COMS W3261
Computer Science Theory
Chapter 3 Notes

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Regular Expressions

Regular expressions can define exactly the same languages that the various forms of automata describe: the regular languages. However, regular expressions offer something that automata do not: a declarative way to express the strings we want to accept. Thus regular expressions serve as the input language for many systems that process strings.

The Operators of Regular Expressions

Regular expressions denote languages. There are three operations on languages that the operators of regular expressions represent. These operators are:

1. The *union* of two languages L and M, denoted by $L \cup M$, is the set of strings that are in either L or M or both.

2. The *concatenation* of languages L and M is the set of strings that can be formed by taking any string in L and concatenating it with any string in M. The concatenation operator is frequently called "dot".

3. The *closure* (or *star*, or *Kleene closure*) of a language L is denoted L^* and represents the set of those strings that can be formed by taking any number of strings from L, possibly with repetitions and concatenating all of them.

Finite Automata and Regular Expressions

While the regular-expression approach to describing languages is fundamentally different from the finite-automaton approach, these two notations turn out to represent exactly the same set of languages, which we have termed the "regular languages". In order to show that the regular expressions define the same class, we must show that:

1. Every language defined by one of these automata is also defined by a regular expression. For this proof, we can assume the language is accepted by some DFA.

2. Every language defined by a regular expression is defined by one of these automata. For this part of the proof the easiest is to show that there is an NFA with ϵ-transitions accepting the same language.

From DFA's to Regular Expressions

The construction of a regular expression to define the language of any DFA is surprisingly tricky. Roughly, we build expressions that describe sets of strings that label certain paths in the DFA's transition diagram. However, the paths are allowed to pass through only a limited subset of states.

Theorem 1. *If $L = L(A)$ for some DFA A, then there is a regular expression R such that $L = L(R)$.*

Converting DFA's to Regular Expressions by Eliminating States

The construction of a regular expression is expensive. Not only do we have to construct about n^3 expressions for an n-state automaton, but the length of the expression can grow by a factor of 4 on the average, with each of the n inductive steps., if there is no simplification of the expressions. Thus, the expressions themselves could reach on the order of 4^n symbols.

The approach to constructing regular expressions that we shall now learn involves eliminating states. When we eliminate a state s, all the paths that went through s no longer exist in the automaton. If the language of the automaton is not to change, we must include, on an arc that goes directly from q to p, the labels of paths that went from state q to p, through s. Since the label of this arc may now involve strings, rather than single symbols, and the label of there may even be an infinite number of such strings, we cannot simply list the strings as a label. Fortunately, there is a simple, finite way to represent all such strings: use a regular expression.

Thus, we are led to consider automata that have regular expressions as labels. The language of the automaton is the union over all paths from the start state to an accepting state of the language formed by concatenating the languages of the regular expressions along that path. Each symbol a, or ϵ if it is allowed, can be thought of as a regular expression whose language is a single string, either $\{a\}$ or $\{\epsilon\}$.

We suppose that the automaton of which s is a state has predecessors states

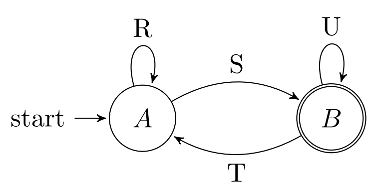
$q_1, q_2, \ldots, q_k$ for s and successor states $p_1, p_2, \ldots, p_m$ for s. It is possible that some of the q's are also p's, but we assume that s is not among the q's or p's, even if there is a loop from s to itself. We also show a regular expression on each arc from one of the q's to s; expression Q_i labels the arc from q_i. Likewise, we show a regular expression P_i labeling the arc from s to p_i, for all i. We show a loop on s with label S. Finally, there is a regular expression R_{ij} on the arc from q_i to p_j, for all i and j. Note that some of these arcs may not exist in the automaton, in which case we take the expression on that arc to be $\emptyset$.

All arcs involving state s are deleted. To compensate, we introduce, for each predecessor q_i of s and each successor p_j of s, a regular expression that represents all the paths that start at q_i, go to s, perhaps loop around s zero or more times, and finally go to p_j. The expression for these paths is $Q_i S^* P_j$. This expression is added (with the union operator) to the arc from q_i to p_j. If there was no arc $q_i \to p_j$, then first introduce one with regular expression $\emptyset$.

The strategy for constructing a regular expression from a finite automaton is as follows:

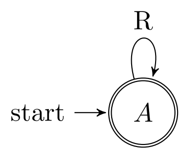
1. For each accepting state q, apply the above reduction process to produce an equivalent automaton with regular-expression labels on the arcs. Eliminate all states except q and the start state q_0.

2. If $q \neq q_0$, then we shall be left with a two-state automaton that looks like this:

   The regular expression for the accepted strings can be described in various ways. One is $(R + SU^*T)^*SU^*$. In explanation, we can go from the state state to itself any number of times, by following a sequence of paths whose labels are in either $L(R)$ or $L(SU^*T)$. The expression SU^*T represents paths that go to accepting state via a path in $L(S)$, perhaps return to the accepting state several times using a sequence of paths with labels in $L(U)$, and then return to the start state with a path whose label is in $L(T)$. Then we must go to the accepting state, never to return to the start state, by following a path with a label in $L(S)$. Once in the accepting state, we can return to it as many times as we like, by following a path whose label is in $L(U)$.

3. If the start state is also an accepting state, then we must also perform a state-elimination from the original automaton that gets rid of every state but the start state. That will look something like this:



The regular expression denoting the strings that it accepts is R^*.

4. The desired regular expression is the sum (union) of all the expressions derived from the reduced automata for each accepting state, by rules (2) and (3).

Converting Regular Expresions to Automata

Every language L, that is $L(R)$ for some regular expression R, is also $L(E)$ for some ϵ-NFA E. The proof is a structural induction on the expression R. We start by showing how to construct automata for the basis expressions: single symbols, ϵ, and $\emptyset$. We then show how to combine these automata into larger automata that accept the union, concatenation, or closure of the language accepted by smaller automata.

Ordering the Elimination of States

When a state is neither the start state nor an accepting state, it gets eliminated in all the derived automata. Thus, one of the advantages of the state-elimination process compared with the mechanical generation of regular-expressions is that we can start by eliminating all the states that are neither start nor accepting, once and for all. We only have to begin duplicating the reduction effort when we need to eliminate some accepting states.

Theorem 2. *Every language defined by a regular expression is also defined by a finite automaton.*

Applications of Regular Expressions

We shall consider two important classes of regular-expression-based applications: lexical analyzers and text search.

Regular Expressions in UNIX

Before seeing the applications, we shall introduce the UNIX notation for extended regular expressions. This notation gives us a number of additional capabilities. In fact, the UNIX extensions include certain features, especially the ability to name and refer to previous strings that have matched a pattern, that actually allow nonregular languages to be recognized.

The first enhancement to the regular-expression notation concerns the fact that most real applications deal with the ASCII character set. UNIX regular expressions allow us to write *character classes* to represent large sets of characters as succinctly as possible. The rules for character classes are:

- The symbol `.` (dot) stands for "any character".
- The sequence $[a_1a_2 \cdot a_k]$ stands for the regular expression

$$a_1 + a_2 + \cdots + a_k$$

  The notation saves about half the characters, since we don't have to write the $+-$ signs. For example, we could express the four characters used in C comparison operators by `[<>=!]`.

- Between the square braces we can put a range of the form x-y to mean all the characters from x to y in the ASCII sequence. Since the digits have codes in order, as do the upper-case letters and the lower-case letters, we can express many of the classes of characters that we really care about with just a few keystrokes. If we want to include a minus sign among a list of characters, we can place it first or last, so it is not confused with its use to form a character range. Square brackets, or other characters that have special meanings in UNIX regular expressions can be represented as characters by preceding them with a backslash (\).
- There are special notations for server of the most common classes of characters. For instance:

  a) `[:digit:]` is the set of ten digits, the same as `[0-9]`.

  b) `[:alpha:]` stands for any alphabetic character, as does `[A-Za-z]`.

  c) `[:alnum:]` stands for the digits and letters (alphabetic and numeric characters), as does `[A-Za-z0-9]`.

In addition, there are several other operators that are used in UNIX regular expressions. None of these operators extended what languages can be expressed, but they sometimes make it easier to express what we want.

- The operator — is used in place of + to denote union.
- The operator `?` means "zero or one of." Thus, $R?$ in UNIX is the same as $\epsilon + R$ in our notation.
- The operator + means "one or more of." Thus, $R+$ in UNIX is shorthand for RR^* in our notation.
- The operator $\{n\}$ means "n copies of." Thus, $R\{5\}$ in UNIX is shorthand for $RRRRR$.

Lexical Analysis

One of the oldest applications of regular expressions was in specifying the component of a compiler called a "lexical analyzer". This component scans the source program and recognizes all *tokens*, those substrings of consecutive characters that belong together logically. Keywords and identifiers are common examples of tokens, but there are many others.

Commands such as `lex` and `flex` have been found extremely useful because the regular-expression notation is exactly as powerful as we need to describe tokens. These commands are able to use the regular-expression-to-DFA conversion process to generate an efficient function that breaks source programs into tokens.

Finiding Patterns in Text

The general problem for which regular-expression technology has been found useful is the description of a vaguely defined class of patterns in text. The vagueness of the description virtually guarantees that we shall not describe the pattern correctly at first – perhaps we can never get exactly the right description. By using regular-expression notation, it becomes easy to describe the patterns at a high level, with little effort, and to modify the description quickly when things go wrong.

Algebraic Laws for Regular Expressions

Two expressions with variables are *equivalent* if whatever languages we substitute for the variables, the results of the two expressions are the same language.

Like arithmetic expressions, the regular expressions have a number of laws that work for them. Many of these are similar to the laws for arithmetic, if we think of union as addition and concatenation as multiplication. However, there are some operations that do not have analogs to arithmetic.

Associativity and Commutativity

Commutativity is the property of an operator that says we can switch the order of its operands and get the same result. *Associativity* is the property of an operator that allows us to regroup the operands when the operator is applied twice. Here are three laws that hold for regular expressions:

- $L + M = M + L$. This law, the *commutative law for union*, says that we may take the union of two languages in either order.
- $(L + M) + N = L + (M + N)$. This law, the *associative law of union*, says that we may take the union of three languages either by taking the union of the first two initially, or taking the union of the last two initially. Note that, together with the commutative law for union, we conclude that we can take the union of any collection of languages with any order

and grouping, and the result will be the same. Intuitively, a string is in $L_1 \cup L_2 \cup \cdots \cup L_k$ if and only if it is in one or more of the L_i's.

- $(LM)N = L(MN)$. This law, the *associative law for concatenation*, says that we can concatenate three languages by concatenating either the first two or the last two initially.

NOTE: $LM = ML$ is **NOT** true. Concatenation cannot be commutative.

Identities and Annihilators

An *identity* for an operator is a value such that when the operator is applied to the identity and some other value, the result is the other value. An *annihilator* for an operator is a value such that when the operator is applied to the annihilator and some other value, the result is the annihilator.

There are three laws for regular expressions involving these concept:

- $\emptyset + L = L + \emptyset = L$. This law asserts that $\emptyset$ is the identity for union.
- $\epsilon L = L\epsilon = L$. This law asserts that ϵ is the identity for concatenation.
- $\emptyset L = L\emptyset = \emptyset$. This law asserts that $\emptyset$ is the annihilator for concatenation.

Distributive Laws

A *distributive law* involves two operators, and asserts that one operator can be pushed down to be applied to each argument of the other operator individually. The laws are:

- $L(M + N) = LM + LN$. This law, is the *left distributive law of concatenation over union.*
- $(M + N)L = ML + NL$. This law is the *right distributive law of concatenation over union.*

Theorem 3. *If L, M, and N are any languages, then*

$$L(M \cup N) = LM \cup LN$$

The Idempotent Law

An operator is said to be *idempotent* if the result of applying it to two of the same values as arguments is that value. Union and intersection are common examples of idempotent operators. Thus, for regular expressions, we may assert the following law:

- $L + L = L$. This law, the *idempotence law for union*, states that if we take the union of two identical expressions, we can replace them by one copy of the expression.

Laws Involing Closures

The are a number of laws involving the closure operators and its UNIX-style variants $^{+}$ and ?.

- $(L^*)^* = L^*$. This law says that closing an expression that is already closed does not change the language.

- $\emptyset^* = \epsilon$. The closure of $\emptyset$ contains only the string ϵ.

- $\epsilon^* = \epsilon$. It is easy to check that the only string that can be formed by concatenating any number of copies of the empty string is the empty string itself.

- $L^+ = LL^* = L^*L$. Recall that L^+ is defined to be $L + LL + LLL + \cdots$. Also, $L^* = \epsilon + L + LL + LLL + \cdots$. Thus,

  $$LL^* = L\epsilon + LL + LLL + LLLL + \cdots$$

  When we remember that $L\epsilon = L$, we see that the infinite expansions for LL^* are for L^+ are the same. That proves $L^+ = LL^*$.

- $L^* = L^+ + \epsilon$. Since the expansion of L^+ includes every term in the expansion of L^* except ϵ.

- $L^* = \epsilon + L$. This rule is really the definition of the ? operator.

Summary of Chapter 3

Regular Expressions This algebraic notation describes exactly the same languages as finite automata; the regular languages. The regular- expression operators are union, concatenation, and closure.

Regular Expressions in Practice Systems such as UNIX and various of its commands use an extended regular-expression language that provides shorthands for many common expressions. Character classes allow the easy expression of sets of symbols, while operators such as one-or-more-of and at-most-one-of augment the usual regular-expression operators.

Equivalence of Regular Expressions and Finite Automata We can convert a DFA to a regular expression by an inductive construction in which expressions for the labels of paths allowed to pass through increasingly larger sets of states are constructed. Alternatively, we can use a state-elimination procedure to build the regular expression for a DFA. In the other direction, we can construct recursively an ϵ-NFA from regular expressions, and then convert the ϵ-NFA to a DFA, if we wish.

The Algebra of Regular Expressions Regular expressions obey many of the algebraic laws of arithmetic, although there are differences. Union and concatenation are associative, but only union is commutative. Concatenation distributes over union. Union is idempotent.

Testing Algebraic Identities We can tell whether a regular-expression equivalence involving variables as arguments is true by replacing the variables by distinct constants and testing whether the resulting languages are the same.