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Is turbulence ergodic?

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Abstract

We present results of a long in time numerical simulation of the Navier–Stokes equations in a cubic domain with periodic boundary conditions. Comparison of a variety of temporal and spatial statistical properties shows clear similarity between the two, which can be seen as evidence favouring validity of the ergodic hypothesis in turbulent flows.

Though it is not known whether three-dimensional turbulent flows are ergodic, it is common to use the ergodicity hypothesis [1] in turbulence research, e.g., in physical and numerical experiments. According to this hypothesis time (space) and ensemble averages (moments) are equal to one another. In fact, it is assumed that all statistical properties of statistically stationary flows of an ensemble are equivalent to those obtained using time series in one very long realization. A similar property is defined in space for statistically homogeneous flows by replacing time by space coordinate(s) in which the flow domain has an infinite extension, at least in one direction. There is a consensus in the belief that turbulent flows are ergodic. However, there seems to exist no direct evidence regarding the validity of the ergodicity hypothesis in turbulent flows, though some mathematical results regarding the ergodicity for the Navier–Stokes equations were reported recently [2–4]. This communication is a report about an attempt to obtain such evidence via direct numerical simulations of the Navier–Stokes equations without performing a large number of simulations at different initial conditions representing the members of an ensemble.

The main idea is simple and is based on the fact that if a turbulent flow is both statistically stationary in time and homogeneous in space than its temporal and spatial statistical properties should be the same if the ergodic hypothesis is correct. An important consequence is that it is not necessary to perform a large number of time consuming “brutal force” ex-

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periments with different initial conditions in order to compare the time-averaged value of a given observable against the "ensemble" averaged value at a given time. Thus one of the main purposes was to compare various temporal and spatial statistical characteristics of a "simple" turbulent flow which is (approximately) statistically spatially homogeneous and temporally statistically stationary. As mentioned this should be true of all statistical properties. However, for obvious reasons we bring here only some typical examples as diverse as possible with the emphasis on dynamically significant quantities. We used for this purpose a flow in the simplest geometry of a cubic domain with periodic boundary conditions and performed direct numerical simulations of the Navier–Stokes equations [5] with resolution 128^3 uniformly distributed grid points. In order to have comparable information for the time statistics the equations were run for 2 200 000 time steps (cf. with $128^3 = 2\,097\,152$) for 84 points in different locations in the flow domain. We used a simple deterministic forcing in large scales in the form $\mathbf{f} = A\cos z \cos y,\ B\cos x\cos z,\ C\cos y\cos x,\ A = B = C$. The Taylor microscale Reynolds number, $Re_\lambda \approx 145$. Below we give examples of comparison of temporal statistics for one point in space and spatial statistics based on one randomly chosen time snapshot. The results for other points/snapshots are essentially the same. Each figure in the following contains the results for spatial and temporal statistics allowing to make a direct comparison between the two. The spatial integral scale $L = 1.25$ which is not that small as compared to the scale of computational domain 2π. The time integral scale $T_I = 8.5$ and is considerably smaller than the total duration of the time series $T = 8800$.

Since the large scale deterministic forcing breaks the ergodicity on large scales we removed the mean velocity before comparing the temporal and spatial statistics of the velocity field. After such a removal both statistics became similar though some differences remained due to not large enough scale separation between the spatial integral scale and that of the computational box and due to the fact that the flow was only approximately statistically homogeneous, see Figs. 1 and 2.

An example of two point statistics is shown in Fig. 2(a), (b) for the longitudinal and transversal velocity increments $\Delta u'' = \Delta\mathbf{u}\cdot(\mathbf{r}/r)$, $\Delta u^{\perp} = |\Delta u - \Delta u''\mathbf{i}_r|$, where $\Delta\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$ is the velocity increment between the two points and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector connecting the two points. The results shown in this figure exhibit good agreement between the temporal and spatial statistics without the removal of the mean velocities for not large r ($r < 0.1$). The value of r is normalized on the half size of the computational box.

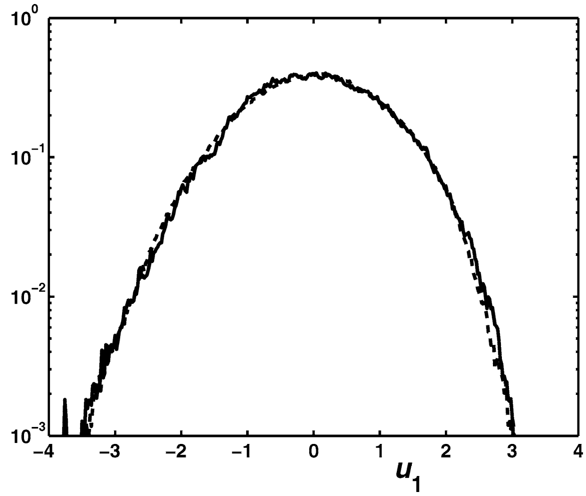

Fig. 1. An example of PDFs for one velocity component. The spatial PDF is made out of one randomly chosen time snapshot and the temporal PDF corresponds to the point located in the proximity of the geometrical center of the computational box. Similar behavior is observed for other points in the flow field. The maximal difference is less than 10%.

As expected the temporal and spatial statistics associated with the field of velocity derivatives exhibit much more similarity than those for the velocity field itself. This is true also both of one and of two-point statistics, e.g., for the longitudinal and transversal vorticity increments $\Delta\omega'' = \Delta\boldsymbol{\omega}\cdot(\mathbf{r}/r)$, $\Delta\omega^{\perp} = |\Delta\boldsymbol{\omega} - \Delta\omega''\mathbf{i}_r|$, where $\Delta\boldsymbol{\omega} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1$ is the vorticity increment between the two points and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ (see Fig. 2(c), (d)). Here the mean velocity does not have any influence on the statistics just like in the case of small distances for the results shown in Fig. 2(a), (b). In other words very good agreement between temporal and spatial statistics was observed without removal of the mean velocity for such basic quantities as vorticity, rate of strain and related. It seems that the main reason for better agreement between the temporal and spatial statistics associated with the field of velocity

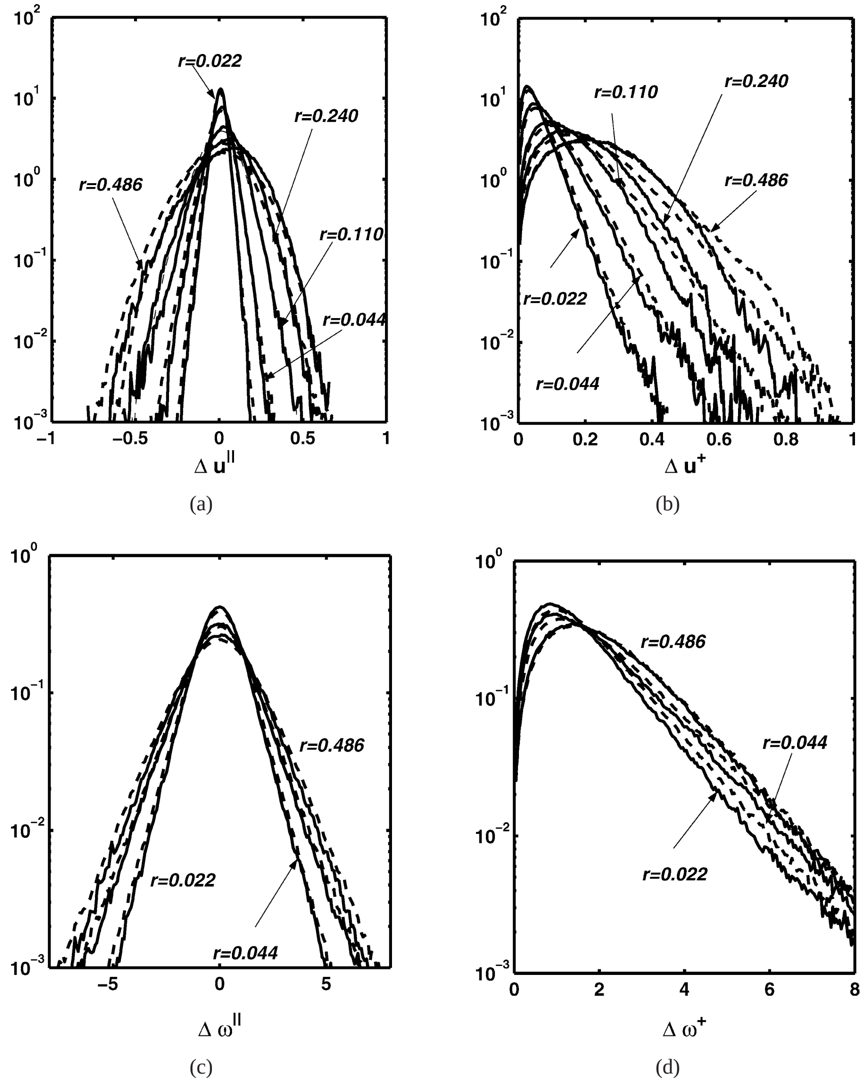

Fig. 2. Two-point statistics. PDFs of longitudinal $\Delta u''$ (a), and transversal $\Delta u^{\perp}$ (b), velocity increments for several distances between the points; longitudinal $\Delta\omega''$ (c), and transversal $\Delta\omega^{\perp}$ (d), vorticity increments for several distances between the points. The temporal statistics is shown by continuous lines.

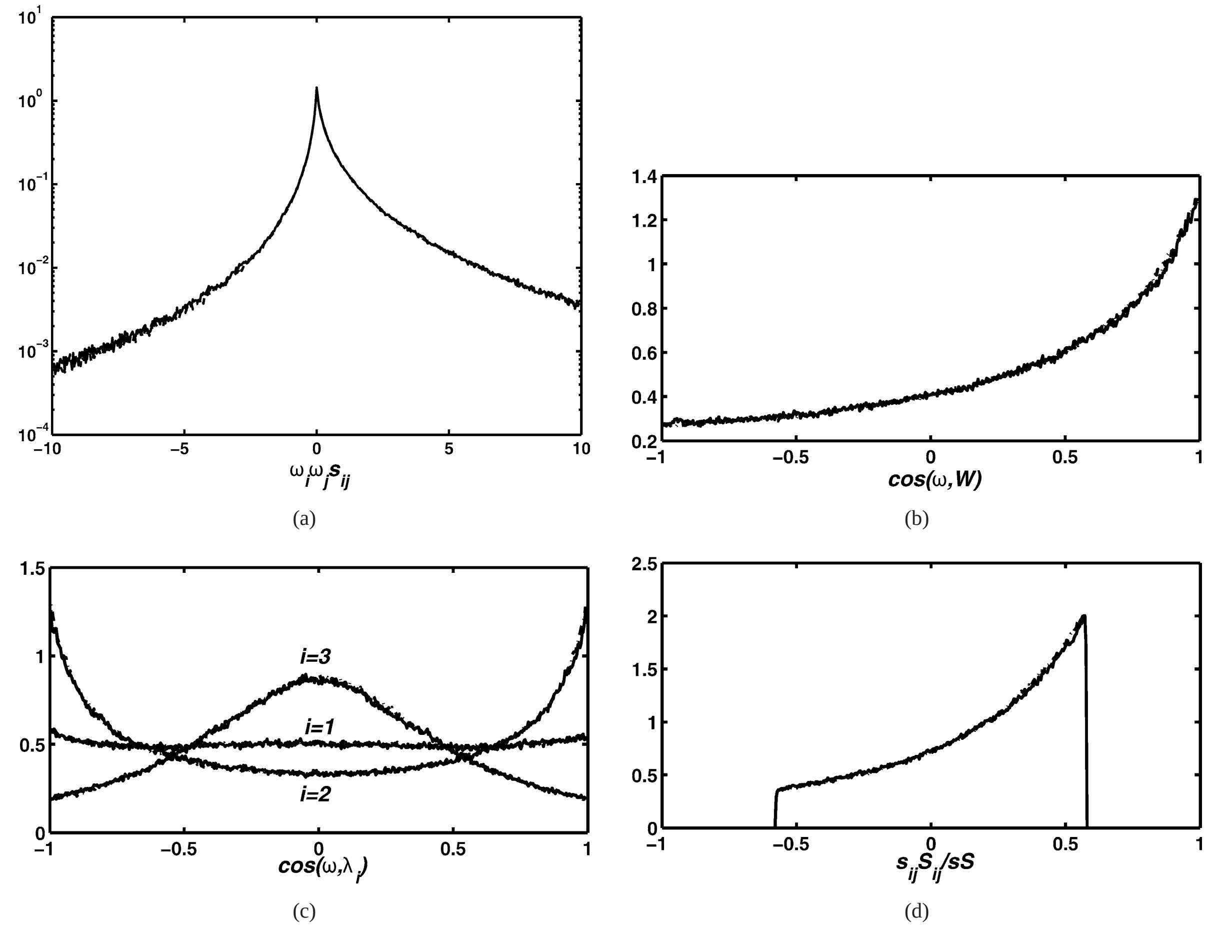

Fig. 3. Univariate probability density functions (PDFs) of dynamically significant quantities associated with the field of velocity derivatives–vorticity vector, ω_i and rate of strain tensor, s_{ik}: (a) entropy production $\omega_i\omega_k s_{ik}$; (b) cosine $\cos(\boldsymbol{\omega},\mathbf{W})$ of the angle between vorticity ω_i and vortex stretching vector $W_i=\omega_k s_{ik}$; (c) cosine $\cos(\boldsymbol{\omega},\boldsymbol{\lambda}_i)$ of the angle between vorticity ω_i and the eigenframe $\boldsymbol{\lambda}_i$ of the rate of strain tensor s_{ij}; (d) cosine of the "angle" between the rate of strain tensor s_{ij} and its stretching tensor $S_{ij}=s_{jk}s_{ki}$; this angle is defined via the relation $\cos(s,S)=s_{ij}S_{ij}/sS$; $s^2\equiv s_{ij}s_{ij}$, $S^2\equiv S_{ij}S_{ij}$, $S_{ij}=-s_{ik}s_{kj}$.

derivatives (and those for velocities for small distances between the two points) is the phenomenon of self-amplification of the field of velocity derivatives in three-dimensional turbulence [6], in which the forcing plays a negligible role.

The similarity between the temporal and spatial statistics is very good not only for univariate statistics (Fig. 3), but also for joint PDFs of two quantities (Fig. 4) and for joint PDFs of three quantities (Fig. 5). In Fig. 3 the temporal statistics is denoted in blue color.

Finally, we present also two examples of triple-point statistics for velocity and for vorticity, Fig. 6. Among many possible such quantities we have chosen the representative ones as follows $(\Delta u^{\perp})^2_{21}(\Delta u^{\perp})_{31}$, $(\Delta u'')^2_{21}(\Delta u'')_{31}$. Here $(\Delta u^{\perp})_{21}=\Delta\mathbf{u}_{21}\cdot(\mathbf{r}^{\perp}/r^{\perp})$, $(\Delta u^{\perp})_{31}=\Delta\mathbf{u}_{31}\cdot(\mathbf{r}^{\perp}/r^{\perp})$, $(\Delta u'')_{21}=|\Delta\mathbf{u}_{21}-(\Delta u^{\perp})_{21}\mathbf{i}_{r^{\perp}}|$, $(\Delta u'')_{31}=|\Delta\mathbf{u}_{31}-(\Delta u^{\perp})_{21}\mathbf{i}_{r^{\perp}}|$ and $\Delta\mathbf{u}_{21}=\mathbf{u}_2-\mathbf{u}_1$, $\Delta\mathbf{u}_{31}=\mathbf{u}_3-\mathbf{u}_1$, $\mathbf{r}^{\perp}=\mathbf{r}_{21}\times\mathbf{r}_{31}$, $\mathbf{r}_{21}=\mathbf{r}_2-\mathbf{r}_1$, $\mathbf{r}_{31}=\mathbf{r}_3-\mathbf{r}_1$, $\mathbf{i}_{r^{\perp}}=(\mathbf{r}^{\perp}/r^{\perp})$. Similar two quantities where chosen for vorticity. Again a good agreement is observed between the spatial and

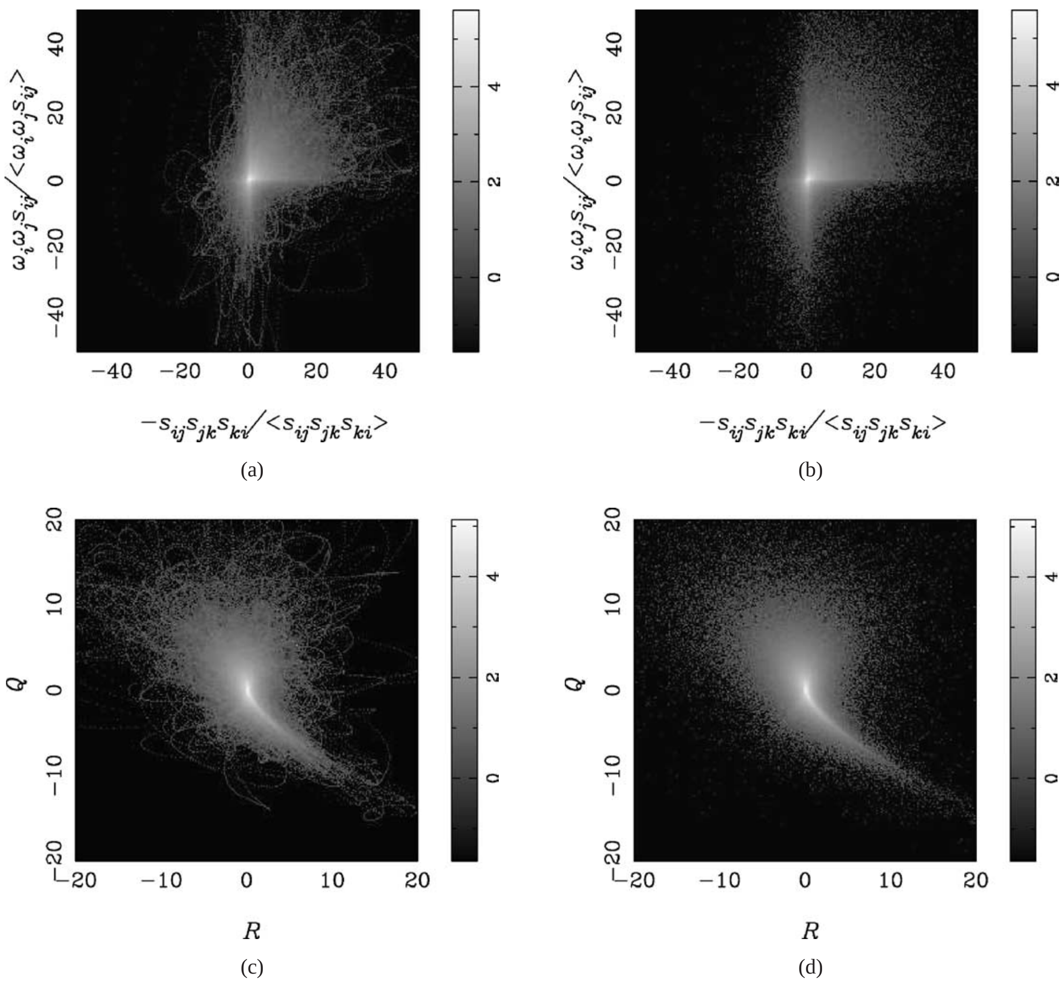

Fig. 4. Examples of joint statistics of two quantities: (a) and (b) joint PDF of $\omega_i\omega_j s_{ij}$ and $-s_{ij}s_{jk}s_{ki}$; (c) and (d) the "tearing drop" for the invariants R, Q of the velocity gradient tensor $\partial u_i/\partial x_j$, $R = -1/3\{s_{ij}s_{jk}s_{ki} + (3/4)\omega_i\omega_j s_{ij}\}$, and $Q = (1/4)\{\omega^2 - 2s^2\}$. Plots (a) and (c) correspond to a time series at the point located at $p(33)$. Plots (b) and (d) are based on 128^3 points at one time snapshot, i.e., correspond to spatial statistics. The similarity was checked for a variety of other quantities associated with vorticity and strain tensor.

temporal statistics, which is better for small **r** and not too large increments and for vorticity increments.

An interesting feature is seen in the examples in Figs. 4 and 5 corresponding to time statistics which show traces of time evolution, whereas corresponding examples associated with spatial statistics have nothing to do with the time evolution. The similarity between the two can be seen as an indication of equivalence of two formulations of the ergodic hypothesis. The first one corresponds to the "evolutionary" view on ergodicity, i.e., that the long enough trajectory will sample almost all of attractor in the phase space. Therefore the statistical properties of statistically stationary flows of an ensemble are equivalent to those obtained using a time series in one very long realization. Another formulation does not involve the evolutionary aspects and merely states the equivalence of statistical properties of the two.

In summary, the reported results from a long enough in time numerical simulation provides clear evidence that if a turbulent flow which is both statistically stationary in time and homogeneous in space than its temporal and spatial statistical properties are

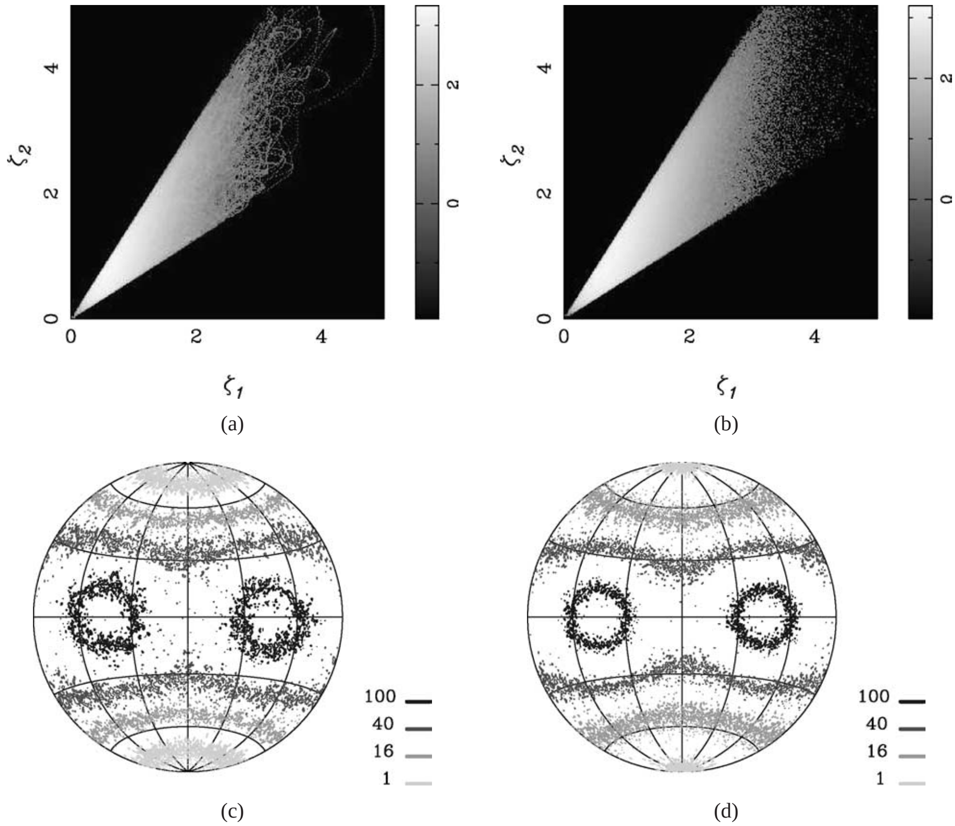

Fig. 5. Examples of joint statistics of three quantities related by a simple kinematic or geometric relation: (a) and (b) joint PDF of the eigenvalues of the rate of strain tensor $\Lambda_1, \Lambda_2, \Lambda_3$ in the plane $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$; (c) and (d) joint PDF of $\cos(\omega, \lambda_1)$, $\cos(\omega, \lambda_2)$ and $\cos(\omega, \lambda_3)$ on a sphere $\cos^2(\omega, \lambda_1) + \cos^2(\omega, \lambda_2) + \cos^2(\omega, \lambda_3) = 1$; in (c) and (d) we used the Hummer–Aitoff equal-area projection. Plots (b) and (d) are based on 128^3 points at one time snapshot, i.e., correspond to spatial statistics. Plots (a) and (c) correspond to a time series at the point located at $p(33)$. Joint statistics of three quantities not related by a simple kinematic or geometric relation exhibit strong similarity as well (not shown). We checked, for example, the joint PDFs of $\omega_i \omega_j s_{ij}$, ω^2 and s^2; $R = -1/3\{s_{ij} s_{jk} s_{ki} + (3/4)\omega_i \omega_j s_{ij}\}$, $Q = (1/4)\{\omega^2 - 2s^2\}$ and ω^2; $R = -1/3\{s_{ij} s_{jk} s_{ki} + (3/4)\omega_i \omega_j s_{ij}\}$, $Q = (1/4)\{\omega^2 - 2s^2\}$, and s^2; and some others.

the same. This in turn can be seen as evidence in favor of validity of the ergodic hypothesis, though we do not claim more than that.

It is naturally to expect that non-linear systems driven by a random force should be ergodic, see for example Ref. [4]. We would like to stress that our simulation was made with purely deterministic and time independent forcing. Nevertheless, the flow clearly exhibited strong similarity between its temporal and spatial statistical properties with the exception of the largest scales. A natural explanation is that this happens due to the property of self-randomization of fluid-dynamical turbulence.

A natural question concerns the nonhomogeneous flows. One can expect similar results as obtained above for flows with homogeneous coordinates, such as the flow in a plane channel [7]. An obvious conjecture is that the temporal and spatial statistical properties of such a flow will be the same for fixed values of the distance from the wall.

Acknowledgements

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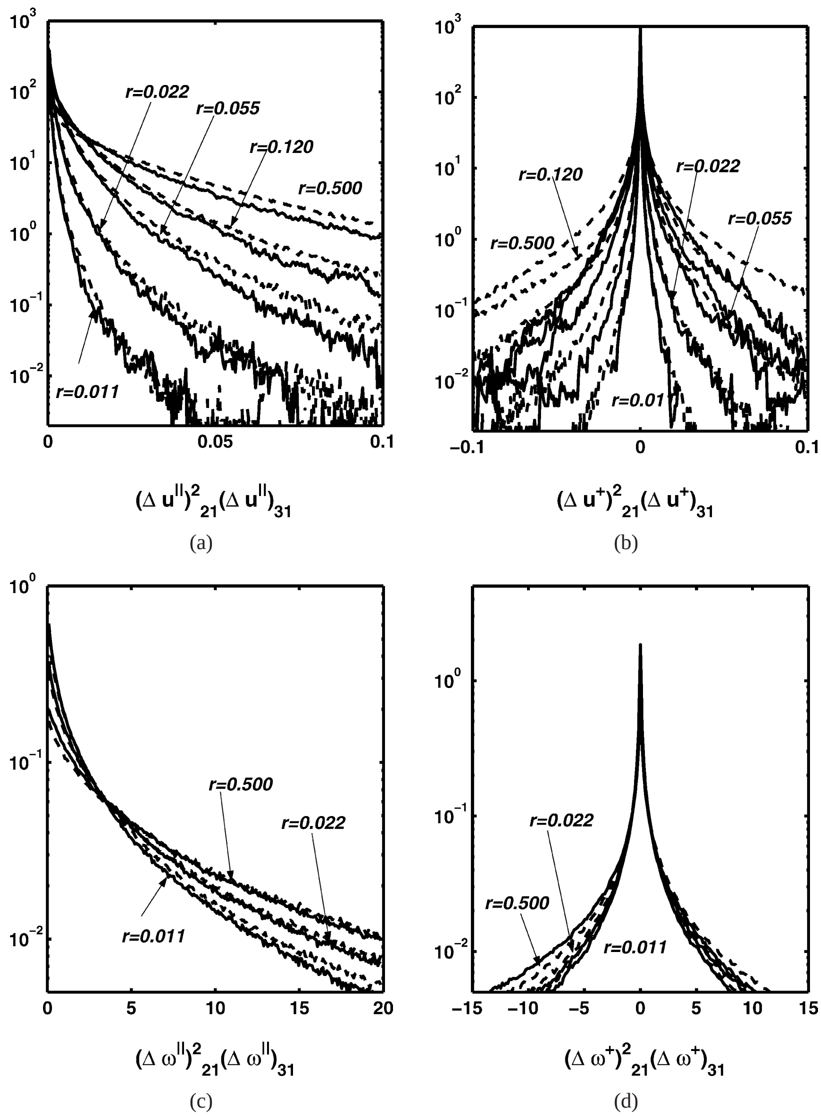

Fig. 6. Three-point statistics for velocity and vorticity increments. PDFs of longitudinal $\Delta u_n^{\prime\prime}$ (a) and transversal $\Delta u_n^{\perp}$ (b) velocity increments for several distances between the points; PDFs of longitudinal $\Delta\omega_n^{\prime\prime}$ (c) and transversal $\Delta\omega_n^{\perp}$ (d) vorticity increments for several distances between the points. The temporal statistics is shown by continuous lines.

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