Practical DFT – Choosing N_0, T or T_0 – How Do We Determine What These Should Be?

That is, how do we decide the number of samples needed and the interval to sample our signal at?

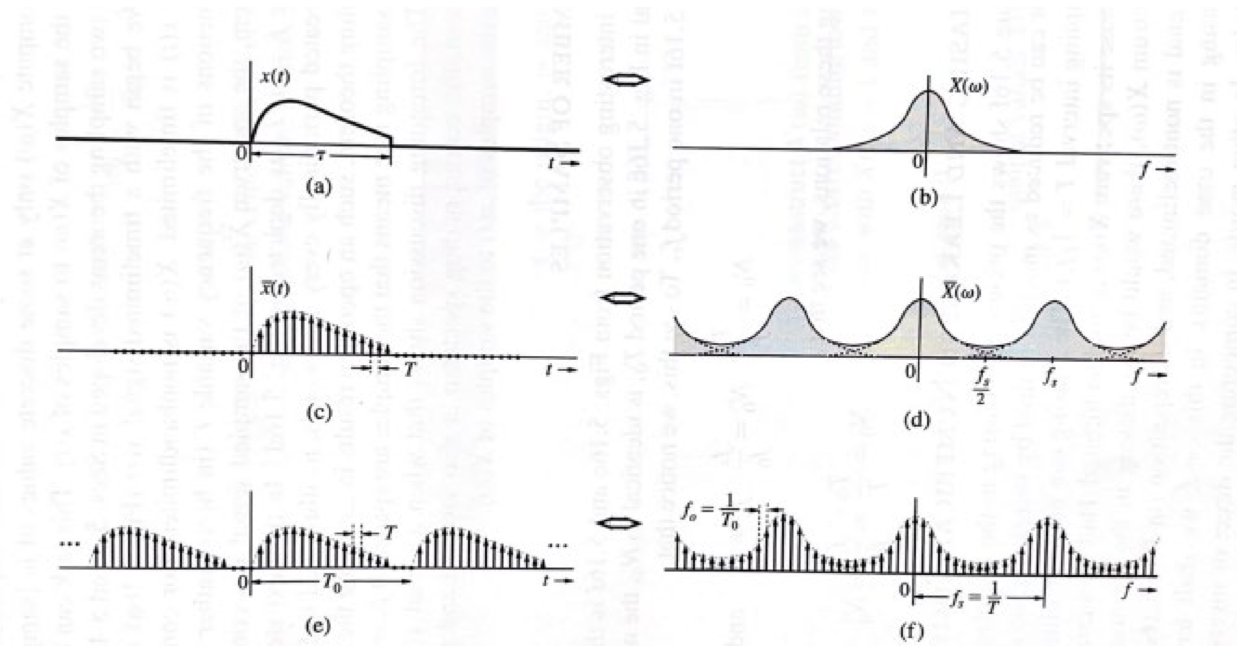


Figure 5.16 Relationship between samples of $x(t)$ and $X(\omega)$.

Before selecting these, let's look at the sampling theorem logic above to determine how these terms are related

The figure above demonstrates, according to the sampling theorem, how the sampling rate of the time-domain signal will dictate the number of frequency samples that exist in the DFT spectrum.

We start off with our normal time domain signal x(t) and its corresponding Fourier Transform, X(w). Then, multiplying x(t) by that unit impulse train results in the sampled signal $x_p(t)$. According to the **sampling theorem**, this sample signal's spectrum is just going to be the original signal's spectrum X(w) periodically replicated and scaled. It is replicated every f_s hertz in the frequency domain; that is, the separation between each replication in Hz is == the frequency that x(t) was sampled at.

Now, we know computers can't truly work with a continuous spectrum. So, what can we do? Sample the spectrum, effectively! Like we just observed, sampling a signal would result in its transform's periodic replication. Keep this in mind. We don't actually directly sample some continuous spectrum, as we don't even have that continuous spectrum. Rather, we have to use this logic in reverse to obtain that sampled spectrum.

So, what we can do is **treat the finite sampled signal** $x_p(t)$ == x[n] as a **single period** of a periodic signal. That is, we can periodically replicate whatever finite number of samples we have every multiple of that number of samples.

In terms of samples, if we have N_0 samples, then we would repeat the values of our signal every N_0 samples! In terms of seconds, if we have a signal x(t) that was T_0 seconds long in duration, then we'd repeat it every T_0 seconds! (and if we sampled it every T seconds, then we'd have T_0/T samples, just for reference).

Now, remember: the sampling theorem tells us that the distance/separation between each periodic replication of a *signal's transform* (DFT or IFT, applies in both directions) is equal to the rate (samples/unit) that the other signal was sampled at.

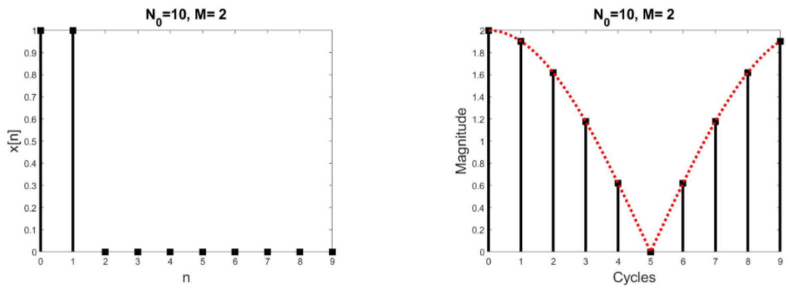
Therefore, using this logic, if the time-domain signal had a separation of T_0 seconds between each replication, then the signal's transform must have been sampled at a RATE of T_0 samples "per HZ." **If you take the reciprocal of this RATE T_0 samples per Hz, you get the frequency sampling interval, f_0, with units Hz/sample!**

Additionally, because we know that the spectrum of $x_p(t)$ is periodic (before and after sampling), we really only care about a single period, as it'll just repeatedly be the same. Therefore, if we know that the spacing per sample is f_0 Hz and we know the bandwidth (the maximum frequency of) appearing in our signal x(t)'s spectrum, then we can deduce that the number of samples would be bandwidth 2B / f_0. 2B because we're only looking at a single period.

B is the bandwidth of x(t), and should be half the sampling frequency as we'll soon see (or because it's the Nyquist rate). Therefore B should be equal to the sampling frequency of the time domain signal, f_s, divided by 2. So 2B == $2*f_s/2$ == $1/f_s$. f_s is the number of samples per second for x(t), == 1/T. Then, f_0 equals the number of Hz per sample, == $1/T_0$. Therefore, $(1/T)/(1/T_0)$ gives us T_0/T, which if we look at the sampled x(t) in the time domain, we see that this is equal to the **number of samples!** SO, the conclusion here is that the **number of samples N_0 in the frequency domain is the same as the number of samples N_0 in the time domain**.

Okay—but what makes the DFT spectrum symmetric, then?

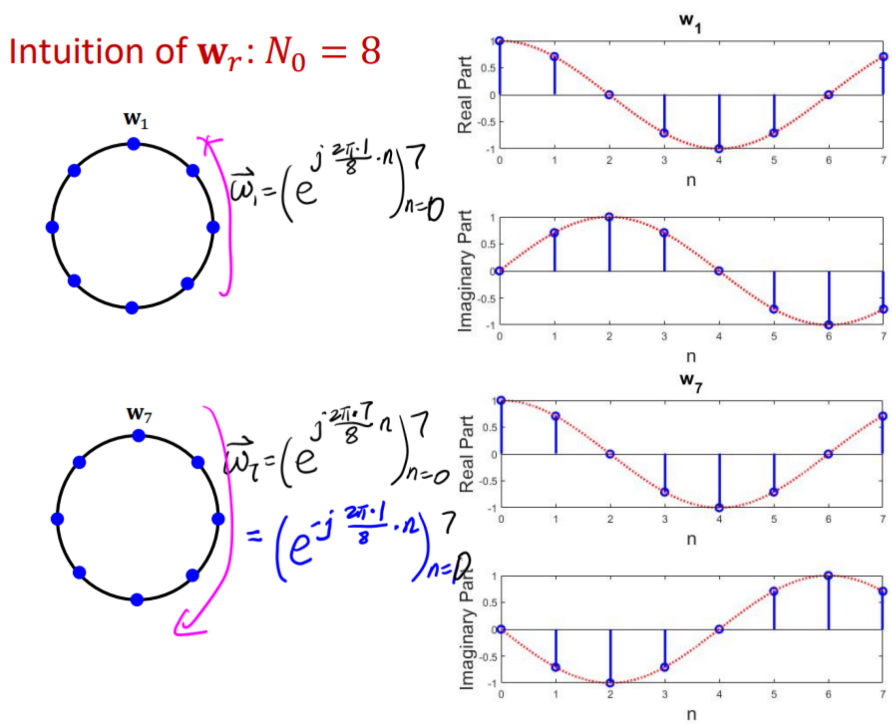
If you start looking at some DFT spectrums, like the one to the right, you'll notice that the magnitude spectrum is mirrored about some sample point. And if you look closer, you'll notice that the point about which it's symmetric is the number of samples divided by 2. N_0 / 2.



Why does this occur? Well, remember that the Fourier expansion or representation of a signal is the sum of all complex exponential signals of different frequencies, amplitudes, and phases that can be added together to form a signal that looks like the original signal.

Now, the discrete spectrum on the right shows us the magnitude of the different frequency complex exponentials, where r = the cycle number, and the frequency of the complex exponential would be r * the spacing in Hz between each frequency in this spectrum! I.e., $f_r = r*f_0$.

Okay, getting to why it becomes symmetric: Certain frequency complex exponentials end up having the same magnitudes at the same times in their rotations as other frequency complex exponentials. This is because their real components match up, and their imaginary components are opposite, but their magnitudes are the same at the same times therefore. This figure helps better explain the phenomenon.



THUS, we observe that, after the r == $N_0/2$ frequency complex exponential, the magnitude of these signals starts to be the same as their counterpart frequency signal $N_0 - r$. For this reason, our spectrum is mirrored about this point!

So, what does this symmetric spectrum mean for us practically?

Well, if we have these N_0 frequency samples in our spectrum, but half of them are just **always literal reflections** of lower frequency component signals (complex exponentials), this really just means that we can only represent frequencies up to that $r = N_0/2$!

That is, if we have N_0 samples in our discrete frequency spectrum, and f_0 Hz between each rth frequency signal, then the maximum that we can maintain in our spectrum is going to be $r=N_0/2 * f_0$. That is, the **maximum representable frequency is $f_s / 2$!**

Remember, f_0 is determine by $1/T_0$, where T_0 is the total amount of time in the original signal. AND, $T_0 = T*N_0$, $T = 1/f_s$, so $T_0 = N_0/f_s$, which means $\mathbf{f_0 = f_s / N_0}$. Practically, then, the highest frequency that will be part of our spectrum will be $\mathbf{f_s}$! Therefore, the sampling rate is the bandwidth of the DFT spectrum (but not necessarily the highest ***observable*** frequency, as we'll find below). Okay, with this refresher in mind:

Example:

> If I sample my original at a sampling rate of f_s = 10 samples or cycles/second == 10 Hz, and my signal has the duration T_0 = 1s, then I know that my sampling interval $T_0 = 1/10 = 0.1$ seconds per sample, and that I'll end up with 1/.1 == 10 samples == N_0. Using the sampling theorem logic from before, we know that the number of samples in our spectrum will also therefore be N_0 == 10 samples.
>
> Now, $f_0 == 1/T_0$ Hz per sample == 1 Hz/sample (could also use f_s/N_0 == 10/10 == 1Hz). Okay, so that means that, with 10 samples, we should, in theory, be able to show the magnitude of complex exponential signals from 0Hz → r = 9*1Hz = 9 Hz (r=0→9 == 10 samples).
>
> However, because of the symmetry that occurs, we really only end up only getting insight into the signals with frequencies that are half of the 10Hz bandwidth. That is, we only really care about or can see the magnitudes of half of the signals that are on this symmetric spectrum, as past $r = N_0/2$, they are just duplicates of earlier frequencies, and therefore don't tell us anything new! Just repeated information!
>
> **Therefore**, the highest frequency that we can observe the magnitude of is $N_0/2 * f_0$ == 5*1 = 5Hz.

So what does this mean for choosing sampling rate f_s?

Well, imagine your original signal x(t) is a sinusoid with frequency = 20 Hz. If you sampled it at f_s = 10Hz as in the above example, the maximum frequency that you'd be able to see the magnitude of is its 5Hz component signal (complex exponential).

Therefore, because **maximum representable frequency in a DFT is half your sampling rate f_s / 2**, you must **choose a sampling rate that is <u>at least</u> 2 times as large as your signals max frequency!** I.e.,

$$\mathbf{f_s > 2*f_{max} \text{ of } x(t)} == 2*BW\ x(t).$$

So for the 20Hz signal above, you'd need a sampling rate of 40Hz to be able to accurately see represent all of the frequency signals of x(t). This is another awesome motivation for the Nyquist rate of $f_s = 2*f_{max}$!

Practical Audio Example

A cool example of this might crop up when working with music. If the human ear can really only hear signals of frequency up to 20 KHz, then the sampling rate of audio might only need to be around 44 KHz (4KHz extra for practical overhead, perhaps?).

Often, however, you'll find that audio equipment (like audio interfaces) will sample the analogue input audio signal at some ludicrous rate, like 196 KHz, however. Why might this be?

Well, if you recall from the sampling theorem and Nyquist discussions, a larger sampling rate will still offer more definition in the signal. While this isn't necessarily to accommodate higher frequencies, it might be done in order that applying affects and processing different sound signals can be done with greater precision!

So how do we actually go about selecting the rest of the parameters for our transform (N_0, T, etc.).

In truth, most of the parameters you'll need to figure out for your transform will all come from that required sampling rate! The required sample rate must be twice as large as the maximum frequency of the signal being sample == twice the bandwidth of the signal being sample:

$$f_s > 2*f_{max} \text{ of } x(t) == 2xBW \text{ of } x(t)$$

Then, once you've determined what the sampling rate must be, the number of seconds or time between each sample of x(t) is simply

$$T = 1/f_s$$

Now, T_0 could be decided from a few different perspectives:

a. You have recorded a finite amount of data in a measured amount of time == T_0.
    a. In this case, you already know what you're T_0 is, and therefore you can determine the number of samples you'll have provided your sampling rate and sampling interval as

$$N_0 = T_0/T$$

b. You have a certain desired f_0 == Hz per sample resolution you want for your transform's spectrum
    a. In this case, the interval of time you'd need to collect data for (or pad up to) is

$$T_0 = 1/f_0.$$

    b. And the number of samples you'd have as a result would be (same as above)

$$N_0 = T_0/T$$