be even less than the d points needed to solve the problem in the deterministic case.*
- *In practice one should repeat the computation with neighborhoods of various sizes and compare the results.*
- *We refer to [139] for the theory and to [67] [74], [97], [158] and [159] for more details on the numerical implementation and discussion of the results.*

6.4 Reconstruction.

Contrary to the case of numerical simulations, when one deals with experimental data, the phase space is not known. For partial differential equations, for example, the phase space is often infinite dimensional but the dynamical system may have a finite dimensional attractor. As explained above, the system is observed through the time evolution of a real (or vector valued) observable, namely one has a recording of a real (or vector valued) sequence (61)

$$\big(X_n\big)_{0\le n\le N} = \big(g\big(T^n(x)\big)\big)_{0\le n\le N}.$$

The initial condition x is of course unavailable and the observation (also the dynamics itself) may be corrupted by noise ($y_j = T(x_j) + \epsilon_j$). What can be done in this apparently adverse context?

A reconstruction method has been developed by F.Takens which is based on the shift. One first fixes an integer d for the putative dimension of the space where the attractor and the dynamics will be reconstructed (we will discuss the best choice later on). From the data (61): $\big(X_n\big)_{0\le n\le N} = \big(g\big(T^n(x)\big)\big)_{0\le n\le N}$ one constructs a sequence of d dimensional vectors $Z_0, Z_1, \dots, Z_{N-d}$ defined by

$$Z_0 = \begin{pmatrix} X_{d-1} \\ \vdots \\ X_0 \end{pmatrix},\ Z_1 = \begin{pmatrix} X_d \\ \vdots \\ X_1 \end{pmatrix}, \cdots,\ Z_{N-d} = \begin{pmatrix} X_N \\ \vdots \\ X_{N-d+1} \end{pmatrix}. \tag{67}$$

In other words, we define a map Φ from the (unknown) phase space Ω to $\mathbb{R}^d$ by

$$\Phi(x) = \begin{pmatrix} g\big(T^{d-1}(x)\big) \\ \vdots \\ g(x) \end{pmatrix}. \tag{68}$$

If the observable g is regular, Φ is regular but in general not invertible. This map (semi) conjugates the time evolution T and the shift $\mathcal{S}$, namely $\mathcal{S}\circ\Phi = \Phi\circ T$. Indeed

$$Z_{n-1} = \begin{pmatrix} X_{n+d-2} \\ \vdots \\ X_{n-1} \end{pmatrix} \Phi\big(T^{n-1}(x)\big) \begin{pmatrix} g\big(T^{n+d-2}(x)\big) \\ \vdots \\ g\big(T^{n-1}(x)\big) \end{pmatrix}$$

and therefore

$$\mathcal{S}Z_{n-1} = \mathcal{S}\Phi\big(T^{n-1}(x)\big) = \begin{pmatrix} X_{n+d-1} \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} g\big(T^{n+d-1}(x)\big) \\ \vdots \\ g\big(T^{n}(x)\big) \end{pmatrix}$$

$$= \begin{pmatrix} g\big(T^{n+d-2}\big(T(x)\big)\big) \\ \vdots \\ g\big(T^{n-1}\big(T(x)\big)\big) \end{pmatrix} = \Phi\big(T\big(T^{n-1}(x)\big)\big)\ .$$

Do we obtain a faithful image of the dynamics by this method?

Remark 6. *One cannot expect to get information outside the attractor. Indeed, after a transient time (which may be quite short), the orbit of the initial condition x is very near to the attractor, and will stay near (approach) forever.*

There are well known results about the reconstruction of manifolds. A simplified formulation is as follows. Given a manifold H of dimension d, what is the minimal dimension m such that one can reconstruct accurately H in $\mathbb{R}^m$? More precisely, find a map Φ from H to $\mathbb{R}^m$ which is regular, injective and with the inverse map (defined only on the image) regular (and injective). This is called an embedding.

An important theorem of Whitney says that this is always possible if $m \geq 2d+1$, note that this number does not depend on n (see for example [84]).

Another important (and more difficult) theorem of Nash says that a compact (bounded) Riemannian manifold (i.e. a hyper-surface with a metric) can be embedded in $\mathbb{R}^m$ with its metric if $m \geq d(3d+11)/2$.

There is a result of Mané analogous to Whitney's theorem for the projections (which extends to fractal sets). If K is a set of dimension D (not necessarily integer), almost all projections over subspaces of (integer) dimension $m > 2D+1$ are injective (they separate the points). The difference with Whitney's theorem is that here we use projections, Whitney's theorem says there is an embedding. In the preceding example of a curve one gets $m = 4$

Unfortunately, Whitney's theorem, Nash theorem, Mané's theorem for Cantor sets etc. are not very useful in the present context because we require a mapping of the special form (68). Such special mappings have sometimes some unpleasant behaviours that should be avoided. For example if a and b are two fixed points in Ω (namely $T(a) = a$ and $T(b) = b$) such that $g(a) = g(b)$, then it follows immediately from the definition (68) that $\Phi(a) = \Phi(b)$, hence Φ is not injective. Similar anomalies occur for periodic orbits and this lead F.Takens to consider the generic case for g **and** T.

We now state the Takens reconstruction theorem.

Theorem 6.1. *For a generic transformation T with an attractor $\mathscr{A}$ of (Hausdorff) dimension D, and for a generic real valued observable g, the map Φ (68):*

$$\Phi(x) = \begin{pmatrix} g\big(T^{d-1}(x)\big) \\ \vdots \\ g(x) \end{pmatrix}$$

is an embedding of $\mathscr{A}$ in $\mathbb{R}^d$ if $d > 2D$.

Recall that an embedding Φ of a set M in a manifold M' is a homeomorphism between M and $\Phi(M)$ such that $D\Phi$ is invertible at each point of M.

Remark 7.

- *We will see later on that attractors may have non integer dimension.*
- *In the original work generic was in the sense of Baire's second category (countable intersection of open dense sets).*
- *The same result has been proved for other genericity conditions for example of probabilistic nature (this has to be defined with some care because the spaces of maps and observables are in general infinite dimensional see [143]).*

- *The genricity conditions avoid the problems we have mentioned above for the maps of the form (68). One can list a number of conditions so that Theorem 6.1 holds, they are however hard to check in practice. Theorem 6.1 states that these conditions are generic.*

- *Recall that symmetry properties for example of the observable g may break genericity.*

In practice, one has no way to check the genericity assumption, so one assumes it! Also one does not know a-priori the dimension of the attractor, and therefore the dimension d of the reconstruction space. One tries several values for d and apply various criteria to select an adequate one.

One of these criteria is the false neighbors criteria. If d is too small, then by accident two far away points a and b of the phase space may be mapped by Φ to two nearby points. However this accidental effect should not persist for

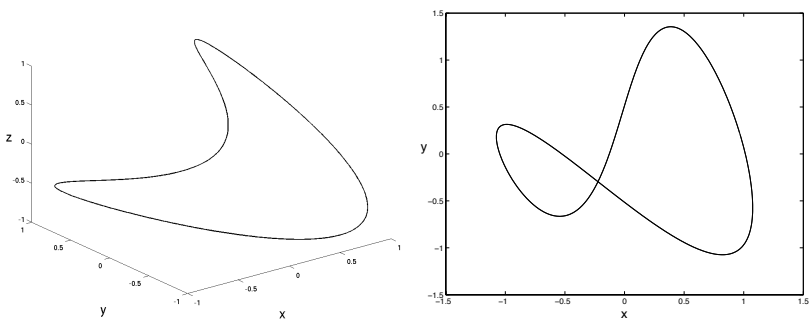


Figure 88: A curve non self-intersecting whose projection has a double point.

the iterates by the map T of the points a and b. At least some of the first few iterates should be reasonably far away one from the other.

In practice, one selects a number $\delta > 0$ much smaller than the size of the image of the attractor, and considers all the pairs of points at a distance less than δ. For each such pair, one looks at the image pair, and if for one pair one gets a large distance, then the reconstruction dimension is suspected to be too small.

Remark 8. *Note that by continuity of Φ and T, if we consider two nearby initial conditions, their first few iterates should be close.*

Therefore if we have a pair of points at distance less than δ, the images should not be far away.

If they are we have the above problem of a too small embedding dimension.

In principle there is no problem in using a large reconstruction dimension. However the statistics of the different measurements will be poorer.

We will see later on when discussing the measurement of dimension another criteria to select a reconstruction dimension. It is wise to use as many criteria as available (see also the Kaplan-Yorke formula later). We refer to [151], [143], [103] and [148] for proofs and references and to [75], citeks fore more on the implementation of these techniques.

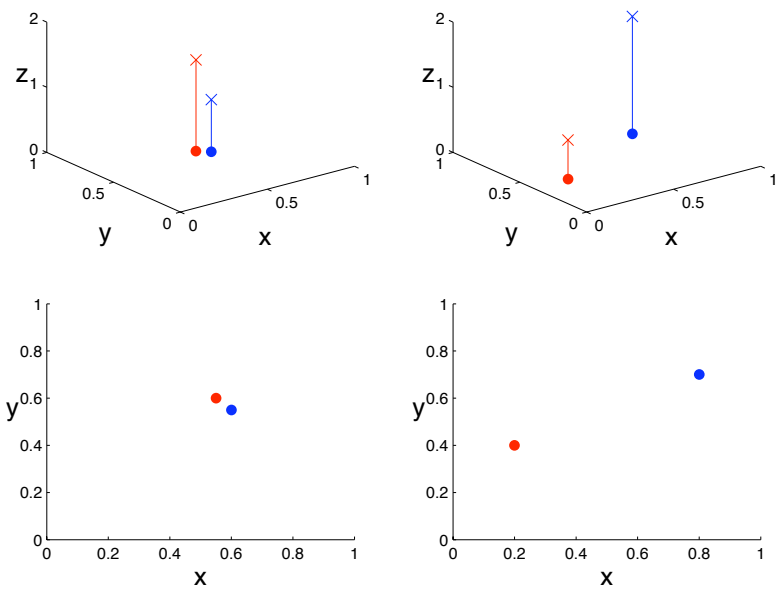


Figure 89: Two (not so close) points whose projection on the $x - y$ plane are close but the images are separated even in projection.

For partial differential equations, there is a reconstruction involving points in the (physical) space. Consider a non linear PDE in a bounded space domain $\mathcal{D}$. For example

$$\partial_t A = \Delta A + A - A\big(|A|^2 - 1\big)$$

with A complex, and some boundary condition on $\partial\mathcal{D}$. The phase space Ω is infinite dimensional. For example the space of regular functions (with square summable Laplacian). There is a well defined semi-flow (but going backward in time is mostly explosive). One can prove that there is a finite dimensional attracting set (in the infinite dimensional phase space). The dimension of this attracting set is proportional to the size of the domain $\mathcal{D}$ (provided there is some uniformity on the boundary condition). Moreover there is an inertial manifold. This is an invariant manifold (hyper-surface) of finite (integer) dimension which attracts transversally all the orbits. In particular it contains the attracting set (see [156]). Points of the attractor can be distinguished by a (fine enough) discrete set in $\mathcal{D}$ (see [90]). In other words, there is a number $\eta > 0$, such that if a set of points M in $\mathcal{D}$ does not leave any hole of size η, then if two functions A_1 and A_2 of the attractor are equal at each point of M, they are equal everywhere in $\mathcal{D}$. This result is known under certain conditions on the equation and for regular domains. It is still an open problem for the Navier Stokes equation, even in dimension two. The number η depends on the equation, and in particular of the size of the coefficients. The cardinality of M is proportional to the volume.

6.5 Measuring the Lyapunov exponents.

One applies first the technique explained above to reconstruct the system. Note that because of the special form of the reconstruction, the tangent (reconstructed) map has a special form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_d \end{pmatrix}$$

Remark 9.

- *We have seen that fake exponents may appear as multiple of true exponents if the radius of the neighborhood used to compute the differential of the map ,is too large so that nonlinearities are not negligible.*
- *In the reconstruction, fake exponents can also appear due to the numerical noise in the transverse directions of the reconstructed attractor.*
- *The reconstruction dimension is often larger than the dimension of the attractor, and the negative Lyapunov exponents are polluted by noise (also fake zero exponents may appear).*
- *These fake exponents are in general unstable by time inversion (if the map is invertible one can look for the exponents of the inverse). Also it is wise to explore changes in the various parameters of the algorithms.*
- *It is in general difficult to determine several exponents, especially the negative ones.*
- *For the positive exponents, one can try to reduce the dimension to the number of positive exponents. For this purpose, one can use only some components of the reconstruction. For example if one uses a reconstruction dimension $d = pq$ with p and q integers, one can consider the subspaces $\big(g(x), g(T^{q-1}(x)), \cdots, g(T^{pq-1}(x))\big)$, when trying to measure the p largest exponents.*

We refer to [2], [97] and [144] for more results and references.

6.6 Dimensions.

Because of the sensitive dependence on initial condition and the compactness of the attractors, these sets have often a complicated (fractal) structure. In that case they are called strange attractors. They often are locally the product of a Cantor set (in the stable direction) by regular leaves (in the unstable direction).

One way to capture (measure) the complexity of the geometry of the (strange) attractors is to determine their dimension.

There are many definitions for the dimension of a set. Two are most often used in the study of dynamical systems: the Hausdorff dimension d_{H} and the box counting dimension d_{Box} (also called capacity). They are not equivalent,

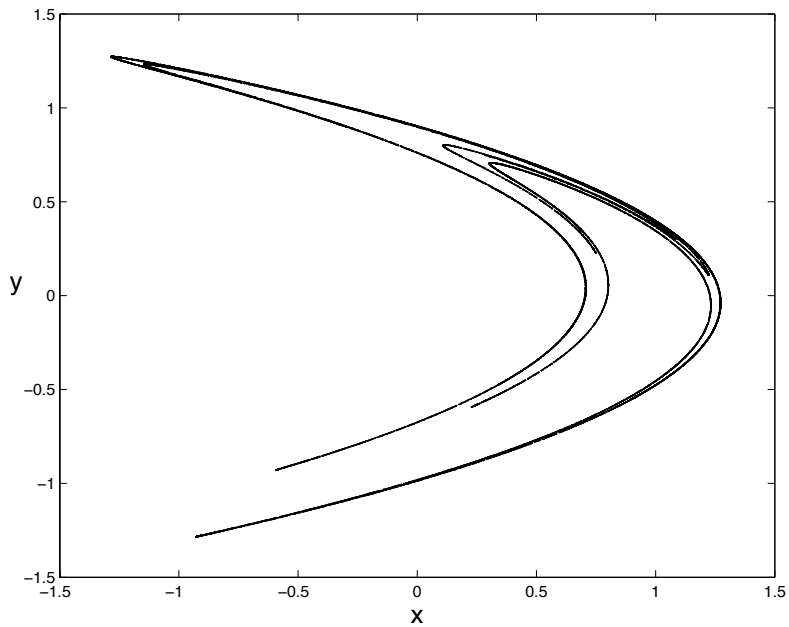


Figure 90: The attractor for the Hénon map.

the Hausdorff dimension is more satisfactory from a theoretical point of view, the box counting dimension (or box dimension) is easier to compute numerically.

Recall (see [95], [71] or [124]) that the Hausdorff dimension of a set A is defined as follows. Let $B_r(x)$ denote the ball of radius r centered in x. Let $B_{r_j}\left(x_j\right)$ be a sequence of balls covering A, namely

$$A \subset \bigcup_j B_{r_j}\left(x_j\right) .$$

For a numbers $d > 0$ and $\epsilon > 0$, define

$$\mathcal{H}_d(\epsilon, A) = \inf_{A \subset \cup_j B_{r_j}(x_j)\,,\, \sup_j r_j \leq \epsilon} \sum_j r_j^d . \tag{69}$$

This is obviously a non increasing function of ϵ which may diverge when ϵ tends to zero. Moreover, if the limit when $\epsilon \searrow 0$ is finite for some d, it is equal to zero for any larger d. This limit is also non increasing in d. Moreover, if it is finite and non zero for some d, it is infinite for any smaller d. The Hausdorff dimension of A, denoted below by $d_{\mathrm{H}}(A)$ is defined as the infimum of the positive numbers d such that the limit vanishes (for this special d the limit may be zero, infinite, finite or does not exist), namely

$$d_{\mathrm{H}}(A) = \inf \left\{ d \,\middle|\, \lim_{\epsilon \to 0} \mathcal{H}_d(\epsilon, A) = 0 \right\}$$

This is also the supremum of the set of d positive numbers such that the limit is infinite.

The box counting dimension of a subset A of $\mathbb{R}^n$ is defined as follows. For a positive number r, let $N_A(r)$ be the smallest number of balls of radius r needed to cover A. If for (all) small r we have $N_A(r) \approx r^{-d}$, we say that the box dimension of A is d, and we denote it by $d_{\mathrm{Box}}(A)$.

Exercise 69. *Show that a line has box dimension one, a piece of regular surface has box dimension two etc.*

How to measure the dimension of an attractors from our data? A difficulty is that in general one does not have access to the whole attractor. As explained at the beginning of this chapter, it is often the case that one knows only one trajectory which is not necessarily uniformly spread over all the attractor, even though it may be dense. We will however assume that this trajectory is typical for a certain measure μ (for example a Physical measure), and this leads to the definition of the dimension of a probability measure, which is used instead of the notion of dimension of a set.

Definition 23. *The dimension of a (probability) measure is the infimum of the dimensions of the (measurable) sets of measure one.*

Remark 10. *In practice one should of course say which dimension is used (Hausdorff, box counting etc.) Note also that the infimum may not be attained. If a measure is supported by an attractor, its dimension is of course at most the dimension of the attractor (the dimension of a measure is at most the dimension of its support). This inequality may be strict.*

The dimension of a measure is somehow related with the behaviour of the measure of small balls as a function of the diameter. We start by giving a somewhat informal argument. Let μ be a measure, and let $d_{\mathrm{Box}}(\mu)$ be its box counting dimension. This roughly means that we need about $N_A(r) \approx r^{-d_{\mathrm{Box}}(\mu)}$ balls of radius r to cover (optimally) the "smallest" set A of full measure. Let $B_1,\ B_2, \dots,\ B_{N_A(r)}$ be such balls. In order to have as few balls as possible, they should be almost disjoint, and to simplify the argument we will indeed assume that they are disjoint. Since the set A is of full measure we have

$$1 = \mu(A) = \mu\left(\bigcup_{j=1}^{N_A(r)} B_j\right) = \sum_{j=1}^{N_A(r)} \mu\big(B_j\big)\ .$$

In the simplest homogeneous case, all the balls have the same measure, and we get for $1 \le j \le N_A(r)$

$$\mu\big(B_j\big) = \frac{1}{N_A(r)} \approx r^{d_{\mathrm{Box}}(\mu)}\ .$$

A rigorous argument along these lines leads to a lower bound for the Hausdorff dimension (Frostman lemma).

Lemma 6.2. *Assume there are two constants $C > 0$ and $\delta > 0$ such that a probability measure μ on $\mathbf{R}^n$ satisfies for any $x \in \mathbf{R}^n$ and any $r > 0$ the inequality*

$$\mu\left(B_r(x)\right) \le C r^{\delta}\ .$$

Then

$$d_{\mathrm{H}}(\mu) \ge \delta\ .$$

Proof. Let A be a set of measure one, and let $B_{r_j}(x_j)$ be any sequence of balls covering A. Since

$$A \subset \bigcup_j B_{r_j}(x_j) \ .$$

we have (since the balls are not necessarily disjoint)

$$1 = \mu(A) \leq \sum_j \mu\big(B_{r_j}(x_j)\big) \ .$$

Therefore from the assumption of the theorem, we have

$$1 \leq C \sum_j r_j^\delta \ .$$

In other words, for any $\epsilon > 0$ and any set A of full measure we have

$$\mathcal{H}_\delta(\epsilon, A) \geq \frac{1}{C} \ .$$

The theorem follows immediately from the definition of the Hausdorff dimension. □

We refer to [95] for more on this so called Frostman lemma, and in particular for the converse.

It is natural to ask if there are relations between the dimension of a measure and the other quantities like entropy and Lyapunov exponents. In dimension two, L.S.Young proved an important relation.

Theorem 6.3. *For a regular invertible map of a two dimensional compact manifold and an ergodic invariant probability measure μ with two exponents satisfying $\lambda_1 > 0 > \lambda_2$, we have*

$$d_{\mathrm{H}}(\mu) = h(\mu) \left(\frac{1}{\lambda_1} + \frac{1}{|\lambda_2|} \right) \ .$$

In the case of an SRB measure, we have under the hypothesis of the above Theorem and using Theorem 5.9, $h(\mu) = \lambda_1$, hence

$$d_{\mathrm{H}}(\mu) = 1 + \frac{\lambda_1}{|\lambda_2|} \ . \tag{70}$$

In the particular case of the Hénon map, for the "good" parameter values, and for the Physical measure, $h(\mu) = \lambda_1$, and since $\lambda_1 + \lambda_2 = \log b$, we get

$$d_{\mathrm{H}}(\mu) = 1 + \frac{\lambda_1}{|\log b - \lambda_1|} \ .$$

We refer to [164] for a proof of the Theorem, but we now give an intuitive argument for this result. From the Katok result 5.4, we have to count the maximal number of typical trajectories ϵ distinct on a time interval of length k. For that purpose, we will make the counting at an intermediate time $0 < k_1 < k$ (k_1 optimally chosen later). At time k_1, we cover a set of positive measure by

balls of radius $\epsilon' \ll \epsilon$ (ϵ' will be optimally chosen below). From k_1 to k the balls of radius ϵ' are stretched. Therefore

$$\epsilon = \epsilon' e^{k_2 \lambda_1}$$

where $k_2 = k - k_1$ (one looks at the worse case: separation takes place at the last time k).

From k_1 to 0, the balls of radius ϵ' are stretched by the inverse map, hence

$$\epsilon = \epsilon' e^{-k_1 \lambda_2} \ .$$

Therefore

$$\epsilon = \epsilon' e^{-k_1 \lambda_2} = \epsilon' e^{k_2 \lambda_1}$$

which implies $k_1 = k\lambda_1/\big(\lambda_1 - \lambda_2\big)$, from which we derive ϵ'. However the number of trajectories we are looking for is the minimal number of balls of radius ϵ' necessary to make a set of (almost) full measure. Since the balls have measure ${\epsilon'}^{d_{\mathrm{Box}}(\mu)}$, we need ${\epsilon'}^{-d_{\mathrm{Box}}(\mu)}$ of them. The maximum number of pairwise ϵ separated trajectories is therefore

$${\epsilon'}^{-d_{\mathrm{Box}}(\mu} = \epsilon^{-d} e^{d_{\mathrm{Box}}(\mu) k_2 \lambda_1} \approx e^{kh(\mu)} \ ,$$

and therefore

$$h(\mu) = \frac{d_{\mathrm{Box}}(\mu)\ k_2\ \lambda_1}{k} = \frac{d_{\mathrm{Box}}(\mu)\ \lambda_1\ \big|\lambda_2\big|}{\lambda_1 + \big|\lambda_2\big|} \ .$$

Another relation between dimension and Lyapunov exponents is the so called Kaplan-Yorke formula. Let $\lambda_0 \geq \lambda_2 \geq \ldots \geq \lambda_d$ be the decreasing sequence of Lyapunov exponents with multiplicity (in other words the number of terms is equal to the dimension of the phase space). Let

$$k = \max\big\{\, i \,\big|\, \lambda_0 + \ldots + \lambda_i \geq 0 \big\} \ .$$

Kaplan and Yorke defined the Lyapunov dimension by

$$d_{\mathrm{L}}(\mu) = k + \frac{\big|\lambda_{k+1}\big|}{\lambda_0 + \ldots + \lambda_k} \ .$$

Kaplan and Yorke conjectured that $d_{\mathrm{L}}(\mu) = d_{\mathrm{H}}(\mu)$ (the now called Kaplan Yorke formula) but counter examples were discovered later on. Note however that in the hypothesis of Theorem 6.3 (namely diffeomorphisms of surfaces) the equality holds. In the general case, Ledrappier proved the following inequality.

Theorem 6.4. *For any ergodic invariant measure μ we have $d_{\mathrm{H}}(\mu) \leq d_{\mathrm{L}}(\mu)$.*

We now give a rough idea for the "proof" of the Kaplan-Yorke formula using the box dimension. Take a ball B of radius ϵ (small) in the attractor. Its measure is about $\approx \epsilon^{d_{\mathrm{Box}}}(\mu)$. If one iterates this ball k times, (ϵ small, k not too large), one obtains an ellipsoid $\mathcal{E} = T^k(B)$, elongated in the unstable directions, contracted in the stable directions. We now cover this ellipsoid $\mathcal{E}$ by balls of radius

$$\epsilon' = e^{k\lambda_j} \epsilon$$

with j chosen (optimally) later on but such that $\lambda_j < 0$. A ball of radius ϵ' has

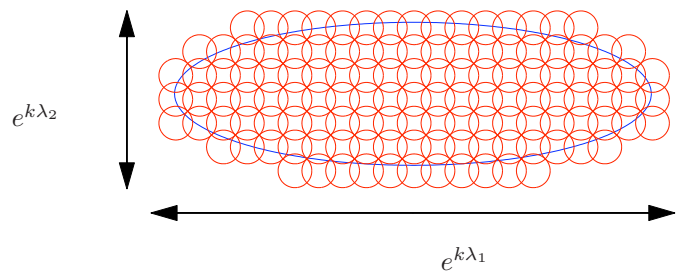


Figure 91: Covering $T^k(B)$ by balls of radius ϵ'. Only expanded directions are shown.

measure $\sim \epsilon'^{d_{\rm Box}(\mu}$), and since the measure μ is invariant, we get

$$\epsilon^{d_{\rm Box}(\mu)} = \mu\big(B\big) = \mu\big(\mathcal{E}\big) = N\ \epsilon'^{d_{\rm Box}(\mu)}$$

where N is the number of balls of radius ϵ' necessary to cover the ellipsoid $\mathcal{E} = T^k(B)$.

We now evaluate N. In the direction corresponding to the exponent λ_0, the original ball is stretched ($\lambda_0 > 0$), and its size becomes $\epsilon \exp(\lambda_0 k)$. We need therefore

$$\frac{\epsilon e^{\lambda_0 k}}{\epsilon'}$$

balls to cover $\mathcal{E}$ in that direction. The same argument holds the in the directions corresponding to any exponent λ_l with $l < j$ (with multiplicities). In the direction corresponding to the exponent λ_j one needs only one ball, which also covers in all directions ($l > j$) since they are contracted.

Summarizing, since

$$N = \prod_{l=0}^{j-1} \frac{\epsilon e^{\lambda_l k}}{\epsilon'} = \left(\frac{\epsilon}{\epsilon'}\right)^j e^{kS_j} = e^{-k\,j\,\lambda_j} e^{kS_j}\ ,$$

where $S_p = \lambda_0 + \cdots + \lambda_{p-1}$. Using $\epsilon^{d_{\rm Box}(\mu)} = N\epsilon'^{d_{\rm Box}(\mu)}$, we get

$$\epsilon^{d_{\rm Box}(\mu)} = N\epsilon^{d_{\rm Box}(\mu)} e^{k\lambda_j d_{\rm Box}(\mu)} = \epsilon^{d_{\rm Box}(\mu)} e^{kS_j} e^{-k\,j\,\lambda_j} e^{k\lambda_j d_{\rm Box}(\mu)}$$

This must be true for any integer k, hence

$$S_j + \lambda_j (d_{\rm Box}(\mu) - j) = 0\ .$$

It seems that we have only one equation for two unknowns j and $d_{\rm Box}(\mu)$.

Nevertheless there is only one solution! To see this it is convenient to plot the piecewise linear interpolation of S_p as a function of p. Between $p = k+1$ and $p = k+2$ the linear interpolation is $S_{k+1} + \lambda_{k+1}(x-k-1)$. $d_{\rm L}$ is the unique zero of this function, as seen in the next figure

We now come back to the question of measuring the dimension of an ergodic invariant measure μ. The idea is the following. Assume that for most points x in the support of the measure

$$\mu\big(B_r(x)\big) \approx r^{{\rm D}(\mu)}\ ,$$

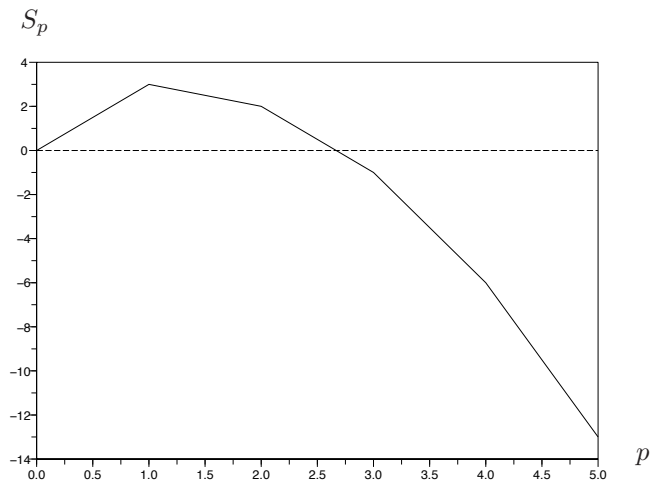


Figure 92: Piecewise linear interpolation of the numbers S_p as a function of p.

then obviously

$$\mathrm{D}(\mu) \approx \frac{\log \mu\big(B_r(x)\big)}{\log r} \ .$$

Remark 11. *There are cases where the above assumption fails, see [115].*

In order to increase the statistics, a better formula would be something like

$$\mathrm{D}(\mu) = \lim_{r \searrow 0} \int \frac{\log \mu\big(B_r(x)\big)}{\log r} d\mu(x) \ .$$

Numerical simulations performed on several systems suggested that this quantity converges slowly to its limit. Moreover this quantity is not easy to compute from a time series (one finite orbit). For these reasons people have adopted another definition. Consider the function of $r > 0$

$$C_2(r) = \int \mu\big(B_r(x)\big) \ d\mu(x) \ . \tag{71}$$

The first guess is that this quantity should again behave roughly as $r^{\mathrm{D}(\mu)}$, since the integrand does. However, it turns out that this quantity behaves like $r^{d_2(\mu)}$ with $d_2(\mu) \leq \mathrm{D}(\mu)$, and often the inequality is strict. This is because of Jensen's inequality (the logarithm of the integral is in general larger than the integral of the logarithm).

The number $d_2(\mu)$ is called the correlation dimension of the measure μ. The quantity $C_2(r)$ is observed to be statistically rather stable and easy to compute from a numerical time series as we will see now.

Assume that we have a trajectory in the phase space $x_1, x_2, \ldots$ (an orbit). We will apply twice the ergodic theorem to obtain a (asymptotic) formula for

the function $C_2(r)$. By the Birkhoff ergodic theorem we have almost surely

$$\int \mu\big(B_r(x)\big)\, d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mu\big(B_r(x_j)\big) \ .$$

We can now use again the ergodic theorem to compute $\mu\big(B_r(x_j)\big)$. Let θ denote the Heaviside function on $\mathbb{R}$ (the function equal to one on the positive real numbers and equal to zero on the negative ones). We have

$$\mu\big(B_r(x_j)\big) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \theta\big(r - d(x_k, x_j)\big)$$

where d is the distance on the phase space.

We can now put these two formulas together and obtain the following algorithm

$$C_2(r) = \int \mu\big(B_r(x)\big)\, d\mu(x) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{p=1,\, q=1}^{p=N,\, q=N} \theta\big(r - d(x_p, x_q)\big) \ . \tag{72}$$

We mention a rigorous result in the context of the above formal manipulation.

Theorem 6.5. *For values of r where $C_2(r)$ is continuous, we have μ almost surely*

$$C_2(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{p=1,\, q=1}^{p=N,\, q=N} \theta\big(r - d(x_p, x_q)\big) \ .$$

Note that the function $C_2(r)$ is a non decreasing function of r. Therefore it is continuous except in at most a countable number of points. We refer to [121] and [145] for a proof.

We now discuss some of the the practical issues in using the above formula (72)

$$r^{d_2(\mu)} \sim C_2(r) \sim \frac{1}{N^2} \sum_{p=1,\, q=1}^{p=N,\, q=N} \theta\big(r - d(x_p, x_q)\big) \tag{73}$$

to measure the dimension. To give a first idea, we show in the next figure the function $C_2(r)$ for the Hénon map in log-log scale where a power law $C_2(r) \sim r^{d_2(\mu)}$ should appear as a straight line. There are several practical difficulties in measuring $d_2(\mu)$. Strong local correlations may occur.

If one samples a flow $x_j = \phi_{j\tau}\big(x_0\big)$. If τ is too small, by continuity there is a strong correlation between x_j and x_{j+1}. To avoid these problems, the summation in 73 is extended only on those p and q with $|p - q| > K$, K large enough, of order $1/s$ where s is an estimate of the decorrelation time. This time scale can be obtained by looking at the signal (for example the change of side in the Lorenz system), by looking at the decay of correlation function or by other means.

Another difficulty is the presence of noise. A white noise corresponds to an attractor of infinite dimension. If the noise has amplitude ϵ, its influence will only be noticeable if r is of the order ϵ or smaller. Above this value, one expects that the noise averages out. One can hope for a power law behaviour of $C_2(r)$ only for a range of values of r which must give the true dimension. r must be

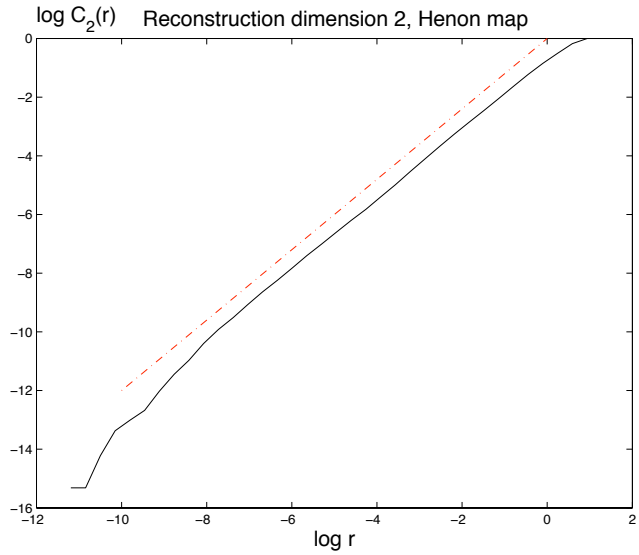


Figure 93: The function $C_2(r)$ for the Hénon map in log-log plot (black line). The red line is a reference line with the "true" slope.

- small enough for the (dynamical) fluctuations of $\log \mu\big(B_r(x)\big)/\log r$ to be small,
- large enough to be above the noise.

One may also use other kernels than the Heaviside function, like for example a Gaussian.

From a practical point of view, one looks in the graph of $C_2(r)$ for a range of intermediate values of r where this function looks like a power of r. Logarithmic scales are convenient in a first approach.

For some "good" systems there is a theoretical bound on the fluctuations of

$$C_2(r,N) = \frac{1}{N^2} \sum_{p=1,\,q=1}^{p=N,\,q=N} \theta\big(r - d(x_p, x_q)\big) \ .$$

[32] In the general case one expects worse results. For some systems where exponential or Devroye estimates are available, the standard deviation of $C_2(r,N)$ can be bounded by $N^{-1/2}r^{-1}$. Using this estimate, if one wants to see the power law behaviour above the dynamical fluctuations, one needs

$$r^{d_2(\mu)} \gg \frac{1}{N^{1/2}r} \ .$$

That is to say

$$r \gg \frac{1}{N^{1/(2d_2(\mu)+2)}} \qquad \text{or} \qquad N \gg \frac{1}{r^{2d_2(\mu)+2}}$$

If one has an a-priori idea on the value of $d_2(\mu)$, one can guess a value for N. For the Hénon map this gives $r \gg N^{-.21}$, which leads to $N \sim 60000$ for $r > .1$.

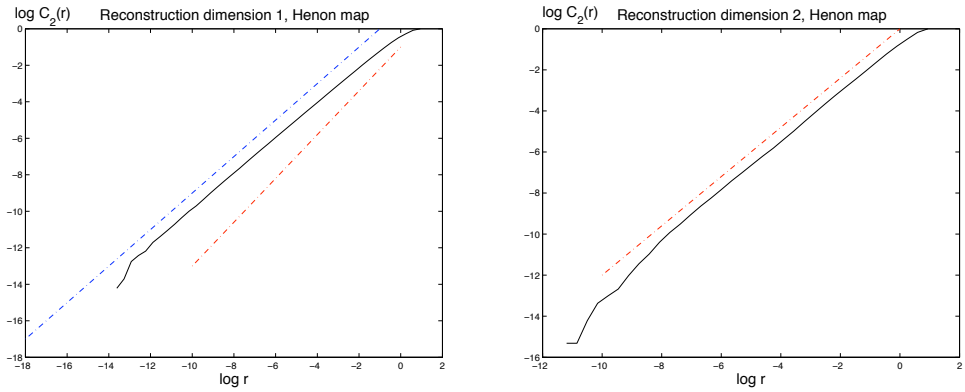


Figure 94: The function $C_2(r)$ for the x component of the Hénon map and reconstruction dimensions 1 and 2. In the left figure the result is the black curve, the blue line is a reference line of slope 1, the red line a reference line of slope the "true" dimension. In the left figure the result is the black curve, the red line is a reference line of slope the "true" dimension.

These bounds are often pessimistic (they are for the worse cases). But they may provide a helpful order of magnitude. We refer to [19], [20], [22], [55], [101], [102], [149], [150], [153], [154], [126] for more results and references.

Except in numerical simulations, one works with reconstructed attractors. It is wise to look at the results for different values of the reconstruction dimension. If the observed dimension (slope of the log-log plot of $C_2(r)$) is equal to the reconstruction dimension, this means that the dimension of the attractor has not yet been reached. One should increase the reconstruction dimension. This is illustrated in figure 95.

It is useful to plot the measured dimension as a function of the reconstruction dimension to look for a saturation when a reconstruction dimension is large enough to obtain an embedding. Also one should verify the coherence of the different estimates of the reconstruction dimension: false neighbors etc.

The influence of the noise at small r is clearly visible in the next figure 96 which shows the function $C_2(r)$ for the x component of the Hénon map and several reconstruction dimensions with noise of standard deviation .05. The "true" slope is indicated in red.

Estimating the slope of the function $C_2(r)$ is not an easy task, especially if the power law is only valid on a restricted range (limited by dynamical fluctuations at small $-\log r$ and by noise at large $-\log r$).

Several statistical works have dealt with this problem using linear interpolation, maximum likelihood, derivative estimates etc.

Here is one of the proposed formula (with no claim that it is universally better than others).

$$d_2(\mu) = \frac{-N_s}{\sum_{i=1}^{N_s} \log(r_i) + N_l \log r_l}$$

where the power law range is $[r_l, r_u]$, and it contains N_s points $r_1, \cdots, r_{N_s}$, while

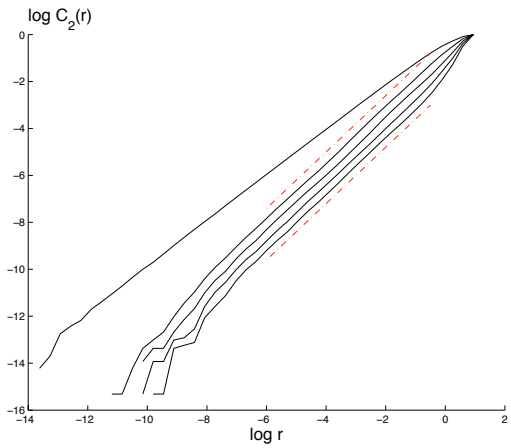


Figure 95: The function $C_2(r)$ (log-log plot) for the x component of the Hénon map and reconstruction dimensions up to 5. The "true" slope is indicated in red.

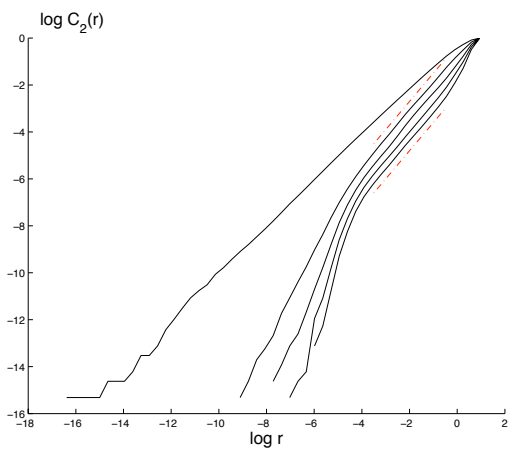


Figure 96: The function $C_2(r)$ (log-log plot) for the x component of the Hénon map and five reconstruction dimensions with noise of standard deviation .05.

the range $[0, r_l]$ contains N_l points.

Statistical studies of this estimator or of similar ones allow to construct confidence intervals.

6.7 Entropy

To measure the Kolmogorov-Sinai entropy one uses the Brin-Katok formula and the ergodic theorem to compute the measure. One gets

$$h_\mu(T) = \lim_{\epsilon \searrow 0} \liminf_{n \to \infty}$$

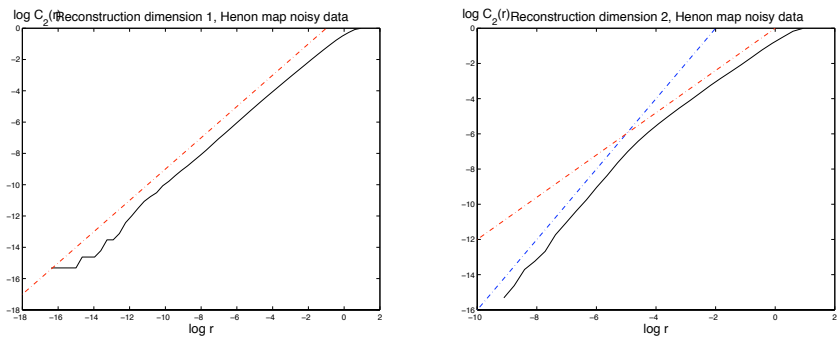


Figure 97: The function $C_2(r)$ (log-log plot) for the x component of the Hénon map and reconstruction dimensions 1 and 2 with noise of standard deviation .05. On the left (reconstruction dimension one), the result is the black curve, the red line is a reference line with slope 1. On the right (reconstruction dimension two), the result is the black curve, the red line is a reference line with slope the "true" dimension, the blue line is a reference line of slope two.

$$-\frac{1}{n}\log\left(\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\prod_{l=0}^{n-1}\theta\big(\epsilon-d\big(T^l(x),T^{l+j}(x)\big)\big)\right) .$$

This formula is analogous to the formula used to compute the correlation dimension and similar algorithms are used in practice.

6.8 Multifractal

Even if the functions $x \to \log\mu\big(B_r(x)\big)/\log r$ converge almost surely when r tends to zero, this quantity can show for non zero r interesting (and even measurable) fluctuations (recall that $B_r(x)$ is the ball centered in x of radius r). The study of these fluctuations is called the multifractal analysis. More precisely, for any $\alpha \geq 0$, let

$$\mathcal{E}_\alpha^+ = \left\{ x \,\middle|\, \limsup_{r\to 0}\frac{\log\mu\big(B_r(x)\big)}{\log r} = \alpha \right\} ,$$

and

$$\mathcal{E}_\alpha^- = \left\{ x \,\middle|\, \liminf_{r\to 0}\frac{\log\mu\big(B_r(x)\big)}{\log r} = \alpha \right\} .$$

One would like to characterise the "size" of these sets. This can be done for example using another measure (or family of measures), or using a dimension. The so called multifractal spectrum is usually defined a the function $\alpha \to d_{\mathrm{H}}\big(\mathcal{E}_\alpha^+\big)$ (or $\alpha \to d_{\mathrm{H}}\big(\mathcal{E}_\alpha^+\big)$).

Under suitable hypothesis, one can relate these quantities to some large deviation functions. We refer to [46], [66], [136], [161] [152], [157] for more results and references. We will explain here how this works on a simple example. Consider the dissipative Baker map (2). As we have seen in exercise 15, the

attractor is the product of a segment by a Cantor set $\mathcal{K}$. This Cantor set can be constructed as follows. Consider the two contractions f_1 and f_2 of the interval given by

$$f_1(y) = \frac{y}{4} \quad \text{and} \quad f_2(y) = \frac{2+y}{3} \ .$$

The Cantor set is obtained by applying all the infinite products of f_1 and f_2 to the interval. Namely, one defines for each n a set $\big(\mathcal{K}_n\big)$ consisting of 2^n disjoint intervals, and constructed recursively as follows. First $\mathcal{K}_0 = [0,1]$, and then for any $n \geq 1$,

$$\mathcal{K}_{n+1} = \big\{ f_1\big(I\big) \,\big|\, I \in \mathcal{K}_n \big\} \cup \big\{ f_2\big(I\big) \,\big|\, I \in \mathcal{K}_n \big\} \ . \tag{74}$$

The transverse measure ν of the SRB measure (see formula 60) satisfies $\nu(I) = 2^{-n}$ for any $I \in \mathcal{K}_n$.

Exercise 70. *Show that this construction uniquely defines the measure ν.*

Exercise 71. *Consider the map f of the unit interval defined by*

$$f(x) = \begin{cases} 4x & \text{for } 0 \leq x \leq 1/4 \ , \\ 12(x-1/4)/5 & \text{for } 1/4 < x < 2/3 \ , \\ 3(x-2/3) & \text{for } 2/3 \leq x \leq 1 \ . \end{cases}$$

Show that the Cantor set $\mathcal{K}$ is invariant by f. Show that f_1 and f_2 are two inverse branches of f. Show that ν is an ergodic invariant measure for this map (one can use a coding).

We will need a large deviation estimate for the sizes of the intervals in each $\mathcal{K}_n$. In the present situation, this is easy since the maps f_1 and f_2 have constant slope. Indeed, we have for any integer $0 \leq p \leq n$

$$\Big| \Big\{ I \in \mathcal{K}_n \,\Big|\, |I| = 4^{-p}\, 3^{p-n} \Big\} \Big| = \begin{pmatrix} n \\ p \end{pmatrix} \ .$$

For later purposes it is convenient to use the Stirling approximation. Let φ be the function defined for $s \in [-\log 4, -\log 3]$ by

$$\varphi(s) = \frac{s + \log_2 3}{\log_2(4/3)} \log_2 \left(-\frac{s + \log_2 3}{\log_2(4/3)} \right) - \frac{s + \log_2 4}{\log_2(4/3)} \left(\frac{s + \log_2 4}{\log_2(4/3)} \right) \ ,$$

then for $s \in [-\log_2 4, -\log_2 3]$

$$\Big| \Big\{ I \in \mathcal{K}_n \,\Big|\, |I| \approx 2^{ns} \Big\} \Big| \approx 2^{n\,\varphi(s)} \ . \tag{75}$$

Exercise 72. *Consider the sequence of functions*

$$Z_n(\beta) = \sum_{I \in \mathcal{K}_n} |I|^\beta \ .$$

Compute $Z_n(\beta)$, and derive that

$$F(\beta) = \lim_{n \to \infty} \frac{1}{n} \log_2 Z_n(\beta) = \log_2 \big(4^{-\beta} + 3^{-\beta} \big) \ .$$

Show that φ is the Legendre transform of F. Prove relation (75) using steepest descent. Show that $\varphi(s)$ is concave, and its maximum is attained at $s = -\log_2 \sqrt{12}$.

We can now state the result.

Theorem 6.6. *For any* $\alpha \in [1/\log_2 4, 1/\log_2 3]$ *we have*

$$d_{\mathrm{H}}\big(\mathcal{E}_\alpha^-\big) = d_{\mathrm{H}}\big(\mathcal{E}_\alpha^+\big) = \alpha\,\varphi(-1/\alpha)\ .$$

Proof (*sketch*). We will only give a proof for $\alpha \le \alpha_c = 1/\log_2\sqrt{12}$. We refer the reader to the literature for the other range of α. Let

$$\tilde{\mathcal{E}_\alpha} = \left\{ x \,\middle|\, \alpha \text{ is an accumulation point of } \frac{\log \mu\big(B_r(x)\big)}{\log r} \text{ when } r \to 0 \right\}\ .$$

Note that $\mathcal{E}_\alpha^\pm \subset \tilde{\mathcal{E}_\alpha}$. To construct the upper bound for $d_{\mathrm{H}}\big(\tilde{\mathcal{E}_\alpha}\big)$ we will use a particular covering by balls. Indeed, let $x \in \tilde{\mathcal{E}_\alpha}$. Then by definition, there exists an infinite sequence $\big(r_j\big)$ tending to zero such that

$$\lim_{j\to\infty} \frac{\log \mu\big(B_{r_j}(x)\big)}{\log r_j} = \alpha\ .$$

Since $x \in \mathcal{K}$, for any j we can find an integer n_j such that there is an interval $I \in \mathcal{K}_{n_j}$ satisfying

$$\mathcal{K} \bigcap B_{r_j}(x) \subset I$$

and an interval $J \in \mathcal{K}_{n_j+2}$ such that

$$J \subset B_{r_j}(x)\ .$$

Exercise 73. *Prove this statement, for example by drawing a local picture picture around* x *of* $\mathcal{K}_n$.

In particular, we have

$$2^{-n_j-2} \le \mu\big(B_{r_j}(x)\big) \le 2^{-n_j}\ .$$

From the condition $x \in \tilde{\mathcal{E}_\alpha}$, we derive

$$2^{-n_j} \approx \mu\big(B_{r_j}(x)\big) \approx r_j^\alpha\ ,$$

and therefore

$$r_j \approx 2^{-n_j/\alpha}\ .$$

This implies

$$\big|I\big| \lessapprox 2^{-n_j/\alpha}$$

that for any fixed N we have

$$\tilde{\mathcal{E}_\alpha} \subset \bigcup_{n=N}^{\infty} \bigcup_{I\in\mathcal{K}_n\ |I|\le 2^{-n/\alpha}} |I|\ .$$

If we assume $\alpha \le \alpha_c = 1/\log_2\sqrt{12}$, we have $1\alpha \ge \log_2\sqrt{12}$ and therefore $s = -1\alpha \le -\log_2\sqrt{12}$.

If we use the above intervals I as a covering for $\tilde{\mathcal{E}_\alpha}$, we get from the large deviation estimate (75) and the definition (69) that for any $d > \alpha\,\varphi(-1/\alpha)$

$$\mathcal{H}_d\big(3^{-N}, \tilde{\mathcal{E}_\alpha}\big) \leq \sum_{n=N}^{\infty} 2^{-n\,d/\alpha} 2^{n\,\varphi(-1/\alpha)} \ ,$$

which implies

$$\lim_{N\to\infty} \mathcal{H}_d\big(3^{-N}, \tilde{\mathcal{E}_\alpha}\big) = 0 \ ,$$

and therefore

$$d_{\mathrm{H}}\big(\tilde{\mathcal{E}_\alpha}\big) \leq \alpha\,\varphi(-1/\alpha) \ . \tag{76}$$

We now prove a lower bound on $d_{\mathrm{H}}((\mathcal{E}_\alpha^- \cap \mathcal{E}_\alpha^+)$. For this purpose we will use Frostman Lemma 6.2 and we start by constructing a measure with suitable properties. This can be done using a coding and the measure appears as a Bernoulli measure, we will however construct it directly. We will in fact construct a one parameter family of measures ν_p, $p \in]0,1[$ (and we define as usual $q = 1-p$). The construction is recursive using the sequence of intervals in $\mathcal{K}_n$. The measure μ_p will be a probability measure and we therefore set $\nu_p\big([0,1]\big) = 1$. Assume now that $\nu_p(I)$ has already been defined for any $I \in \mathcal{K}_n$. As we have seen in the recursive construction (74), the intervals in $\mathcal{K}_{n+1}$ are obtained from those of $\mathcal{K}_n$ by applying f_1 or f_2. We now define ν_p on the intervals of $\mathcal{K}_{n+1}$ by

$$\nu_{n+1}(J) = \begin{cases} p\,\nu_n(I) & \text{if } J = f_1(I) \ , \\ q\,\nu_n(I) & \text{if } J = f_2(I) \ . \end{cases}$$

We leave to the reader that this recursive construction defines a Borel measure on $[0,1]$ with support $\mathcal{K}$. In particular, $\mu = \nu_{1/2}$. Consider now the map f of exercise (71). It is easy to verify that its Lyapunov exponent for the measure ν_p is equal to

$$\lambda_p = p\log 4 + q\log 3 \ .$$

We define a set $A \subset \mathcal{K}$ by

$$A = \left\{ x \,\middle|\, \lim_{r\to 0} \frac{\log \nu_p\big(B_r(x)\big)}{\log r} = \frac{-p\log p - q\log q}{\lambda_p} \right\} \ .$$

It follows from the ergodic Theorem (3.1) and the Shannon-McMillan-Breiman Theorem (5.2) that $\nu_p(A) = 1$.

Exercise 74. *Prove this relation using formula (59).*

On the other hand, we see that if we choose p such that

$$\alpha = \frac{-\lambda_p}{\log 2} \ ,$$

since $A \subset \mathcal{E}_\alpha^- \cap \mathcal{E}_\alpha^+$, we have

$$\nu_p\left(\mathcal{E}_\alpha^- \cap \mathcal{E}_\alpha^+\right) = 1 \ .$$

Therefore, if

$$p = -\frac{\alpha + \log_2 3}{\log_2(4/3)}$$

we get using Frostman Lemma 6.2

$$d_{\mathrm{H}}((\mathcal{E}_\alpha^- \cap \mathcal{E}_\alpha^+) \geq \frac{-p\log p - q\log q}{\lambda_p} = \alpha\,\varphi(-1/\alpha)\ .$$

Since this quantity is equal to the upper bound (76), the result follows. □

Exercise 75. *Write a complete proof of the above result by putting all the necessary ϵ and δ.*

The maximal value for the Hausdorff dimension of $\mathcal{E}_\alpha^\pm$ is the dimension of the invariant set. On the other hand, if α is equal to the Lyapunov exponent (divided by $\log 2$), then $\mathcal{E}_\alpha^\pm$ have full measure. This implies that the curves $\alpha \to \mathcal{E}_\alpha^\pm$ should be below the first diagonal and tangent to it in one point. This is illustrated in figure 98 which shows the graph of $d_{\mathrm{H}}(\mathcal{E}_\alpha^-) = d_{\mathrm{H}}(\mathcal{E}_\alpha^+)$ when α varies. This figure was produce with the two maps

$$f_1(y) = \frac{y}{29} \quad \text{and} \quad f_2(y) = \frac{2+y}{3}\ . \tag{77}$$

so that one can see more clearly the curves.

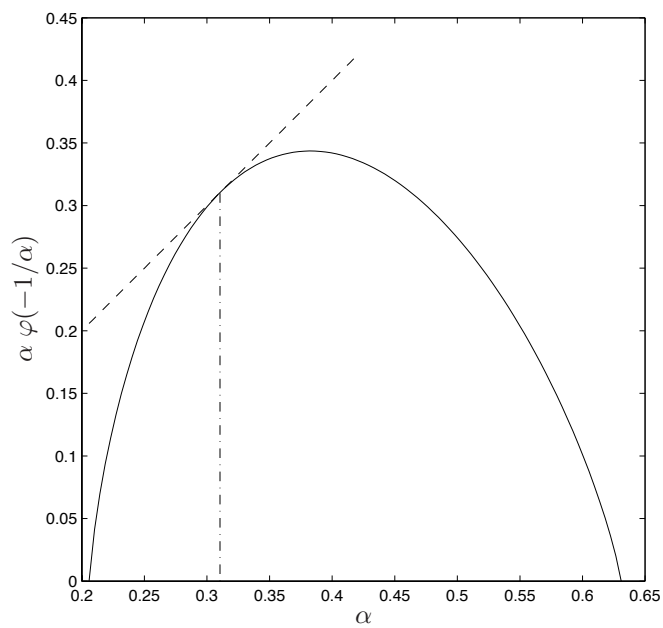


Figure 98: Hausdorff dimension of the sets $\mathcal{E}_\alpha^\pm$ as a function of α for the maps (77) .

6.9 Statistics of the invariant measure

As we have seen, it is often the case that one does not have access to the phase space but only to an image of it. In particular, as was emphasised several times, we may only have a time series corresponding to the observation of the system

through an observable g, namely a finite sequence of real numbers $y_1, \dots, y_n$ satisfying

$$y_j = g\big(T^{j-1}(x)\big)$$

for an initial condition x on the phase space which in general is unknown.

Moreover, we only have an image μ_g of the invariant measure μ through g, namely

$$\mu_g = \mu \circ g^{-1} \ .$$

In order to determine μ_g, one can use the empirical distribution function

$$F_{n,g}(t) = \frac{1}{n}\sum_{j=1}^{n} \theta\big(t - y_j\big) = \frac{1}{n}\sum_{j=1}^{n} \theta\big(t - g\big(T^{j-1}(x)\big)\big) \ .$$

If the measure μ is ergodic, one concludes immediately from Birkhoff's ergodic theorem that for μ almost every x, this sequence of random functions converges to the distribution function of μ_g, $F_g(t) = \mu_g(]-\infty, t])$. In the case of independent random variables, this is the Glivenko Cantelli theorem (see for example [18]).

Again in the case of independent random variables (under some minor hypothesis), the fluctuations are known. More precisely, a celebrated theorem of Kolmogorov establishes that the L^∞ norm of the difference between the empirical distribution function and the true one, multiplied by $n^{1/2}$ converges in law to a random variable whose distribution is known explicitly (see [18]). This leads to the well known Kolmogorov Smirnov non parametric test (see [18]).

One may wonder if similar results hold for dynamical systems. For piecewise expanding maps of the interval equipped with the absolutely continuous invariant measure, and the observable $g(x) = x$ (which is enough in that case of dimension one to get the measure), this question was considered by S.Martínez , B.Schmitt and myself [49] and we proved that the process

$$\sqrt{n}\big(F_{n,g}(t) - F_g(t)\big)$$

converges in law to a Gaussian bridge (in general not a Brownian bridge even for the map $2x \pmod 1$). From this result one can derive non parametric tests. We refer to [48] for the details. In [32] this question was also considered for some non uniformly hyperbolic systems.

We derive here for the case of piecewise expanding maps of the interval a general estimate following from the exponential estimate (see also [49]).

Theorem 6.7. *Let T be a dynamical system on a phase space Ω, and let μ be an ergodic invariant measure satisfying the exponential estimate (51). Then there are two positive constants Γ_1 and Γ_2 such that for any real valued Lipschitz observable g with $\|g\|_{L^\infty} \leq 1$ such that $g^*\mu$ is absolutely continuous with bounded density, we have*

$$\mathbb{P}\Big(\sup_t \big|F_{n,g}(t) - F_g(t)\big| > sn^{-1/4}\Big) \leq\leq \Gamma_1\sqrt{n}\, s^{-2} e^{\Gamma_2 s^4}.$$

Remark 12. *The assumption $\|g\|_{L^\infty} \leq 1$ is only for convenience, one can rescale the result by replacing g by $g/\|g\|_{L^\infty}$.*

Proof. The difficulty here is that the function θ entering the definition of the empirical distribution $F_{n,g}$ is not Lipschitz. In order to apply nevertheless the exponential inequality, we will sandwich the function θ between two Lipschitz functions. For a positive number γ (small in the application), we define the function θ_γ by

$$\theta_\gamma(s) = \begin{cases} 0 & \text{if } s < 0 \\ s/\gamma & \text{if } 0 \le s \le \gamma \\ 1 & \text{if } s \ge \gamma \ . \end{cases}$$

We now define the random function $F_{n,\gamma,g}$ by

$$F_{n,\gamma,g}(t) == \frac{1}{n} \sum_{j=1}^{n} \theta_\gamma\big(t - g \circ T^{j-1}\big) \ .$$

It is easy to verify that for any real s we have

$$\theta_\gamma(s) \le \theta(s) \le \theta_\gamma(s+\gamma)$$

which immediately implies for any t

$$F_{n,\gamma,g}(t) \le F_{n,g}(t) \le F_{n,\gamma,g}(t+\gamma) \ . \tag{78}$$

Note that $F_{n,\gamma,g}(t)$ is now a Lipschitz function in t with Lipschitz constant γ^{-1}. Therefore, if $\delta > 0$ is a (small) number to be optimally chosen later on, we have

$$\left| \sup_t \Big(F_{n,\gamma,g}(t) - F_{\gamma,g}(t) \Big) - \sup_p \Big(F_{n,\gamma,g}(p\delta) - F_{\gamma,g}(p\delta) \Big) \right| \le \gamma^{-1}\delta \ , \tag{79}$$

where

$$F_{\gamma,g}(t) = \mathbb{E}\big(\theta_\gamma\big(t - g(\,\cdot\,)\big)\big) \ .$$

This estimate is of course interesting only if $\gamma^{-1}\delta < 1$.

We now consider the function $K_{n,\gamma,g}$ of n variables $z_1, \ldots, z_n$ given by

$$K_{n,\gamma,g}(t) = \frac{1}{n} \sum_{j=1}^{n} \theta_\gamma\big(t - g(z_j)\big) \ .$$

This function is obviously componentwise Lipschitz, and it is easy to verify that for each variable its Lipschitz constant defined in (50) is bounded by

$$L_j\big(K_{n,\gamma,g}\big) \le \frac{\gamma^{-1} L_g}{n} \ ,$$

where L_g is the Lipschitz constant of the observable g. Therefore, from Pisier's inequality, we get for any real number β

$$\mathbb{E}\left(e^{\beta \sup_{-\delta^{-1} \le p \le \delta^{-1}(1+\gamma)} \left(F_{n,\gamma,g}(p\delta) - F_{\gamma,g}(p\delta) \right)} \right)$$

$$\le \sum_{-\delta^{-1} \le p \le \delta^{-1}(1+\gamma)} \mathbb{E}\left(e^{\beta \left(F_{n,\gamma,g}(p\delta) - F_{\gamma,g}(p\delta) \right)} \right) \ ,$$

where as we have seen already several times, the expectation is taken over the measure

$$d\mu(z_1)\prod_{j=1}^{n-1}\delta\big(z_{j+1}-T(z_j)\big)\ .$$

Using the exponential inequality (51) we get

$$\mathbb{E}\left(e^{\beta\sup_{-\delta^{-1}\le p\le\delta^{-1}(1+\gamma)}\left(F_{n,\gamma,g}(p\delta)-F_{\gamma,g}(p\delta)\right)}\right)$$

$$\le\delta^{-1}(2+\gamma)C_1e^{C_2\beta^2\gamma^2L_g^2/n}\ .$$

If ϵ_1 is a (small) positive number, we have using Chebychev's inequality

$$\mathbb{P}\left(\sup_{-\delta^{-1}\le p\le\delta^{-1}(1+\gamma)}\big(F_{n,\gamma,g}(p\delta)-F_{\gamma,g}(p\delta)\big)>\epsilon_1\right)\le\delta^{-1}(2+\gamma)C_1e^{-\beta\epsilon_1}e^{C_2\beta^2\gamma^{-2}L_g^2/n}\ ,$$

and this inequality holds for any real β. Using the estimate (79) this implies for any $\epsilon_1>0$

$$\mathbb{P}\left(\sup_t\big(F_{n,\gamma,g}(t)-F_{\gamma,g}(t)\big)>\epsilon_1+\gamma^{-1}\delta\right)\le\delta^{-1}(2+\gamma)C_1e^{-\beta\epsilon_1}e^{C_2\beta^2\gamma^{-2}L_g^2/n}\ .$$

Let the function $\omega_g(\gamma)$ be defined by

$$\omega_g(\gamma)=\sup_t\big(F_g(t)-F_{\gamma,g}(t)\big)\ .$$

We then have immediately from inequality (78)

$$\mathbb{P}\left(\sup_t\big(F_{n,\gamma,g}(t)-F_{\gamma,g}(t)\big)>\epsilon_1+\gamma^{-1}\delta+\omega_g(\gamma)\right)\le\delta^{-1}L_gC_1e^{-\beta\epsilon_1}e^{C_2\beta^2\gamma^{-2}L_g^2/n}\ .$$

If we assume

$$\omega_g(\gamma)\le C_1\gamma\ ,$$

the optimal choice of δ and γ in the quantity

$$\epsilon_1+\gamma^{-1}\delta+\omega_g(\gamma)$$

are (within constant factors)

$$\gamma=\delta^{1/2}=\epsilon_1\ .$$

The optimal choice of β (again within constant factors) is now

$$\beta=n\epsilon_1^3\ ,$$

and we get for some positive constants Γ_1 and Γ_2

$$\mathbb{P}\Big(\sup_t\big|F_{n,g}(t)-F_g(t)\big|>sn^{-1/4}\Big)\le\Gamma_1\sqrt{n}\ s^{-2}e^{\Gamma_2s^4}.$$

□

Although this estimate is asymptotically weaker than the convergence in law to a Gaussian bridge as in the Komomogorov Smirnov test, it allows however to obtain rather strong tests.

There are also results on kernel density estimates in the case of absolutely continuous invariant measures, see [32], [23] and [24], [135].

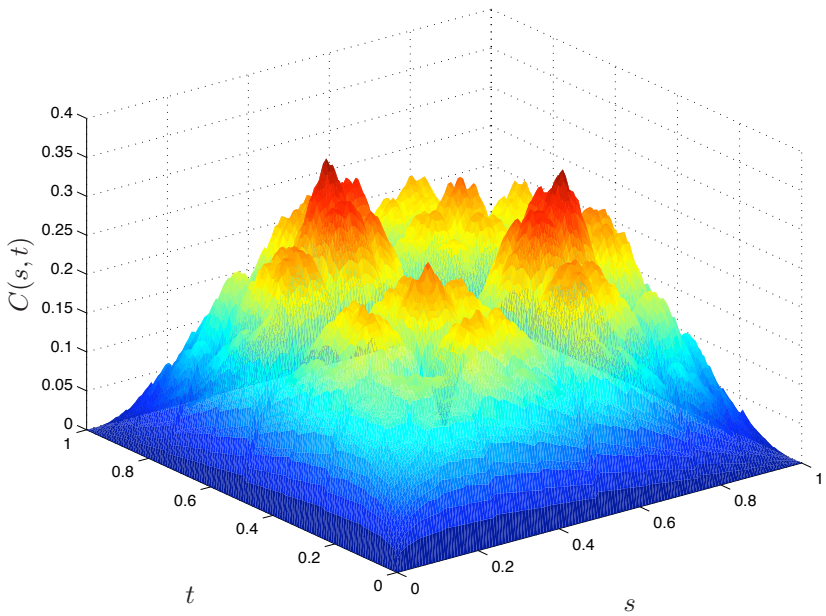


Figure 99: Covariance of the Gaussian bridge for the map $2x \pmod 1$.

7 Other results.

7.1 Entrance and recurrence times in small sets.

One of the oldest question in ergodic theory is the problem of entrance time in a (small) set. For a (measurable) subset A of the phase space Ω, one defines the (first) entrance time $\tau_A(x)$ of the trajectory of x in the set A by

$$\tau_A(x) = \inf \left\{ n > 0 \,\middle|\, T^n(x) \in A \right\} .$$

For x in A this number is called the return time. Note that the function τ_A is a measurable function on the phase space with values in the integers.

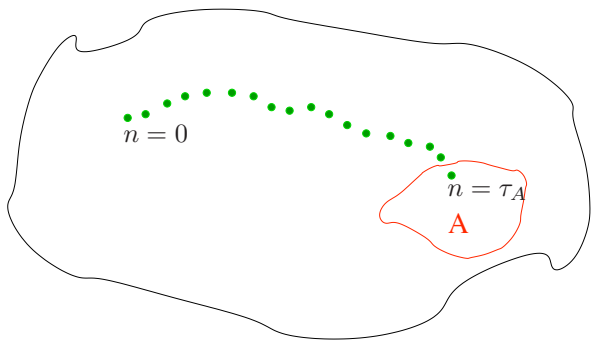


Figure 100: A sketch of the definition of τ_A.

Historically, this quantity appeared in Boltzmann's ideas about ergodic theory, when he asked about the time it would take for all the molecules of a room to concentrate in only half of the available volume. His idea was that events of small probability occur only on a very large time scale. One of the first few general results around these questions is the Poincaré recurrence theorem which dates from the same period of heated discussions about the foundations of statistical mechanics.

Theorem 7.1. *Let T be a transformation on a phase space Ω, and μ be an invariant probability measure for T. Then for each measurable set A of positive measure, there is a finite integer $n \geq 1$ such that $A \cap T^n(A) \neq \emptyset$ and moreover $\mu\big(A \cap T^n(A)\big) > 0$.*

The basic idea of the proof is that all the images $T^n(A)$ cannot be disjoint because they have the same (non-zero) measure.

Proof. The proof is by contradiction. Assume there exists a measurable set A with $\mu(A) > 0$ and such that for any n we have $\mu\big(A \cap T^n(A)\big) = 0$. By the invariance of the measure we have for any integer n,

$$\mu\big(T^{-n}(A) \cap A\big) \subset \mu\big(T^{-n}\big(A \cap T^n(A)\big)\big) = \mu\big(A \cap T^n(A)\big) = 0 .$$

Therefore, since for any integer m, we have

$$T^{-m}\big(A \cap T^{-n}(A)\big)\big) = T^{-m}(A) \cap T^{-n-m}(A) ,$$

we conclude that for any integers m and n

$$\mu\big(T^{-m}(A) \cap T^{-n-m}(A)\big) = 0; .$$

Let

$$B_p = \cup_{n=0}^{p} T^{-n}(A) \ ,$$

since the sets $T^{-n}(A)$ have an intersection of measure zero and the same measure $\mu(A)$, we have

$$\mu\big(B_p\big) = p\ \mu(A) \ .$$

This is only possible as long as $p\ \mu(A) \leq 1$ since A is of positive measure. Hence we have a contradiction with our assumptions on A, and the theorem is proved. □

Remark 13. *It follows from the proof of the theorem that $A \cap T^n(A) \neq \emptyset$ (and in fact $\mu\big(A \cap T^n(A)\big) > 0$) for at least one n such that $1 \leq n \leq 1/\mu(A)$.*

Another important result about entrance time follows from the ergodic theorem. It says that the entrance time is a well defined function.

Theorem 7.2. *If A is such that $\mu(A) > 0$, and μ is ergodic, then τ_A is almost surely finite.*

Proof. The proof is by contradiction. Let B be a measurable set of positive measure where $\tau_A = \infty$ ($\tau_A(x) = \infty$ for any $x \in B$). We have for almost every x in B

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_A\big(T^n(x)\big) = 0 \ .$$

On the other hand by ergodicity, this quantity should be almost surely equal to $\mu(A) > 0$, hence a contradiction. □

Exercise 76. *Show that almost surely*

$$\tau_A(x) = 1 + \sum_{j=1}^{\infty} \prod_{m=1}^{j-1} \chi_{A^c}\big(T^j(x)\big) \ .$$

The ergodic theorem tells us that if one considers a large time interval [0,N], and a (measurable) set A of positive measure, then the number of times a typical trajectory has visited A (the event A has occurred) is of the order $\mu(A)N$. In other words, $1/\mu(A)$ is a time scale associated to the set A, and one may wonder if this time scale is related to the entrance time. There is a general theorem in this direction due to Mark Kac.

Theorem 7.3. *Let T be a transformation on a phase space Ω, and μ be an ergodic invariant probability measure for T. Then for each measurable set A of positive measure,*

$$\int_A \tau_A d\mu = 1.$$

Proof. We give a proof in a particular case, and refer to the literature (for example [107]) for the general case. It immediately follows from the definition of τ_A (see also exercise 76) that almost surely

$$\chi_{A^c}\big(T(x)\big)\tau_A\big(T(x)\big) = \tau_A(x) - 1 = \chi_A(x)\tau_A(x) + \chi_{A^c}(x)\tau_A(x) - 1 \ .$$

Exercise 77. *Prove this equality. Hint: consider the two cases* $\tau_A = 1$ *and* $\tau_A > 1$.

If we now assume that the function τ_A is integrable, the result follows by integrating both sides of this equality and using the invariance of the measure. $\square$

Figure 101: Mark Kac (1914-1984).

There is also a general lower bound on the expectation of τ_A.

Theorem 7.4. *For any measurable set* A *with* $\mu(A) > 0$ *we have*

$$\mathbb{E}(\tau_A) \geq \frac{1}{2\mu(A)} - 3/2.$$

Note that the bound is not very interesting for $\mu(A) > 1/5$ since by definition $\tau_A \geq 1$.

Proof. Recall that

$$\mathbb{E}\big(\tau_A\big) = \sum_{p=1}^{\infty} \mathbb{P}\big(\tau_A \geq p\big) \ .$$

We now estimate from below each term in the sum. We have

$$\mathbb{P}\big(\tau_A \geq p\big) = \int d\mu(x) \prod_{j=1}^{p-1} \chi_{A^c}\big(T^j(x)\big) = \int d\mu(x) \prod_{j=1}^{p-1} \Big(1 - \chi_A\big(T^j(x)\big)\Big) \ .$$

It is left to the reader to verify that

$$\prod_{j=1}^{p-1} \Big(1 - \chi_A\big(T^j(x)\big)\Big) \geq 1 - \sum_{j=1}^{p-1} \chi_A\big(T^j(x)\big) \ .$$

Therefore for any $p \geq 1$ we have

$$\mathbb{P}\big(\tau_A \geq p\big) \geq 1 - (p-1)\mu(A) \ .$$

Note that this bound is not very useful if $p > 1 + 1/\mu(A)$. Let q be a positive integer to be chosen optimally later on. We have

$$\mathbb{E}\big(\tau_A\big) \geq \sum_{p=1}^{q} \mathbb{P}\big(\tau_A \geq p\big) \geq \sum_{p=1}^{q} \big(1 - (p-1)\mu(A)\big) = q - \frac{q(q-1)\mu(A)}{2} \ .$$

Since this estimate is true for any integer q, we can look for the largest right hand side. We take $q = [1/\mu(A)]$ which is near the optimum and get the estimate using $1/\mu(A) - 1 \leq q \leq 1/\mu(A)$. □

In general one cannot say much more on the random variable τ_A without making some hypothesis on the dynamical system or on the set A. We now give an estimate on the tail of the distribution of τ_A in the case of dynamical systems with decay of correlations satisfying

$$\begin{aligned} \big|C_{f,g}(n)\big| &= \left| \int f(x)\, g\big(T^n(x)\big)\, d\mu(x) - \int f\, d\mu \int g\, d\mu \right| \\ &\leq C_{\mathcal{B}_1,\mathcal{B}_2}(n)\ \|f\|_{\mathcal{B}_1}\ \|g\|_{\mathcal{B}_2} \ . \end{aligned} \tag{80}$$

Theorem 7.5. *Assume the dynamical system defined by the map T and equipped with the ergodic invariant measure μ satisfies the above estimate (80) with $C_{\mathcal{B}_1,\mathcal{B}_2}(n)$ decaying exponentially fast, namely there are two constants $C > 0$ and $0 < \rho < 1$ such that for any integer n*

$$C_{\mathcal{B}_1,\mathcal{B}_2}(n) \leq C\rho^n \ .$$

Let A be a measurable set such that $0 < \mu(A) < 1$, and such that the characteristic function of A belongs to $\mathcal{B}_1$. Assume also that $\mathcal{B}_2 = L^1(d\mu)$. Then

$$\mathbb{P}\big(\tau_A > n\big) \leq e^{\gamma_A\, n\, \mu(A)/\log \mu(A)} \ ,$$

where γ_A depends only on the $\mathcal{B}_1$ norm of the characteristic function of A.

Note that $\log \mu(A)$ is negative and therefore the upper bound decays exponentially fast with n. This estimate says that τ_A is unlikely to be much larger than $-(\log \mu(A))/\mu(A)$. Note also that in this estimate there is no restriction on the set A except that its characteristic function belongs to $\mathcal{B}_1$.

All these assumptions are satisfied in the case of piecewise expanding maps of the interval when A is an interval and μ the Lebesgue measure (see the Theorem on the spectrum of the Perron-Frobenius operator). In this case $\mathcal{B}_1$ is the set of functions of bounded variations, and the norm of the characteristic function of an interval is independent of the interval.

Proof. We have from the definition

$$\mathbb{P}\big(\tau_A > n\big) = \int \prod_{j=1}^{n} \chi_{A^c} \circ T^j(x)\ d\mu(x) \ .$$

Let k be an integer to be fixed later on, and let $m = [n/k]$. We have obviously

$$\mathbb{P}\big(\tau_A > n\big) \leq \int \prod_{l=1}^{m} \chi_{A^c} \circ T^{lk}(x)\ d\mu(x) \ .$$

Using (36) with our hypothesis (and the invariance of the measure) we get

$$\mathbb{P}\big(\tau_A > n\big) \leq\leq \int \big(1-\chi_A\big) \prod_{l=1}^{m-1} \chi_{A^c} \circ T^{lk}(x)\, d\mu(x) \leq$$

$$\Big(1-\mu(A) + C\rho^k \big\|\chi_A\big\|_{\mathcal{B}_1}\Big) \int \prod_{l=1}^{m-1} \chi_{A^c} \circ T^{lk}(x)\, d\mu(x)\ .$$

We now choose k as the smallest integer such that

$$C\rho^k \big\|\chi_A\big\|_{\mathcal{B}_1} \leq \frac{1}{2}\mu(A)\ .$$

Iterating the estimate we obtain

$$\mathbb{P}\big(\tau_A > n\big) \leq \big(1-\mu(A)/2\big)^m \leq e^{-m\,\mu(A)/2}\ ,$$

and the result follows. □

In the case of small sets A, a lot of works have been devoted to the study of asymptotic distribution of the entrance time in A or to the stochastic process of successive returns to the set A. One can expect to see emerging something similar to the exponential law, or a Poisson point process for the times of successive visits. The intuition is that if A has a small measure, the entrance in A is a rare event. Of course the successive returns to A are not independent events. However a decay of correlation should make these successive events almost independent (loss of memory). We only give a detailed result in the case of piecewise expanding maps of the interval.

Theorem 7.6. *Let T be a piecewise expanding map of the interval, and μ a mixing a.c.i.p.m. with density h. There is a set B of full measure such that if A_n be a sequence of intervals of length tending to zero and accumulating to a point $b \in B$, then the sequence of random variables $\mu\big(A_n\big)\tau_{A_n}$ converges in law to an exponential random variable of parameter one. In other words, for any fixed number $s > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}\big(\tau_{A_n} > s/\mu\big(A_n\big)\big) = e^{-s}\ .$$

We give below the sketch of a proof of this theorem modulo a technical point for which we refer the reader to the literature. This proof is based on an idea of Kolmogorov for proving the central limit theorem in the i.i.d. case (see [17]).

Proof. In order to alleviate the notation, we will denote by ϵ the positive number $\mu(A)$. Let s be a positive number and let $n = [s/\epsilon]$. We have obviously

$$\mathbb{P}\big(\tau_A > n\big) = (1-\epsilon)^n + \sum_{q=0}^{n-1} \Big((1-\epsilon)^{n-q-1}\mathbb{P}\big(\tau_A > q+1\big) - (1-\epsilon)^{n-q}\mathbb{P}\big(\tau_A > q\big)\Big)\ .$$

If ϵ tends to zero, the first term converges to e^{-s}, and we only need to prove that the second term converges to zero. Since for $q \geq 1$

$$\mathbb{P}\big(\tau_A > q\big) = \mathbb{E}\big(\prod_{j=0}^{q} \chi_{A^c} \circ T^j\big)\ ,$$

we have from the invariance of the measure

$$
\begin{aligned}
&(1-\epsilon)^{n-q-1}\mathbb{P}\big(\tau_A > q+1\big) - (1-\epsilon)^{n-q}\mathbb{P}\big(\tau_A > q\big) \\
&= (1-\epsilon)^{n-q-1}\left(\mathbb{E}\Big(\prod_{j=0}^{q+1}\chi_{A^c}\circ T^j\Big) - (1-\epsilon)\mathbb{E}\Big(\big(\chi_A+\chi_{A^c}\big)\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big)\right) \\
&= (1-\epsilon)^{n-q-1}\left(\epsilon\,\mathbb{E}\Big(\prod_{j=0}^{q+1}\chi_{A^c}\circ T^j\Big) - (1-\epsilon)\mathbb{E}\Big(\chi_A\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big)\right) \\
&= (1-\epsilon)^{n-q-1}\left(\epsilon\,\mathbb{E}\Big(\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big) - \mathbb{E}\Big(\chi_A\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big)\right) + (1-\epsilon)^{n-q-1}\mathcal{O}(1)\epsilon^2\,,
\end{aligned}
\tag{81}
$$

where the last term comes from the estimation

$$
0 \le \epsilon\mathbb{E}\Big(\chi_A\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big) \le \epsilon\mathbb{E}\Big(\chi_A\Big) = \epsilon^2\,.
$$

Note that the modulus of the whole term is less than $2\epsilon+\epsilon^2$ (drop the product of the characteristic functions). Let k_A be an integer such that $A\cap T^j(A)=\emptyset$ for $j=1,\dots,k_A$ (it is enough to assume that the intersection has measure zero). Therefore for $q>k_A$ we have

$$
\mathbb{E}\Big(\chi_A\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big) = \mathbb{E}\Big(\chi_A\prod_{j=k_A+1}^{q+1}\chi_{A^c}\circ T^j\Big)\,.
$$

For the other term, we use Bonferoni's inequality, namely

$$
1-\sum_{j=1}^{k_A}\chi_A\circ T^j \le \prod_{j=1}^{k_A}\chi_{A^c}\circ T^j \le 1-\sum_{j=1}^{k_A}\chi_A\circ T^j + \sum_{1\le r\ne s\le k_A}\chi_A\circ T^r\chi_A\circ T^s\,.
$$

Exercise 78. *Prove these inequalities.*

This implies immediately for $q>k_A$

$$
\left|\mathbb{E}\Big(\prod_{j=1}^{q+1}\chi_{A^c}\circ T^j\Big) - \mathbb{E}\Big(\prod_{j=k_A+1}^{q+1}\chi_{A^c}\circ T^j\Big)\right| \le \epsilon k_A^2\,.
$$

We can now use the decay of correlations in the form of equation (36) for $\chi_A\in\mathcal{B}_1$ and $\prod_{j=k_A+1}^{q+1}\chi_{A^c}\circ T^j\in\mathcal{B}_2$. For example for a mixing a.c.i.p.m. of a piecewise expanding maps of the interval we can take for A an interval which belongs the space of functions of bounded variations ($\mathcal{B}_1$) and for $\mathcal{B}_2$ it is enough to take L^∞. We get

$$
\left|\epsilon\mathbb{E}\Big(\prod_{j=k_A+1}^{q+1}\chi_{A^c}\circ T^j\Big) - \mathbb{E}\Big(\chi_A\prod_{j=k_A+1}^{q+1}\chi_{A^c}\circ T^j\Big)\right| \le C_{\mathcal{B}_1,L^\infty}(k_A)\,\|\chi_A\|_{\mathcal{B}_1}\,.
$$

Combining all the above estimates we get for $n-1 \geq q > k_A$

$$\big|(1-\epsilon)^{n-q-1}\mathbb{P}\big(\tau_A > q+1\big) - (1-\epsilon)^{n-q}\mathbb{P}\big(\tau_A > q\big)\big|$$

$$\leq (1-\epsilon)^{n-q-1}\left(\epsilon^2(1+k_A^2) + C_{\mathcal{B}_1,L^\infty}(k_A)\big\|\chi_A\|_{\mathcal{B}_1}\right) .$$

On the other hand, as was already observed, we have immediately from equation (81)

$$\big|(1-\epsilon)^{n-q-1}\mathbb{P}\big(\tau_A > q+1\big) - (1-\epsilon)^{n-q}\mathbb{P}\big(\tau_A > q\big)\big| \leq 2\epsilon .$$

Summing over q (from 1 to k_A using the rough bound, and from k_A to n using the more elaborated one) we get for $n > k_A$

$$\left|\sum_{q=0}^{n-1}\Big((1-\epsilon)^{n-q-1}\mathbb{P}\big(\tau_A > q+1\big) - (1-\epsilon)^{n-q}\mathbb{P}\big(\tau_A > q\big)\Big)\right|$$

$$\leq \mathcal{O}(1)\Big(\epsilon k_A + \epsilon k_A^2 + \epsilon^{-1}C_{\mathcal{B}_1,L^\infty}(k_A)\big\|\chi_A\|_{\mathcal{B}_1}\Big) .$$

The theorem follows for any sequence $\big(A_n\big)$ of sets for which the right hand side of the above estimate trends to zero. For a mixing a.c.i.p.m of a piecewise expanding map of the interval, $\big\|\chi_A\|_{\mathbf{BV}} = 2$, and $C_{\mathcal{B}_1,L^\infty}(k_A)$ decays exponentially fast in k_A (recall $\mathcal{B}_1 = \mathbf{BV}$ and see estimate (38). It is therefore enough to ensure that $k_A = \mathcal{O}(1)(-\log\mu(A))$. There turns out to be many such intervals. We refer to [43] and exercise 79 for the details. □

Exercise 79. *Consider the full shift on two symbols* $\{0,1\}$ *and the product measure* $(1/2,1/2)$ *(or the map* $2x \pmod 1$ *with the Lebesgue measure). Let* $\mathcal{C}_{n,k}$ *(*$1 \leq k/ \leq n-1$*) be the set of cylinders* C *of length* n *(starting at position one) such that* $C \cap \mathcal{S}C \neq \emptyset$ *(recall that* $\mathcal{S}$ *is the shift). For* $C \in \mathcal{C}_{n,k}$ *show that the last* $n-k$ *symbols are determined by the* k *first ones. Show that* $\mu\big(\mathcal{C}_{n,k}\big) \leq 2^{k-n}$ *(*$\big|\mathcal{C}_{n,k}\big| \leq 2^k$*). For a cylinder set* C *of length* n*, define for* $1 \leq q \leq n-1$ *the function*

$$f_{C,\,q} = \chi_C \prod_{j=q}^{n-1}\left(1 - \chi_C \circ S^j\right) .$$

Show that $f_{c,q}$ *is equal to zero if* $C \in \mathcal{C}_{n,k}$ *for some* $q \leq k \leq n-1$ *and is equal to* χ_C *otherwise. Derive that*

$$\mu\left(\bigcap_{k=q}^{n-1}\mathcal{C}_{n,k}^c\right) = \int \sum_C f_{C,q}\, d\mu.$$

Show that (see Bonferoni inequality)

$$f_{C,\,q} \geq \chi_C - \sum_{j=q}^{n-1}\chi_C\chi_C \circ S^j .$$

Conclude that if $C \in \mathcal{C}_{n,k}$ *for some* $q \leq k \leq n-1$*, then*

$$\int f_{C,\,q}\, d\mu \geq \mu(C) - n2^{-n-q} .$$

Using the previous estimate of $\mu(C)$ for $C \in \mathcal{C}_{n,k}$ with $k < n/2$, show that there is a constant $c > 0$ such that for any n

$$\mu\left(\bigcup_{k=1}^{n-1} \mathcal{C}_{n,k}\right) \leq c\, n\, 2^{-n/2}\ .$$

For $k = 2m$, let K be a cylinder of length m which does not belong to $\mathcal{C}_{m,j}$ for any $1 \leq j < m$. Show that the cylinder $K \cap \mathcal{S}^m K$ belongs to $\mathcal{C}_{k,m}$. Show that $\mu\big(\mathcal{C}_{k,k/2}\big) \geq 2^{-k/2}$.

The preceding result involves some condition on the small sets (A_n), namely they accumulate to a "non exceptional" point b. These "exceptions" are points which in some sense recur too fast, so that the decay of correlation has not allowed a sufficient loss of memory before recurrence occurs. Periodic points are examples of such points. For example if A contains a fixed point we have $A \cap T(A) \neq \emptyset$. Those sets which recur too fast prevent in some sense enough loss of memory to ensure a limiting exponential law. Some results can nevertheless be obtained in these situations. We refer to [82], and [33] for some examples.

By looking at the successive entrance times one can under adequate hypothesis prove that the limiting process when the set becomes smaller and smaller is a Poisson point process (or a marked Poisson point process). We refer to [82], and [33], [36], [42], [44], [83], [1], [58] for more details, results and references.

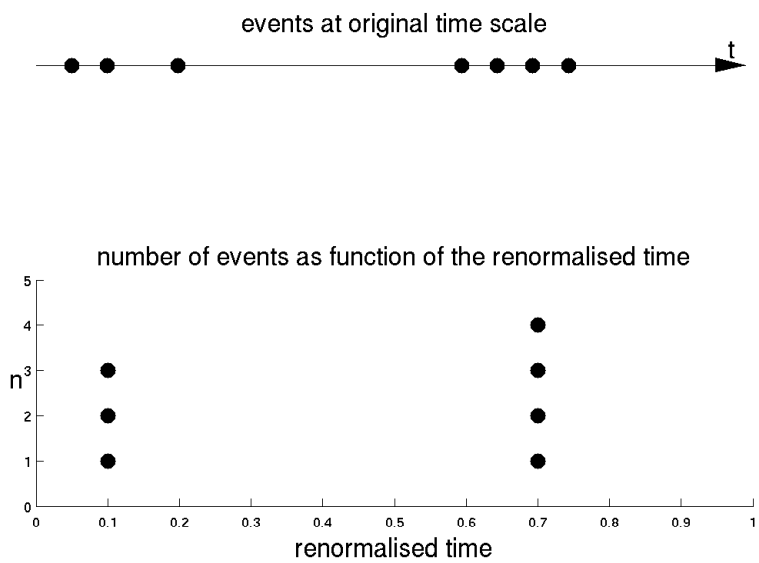


Figure 102: Sketch of a renormalisation leading to a Marked Poisson process.

The approach of two typical trajectories, namely the entrance time in a neighborhood of the diagonal for the product system leads to a marked Poisson process at least for piecewise expanding maps of the interval. One considers two initial conditions x and y chosen at random, independently. For a (small) $\epsilon > 0$, one defines the successive sequence of times where the two orbits are at

a distance less that ϵ. Namely $\tau_{0,\epsilon}(x,y)=0$, and

$$\tau_{k+1,\epsilon}(x,y)=\inf\left\{\ell>\tau_{k,\epsilon}(x,y)\;\middle|\; d\big(T^\ell(x),T^\ell(y)\big)<\epsilon\right\}\,.$$

For piecewise expanding maps of the interval, the process $(\epsilon\ \tau_{k,\epsilon})$ converges in law to a Marked Poisson process. We refer to [33] for details and to [105] for related results.

We have seen before an interesting case for the entrance time in a set A which is the first cylinder of length n of a coded sequence. One then defines τ_n by

$$\tau_n(x)=\inf\left\{\ell>0\;\middle|\; x_1^n=x_\ell^{n+\ell}\right\}.$$

In other words this is the first time in the sequence x_1^∞ where the first chunk x_1^n is seen again. In this situation, the return time is related to the entropy of the system by the Theorem of Ornstein-Weiss 5.5 which states that almost surely

$$\lim_{n\to\infty}\frac{1}{n}\log\tau_n=h_\mu(\mathcal{S})\,.$$

We refer to [128], [41], [105] and [31] for more results on this case and the study of the fluctuations.

The possible asymptotic laws for entrance times are discussed in [64] and [112].

7.2 Number of visits to a set.

Given a measurable set A of positive measure, it is natural to ask how many times a typical orbit visits A during a time interval $[0,S]$ (it is convenient to allow S to be any positive real number). This number of visits denoted by $N_A[0,S]$ is obviously given by

$$N_A[0,S](x)=\sum_{0\le j\le S}\chi_A\big(T^j(x)\big) \tag{82}$$

for the orbit of the point x.

Birkhoff's ergodic tells us that if one considers a large time interval [0,S], and a (measurable) set A of positive measure, then the number of times a typical trajectory has visited A (the event A has occurred) is of the order $\mu(A)S$. In other words, $1/\mu(A)$ is a time scale associated to the set A. For a fixed A, we have discussed at length the behaviour of $N_A[0,S]$ for large S (law of large numbers, CLT etc.).

Another type of asymptotic is concerned with the case where $\mu(A)$ is small. In that case, on a "large" interval of time of scale $1/\mu(A)$ one expects to see only a few occurrences of the event A. It turns out that for certain classes of dynamical systems, this can be quantified in a precise way. It follows for example that (under some hypothesis), if the process of successive entrance times converges in law to a Poisson point process, the number of visits suitably scaled converges in law to a Poisson random variable.

7.3 Records.

Another interesting quantity to look at is the distance of a typical finite orbit to a given point in the phase space. Consider a dynamical system with

a metric d on the phase space. Assume a point x_0 has been chosen once for all. One can look at the successive distances to x_0 of the orbit of x, namely $d\big(x_0, x\big), \ldots, d\big(x_0, T^n(x)\big)$. This is a sequence of positive numbers and the record is the smallest one. This number depends of course on x_0, n and x. We can define this record more precisely by

$$R_{n,x_0}(x) = -\inf_{1\leq j\leq n} \log d\big(x_0, T^j(x)\big) \ .$$

The logarithm is a convenient way to transform a small number into a large one but other means can be used to do that (for example the inverse).

If we consider a measure μ on the phase space for the distribution of x (for example an invariant measure), then R_{n,x_0} becomes a random variable. The law of R_{n,x_0} is related to entrance times. Indeed, it follows at once from the definition that if the record satisfies $R_{n,x_0}(x) > s$, then $d\big(x_0, T^j(x)\big) > e^{-s}$ for $0 \leq j \leq n$. In other words

$$\big\{x \,\big|\, R_{n,x_0}(x) > s\big\} = \Big\{x \,\Big|\, \tau_{B_{e^{-s}}(x_0)}(x) > n\Big\} \bigcap B_{e^{-s}}(x_0)^c \ .$$

This allows to pass from results on entrance times to records and vice versa. We now give a simple estimate on the record related to the capacitary dimension.

Definition 24. *Let μ be a measure on a metric space Ω with a metric d, and let x_0 be a point in Ω. The capacitary dimension d_{Cap} of μ at x_0 is defined by*

$$d_{\mathrm{Cap}}^{x_0}(\mu) = \sup\left\{\beta \geq 0 \,\middle|\, \int \frac{1}{d(x,x_0)^\beta}\, d\mu(x) < \infty\right\} \ .$$

We refer to [95] and [124] for more results on this quantity. Note that it can be infinite (for example if x_0 does not belong to the support of μ).

Theorem 7.7. *For a fixed x_0 and an invariant measure μ we have*

$$\limsup_{n\to\infty} \frac{1}{\log n} \mathbb{E}\big(R_{n,x_0}\big) \leq \frac{1}{d_{\mathrm{Cap}}^{x_0}(\mu)} \ .$$

This result is somewhat similar to the general estimate (7.4).

Proof. We first apply Pisier inequality, namely for a fixed number $\beta > 0$ we have

$$e^{\beta R_{n,x_0}(x)} \leq \sum_{j=0}^{n} e^{-\beta \log d\left(T^j(x), x_0\right)} = \sum_{j=0}^{n} \frac{1}{d\big(T^j(x), x_0\big)^\beta} \ .$$

Integrating over μ and using the invariance of this measure we get

$$\mathbb{E}\left(e^{\beta R_{n,x_0}}\right) \leq n \int \frac{1}{d\big(T^j(x), x_0\big)^\beta}\, d\mu(x) \ .$$

This bound is only interesting if $\beta < d_{\mathrm{Cap}}^{x_0}(\mu)$ which is the condition for the right hand side to be finite. Using Jensen's inequality we get

$$e^{\beta \mathbb{E}\left(R_{n,x_0}\right)} \leq n \int \frac{1}{d\big(T^j(x), x_0\big)^\beta}\, d\mu(x) \ ,$$

and taking the log we obtain

$$\mathbb{E}\left(R_{n,x_0}\right) \leq \frac{\log n}{\beta} + \frac{1}{\beta}\log\left(\int \frac{1}{d\big(T^j(x),x_0\big)^\beta}\, d\mu(x)\right) .$$

This implies for any $\beta < d^{x_0}_{\mathrm{Cap}}(\mu)$

$$\limsup_{n\to\infty} \frac{1}{\log n}\mathbb{E}\big(R_{n,x_0}\big) \leq \frac{1}{\beta}$$

and the result follows. □

It is in general more difficult to obtain a lower bound for the record. We observe that if we know that the entrance time in a ball $B_r(x_0)$ of radius r around a point x_0 converges in law when $r \to 0$ (after suitable normalisation) to an exponential distribution, then we can often say something on the asymptotic distribution of the record (suitably normalised). Namely, if there is a sequence Λ_r such that

$$\lim_{r\to\infty} \mathbb{P}\big(\tau_{B_r(x_0)} > \Lambda_r s\big) = e^{-s}$$

then we get

$$\begin{aligned}\mathbb{P}\big(\tau_{B_r(x_0)} > \Lambda_r s\big) &= \mathbb{P}\left(-\log\big(d\big(T^j(x),x_0\big)\big) < -\log r \;\middle|\; 1 \leq j \leq \Lambda_r s\right)\\ &= \mathbb{P}\left(R_{[\Lambda_r s],x_0} < -\log r\right) = e^{-s} . \qquad (83)\end{aligned}$$

where we have assumed that the record does not occur at the initial time. Assume now, as it is often the case that

$$\Lambda_r \approx r^{-a} .$$

Note that if we are in a regular situation,

$$\mu\big(B_r(x_0)\big) \approx r^{d_{\mathrm{H}}(\mu)}$$

and

$$\Lambda_r \approx \frac{1}{\mu\big(B_r(x_0)\big)} \approx r^{-d_{\mathrm{H}}(\mu)} .$$

It follows from (83) by defining $n = [\Lambda_r s]$, that

$$\lim_{n\to\infty} \mathbb{P}\left(R_{n,x_0} < \frac{1}{a}\log n + v\right) = e^{-e^{-av}}$$

which is Gumble's law (see [73]). We refer to [37] for more details.

7.4 Quasi stationary measures.

Another notion related to entrance times is that of dynamical systems with holes. Let A be a measurable subset (not necessarily small) of the phase space Ω and imagine that the trajectory under the map T is killed when it enters A for the first time (the system leaks out of A). Various interesting questions arise for the trajectories which have survived up to a time n. In particular the set of

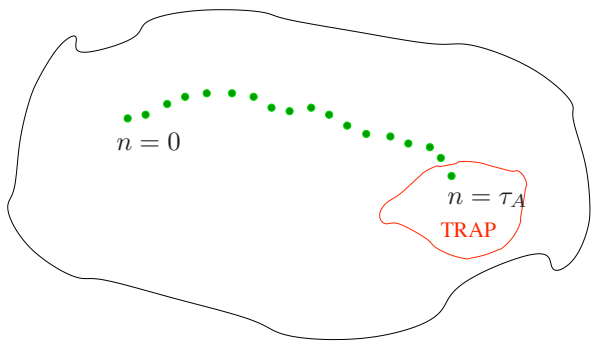


Figure 103: A discrete time evolution with a trap.

initial conditions which never enter A (in other words which survive for ever) is given by

$$\mathscr{R} = \bigcap_{j=0}^{\infty} T^{-j}\big(A^c\big) \ .$$

Recall that T^{-1} denotes the set-theoretic inverse of the map T.

It often occurs that this set is small, for example of measure zero for some interesting measure. A simple example is given by the one parameter family of maps of the interval $[0, s]$ (for $s > 1$).

$$T_s(x) = \begin{cases} s(1 - |1 - 2x|) & \text{for } 0 \le x \le 1 \\ x & \text{for } 1 < x \le s \end{cases} \tag{84}$$

The image of the interval $]1/2s, 1 - 1/2s[$ falls outside the interval $[0, 1]$.

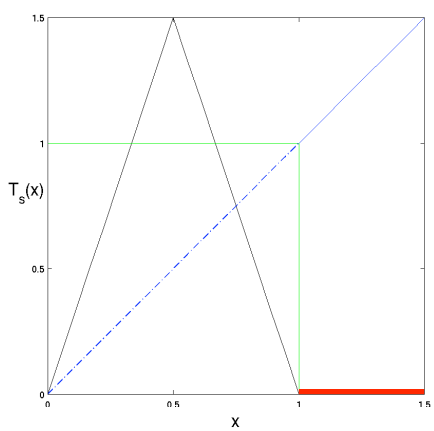


Figure 104: The graph of the map $x \to T_{3/2}(x)$ of formula (84).

Exercise 80. *Show that for the above maps T_s and $A =]1, s]$, the set $\mathscr{R}$ is a Cantor set of Lebesgue measure zero.*

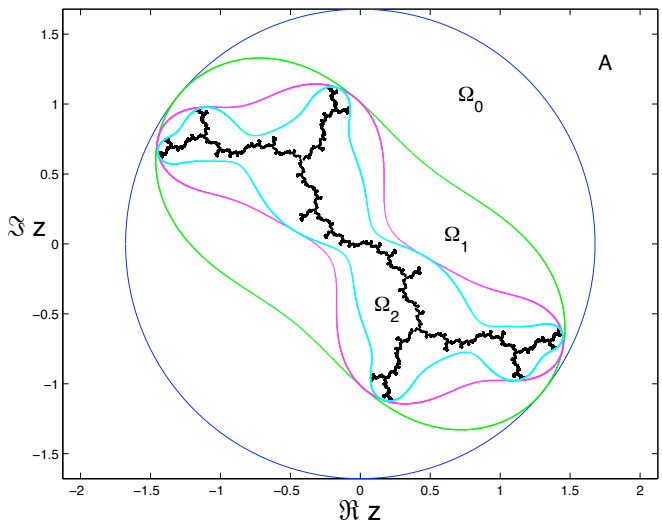


Figure 105: Example of sets A, Ω_0, Ω_1, Ω_2 and Ω_∞ for the map $T(z) = z^2 - 0.2282 + 1.1151i$

It may sometimes happen that the set $\mathscr{R}$ is empty. From now on we will always assume that this is not the case.

Assume now we have a given measure μ on the phase space Ω. We will denote by Ω_n the set of initial conditions which have survived up to time n (in other words whose trajectory has not met A up to time n). Namely

$$\Omega_n = \bigcap_{j=0}^{n} T^{-j}\big(A^c\big) \ .$$

This is obviously the subset of A^c where $\tau_A > n$. As mentioned above, it is often the case that $\mu\big(A_n\big)$ tends to zero when n tends to infinity. Note that $\Omega_\infty = \mathscr{R}$. The sets A, Ω_0, Ω_1, Ω_2 and Ω_∞ (the so called Julia set) for the map of the complex plane $T(z) = z^2 - 0.2282 + 1.1151i$ One can look at the restriction of μ to these initial conditions whose orbit has survived up to time n, namely we define a sequence of probability measures $\big(\mu_n\big)$ by

$$\mu_n(B) = \frac{\mu\left(B \cap \Omega_n\right)}{\mu\left(\Omega_n\right)} \ ,$$

which is just the conditional measure on the set of initial conditions which have not escaped up to time n (here and in the sequel we assume that for any n, $\Omega_n \neq \emptyset$).

The limiting measure if it exists is supported by the set $\mathscr{R}$ of initial conditions whose orbit survives forever. Such measures have been studied extensively in particular in the case of complex dynamics (see (see [29]) for example), and more generally in the case of repulsors which do not intersect the hole. We refer to [50], [34] and references therein.

Another interesting notion was introduce by Yaglom and is often called the Yaglom limit. It is connected with the notion of quasi invariant measure (or

quasi invariant distribution). Consider the initial conditions which have survived up to time n, namely which belong to Ω_n. The nth iterate of many of these initial conditions are points which for most of them are on the verge of escaping. Can one say something about their distribution?

This leads to consider a sequence $\big(\nu_n\big)$ of probability measures given by

$$\nu_n(B) = \frac{\mu\left(T^{-n}(B) \cap \Omega_n\right)}{\mu\left(\Omega_n\right)} \ .$$

This is different from the measures μ_n seen before which were measures on initial conditions. Here we have distributions on the survivors at time n. If $x \in T^{-n}(B) \cap \Omega_n$, its orbit up to time n avoids A, and $T^n(x) \in B$. Again one can ask if there is a limit for this sequence (the so called Yaglom limit of the initial distribution $\nu_0 = \mu$). It turns out that such limiting measures, if they exits, satisfy an interesting relation.

Lemma 7.8. *For any measurable set B and any integer n we have*

$$\nu_n\big(T^{-1}(A^c \cap B)\big) = \frac{\mu\big(\Omega_{n+1}\big)}{\mu\big(\Omega_n\big)} \nu_{n+1}(B) \ .$$

Exercise 81. *Give a proof of this lemma.*

Iterating this relation, one gets for any integers n and p

$$\nu_n\big(\Omega_p \cap T^{-p}(B)\big) = \frac{\mu\big(\Omega_{n+p}\big)}{\mu\big(\Omega_n\big)} \nu_{n+p}(B) \ .$$

In particular, if the ratio $\mu\big(\Omega_{n+1}\big)/\mu\big(\Omega_n\big)$ converges to a number $\rho > 0$, and the sequence $\big(\nu_n\big)$ converges in an adequate sense to a measure ν, this measure satisfies the relation

$$\nu\big(T^{-1}(A^c \cap B)\big) = \rho\, \nu(B) \ . \tag{85}$$

Definition 25. *A (probability) measure satisfying for some $\rho \in [0, 1[$*

$$\nu\big(T^{-1}(A^c \cap B)\big) = \rho\, \nu(B) \ ,$$

for any measurable set B is called a quasi stationary distribution (q.s.d.).

If $\rho = 1$ and $A = \emptyset$, ν is an invariant measure. Note that by taking $B = A^c$ we get

$$\rho = \frac{\nu\big(T^{-1}(A^c)\big)}{\nu\big(A^c\big)} \ .$$

Iterating equation (85) one gets

$$\nu\big(\Omega_p \cap T^{-p}(B)\big) = \rho^p \nu(B) \ .$$

Taking $B = \Omega$ and using that ν is a probability measure we get

$$\nu\big(\Omega_p\big) = \rho^p \ .$$

In other words, since

$$\nu\big(\Omega_p\big) = \nu\big(\tau_A > p\big)$$

we see that in the measure ν, the entrance time (which is here the time of death) is exactly exponentially distributed. Note that here there is no assumption on the size of the set A.

As an example, we come back to the case of the map T_s of example (84), with the set $A =]1, s]$, namely

$$T_s(x) = \begin{cases} s(1 - |1 - 2x|) & \text{for } 0 \le x \le 1 \\ x & \text{for } 1 < x \le s \end{cases}$$

Let ν be the measure on $[0, s]$ which is the Lebesgue measure on the interval $[0, 1]$ and vanishes on $]1, s]$. In order to check that ν is a quasi stationary distribution, it is enough to take $B \in A^c = [0, 1]$ and by definition (85) to verify that

$$\frac{\nu\big(T_s^{-1}(B)\big)}{\nu\big(B\big)}$$

does not depend on B. Since ν is a Borel measure, it is enough to verify this assumption for finite unions of intervals. Moreover one readily checks that if this relation holds for any interval (contained in $[0, 1]$), it will also hold for any finite union of intervals.

If $B = [a, b]$ $(0 \le a < b \le 1)$, we have at once

$$T_s^{-1}(B) = [a/(2s), b/(2s)] \cup [1 - b/(2s), 1 - a/(2s)] \ .$$

Note that $T_s^{-1}(B) \subset A^c = [0, 1]$. Therefore

$$\nu\big(T_s^{-1}(B)\big) = \frac{(b - a)}{s}$$

and

$$\frac{\nu\big(T_s^{-1}(B)\big)}{\nu\big(B\big)} = \frac{1}{s}$$

which does not depend on B.

Under the condition $T\big(\Omega_0\big) \supset \Omega_0$ and some conditions on T, Pianigiani and Yorke have established the existence of a q.s.d. absolutely continuous with respect to the Lebesgue measure (see [50] and references therein).

7.4.1 An application to the Newton method.

As an other application, we consider the problem of solving the polynomial equation $z^3 = 1$. The roots are of course well known (the three cube roots of unity 1, j, j^2), but one can investigate on this simple example the behaviour of the Newton method which is often used to find roots of more complicated equations. The Newton method consists in iterating a map (see for example [81]) which in this case is given by

$$R(z) = \frac{2z}{3} + \frac{1}{3z^2} \ .$$

It is easy to verify that the three roots of unity are the only fixed points ($R(z) = z$), and the derivative of R vanishes in these points. It is known that if one starts with an initial condition z_0 near one of the roots, the iteration

$$z_{n+1} = R\big(z_n\big) \ ,$$

converges very fast (doubly exponentially fast) to the root. However there are starting points whose orbit never converges. We refer to [65], [88] for more information.

Let Ξ_∞ denote the set of initial conditions whose orbit does not converge to any of the fixed points 1, j or j^2. This set turns out to be is in this case the Julia set of the map R (see for example [13] and [120] for definitions and results).

The part of this set near the origin is drawn in black in figure (7.4.1)

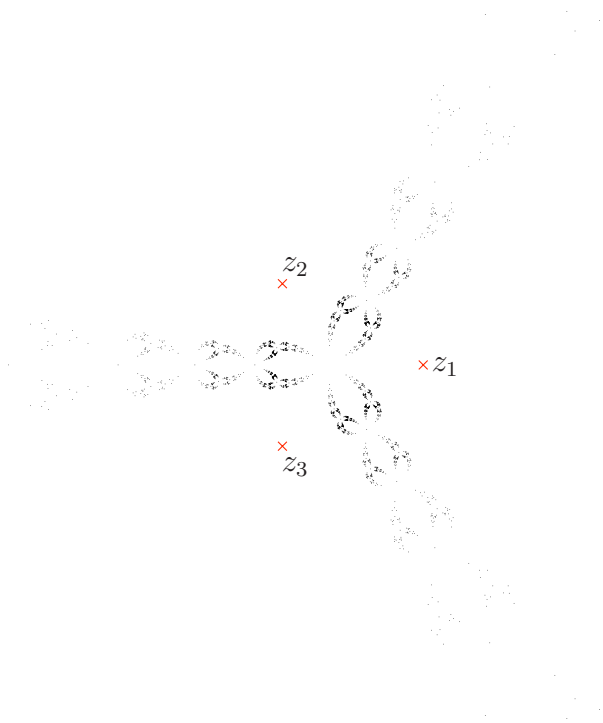


Figure 106: The Julia set for the Newton method of $z^3 = 1$

In order to apply the preceding results about the existence and properties of quasi stationary distributionss, we first conjugate the transformation by a map sending the fixed point one to infinity. One can use for example the (bijective) map

$$H(z) = \frac{1}{z-1} \ .$$

Instead of using R, it is therefore equivalent to use the map

$$f(z) = H \circ R \circ H^{-1}(z) = \frac{2z^3 + 6z^2 + 3z}{3z+2}$$

This map has two fixed points with zero derivative at $1 + j$ and $1 + j^2$ (and a third one at infinity).

The map f is again a rational map. We will denote by J its Julia set which is now a compact set. We have also conjugated the map by a rotation to obtain nicer pictures.

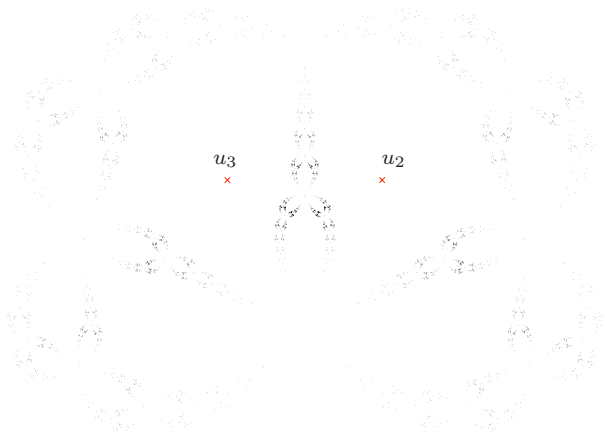


Figure 107: The Julia set for the map f.

Recall that a critical point of a rational map is a point where the derivative vanishes. The map R and hence the map f have only three critical points which are all fixed points. Therefore we conclude by Theorem 8.1 in [13] that the Julia set J is hyperbolic, namely there is a number $K_0 > 1$ and an integer n_0 such that

$$\inf_{z \in J} \left| f^{n_0\prime}(z) \right| > K_0 \ .$$

By the chain rule, we have for any z

$$f^{n_0\prime}(z) = \prod_{j=0}^{n_0-1} f'\big(f^j(z)\big) \ .$$

Since -2/3 $\notin J$, f' is continuous (and therefore bounded in J). This implies that $f^{n_0\prime}$ is continuous on a neighborhood of the compact set J. Therefore there exists a number $\epsilon_0 > 0$ and a number $K > 1$ such that

$$\inf_{z, d(z,J) \leq \epsilon_0} \left| f^{n_0\prime}(z) \right| > K \ ,$$

where $d(z, J)$ denotes the distance from z to J, namely

$$d(z, J) = \inf_{u \in J} |u - z| \ .$$

Let

$$\epsilon_1 = \frac{\epsilon_0}{\sup z, d(z, J) \leq \epsilon_0 |f^{n_0\prime}(z)|} \ .$$

Since $f^{n_0\prime}$ is continuous on the compact set J, we have $\epsilon_1 > 0$.

Lemma 7.9. *There exists a number $\epsilon_1 > 0$, $\epsilon_1 < \epsilon_0$, such that if we define*

$$\Omega_0 = \big\{ z \,\big|\, d(z, J) < \epsilon_1 \big\} \ ,$$

then

$$f^{n_0}\big(\Omega_0\big) \supset \Omega_0 \ .$$

Proof. Let $z \in J$. Since $\big|f^{n_0\prime}(z)\big| > K$, we can apply the inverse function theorem to $f^{n_0\prime}$ (see [63]). This implies that there exists a number $\tilde{\epsilon}_1 > 0$, which may depend on z such that $f^{n_0\prime}$ is a bijection from $B_{\tilde{\epsilon}_1}(z)$ to its image. Moreover, we can assume $\tilde{\epsilon}_1 > 0$ small enough so that $\epsilon_0 > \tilde{\epsilon}_1 > 0$ and for any $\epsilon \leq \tilde{\epsilon}_1$

$$f^{n_0}\big(B_\epsilon(z)\big) \supset B_{(1+K)\epsilon/2}\big(f^{n_0}(z)\big) \ .$$

Moreover, it follows from the inverse function theorem that the numbers $\tilde{\epsilon}_1$ are bounded away from zero uniformly in $z \in J$. We can take for ϵ_1 the infimum of these numbers and the result is proved. □

One can now apply the results of Pianigiani and Yorke (see [50] and references therein) to prove that there exists a quasi stationary distribution. We refer to [52] for more details.

We can now apply the inverse of the conjugation H to come back to our initial picture.

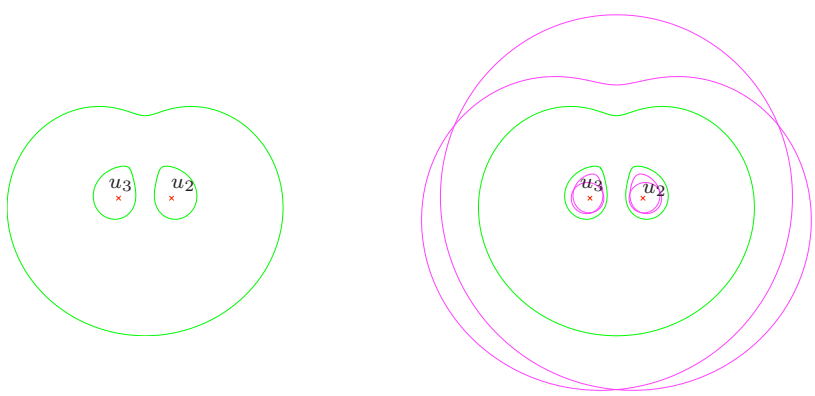


Figure 108: The boundary of a neighbourhood of the three critical points of the map f in green on the left, together with their image in red on the right.

Going back to the initial picture we get

7.5 Stochastic perturbations.

As we already mentioned several times, in concrete experiments (and to some extent also in numerical simulations) one obtains a unique record of a (finite piece of) trajectory of a dynamical (or its image through an observable), often corrupted by observational noise.

There may be also intrinsic noise in the system coming for example from thermal fluctuations. There are many models for this effect and we will only deal here with the simplest case.

We will assume that the phase space is $\mathbb{R}^d$ and that there is a sequence of i.i.d random variables $\xi_0, \xi_1, \ldots$ such that instead of iterating the orbit of an

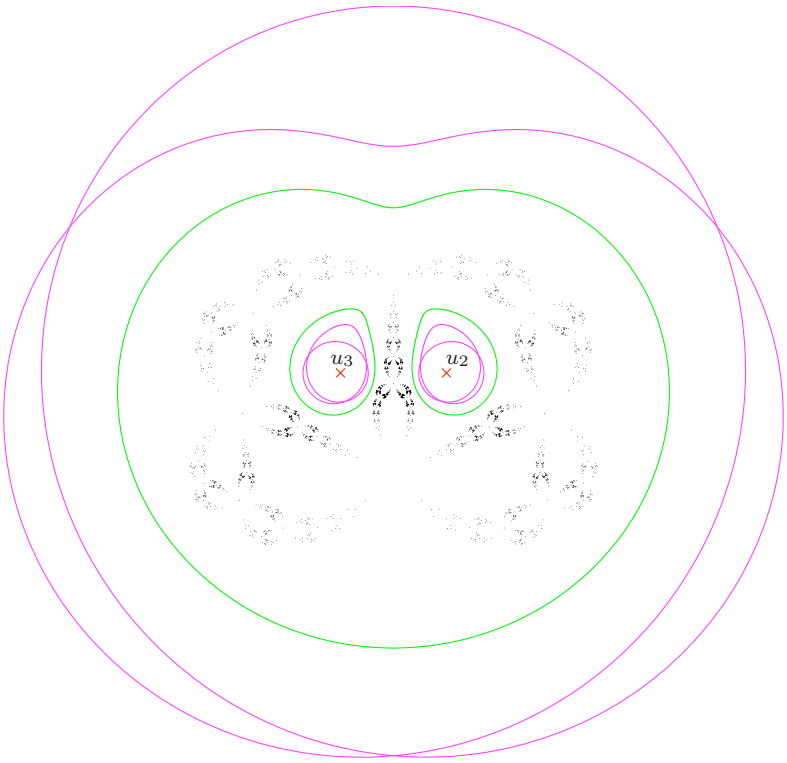


Figure 109: Julia set, boundary of a neighbourhood of the three critical points of the map f in green, together with their image in red.

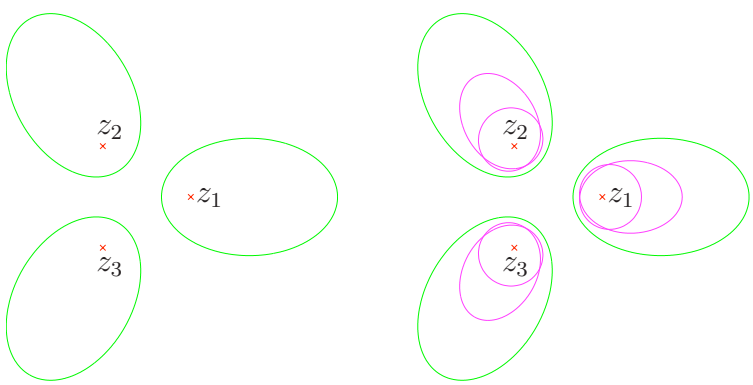


Figure 110: The boundary of a neighbourhood of the three critical points of the map R in green on the left, together with their image in red on the right.

initial condition x, one observes a sequence of points $\big(x_n\big)$ in the phase space given by

$$x_{n+1} = T(x_n) + \epsilon\, \xi_n \ , \tag{86}$$

where ϵ is a fixed parameter. The process $\big(x_n\big)$ is called a stochastic perturbation

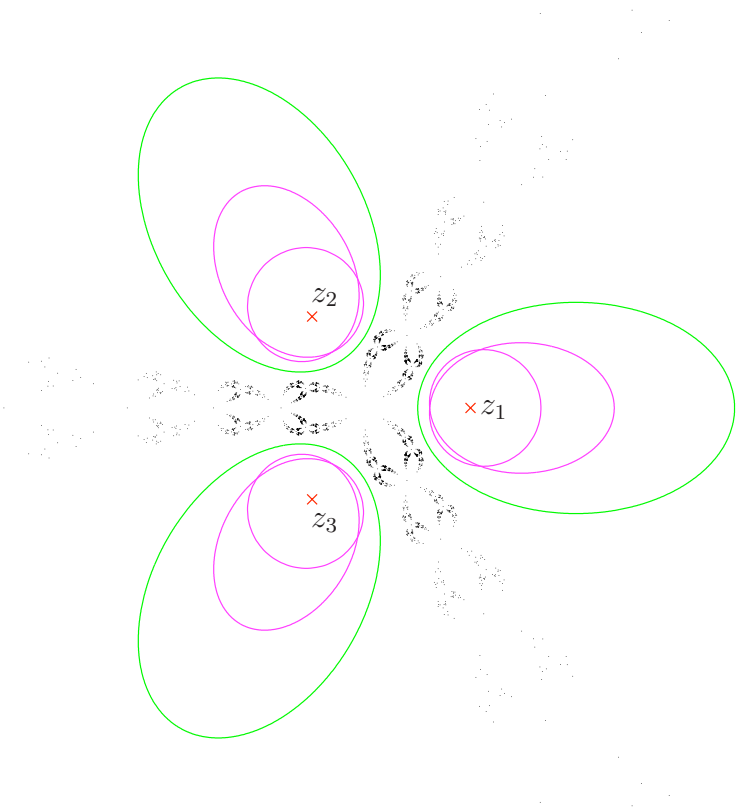


Figure 111: Julia set, boundary of a neighbourhood of the three critical points of the map R in green, together with their image in red.

of the dynamical system T.

In most cases, the noise (ξ_n) is small and it is convenient to normalize its amplitude by this parameter ϵ. For example one can impose that the variance of the ξ_n is equal to one. Physically, this noise reflects the fact that (mostly unknown or hard to model) extra effects neglected in the description of the experiment constantly perturb the system.

A good example are temperature fluctuations. Note also that formula (86)

$$x_{n+1} = T(x_n) + \epsilon\, \xi_n \ ,$$

describes a non-autonomous situation since the map also depends on time through the sequence (ξ_n).

If the ξ_n are independent, this is a Markov chain. Experimentally, one is of course very likely to meet a combination of stochastic perturbation and random measurement errors. Namely one gets a random iteration

$$x_{n+1} = T(x_n) + \epsilon\, \xi_n \ ,$$

and an observation corrupted by noise

$$y_n = x_n + \zeta_n \ .$$

Note that the two effects have in general a very different origin. The first one is in a sense more basic (neglecting in the modeling of some degrees of freedom), while the other is more observational (noise of the electronic equipment for example).

In the presence of noise there are two basic questions.

i) Can one recover to some extent the true trajectory out of the noisy one?

ii) Can one recover the statistical properties?.

Concerning the first question we have already mentioned the existence of denoising algorithms (see remark 4). We also recall that the shadowing lemma allows to construct a true orbit in the vicinity of a noisy one (see [21]). The more ambitious complete reconstruction of a trajectory out of a noisy perturbation (of infinite length) was discussed in several papers by Lalley and Nobel (see [108], [109], and [110]). We refer to the literature for the results

We will discuss now the consequences of the first effect (namely assume $\epsilon_n = 0$), and be concerned with recovering the statistical properties of the dynamical system out of noisy data. We will always assume below that the average of ξ_n is equal to zero. This is a natural assumption, for example the thermal noises are often white noises of zero average. If the average m of ξ_n is not zero, one can subtract it from the noise and add it to the transformation, namely replace ξ_n by $\xi_n - m$ and T by the map $x \to T(x) + \epsilon\, m$. In formula (86) this produces the same sequence $\big(x_n\big)$.

We now want to make a connection between the statistical properties of the (unperturbed, deterministic) dynamical system, and the statistical properties of the stochastically perturbed system. Recall that due to the hypothesis of independence of the ξ_n the sequence $\big(x_n\big)$ defined in formula (86)

$$x_{n+1} = T(x_n) + \epsilon\, \xi_n \ ,$$

is a Markov process. Adding the measurement errors would lead to a hidden Markov process. Assume moreover that the random variables ξ_n have a density g with respect to the Lebesgue measure. Then the transition probability is given by

$$p_\epsilon\big(x_{n+1} \,\big|\, x_n\big) = \frac{1}{\epsilon}\, g\left(\frac{x_{n+1} - T\big(x_n\big)}{\epsilon}\right) \ . \tag{87}$$

Exercise 82. *Prove this formula.*

Formally, when ϵ tends to zero, the right hand side converges to $\delta\big(x_{n+1} - T\big(x_n\big)\big)$ (where δ is the Dirac distribution) and we recover the original dynamical system. This limit is however quite singular and requires some care. It follows at once from formula (87) that in the case where the noise ξ_n has a density, any invariant measure for the Markov chain $\big(x_n\big)$ is absolutely continuous.

A natural question is what are the accumulation points of the invariant measures of the chain when the amplitude ϵ of the noise tends to zero. One should of course as usual avoid the phenomenon of escape of mass to infinity,

although several works have been devoted to this interesting question in the context of quasi stationary measures, see [54], [51] and [137].. This can be done for example by imposing a compact phase space (in that case some correction to the noise has to be made near the boundary).

Theorem 7.10. *Assume the phase space is compact, that the map T of the dynamical system is continuous and the noise is bounded. Then any (weak) accumulation point of invariant measures of the Markov chain when the amplitude ϵ of the noise tends to zero is an invariant measure of T.*

Proof. Let (ϵ_n) be a sequence tending to zero, and assume we have an associated sequence (μ_{ϵ_n}) of invariant measures (μ_{ϵ_n} is an invariant measure of the chain p_{ϵ_n}) which converges weakly to a measure μ. To show that μ is invariant, it is enough to prove (see exercise III.1) that for any continuous function g we have

$$\int g \circ T \, d\mu = \int g \, d\mu \, .$$

From the invariance of μ_{ϵ_n}, we have for any fixed continuous function g

$$\mathbb{E}\left(\int g\big(T(x) + \epsilon_n \xi\big) \, d\mu_{\epsilon_n}(x) \right) = \int g(x) \, d\mu_{\epsilon_n}(x) \, ,$$

where the expectation is over ξ. Since the phase space is compact, g is uniformly continuous, and therefore since the noise is bounded and ϵ_n tends to zero, for any $\eta > 0$ we can find an integer N such that for any $n > N$ we have

$$\big| g\big(T(x) + \epsilon_n \xi\big) - g\big(T(x)\big) \big| < \eta \, .$$

This implies for any $n > N$

$$\left| \int g\big(T(x)\big) \, d\mu_{\epsilon_n} - \int g(x) \, d\mu_{\epsilon_n}(x) \right| < \eta \, ,$$

and the result follows. □

This result leads naturally to the following definition.

Definition 26. *The invariant measure μ of a dynamical system is stochastically stable (with respect to the noise (ξ_n)) if it is an accumulation point of invariant measures of the stochastic processes (86) when the amplitude ϵ of the noise tends to zero.*

Even if the random variables ξ_n have a density (and we have seen that in this case the invariant measures of the chain are absolutely continuous), the accumulation points may not be absolutely continuous. This phenomenon occurs if the map T has only attracting sets of Lebesgue measure zero. The nearest we have seen to an absolutely continuous invariant measure is an SRB measure (see definition IV.3) whose conditional measures on the local unstable manifolds are absolutely continuous. The stochastic stability of SRB measures has been indeed proved for axiom A systems and in some non uniformly hyperbolic cases (see [165] and references therein).

Theorem 7.11. *Let (Ω, T) be an axiom A dynamical system. Let Λ be an attractor of T with a basin of attraction U. Let (x_n) be a stochastic perturbation as in (86) with a bounded noise with continuous density. Then for ϵ small enough there is a unique invariant measure μ_ϵ for the process (86) supported in U, and this measure converges weakly when ϵ tends to zero to the SRB measure.*

We will not prove this theorem, instead we mention the stochastic stability of a.c.i.p.m. for piecewise expanding maps of the circle whose proof is given below, see also [68].

The theory is analogous to the case of the interval, and in particular the spectral Theorem 4.2 for the Perron-Frobenius operator applies.

Considering the circle in spite of the interval avoids unnecessary complications due to the boundary (which do not carry any interesting concepts). The problem is that on the interval if we use formula (86) for the definition of the stochastic perturbation, even if ξ_n is bounded, if the point $T(x_n)$ is too near to the boundary, the point x_{n+1} can be outside of the interval. This may be an interesting leaking problem somewhat related to the questions of quasi stationary distributions but we do not want to add this supplementary complication here.

Theorem 7.12. *Let T be a piecewise expanding map of the circle with a unique mixing absolutely continuous invariant measure with density h. Assume the noise ξ_n has a density g which is continuous and with compact support. Then for any $\epsilon > 0$ small enough, the stochastic process*

$$\theta_{n+1} = f(\theta_n) + \epsilon\xi_n \pmod{2\pi} \tag{88}$$

has a unique invariant measure μ_ϵ which is absolutely continuous with density h_ϵ (which is a continuous function). When ϵ tends to zero, μ_ϵ converges weakly to $h\,dx$ and more precisely h_ϵ converges to h in the Banach space **BV** *of functions of bounded variations.*

The theorem can be proved using perturbation theory for the Perron-Frobenius operator. A typical perturbed spectrum looks as in the right figure.

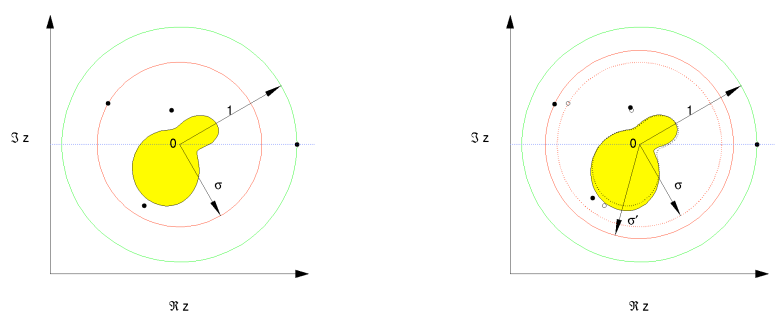


Figure 112: On the left, the spectrum of an unperturbed Perron-Frobenius operator. On the right, the spectrum under a small perturbation. Note that one is still an eigenvalue, the eigenvector being the density h_ϵ.

Proof. Since the support of ρ is bounded, there is a number $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0[$ we have almost surely and for any integer n the bound $\left|\xi_n\right| < \pi$.

In particular, since for such an ϵ the function $\rho(x/\epsilon)$ vanishes outside the interval $]-\pi,\pi[$, we can consider this function as a function defined on the circle. We will only consider below amplitudes ϵ of the noise belonging to the interval $\epsilon \in [0,\epsilon_0[$. Let R_ϵ be the operator defined on integrable functions of the circle by

$$R_\epsilon g(\theta) = \frac{1}{\epsilon}\int \rho\left(\frac{\theta-\phi}{\epsilon}\right) g(\phi)d\phi \ .$$

Using Young's Theorem, it follows at once that this operator is bounded in $L^1(d\,\theta)$ and $L^\infty(d\,\theta)$ with norm one. It is also easy to verify that it is bounded in the Banach space **BV** also with norm one.

Exercise 83. *Prove this last statement using the equivalent formula*

$$R_\epsilon g(\theta) = \frac{1}{\epsilon}\int \rho\big(\psi/\epsilon\big) g(\theta-\psi)d\psi \ .$$

It is also easy to verify that R_ϵ tends strongly to the identity in the Banach spaces $L^1(d\,\theta)$, $L^\infty(d\,\theta)$ and **BV**. Namely for any $g \in \mathcal{B}$ we have

$$\lim_{\epsilon\to 0}\big\|R_\epsilon(g)-g\big\|_{\mathcal{B}} = 0 \ .$$

Exercise 84. *Prove this assertion. Hint: observe that*

$$R_\epsilon(g)(\theta) - g(\theta) = \frac{1}{\epsilon}\int \rho\big(\psi/\epsilon\big)\big(g(\theta-\psi)-g(\theta)\big)d\psi \ ,$$

and approximate g by piecewise continuous functions.

As we have explained before, the transition kernel of the stochastic process (88) is given by (87), and therefore any invariant measure $d\mu_\epsilon$ has a density h_ϵ given by

$$h_\epsilon(\theta) = \frac{1}{\epsilon}\int \rho\left(\frac{\theta-f\big(\phi\big)}{\epsilon}\right) d\mu_\epsilon(\phi) \ .$$

In particular, this function h_ϵ should satisfy the equation

$$h_\epsilon(\theta) = \frac{1}{\epsilon}\int \rho\left(\frac{\theta-f\big(\phi\big)}{\epsilon}\right) h_\epsilon(\phi)\ d\phi \ .$$

This relation implies immediately that h_ϵ should be continuous and with support the whole circle.

Exercise 85. *Prove these statements (use that ρ is continuous).*

Performing a change of variables as in equation (13), one obtains immediately that h_ϵ should satisfy the equation

$$h_\epsilon = R_\epsilon P h_\epsilon \ ,$$

where P is the Perron-Frobenius operator (14). Had R_ϵ converged in norm to the identity, we could have used perturbation theory to finish the proof. This is however not true.

Exercise 86. *Prove that in the three Banach spaces $\mathcal{B}$ above, for any $\epsilon > 0$*

$$\left\| R_\epsilon - I \right\|_{\mathcal{B}} = 2 \ .$$

In order to control the perturbation in a weaker sense, we recall that from Theorem 4.2 we can write $P = P_0 + Q$ where P_0 is a rank one operator, and Q is an operator with spectral radius $\sigma < 1$. Therefore

$$R_\epsilon P = R_\epsilon P_0 + R_\epsilon Q \ ,$$

and here again, $R_\epsilon P_0$ is an operator of rank one. We now analyse the resolvent of this operator. We first observe that the rank one operator $R_\epsilon P_0$ has eigenvalue one. Indeed, we have (see Theorem 4.2)

$$\big(P_0 g\big)(x) = h(x) \int g(y)\, dy \ ,$$

and therefore since R_ϵ preserves the integral

$$\int \big(R_\epsilon P_0 g\big)\ (x) dx = \int \big(P_0 g\big)(x)\ dx = \int g(y)\, dy \ .$$

In other words, the integration with respect to the Lebesgue measure is a left eigenvector of eigenvalue one. This is not surprising and reflects the fact that the transition kernel is a probability. By a similar argument one also checks that for any integrable function g

$$\int \big(R_\epsilon Q g\big)\ (x) dx = 0 \ ,$$

namely $P_0 R_\epsilon Q = 0$. Since $R_\epsilon P_0$ is an operator of rank one with eigenvalue one, it is a projection.

Exercise 87. *Prove directly that $\big(R_\epsilon P_0\big)^2 = R_\epsilon P_0$.*

Therefore, for any complex number z with $z \neq 0, 1$ we have

$$\big(R_\epsilon P_0 - z\big)^{-1} = \frac{R_\epsilon P_0}{1-z} + \frac{I - R_\epsilon P_0}{z} \ .$$

This implies that, for $z \neq 0, 1$, we can write

$$R_\epsilon P - z = \big(R_\epsilon P_0 - z\big)\Big(I - \big(R_\epsilon P_0 - z\big)^{-1} R_\epsilon Q\Big)$$

$$= \big(R_\epsilon P_0 - z\big)\left(I - \frac{R_\epsilon P_0}{1-z} R_\epsilon Q - \frac{I - R_\epsilon P_0}{z} R_\epsilon Q\right) \ .$$

As we have seen above, $P_0 R_\epsilon Q = 0$, and we get

$$R_\epsilon P - z = \big(R_\epsilon P_0 - z\big)\left(I - \frac{R_\epsilon Q}{z}\right) \ .$$

Assume for the moment $\left\| Q \right\|_{\mathbf{BV}} < 1$ (recall that it is only the spectral radius which is known to be smaller than one). We can try to obtain the resolvent of $R_\epsilon P$ by using a Neuman series for the inverse of the last factor in the above

formula. In other words, we have to investigate the convergence of the sum of operators

$$K_\epsilon(z) = \sum_{j=0}^{\infty} \left(\frac{R_\epsilon Q}{z} \right)^j \ . \tag{89}$$

This series converge for any complex number z such that $|z| \geq \upsilon(\epsilon) = \left\| R_\epsilon Q \right\|_{\mathbf{BV}}$. In particular, if $|z| > \upsilon(\epsilon)$ and $z \neq 1$, the operator $R_\epsilon P - z$ is invertible, and its inverse is given by

$$(R_\epsilon P - z)^{-1} = K_\epsilon(z) \left(\frac{R_\epsilon P_0}{1-z} + \frac{I - R_\epsilon P_0}{z} \right) \ .$$

In particular, if $\upsilon(\epsilon) < 1$, we conclude that the operator $R_\epsilon P$ has one as a simple eigenvalue, and this is the only point in the spectrum outside the disk $|z| \leq \upsilon(\epsilon)$. We can therefore express the (rank one) spectral projection P_ϵ on this eigenvalue one by the formula (see [96])

$$P_\epsilon = \frac{1}{2\pi i} \int_{|z-1|<(1-\upsilon(\epsilon))/2} (R_\epsilon P - z)^{-1} \, dz =$$

$$\frac{1}{2\pi i} \int_{|z-1|<(1-\upsilon(\epsilon))/2} K_\epsilon(z) \left(\frac{R_\epsilon P_0}{1-z} + \frac{I - R_\epsilon P_0}{z} \right) dz \ .$$

The function $P_\epsilon h$ is proportional to h_ϵ since P_ϵ is a rank one projection on this function. Moreover, since h is of integral one and the integration with respect to the Lebesgue measure is a left eigenvector of P_ϵ of eigenvalue one, we have

$$\int \big(P_\epsilon h\big)(x) \, dx = \int h(x) \, dx = 1 \ .$$

This implies immediately

$$P_\epsilon h = h_\epsilon \ .$$

Using the above formula for P_ϵ and the fact that $P_0 h = h$ we get from Cauchy's formula

$$h_\epsilon = \frac{1}{2\pi i} \int_{|z-1|<(1-\upsilon(\epsilon))/2} K_\epsilon(z) R_\epsilon h \, \frac{dz}{1-z} = K_\epsilon(1) R_\epsilon h \ .$$

We now observe that

$$h_\epsilon = K_\epsilon(1) R_\epsilon h = R_\epsilon h + \sum_{n=0}^{\infty} \big(R_\epsilon Q\big)^n R_\epsilon Q R_\epsilon h \ .$$

Since R_ϵ converges strongly to the identity in $\mathbf{BV}$ and since $Qh = 0$, we have

$$\lim_{\epsilon \to 0} Q R_\epsilon h = 0 \ .$$

Since we have assumed that $\left\| Q \right\|_{\mathbf{BV}} < 1$, and since we have established that $\left\| R_\epsilon \right\|_{\mathbf{BV}} = 1$, this implies

$$\lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \big(R_\epsilon Q\big)^n R_\epsilon Q R_\epsilon h = 0 \ .$$

It then follows again from the strong convergence of R_ϵ to the identity that

$$\lim_{\epsilon \to 0} h_\epsilon = h \ .$$

If $\|Q\|_{\mathbf{BV}} \geq 1$, since the spectral radius of Q is strictly smaller than one, there is an equivalent norm where Q is a strict contraction (see [96]). One equips $\mathbf{BV}$ with this new equivalent norm, and repeats the above argument. The details are left to the reader. $\square$

We refer to [165] and [16] for references and more results on this subject.

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