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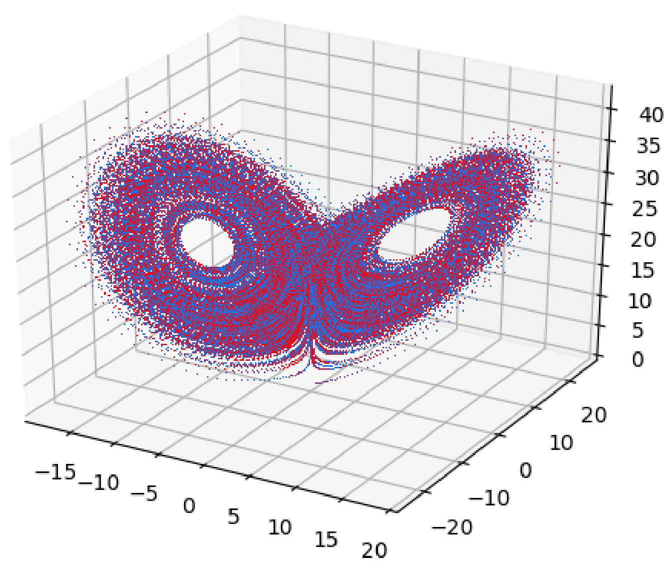
Numerical Simulation of the Butterfly Effect Through Generalized Lorenz Models



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1 Motivation

Weather forecasts influence the day-to-day of countless lives around the world. So much so that practically every cable news show will have multiple weather segments in a day with polished individuals presenting them, looking confident in their predictions. And most of the time they are! According to NOAA's SciJinks educational website [1], five-day forecasts and weekly forecasts are correct 90% and 80% of the time respectively. However, the true challenge lies in predicting weather on longer time scales. As an example, ten-day forecasts are only correct 50% of the time [1], and the downwards trend only gets worse as we try to predict further out into the future. These inaccuracies in long-term forecasting models were first observed by Edward Lorenz in his 1963 paper titled *Deterministic Nonperiodic Flow* [2]. In it, he used a three-parameter model with three ordinary differential equations to simulate a simplified convective system. The major takeaway was that these systems had extreme sensitivity to initial conditions, meaning two systems starting at near-identical states could end up in drastically different configurations after a certain time. The implications of this were of importance to weather forecasting, since the author concluded that it would be practically impossible to accurately predict long-term weather patterns due to these discrepancies and the limits of human measurement. Thus, the goal of this paper will be to explore the theory introduced by Lorenz and reproduce the conclusions of his landmark paper, as well as expand on it using more refined models put forth by Stein [3] and Shen [4].

2 Theory

The most commonly used forecasting approach is "Numerical Weather Prediction" or NWP for short [5]. It involves a computer-based modeling of the atmosphere which represents the current atmospheric state on a three-dimensional grid. Using the partial differential equations that govern how the atmosphere will change in time at each point, a numerical procedure is used in a time-dependent feedback loop to analyze the evolution of the modelled atmosphere.

2.1 The Lorenz Model

In regards to Lorenz's work, his NWP method used the equations developed by Saltzmann [6] to represent finite amplitude convection. Assuming all motions are parallel to the x-z plane, Saltzmann's equations used a stream function ψ to represent the x-z motion and a function θ to represent the "departure from temperature from that occurring in the state of no convection" :

$$\frac{\partial}{\partial t} \bigtriangledown^2 \psi = -\frac{\partial(\psi, \bigtriangledown^2 \psi)}{\partial(x,z)} + \nu \bigtriangledown^2 \psi + g\alpha \frac{\partial \theta}{\partial x} \ , \tag{1}$$

$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi, \theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial \psi}{\partial x} + \kappa \bigtriangledown^2 \theta \ . \tag{2}$$

The constants g, α, ν and κ represent the gravitational acceleration, the coefficient of thermal expansion, the kinematic viscosity and the thermal conductivity respectively.

Prior to Saltzmann, Rayleigh [7], while studying the flow occurring in a layer of fluid of uniform depth H, had found that equations of the form

$$\psi = \psi_0 \sin(\pi a H^{-1} x) \sin(\pi H^{-1} z) \tag{3}$$

$$\theta = \theta_0 \cos(\pi a H^{-1} x) \sin(\pi H^{-1} z) \tag{4}$$

would arise if the quantity called the Rayleigh Number

$$R_a = g\alpha H^3 \Delta T \nu^{-1} \kappa^{-1} \tag{5}$$

exceeded a critical value

$$R_c = \pi^4 \alpha^{-2} (1 = \alpha^2)^3 \ . \tag{6}$$

What Saltzmann did was to expand ψ and θ using a double Fourier series in x and z with only functions of time as coefficients. Substituting these into 3 and 2, he would proceed to reduce the resulting infinite system into a finite one containing only certain equations of time and finally performing numerical integration to get time-dependent solutions.

Simplifying Saltzmann's equations, Lorenz obtained three equations describing the motion of a particle undergoing convection :

$$\dot{X} = -\sigma X + \sigma Y \ , \tag{7a}$$

$$\dot{Y} = -XZ + rX - Y \ , \tag{7b}$$

$$\dot{Z} = -XY + bZ \ . \tag{7c}$$

Here X represents the intensity of the fluid motion, while Y and Z are the temperature variations in the horizontal and vertical components of the convection respectively.$\dot{X}$, $\dot{Y}$ and $\dot{Z}$ denote derivatives with respect to the dimensionless time $\tau = \pi^2 H^{-1}(1+a^2)\kappa t$ which is proportional to the actual time t. Therefore the dimensionless temporal evolution of the model will mirror the temporal evolution of the model to within a multiplicative constant. $\sigma = \kappa^{-1}\nu$ is the Prandtl number, $r = R_a R_c^{-1}$ and $b = 4(1+a^2)^{-1}$ (a is the ratio of the vertical scale of the convection cell to its horizontal counterpart). It is these dynamical equations that shall be used in our code. As a final note on the Lorenz model, it is important to note that the variable r is proportional to $\Delta T_{vertical}$. Therefore, if one desires to model the simple convection of a system (e.g. the atmosphere), by finding the temperature difference between the top and the bottom of the system, one could determine the r value which will best represent said system.

2.2 Higher Dimensional Models

As an extension of our experimentation on the 3D Lorenz model, 4D and 5D models by Stein [3] and Shen [4] were used to augment the Lorenz model in the hopes of improving outcome stability when fed similar initial points.

In the case of the 4D model, Stein added another variable $\dot{W}$ such that the Lorenz equations become

$$\dot{X} = \sigma(Y - X + dW) \ , \tag{8a}$$

$$\dot{Y} = -XZ + rX - Y \ , \tag{8b}$$

$$\dot{Z} = XY - bZ \ , \tag{8c}$$

$$\dot{W} = \sigma(-dX - W) \ . \tag{8d}$$

Here the new constant d is to take into account the vertical vorticity of the system. In his experimentation, Stein decided to focus on runs where $sigma = 0.5$, $b = 2$ and $d = 3$ since it was the regime where the conduction solution first loses stability to oscillatory convection. As such, our experimentation will use these same values. The previous Lorenz model can be thought of as representing a fluid which isn't rotating horizontally, whereas Stein does take into account this component of the movement. It can therefore be concluded that Stein wished to model a slightly different phenomena than Lorenz, and as such we should expect that for identical initial conditions and constants, both systems will most likely have different outcomes.

In regards to Shen's 5D model, it does not build upon Stein's 4D one. In fact, both models adapt the Lorenz model with no real relationship between them. What Shen decided to do was take into consideration more Fourier modes than Lorenz did. One can think that Lorenz decided to limit himself to 3 modes in his original paper due to the technological limits of computers at that time. As a result of including more Fourier modes, Shen obtained the 5 differential equations

$$\dot{X} = \sigma(Y - X) \ , \tag{9a}$$

$$\dot{Y} = -XZ + rX - Y \ , \tag{9b}$$

$$\dot{Z} = XY - XY_1 - bZ \ , \tag{9c}$$

$$\dot{Y_1} = XZ - 2XZ_1 - d_oY_1 \ , \tag{9d}$$

$$\dot{Z_1} = 2XY_1 - 4bZ_1 \ . \tag{9e}$$

Here the constants σ, r, and b are identical to those introduced in the Lorenz 3D model, while $d_0 = (1+9^2)/(1+a^2)$ was set to the same value as the d parameter in the 4D model so as to compare them. Shen explains his new model by stating that "the 5D Lorenz model may be viewed as a coupled system that consists of a forced dissipative system with low-wavenumber modes [i.e., Eqs. (9a)–(9c)] and a (nonlinear) dissipative-only system with high-wavenumber modes [i.e., Eqs. (9d)-(9e)]."

Armed with these three models and the parameters that govern them, we can now proceed to analyse them in depth.

3 Methods

3.1 Numerical Model Implementation

The numerical simulations used in this study were evaluated by a double-approximated Forward Euler method. The Forward Euler method alone uses linear approximation in order to solve differential equations for which the initial conditions are known [8]. If the position at time step n, X_n, moves along a curve satisfying the differential equation $F(t_n, X_n)$, then the new position can be calculated as:

$$X_{n+1} = X_n + \Delta t \cdot F(t_n, X_n) \tag{10}$$

after a time step of size Δt. This relies upon the approximation of a curve as a straight line over a sufficiently small Δt, which allows extrapolation along the tangent line of the curve. This method is a first-order method, meaning that it has local error proportional to $(\Delta t)^2$ and global error proportional to Δt [8]. Accuracy can be improved using a predictor-corrector algorithm, calculating a "first-guess" of X_{n+2} which is then used alongside the known value X_n in order to interpolate the position of interest according to Eq 11:

$$X_{n+1} = \frac{1}{2} \cdot (X_n + X^g_{n+2}) \tag{11}$$

where the prediction X^g_{n+2} is calculated using two steps of the Forward Euler method:

$$\begin{aligned} X^g_{n+1} &= X_n + \Delta t \cdot F(t_n, X_n) \\ X^g_{n+2} &= X^g_{n+1} + \Delta t \cdot F(t_{n+1}, X_{n+1}) \end{aligned}$$

This predictor-correction method can achieve local error proportional to $(\Delta t)^3$ and global error proportional to Δt^2 [8]. The double-approximated Forward Euler method comes at a higher computational cost due to the two additional calculation steps required to perform each iteration. However, this is deemed acceptable for the current investigation since computational times using a standard laptop computer are only on the order of 5 seconds when run over simulation lengths of 10000 dimensionless time units using $dt = 0.01$ for all tests. To put that performance into historical perspective, laptop computers regularly used by high school students today offer a computation rate of approximately 0.005 seconds per iteration including output time for this simulation method, 3 orders of magnitude better than Lorenz' original Royal McBee LGP-30 "electronic computing machine,"[1] which could perform one iteration per second not including output time.

3.2 Exploration of Parameter Spaces

Two variables were identified for an investigation of their effects on observed trajectories across all three of our models: the Prandtl number, σ, and the ratio of the Rayleigh number to its critical value, $r = \frac{R_a}{R_c}$, as defined in section 1.2. These variables were chosen because of their important physical interpretations and because the other multiplicative constants present are not consistently used or defined across all three models, causing difficulty in comparing the results between models.

While holding the initial conditions constant, simulations were conducted for both low (0.5) and high (10) Prandtl numbers. These numbers were chosen because of the significant changes in trajectories as well as their proximity to the values for air and water, respectively: air has a Prandtl number of around 0.71 while the value for water varies between 7.2 at 20°C and 13.4 at 0°C. [9]

When varying the Rayleigh number, a range of interesting values were examined due to the trajectory regimes they induce. In the Lorenz model for a hydrodynamical system, an r value greater than 1 indicates that convection is the dominant mode of heat transfer, while the critical value r_{crit} marks the transition from steady convection to unstable convection for a given system. Rayleigh numbers from 1.0 up to just below each system's r_{crit} were

[1] Weighing 800lbs and retailing for 47,000 USD, equivalent to over 400,000 USD today [10]

used to simulate trajectories which converge into a state of steady convection, then r_{crit} was determined by finding the value at which trajectories enter the unstable convection regime. Some of the systems tested never reached r_{crit} before [overflow problem as explained by Bill Frost]. The systems for which r_{crit} was found were also tested with higher Rayleigh numbers in order to observe the resultant unstable convection trajectories.

With regards to applications in long-range weather forecasting, a crucial characteristic of this prediction model is its sensitivity to initial conditions. In order to test this sensitivity, a simulation was run twice of the same system using the same parameters (σ, r, dt, etc.) with two slightly different initial states. The two trajectories were plotted on the same axes and the final position after 1000 time steps for each was marked in order to show the discrepancy in predicted value between the two simulation runs. This comparison was done using each of our three models for systems just below and just above r_{crit}.

We also performed a similar comparison between models using a sub-critical r value and identical σ, b, and initial conditions. For the 4DLM and 5DLM the d value was also held constant. This cross-model testing was conducted in the stable convection regime because the asymptotic behaviour observed here allows better quantitative comparison between final convergent states relative to the non-converging states in the unstable convection regime.

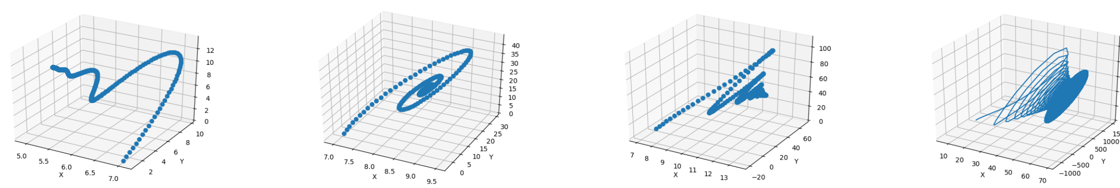
4 Results

4.1 Trajectories and Their Locations In Phase Space

In this section identical initial conditions were used throughout, namely $X = 7$, $Y = 1$, while Y_1, and Z_1 were initially set to zero. A b value of $8/3$ was used with all 3 models and a d value of 3 for the 4DLM was used alongside an identical d_0 value of 3 for the 5DLM.

4.1.1 3DLM

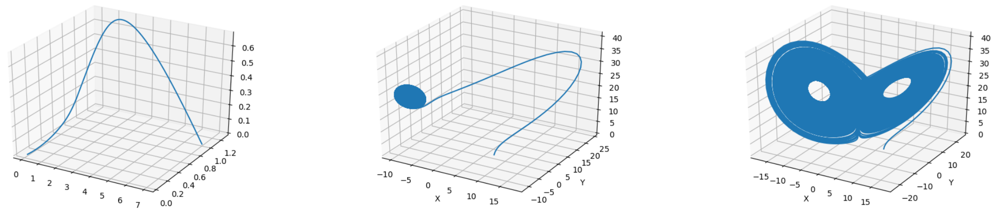
For systems with a Prandtl number of 0.5, convergence to steady convection was found for very high r, with a critical value not found until r_{crit} ≈1450. At low r values (less than 20), the trajectories converge rapidly to an asymptotic steady state. As r is increased, the spiralling trajectories demonstrate wider looping before they converge and the conical shape of the convergence is exaggerated until r_{crit} is reached; at this point the seemingly asymptotic cone suddenly spins out into a wide, flat disc and no longer converges to a steady state.



(a) r=10 (b) r=30 (c) r=70 (d) r=1450

Figure 1: Trajectories for low Prandtl number. r_{crit} observed at 1450. It is difficult to effectively portray the unsteady convection occurring in Fig. 1d from a single angle; refer to Fig. 8a for a better view of the trajectory after transitioning from the early conical convergence behaviour to the unsteady disk orbit.

Unstable convection is observed much earlier when a Prandtl number of 10 is used. This is the set of parameters originally examined by Lorenz [2] since it is relevant to the experiments examining rotating heated water upon which his model was based. For r=1, the energy of the system is quickly dissipated as all variables converge to zero. Closer to the critical value, the system spirals in asymptotically to a steady convection state, but once it reaches r_{crit} = 24.049 the system transitions sharply into the famous "butterfly" shaped unsteady convection trajectory from which the butterfly effect derives its name. Trajectories of this approximate shape persist throughout the super-critical regime as r is then increased past this point. This behaviour is characterized by large oscillations which repeat their path closely but never exactly.

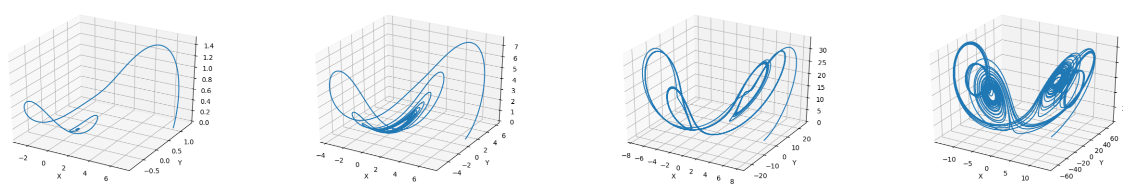


(a) r=1 (b) r=24.0 (c) r=24.049

Figure 2: Trajectories for high Prandtl number. r_{crit} observed at 24.049

4.1.2 4DLM

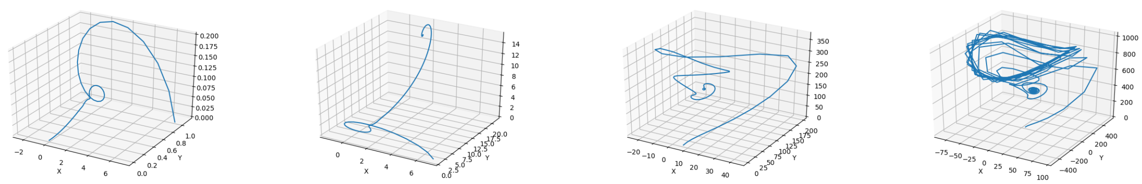
When testing Stein's 4 dimensional model at low Prandtl number, the transition to unsteady convection was found at $r_{crit} \approx 7.0$. Below this number asymptotic spiraling was observed, while for super-critical values a butterfly trajectory similar to that observed in Lorenz' study was produced. As r is increased past its critical value the trajectory spreads out and repeats itself less closely, but the general butterfly shape is maintained.



(a) r=0.5 (b) 6 (c) 24 (d) r=70

Figure 3: Trajectories for low Prandtl number. r_{crit} observed at 7.

At high Prandtl number, Stein's model does not exhibit a regime of unsteady convection, as we reach state values exceeding the maximum representable value by our computers before any trajectories cease to converge to a state of steady convection. For $r = 1$, the energy of the system dissipates and all variables converge to zero, but above this value the trajectories follow a few bends before eventually spiralling in to a final steady state. This spiral steadily grows into a conical asymptote as observed for the 3DLM at low Prandtl number but never transitions to an unsteady convection regime.

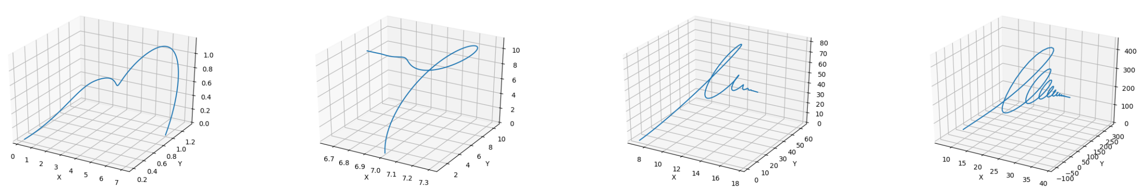


(a) r=1 (b) r=24 (c) r=200 (d) r=430.9

Figure 4: Trajectories for high Prandtl number. Convergence to steady convection is observed for all r values here.

4.1.3 5DLM

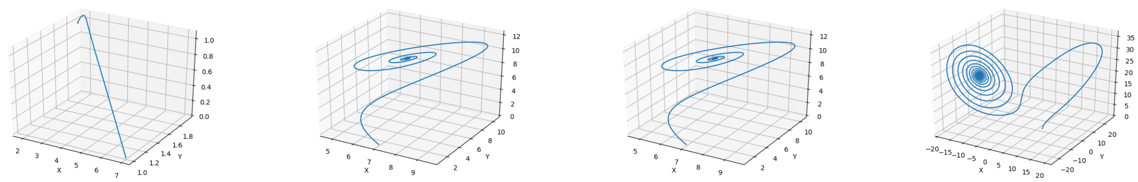
Using Shen's 5 dimensional model at low Prandtl number, no unsteady convection was observed before state values reached the maximum representable value by our computers, causing overflow. Energy dissipation resulting in convergence of all variables to zero occurs at $r = 1$ while higher r values exhibit asymptotic spiralling in to steady convection states. This spiral, much like the trajectories in Fig. 1 and Fig. 4, becomes increasingly conical as r increases.



(a) r=1 (b) r=10 (c) r=60 (d) r=280.4

Figure 5: Trajectories for low Prandtl number. Convergence to steady convection is observed for all r values here.

For high Prandtl number, Shen's model transitions to unsteady convection following a butterfly trajectory above $r_{crit} \approx 38.6$. As observed with both other models at high Prandtl number as well as the 5DLM at low Prandtl number, energy dissipation and convergence to zero occurs for $r = 1$. Above this value spiralling (but not conical) convergence occurs, eventually forming what resembles one half of the unstable butterfly trajectory. This shape transitions sharply into the familiar butterfly shape with unstable convection at r_{crit}, maintaining roughly this same shape until r reaches 212.22 causing the state values to overflow.



(a) r=2 (b) r=10 (c) r=24 (d) r=38.6

Figure 6: Trajectories for high Prandtl number. r_{crit} observed at 38.6.

4.2 Sensitivity to Initial Conditions

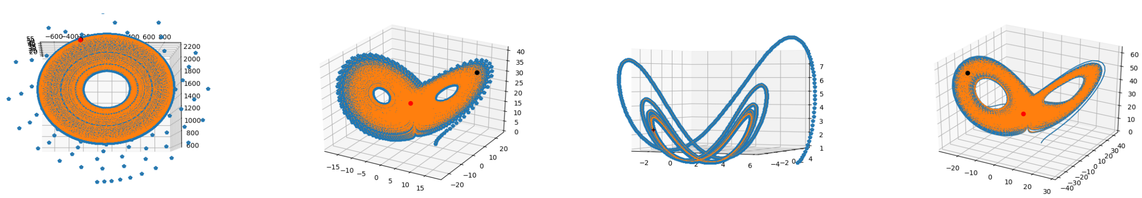
By carrying out this experiment testing sensitivity to initial conditions, we found that the trajectory converges to the same asymptote even for very large disturbances (at least an order of magnitude larger than the initial coordinates themselves). Interesting results were only found in systems where unstable convection occurs, i.e. above the critical r value. In these systems the trajectory of two simulations with slightly different initial conditions were visually indistinguishable, however the final state after 1000 time steps varied greatly for sufficiently large IC perturbations. The difference in initial Y value required for visible differences in final state will be referred to as dI_{crit} and appears to depend strongly upon Prandtl number as shown in Table 1.

Model Dimension	Prandtl Number	dI_{crit}
3DLM	0.5	1.0
3DLM	10	$5 \cdot 10^{-15}$
4DLM	0.5	1.0
5DLM	10	$1.45 \cdot 10^{-15}$

Table 1: Critical deviation in initial conditions for each of the unstable convection regimes examined.

For the two systems examined at low Prandtl number, a dI_{crit} on the order of magnitude of the initial conditions themselves is needed before visible differences in final state occur.

Furthermore, this transition is not very rapid as slightly super-critical dI values lead to only small differences in final state. On the other hand, for high Prandtl number extremely small perturbations are sufficient to cause visible deviations in final state and this transition is much more sharp. The difference in transition can be seen in Fig 7. It is interesting to note that for the two low-σ trajectories, testing at higher r values required smaller dI values to acheive a visible deviation. However these higher r values still result in the gradual transition observed for slightly super-critical r systems.

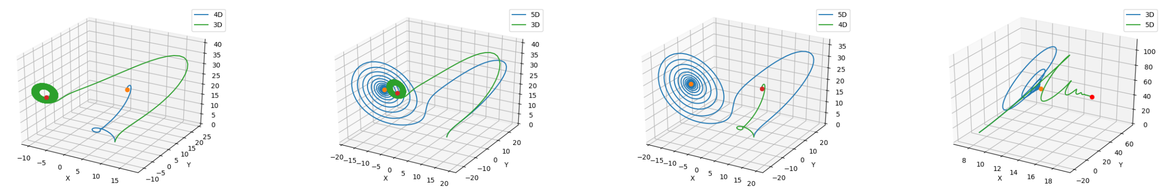


(a) r=1500 (b) r=24.049 (c) r=7.0 (d) r=38.6

Figure 7: Trajectories for critical r values resulting in unstable convection with visible deviation between different initial conditions. The dI values from Table 1 were used. The difference in transition

4.3 Comparison of Model Predictions

Based upon the results of Section 3.1, phase space locations for this model comparison were chosen by finding r and σ values which appeared to exhibit similarly stable convection trajectories under similar conditions. A selection of the resultant comparison plots are shown in Fig. 8, displaying our finding that no systems were found for which the 4D model closely mimics the trajectory of either of the other two models. On the other hand, the 3D and 5D models can be seen to share similar trajectories for some of the systems tested. This is also observed in Fig. 7 (b) and (d), where the unstable convection trajectories of both systems exhibit Lorenz' butterfly behaviour at high Prandtl number.



(a) 3D vs 4D for r=24.0 at high Prandtl number (b) 3D vs 5D for r=24.0 at high Prandtl number (c) 4D vs 5D for r=24.0 at high Prandtl number (d) 3D vs 5D for r=70.0 at low Prandtl number

Figure 8: Comparison between models of trajectories for stable convection under identical initial conditions and parameters. Of these, only comparisons between the 3DLM and the 5DLM exhibit similar behaviour between models while the 4DLM does not seem to follow the trajectories of either.

5 Discussion

5.1 Phase Space Trajectories

Our results show that, in general, higher dimensional models exhibit a state of steady convection up to higher r values. This is clearly shown when Prandtl number is 5. The Lorentz model is stable only to an r value of ∼23 while the 4DLM and 5DLM models have values of ∼131 and ∼163 respectively. We see that there is a large increase in the range of r values for which the system exhibits steady convection between Lorentz model and the 4DLM while a much smaller increase is observed between the 4DLM and the 5DLM. One could speculate that adding further modes would eventually increase the range only by insignificant amounts, suggesting that there is a limit to how stable a model can become.

In some cases we could not find a values for r_{crit} at a given Prandtl number. When this occurred with the Lorentz model it could have been because we had $\sigma < b+1$, a conditioning specifying that steady convection was always stable and there existed no positive r_{crit} value as discussed by Lorenz in his paper‘s section on applications of linear theory [2]. This would be the case for $\sigma = 0.5$ since we used a b value of 8/3 and would explain why we found no r_{crit} value in the 5DLM even after trying arbitrarily large values. Critical values were in fact found in the other two models; in the case of the 3DLM this occurs at very high (r=1450) values which is 1-2 orders of magnitude greater than we found for other systems.

We believe this to be a computational artifact arising from the large Y derivatives produced by extremely large r; this could push the limits of accuracy when using the Euler method which relies upon a linear approximation of differential equations at small time steps. If this approximation worsens significantly the resultant simulation could vary widely from the actual physical phenomenon it is meant to model, potentially explaining why this system seems to violate the requirement for unstable convection mentioned above. Stein's 4DLM on the other hand is not necessarily bound by the requirements for unstability laid out by Lorenz, as it is modelling a rotating fluid whereas Lorenz originally modelled a fluid without vertical vorticity. This is unlike Shen's 5DLM which simply extends Lorenz'approach of expanding the Fourier modes of the system, thus modelling the same phenomenon with more modes as opposed to modelling a different system with an additional degree of freedom.

5.2 Sensitivity

For trajectories which converge into a state of steady convection, making even fairly large changes to the initial state did not produce any change in the asymptotic final state to which the system converges. On the other hand when trajectories enter the unstable convection regime small changes in state could evolve into very different final states showing that predictions of a moderately distant future is unfeasible due to the impossibility of measuring initial conditions to an exact value.

When the r value is positive but smaller than 1, the origin is a stable point and all trajectories converge to zero as in Fig. 4(a) and Fig. 5(a). When the r value increases above r=1 the origin is no longer stable but there are two stable point at locations that depend on the r value and the b value, this is why trajectories converged to the same point for different initial states when $1 < r < r_{cit}$. Above r_{crit} the two points become unstable. The instabilities observed at high Prandtl number required much smaller perturbations and produced sharper transitions from matched trajectories to significantly different trajectories as a function of the difference in initial conditions. This matches the behaviour observed by Lorenz in his paper, which is paralleled by Shen. We believe that this is a result of the different conditions specified by a high Prandtl number, namely that the dominant mode of diffusion in the system is viscous whereas thermal diffusion dominates for low Prandtl numbers. When thermal diffusion dominates, the temperature gradients which cause the convection and flow of fluid can achieve thermal relaxation much more rapidly which damps out unstable behaviour. When viscous diffusion dominates, the fluid flow behaviour is much more important and so there is less damping in the system which allows instabilities to grow more rapidly. This is analogous to how a driven spring immersed in oil or water has smaller amplitude for a given driving force than it would if it was instead immersed in air which is a less effective damping medium.

5.3 Comparison of Models

When we compared the stable convection trajectories of different models we saw that they all converged to a different state even if they had the same initial conditions and parameters. Despite this there was some resemblance in the paths of the Lorentz Model and the 5DLM while there is none between the 4DLM and the two other models. This is likely because the 5DLM merely extends the original model by including higher-order Fourier series terms in order to represent the same system with more detail, whereas the 4DLM is meant to represent a system with an extra degree of freedom, vertical vorticity. This can also explain why we do not see a r_{crit} value at $\sigma = 0.5$ for the 5DLM, just like for the Lorenz model, but we do find one for the 4DLM. This means that while we cannot meaningfully compare results between the Lorenz and Stein models, further investigation into whether Shen's higher dimensional model improves upon the accuracy of the original method would certainly be feasible. An important question for deeper work in this problem is whether the more stable results of Shen's model actually improve weather predictions. We can see that solutions remain stable for higher r values but this does not tell us which model better imitates the behaviour of real-world weather systems; a more predictable long-range simulation is not very useful if it no longer closely matches the physical phenomenon which we seek to predict.

5.4 Applications to Modern Weather Prediction

Adding dimensions to the model should hypothetically better describe small-scale features with ever-increasing accuracy, however atmospheric scientists have found very strict limits on forecasting feasibility despite highly sophisticated models benefiting from modern computing and data-taking capabilities. Why can we not achieve arbitrarily long-term forecasting by creating higher-dimensional models? The major reason lies of course in the sensitivity to initial conditions of our models. Lorenz's work showed that even with the acquisition of precise measurements using sophisticated equipment and techniques, the sensitivity to initial conditions of weather patterns makes it so that after a certain time, any model regardless of precision will suffer accuracy losses. One way forecasters use the measurement uncertainties is by compiling weather predictions spanning the range of the measurement uncertainty and looking for tendencies in the data [11] . If a large enough tendency is observed, then forecasters can have increased confidence that the observed trend will actually happen. Forecasters can also use statistical techniques to correct for systematic biases and thus quantify expected errors. However, the ever-present sensitivity to initial conditions will always remain a corrupting force to weather predictions, regardless of the attempts by the forecasters to get it right. Therefore, it is in the weatherman's best interest to understand these limitations and work within them instead of trying to suppress them outright, because it is a fight he or she will never win.

6 Conclusions

In this paper we have outlined three models of increasing complexity which seek to model the deterministic nonperiodic flow of fluids driven by an external heat source, a system which serves as the basis for modern forecasting models for our atmosphere. We used these Lorenz Models to simulate behaviour for Prandtl numbers similar to those of air and of water, across a wide range of Rayleigh numbers, in order to characterize the resultant trajectories observed over this phase space. We attempted to reproduce Lorenz' finding that there exists for a given system a Prandtl number below which no Rayleigh number will produce unstable convection; we in fact were successful at finding a critical value for the transition to instability but the extremely large r at which this occurs as well as the nearly asymptotic behaviour which eventually collapses into instability leads us to believe that this violation of the differential equations' prediction is due to numerical inaccuracy errors and no longer mirrors the physical behaviour of the system. The 4DLM also violates this expectation, however this is less surprising since the added term was not found through an extension of the Fourier series method used by Lorenz and instead adds a new degree of freedom in the motion of the system through vertical vorticity. The resultant new equations can not be expected to obey the same regime constraints since they no longer seek to model exactly the same system. Similarly, a comparison of predictions between models shows that the 3 and 5 dimensional models produce similar results under identical conditions, while the 4 dimensional model exhibits significantly different trajectories from both of the other two simulations.

Strong sensitivity to initial conditions was observed across the models when unstable convection occurs, however a large correlation between Prandtl number and sensitivity was observed. High Prandtl number systems exhibited extreme sensitivity whereas low Prandtl number systems exhibited smaller but still significant sensitivity; this difference may be attributable to the dominance of thermal diffusion at low Prandtl number which serves as a damping mechanism on the system and smooths out instabilities. Further investigation could be done using a more powerful computer into the size of these stabilities when simulating time-scales on the order of days or weeks in order to give a sense of the uncertainty inherent on the scales necessary for useful weather prediction. Based upon our replication and extension of Lorenz‘results, it appears that adding additional dimensions to our models will likely provide diminishing returns and that the non-zero uncertainties inherent in real-world weather measurements, even with today‘s highly precise sophisticated measurement techniques, will necessarily lead to significant accuracy losses for any arbitrarily long-range forecast. We also briefly addressed a common method used to improve reliability in these forecasts by compiling predictions spanning the range of input measurement uncertainty and looking for statistically significant agreements in the data. However, even this method paired with a global network of weather stations, satellites, and balloon probes modern weather prediction is extremely unreliable past around 5-10 days. In conclusion, systems exhibiting unsteady convection suffer from extreme sensitivity to initial conditions which is a crucial barrier to long-term weather forecasting; many models for Numerical Weather Prediction have been built upon Lorenz‘ 1963 paper, however even after almost 60 years his groundbreaking assessment that the chaotic instabilities observed in his simulations would rule out the possibility of long-term predictions remains one of the most important discoveries of the 20th century.

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A Additional Graphs

The following graphs show the trajectories which converge into a state of steady convection, for the 4D and 5D Lorenz models. The two trajectories shown are visually indistinguishable because for $1 < r < r_{crit}$ the system's final state does not depend on its initial conditions as the trajectories converge to the same final state for a give r value and b value even with relatively large (at or above the order of magnitude of the initial coordinates themselves) differences in initial conditions.

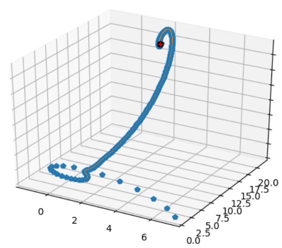
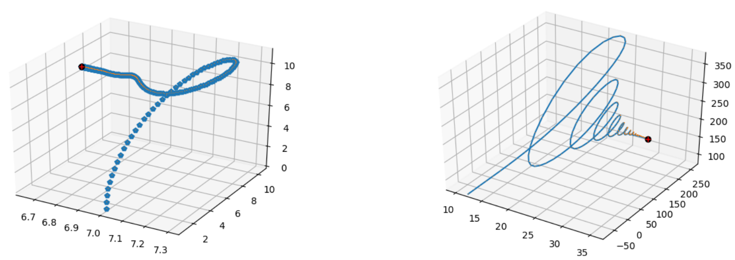


Figure 9: A 4DLM trajectory for high Prandtl number and $r = 24$.



(a) (b)

Figure 10: 5DLM trajectories for low Prandtl number and $r = 10$ for a) and $r = 280$ for b).