Deterministic and stochastic views of turbulence.

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Abstract

I want to understand the connection between dynamical systems and stochastic processes.

1 Introduction

This presentation is about two questions that I have always asked myself (and a few other people), for which I never obtained statisfactory answers:

1. Why do we generate random numbers in a computer in a completely deterministic way, and still "get away" with it?

2. Why do we take averages and moments of the quantitities in the Navier-Stokes and scalar transport equations, and, again, "get away" with the statistics?

It turns out that fairly reasonable answers for these questions are available, even though there is much more open to learning and researching. The answers lie in the intertwined fields of *stochastic processes*, *dynamical systems*, and *ergodic theory*.

I am not qualified to talk about those things at any decent level of knowledge and depth. Therefore, this talk is little more than a sketch of what I can *glimpse* to be answers.

In the most basic and perhaps brutal sense, it is profitable to *regard* turbulence data as realizations of an underlying stochastic process:

- We can do all kinds of statistical manipulations: means, standard deviations, covariances, correlation functions, structure functions, spectra.

- This is in principle no different than modeling streamflow or precipitation as stochastic processes, or fluctuations in the stock market.

In this very simple approach, statistics is a *tool* used to analyze a very complicated phenomenon. Rather than go into its intricacies, we are content with a statistical description. Therefore, this approach can be called "pragmatic": we forfeit any hope of discussing the underlying *nature* of the phenomenon, and instead use descriptors of its variability: all we want is to make statements about the probability of finding values related to the phenomenon within a specified range.

In many cases, such as the throw of a coin or of a die, although we *know* that there are equations of motion governing the process, we also think that the complications of the motion and (most often) the initial conditions are so big that we are justified in *regarding* the outcome as stochastic: see the nice figure 1. There, the statistical description boils down to (perhaps) some experiments at actually throwing dice, and then summing up with the statement that, for a balanced or fair die, we must have:

$$
\begin{aligned}
&P\{X=1\} = P\{X=2\} = P\{X=3\} = \\
&P\{X=4\} = P\{X=5\} = P\{X=6\} = 1/6.
\end{aligned}
\tag{1}
$$

Note the slight circular argument here: a fair die is one that lands on each face with the same probability! From (1), then, we can make statements such as

$$
\mathrm{E}\{X\} = \sum_{i=1}^{6} \frac{i}{6} = \frac{21}{6} = \frac{7}{2}, \tag{2}
$$

$$
\mathrm{Var}\{X\} = \sum_{i=1}^{6} (i - {}^{7}\!/_{2})^2 \, \frac{1}{6} = \frac{35}{12}. \tag{3}
$$

At this point, therefore, we identify two completely separated ways to approach a problem:

Deterministically, by solving the equations for its dynamics, and analyzing a certain number of results.

Stochastically, by *not even employing dynamical equations*, and proposing instead statistical descriptors, parametric or not, based on which probabilistic statements can be made.

Statistical mechanics, on the other hand, tries to employ both approaches: starting with dynamical equations, we try to derive the probabilistic laws for the phenomenom that somehow are compatible, from the beginning, with the dynamics.

Most of the successful approaches to turbulence problems are of this "mixed" type. Yet, very seldom do we find in the books and papers an explicit justification for the approach. More specifically,

1. We almost never find a direct mention to the *source* of randomness in a statistical mechanical description: it is usually assumed just to "be there".

2. We almost never find a justification for treating the outcome of a deterministic dynamical system stochastically. We take averages of the equations, in many senses, and "get used to it".

3. The emergence of irreversibility and therefore the second law of thermodynamics from deterministic and time-reversible equations (such as the laws of motion of classical mechanics) remains a mystery.

A complete solution to these shortcomings is not in sight, and in the current notes we cannot promise really anything: at the end of the day, we will be essentially back to 1–3 above. However, we hope to give enough background here for one to *undertsand* better the shortcomings of our current approaches to both the deterministic and the stochastic views of turbulence.

2 Cases where we mix up random and deterministic

Einstein (1905)'s theory of Brownian motion is a good example.

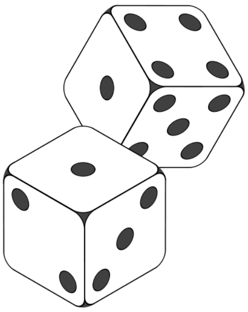

Figure 1: Physical dice thrown on a table obey the deterministic equations of motion: how come the outcome is "random"?

3 Why do we need an axiomatic approach

Very good discussion in

http://scientopia.org/blogs/goodmath/2013/08/24/kolmogorovs-axioms-of-probability/

and
http://www.ma.utexas.edu/users/mks/statmistakes/probability.html

4 State space

So physicists talk about the space state Ω. In our case, the state space is the whole field of fluid velocities U_i, temperature T, density $\wp$, and pressure P in a region of physical space where the flow takes place.

The time evolution of the system configuration in state space is a really interesting thing. Let the state of the system at $t = 0$ be $\boldsymbol{x}_0 \in \Omega$. Although mathematicians don't bother, I am assuming that $\boldsymbol{x}_0$ is some kind of vetor in state space. In a discrete-time evolution, the dynamical system will evolve according to some rule $T : \Omega \to \Omega$, so that:

$$\begin{aligned}
\boldsymbol{x}_1 &= T(\boldsymbol{x}_0),\\
\boldsymbol{x}_2 &= T(\boldsymbol{x}_1) = T(T\boldsymbol{x}_0)),\\
\boldsymbol{x}_n &= T(T(\ldots T(\boldsymbol{x}_0))) = T^n(\boldsymbol{x}_0).
\end{aligned}$$

In a continuous-time evolution, we must have

$$\begin{aligned}
T_0(\boldsymbol{x}_0) &= \boldsymbol{x}_0,\\
T_s(T_t(\boldsymbol{x}_0) &= \boldsymbol{x}_{s+t} = T_{s+t}(\boldsymbol{x}_0).
\end{aligned}$$

The simplest example, and probably the most interesting to us, is the system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{x}, t).$$

Numerical solutions of partial differential equations, like the Navier-Stokes equations, can in principle always be cast in this form.

5 Examples, from Collet

Let $\Omega = [0, 1]$ and

$$f(x) = 2x \bmod 1 \tag{4}$$

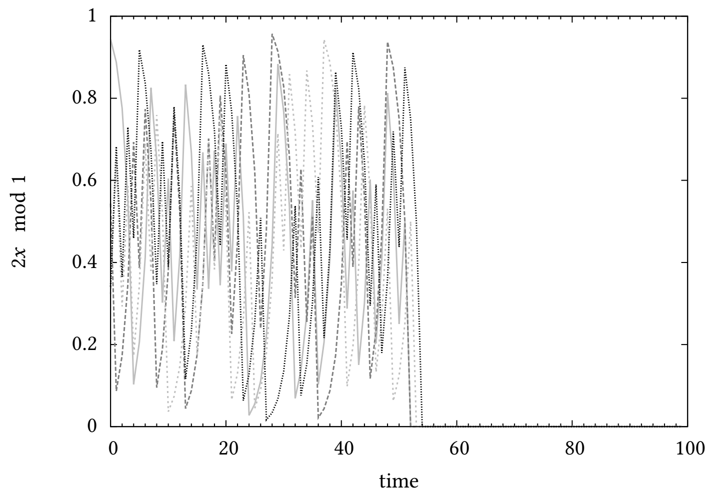

Figure 2: The dynamical system (4). For any initial condition (4 shown), it locks to 0 after $t \sim 50$.

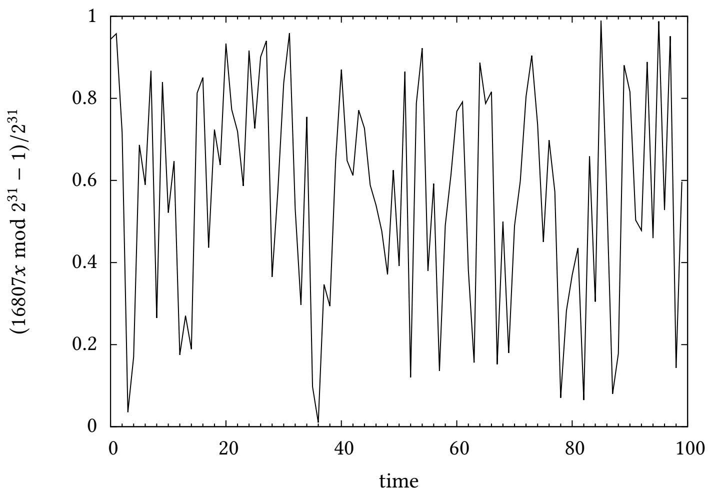

Figure 3: The dynamical system (5), a simple random number generator.

We can actually write a simple program, and plot the result. It is shown in figure 2.

Our second example is the map

$$f(x) = 16807x \mod 2^{31} - 1. \tag{5}$$

It is shown in figure 3, normalized by 2^{31}. It is a random number generator!

In both examples above, T (or f) makes $\boldsymbol{x}$ re-visit the points of Ω on and on. In some loose sense, the relative frequency with which different regions of Ω are visited will produce a probability measure on Ω (I think).

Then, on p. 8 of Collet: the Navier-Stokes equations are a dynamical system!

So what is the difference between the NS equations and the simple random number generator (5)? Maybe not that great!

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According to Collet, there are basically to approaches to understand the time evolution of a dynamical system:

The topological, or geometric description, concerned with orbits, attractors, bifurcations, etc.

The ergodic description, concerned with measures, (and I add: probabilities, averages, etc.)

Let us proceed slowly, because Collet is a good source!

Given $A \subset \Omega$, let us define the indicator function $\mathcal{X}_A(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega$:

$$\mathcal{X}_A(\boldsymbol{x}) \equiv \begin{cases} 1, & \boldsymbol{x} \in A, \\ 0, & \boldsymbol{x} \notin A. \end{cases}$$

Now let a (discrete-time) dynamical system start at $\boldsymbol{x}_0$. Note that I don't know whether $\boldsymbol{x}_0$ is in A. The average time that it spends in A until time N is

$$m_N(\boldsymbol{x}_0, A) = \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^j(\boldsymbol{x}_0)). \tag{6}$$

Then define, assuming its existence:

$$\mu_{\boldsymbol{x}_0}(A) = \lim_{N\to\infty} m_N(\boldsymbol{x}_0, A). \tag{7}$$

First I would like to verify that the limit also exists for $T(\boldsymbol{x}_0)$: how do I do it?

$$\begin{aligned} m_N(T(\boldsymbol{x}_0), A) &= \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^j(T(\boldsymbol{x}_0))) \\ &= \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^{j+1}(\boldsymbol{x}_0)) \end{aligned}$$

I want to prove $\mu_{\boldsymbol{x}_0}(A) = \mu_{T(\boldsymbol{x}_0)}(A)$. I make the observation that removing finitely many elements from an infinite sum should not change the limit. Let's try:

$$\begin{aligned} \mu_{\boldsymbol{x}_0}(A) &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) \\ &= \cancelto{0}{\lim_{N\to\infty} \frac{1}{N+1} \mathcal{X}_A(\boldsymbol{x}_0)} + \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=1}^{N} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) \end{aligned}$$

In the same vein,

$$\begin{aligned}
\mu_{T(\boldsymbol{x}_0)}(A) &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^j(T(\boldsymbol{x}_0))) \\
&= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^{j+1}(\boldsymbol{x}_0)) \\
&= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=1}^{N} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) + \cancelto{0}{\lim_{N\to\infty} \frac{1}{N+1} \mathcal{X}_A(T^{N+1}}(\boldsymbol{x}_0))
\end{aligned}$$

Hence, $\mu_{\boldsymbol{x}_0}(A) = \mu_{T(\boldsymbol{x}_0)}(A)$ ∎

Things are bound to become more difficult, however. First, it is *also* OK to write

$$\begin{aligned}
\mu_{\boldsymbol{x}_0}(A) &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) \\
&= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N-1} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) + \cancelto{0}{\lim_{N\to\infty} \frac{1}{N+1} \mathcal{X}_A(T^N}(\boldsymbol{x}_0))
\end{aligned}$$

Now,

$$\begin{aligned}
\mu_{T^{-1}(\boldsymbol{x}_0)}(A) &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_A(T^j(T^{-1}(\boldsymbol{x}_0))) \\
&= \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=-1}^{N-1} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) \\
&= \cancelto{0}{\lim_{N\to\infty} \frac{1}{N+1} T^{-1}}(\boldsymbol{x}_0) + \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N-1} \mathcal{X}_A(T^j(\boldsymbol{x}_0)) = \mu_{\boldsymbol{x}_0}(A) \ \blacksquare
\end{aligned}$$

Now change gear, and look at

$$m_N(\boldsymbol{x}_0, T^{-1}(A)) = \frac{1}{N+1} \sum_{j=0}^{N} \mathcal{X}_{T^{-1}(A)}(T^j(\boldsymbol{x}_0)).$$

The main point here is

$$\begin{aligned}
T^j(\boldsymbol{x}_0) \in T^{-1}(A) &\Rightarrow T(T^j(\boldsymbol{x}_0)) \in T(T^{-1}(A)), \\
&\Rightarrow T^{j+1}(\boldsymbol{x}_0) \in A.
\end{aligned}$$

Hence,

$$\mathcal{X}_{T^{-1}(A)}(T^j(\boldsymbol{x}_0)) = \mathcal{X}_A(T^{j+1}(A)).$$

The fact that $\mu_{\boldsymbol{x}_0}(A) = \mu_{\boldsymbol{x}_0}(T^{-1}(A))$ follows from the same line of reasoning as above.

From Collet:

> If one assumes that $\mu_{\boldsymbol{x}_0}(A)$ does not depend on $\boldsymbol{x}_0$ at least for Borel sets A (or some other sigma algebra but we will mostly consider the Borel sigma algebra below), one is immediately lead to the notion of invariant measure.

A measure μ on a sigma-algebra $\mathcal{B}$ is invariant by the measurable map T if for any measurable set A

$$\mu\left(T^{-1}(A)\right) = \mu(A). \tag{8}$$

This may be very important! This may assure that the dynamical system is a "maker" of random variables! Let's find out!

Nomenclature: $(\Omega, T, \mathcal{B}, \mu)$ is a dynamical system with state space Ω, discrete time evolution T, $\mathcal{B}$ is a sigma-algebra on Ω such that T is measurable with respect to $\mathcal{B}$ is μ is a measure on $\mathcal{B}$ that is invariant by T.

A broken theorem!

μ is invariant if and only if for every measurable g we have

$$\int g \circ T \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu. \tag{9}$$

Now, since g defines a random variable in Ω, the collection $g \circ T^n$ for all n defines a stochastic process. On account of (9), this is a stationary stochastic process.

> All the results from the theory of stochastic processes apply to a dynamical system equipped with an invariant measure.

It is important to mention that the very notion of what is deterministic, and what is stochastic, is fuzzy. For instance, in this passage Breuer and Petruccione (1992), we read, right at the introduction:

> It is well known that models of homogeneous turbulence often rely upon statistical tools [1,2]. In principle, statistical concepts are introduced in the theory only by considering random initial ensembles of velocity fields. However, the time evolution of each member is governed by the deterinistic Navier-Stokes equation.

In this view, the time evolution is deterministic, and randomness enters via the initial conditions. But the general definition of a *stochastic process* is very similar. It is: given a triplet $(\Omega, \mathcal{F}, P)$, a stochastic process is a *measurable function*

$$\begin{aligned} X : (\Omega, T) &\to \mathbb{R}, \\ (\omega, t) &\mapsto x = X(\omega, t) \end{aligned}$$

T is the indexing set. T can be the natural numbers, the integers, the positive reals, etc.

References

H. P. Breuer and F. Petruccione. Burger's model of turbulence as a stochastic process. *Journal of Physics A: Mathematical and General*, 25(11):L661–L667, 1992.

A. Einstein. Über die von der molekularkinetischen theorie der wärme geforderte bewegung von in ruhenden flüssigkeiten suspendierten teilchen. *Annalen der Physik*, 322(8):549–560, 1905.