Project 1: Numerical Solution of the Linear Wave Equation

MAE 456: Computational Methods in Aerodynamics

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Introduction

Various discretizations for the linear wave equation were employed using different computational methods. Included was the implementation of an explicit backwards spatial difference, an explicit central spatial difference, and an implicit central special difference, all of which were first order in time. In addition, the second order (in both time and space) explicit Lax-Wendroff scheme was compared and analyzed.

Backward spatial explicit

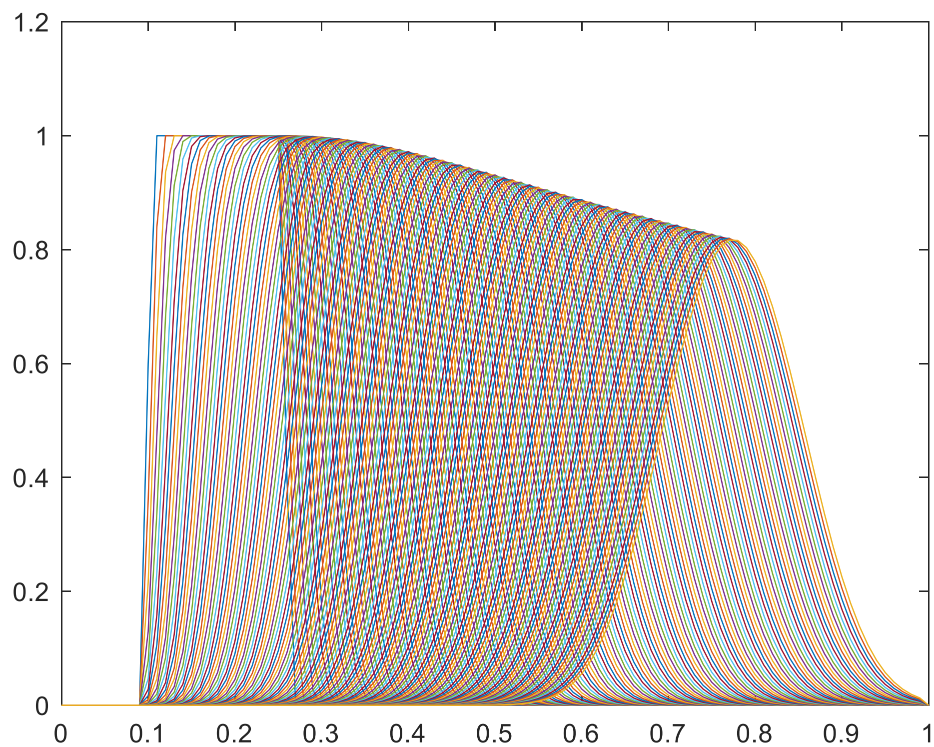

Figure 1: Explicit Backward Spatial at CFL = 0.4. Dispersive error gradually deforms the wave and decreases the amplitude as it propagates.

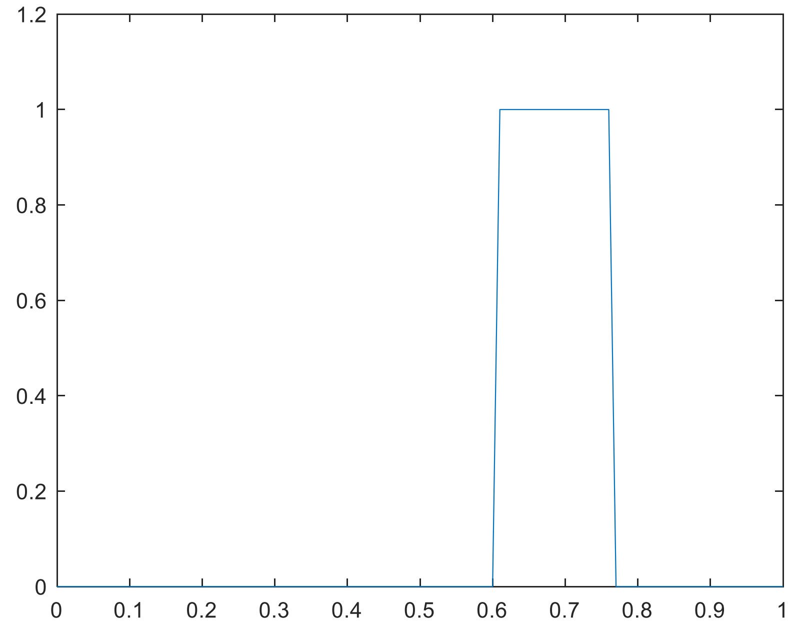

Figure 2: Explicit Backward Spatial at CFL = 1.0. The square wave was represented perfectly. Shown here is the twentieth step (out of fifty total). The wave maintained this stable shape throughout all the steps.

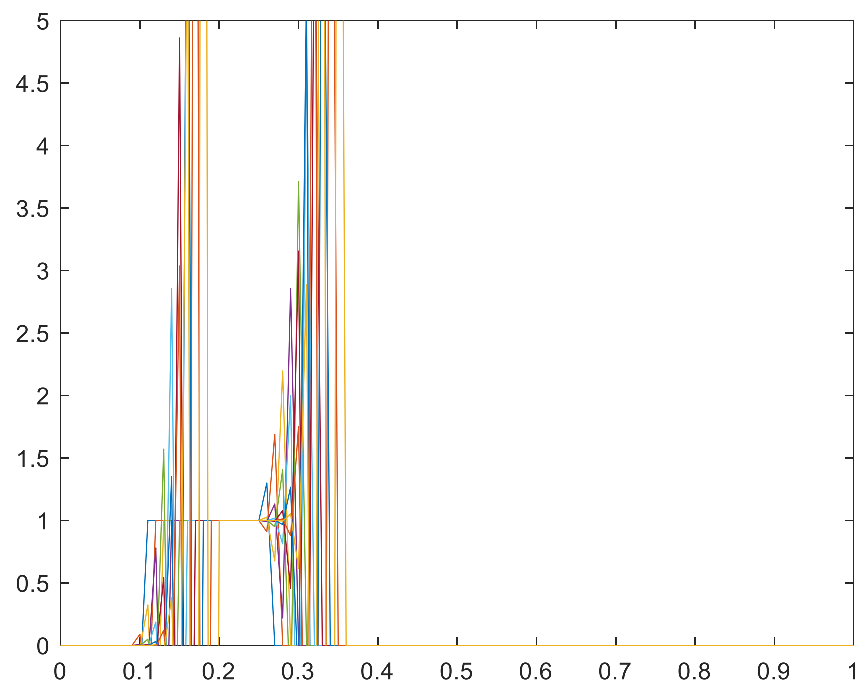

Figure 3: Explicit Backward Spatial at CFL = 1.3. Extreme dissipative error occurs almost immediately, here shown is the tenth step where the instability becomes undisputable.

At a CFL of 1.0, the linear wave equation is stable when approximated by the explicit backward spatial method, as seen in Fig.2, where the shape clearly matches that of a square wave. However, at a CFL number of 0.4, dispersive error takes hold and gradually deforms the wave, making the approximation inaccurate. Additionally, at a CFL number of 1.3, the wave form also is unstable. In Fig.3, the solution is shown to diverge by dissipative error rather than dispersive.

Central Spatial explicit

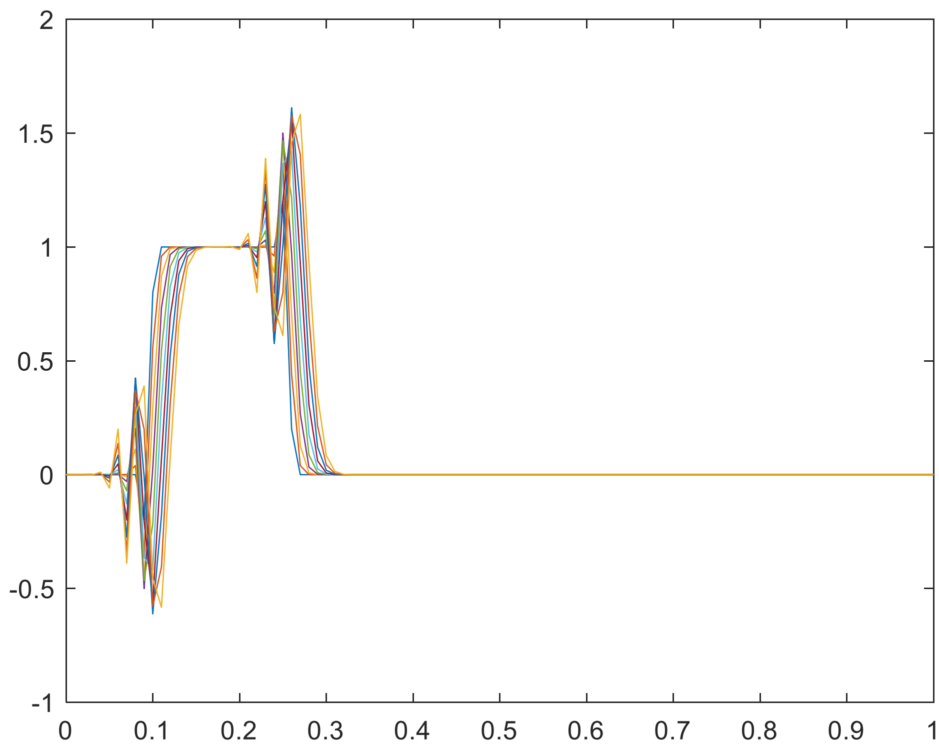

Figure 4: Explicit Central Spatial at CFL = 0.4. Dissipative error occurs around the tenth step (shown here) and continues to increase thereafter.

All values for the CFL number proved to provide unstable solutions for the explicit central spatial method. Fig. 4 illustrates the shared dissipative error pattern by all three tested CFL numbers. For the CFL number of 1.3, the error accumulated much more rapidly, and Fig. 4 would represent more so the fifth step rather than the tenth step for that CFL number.

Central Spatial implicit

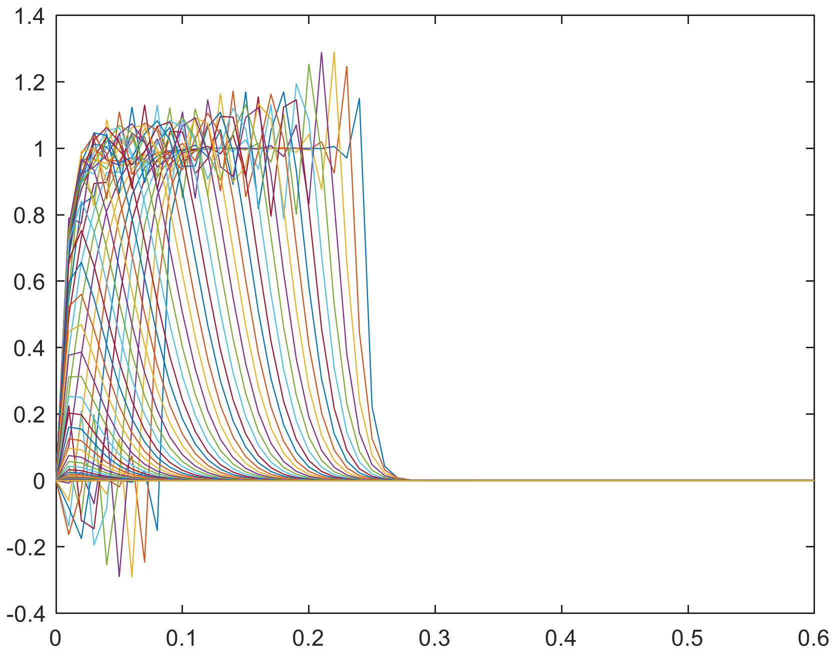

Figure 5: Implicit Central Spatial at CFL = 0.4. Dissipative error begins immediately, and the wave begins to propagate backwards.

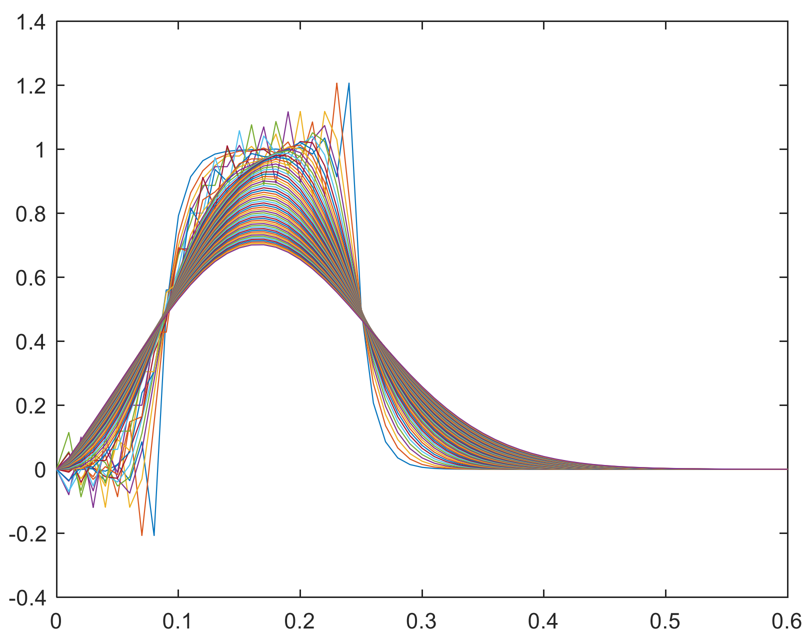

Figure 6: Implicit Central Spatial at CFL = 1.0. The wave does not propagate. Initially the wave has a mixture of dispersive and dissipative error, but later the error is predominantly dispersive.

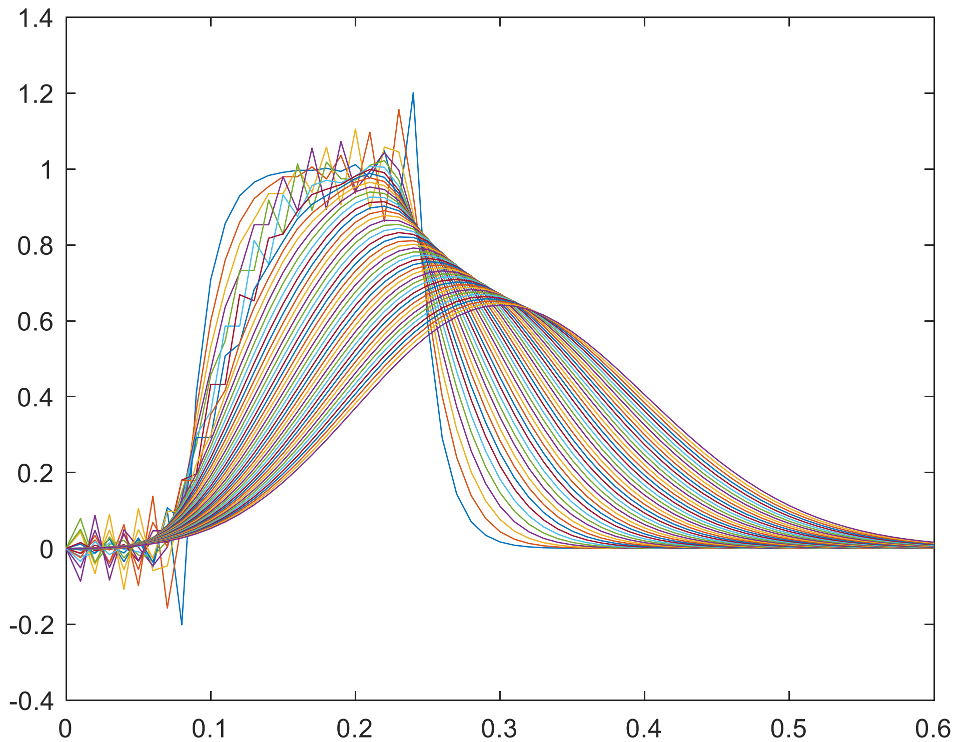

Figure 7: Implicit Central Spatial at CFL = 1.3. Follows the same error transition pattern as in Fig. 6, showing the CFL at 1.0, but here the wave propagates forward to a degree.

The motion behavior of the implicit central spatial method depended heavily on the CFL number. For the 0.4 value, the wave propagated backwards, while the 1.0 value showed no motion, and the 1.3 value propagated the wave forward. This suggests that a value greater than one is required to predict proper wave motion. Error in all cases displayed an interesting pattern of initially mixing dissipative and dispersive error and then transitioning into an overwhelmingly amplitude-decaying dispersive error.

Lax-Wendroff explicit scheme

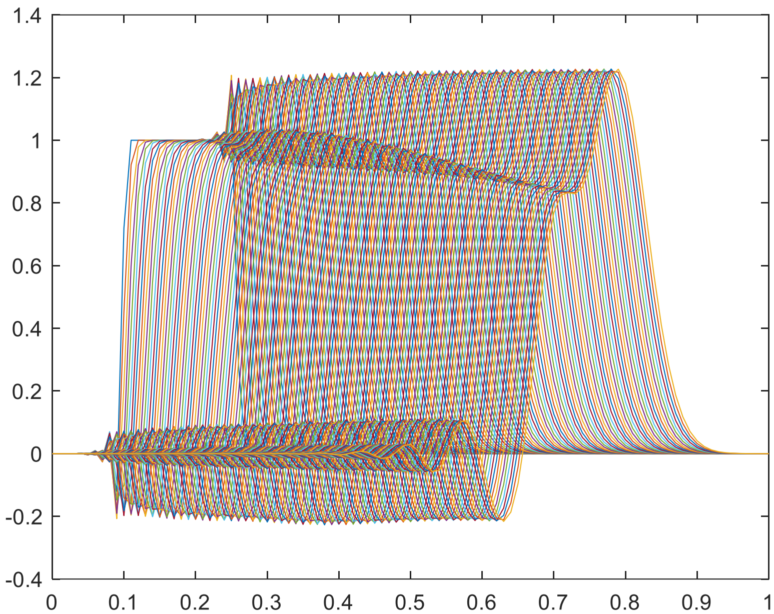

Figure 8: Explicit Lax-Wendroff Scheme at CFL = 0.4. Initially shows stability, then transitions into dissipative error that gradually increases and deforms the wave. There is potentially a mixture of dispersive error as well, as the curve begins to round out towards the end.

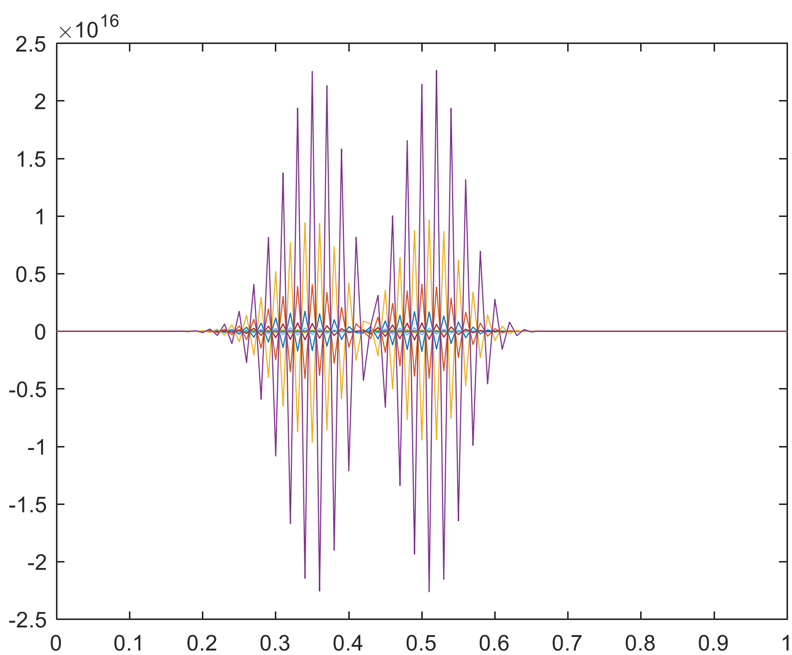

Figure 9: Explicit Lax-Wendroff Scheme at CFL = 1.3. Significant and immediate dissipative error is present, denoting instability of the Lax-Wendroff Scheme at this CFL.

The Lax-Wendroff scheme was stable for the CFL number of 1.0, just like the explicit backward spatial difference method. Refer to Fig. 2 to see the shape of the wave, as it also maintained the square shape throughout the distance. Problems occurred for the other CFL numbers tested, as shown in the two figures above. At a CFL number of 0.4, the Lax-Wendroff scheme held the shape of the wave temporarily before initially diverging due to dissipative error and subsequently diverging due to the cumulative dissipative and dispersive error that distorted the wave shape away from the expected square. At a CFL number of 1.3, the wave immediately demonstrated significant dissipative error that only worsened with time, therefore it is likely that the Lax-Wendroff method is not stable for CFL values greater than one.

Code

```
%% CFD Project 1
% LWE

%% Preallocation/Input section

%%%%%% Run this section before running any other sections %%%%%%

% Inputs
length = 1; % from 0.25 to 0.85
dist = 0.6; % distance to travel
deltaX = 0.01;
wave_speed = 1; % variable c
nu = 1; % cfl/vN number, 0.4, 1.0, 1.3
time = dist/wave_speed; % total time to run

% Dependent inputs
deltaT = nu*deltaX/wave_speed;
nsteps = time/deltaT;

% Number of points
imx = length/deltaX + 1;

% Preallocation
x = ones(1,imx);
u = zeros(1,imx);
utmp = zeros(1,imx);
% Generate 1D mesh
for i = 1:imx
    x(i) = deltaX*(i-1);
end

% Initialize solution
for i = 1:imx
    % This if sequence is slightly redundant
    if x(i) < 0.1
        u(i) = 0;
    elseif x(i) >= 0.1 && x(i) <= 0.25
        u(i) = 1;
    else
        u(i) = 0;
    end
end
u = u'; % transpose for implicit method

%% Explicit backward spacial difference, first order in time
% Loop over time
```

```
for n = 1:10
    for i = 2:imx-1
       utmp(i) = u(i) - nu*(u(i) - u(i-1));
    end
    u(:,:) = utmp(:,:);
    plot(x(:,:),u(:,:));
    hold on
    pause(0.1)
    ylim([0,5])
end

%% Explicit central spacial difference, first order in time
% From class --> for any nu, will never be stable (von neumann analysis)
% Loop over time
for n = 1:5
    for i = 2:imx-1
       utmp(i) = u(i) - nu/2*(u(i+1) - u(i-1));
       %utmp(i) = u(i+1) - nu*(u(i+1) - u(i-1)); % this works but thats weird, nu = 1
        %utmp(i) = u(i-1); % this is why
    end
    u(:,:) = utmp(:,:);
    %u(1,2:imx-1) = utmp(1,2:imx-1);
    plot(x(:,:),u(:,:));

    hold on
    pause(0.1)
end

%% Implicit central spacial difference, first order in time
% Set Boundary Values
BV1 = 0; % Start boundary value for (u_i)^(n+1)
BV2 = 0; % End boundary value

% Preallocate
a = zeros(1,imx-2);
d = a;
c = a;
b = a;
X = a;
% Create tridiagonal matrix vectors
%{
for i = 1:imx-2
    a(i) = -nu/2;
    d(i) = 1;
    c(i) = -a(i);
    for i = 2:imx-1
        b(i-1) = u(i);
    end
end
% Assign boundary values to b vector
b(1) = b(1) + nu/2*BV1;
b(imx-2) = b(imx-2) + nu/2*BV2;
%}
for n = 1:nsteps
    for i = 1:imx-2
        a(i) = -nu/2;
        d(i) = 1;
        c(i) = -a(i);
        for i = 2:imx-1
            b(i-1) = u(i);
        end
    end
    % Assign boundary values to b vector
    %b(1) = b(1) + nu/2*BV1;
```

```
    %b(imx-2) = b(imx-2) + nu/2*BV2;
    b(1) = 0;
    b(imx-2) = 0;

    % Solve tridiagonal matrix using thomas algorithm
    for i = 2:(imx-2)
        w = a(i)/d(i-1);
        d(i) = d(i) - w*c(i-1);
        b(i) = b(i) - w*b(i-1);
    end
    X(imx-2) = b(imx-2)/d(imx-2);
    for i = (imx-3):-1:1
        %X(i+1) = b(i+1)/d(i+1);
        X(i) = (b(i) - c(i)*X(i+1))/d(i);
    end

    for a = 2:imx-3
    u(a) = X(a);
    end

    u(1) = 0;
    u(imx-2) = 0;
    plot(x,u)
    xlim([0,0.6]);
    hold on
    pause(.1)
end
%% Explicit Lax-Wendroff, second order in time and space
% Loop over time
for n = 1:nsteps
    for i = 2:imx-1
       utmp(i) = u(i) - 1/2*nu*(u(i+1) - u(i-1)) + 1/2*nu^2*(u(i+1) - 2*u(i) + u(i-
1));
    end
    u(:,:) = utmp(:,:);
    plot(x(:,:),u(:,:));
    hold on
    pause(0.1)
end

%% Implicit method using backslash solve
tridiag = full(gallery('tridiag',imx,-nu/2,1,nu/2));
for i = 1:nsteps
    utmp = tridiag\u;
    u = utmp;
    plot(x,u)
    hold on
    pause(0.1)
end
```