Computer Science Theory
COMS W3261
Homework 5

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Problems

1. Using the following grammar for lambda calculus expressions

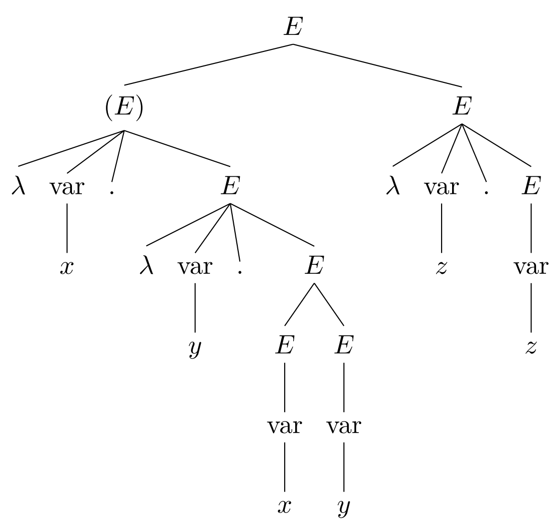
$$E \to \lambda \text{var}.E \mid E\, E \mid (E) \mid \text{var}$$

constructs a parse tree for the expression

$$(\lambda x.\lambda y.x\, y)\lambda z.z$$

Use the standard disambiguating conventions for lambda expressions in constructing your parse tree.

Solution:



2. Consider the lambda-calculus expression $(\lambda x.(\lambda y.x)x)((\lambda z.z)(\lambda w.(\lambda v.v)w))$.

   (a) Identify all redexes in this expression.

Solution: There are 4 redexes.

1. $(\lambda x.(\lambda y.x)x)((\lambda z.z)(\lambda w.(\lambda v.v)w))$
2. $(\lambda y.x)x$
3. $(\lambda z.z)(\lambda w.(\lambda v.v)w)$
4. $(\lambda v.v)w$

(b) Evaluate this expression using normal order evaluation.

Solution:

$$\begin{aligned}
(\lambda x.(\lambda y.x)x)((\lambda z.z)(\lambda w.(\lambda v.v)w)) &\underset{\beta}{\rightarrow} (\lambda y.((\lambda z.z)(\lambda w.(\lambda v.v)w))((\lambda z.z)(\lambda w.(\lambda v.v)w))) \\
&\underset{\beta}{\rightarrow} (\lambda z.z)(\lambda w.(\lambda v.v)w) \\
&\underset{\beta}{\rightarrow} (\lambda w.(\lambda v.v)w) \\
&\underset{\alpha}{\rightarrow} (\lambda a.(\lambda v.v)w) \\
&\underset{\beta}{\rightarrow} \lambda v.v
\end{aligned}$$

(c) Evaluate this expression using applicative order evaluation.

Solution:

$$\begin{aligned}
(\lambda x.(\lambda y.x)x)((\lambda z.z)(\lambda w.(\lambda v.v)w)) &\underset{\alpha}{\rightarrow} (\lambda x.(\lambda y.x)x)((\lambda z.z)(\lambda a.(\lambda v.v)w)) \\
&\underset{\beta}{\rightarrow} (\lambda x.(\lambda y.x)x)((\lambda z.z)(\lambda v.v)) \\
&\underset{\beta}{\rightarrow} (\lambda x.(\lambda y.x)x)(\lambda v.v) \\
&\underset{\alpha}{\rightarrow} (\lambda q.(\lambda y.q)x)(\lambda v.v) \\
&\underset{\beta}{\rightarrow} (\lambda y.(\lambda v.v))x \\
&\underset{\beta}{\rightarrow} \lambda v.v
\end{aligned}$$

3. Evaluate the lambda expression $(\lambda x.(\lambda y.(x(\lambda x.x\,y))))y$. Describe all the steps in your evaluation.

Solution:

$$\begin{aligned}
(\lambda x.(\lambda y.(x(\lambda x.x\,y))))y &\underset{\alpha}{\rightarrow} (\lambda z.(\lambda y.(z(\lambda x.x\,y))))y && (1) \\
&\underset{\alpha}{\rightarrow} (\lambda z.(\lambda q.(z(\lambda x.x\,y))))y && (2) \\
&\underset{\beta}{\rightarrow} (\lambda q.(y(\lambda x.xy))) && (3)
\end{aligned}$$

(1) Alpha reduction to remove ambiguity between bound variables and free variables. The outermost λx is converted to λz along with the bound x variable.

(2) Alpha reduction to remove ambiguity between bound variables and free variables. λy is converted to λq to avoid ambiguity with the free y on the outside of the equation.

(3) Beta reduction. Substitute the y argument for any z's in the equation. We cannot reduce anymore from here so we are left with the equation $\lambda q.(y(\lambda x.x\,y))$.

4. Let G be the lambda abstraction

$$G = (\lambda f.\lambda x.f(f\,x))$$

Evaluate the lambda expression $G\,G$.

Solution:

$$\begin{aligned}
GG &= (\lambda f.\lambda x.f(f\,x))(\lambda f.\lambda x.f(f\,x)) \\
&\underset{\beta}{\to} \lambda x.(\lambda f.\lambda x.f(f\,x))((\lambda f.\lambda x.f(f\,x))\,x) \\
&\underset{\alpha}{\to} \lambda x.(\lambda f.\lambda z.f(f\,z))((\lambda f.\lambda z.f(f\,z))\,x) \\
&\underset{\beta}{\to} \lambda x.\lambda z.((\lambda f.\lambda z.f(f\,z))\,x)(((\lambda f.\lambda z.f(f\,z))x)x) \\
&\underset{\alpha}{\to} \lambda x.\lambda z.((\lambda f.\lambda v.f(f\,v))\,x)(((\lambda f.\lambda v.f(f\,v))x)x) \\
&\underset{\beta}{\to} \lambda x.\lambda z.(\lambda v.x(x\,v))((\lambda v.x(x\,v))x) \\
&\underset{\beta}{\to} \lambda x.\lambda z.x(x(\lambda v.x(x\,v))x) \\
&\underset{\beta}{\to} \lambda x.\lambda z.x(x(x(x\,v)))
\end{aligned}$$

5. Let add, one, and two be the following lambda expressions:

$$\begin{aligned}
\text{add} &= \lambda m.\lambda n.\lambda f.\lambda x.m\,f(n\,f\,x) \\
\text{one} &= \lambda f.\lambda x.f\,x \\
\text{two} &= \lambda f.\lambda x.f(f\,x)
\end{aligned}$$

Evaluate (add one two).

Solution:

$$
\begin{aligned}
(\text{add one two}) &= ((\lambda m.\lambda n.\lambda f.\lambda x.mf(n\,f\,x))\text{one two}) \\
&\underset{\beta}{\rightarrow} (\lambda n.\lambda f.\lambda x.\text{one}\,f(n\,f\,x))\,\text{two} \\
&\underset{\beta}{\rightarrow} (\lambda f.\lambda x.\text{one}\,f(\text{two}\,f\,x)) \\
&\underset{\alpha}{\rightarrow} (\lambda f.\lambda x.(\lambda f'.\lambda x\prime.f'\,x')f(\text{two}\,f\,x)) \\
&\underset{\beta}{\rightarrow} (\lambda f.\lambda x.(\lambda x'.f\,x')(\text{two}\,f\,x)) \\
&\underset{\beta}{\rightarrow} \lambda f.\lambda x.f\,(\text{two}\,f\,x) \\
&\underset{\alpha}{\rightarrow} \lambda f.\lambda x.f\,((\lambda f'.\lambda x'.f'(f'\,x'))\,f\,x) \\
&\underset{\beta}{\rightarrow} \lambda f.\lambda x.f(\lambda x'.f(f\,x')x) \\
&\underset{\beta}{\rightarrow} \lambda f.\lambda x.f(f(f\,x))
\end{aligned}
$$