Ten Quick Tips for Effective Dimensionality Reduction

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Introduction

Dimensionality reduction (DR) is frequently applied during the analysis of high-dimensional data. Both a means of denoising and simplification, it can be beneficial for the majority of modern datasets, where hundreds to millions of variables are measured on the same biological samples. Many statistical methods lack power when applied to high dimensional data due to *the curse of dimensionality*. This challenge is due to the fact that available data points, even in the case of big data, are sparsely submerged in a voluminous high dimensional space that is practically impossible to explore exhaustively (see Chapter 12 [1]). By reducing the dimensionality of the data, one can often alleviate this troublesome phenomenon. Low-dimensional data representations that remove the noise but retain the signal of interest can be instrumental in understanding hidden structures and patterns. The original data might contain measurements of too many variables, some of which might be either uninformative or redundant. Dimensionality reduction can be viewed as a method for latent feature extraction. It is frequently applied for data compression, denoising, exploration, or visualization. Although a multitude of techniques have been developed and many are widely used in standard data analytic pipelines, in practice, DR methods are easy to misuse, and their results are often misinterpreted. This article presents a set of useful guidelines for practitioners specifying how to correctly perform DR, interpret its output, and communicate results. Note that this is not a review article and we recommend some important reviews in the references.

Tip 1: Choose an appropriate method

The abundance of available DR methods can seem intimidating when you want to pick one out of the existing bounty for your analysis. The truth is, you don't really need to commit to only one tool, but you must recognize which methods are appropriate for your application.

The choice of the DR method depends on the nature of the input data. For example, different methods apply to continuous, categorical, count or distance data. You should consider your beliefs about the observed measurements; for example, it is often the case that the data can adequately capture only the relationships between nearby (similar) data points, but not the long-range interactions between distant observations. Considering the nature and the resolution of your data is important, as DR methods can be focused on recovering either global or local structures in the data. In general, linear methods such as principal component analysis (PCA) [2, 3], correspondence analysis (CA) [4], multiple correspondence analysis (MCA) [5] or classical multidimensional

scaling (cMDS), also referred to as principal correspondence analysis (PCoA) [6], are more adept at preserving global structure, whereas non-linear methods such as kernel PCA [7, 8], non-metric multidimensional scaling (NMDS) [9, 10], Isomap [11], diffusion maps [12], and varieties of neighbor embedding (NE) techniques [13] such as t-Stochastic Neighbor Embedding (t-SNE) [14] are better at representing local interactions. Neighbor embedding approaches do not preserve long-range interactions between datapoints, and generate visualization where the arrangement of non-neighboring groups of observations is not informative. As a consequence, inferences should not be made based on large-scale structures observed in NE plots. Reviews of linear and non-linear DR methods are provided in [15] and [16] respectively.

If observations in your data have labels assigned and your goal is to obtain a reduced representation that best separates the given classes, you might consider using supervised dimensionality reduction techniques. Examples of supervised DR methods include partial least squares (PLS) [17], linear discriminant analysis (LDA) [18], neighborhood component analysis (NCA) [19], and the bottleneck neural network classifier [20]. Unlike the unsupervised techniques listed above, which are are blind to class membership, the supervised DR methods explicitly use the information on categories to group observations from the same class together.

In situations where multi-domain data is collected, e.g. gene expression together with proteomics and methylation data, you might apply DR to each data table separately, and then align them using a Procrustes transformation [21], or, instead consider methods that allow for integrating multiple datasets such as STATIS [22, 23] and DISTATIS [24], discussed in tip 9. Table 1 gives a classification and a summary of the basic properties of the DR techniques.

Table 1. Dimensionality reduction methods. Basic properties: type of input required, whether supervised or unsupervised, linear or nonlinear, whether feature representation is provided (in addition to sample representation). We also give the order of computation time (complexity) in terms of: n – the number of observations, p – the number of features in the original data, k – the selected number of nearest neighbors and h – the number of iterations and P is the total number of variables in all available datasets collected on n samples in case of multi-domain data.

Method	**Input Data**	**Method Class**	**Nonlinear**	**Complexity**
PCA	continuous data	unsupervised		$\mathcal{O}(\max(n^2p, np^2))$
CA	categorical data	unsupervised		$\mathcal{O}(\max(n^2p, np^2))$
MCA	categorical data	unsupervised		$\mathcal{O}(\max(n^2p, np^2))$
PCoA (cMDS)	distance matrix	unsupervised		$\mathcal{O}(n^2p)$
NMDS	distance matrix	unsupervised		$\mathcal{O}(n^2h)$
Isomap	continuous*	unsupervised	✓	$\mathcal{O}(n^2(p + \log n))$
Diffusion Map	continuous*	unsupervised	✓	$\mathcal{O}(n^2p)$
Kernel PCA	continuous*	unsupervised	✓	$\mathcal{O}(n^2p)$
t-SNE	continuous/distance	unsupervised	✓	$\mathcal{O}(n^2p + n^2h)$
Barnes-Hut t-SNE	continuous/distance	unsupervised	✓	$\mathcal{O}(nh \log n)$
LDA	continuous (X & Y)	supervised		$\mathcal{O}(np^2 + p^3)$
PLS (NIPALS)	continuous (X & Y)	supervised		$\mathcal{O}(npd)$
NCA	distance matrix	supervised	✓	$\mathcal{O}(n^2h)$
Bottleneck NN	continuous/categorical	supervised	✓	$\mathcal{O}(nph)$
STATIS	continuous	multi-domain		$\mathcal{O}(n^2P, nP^2)$
DiSTATIS	distance matrix	multi-domain		$\mathcal{O}(n^2P, nP^2)$

* Commonly, Isomap estimates geodesic distances between datapoints from Euclidean distances, and Diffusion Map and Kernel PCA compute Gaussian Kernels, and thus require continuous data input. However, it is possible to use categorical data if other dissimilarities or kernels are used.

To assist practitioners, in the table below we include a list of stable implementations of methods discussed in this article.

Table 2. Example implementations. Software packages and function performing specified dimensionality reduction techniques available in R and python. R implementations are given as: `package_name::function_name`; listed python functions come from `sklearn` and `scipy` libraries.

Method	R function	python function
PCA	`stats::prcomp`	`sklearn.decomposition.PCA`
CATPCA	`gifi::princals`	
CA	`FactoMineR::CA`	
MCA	`FactoMineR::MCA`	
PCoA (cMDS)	`stats::cmdscale`	`sklearn.manifold.MDS`
NMDS	`ecodist::nmds`	`sklearn.manifold.MDS`
Isomap	`vegan::isomap`	`sklearn.manifold.Isomap`
Diffusion Map	`diffusionMap::diffuse`	
(Barnes-Hut) t-SNE	`Rtsne::Rtsne`	`sklearn.manifold.TSNE`
LDA	`MASS::lda`	`sklearn.discriminant_analysis.LinearDiscriminantAnalysis`
PLS (NIPALS)	`mixOmics::pls`	`sklearn.cross_decomposition.PLSRegression`
DiSTATIS	`DistatisR::distatis`	
Procrustes	`vegan::procrustes`	`scipy.spatial.procrustes`

The outputs of most linear DR methods can be visualized in R with `factoextra` package [25], used to generate a number of the plots in this article.

Tip 2: Pre-processing continuous and count input data

Before applying dimensionality reduction, suitable data pre-processing is often necessary. For example, data centering – subtracting variable means from each observation – is a required step for PCA on continuous variables and is applied by default in most standard implementations. Another commonly employed data transformation is scaling – multiplying each measurement of a variable by a scalar factor so that the resulting feature has a variance of one. The scaling step ensures equal contribution from each variable, which is especially important for datasets containing heterogeneous features with highly variable ranges or distinct units, e.g. the patient clinical data, or environmental factors data.

When the units of all variables are the same, e.g. in high throughput assays, normalizing feature variances is not advised since it results in shrinkage of features containing strong signals and inflation of features with no signal. Other data transformations are required, depending on the application, the type of input data, and the DR method used. For example, if changes in your data are multiplicative, e.g. your variables measure percent increase/decrease, you should consider applying a log-transform before applying PCA. When working with genomic sequencing data, two issues need to be addressed before you can apply DR. First, each sequencing sample has a different library size (sequencing depth) – a nuisance parameter that artificially differentiates observations. In order to make observations comparable to each other, samples need to be normalized by dividing each measurement by a corresponding sample size factor estimated using specialized methods (e.g. `DESeq2` [26], `edgeR` [27]). Additionally, the assay data exhibit a mean-variance trend where features with higher means have higher variances. A variance stabilization transformation (VST) is needed to adjust for this effect and to avoid a bias towards highly abundant features. For

counts following a negative-binomial distribution, such as the sequencing read counts, an inverse hyperbolic sine transformation or similar techniques are recommended [28–30]. Sample normalization and variance stabilization together are effective and sufficient pre-processing steps for high throughput data.

Tip 3: Categorical input data

In many cases, the collected measurements are not numerical but qualitative or categorical. Such variables represent categories – non-numeric quantities, e.g. phenotypes, cohort memberships, sample sequencing runs, survey respondent ratings. When the relationship between the levels (distinct values) of two categorical variables is of interest, correspondence analysis (CA) is applied to a contingency table (constructed from the data), whose entries are the categories' co-occurrence frequencies. If more than two categorical variables are available, multiple correspondence analysis (MCA) enables the study both the relationship between the observations, and the associations between variable categories. MCA is a generalization of CA and is simply CA applied to an indicator matrix formed by dummy (one-hot) encoding of the categorical variables [5]. When the input data contains both numerical and categorical variables, two strategies are available. If there are only a few categorical variables, PCA is used on numerical variables, and the group means for the levels of the categorical variables can be projected as supplementary (unweighted) points (see chapter 9 of [1] for details). If the mixed data contains a large number of categorical variables, multiple factor analysis (MFA) [31] can be used. The method applies PCA on numerical and MCA on categorical variables and combines the results by weighting of the variable groups.

Another approach to working with categorical or mixed data is to perform PCA on variables transformed using *an optimal quantification*. Traditional PCA cannot be applied to categorical variables, because its objective is to maximize the variance accounted for, a concept that exists only for numerical variables. For categorical variables, *nominal* (unordered) or *ordinal* (ordered), variance can be replaced by a Chi-squared distance on the category frequencies (as in CA), or an appropriate variable transformation made before doing a PCA. Converting categorical variables to dummy binary features is one method, another approach is to use optimal scaling *categorical* PCA (CATPCA) [32–34]. Optimal scaling replaces original levels of categorical variables with category quantifications such that the variance in the new variables is maximized [35]. Categorical PCA is then formulated as an optimization problem, where the squared difference between the quantified data and the principal components is minimized iteratively alternating between the component scores, the component loadings and the variable quantification.

An advantage of optimal scaling is that it does not assume linear relationships between variables. Actually, the ability of CATPCA to handle nonlinear relations between variables is important even when the input data are all numeric. For example, as pointed in [35], the younger and the older individuals tend to have lower income than the middle-aged ones, indicating that age and income are two nonlinearly related variables. When a linear PCA explains only a low proportion of the variance, optimal scaling provides a possible remedy.

Tip 4: Similarity and dissimilarity input data

When neither quantitative nor qualitative features are available, the relationships between datapoints, measured as dissimilarities (or similarities), can be the basis of dimensionality reduction performed as a low-dimensional embedding. This is useful even

when the datasets are initially comprised of measurements. When working with derived distances, you should check how they were computed and make sure that the chosen dissimilarity provides the best summary of your data, e.g. if the original data is binary, the Euclidean distance is not appropriate and the Manhattan distance is better, except if the features are sparse then the Jaccard distance is preferred.

cMDS/PCoA and NMDS use pairwise dissimilarities between datapoints to find an embedding in Euclidean space that provides the best approximation to the supplied distances. While cMDS is an matrix decomposition method akin to PCA, NMDS is an optimization technique that strives to retain only the ordering of the dissimilarities [36]. The latter approach is more applicable when the user has low confidence in the values of the input distances. When the dissimilarity data is only available in non-standard, qualitative formats, more specialized ordinal embedding methods are available, discussed in detail by Kleindessner and von Luxburg in [37, 38]. When using optimization based MDS, you can choose to preserve only the local interactions by restricting the minimization problem to only the distances from datapoints to their neighbors, e.g. the k-nearest neighbors. This approach can be referred to as *local* MDS.

Dissimilarities can also be used as input to t-SNE. Similarly to local MDS, t-SNE is only focused on representing the short-range interactions. However, the method achieves locality in a different way, by converting the supplied distances into proximity measures using a small-tail Gaussian kernel. A collection of neural-network based approaches, called `word2vec` [39], have been developed that also use similarity data (the co-occurrence data) to generate vector embeddings of objects in a continuous Euclidean space. These techniques have proven highly effective at generating word embeddings from text corpus-derived-data, and have since been adapted for gene co-expression data in `gene2vec` program by Du et al. [40]. The robustness of these highly-computational methods has not been yet extensively tested on many biological datasets.

Tip 5: Consciously decide on the number of dimensions to retain

When performing DR, choosing a suitable number of new dimensions to compute is crucial. This step determines whether the signal of interest is captured in the reduced data, especially important when DR is applied as a preprocessing step preceding to statistical analyses or machine learning tasks (e.g. clustering). Even when your primary goal is data visualization, where only two or three axes can be displayed at a time, you still need to select a sufficient number new features to generate. For example, the first two or three principal components might explain an insufficient fraction of the variance, in which case the higher order components should be retained, and multiple combinations of the components should be used for visualizations (e.g. PC1 vs PC2, PC2 vs PC4, and PC3 vs PC5 etc.) In some cases, the strongest signal is a confounding factor, and the variation of interest is captured by higher order PCs. If this is the case, you must use higher order components to expose the desired patterns.

The optimal choice for the number of dimensions to keep depends largely on the data itself. You cannot decide on the right dimension for the output before consulting the data. Remember that the number of dimensions can be at most the minimum of the number of observations (rows) and the number of variables (columns) in your dataset. For example, if your dataset contains expression of $10,000$ genes but for only 10 samples, there could not be more than 10 (or even 9 if the input data have been centered) axes in your reduced data representation. For DR methods based on spectral decompositions, such as PCA or PCoA, you could use the distribution of the eigenvalues to guide your choice of dimensions. In practice, people usually rely on *scree plots* (example in Fig. 1)

and *the elbow rule* when making decisions. A scree plot simply shows the eigenvalues corresponding to each of the axis in the output representation, or equivalently, the proportion of the variance each axis (e.g. a principal component) explains. When viewing the plot you should look for a cut-off point, where an eigenvalue drops significantly below the level on the one immediately preceding it – the"elbow" point. Alternatively, you can inspect a histogram of the eigenvalues and search for the large values that "stand out" from the bulk. Formally, the Marchenko-Pastur distribution asymptotically models the distribution of the singular values of large random matrices. Therefore, for datasets large in both the number of observations and features, you use a rule of retaining only eigenvalues outside the support of the fitted Marchenko-Pastur distribution; however, remember that this applies only when your data has at least thousands of samples and thousands of features.

For non-spectral, optimization-based methods, the number of components is usually pre-specified before DR computations. When using these approaches, the number of components can be chosen by repeating the DR process using an increasing number of dimensions and evaluating whether incorporating more components achieves a significantly lower value of the loss function that the method minimizes e.g. the KL divergence between transition probabilities defined for the input and the output data in case of t-SNE. Ideally, you would like your findings (e.g. patterns seen in visualizations) to be robust to the number of dimensions you choose.

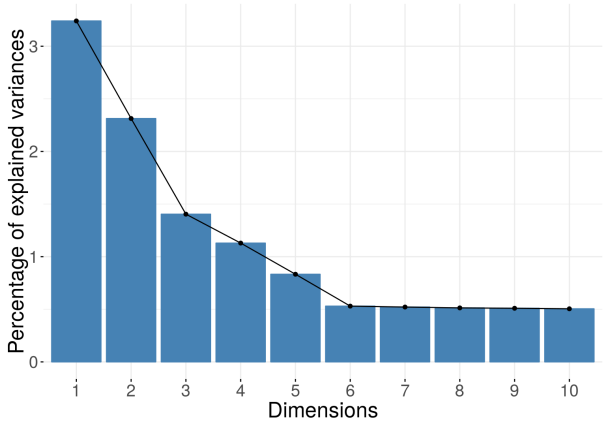
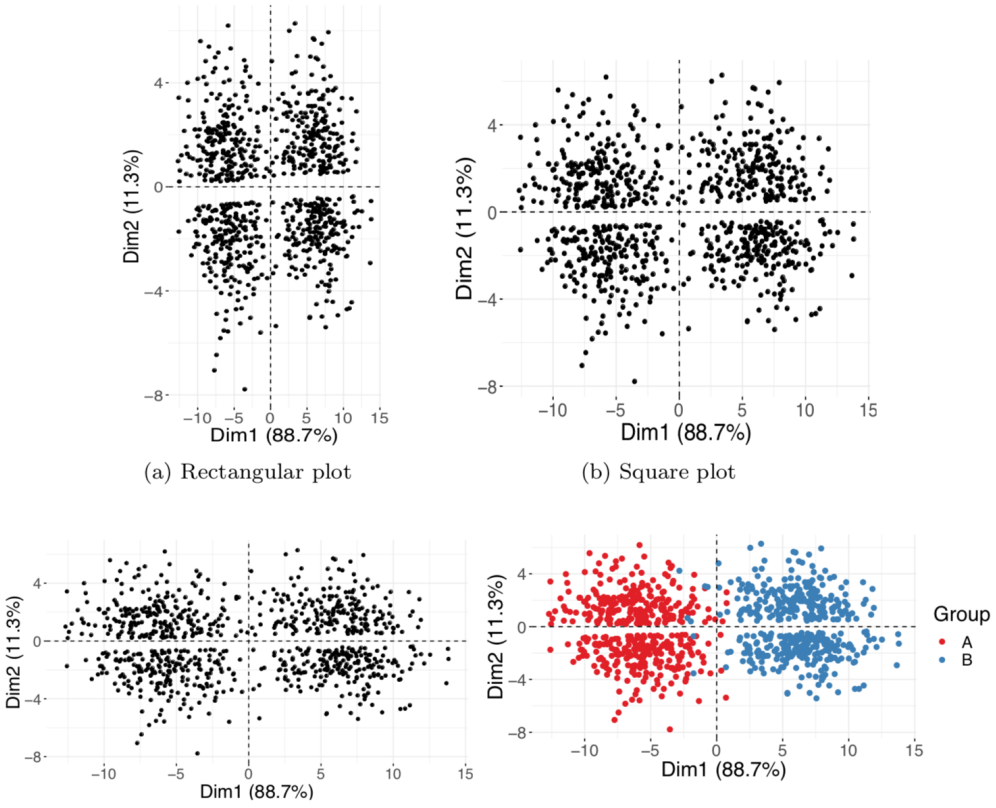


Fig 1. Scree plot. For spectral methods, the eigenvalues can be used to decide how many dimensions are sufficient. The number of dimensions to keep can be selected based on an "elbow rule". In the example shown, you should keep the first five principal components.

Tip 6: Apply the correct aspect ratio for your visualizations

Visualization is an important part of the data exploration process. Therefore, it is crucial that the DR plots you generate accurately reflect the output of the DR methods you use. An important, but frequently overlooked attribute of a visualization, is its aspect ratio. The proportional relationship between the height and the width (and also the depth) of a 2D (and 3D) plot can strongly influence one's perception of the data; therefore, the DR plots should obey the aspect ratio consistent with the relative amount of information explained by the output axes displayed.

In case PCA or PCoA, each output dimension has a corresponding eigenvalue proportional to the amount of variance it explains. If the relationship between the height and the width of a plot is arbitrary, an adequate picture of the data cannot be attained. Two dimensional PCA plots with equal height and width are misleading but frequently encountered, since popular software programs for analyzing biological data, often produce square (2D) or cubical (3D) graphics by default. Instead, the height-to-width ratio of a PCA plot should be consistent with the ratio between the corresponding eigenvalues. Since eigenvalues reflect the variance in coordinates of the associated principal components, you only need to to ensure that in the plots, one "unit" in direction of one PC has the same length as one "unit" in direction of another PC [1].



(a) Rectangular plot (b) Square plot

(c) Adjusted ratio unlabeled (d) Adjusted ratio labeled

Fig 2. Aspect ratio for PCA plots. Two simulated Gaussian clusters projected on the first and the second principal components. Incorrect aspect ratio in a rectangular (a) and square (b) plot. Correct aspect ratio in (c, d) where the plot's height and width are adjusted to match the variances in PC1 and PC2 coordinates. Colors shown in (d) indicate the true Gaussian group membership.

The aspect ratio issue is illustrated with a simulated example, depicted in Fig. 2.In the rectangular (a) and the square (b) plots, the aspect ratio is inconsistent with variance of the PC1 and PC2 coordinates; the result is an (incorrect) apparent grouping of the datapoints into a top and a bottom cluster. In contrast, Fig. 2(c) with lengths of the two axes set to respect the ratio between the corresponding eigenvalues, shows correct clustering, consistent with the true class assignment. For more examples of how the aspect ration can affect the plot interpretation, see chapter 7 and 9 of [1].

[1] If you use `ggplot2` R package for generating plots, adding `+ coords_fixed(1)` will ensure a correct aspect ratio.

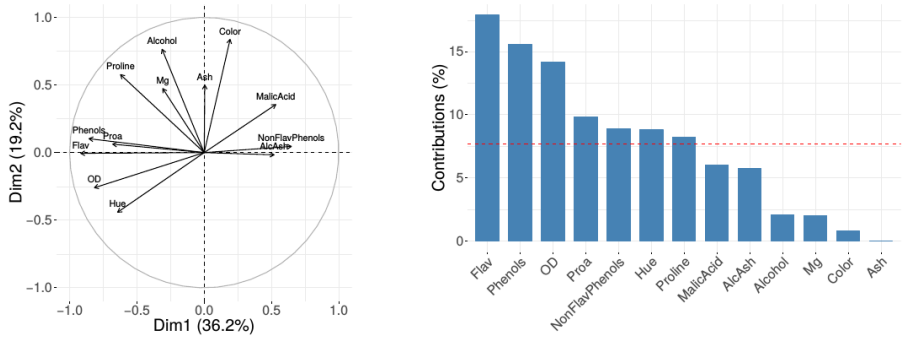
The ordering of the dimensions is not meaningful in many optimization-based DR methods. For example, in the case of t-SNE, you can choose the number of output dimensions (usually 2 or 3) before computing the new representation. Unlike the principal components, the t-SNE dimensions are unordered and equally important, since they have the same weight in the loss function minimized by the optimization algorithm. Thus for t-SNE, the convention is to make the projection plots square or cubical.

Tip 7: Understand the meaning of the new dimensions

Many linear DR methods, including PCA and CA, provide a reduced representation both for the observations and for the variables. Features maps or correlation circles can be used to determine which original variables are associated with each other or with the newly generated output dimensions. The angles between the feature vectors or with the PC axes are informative: vectors at approximately 0°(180°) with each other indicate that the corresponding variables are closely, positively (negatively) related, whereas vectors with a 90°angles indicate rough independence.

Fig. 3(a) shows a correlation circle with scaled coordinates of the variable's projection. The plot indicates, e.g. that high values of PC1 indicate low values in "Flav" (Flavanoids) and "Phenols" (Total phenols), and high values in "Malic Acid" and "AlcAsh"(Alcalinity of ash). Additionally, "AlcAsh" (Alcalinity of ash) seem to be closely negatively correlated with "NonFlav Phenols" (Non-flavanoid phenols) and independent of "Alcohol" levels.

Visualization of the original variables' importance for the new dimensions can be performed through a contribution bar plot. A variable's contribution to a given new axis is computed as the ratio between its squared coordinate (in this axis) over the corresponding sum over all variables; this fraction is most often converted to percentages. Many programs provide variable contribution as a standard output, and can also be defined not only for a single but multiple DR axes by summing the values corresponding to selected components. Fig. 3(b) shows variables' percent contribution to PC1; note that the percent contribution does not carry information on the direction of the correlation. When working with high-dimensional datasets such as high throughput assays, visualizing the contribution of thousands or more original variables to the PC's is not possible. In these cases, you can generate a contribution plot with only a top few (e.g. 20) features with highest weights.



(a) Correlation circle **(b)** Variable Contribution

Fig 3. Variables' projection. PCA on wine dataset shows how variables' representation can be used to understand the meaning of the new dimensions. Correlation circle (a) and PC1 contribution plot (b).

Variables and observations can be included in the same graphic – referred to as a *biplot*. The term was coined by Kuno Ruben Gabriel [41] in 1971, but similar ideas were proposed by Jolicoeur and Mosimann already in 1960 [42]. Biplots such as the one in Fig. 4 allow viewers to explore the trends in the data samples and features simultaneously; looking at both at the same time, you might discover groups of similar (close by) observations that have high or low values for certain measured variables (see tip 8 for further details).

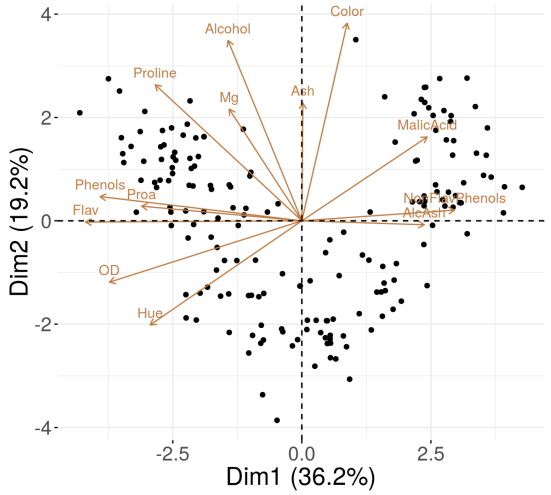


Fig 4. Biplot. PCA projection of samples and variables on the same plot.

Tip 8: Find the hidden signal

The primary objective of dimensionality reduction is to compress data while preserving most of the meaningful information. Compression facilitates the process of data understanding, because the reduced data representation is expected to capture the dominant sources of variation more efficiently, exposing the underlying structure of the data. Interpretation of the main lower dimensional signal can be easier if you can uncover the "hidden variables". The most frequently encountered latent patterns are clustering of datapoints into discrete groups or situation along a continuous gradient.

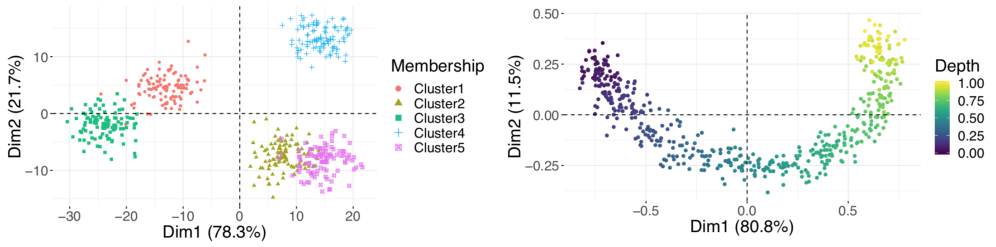
In the former case, one's goal is to bundle similar observations together; an example of simulated clustered data is shown in Fig. 5(a). Researchers often apply PCA before clustering, i.e. they use a set of the top (e.g. 50) principal components as input to a clustering algorithm. PCA and cMDS provide Euclidean approximations to dissimilarities that may or may not obey a triangle inequality.

PCA-reduction is intended as a noise reduction step, since the top eigenvectors are expected to contain all signals of interest [43]. Regrettably, this property does not extend to all DR methods. The output generated by neighborhood embedding techniques, such as t-SNE, should not be used for clustering as they preserve neither distances nor densities – both quantities highly important in the interpretation of clustering output.

Unlike discrete clusters, continuous changes in the data are less widely recognized. It is important to know how to identify and accurately interpret latent gradients, as they often appear in biological data governed by unknown processes. Gradients are present when data points do not separate into distinct tightly packed clusters, but instead

exhibit a gradual shift from one extreme to another; they often emerge as smooth curves in DR-visualizations. It is worth noting that, data points are often arranged in horseshoes or arch-shaped configurations when PCA and cMDS (PCoA) is applied to data involving a linear gradient. *A horseshoe effect* can appear in PCA and cMDS plots when the associated eigenvectors take on a specific form [44] due to the properties of the data covariance or distance matrices used for computations, in particular when these matrices can be expressed as centrosymmetric Kac-Murdock-Szego matrices [45].

You can see an example of this pattern for simulated data with a latent gradient in Fig. 5(b). Continuous transitions are frequently encountered when measurements are taken over time; for example, the cell development literature is rich with publications introducing methods for analyzing pseudo-time, a gradient observed during cell differentiation or development [46, 47]. There can be multiple gradients affecting the data, and a steady change can be recorded in different directions [48]. However, the variable behind the observed continuous gradient could be unknown. In this case, you should focus on finding the discrepancies between the observations at the endpoints (extremes) of the gradients by inspecting the differences between their values for any available external covariates [49], if collected (see tip 7); otherwise, you might need to gather additional information on the samples in your dataset to investigate the explanation of these differences.



(a) Clusters (b) Gradient

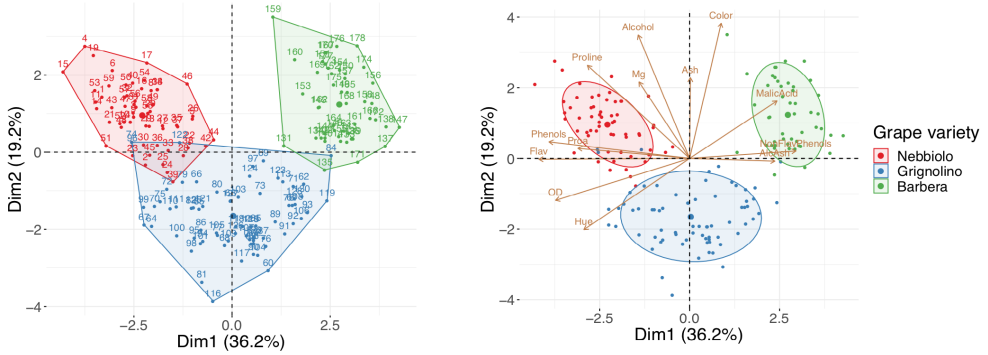
Fig 5. Latent structure. Observations in PCA plots may cluster into groups (a) or follow a continuous gradient (b).

Additional measurements – the ones not used for DR computations – are frequently collected on observations or features included in the dataset. The extra information can be used to improve the understanding of the data. The most simple and common way to utilize the external covariates is to include them in DR visualizations – with their values encoded as color, shape, size or even transparency of corresponding points on the plot. An example of this is shown in Fig. 6(a): the PCA embedding for a dataset on wine properties [50], where the datapoints are colored by wine class, a variable that the DR was blind to. The observed grouping of the wines suggests that thirteen wine properties used for DR can characterize the wine categories well. The "Wine Data Set" is accessible from the UCI Machine Learning Repository [51].

Sometimes, directly plotting the external variable against the newly computed features is an effective way to expose trends present in the data. For example, a scatter plot of a continuous variable, e.g. a patient's age or weight, versus coordinates of a selected output dimension shows correlation between the selected covariate and the new feature. If the external information is categorical instead of continuous, a boxplot of the principal component coordinates (e.g. PC1, PC2 or others) can be generated for each level of the variable.

External information can also be incorporated in biplots. Fig. 6(b) shows how

combining the external information on the observations with the interpretation of the new axes in terms of the original variables (as described in tip 7) allows you to discover that *Barbera* wines tend to have higher values of "Malic Acid" and lower "Flavanoids", and *Grignolinos* tend to have low "Ash" and "Alcohol" content.



(a) Sample Projection (b) Biplot

Fig 6. Using external information. (a) A PCA sample projection on wine dataset shows that based on their properties, wines tend to cluster in agreement with the grape variety classification: Nebbiolo, Grignolino and Barbera. (b) A PCA biplot can be used to find what groups of wines tends to have higher levels of which property.

Additionally, external information can be used to discover batch effects. Batch effects are technical or systemic sources of variation that obscure the main signal of interest. They are frequently encountered in sequencing data, where samples from the same sequencing run (lane) cluster close together. Since batch effects can confound the signal of interest, it is a good practice to check for their presence, and if found, to remove them before proceeding with further downstream analysis. You can detect technical or systemic variations by generating a DR embedding map with the data points colored by their batch membership, e.g by the sequencing run, the cage number, the study cohort. If a batch effect is discovered, you can remove it by shifting all observations in such a way that each batch has its centroid (the group's barycenter) located at the center of the plot (usually the origin of the coordinate system).

Tip 9: Take advantage of multi-domain data

Sometimes, more than one set of measurements is collected for the same collection of samples; for example, we often encounter high-throughput genomic studies involving data from multiple domains. For the same biological sample one can gather microarray gene expression, miRNA expression, proteomics, and DNA methylation data [52]. Integrating multiple datasets allows you to both obtain a more accurate representation of higher order interactions, and evaluate the associated variability. Samples often exhibit have varying levels of uncertainty, as different regions of the data can be subject to different rates of changes or fluctuations.

One way of dealing with *multi-domain*, also referred to as *multi-modal*, *multi-way*, *multi-view* or *multi-omics* data is to perform DR for each dataset separately, and then align them together using a Procrustes transformation – a combination of translation, scaling and rotation to align one configuration with another as closely as possible ((see [21] and [36]). A number of more advanced methods have also been developed, for instance, STATIS [22] and DISTATIS [24, 53] – generalizations of PCA and classical MDS respectively. Both methods are used to analyze several sets of data tables

collected on the same set of observations, and both are based on an idea of combining datasets into a common consensus structure called the *compromise* [54].

The datasets can all be projected onto this consensus space. The projections of individual datasets can be helpful for observing different patterns in observations characterized by data from different domains. Fig. 7 shows an example of the use of DiSTATIS on five simulated distance tables for 20 synthetic datapoints. Different colors correspond to different datapoints, and different shapes correspond to different distance tables. The compromise points between the tables are denoted with larger diamond-shape-markers. For a detailed survey on the analysis of multitable data, with a focus on biological multiomics datasets, refer to Meng et al. [55].

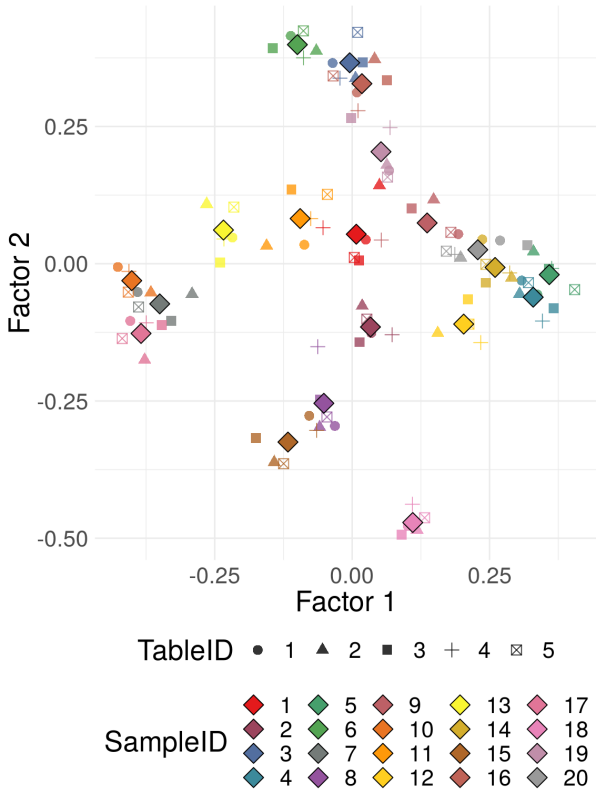


Fig 7. Multi-domain data. DiSTATIS on multiple distance tables defined for the same observations. Multiple distances can be computed from different data modalities e.g. gene expression, methylation, clinical data, or from data resampled from known data-generating distribution.

Tip 10: Check the robustness of your results and quantify uncertainties

For some datasets, the PCA principal components are ill-defined i.e. two or more successive PCs may have very similar variances, and the corresponding eigenvalues are almost exactly the same, as in Fig. 8. Although a subspace spanned by these components together is meaningful, the eigenvectors (PCs) are not informative individually, and their loadings cannot be interpreted separately, since a very slight change in even one observation can lead to a completely different set of eigenvectors [1].

In these cases, we say that these PCs are unstable. The dimensions corresponding to similar eigenvalues need to be kept together, and not individually interpreted.

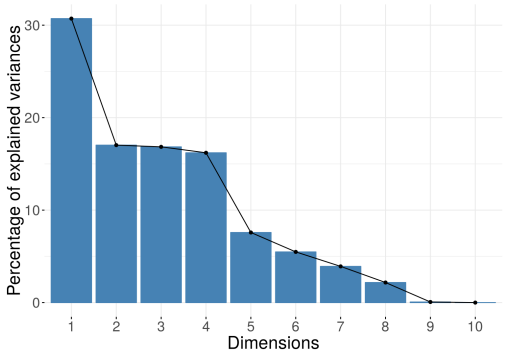


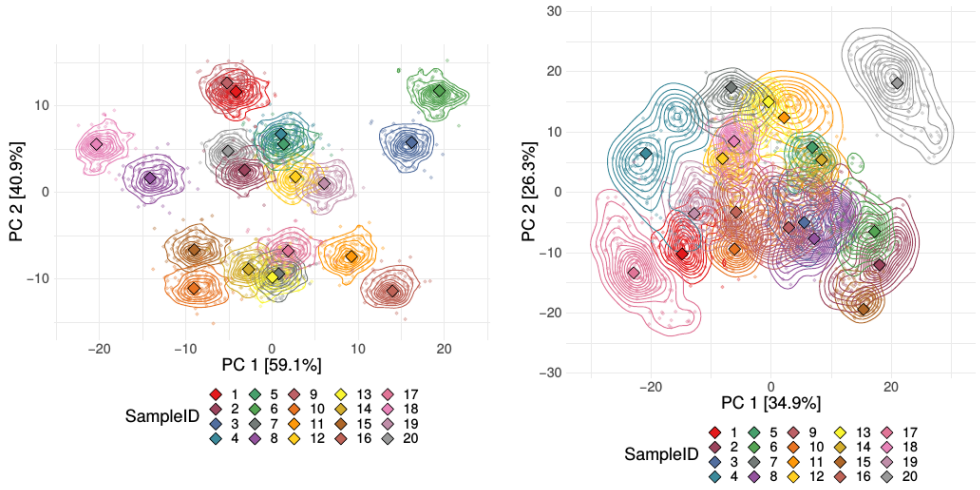
Fig 8. Unstable eigenvalues. When subsequent eigenvalues have close to equal values, PCA representation is unstable.

When working with techniques that require parameter specification, you should also check the stability of your results against different parameter settings. For example, when running t-SNE, you need to pick a value for perplexity, and different settings can alter the results obtained even qualitatively. It has been frequently observed that when the perplexity is set to a very small value, *artificial clusters* start forming in t-SNE plots. You should not use the values of the t-SNE objective function, the Kullback-Divergence (KL) divergence, as a criterion to choose an "optimal perplexity". This is due to the fact that the KL divergence always decreases (monotonically) as perplexity values increase. For t-SNE, a BIC-type rule for selecting perplexities was proposed by Cao and Wang in [56]. However, in practice you should repeat DR computations for a range of input parameters and visually evaluate whether the patterns discovered are consistent across varying specifications, as formal guarantees and stability theory for t-SNE has not yet been developed. In particular, if the clustering pattern disappears with only a slight increase of the perplexity value, the grouping you observed might be only an artifact due to an unsuitably small choice of the parameter.

A separate concern is a method's stability against outliers. In general, it is known that observations far from the origin have more influence on the PCs than the ones close to the center; sometimes it is possible that only a small fraction of the samples in the data almost fully determines the PCs. One should be mindful of these situations and verify that the structure captured by DR represents the bulk of the data and not just a few outliers. In DR maps, the outliers are the remote points, distant from the majority of the observations. In case of PCA and other linear methods, if all of the points in a sample projection plot are located closely to the origin (the center of the plot), whereas only one or a few points lie very far away, the DR solution is said to be dominated by the outliers. You should inspect suitable data-specific quality control metrics for these points and consider their removal. If samples are removed, the DR needs to be recomputed and the changes in the output representation should be noted. Observe how observations shifted by comparing the DR visualizations before and after the removal of the outliers. You should consider removing not only the technical outliers, but also the "outgroups", the aberrant groups known to be extensively different from the majority of the data. Eliminating the outgroups and recomputing the DR, allows for patterns in the bulk of the data to emerge. On the other hand, if a dataset contains many aberrant observations, stable methods such as robust Kernel PCA [57] should be used.

Additionally, you can estimate the uncertainties associated with observations, by constructing a collection of "bootstrap" datasets, i.e. random subsets of the data

generated by re-sampling observations with replacement. The bootstrap set can be treated as multi-way data and the STATIS or Procrustes aligning method described in tip 8 can be applied to "match" the random subsets together. When a realistic noise model for the data is available, instead of using bootstrap subsamples, you can generate copies of all data points by perturbing the measurement values for each sample and then applying STATIS or DiSTATIS method as described in the previous tip, to generate the coordinates for the *compromise* and for each individual perturbed copy of the data. Obtaining multiple coordinates estimates per datapoint, allows you to estimate the corresponding uncertainty. You can visualize each sample's uncertainty on a DR embedding map using density contours, or by plotting all data points from each bootstrap's projection onto the compromise. Fig. 9 shows the Procrustes alignments of PCA projections for two simulated datasets. The colored lines indicate density contours for the output coordinates of the bootstrap subsets, and the diamond markers correspond to the coordinates of the projection of the full data. Plots were produced for 20 synthetic data points from a true 2-dimensional and 5-dimensional Gaussian both orthogonally projected to 10 dimensions. We can observed that uncertainties for points in the lower rank data are much smaller i.e. the first 2 PCs represent the first dataset better than the second one.



(a) Low-rank data (b) High-rank data

Fig 9. Datapoint uncertainties. Stability in the DR output coordinates for each datapoint. Projections of bootstrap samples for two 10D simulated datasets with rank 2 (a) and rank 5 (b), onto the first two principal components aligned using a Procrustes transformation. Smaller, circular markers correspond to each bootstrap trial, and larger, diamond markers are coordinates of the full dataset.

Conclusion

Dimensionality reduction is very useful and sometimes essential when analyzing high dimensional data. Despite their widespread adoption, DR methods are often misused or misinterpreted. Researchers performing DR might find the sheer number of available methods already intimidating, not to mention a wide variety of different distance metrics or parameter settings whose choice some of these methods might require. This

set of ten tips serves as a checklist or informal guideline for practitioners. We described a general step-by-step procedure for performing effective dimensionality reduction, and gave pointers for correctly interpreting and adequately communicating the output of DR algorithms. Most of the recommendations discussed here apply to any DR method, but some were instructions directed towards specific reduction approaches.

In addition to everything discussed earlier, we would like to offer one extra piece of advice: keep track of all the decisions you make, including the method you select, the distances or kernels you choose, and the values of parameters you use. The most convenient way to save all steps of your work together with the results obtained is through an R, an IPython or a Jupyter notebook; these applications allow you to generate a full analysis report containing narrative text, code, and its output. Recording your choices is a crucial part of reproducible research [58]; it allows others to replicate the same results you obtained, and speeds up your analysis process next time you work with similar data.

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