ASSIGNMENT #1

FE630 Portfolio Theory and Application

Napat Loychindarat
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Question 1: Basic Matrix Operations, Eigenvalues and Eigenvectors

$$\text{Matrix } A = \begin{pmatrix} 1 & -2 & 6 \\ -2 & 3 & 0 \\ 6 & 0 & 5 \end{pmatrix}$$

1.1) Compute the Characteristic Polynomial of A, and the Eigenvalues and Eigenvectors of A

To find the Characteristic Polynomial of A, we need to solve $\det(A - \lambda I) = 0$

Then, substitute values:

$$\begin{vmatrix} 1-\lambda & -2 & 6 \\ -2 & 3-\lambda & 0 \\ 6 & 0 & 5-\lambda \end{vmatrix} = 0$$

We get:

$$[(1-\lambda)(3-\lambda)(5-\lambda) + (-2)(0)(6)+(6)(-2)(0)] - [(6)(3-\lambda)(6)+(0)(0)(1-\lambda)+(5-\lambda)(-2)(-2)] = 0$$

$$(1-\lambda)(3-\lambda)(5-\lambda) - [(6)(3-\lambda)(6) + (5-\lambda)(-2)(-2)] = 0$$

$$[(15 - 23\lambda + 9\lambda^2 + \lambda^3)] - [128 - 40\lambda)] = 0$$

$$-113 + 17\lambda + 9\lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 9\lambda^2 - 17\lambda + 113 = 0$$

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 9.540400 \\ 3.181955 \\ -3.722355 \end{pmatrix}$$

Therefore, we have eigenvalues: $\lambda = 9.540400, 3.181955, -3.722355$.

To find eigenvector associated with eigenvalue $\lambda_1 = 9.540400$, we have that

$$\begin{bmatrix} 1 - 9.540400 & -2 & 6 \\ -2 & 3 - 9.540400 & 0 \\ 6 & 0 & 5 - 9.540400 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} -8.540400 & -2 & 6 \\ -2 & -6.540400 & 0 \\ 6 & 0 & -4.540400 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$

Solve this matrix, then we have:

$$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 1.51800951 \\ -0.464194701 \\ 2.006003229 \end{bmatrix}$$

We normalize the matrix:

$$\sqrt{1.51800951^2 + (-0.464194701)^2 + 2.006003229^2} = 2.558100575$$

$$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \frac{1}{2.558100575} \begin{bmatrix} 1.51800951 \\ -0.464194701 \\ 2.006003229 \end{bmatrix}$$

$$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0.5934128 \\ -0.1814607 \\ 0.7841768 \end{bmatrix}$$

To find eigenvector associated with eigenvalue $\lambda_2 = 3.181955$, we have that

$$\begin{bmatrix} 1 - 3.181955 & -2 & 6 \\ -2 & 3 - 3.181955 & 0 \\ 6 & 0 & 5 - 3.181955 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} -2.181955 & -2 & 6 \\ -2 & -0.181955 & 0 \\ 6 & 0 & 1.818045 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

Solve this matrix, then we have:

$$\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} -0.412815586 \\ 4.53755693 \\ 1.362393953 \end{bmatrix}$$

We normalize the matrix:

$$\sqrt{(-0.412815586)^2 + 4.53755693^2 + 1.362393953^2} = 4.755623711$$

$$\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \frac{1}{4.755623711} \begin{bmatrix} -0.412815586 \\ 4.53755693 \\ 1.362393953 \end{bmatrix}$$

$$\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} -0.08680581 \\ 0.95414545 \\ 0.28648075 \end{bmatrix}$$

To find eigenvector associated with eigenvalue $\lambda_3 = -3.722355$, we have that

$$\begin{bmatrix} 1 + 3.722355 & -2 & 6 \\ -2 & 3 + 3.722355 & 0 \\ 6 & 0 & 5 + 3.722355 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 4.722355 & -2 & 6 \\ -2 & 6.722355 & 0 \\ 6 & 0 & 8.722355 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0$$

Solve this matrix, then we have:

$$\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 1.46883 \\ 0.436999 \\ -1.01039 \end{bmatrix}$$

We normalize the matrix:

$$\sqrt{01.46883^2 + 0.436999^2 + (-1.01039)^2} = 1.835570115$$

$$\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \frac{1}{1.835570115} \begin{bmatrix} 1.46883 \\ 0.436999 \\ -1.01039 \end{bmatrix}$$

$$\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0.8002038 \\ 0.2380724 \\ -0.5504502 \end{bmatrix}$$

Hence, we have normalized eigenvectors associated with eigenvalues $\lambda = 9.540400, 3.181955, -3.722355$ as follows:

$$v = \begin{bmatrix} 0.5934128 & -0.08680581 & 0.8002038 \\ -0.1814607 & 0.95414545 & 0.2380724 \\ 0.7841768 & 0.28648075 & -0.5504502 \end{bmatrix}$$

1.2) Using the Eigenvalues and Eigenvectors of A, find an orthonormal matrix P and a diagonal matrix D such that A = PDP^{-1}.

To find orthogonal matrix, we use eigenvalues and eigenvectors:

Since $A^T = A$, A is therefore a symmetric matrix. When we perform eigenvalue decomposition, our eigenvectors are orthogonal.

Therefore,

$$P = v = \begin{bmatrix} 0.5934128 & -0.08680581 & 0.8002038 \\ -0.1814607 & 0.95414545 & 0.2380724 \\ 0.7841768 & 0.28648075 & -0.5504502 \end{bmatrix}$$

$$D = \lambda I = \begin{bmatrix} 9.540400 & 0 & 0 \\ 0 & 3.181955 & 0 \\ 0 & 0 & -3.722355 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & -2 & 6 \\ -2 & 3 & 0 \\ 6 & 0 & 5 \end{bmatrix}$$

Hence, $A = PDP^{-1}$.

1.3) Is the matrix A invertible? If yes, provide an expression of its inverse A^{-1} in terms of *P* and *D*, and provide also the eigenvalues of A^{-1}.

$$det(A) = \begin{vmatrix} 1 & -2 & 6 \\ -2 & 3 & 0 \\ 6 & 0 & 5 \end{vmatrix} = -113$$

Since $det(A) \neq 0$, matrix A is therefore invertible.

$$A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$$

Compute eigenvalues of A^{-1} by looking at D^{-1}

$$D^{-1} = \begin{bmatrix} \frac{1}{9.540400} & 0 & 0 \\ 0 & \frac{1}{3.181955} & 0 \\ 0 & 0 & \frac{1}{-3.722355} \end{bmatrix} = \begin{bmatrix} 0.1048174 & 0 & 0 \\ 0 & 0.3142722 & 0 \\ 0 & 0 & -0.2686471 \end{bmatrix}$$

Hence, eigenvalues of A^{-1}= 0.3142722, 0.1048174, -0.2686471.

1.4) Use R, Matlab or Python to verify your results from questions 1, 2 and 3.

For question 1.1)

Find eigenvalues and eigenvectors of matrix A.

```
A <- matrix(c(1,-2,6,-2,3,0,6,0,5), nrow = 3, ncol = 3)
A

##      [,1] [,2] [,3]
## [1,]    1   -2    6
## [2,]   -2    3    0
## [3,]    6    0    5

eigen(A)

## eigen() decomposition
## $values
## [1]  9.540400  3.181955 -3.722355
##
## $vectors
##            [,1]        [,2]       [,3]
## [1,]  0.5934128 -0.08680581  0.8002038
## [2,] -0.1814607  0.95414545  0.2380724
## [3,]  0.7841768  0.28648075 -0.5504502
```

For question 1.2)

Check symmetric matrix, get orthonormal matrix P and find diagonal matrix D.

```
t(A) == A

##      [,1] [,2] [,3]
## [1,] TRUE TRUE TRUE
## [2,] TRUE TRUE TRUE
## [3,] TRUE TRUE TRUE

P <- eigen(A)$vectors
P

##            [,1]        [,2]       [,3]
## [1,]  0.5934128 -0.08680581  0.8002038
## [2,] -0.1814607  0.95414545  0.2380724
## [3,]  0.7841768  0.28648075 -0.5504502

D <- diag(eigen(A)$values)
D

##        [,1]     [,2]      [,3]
## [1,] 9.5404 0.000000  0.000000
## [2,] 0.0000 3.181955  0.000000
## [3,] 0.0000 0.000000 -3.722355
```

For question 1.3)

Check invertible property, get matrix A^{-1} and find eigenvalues of A^{-1}.

```
det(A)

## [1] -113

solve(A)
```

```
##              [,1]        [,2]        [,3]
## [1,] -0.13274336 -0.08849558 0.159292035
## [2,] -0.08849558  0.27433628 0.106194690
## [3,]  0.15929204  0.10619469 0.008849558
```

```r
P%*%solve(D)%*%solve(P)
```

```
##              [,1]        [,2]        [,3]
## [1,] -0.13274336 -0.08849558 0.159292035
## [2,] -0.08849558  0.27433628 0.106194690
## [3,]  0.15929204  0.10619469 0.008849558
```

```r
eigen(solve(A))$values
```

```
## [1]  0.3142722  0.1048174 -0.2686471
```

Question 2: Non Constrained Optimization and Convexity

2.1) Find (on the paper) the extrema of f : (x_1; x_2; x_3) → f(x_1; x_2; x_3) = x_1^2+(x_1+x_2)²+(x_1+x_3)². Justify your answers. Use an optimizer in R, Matlab or Python to verify your answer.

Take derivative of f(x)

$$\nabla f(x) = \begin{bmatrix} 2x_1 + 2(x_1 + x_2) + 2(x_1 + x_3) \\ 2(x_1 + x_2) \\ 2(x_1 + x_3) \end{bmatrix}$$

Set matrix equals to 0.

$$\begin{bmatrix} 6x_1 + 2x_2 + 2x_3 \\ 2x_1 + 2x_2 \\ 2x_1 + 2x_3 \end{bmatrix} = 0$$

Then, we have:

$$x_1 = 0, x_2 = 0, x_3 = 0 \text{ or } (0, 0, 0)$$

Take second-order partial derivative or Hessian of f(x)

$$\nabla^2 f(x) = H(x_1,\ x_2,\ x_3) = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

First principal minor H_1(0, 0, 0) is 6 > 0

Second principal minor H_2(0, 0, 0) is $\begin{vmatrix} 6 & 2 \\ 2 & 2 \end{vmatrix} = 12 - 4 = 8 > 0$

Third principal minor H_3(0, 0, 0) is $\begin{vmatrix} 6 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix}$ = (6)(2)(2)+0+0-(2)(2)(2)-0-(2)(2)(2)=24-8-8= 8 > 0

Moreover,

$$\det(H - \lambda I) = \begin{vmatrix} 6 - \lambda & 2 & 2 \\ 2 & 2 - \lambda & 0 \\ 2 & 0 & 2 - \lambda \end{vmatrix} = 0$$

$$(6 - \lambda)(2 - \lambda)^2 - 4(2 - \lambda) - 4(2 - \lambda) = 0$$

$$(6 - \lambda)(2 - \lambda)^2 - 8(2 - \lambda) = 0$$

$$(2 - \lambda)[(6 - \lambda)(2 - \lambda) - 8] = 0$$

We have that $\lambda = 7.4641, 2, 0.5359$. This means that the eigenvalues of Hessian are strictly positive.

Since H(0, 0, 0) and eigenvalues are positive definite, then (0, 0, 0) is therefore a local minimum.

```
library(nloptr)

## Warning: package 'nloptr' was built under R version 3.4.2

F1<-function(x){
x[1]^2+(x[1]+x[2])^2+(x[1]+x[3])^2
}
bobyqa(c(0,0,0),f2,lower=c(-1,-1,-1),upper=c(2,2,2))
```

```
## $par
## [1] 0 0 0
##
## $value
## [1] 0
##
## $iter
## [1] 23
##
## $convergence
## [1] 4
##
## $message
## [1] "NLOPT_XTOL_REACHED: Optimization stopped because xtol_rel or xtol_abs (above) was reached."
```

```r
H <- matrix(c(6, 2, 2, 2, 2, 0, 2, 0, 2), nrow = 3, byrow = T)
eigen(H)
```

```
## eigen() decomposition
## $values
## [1] 7.4641016 2.0000000 0.5358984
##
## $vectors
##           [,1]       [,2]       [,3]
## [1,] 0.8880738  0.0000000  0.4597008
## [2,] 0.3250576 -0.7071068 -0.6279630
## [3,] 0.3250576  0.7071068 -0.6279630
```

2.2) Is the function defined by function f : (x1; x2; x3) → f(x1; x2; x3) = x_1^4+$(x_1+x_2)^2$+$(x_1+x_3)^2$ convex or concave over $\mathbb{R}^3$?. Write a computer program in R, Matlab or Python to verify your answer.

Take derivative of f(x)

$$\nabla f(x) = \begin{bmatrix} 4x^3{}_1 + 2(x_1 + x_2) + 2(x_1 + x_3) \\ 2(x_1 + x_2) \\ 2(x_1 + x_3) \end{bmatrix}$$

Set matrix equals to 0.

$$\begin{bmatrix} 12x^2{}_1 + 2x_2 + 2x_3 \\ 2x_1 + 2x_2 \\ 2x_1 + 2x_3 \end{bmatrix} = 0$$

Then, we have:

$$x_1 = 0, x_2 = 0, x_3 = 0 \text{ or } (0, 0, 0)$$

Take second-order partial derivative or Hessian of f(x)

$$\nabla^2 f(x) = H(x_1,\, x_2,\, x_3) = \begin{bmatrix} 12x^2{}_1 + 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

First principal minor H_1 is $12x^2{}_1 + 4 > 0$.

Second principal minor H_2 is $\begin{vmatrix} 12x^2{}_1 + 4 & 2 \\ 2 & 2 \end{vmatrix} = 24x^2{}_1 + 8 - 4 = 24x^2{}_1 + 4 > 0$.

Third principal minor H_3 is $\begin{vmatrix} 12x^2{}_1 + 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix} = (12x^2{}_1 + 4)(2)(2)+0+0-(2)(2)(2)-0-(2)(2)(2)= 48x^2{}_1 \geq 0$.

Moreover,

$$\det(H(0,0,0) - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 2-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda)^2 - 4(2-\lambda) - 4(2-\lambda) = 0$$

$$(4-\lambda)(2-\lambda)^2 - 8(2-\lambda) = 0$$

$$(2-\lambda)[(4-\lambda)(2-\lambda) - 8] = 0$$

We have that $\lambda = 6, 2, 0$. This means that the eigenvalues of Hessian are positively non-negative.

Since H_1, H_2, H_3 are more than 0 but H_3 can be zero as well as the eigenvalues are positively non-negative, hence hessian is positive semi-definite and f(x) is convex function over $\mathbb{R}^3$.

```r
library(nloptr)
f2<-function(x)
{
x[1]^4+(x[1]+x[2])^2+(x[1]+x[3])^2
}
bobyqa(c(0,0,0),f2,lower=c(-1,-1,-1),upper=c(2,2,2))

## $par
## [1] 0 0 0
##
## $value
## [1] 0
##
## $iter
## [1] 24
##
## $convergence
## [1] 4
##
## $message
## [1] "NLOPT_XTOL_REACHED: Optimization stopped because xtol_rel or xtol_abs (above) was reached."

H <- matrix(c(4, 2, 2, 2, 2, 0, 2, 0, 2), nrow = 3, byrow = T)
eigen(H)

## eigen() decomposition
## $values
## [1] 6.000000e+00 2.000000e+00 3.552714e-15
##
## $vectors
##           [,1]       [,2]       [,3]
## [1,] 0.8164966  0.0000000  0.5773503
## [2,] 0.4082483 -0.7071068 -0.5773503
## [3,] 0.4082483  0.7071068 -0.5773503
```

Question 3: Optimization with Equality Constraints

3.1) Solve

$$\begin{cases} \min\limits_{x_1, x_2, x_3} \ \boldsymbol{x_1 + x_2 + 2x_3^2} \\ \textit{subject to} \ \begin{matrix} \boldsymbol{x_1 = 1} \\ \boldsymbol{x_1^2 + x_2^2 = 1} \end{matrix} \end{cases}$$

And use optimizer to verify your answer.

Define Lagrangian function of problem.

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1 + x_2 + 2x_3^2 + \lambda_1(1 - x_1) + \lambda_2(1 - x_1^2 + x_2^2)$$

Take first partial derivative and set equal to zero.

$$\frac{\partial L}{\partial x_1} = 1 + \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 4x_3 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 1 - x_1 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - x_1^2 - x_2^2 = 0$$

We have that $x_1 = 1, x_2 = 0, x_3 = 0, \lambda_1 = 0, \lambda_2 = \frac{1}{2}$.

Then substitute values, we have: $f(1, 0, 0) = 1 + 0 = 1$.

Then consider Hessian matrix of f$(x_1, x_2, x_3) = x_1 + x_2 + 2x_3^2$

$$H(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

We have that $\lambda = 4, 0, 0$. This means that the eigenvalues of Hessian are positively non-negative.

Since hessian matrix is positive semi-definite, f(x) is a convex function and constraint functions are linear, the solution $(x_1, x_2, x_3, \lambda_1, \lambda_2) = (1, 0, 0, 0, \frac{1}{2})$ minimizes f(x_1, x_2, x_3)= f(1, 0, 0)=1 subject to the constraints.

```r
library(nloptr)
f1<-function(x) {
    x[1]+x[2]+2*x[3]^2
}
f2<-function(x){
    h <- numeric(4)
    h[1] <- x[1]-1
    h[2] <- 1-x[1]
    h[3] <- x[1]^2+x[2]^2-1
    h[4] <- 1-x[1]^2-x[2]^2
    return(h)
}
opt1 <-cobyla(c(0,-0.0000001,-0.0000001), f1, lower = NULL, upper = NULL,hin = f2)
opt1$par
```

```
## [1]  1.000000e+00 -7.211365e-08 -1.276975e-07

opt1$value

## [1] 0.9999999
```

3.2) Solve

$$\begin{cases} \min\limits_{x_1, x_2} x_1^2 + 2x_2^2 \\ subject\ to\ x_1 + x_2 = 1 \end{cases}$$

And use optimizer to verify your answer.

Define Lagrangian function of problem.

$$L(x_1, x_2, \lambda) = x_1^2 + 2x_2^2 + \lambda(1 - x_1 - x_2)$$

Take first partial derivative and set equal to zero.

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - x_1 - x_2 = 0$$

We have that $x_1 = \frac{2}{3}, x_2 = \frac{1}{3}, \lambda = \frac{4}{3}$

Then substitute values, we have: $f\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{4}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}$.

Then consider Hessian matrix of f$(x_1, x_2) = x_1^2 + 2x_2^2$

$$H(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

We have that $\lambda = 4, 2$. This means that the eigenvalues of Hessian are strictly positive.

Since hessian matrix is positive definite, f(x) is a convex function and constraint function is linear, the solution $(x_1, x_2, \lambda) = \left(\frac{2}{3}, \frac{1}{3}, \frac{4}{3}\right)$ minimizes f(x_1, x_2)= f$\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3}$ subject to the constraints.

```
f1 <- function(x){
    x[1]^2+2*x[2]^2
}
f2 <- function(x){
    h <- numeric(2)
    h[1] <- x[1]+x[2]-1
    h[2] <- 1-x[1]-x[2]
    return(h)
}
opt2 <- cobyla(c(1,1), f1, lower = NULL, upper = NULL,hin = f2)
opt2$par

## [1] 0.6666676 0.3333324

opt2$value

## [1] 0.6666667
```

3.3) Suppose you want to place a book of orders to buy or sell stocks using 2 brokers Broker A and Broker B. The fee charged by Broker A is ax^2 per share x and the fee charged by Broker B is by^2 per share y, where $a > 0$ and $b > 0$ are given. If you want to place an order for a total of Q shares using both brokers, with an objective of minimizing the cost, how can you slip optimally Q into x and y? What happens to the optimal cost if Q increases by r%?

Set objective function:

$$\begin{cases} \min\limits_{x,y} ax^2 + by^2 \,; a > 0, b > 0 \\ \quad subject\ to\ x + y = Q \end{cases}$$

Define Lagrangian function of problem.

$$L(x, \mathrm{y}, \lambda) = ax^2 + by^2 + \lambda(Q - x - y)$$

Take first partial derivative and set equal to zero.

$$\frac{\partial L}{\partial x} = 2\mathrm{a}x - \lambda = 0$$

$$\frac{\partial L}{\partial y} = 2by - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = Q - x - y = 0$$

We have that $x = \frac{Qb}{(a+b)}, y = \frac{Qa}{(a+b)}, \lambda = \frac{2Qab}{(a+b)}$

Then substitute values, we have: $f(\frac{Qb}{(a+b)}, \frac{Qa}{(a+b)}) = \frac{Q^2ab}{(a+b)}$ $meaing\ the\ cost\ equal\ to\ \frac{Q^2ab}{(a+b)}$.

Then consider Hessian matrix of $\mathrm{f}(x, \mathrm{y}) = ax^2 + by^2$

$$H(x, \mathrm{y}) = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix} ; where\ a > 0, b > 0$$

We have that $\lambda = 2a, 2b$. This means that the eigenvalues of Hessian are strictly positive.

Since hessian matrix is positive definite, f(x) is a strictly convex function and constraint function is linear, the solution $(x, y, \lambda) = (\frac{Qb}{(a+b)}, \frac{Qa}{(a+b)}, \frac{2Qab}{(a+b)})$ minimizes $\mathrm{f}(x, \mathrm{y}) = \mathrm{f}(\frac{Qb}{(a+b)}, \frac{Qa}{(a+b)}) = \frac{Q^2ab}{(a+b)}$ subject to the constraints.

If Q increases by r%, $(1+r)Q$, the minimum will get changed.

$$rQ\lambda = rQ * \frac{2Qab}{(a+b)} = \frac{2Q^2abr}{(a\ +\ b)}$$

$$\lambda[(1+r)Q - x - y] = \frac{2Qab}{(a+b)}\left[(1+r)Q - \frac{Qb}{(a\ +\ b)} - \frac{Qa}{(a\ +\ b)}\right]$$

$$= \frac{2Qab}{(a+b)}\left[\frac{2Qar + 2Qbr}{(a+b)}\right] = \frac{2Q^2abr}{(a\ +\ b)}$$

Hence, the minimum cost is increased by

$$\frac{\frac{2Q^2abr}{(a\ +\ b)}}{\frac{Q^2ab}{(a\ +\ b)}} = 2r\%$$

Question 4: Optimization with Inequality Constraints

4.1) Solve

$$\begin{cases} \max\limits_{x_1, x_2} \boldsymbol{x^3 - 3x} \\ \boldsymbol{subject\ to\ x \leq 2} \end{cases}$$

And use an optimizer to verify your answer.

Define Lagrangian function of problem.

$$L(x, \lambda) = x^3 - 3x + \mu(2 - x)$$

Take first partial derivative and set equal to zero.

$$\frac{\partial L}{\partial x} = 3x^2 - 3 - \mu = 0$$

$$\frac{\partial L}{\partial \mu} = 2 - x = 0$$

With $x \leq 2, \mu(2 - x) = 0, \mu \geq 0$

Consider cases depending on the complementarity conditions;

If $\mu = 0, then$

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$(x - 1)(x + 1) = 0;\ x = 1, -1$$

Since 1, -1 < 2, they are feasible. f(1) = -2, f(-1) = 2.

If x = 2, then

$$3(2)^2 - 3 - \mu = 0$$

$$\mu = 9$$

Then $L(2, 9) = (2)^3 - 6 - 9(2 - 2) = 2$

Since f(-1) and f(2) = 2, we have 2 solutions to maximize the function which x = -1 and x = 2.

```
f1<-function(x){
    -x^3 +3*x
}
f2<-function(x){
    2 - x
}
opt3<-cobyla(c(1), f1, lower = NULL, upper = NULL,hin = f2)
opt3$par

## [1] 2

opt3$value

## [1] -2
```

4.2) Solve

$$\begin{cases} \min_{x_1, x_2, x_3} (x_1 - 2)^2 + 2(x_2 - 1)^2 \\ \text{subject to} \quad \begin{matrix} x_1, \; x_2, \; x_3 \geq 0 \\ 2x_1 + 2x_2 + 4x_3 \leq a \end{matrix} \end{cases}$$

Define Lagrangian function of problem.

$$L(x_1, x_2, \mu) = (x_1 - 2)^2 + 2(x_2 - 1)^2 + \mu(a - 2x_1 - 2x_2 - 4x_3)$$

Take first partial derivative and set equal to zero.

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 2) - 2\mu = 0$$

$$\frac{\partial L}{\partial x_2} = 4(x_2 - 1) - 2\mu = 0$$

$$\frac{\partial L}{\partial \mu} = (a - 2x_1 - 2x_2 - 4x_3) = 0$$

With $x_1, \; x_2, \; x_3 \geq 0, \mu \geq 0$

Consider cases depending on the complementarity conditions;

If $\mu = 0, then$ we have $x_1 = 2, x_2 = 1$

Since $1, 2 > 0$, they are feasible. Then, we solve that $x_3 \leq \frac{a-6}{4}$.

From $x_3 \geq 0$, then $\frac{a-6}{4} \geq 0 \; and \; a \geq 6 \; with \; minimun \; of \; f(x_1, x_2) = 0$.

Hence, the solution is $x_1 = 2, x_2 = 1, x_3 \leq \frac{a-6}{4}$.

Question 5: Mean-Variance Optimization

Consider an Investment Universe made of 3 stocks S1, S2 and S3 with the following characteristics:

$$\Sigma = \begin{pmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{pmatrix}, \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 4.27 \\ 0.15 \\ 2.85 \end{pmatrix}$$

5.1) Find the Global Minimal Variance Portfolio. Find the Minimum Variance Portfolio (P1) with Expected Return equal to $\boldsymbol{\rho_1}$:

Define $\mathrm{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then calculate:

$$\mathrm{A} = e^T \Sigma^{-1} \rho = [1 \quad 1 \quad 1] \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 4.27 \\ 0.15 \\ 2.85 \end{bmatrix} = 467.6416$$

$$B = \rho^T \Sigma^{-1} \rho = [4.27 \quad 0.15 \quad 2.85] \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 4.27 \\ 0.15 \\ 2.85 \end{bmatrix} = 2225.294$$

$$C = e^T \Sigma^{-1} e = [1 \quad 1 \quad 1] \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 185.1305$$

$$D = BC - A^2 = 2225.294 * 185.1305 - 467.6416^2 = 193281.1$$

$$\mathrm{g} = \frac{1}{D}(B\Sigma^{-1}e - A\Sigma^{-1}\rho) = \begin{bmatrix} -0.0917 \\ 1.00733 \\ 0.08437 \end{bmatrix}$$

$$h = \frac{1}{D}(C\Sigma^{-1}\rho - A\Sigma^{-1}e) = \begin{bmatrix} 0.21424 \\ -0.2577 \\ 0.0435 \end{bmatrix}$$

Find expectation:

$$E[r_p] = \frac{A}{C} = \frac{467.46}{185.1305} = 2.526$$

Find optimal weight:

$$w_p = g + hE[\mathrm{r}_p] = \begin{bmatrix} -0.0917 \\ 1.00733 \\ 0.08437 \end{bmatrix} + \begin{bmatrix} 0.21424 \\ -0.2577 \\ 0.0435 \end{bmatrix} * 2.526 = \begin{bmatrix} 0.4495 \\ 0.3464 \\ 0.1941 \end{bmatrix}$$

$$\sigma_p^2 = w_p^T \Sigma w_p = [0.4495 \quad 0.3461 \quad 0.1941] \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 0.4495 \\ 0.3464 \\ 0.1941 \end{bmatrix} = 0.005401596$$

Hence, the standard deviation is:

$$\sigma_p = \sqrt{0.005401596} = 0.07349555$$

Change the expectation of return to 4.27%:

$$E[r_p] = 4.27\%$$

Compute new optimal weight

$$w_p = g + hE[\mathrm{r}_p] = \begin{bmatrix} -0.0917 \\ 1.00733 \\ 0.08437 \end{bmatrix} + \begin{bmatrix} 0.21424 \\ -0.2577 \\ 0.0435 \end{bmatrix} * 4.27 = \begin{bmatrix} 0.8230969 \\ -0.0930379 \\ 0.2699410 \end{bmatrix}$$

$$\sigma_p^2 = w_p^T \Sigma w_p = [0.8230969 \quad -0.0930379 \quad 0.2699410] \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 0.8230969 \\ -0.0930379 \\ 0.2699410 \end{bmatrix} = 0.008314835$$

Hence, the standard deviation is:

$$\sigma_p = \sqrt{0.008314835} = 0.09118572$$

Hence the minimum variance portfolio (P1) has

$$w_p = \begin{bmatrix} 0.8230969 \\ -0.0930379 \\ 0.2699410 \end{bmatrix}, \sigma_p = 0.09118572$$

5.2) Find the Minimum Variance Portfolio (P2) with Expected Return equal to $\rho_2 + \rho_3$:

Change the expectation of return to $(0.15 + 2.85) = 3\%$:

$$E[r_p] = 3\%$$

Compute new optimal weight

$$w_p = g + hE[\mathrm{r}_p] = \begin{bmatrix} -0.0917 \\ 1.00733 \\ 0.08437 \end{bmatrix} + \begin{bmatrix} 0.21424 \\ -0.2577 \\ 0.0435 \end{bmatrix} * 3 = \begin{bmatrix} 0.55102 \\ 0.23423 \\ 0.21487 \end{bmatrix}$$

$$\sigma_p^2 = w_p^T \Sigma w_p = [0.55102 \quad 0.23423 \quad 0.21487] \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 0.55102 \\ 0.23423 \\ 0.21487 \end{bmatrix} = 0.00561814$$

Hence, the standard deviation is:

$$\sigma_p = \sqrt{0.00561814} = 0.07495428$$

Hence the minimum variance portfolio (P2) has

$$w_p = \begin{bmatrix} 0.55102 \\ 0.23423 \\ 0.21487 \end{bmatrix}, \sigma_p = 0.07495428$$

5.3) Using the Portfolios (P1) and (P2) previously found, apply the Two-fund Theorem to find the Minimum Variance Portfolio with Expected Return equal to $\frac{\rho_1+\rho_2+\rho_3}{3}$:

Use two fund theorem to $(4.27+0.15+2.85)/3 = 2.4233\%$:

$$E[r_q] = 2.4233\%$$

From

$$E[r_q] = \alpha\left[E[r_{p1}]\right] + (1-\alpha)\left[E[r_{p2}]\right]$$

$$2.4233 = \alpha[4.27] + (1-\alpha)3$$

$$\alpha = -0.454094488$$

Compute new optimal weight

$$w_p = g + hE[r_p] = \begin{bmatrix} -0.0917 \\ 1.00733 \\ 0.08437 \end{bmatrix} + \begin{bmatrix} 0.21424 \\ -0.2577 \\ 0.0435 \end{bmatrix} * 2.4233 = \begin{bmatrix} 0.4274678 \\ 0.3828456 \\ 0.1897835 \end{bmatrix}$$

or

$$w_p = \alpha w_{p1} + (1-\alpha) w_{p2}$$

$$= -0.454094488 * \begin{bmatrix} 0.8230969 \\ -0.0930379 \\ 0.2699410 \end{bmatrix} + (1+0.454094488)\begin{bmatrix} 0.55102 \\ 0.23423 \\ 0.21487 \end{bmatrix} = \begin{bmatrix} 0.4274678 \\ 0.3828456 \\ 0.1897835 \end{bmatrix}$$

$$\sigma_p^2 = w_p^T \Sigma w_p = \begin{bmatrix} 0.4274678 & 0.3828456 & 0.1897835 \end{bmatrix} \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.011 & 0.003 \\ 0.001 & 0.003 & 0.02 \end{bmatrix} \begin{bmatrix} 0.4274678 \\ 0.3828456 \\ 0.1897835 \end{bmatrix} = 0.005411694$$

Hence, the standard deviation is:

$$\sigma_p = \sqrt{0.005411694} = 0.07356422$$

Hence the minimum variance portfolio (P2) has

$$w_p = \begin{bmatrix} 0.4274678 \\ 0.3828456 \\ 0.1897835 \end{bmatrix}, \sigma_p = 0.07356422$$

5.4) Write a computer program in R, Matlab or Python to solve the 2 previous questions. Apply the Two-fund Theorem to generate and plot the Mean-Variance efficient frontier (the graph should also display the Expected Returns and Volatilities of securities S1, S2 and S3).

For 5.1,

Find Global Min Variance

```
Sig <- matrix(c(0.01,0.002,0.001,0.002,0.011,0.003,0.001,0.003,0.02),ncol=3)
e <- matrix(c(1,1,1),nrow=3)
rho <- matrix(c(4.27,0.15,2.85),ncol=1)

GetGMinV <- function(Sig,e,rho){
```

```
    A <- as.numeric(t(e)%*%solve(Sig)%*%rho)
    B <- as.numeric(t(rho)%*%solve(Sig)%*%rho)
    C <- as.numeric(t(e)%*%solve(Sig)%*%e)
    D <- B*C-A^2
    g <- 1/D*(B*solve(Sig)%*%e-A*solve(Sig)%*%rho)
    h <- 1/D*(C*solve(Sig)%*%rho-A*solve(Sig)%*%e)
    Erp <- A/C
    wp <- g+h*Erp
    VarP <- t(wp)%*%Sig%*%wp
    SigP <- sqrt(VarP)

    return(list(weight = wp,sd = SigP))
}
GetGMinV(Sig,e,rho)

## $weight
##           [,1]
## [1,] 0.4494681
## [2,] 0.3563830
## [3,] 0.1941489
##
## $sd
##            [,1]
## [1,] 0.07349555
```

By using optimization function,

```
library(quadprog)
dvec <- c(0,0,0)
bvec <- c(1,1.5)
Amat <- matrix(c(1,1,1,4.27,0.154,2.85),3,2,byrow=FALSE)
solve.QP(Dmat=Sig,dvec=dvec,bvec=bvec,Amat=Amat,meq=2)

## $solution
## [1] 0.2290865 0.6214031 0.1495103
##
## $value
## [1] 0.003207396
##
## $unconstrained.solution
## [1] 0 0 0
##
## $iterations
## [1] 3 0
##
## $Lagrangian
## [1] 0.0078940044 0.0009861411
##
## $iact
## [1] 1 2
```

Find minimum variance using analytics solution.

```
GetminV <- function(Sig,e,rho,Erp){

    A <- as.numeric(t(e)%*%solve(Sig)%*%rho)
    B <- as.numeric(t(rho)%*%solve(Sig)%*%rho)
    C <- as.numeric(t(e)%*%solve(Sig)%*%e)
    D <- B*C-A^2
    g <- 1/D*(B*solve(Sig)%*%e-A*solve(Sig)%*%rho)
    h <- 1/D*(C*solve(Sig)%*%rho-A*solve(Sig)%*%e)
    wp <- g+h*Erp
    VarP <- t(wp)%*%Sig%*%wp
    SigP <- sqrt(VarP)
    return(list(weight = wp,sd = SigP))
}
P1 <- GetminV(Sig,e,rho,4.27)
P1
```

```
## $weight
##             [,1]
## [1,]  0.8230969
## [2,] -0.0930379
## [3,]  0.2699410
##
## $sd
##             [,1]
## [1,] 0.09118572
```

For 5.2,

Using optimization function,

```
library(quadprog)
dvec <- c(0,0,0)
bvec <- c(1,3)
Amat <- matrix(c(1,1,1,4.27,0.154,2.85),3,2,byrow=FALSE)
solve.QP(Dmat=Sig,dvec=dvec,bvec=bvec,Amat=Amat,meq=2)
```

```
## $solution
## [1] 0.5508314 0.2344884 0.2146802
##
## $value
## [1] 0.002807968
##
## $unconstrained.solution
## [1] 0 0 0
##
## $iterations
## [1] 3 0
##
## $Lagrangian
## [1] 0.0042552256 0.0004535704
##
## $iact
## [1] 2 1
```

Use analytics solution.

```
P2 <- GetminV(Sig,e,rho,(0.15+2.85))
P2
```

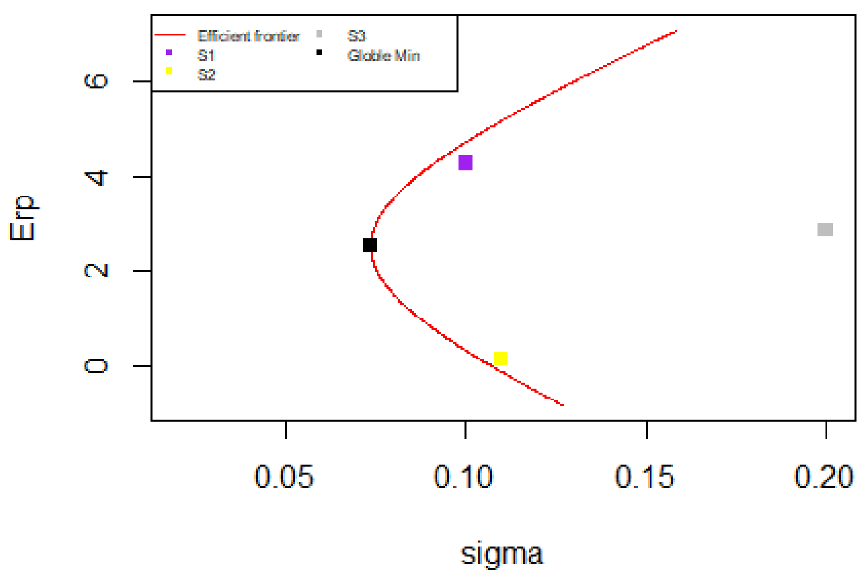
```
## $weight
##           [,1]
## [1,] 0.5510146
## [2,] 0.2342373
## [3,] 0.2147480
##
## $sd
##             [,1]
## [1,] 0.07494523
```

For 5.3,

```
P3 <- GetminV(Sig,e,rho,(4.27+0.15+2.85)/3)
P3
```

```
## $weight
##           [,1]
## [1,] 0.4274707
## [2,] 0.3828426
## [3,] 0.1896867
##
## $sd
```

```
##              [,1]
## [1,] 0.07356422

ran <- seq(from = -3, to = 3.2, by = 0.01)
w1 <- as.vector(P1$weight)
w2 <- as.vector(P2$weight)
sig <- matrix(c(0.01,0.002,0.001,0.002,0.011,0.003,0.001,0.003,0.02),3,3,byrow = T)
Rho <- c(4.27,0.15,2.85)

Erp <- NULL
sigma <- NULL

for(i in 1:length(ran)){
    wp <- ran[i]*w1+(1-ran[i])*w2
    Erp[i] <- t(wp)%*%Rho
    sigma[i] <- sqrt(t(wp)%*%sig%*%wp)
}
plot(sigma,Erp,type = 'l', xlim = c(0.02, 0.201), col="red")
points(y=4.27, x=0.10,pch = 15,col="purple")
points(y=0.15, x=0.11,pch = 15,col="yellow")
points(y=2.85, x=0.20,pch = 15,col="gray")
points(y=2.526, x=0.07349555,pch=15,col="black")
legend("topleft", c("Efficient frontier","S1","S2","S3","Globle Min"),
                 col=c("red","purple","yellow","gray", "black"), lty = c(1,NA,NA,NA,NA),
                 pch = c(NA,15,15,15,15), ncol = 2, cex=0.5)
```



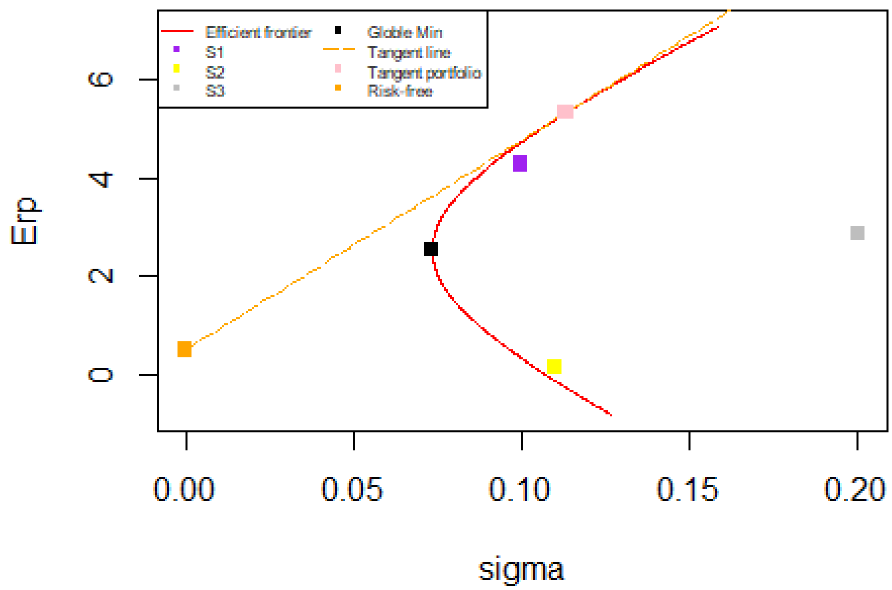
5.5) Assume now that we add a Riskless Asset S0 with return $\rho_0 = 0.5\%$.

a) Find the Tangent Portfolio, P_T.

```
rf <- 0.5
sharp <- (Erp-rf)/sigma
tangent <- max(sharp)
tangent

## [1] 42.47276

signal <- which(sharp==tangent)
signal

## [1] 483

dr <- (Erp[signal]-rf)/100
ds <- (sigma[signal]-0)/100
V <- matrix(c(0.010,0.002,0.001,4.27,1,0.002,0.011,0.003,0.15,1,0.001,
              0.003,0.020,2.85,1,4.27,0.15,2.85,0,0,1,1,1,0,0),5,5,byrow = T)
S <- matrix(c(0,0,0,Erp[signal],1),ncol = 1, nrow =5)
Pt <- solve(V) %*% S
pt <- Pt[-4:-5]
pt

## [1]  1.0462044 -0.3614036  0.3151991

A=as.numeric(t(e)%*%solve(sig)%*%Rho)
B=as.numeric(t(Rho)%*%solve(sig)%*%Rho)
C=as.numeric(t(e)%*%solve(sig)%*%e)
D=B*C-A^2
g=1/D*(B*solve(sig)%*%e-A*solve(sig)%*%Rho)
h=1/D*(C*solve(sig)%*%rho-A*solve(sig)%*%e)
H=as.numeric(t(Rho-rf*e)%*%solve(sig)%*%(Rho-rf*e))
rho_T=A/C-D/C^2/(rf-A/C)
mean <- rho_T
mean

## [1] 5.309515

st<-sqrt(t(pt)%*%Sig%*%pt)
st

##          [,1]
## [1,] 0.113282
```

b) Add the new Efficient Frontier to the graph generated in question 4. Does that efficient frontier intersect with the one obtained with risky asset only? Explain why.

The new efficient frontier intersects the former efficient frontier at tangent portfolio because tangent line or capital market line is created by risk-free rate and the intercept point of former efficient frontier where expected return on a holding equals to the risk-free rate of return.




```
rf <- 0.5
dr <- (Erp[signal]-rf)/100
ds <- (sigma[signal]-0)/100
tr <- seq(rf,Erp[signal]+200*dr,dr)
ts <- seq(0,sigma[signal]+200*ds,ds)

plot(sigma,Erp,type = 'l', xlim = c(0, 0.201), col="red")
points(y=4.27, x=0.10,pch = 15,col="purple")
points(y=0.15, x=0.11,pch = 15,col="yellow")
points(y=2.85, x=0.20,pch = 15,col="gray")
points(y=2.526, x=0.07349555,pch=15,col="black")
lines(x=ts,y=tr,col="orange",lty=5)
points(y=5.3114, x=0.113282,pch=15,col="pink")
points(y=0.5, x=0,pch=15,col="orange")

legend("topleft", c("Efficient frontier","S1","S2","S3","Globle Min", "Tangent line", "Tangent portfolio", "Risk
-free"),
                  col=c("red","purple","yellow","gray", "black", "orange", "pink","orange"), lty = c(1,NA,NA,NA,N
A,5,NA,NA),
                  pch = c(NA,15,15,15,15,NA,15,15), ncol = 2, cex=0.5)
```

c) Using the One-fund Theorem, find the Efficient Portfolio (P3) with target Expected Return equal to 7%.

Combine between risky and risk-free asset.

$$\alpha * \rho_T + (1-\alpha)\rho_0 = \rho_{p3}$$

$$\alpha * 5.3095151 + (1-\alpha)0.5 = 7$$

We get

$$\alpha = 1.35$$

Find the optimal weight

$$w_{p3} = 1.35 * \begin{bmatrix} 1.0462044 \\ -0.3614036 \\ 0.3151991 \end{bmatrix} = \begin{bmatrix} 1.4139323 \\ -0.4884325 \\ 0.4259877 \end{bmatrix}$$

Calculate Variance of portfolio

$$\sigma_{p3}^2 = w_{p3}^T \Sigma w_{p3} = 0.02354876, \sigma_{p3} = 0.153456$$

```r
rho_q <- 7
rho_0 <- 0.5

alpha <- (rho_q-rho_0)/(rho_T-rho_0)

w_3 <- alpha*matrix(pt, nrow = 3)

VarP3 <- t(w_3)%*%Sig%*%w_3
SigP3 <- sqrt(VarP3)
SigP3

##           [,1]
## [1,] 0.153456
```

d) Using the One-fund Theorem, find the Efficient Portfolio (P4) with target Expected Return equal to 2%.

$$\alpha^2 VarP_T = Var[P_4] = 0.02$$

$$\alpha = \sqrt{\frac{0.02}{(0.113282)^2}} = \sqrt{\frac{0.02}{0.01283282}} = 1.248401$$

Calculate optimal weight

$$\rho_{P4} = \alpha * \rho_T + (1-\alpha)\rho_0 = 1.248401 * 5.3095151 + (1 - 1.248401) * 0.5 = 6.5\%$$

```r
VarP4 <- 0.02

alpha <- sqrt(VarP4/(st^2))

rho_p4 <- alpha*rho_T+(1-alpha)*rho_0
rho_p4

##          [,1]
## [1,] 6.504202
```

e)

```r
plot(sigma,Erp,type = 'l', xlim = c(0, 0.201), col="red")
points(y=4.27, x=0.10,pch = 15,col="purple")
points(y=0.15, x=0.11,pch = 15,col="yellow")
points(y=2.85, x=0.20,pch = 15,col="gray")
points(y=2.526, x=0.07349555,pch=15,col="black")
lines(x=ts,y=tr,col="orange",lty=5)
points(y=5.3114, x=0.113282,pch=15,col="pink")
points(y=0.5, x=0,pch=15,col="orange")

points(y=4.27, x=0.09118572,pch = 2,col="red")
points(y=3, x=0.07495428,pch = 2,col="blue")
```

```
points(y=2.433, x=0.07356422,pch = 2,col="green")
points(y=7, x=0.1530993,pch=2,col="yellow")
points(y=6.5, x=sqrt(0.02),pch=2,col="purple")

legend("topleft", c("Efficient frontier","S1","S2","S3","Globle Min", "Tangent line", "Tangent portfolio", "Risk
-free", "P1", "P2", "P3", "P4", "P5"),
                  col=c("red","purple","yellow","gray", "black", "orange", "pink","orange", "red", "blue", "green
", "yellow", "purple"), lty = c(1,NA,NA,NA,NA,5,NA,NA,NA,NA,NA,NA,NA),
                  pch = c(NA,15,15,15,15,NA,15,15,2,2,2,2,2), ncol = 2, cex=0.5)
```