Games and the Minimax algorithm

April 1, 2021

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- We will discuss some aspects of Game Theory
- In particular we will discuss the Minimax Algorithm
- The concepts will be applied to simple examples.

Overview

Formalization of games

- Games can be formalized mathematiclally

Zero sum normal form games

- We will consider games with two players.

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- Both players play simultaneously. Player A does action a, players B does action b.

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- Their action results in a gain $g(a, b)$.

Zero sum normal form games

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- Both players play simultaneously. Player A does action a, players B does action b.
- Their action results in a gain $g(a, b)$.
- We assume that what A wins is what B looses : hence the term **zero-sum** game.
- For instance A wins $g(a, b)$ and B "wins" $-g(a, b)$.

Examples

- Paper, Scissors, Stone
- football penalty

Normal form description

- These games can be represented by a **payoff matrix**.

Zero sum normal form games

		Player B action	
		a	b
Player A action	a	$g(a, b)$	$g(a, b)$
	b	$g(a, b)$	$g(a, b)$

Table: In this game, the players can perform two possible actions. Since the game is zero-sum, it is sufficient to represent the gain of player A.

Example

		Player B action	
		a	b
Player A action	a	2	3
	b	1	8

Table: Example gains.

Concept of *supinf*

		Player B		
		a	b	c
	a	2	0	9
Player A	b	4	4	7
	c	10	1	3

Table: What is the gain the A can be **sure** of obtaining ?

Concept of *supinf*

		Player B		
		a	b	c
	a	2	0	9
Player A	b	4	4	7
	c	10	1	3

Table: What is the gain the A can be sure of obtaining ? Reminder : B wants to **minimize the gain** and acts rationally.

Pure strategy and mixed strategy

- A **pure strategy** is completely deterministic
- A **mixed strategy** assigns a probability distirbution to the set of actions.

Rock Scissors Paper

- We will study mixed strategies in the Rock Scissors Paper game.

Rock Scissors Paper

- How many action tuples are possible ?

Rock Scissors Paper

Exercice 1 : Action probabilities

- **cd minimax_and_games/zero_sum** folder.
- Modify the file **paper_scissors_rock.py** so that the actions performed by the two players are drawn from the relevant distributions (= strategies) **player_X_strategy**.
- Use can use the function **choice** from **numpy** (look for its documentation)

Rock Scissors Paper

Exercice 1 : **Action probabilities**

- Modify the file so that the correct statistics about the victory rate are computed.

Rock Scissors Paper

Exercice 2 : **Action probabilities**

- What happens if you change the strategy of player B ?

Rock Scissors Paper

Exercice 2 : Action probabilities

- What happens if you change the strategy of player B ?
- And the stategy of player A ?
- How can we interpret that result ?

Rock Scissors Paper

Let us define probabilistic **events :**

- V : A wins the game
- Br : B plays rock.
- Bs : B plays scissors.
- Bp : B plays paper.

Rock Scissors Paper

- V : A wins the game
- Br : B plays rock.
- Bs : B plays scissors.
- Bp : B plays paper.

$$\begin{aligned} P(V) &= P(V \cap (Br \cup Bs \cup Bp)) \\ &= P((V \cap Br) \cup (V \cap Bs) \cup (V \cap Bp)) \end{aligned} \tag{1}$$

Symbols :

- P means "probability of".
- $\cup$ means "or"
- $\cap$ means "and"

Rock Scissors Paper

$$
\begin{aligned}
P(V) &= P(V \cap (Br \cup Bs \cup Bp)) \\
&= P((V \cap Br) \cup (V \cap Bs) \cup (V \cap Bp)) \\
&= P(V \cap Br) + P(V \cap Bs) + P(\cap Bp) \text{ (Incompatibility of events)}
\end{aligned} \tag{2}
$$

Rock Scissors Paper : conditional probabilities

$$
\begin{aligned}
P(V) &= P(V \cap (Br \cup Bs \cup Bp)) \\
&= P((V \cap Br) \cup (V \cap Bs) \cup (V \cap Bp)) \\
&= P(V \cap Br) + P(V \cap Bs) + P(V \cap Bp) \\
&= P(V|Br)P(Br) + P(V|Bs)P(Bs) + P(V|Bp)P(Bp)
\end{aligned} \tag{3}
$$

$P(V|Br)$ means "Probability that A wins, given that B plays rock".

Rock Scissors Paper : conditional probabilities

$P(V|Br)$ means "Probability that A wins, given that B plays rock".
If the strategy of A is $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, then

- $P(V|Br) = ?$
- $P(V|Bs) = ?$
- $P(V|Bp) = ?$

Rock Scissors Paper : conditional probabilities

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- $P(V|Bs) = \frac{1}{3}$
- $P(V|Bp) = \frac{1}{3}$

Rock Scissors Paper : conditional probabilities

$P(V|Br)$ means "Probability that A wins, given that B plays rock".
If the strategy of A is $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, then
$P(V|Br) = P(V|Bs) = P(V|Bp) = \frac{1}{3}$.
Hence :

$$\begin{aligned} P(V) &= P(V|Br)P(Br) + P(V|Bs)P(Bs) + P(V|Bp)P(Bp) \\ &= \frac{1}{3}P(Br) + \frac{1}{3}P(Bs) + \frac{1}{3}P(Bp) \\ &= \frac{1}{3}(P(Br) + P(Bs) + P(Bp)) \end{aligned} \tag{4}$$

Rock Scissors Paper : conditional probabilities

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$$\begin{aligned} P(V) &= P(V|Br)P(Br) + P(V|Bs)P(Bs) + P(V|Bp)P(Bp) \\ &= \frac{1}{3}P(Br) + \frac{1}{3}P(Bs) + \frac{1}{3}P(Bp) \\ &= \frac{1}{3}(P(Br) + P(Bs) + P(Bp)) \\ &= \frac{1}{3} \end{aligned} \tag{5}$$

Biased game

Exercice 3 : **Alternative game with the well.**

- Modify the file **paper_scissors_rock_well.py** so that the actions performed by the two players are drawn from the relevant distributions (= strategies) **player_X_strategy**, and so that the statistics are correctly computed.
- find a strategy that gives a better victory rate for A.

Statistics on strategies

Exercice 4 : **Learning a strategy**

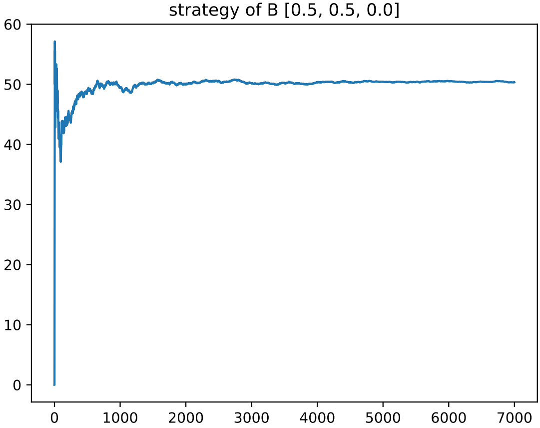
- Modify the file **paper_rock_scissors_learn.py** in order to **learn the strategy of B**, and adapt the strategy of player A in order to have a better victory rate.

Statistics on strategies

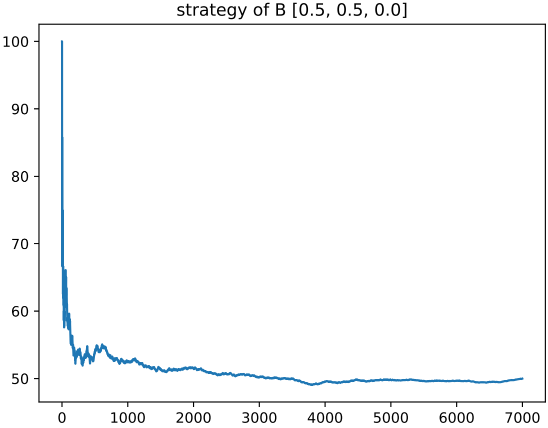
Exercice 5 : **Learning a strategy**

- Modify the file **paper_rock_scissors_learn.py** in order to **learn the strategy of B**, and adapt the strategy of player A in order to have a better victory rate for A.
- Several solutions are possible.
- You can work in groups.

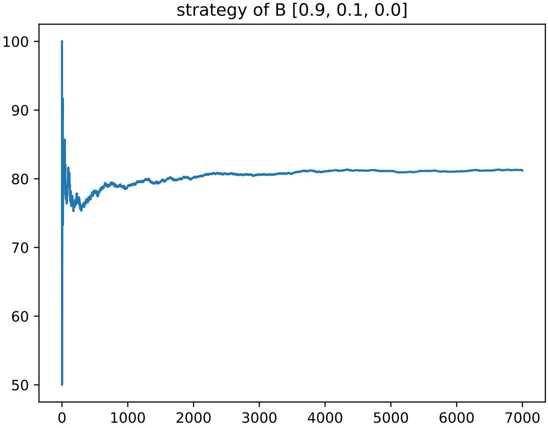
Percentage of victory



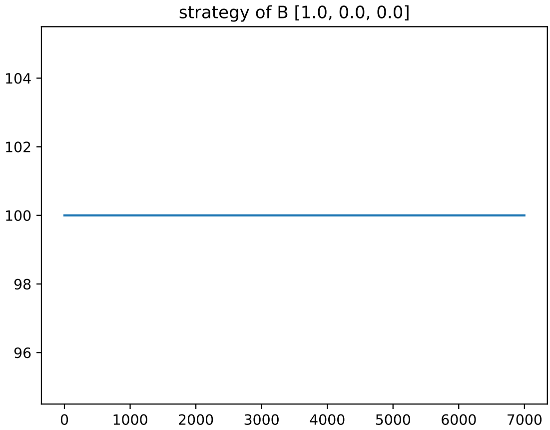
Percentage of victory



Percentage of victory



Percentage of victory



Sequential games

- We will now change the type of games studied
- We still have two players but the game consists in a **sequence of actions**, instead of a **single action**.

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- The two players play successively, taking into acount the previous actions, and also the following actions from their opponent.

Sequential games

- We will now change the type of games studied
- We still have two plkayers but the game consists in a **sequence of actions**, instead of a **single action**.
- The two players play successively, taking into acount the previous actions, and also the following actions from their opponent.
- until the game reaches a **final state**. When the game is in its final state, the players receive a score.

Sequential games

- One player is called the **maximiser**, the other player the **minimiser**.

Sequential games

- One player is called the **maximiser**, the other player the **minimiser**.
- The maximiser tries to get the **highest score**, while the minimiser tries to get the **lowest score**.

The Minimax algorithm

- We will study and implement an algorithm that computes the values of the actions of the two players.

The Minimax algorithm

- We will study and implement an algorithm that computes the values of the actions of the two players.
- **Very important hypotheses :** the agents are assumed to behave **rationally**

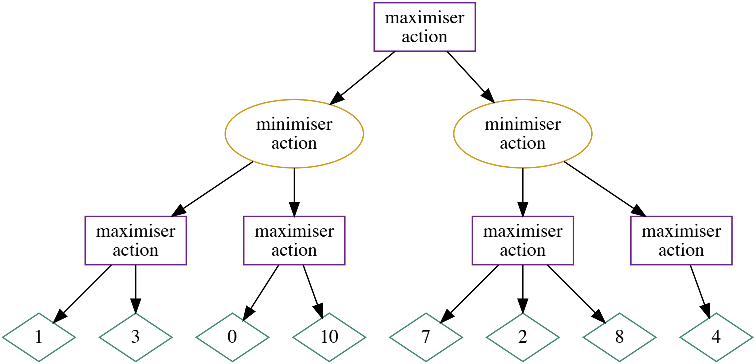

Figure: Representation of the game

Recursion

- The Minimax algorithm is based on a concept called **recursion**.

Recursion

- **Proposed definition** : a method to solve a problem based on smaller instances of the same problem.

First Recursion example

- **cd recursion**
- Please modify **factorial_rec.py** so that it computes the factorial
- $n! = 1 \times 2 \times \ldots \times n$

Recursion

A recursive function always has :

- a base case
- a recursive case

Warning

- Decrease does not mean terminate !
- What happens with the example **bad_recursion** ?
- In python, you can see the recursion limit with **sys.getrecursionlimit()**

Minimax algorithm

- The **Minimax algorithm** is a recursive algorithm that computes the values of all the nodes.

Minimax algorithm

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- It does so by propagating the information from the **leaf nodes** (the **final states of the game**) to the parent nodes.

Minimax algorithm

- The **Minimax algorithm** is a recursive algorithm that computes the values of all the nodes.
- It does so by propagating the information from the leaf nodes (the **final states of the game**) to the parent nodes.
- Let us apply it on an example.

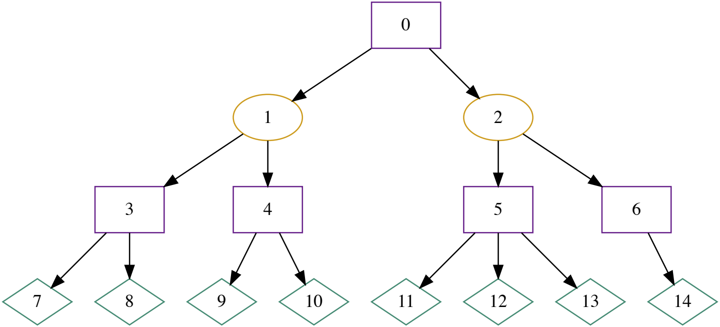
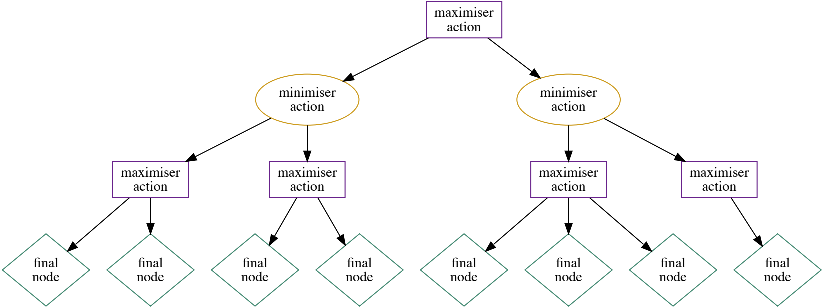


Figure: The numbers do not represent the values here : they represent the index of the node.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minimiser action
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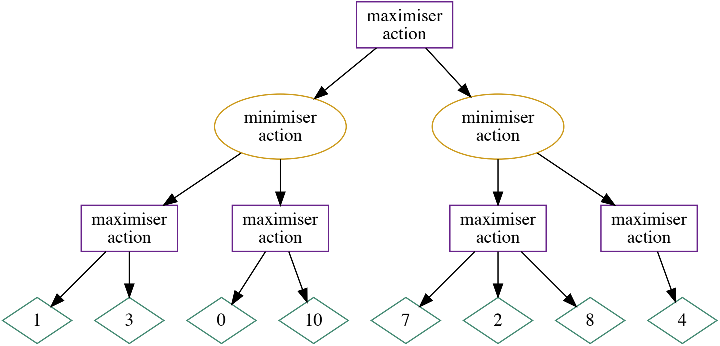
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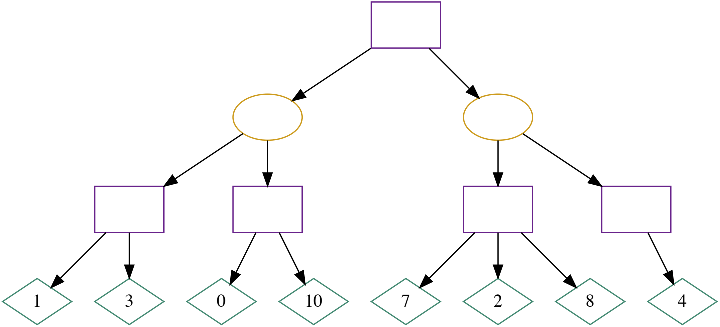
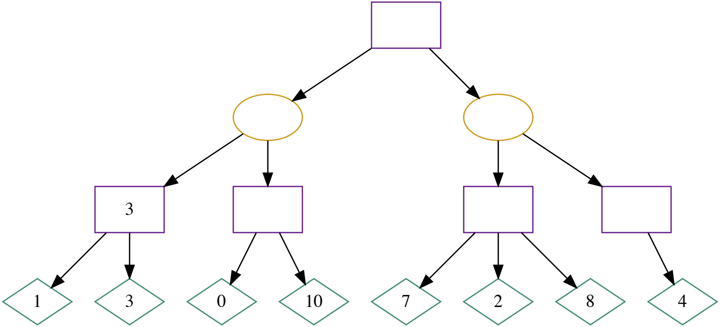


Figure: Values of the final states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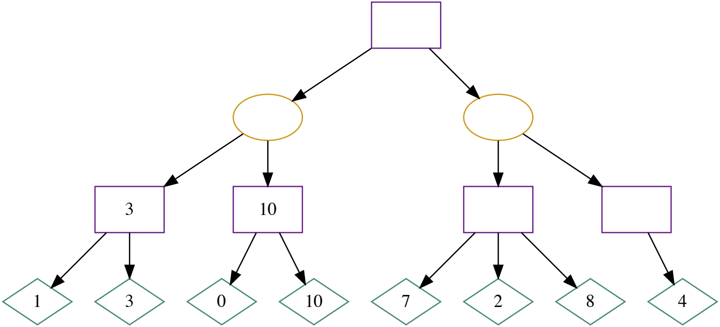
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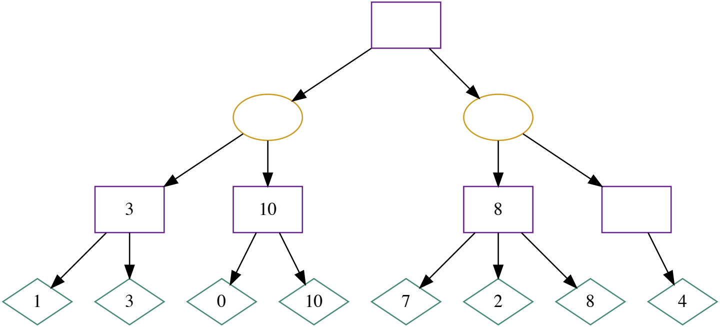
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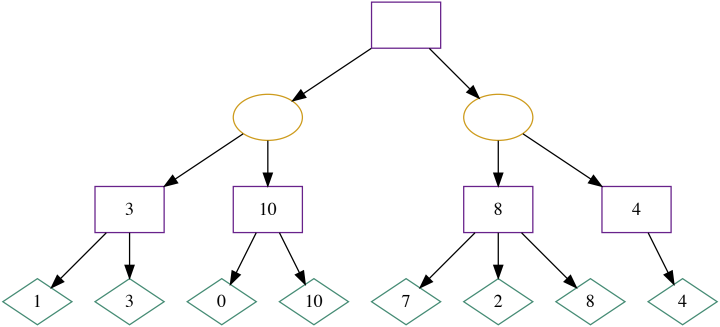
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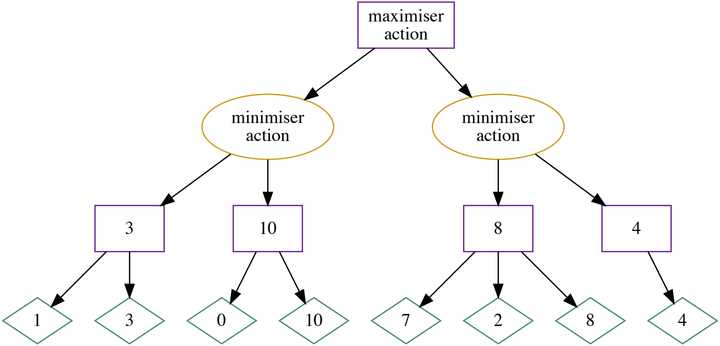
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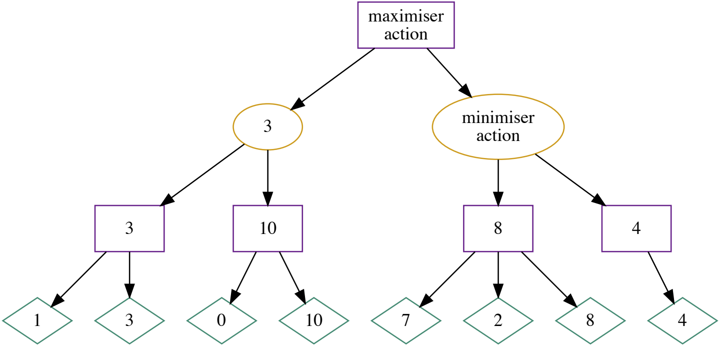
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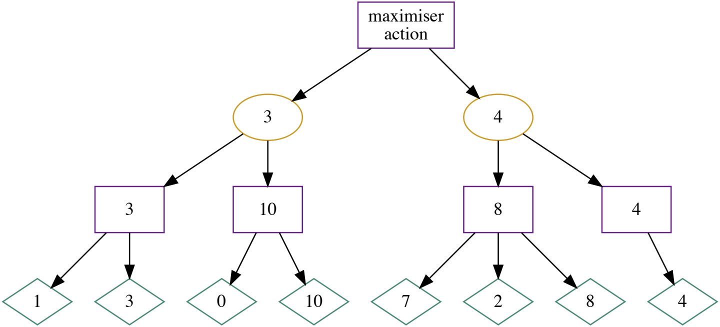
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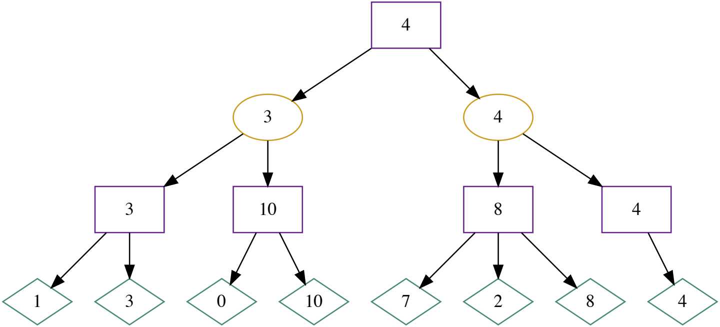
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Python dictionaries

- **dictionaries** are a useful data structure.
- Demo with **ipython**

Minimax algorithm

Exercice 6 : **Implementing the algorithm**

- Please use the file **minimax.py** in order to implement the algorithm.
- I inserted 2 errors in the **minimax** function.
- you can also try with different values for the final states.

Limitations of the method

- What could be the problems with the Minimax algorithm ?

Limitations of the method

- Let p be the **branching factor** of the tree. (Here, the average number of children at each node)
- Let d be the **depth** of the tree.

Limitations of the method

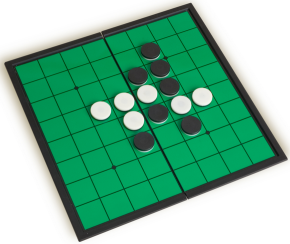
- Let p be the **branching factor** of the tree of actions. (Here, the average number of children at each node)
- Let d be the **depth** of the tree.
- What is the order of magnitude of the number of nodes in the tree ?

Limitations of the method

- Let p be the **branching factor** of the tree of actions. (Here, the average number of children at each node)
- Let d be the **depth** of the tree.
- What is the order of magnitude of the number of nodes in the tree ?
- So in order to run the minimax algorithm needs to perform p^d evaluations.

Exercise

- In the **Othello game**, the average number of actions in each state is around 8.
- We assume that the evaluation for one leaf node takes 1×10^{-6} seconds.



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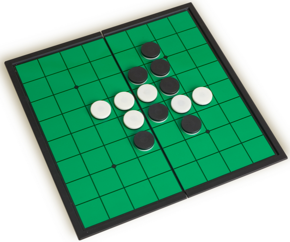
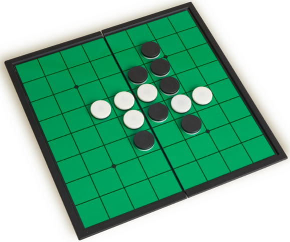
Exercise

- In the **Othello game**, the average number of actions in each state is around 8.
- We assume that the evaluation for one leaf node takes 1×10^{-6} seconds.
- How long would be the search of the minimax if we look 10 actions ahead ?



Exercise

- In the **Othello game**, the average number of actions in each state is around 8.
- We assume that the evaluation for one leaf node takes 1×10^{-6} seconds.
- This duration is too long to be used.



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Conclusion

For real games, we need to use either :

- a smaller tree (look not too many steps ahead)
- a faster algorithm

Alpha-Beta pruning

- We will study a method that opitmizes the Minimax algotihm.

Alpha-Beta pruning

- We will study a method that opitmizes the Minimax algotihm.
- It does so by preventing us from computing **useless nodes**

Alpha-Beta pruning

- We will study a method that opitmizes the Minimax algotihm.
- It does so by preventing us from computing **useless nodes**
- Let us do it on an example.

Alpha-Beta pruning

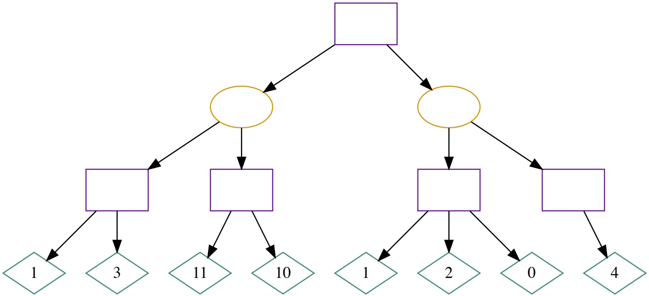
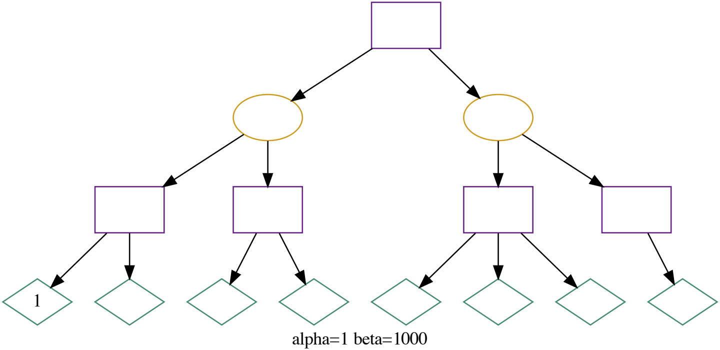
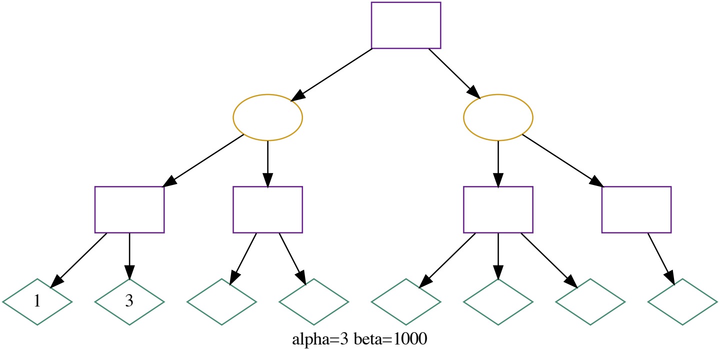


Figure: We use different final state values

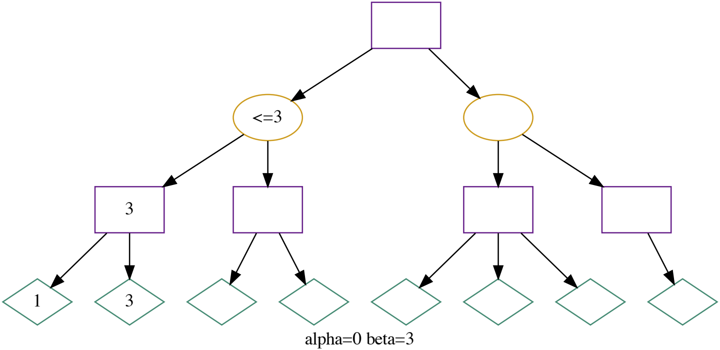
Alpha-Beta pruning



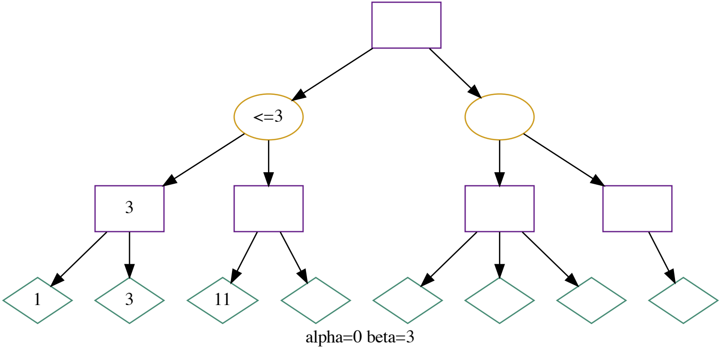
Alpha-Beta pruning



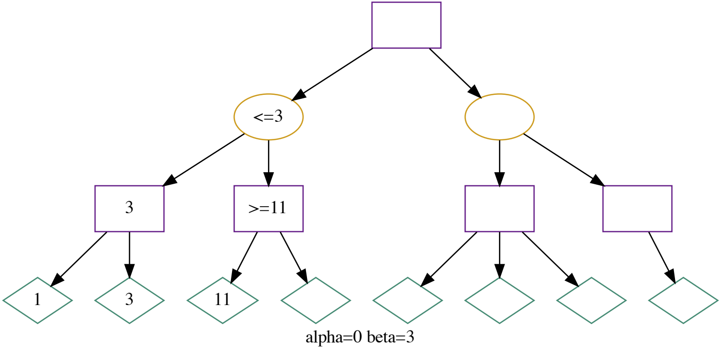
Alpha-Beta pruning



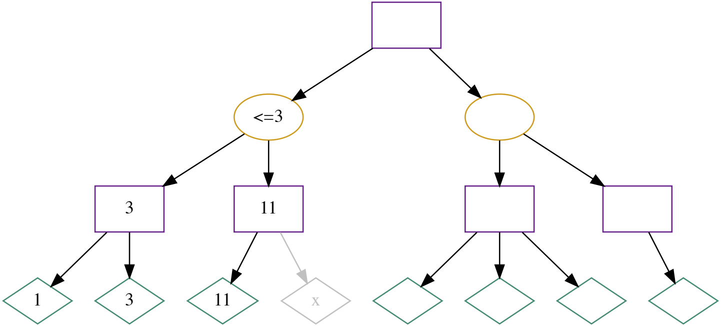
Alpha-Beta pruning



Alpha-Beta pruning

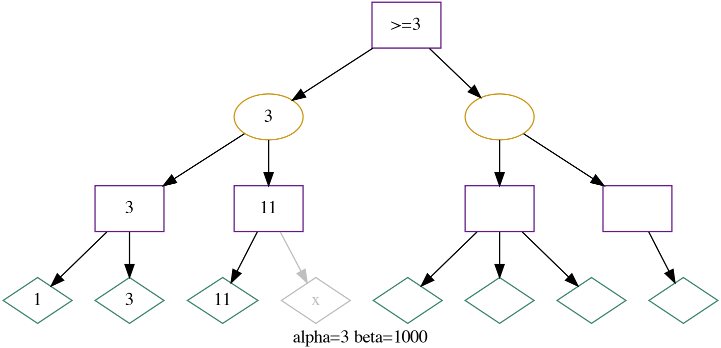


Alpha-Beta pruning

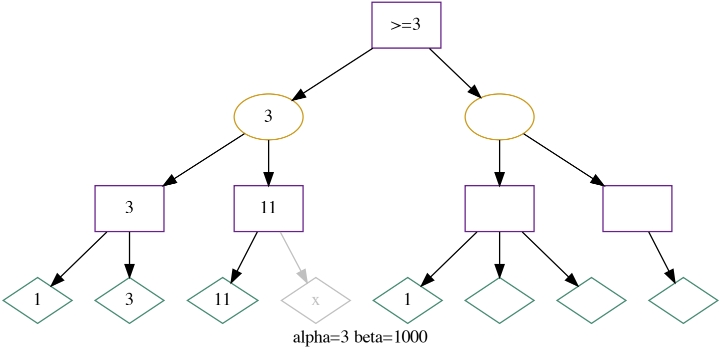


alpha=0 beta=3

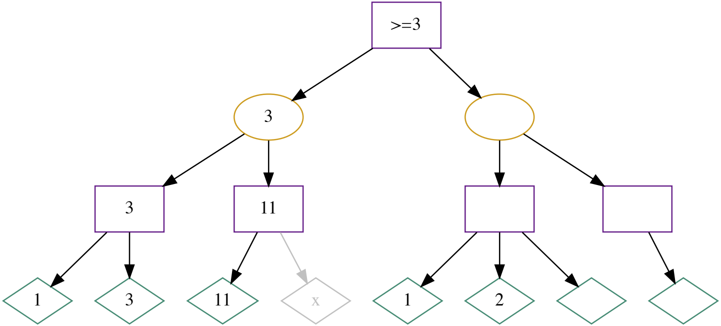
Alpha-Beta pruning



Alpha-Beta pruning

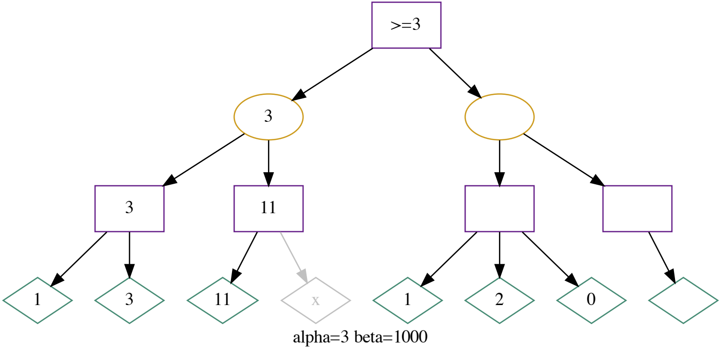


Alpha-Beta pruning

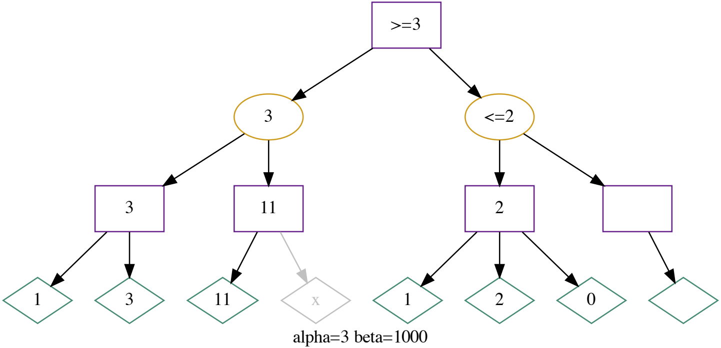


alpha=3 beta=1000

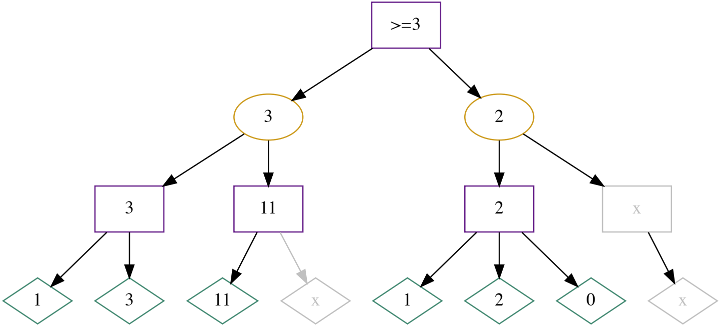
Alpha-Beta pruning



Alpha-Beta pruning

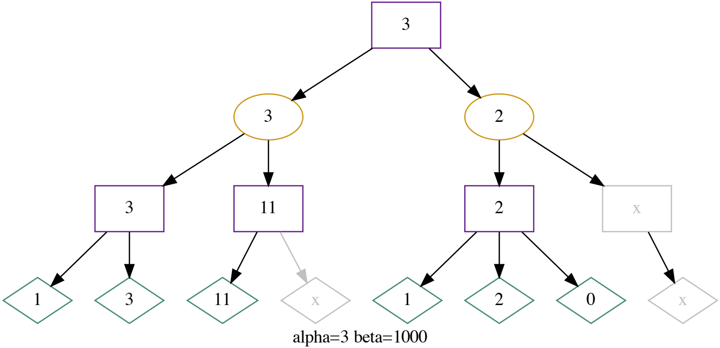


Alpha-Beta pruning



alpha=3 beta=1000

Alpha-Beta pruning



Implementation

Exercice 7 : **Alpha beta pruning**

- Let us now implement the algrorithm
- use the file **alpha_beta.py** in order to implement the algorithm.
- There are several mistakes in the code.

Verification

- Please verify that the AlphaBeta algorithm gives the same result as the Minimax algorithm !

Other values

- Please try to modify the initial values to change the behavior of the algorithm.

Orders of magnitude

- If N is the number of nodes explored by the normal minimax algorithm, the number of nodes explored by Alphabeta can be of order of magnitude $\sqrt{N}$
- This is a great improvement.
- **However**, please note that the improvement depends on the tree. Sometimes the pruning will not accelerate the algorithm that much.

Intermediate scoring

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- In this situation, it is possible to use **intermediate scoring**
- It is a heuristic : there is no theoretical proof that it yields the best solution, but it permits computation

Intermediate scoring and phantom of the opera

- Can you think of an intermediate scoring ?

Intermediate scoring and phantom of the opera

- Can you think of an intermediate scoring ?
- What are the depth and the width of the tree ?